UNKNOTTING VIA NULL-HOMOLOGOUS TWISTS
AND MULTITWISTS

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This collaborative work was based on the proposed problem and prior work of our senior member, Kenan İnce. After several years of work and the writing and submission of this paper, Kenan passed away unexpectedly. They were an exceptional human, mathematician, and advocate. We dedicate this paper to them.

The *untwisting number* of a knot K is the minimum number of null-homologous twists required to convert K to the unknot. Such a twist can be viewed as a generalization of a crossing change, since a classical crossing change can be effected by a null-homologous twist on 2 strands. While the unknotting number gives an upper bound on the smooth 4-genus, the untwisting number gives an upper bound on the topological 4-genus. The *surgery description number*, which allows multiple null-homologous twists in a single twisting region to count as one operation, lies between the topological 4-genus and the untwisting number. We show that the untwisting and surgery description numbers are different for infinitely many knots, though we also find that the untwisting number is at most twice the surgery description number plus 1.

1. Introduction

Given two knot diagrams D_1, D_2 of knots K_1, K_2 which differ only inside small disks $\Delta \subset D_1$, $\Delta' \subset D_2$ containing at least one crossing, a *local move on K_1* is the act of replacing Δ with Δ', and hence converting D_1 to D_2. An *unknotting operation* is a local move such that, for any diagram D of a knot K, we may transform D into a diagram of the unknot via a finite sequence of these local moves. A natural question in knot theory is: given an unknotting operation and a knot K, how many such operations are needed to turn K into the unknot? The most common such unknotting operation is a crossing change, which gives rise to the unknotting number $u(K)$. While the unknotting number is quite simple to define, its computation is frequently difficult. For example, Milnor's conjecture about the unknotting number of torus knots was only proven about 25 years later
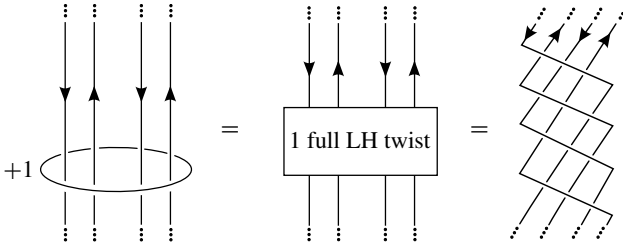
Figure 1. A left-handed null-homologous twist on 4 strands.

by Kronheimer and Mrowka [1993; 1995]. In this paper, we study additional unknotting operations, many of which are generalizations of the crossing change. See Table 1 for an organized list of the related invariants.

One of the primary unknotting operations studied in this paper is a *null-homologous twist*. Mathieu and Domergue [1988] defined this generalization of unknotting number and it was subsequently considered by Livingston [2002]. A null-homologous twist on a knot K is the result of a ± 1-surgery on a null-homologous unknot $U \subset S^3 - K$ bounding a disk D such that $D \cap K = 2k$ points for any $k \in \mathbb{Z}_{\geq 1}$. Diagrammatically, this is the result of adding a full right- or left-handed twist in the twisting region indicated by the unknot U, where $\text{lk}(K, U) = 0$. See Figure 1 for a diagrammatic representation. It is described in [Ince 2016] how a crossing change may be encoded as a null-homologous twist where $D \cap K = 2$. In particular, this implies null-homologous twists are unknotting operations.

The corresponding knot invariant is the *untwisting number* $\text{tu}(K)$, which is defined as the minimum length, taken over all diagrams of K, of a sequence of null-homologous twists beginning at K and resulting in the unknot. This has been the subject of much research in recent years [Baader et al. 2020; Ince 2016; 2017; Livingston 2021; McCoy 2021a; 2021b].

There are many variations of the unknotting number and untwisting number, see Table 1. One variant we will study, due to Nakanishi [2005] (and called the "surgical description number" in that paper), is what we and many other authors call the *surgery description number* $\text{sd}(K)$ of a knot. Again we consider null-homologous twists but now allow any number of full twists to be added in the twisting region; we may call this a *null-homologous m-twist* for $m \in \mathbb{Z}$ to specify the number of twists (with sign) being effected. Then $\text{sd}(K)$ is the minimal number of m-twists necessary to unknot K. (Here, the value of m may change from move to move.)

Another natural variant (due to Murakami [1990]) is the *algebraic unknotting number* $u_a(K)$, the minimum number of crossing changes necessary to turn a given knot into an Alexander polynomial-one knot. Freedman [1982] showed that knots with Alexander polynomial equal to one are topologically slice (in other words, with topological 4-genus $g_4^{\text{top}} = 0$); topologically slice knots are indistinguishable

invariant	definition
$u(K)$	unknotting number of K, i.e., minimal number of crossing changes to unknot
$u_a(K)$	alg. unknotting number, minimal number of crossing changes to Alexander polynomial-one knot
$\mathrm{tu}_a(K)$	alg. untwisting number, minimal number of null-homologous twists to Alexander polynomial-one knot
$\mathrm{tu}(K)$	untwisting number, i.e., the minimal number of null-homologous twists to unknot
$\mathrm{sd}(K)$	surgery description number, i.e., the minimal number of null-homologous multitwists (on the same region of K) to unknot
$\mathrm{sd}_a(K)$	algebraic surgery description number, i.e., the minimal number of null-homologous multitwists (on the same region of K) to Alexander polynomial-one knot
$g_{\mathrm{alg}}(K)$	algebraic genus, i.e., minimal difference in genus between a Seifert surface F for K and a subsurface whose boundary is an Alexander polynomial-one knot

Table 1. Overview of knot invariants appearing in this paper.

from the unknot by *classical invariants*, or knot invariants derived from the Seifert matrix. We consider the similarly defined *algebraic untwisting number* $\mathrm{tu}_a(K)$ and *algebraic surgery description number* $\mathrm{sd}_a(K)$, measuring the number of null-homologous twists or m-twists, respectively, needed to obtain a knot with Alexander polynomial-one, as well.

A tight classical upper bound on the topological 4-genus g_4^{top} of a knot is the *algebraic genus* g_{alg} defined in [Feller and Lewark 2018]. Distinguishing the algebraic genus from other upper bounds on g_4^{top}, such as the algebraic unknotting number, is often achieved by using the bound $g_{\mathrm{alg}} \leq g_3$, where $g_3(K)$ is the 3-genus of K. In Section 3, we provide the first (to our knowledge) known infinite family of knots L_n for which $g_{\mathrm{alg}}(L_n) < u_a(L_n)$ for all $n \in \mathbb{N}$, and since the 3-genus of our examples is large, we do so without using g_3.

The untwisting number connects to recent work of Manolescu and Piccirillo [2023] on candidates for exotic definite 4-manifolds, which uses the concept of *strong H-sliceness* in definite connected sums of $\pm\mathbb{CP}^2$. (See Section 3 for a related definition.) It follows from Proposition 4.1 of [Ince 2017] that, if K can be unknotted using n positive (respectively, negative) nullhomologous twists, then K is strongly topologically H-slice in $X := B^4 \#^n \mp \mathbb{CP}^2 \cong \#^n \mp \mathbb{CP}^2$. We use this fact to obstruct knots from having $\mathrm{sd}_a = 1$ in Section 3.

Results. Our main results involve various relationships between the untwisting number and the surgery description number. To start, we give the first known examples (to the authors' knowledge) such that sd $\neq$ tu. See Section 4 for a description of these knots.

Theorem 1.1. *There are infinitely many knots* $\{K_n\}$ *with* $\mathrm{sd}(K_n) = 1$ *and* $\mathrm{tu}(K_n) = 2$.

This, of course, leads to questions about how far apart the surgery description number and the untwisting number can be.

Question 1.2. Can tu and sd be arbitrarily far apart?

Answering such a question is made more difficult by the close relationships between tu and sd both in definition and in values, demonstrated by the two results:

Theorem 1.3. *Let* $K \subset S^3$ *be a knot. Then* $\mathrm{sd}_a(K) \leq \mathrm{tu}_a(K) \leq 2\,\mathrm{sd}_a(K)$.

Theorem 1.4. *Let* $K \subset S^3$ *be a knot. Then* $\mathrm{sd}(K) \leq \mathrm{tu}(K) \leq 2\,\mathrm{sd}(K) + 1$.

The proof of Theorem 1.3 relies on the work of Duncan McCoy [2021b] relating the untwisting number to the algebraic genus. The proof of Theorem 1.4 is constructive (involving surgery diagrams and Kirby calculus) and allows one to reduce multiple twists in a single region to at most 3 twists in separate regions.

Organization. In Section 2, we give formal definitions of all relevant invariants, as well as some useful prior results. We also prove Theorem 1.3 as a consequence of [McCoy 2021b]. We prove Theorem 1.1 in Section 4 by providing an infinite family of examples where the invariants disagree. Theorem 1.4 is proved in Section 5.

2. Algebraic untwisting invariants

One way to study an unknotting operation is to analyze its impact on the Alexander polynomial of a knot. The effect of an operation on the Alexander polynomial gives rise to *algebraic unknotting operations*:

Definition 2.1. Given an unknotting operation $\mathcal{U}$ and a knot K, the *algebraic $\mathcal{U}$-number* $\mathcal{U}_a(K)$ is the minimal number of $\mathcal{U}$-operations that must be performed in order to convert K into a knot with Alexander polynomial-one.

We certainly have that $\mathcal{U}_a(K) \leq \mathcal{U}(K)$ for any unknotting operation $\mathcal{U}$ and knot K. A lower bound on the algebraic unknotting and untwisting numbers is the topological 4-genus. Another (typically tighter) upper bound on the topological 4-genus is the *algebraic genus*, defined by Feller and Lewark [2018].

Definition 2.2. The *algebraic genus* $g_{\mathrm{alg}}(K)$ of a knot K is the minimum difference in genus $g(F) - g(F')$ between a Seifert surface F for K and a subsurface $F' \subset F$ with the property that $\partial F' = K'$ is a knot with $\Delta_{K'}(t) = 1$.

We note that Definition 2.2 implies that a knot K has $g_{\text{alg}}(K) = 0$ if and only if $\Delta_K(t) = 1$. McCoy proves the following useful characterization of the sensitivity of the algebraic genus to null-homologous twisting.

Theorem 2.3 [McCoy 2021b, Theorem 1.1]. *If K and K' are knots and $m, n \in \mathbb{Z}$ are such that a null-homologous m-twist followed by a null-homologous n-twist on K results in K' and $-mn$ is a square, then*

$$|g_{\text{alg}}(K) - g_{\text{alg}}(K')| \leq 1.$$

Proposition 2.4 [McCoy 2021b, Proposition 3.1]. *Given a knot K with $g_{\text{alg}}(K) > 0$, there exists a knot K' with $g_{\text{alg}}(K') = g_{\text{alg}}(K) - 1$ such that K can be obtained from K' by one right-handed and one left-handed null-homologous twist.*

Feller and Lewark [2018] show that for a knot K the algebraic genus and the algebraic unknotting number are related by $g_{\text{alg}}(K) \leq u_a(K) \leq 2g_{\text{alg}}(K)$. We will show that in fact

(2.5) $$g_{\text{alg}}(K) \leq \text{sd}_a(K) \leq u_a(K) \leq 2g_{\text{alg}}(K) \leq 2\,\text{sd}_a(K)$$

and that $\text{sd}_a(K)$ can provide a better lower bound for $u_a(K)$ than $g_{\text{alg}}(K)$. We begin by showing that the algebraic genus is in fact a lower bound on the algebraic surgery description number.

Proposition 2.6. *Let $K \subset S^3$ be a knot. Then $g_{\text{alg}}(K) \leq \text{sd}_a(K)$.*

Proof. Suppose that K is a knot with $\text{sd}_a(K) = k$. Then there exists a sequence of k null-homologous m_i-twists (for $1 \leq i \leq k$) converting K to a knot with Alexander polynomial-one (which by definition has algebraic genus 0). By Theorem 2.3 (with $n = 0$), each of these m_i-twists decreases the algebraic genus by at most 1, whence $g_{\text{alg}}(K) \leq k$. $\square$

Note that in conjunction with the fact that $g_4^{\text{top}} \leq g_{\text{alg}}$ we have that sd_a and sd are upper bounds on the topological 4-genus. Before proving Theorem 1.3, we need to note the following result of İnce.

Theorem 2.7 [İnce 2016, Theorem 1.1]. *Let $K \subset S^3$ be a knot. Then we have $u_a(K) = \text{tu}_a(K)$.*

We are now ready to prove Theorem 1.3: that, $\text{sd}_a(K) \leq \text{tu}_a(K) \leq 2\,\text{sd}_a(K)$ for any knot K.

Proof of Theorem 1.3 and inequality (2.5). Since any single null-homologous twist is an m-twist with $m = \pm 1$, we have $\text{sd}_a(K) \leq \text{tu}_a(K)$ for any knot K. Combining Proposition 2.6 with Feller and Lewark's [2018] result that $u_a(K) \leq 2g_{\text{alg}}(K)$, we have that $u_a(K) \leq 2\,\text{sd}_a(K)$. Theorem 1.3 and inequality (2.5) now follow from Theorem 2.7. $\square$

Note. Borodzik [2019] showed that the minimal number of null-homologous twists *on two strands* needed to convert a knot K into a knot with Alexander polynomial-one is always less than *three* times the algebraic surgery description number. In fact, our Theorem 1.3, together with the fact that a crossing change is a special case of a null-homologous two-strand twist and the fact that $u_a = \text{tu}_a$, refines this upper bound to twice the algebraic surgery description number.

Even though the algebraic unknotting $u_a(K)$ and untwisting numbers $\text{tu}_a(K)$ coincide, the algebraic surgery description number $\text{sd}_a(K)$ can be strictly less than $u_a(K) = \text{tu}_a(K)$; this is the content of Corollary 4.4.

To conclude that the algebraic surgery description number $\text{sd}_a(K)$ can be a better lower bound on the algebraic unknotting number $u_a(K)$ than the algebraic genus $g_{\text{alg}}(K)$, we should show that there is a knot for which $g_{\text{alg}}(K) \neq sd_a(K)$. We provide infinitely many examples with this property in the next section.

3. Infinite families of knots with $g_{\text{alg}} < \text{sd}_a$

A knot $K \subset S^3$ is called *topologically H-slice* in a closed, smooth 4-manifold M if $K \subset \partial(M \setminus B^4)$ bounds a locally flat, properly embedded, null-homologous topological disk in $M \setminus B^4$. In the context of this paper, if a knot K can be converted to a knot which is topologically slice in B^4 via only left-handed or, respectively, only right-handed nullhomologous m-twists, then K is topologically H-slice in $\#_n \pm \mathbb{CP}^2$ for some n. The following proposition is well known and follows from, for instance, [Conway and Nagel 2020, Theorem 3.8]. To interpret their theorem in our setting, consider a knot K (trivially a colored link) which bounds a null-homologous disk D in $\#_m \mathbb{C}P^2$. Here D can be thought of as an annular cobordism from K to the unknot with no double-points.

Proposition 3.1. *If a knot K is topologically H-slice in $\#_m \mathbb{CP}^2$, then for any $\omega \in S^1$ with $\Delta_K(\omega) \neq 0$,*

$$-2m \leq \sigma_K(\omega) \leq 0,$$

where $\sigma_K(\omega)$ is the Levine–Tristram signature function of K.

In particular, the proposition above implies that if the signature function of a knot takes on both positive and negative values, then $\text{sd}_a \neq 1$.

Theorem 3.2. *If K and K' are knots such that*

- $u_a(K) = u_a(K') = 1$,
- *the signature function of K takes a positive value at a nonroot of $\Delta_K(t)$, and*
- *the signature function of K' takes a negative value at a nonroot of $\Delta_{K'}(t)$,*

then $g_{\text{alg}}(K \# K') = 1$. If, in addition, the signature function of $K \# K'$ takes both positive and negative values at a nonroot of $\Delta_{K \# K'}(t)$, then $g_{\text{alg}}(K \# K') < \text{sd}_a(K \# K')$.

Proof. Suppose K and K' are knots which satisfy the assumptions of Theorem 3.2. Now consider the knot $J = K \# K'$. Note that because K and K' have nontrivial signature functions, neither K nor K' has Alexander polynomial-one. So $\Delta_J(t) \neq 1$ and $g_{\mathrm{alg}}(J) \neq 0$. Because $u_a(K) = u_a(K') = 1$, the knots K and K' can be converted into knots with Alexander polynomial-one via a single crossing change. Recall that a crossing change can change the Levine–Tristram signature function by at most ± 2, where the sign depends on the sign of the crossing change (see, for example, [Conway 2021, Proposition 3(1)]). This implies that the crossing changes converting K and K' to knots with trivial Alexander polynomial can be taken to be of opposite signs. So the knot J can be converted into a knot J' with $\Delta_{J'}(t) = 1$ via a sequence of two crossing changes, one positive and one negative. Since a crossing change can be realized by single null-homologous twist, by Theorem 2.3 we have that

$$|g_{\mathrm{alg}}(J) - g_{\mathrm{alg}}(J')| \leq 1.$$

Because $\Delta_{J'}(t) = 1$, we have that $g_{\mathrm{alg}}(J') = 0$. So $g_{\mathrm{alg}}(J) = 1$ as desired.

Now, suppose that K and K' also satisfy that $\sigma_J(\omega) = \sigma_K(\omega) + \sigma_{K'}(\omega)$ takes both positive and negative values. By Proposition 3.1, $J = K \# K'$ is not topologically H-slice in $\#_m \pm \mathbb{CP}^2$ for any $m \in \mathbb{N}$. In particular, J cannot be converted to a topologically slice knot using a single null-homologous m-twist. Thus we have $\mathrm{sd}_a(J) > 1$. □

Theorem 3.3. *There exists an infinite family $\{K_n\}_{n=2}^{\infty}$ of prime knots such that $g_{\mathrm{alg}}(K_n) < \mathrm{sd}_a(K_n)$ for all $n \geq 2$.*

Note. In fact, since the Levine–Tristram signature and algebraic unknotting number of a knot K are invariants of the S-equivalence class of its Seifert matrix, for any Seifert matrices V, V' satisfying the conditions of Theorem 3.2, there exist infinitely many knots K, K' with Seifert matrices in the S-equivalence classes of V, V', respectively, satisfying the conclusions of the theorem. In particular, our K_n can be chosen to have any adjective (e.g., hyperbolic, quasipositive, ...) for which there are infinitely many representative knots with that property in each S-equivalence class, since our proof relies only on the S-equivalence class of K_n.

In the proof below, we exhibit a concrete family of prime knots via cabling because cabling seems to be of independent interest.

Proof. Suppose that K and K' are knots which satisfy all the assumptions of Theorem 3.2. For example, we can take $K = 10_{32}$ and $K' = -10_{82}$ (see Figure 2). For $n \geq 2$, let $K_n := (K \# K')_{n,1}$, the $(n, 1)$-cable of $K \# K'$. Note that the $(n, 1)$-cable of any knot (where $n \geq 2$) is prime by [Cromwell 2004, Theorem 4.4.1]. Then we have that $g_{\mathrm{alg}}(K_n) \neq 0$ because $\Delta_{K_n}(t) = \Delta_{T(n,1)}(t) \cdot \Delta_{K_1 \# K_2}(t^n) \neq 1$ (by [Lickorish 1997, Theorem 6.15] since $g_{\mathrm{alg}}(K_1 \# K_2) \neq 0$). On the other hand,
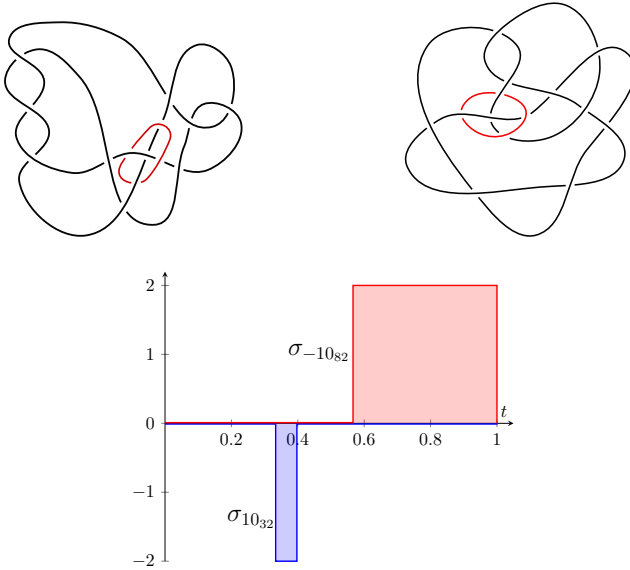
Figure 2. Top: the knots 10_{32} (left) and 10_{82} (right) with red unknots indicating an unknotting crossing change for each. Bottom: the Levine–Tristram signature functions for 10_{32} and -10_{82} (reparametrized so that $\omega = e^{2\pi i t}$).

Feller et al. [2022] tells us how g_{alg} acts under satellite operations. In particular, $g_{\text{alg}}(K_n) \leq g_{\text{alg}}(T(n, 1)) + g_{\text{alg}}(K_1 \# K_2) = 1$, where $T(n, 1)$ denotes the $(n, 1)$-torus knot. So we have that $g_{\text{alg}}(K_n) = 1$.

Also, by [Litherland 1979, Theorem 2], $\sigma_{K_n}(\omega) = \sigma_{T(n,1)}(\omega) + \sigma_{K_1 \# K_2}(\omega^n)$. Since $\sigma_{K_1 \# K_2}(\omega)$ takes both positive and negative values, so does $\sigma_{K_n}(\omega) = \sigma_{K_1 \# K_2}(\omega^n)$. Proposition 3.1 then implies that $\text{sd}_a(K_n) > 1$.   □

We remark that, for any knot K with $g_{\text{alg}}(K) = 1$ and $\text{sd}_a(K) \geq 2$, inequality (2.5) implies that $\text{sd}_a(K) = u_a(K) = 2$. In particular, the knots K_n from Theorem 3.3 satisfy $g_{\text{alg}}(K_n) = 1 < 2 = \text{sd}_a(K_n) = u_a(K_n)$ for all $n \geq 2$. A literature search suggests that $\{K_n\}$ is the first known infinite family of knots for which $g_{\text{alg}} < u_a$. Note that, in [Feller and Lewark 2018], the 3-genus is used to distinguish between g_{alg} and u_a for various knots since $g_{\text{alg}}(K) \leq g_3(K)$ while $u_a \leq 2g_3(K)$. In our case, the 3-genus of the K_n grows large, and we use a different strategy for distinguishing between g_{alg} and u_a.

4. Relationships between the surgery description and untwisting numbers

In the last section, we found an infinite family of knots for which $u_a = \text{tu}_a = \text{sd}_a = 2$. Other examples can be found where $u_a = \text{tu}_a = \text{sd}_a$ are abundant. We now endeavor to find examples where the two quantities (and other similar quantities) disagree.

In particular, in this section, we examine the square of inequalities below, and show that each inequality can be strict for infinitely many knots:

(4.1)
$$\begin{array}{ccc} \mathrm{sd}_a & \leq & \mathrm{tu}_a \\ \mathrm{I}\wedge & & \mathrm{I}\wedge \\ \mathrm{sd} & \leq & \mathrm{tu} \end{array}$$

It is easy to find infinitely many knots such that the vertical inequalities in (4.1) are strict; for example, any nontrivial knots with Alexander polynomial-one satisfy $\mathrm{sd}_a < \mathrm{sd}$ and $\mathrm{tu}_a < \mathrm{tu}$. Finding such examples for the horizontal inequalities in (4.1) is more challenging.

It is known that $u(K)$ and $\mathrm{tu}(K)$ can be arbitrarily different [Ince 2016]. In contrast, we can find no examples of knots in the literature with $\mathrm{sd}(K) \neq \mathrm{tu}(K)$. We provide the first known examples below; in fact, we find an infinite family $\{K_n\}$ of knots satisfying the stronger inequality $\mathrm{sd}(K_n) < \mathrm{tu}_a(K_n)$ for all $n \geq 2$. This is the content of Theorem 1.1. The same family provides infinitely many examples where $\mathrm{sd}_a < \mathrm{tu}_a = u_a$.

For the proof of Theorem 1.1, we employ an obstruction to a knot having *algebraic* unknotting number 1 due to Borodzik and Friedl [2015], which in turn generalizes an unknotting number 1 obstruction due to Lickorish [1985]. The obstruction involves the linking pairing on the first homology of the double-branched cover $\Sigma(K)$; see, e.g., [Gordon 1978] for a discussion of the linking pairing.

Theorem 4.2 [Borodzik and Friedl 2015, Theorems 4.5 and 4.6]. *If a knot K can be **algebraically** unknotted by a single crossing change, then there exists a generator h of $H_1(\Sigma(K); \mathbb{Z})$ such that its linking pairing satisfies*

$$l(h, h) = \frac{\pm 2}{\det(K)} \in \mathbb{Q}/\mathbb{Z}.$$

The proof of Theorem 1.1 follows Lickorish's proof that the knot $P(3, 1, 3)$ does not have unknotting number 1 (the main theorem of Lickorish [1985]).

Theorem 1.1. *There are infinitely many knots $\{K_n\}$ with $\mathrm{sd}(K_n) = 1$ and $\mathrm{tu}(K_n) = \mathrm{tu}_a(K_n) = 2$.*

Proof. Suppose the family K_n is the set of pretzel knots of the form $P(10n+3, 1, 3)$; see Figure 3 for two isotopic diagrams and note that the boxed numbers represent *half*-twists. We first note that $\mathrm{sd}(K_n) = 1$ by performing the $+1/2$-surgery (or equivalently a -2-twist) on the curve C indicated in the figure. After the surgery, we obtain the pretzel knots $P(10n + 3, 1, -1)$, all of which are isotopic to the unknot.
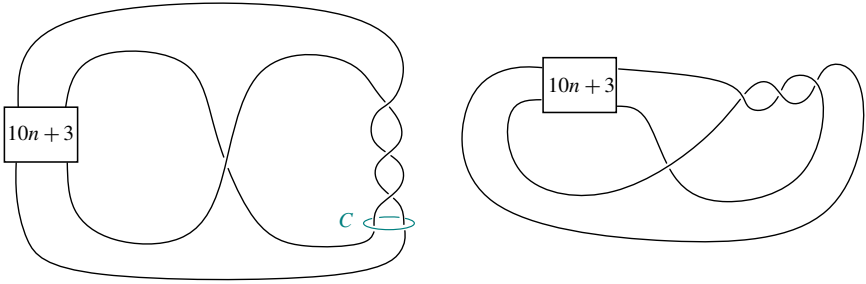
Figure 3. Diagrams of the Pretzel knots $K_n = P(10n + 3, 1, 3)$; the boxed numbers represent *half*-twists. Left: a standard diagram for 3-strand pretzel knots, together with an unknotted curve C where a $+1/2$ Dehn surgery can be applied to convert K_n to the unknot. Right: a diagram for the same knot in which it is more clear that the knots are two-bridge. In fact, they have Conway notation $C(10n + 3, 1, 3)$.

To conclude that $\text{tu}(K_n) = 2$, it is enough to show that $\text{tu}(K_n) \neq 1$; $\text{tu}(K_n) \leq 2$ since the surgery description move can be effected by two (single) null-homologous twists.

To show that $\text{tu}(K_n) \neq 1$, first recall that $\text{tu}_a(K) \leq \text{tu}(K)$ and that $\text{tu}_a(K) = u_a(K)$ for any knot K. Note that $\Delta_{K_n}(t) \neq 1$ for each $n \geq 2$ (see, e.g., [Lickorish 1997, Example 6.9]), so that $\text{tu}_a(K_n) \neq 0$. We then assume that $\text{tu}_a(K_n) = 1$ for contradiction, and prove that the linking pairing on $H_1(\Sigma(K_n); \mathbb{Z})$ does not satisfy the condition in Theorem 4.2 for any $n \geq 1$.

First, note that the knots K_n are 2-bridge; see Figure 3. Each two-bridge knot has a (nonunique) associated fraction p/q with the property that $\Sigma_2(K) \cong L(p, q)$; see, e.g., [Kawauchi 1996, Chapter 2] for a discussion of two-bridge knots. In fact, $\{K_n\}$ are precisely those 2-bridge knots with continued fraction of the form

$$[10n + 3, 1, 3] = 10n + 3 + \frac{1}{1 + \frac{1}{3}} = \frac{40n + 15}{4}.$$

Hence the double-branched covers of these knots $\Sigma(K_n) \cong L(40n + 15, 4)$ are lens spaces. So in particular, $\Sigma(K_n)$ can be obtained as surgery on a knot J (in fact the unknot) via $\frac{40n+15}{4}$-surgery. This implies that $H_1(\Sigma(K_n); \mathbb{Z})$ is cyclic of order $40n + 15$ generated by μ the image of a meridian of J after surgery, and moreover that $l(\mu, \mu) = \frac{4}{40n+15}$ [Lickorish 1985].

Any generator h of $H_1(\Sigma(K_n))$ is of the form $h = t\mu$ for some integer t. Let h be the generator which must exist according to Theorem 4.2 so that

$$(4.3) \qquad \frac{\pm 2}{40n+15} = l(h, h) = l(t\mu, t\mu) = t^2 \cdot l(\mu, \mu) = \frac{4t^2}{40n+15} \in \mathbb{Q}/\mathbb{Z}.$$
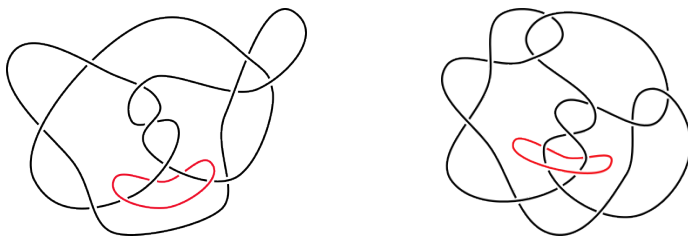
Figure 4. The knots 10_{68} (left) and $11a_{103}$ (right) can be converted to the unknot by inserting two left (resp. right)-handed twists in the regions indicated by the red unknots.

For the two fractions on the far left and far right of (4.3) to be equivalent in $\mathbb{Q}/\mathbb{Z}$, we must have $\pm 2 \equiv 4t^2 \pmod{40n+15}$ so that ± 2 must be a square $\pmod{40n+15}$. We will show that this is not true.

If ± 2 is a square $\pmod{40n+15}$ then it also must be a square $\pmod a$ where a is any factor of $40n+15$. In particular, ± 2 must be a square $\pmod 5$. But neither $-2 \equiv 3$ nor 2 are squares $\pmod 5$. This is a contradiction, and hence $\mathrm{tu}_a(K_n) \neq 1$ for each n, which forces $\mathrm{tu}(K_n) \neq 1$. $\square$

Since $\mathrm{sd}_a(K) \leq \mathrm{sd}(K)$ for all knots K, the next corollary immediately follows.

Corollary 4.4. *There are infinitely many knots $\{K_n\}$ for which $\mathrm{sd}_a(K_n) = 1$ while $\mathrm{tu}_a(K_n) = u_a(K_n) = 2$.*

Note that Corollary 4.4 is the biggest gap we could hope for in the sense that $\mathrm{sd}_a(K) \leq \mathrm{tu}_a(K) = u_a(K) \leq 2\,\mathrm{sd}_a(K)$.

While Theorem 1.1 provides infinitely many examples where $\mathrm{sd} < \mathrm{tu}_a$, one might ask if $\mathrm{sd} \leq \mathrm{tu}_a$ in general. The following theorem provides an answer to this question in the negative.

Theorem 4.5. *The $(p, 1)$-cable of the untwisted Whitehead double of any nontrivial knot, which we denote D_p, has $\mathrm{tu}_a(D_p) = u_a(D_p) = 0 < 1 = \mathrm{sd}(D_p)$ for all $p \in \mathbb{N}$.*

Proof. First, note that the Alexander polynomial of D_p, for any p, is equal to 1 (see the cabling relation in [Lickorish 1997]). Thus $\mathrm{tu}_a(D_p) = 0$. On the other hand, since D_p is not unknotted, we must have that $\mathrm{sd}(D_p) \geq 1$. In fact, one can see that $\mathrm{sd}(D_p) = 1$ by performing a single null-homologous twist about the clasping region in the untwisted Whitehead double. $\square$

Note. To distinguish between sd and tu, obstructions from Heegaard–Floer homology can be used, though this seems feasible only to show that $\mathrm{sd} = 1 < 2 = \mathrm{tu}$ for individual knots. In particular, the sd-moves in Figure 4 show that the knots 10_{68} and $11a_{103}$ have $\mathrm{sd}(K) = 1$, though the facts that $\mathrm{tu}(10_{68})$, $\mathrm{tu}(11a_{103}) = 2$ are results of [Ince 2017, Theorems 1.3, 1.4].

For all examples we produce with sd $\neq$ tu the two invariants in fact only differ by 1. In Section 5 below, we will prove Theorem 1.4 which states that the ratio of *tu* to *sd* is at most 3. This leaves open the following question.

Question 4.6. Does there exist a knot K with $\mathrm{sd}(K) = 1$ but $\mathrm{tu}(K) = 3$, or in general so that $\mathrm{tu}(K) = 2\,\mathrm{sd}(K) + 1$?

Note that the techniques used in the proof of Theorem 1.1 cannot be used to obstruct a knot K with $\mathrm{sd}(K) = 1$ from having $\mathrm{tu}(K) \leq 2$ since the algebraic invariants can differ at most by a factor of 2 by Theorem 1.3. It also seems unlikely that the Floer theoretic techniques of Ince [2017] would alone be enough to answer Question 4.6 given the difficulty in obstructing knots from being H-slice in indefinite 4-manifolds [Kjuchukova et al. 2021]. Thus, new techniques are likely needed to answer the question above.

5. An inequality relating surgery description number and untwisting number

In the previous section we asked whether a knot K with $\mathrm{sd}(K) = 1$ and $\mathrm{tu}(K) = 3$ can exist, or more generally, if a knot with $2\,\mathrm{sd}(K) + 1 = \mathrm{tu}(K)$ exists. In this section, we show that the untwisting number is at most twice the surgery description number plus 1.

The following theorem was inspired by the work of Borodzik [2019] on algebraic k-simple knots. In addition, Duncan McCoy suggested the last portion of the proof of Theorem 1.4, improving the upper bound from an earlier version of the paper.

Theorem 1.4. *For any knot K, we have that* $\mathrm{sd}(K) \leq \mathrm{tu}(K) \leq 2\,\mathrm{sd}(K) + 1$.

Note that while the following proof involves a series of Kirby calculus moves, the moves used are slam dunk moves (away from the knot), and handle slides involving only the added components (never the original knot); thus none of the moves alter the isotopy class of the knot. The result is diagrammatic. For a reference on Kirby calculus, see [Gompf and Stipsicz 1999].

Proof. The first inequality is clear from the definitions. To show the second inequality, we will first show that an unknot of framing $\pm 1/(2k + 1)$ which is null-homologous in the complement of K can be replaced (via careful Kirby calculus) with two unlinked, null-homologous unknots, one with framing $+1$ and one with framing -1. Thus $2k + 1$ full twists in a single twisting region can be realized by a sequence of two full twists (of opposite signs) in some diagram of K. This process (Procedure 1) is described below; an example in the case of five left-handed twists is shown in Figure 5. Throughout, we abuse notation and keep names of unknots unchanged after they have undergone a handle slide.
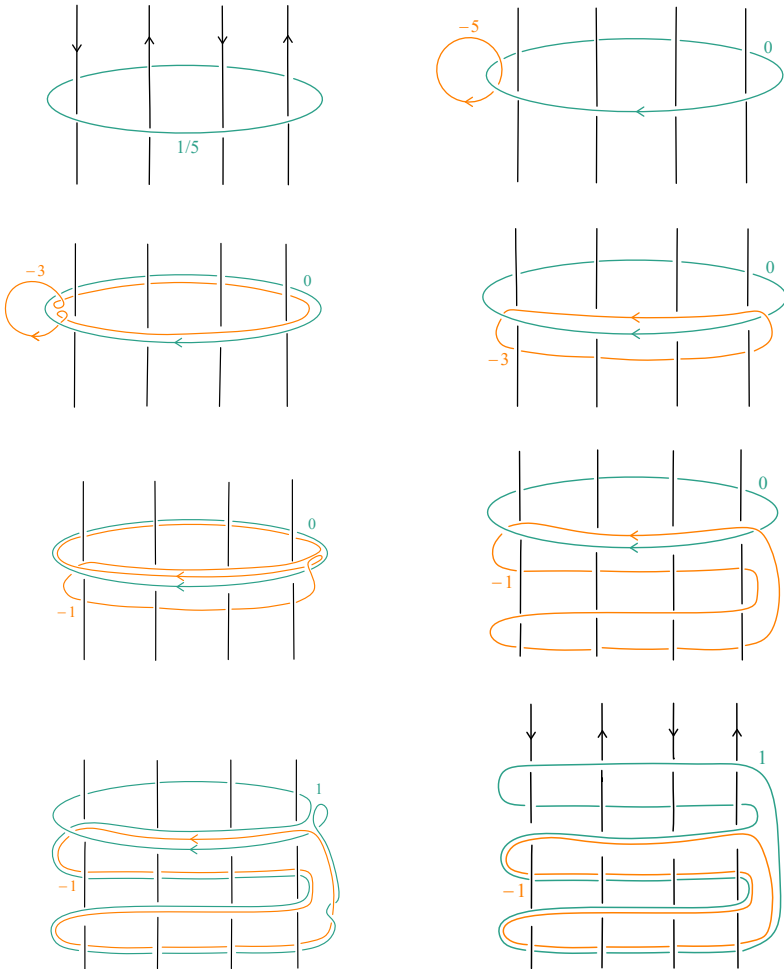
Figure 5. A sequence of Kirby moves which shows that apply-
ing 5 parallel null-homologous twists can be obtained by two null-
homologous twists. Top row: effecting null-homologous twist(s) (left)
and a slam dunk move (right). From second row: handle addition (left)
and isotopy (right).

Procedure 1. (1) Use a reverse slam dunk move to view the $\pm 1/(2k+1)$-framed
unknot as a 0-framed unknot U_1 geometrically linked once with a $\mp(2k+1)$-framed
unknot U_2 as in Figure 5 (top row).

(2) By repeatedly sliding U_2 over U_1, one can change the framing on U_2 to ∓ 1. See
Figure 5 (second and third row). Note that, in each handle slide, only the portion
of U_2 near U_1 is affected. While this changes how K and U_2 are geometrically
linked, the unknots U_1 and U_2 remain linked once.

(3) Finally, slide U_1 over U_2. This has the effect of changing the framing of U_1 by ∓ 1. See Figure 5 (third row, right and bottom row, left). After an isotopy (Figure 5, bottom row, right), it is not hard to see that the resulting U_1 and U_2 are unlinked.

We now show that unknots with framings ± 1 and $\pm 1/(2k)$ which are null-homologous in the complement of K can be replaced (again, via Kirby calculus) with three unlinked, null-homologous unknots, two with framings ± 1 and one with framing ∓ 1. The process is described below; an example is shown in Figure 6.

Procedure 2. (1) Use a reverse slam dunk move to view the $\pm 1/(2k)$-framed unknot as a 0-framed unknot U_1 geometrically linked once with a $\mp(2k)$-framed unknot U_2 as in Figure 6 (top row).

(2) At the beginning of the procedure we assumed we had unknots with framings ± 1 and $\pm 1/(2k)$. Call the unknot with ± 1 framing U_3. Slide U_2 over U_3 with framing ± 1 to change the framing on U_2 by 1. See 6 (middle row, left). At this stage, U_2 is linked with both U_1 and U_3.

(3) Slide U_3 over U_1 in order to unlink U_3 from U_2. The result is that, after an isotopy, U_3 is completely unlinked from U_1 and U_2. In addition, U_1 and U_2 are in position to perform the procedure from the previous paragraph. See Figure 6 (middle row, right and bottom row).

(4) Apply steps (2) and (3) of Procedure 1.
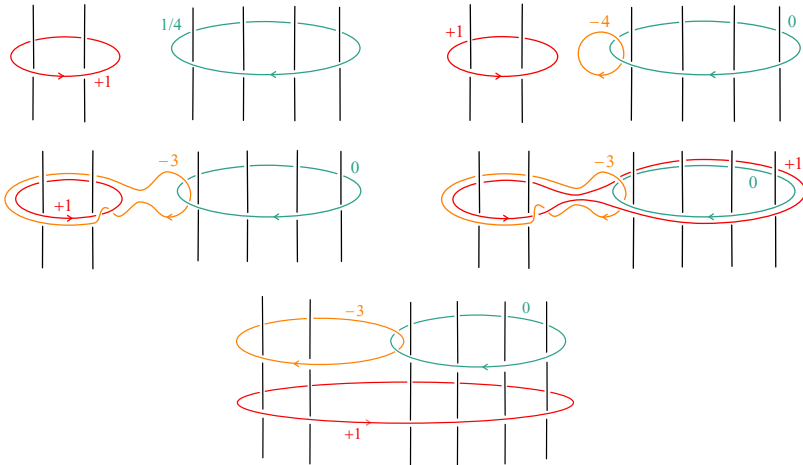
Figure 6. A sequence of Kirby moves to replace $+1$- and $+1/4$-framed null-homologous unknots in the knot complement with an unlinked $+1$-framed component and two components linked once, one with framing 0. Top row: null-homologous unknots in knot complement (left) and a slam dunk move (right). Middle row: handle addition. Bottom: isotopy.

Thus, to see the upper bound, consider the following cases.

• First, if the surgery description number can be realized using only $\pm(2k+1)$-moves (odd numbers of full twists in each twisting region), then we apply Procedure 1 to reduce each $(2k+1)$-move to a $+1$- and -1-move. Thus, in this case, $\mathrm{tu}(K) \le 2\,\mathrm{sd}(K)$.

• Second, if at least one $\pm(2k)$-move (even number of full twists in a single twisting region) is required to realize the surgery description number, then replace one of the $\pm(2k)$-moves with parallel ±1- and $\pm(2k-1)$-framed unknots. Call the ±1-framed unknot U_3 and now use Procedure 2 with U_3 to reduce each remaining $\pm(2k)$-moves to a $+1$- and -1-move. Thus, $\mathrm{tu}(K) \le 2\,\mathrm{sd}(K) + 1$. ☐

Note. In the proof of Theorem 1.4, the upper bound of $2\,\mathrm{sd}(K) + 1$ can only be sharp when *every* minimal sd-sequence for K involves *only* even numbers of full twists. In all other cases, consider a minimal sd-sequence which involves at least one null-homologous $(2k+1)$-twist for some $k \in \mathbb{Z}$. We may use Procedure 1 on all $\pm1/(2k+1)$-framed unknots to convert each into two ±1-framed unknots, then use Procedure 2 on all $\pm1/(2k)$-framed unknots (if one exists) using one of the ±1-framed unknots obtained via Procedure 1 to build an untwisting sequence of length $2\,\mathrm{sd}(K)$.

Acknowledgements

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