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UNKNOTTED CURVES ON GENUS-ONE SEIFERT SURFACES OF WHITEHEAD DOUBLES

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We consider homologically essential simple closed curves on Seifert surfaces of genus-one knots in S^3, and in particular those that are unknotted or slice in S^3. We completely characterize all such curves for most twist knots: they are either positive or negative braid closures; moreover, we determine exactly which of those are unknotted. A surprising consequence of our work is that the figure-eight knot admits infinitely many unknotted essential curves up to isotopy on its genus-one Seifert surface, and those curves are enumerated by Fibonacci numbers. On the other hand, we prove that many twist knots admit homologically essential curves that cannot be positive or negative braid closures. Indeed, among those curves, we exhibit an example of a slice but not unknotted homologically essential simple closed curve. We continue our investigation of unknotted essential curves for arbitrary Whitehead doubles of nontrivial knots, and obtain that there is precisely one unknotted essential simple closed curve in the interior of a double's standard genus-one Seifert surface. As a consequence we obtain many new examples of 3-manifolds that bound contractible 4-manifolds.

1. Introduction

Suppose $K \subseteq S^3$ is a genus g knot with Seifert surface Σ_K. Let b be a curve in Σ_K which is *homologically essential* — that is, it is not separating Σ_K — and *a simple closed curve* — that is, it has one component and does not intersect itself. Furthermore, we will focus on those that are *unknotted* or *slice* in S^3 — that is, each bounds a disk in S^3 or B^4. In this paper we seek to make progress on the following problem:

Problem. *Characterize and, if possible, list all such curves b for the pair (K, Σ_K), where K is a genus-one knot and Σ_K its Seifert surface.*

Our original motivation for studying this problem comes from the intimate connection between unknotted or slice homologically essential curves on a Seifert

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MSC2020: primary 57K30; secondary 57K10.
Keywords: unknotted curves on Seifert surfaces, contractible 4-manifolds.

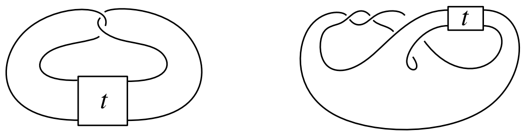


Figure 1. On the left is the twist knot K_t, where the box contains t full right-handed twists if $t \in \mathbb{Z}_{>0}$, and $|t|$ full left-handed twists if $t \in \mathbb{Z}_{<0}$. On the right is the standard Seifert surface for K_t.

surface of a genus-one knot and 3-manifolds that bound contractible 4-manifolds. We defer the detailed discussion of this connection to Section 1B, where we also provide some historical perspective. For now, however, we will focus on getting a hold on the stated problem above for a class of genus-one knots, and as we will make clear in the next few results, this problem is already remarkably interesting and fertile on its own.

1A. ***Main results.*** A well-studied class of genus-one knots is that of the so-called twist knot $K = K_t$, which is described by the diagram on the left of Figure 1 (cf. [Casson and Gordon 1978, page 182]). We note that with this convention K_{-1} is the right-handed trefoil $T_{2,3}$ and K_1 is the figure-eight knot 4_1. We will consider the genus-one Seifert surface Σ_K for $K = K_t$, as depicted on the right of Figure 1.

Theorem 1.1. *Let $t \leq 2$. Then the genus-one Seifert surface Σ_K of $K = K_t$ admits infinitely many homologically essential, unknotted curves if and only if $t = 1$, that is, if and only if K is the figure-eight knot 4_1.*

Indeed, we can be more precise and characterize all homologically essential, simple closed curves on Σ_K, from which Theorem 1.1 follows easily. To state this we recall an essential simple closed curve c on Σ_K can be represented (almost uniquely) by a pair of nonnegative integers (m, n), where m is the number of times $c = (m, n)$ runs around the left band and n is the number of times it runs around the right band in Σ_K. Moreover, since c is connected, we can assume $\gcd(m, n) = 1$. Finally, to uniquely describe c, we call c an ∞ *curve* if its orientation switches from one band to the other or a *loop curve* if it has the same orientation on both bands (see Figure 9).

Theorem 1.2. *Let $K = K_t$ be a twist knot and Σ_K its Seifert surface, as in Figure 1. Then:*

(1) *For $K = K_t$ with $t \leq -1$, we can characterize all homologically essential simple closed curves on Σ_K as the closures of negative braids in Figure 10. In the case of the right-handed trefoil $K_{-1} = T_{2,3}$, exactly 6 of these (see Figure 2) are unknotted in S^3. For $t < -1$, exactly 5 of these (see Figure 4) are unknotted in S^3.*

(2) *For $K = K_1 = 4_1$, we can characterize all homologically essential simple closed curves on Σ_K as the closures of braids in Figure 15. A curve on this surface is unknotted in S^3 if and only if it is*

(a) *a trivial curve* $(1, 0)$ *or* $(0, 1)$*,*
(b) *an* ∞ *curve in the form of* (F_{i+1}, F_i)*, or*
(c) *a loop curve in the form of* (F_i, F_{i+1})*, where* F_i *represents the* i*-th Fibonacci number (see Figure 3).*

For twist knots $K = K_t$ with $t > 1$ the situation is more complicated. Under further hypothesis on the parameters m, n we can obtain results similar to those in Theorem 1.2, and these will be enough to extend the theorem entirely to the case of $K = K_2$, so-called Stevedore's knot 6_1 (here we use the KnotInfo database [Livingston and Moore 2024] for identifying small knots and their various properties). More precisely we have:

Theorem 1.3. *Let* $K = K_t$ *be a twist knot and* Σ_K *its Seifert surface, as in Figure 1. Then*:

(1) *When* $t > 1$ *and* $m < n$, *we can characterize all homologically essential simple closed curves on* Σ_K *as the closures of positive braids in Figure 24*(a),(b). *Exactly* 5 *of these* (*see Figure 4*) *are unknotted in* S^3.

(2) *When* $t > 1$ *and* $m > n$:
   (a) *If* $m - tn > 0$, *then we can characterize all homologically essential simple closed curves on* Σ_K *as the closures of negative braids in Figures 28 and 31. Exactly* 5 *of these* (*see Figure 4*) *are unknotted in* S^3.
   (b) *If* $m - n < n$ *and the curve is an* ∞ *curve*, *then we can characterize all homologically essential simple closed curves on* Σ_K *as the closures of positive braids in Figure 29. Exactly* 5 *of these* (*see Figure 4*) *are unknotted in* S^3.

(3) *For* $K = K_2 = 6_1$, *we can characterize all homologically essential simple closed curves on* Σ_K *as the closures of positive or negative braids. Exactly* 5 *of these* (*see Figure 4*) *are unknotted in* S^3.

What Theorem 1.3 cannot cover is the case $t > 2$, $m > n$ and $m - tn < 0$ or when $m - n < n$ and the curve is a loop curve. Indeed in this range *not* every homologically essential curve is a positive or negative braid closure. For example, when $(m, n) = (5, 2)$ and $t = 3$ one obtains that the corresponding essential ∞ curve, as a smooth knot in S^3, is the knot $m(5_2)$ (see Figure 34 in Section 5 for a verification of this), and for $(m, n) = (7, 3)$ and $t = 3$, the corresponding knot is 10_{132}; both of these are known (e.g., via the KnotInfo database [Livingston and Moore 2024]) not to be positive braid closures — coincidentally, these knots are not unknotted or slice. Moreover we can explicitly demonstrate (see below) that if one removes the assumption of "∞" from part (2)(b) in Theorem 1.3, then the conclusion claimed there fails for certain loop curves when $t > 2$. A natural question is then whether for knots $K = K_t$ with $t > 2$, $m > n$ and $m - tn < 0$ or an $m - n < n$ loop curve,

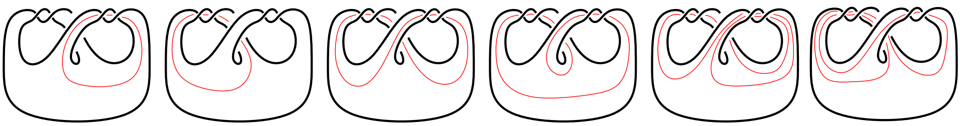


Figure 2. It can easily be shown that these six curves, from left to right, (0, 1), (1, 0), (1, 1) ∞, (1, 1) loop, (1, 2) ∞ and (2, 1) ∞, on Σ_K are unknotted in S^3. One can easily check that the other (1, 2) and (2, 1) curves (that is, the (1, 2) loop and (2, 1) loop curves) both yield the left-handed trefoil $T_{2,-3}$, and hence they are not unknotted in S^3.

there exists unknotted or slice curves on Σ_K other than those listed in Figure 4. A follow-up question will be whether there exists slice but not unknotted curves on Σ_K for some $K = K_t$. We can answer the latter question in the affirmative as follows:

Theorem 1.4. *Let $K = K_t$ be a twist knot with $t > 2$ and Σ_K its Seifert surface, as in Figure 1, and consider the loop curve (m, n) with $m = 3$, $n = 2$ on Σ_K. Then this curve, as a smooth knot in S^3, is the pretzel knot $P(2t-5, -3, 2)$. This knot is never unknotted but it is slice (exactly) when $t = 4$, in which case this pretzel knot is also known as the curious knot 8_{20}.*

Remark 1.5. We note that the choices of m, n values made in Theorem 1.4 are somewhat special in that they yielded an infinite family of pretzel knots, and that it includes a slice but not unknotted curve. Indeed, by using the work of Rudolph [1993], we can show (see Proposition 3.8) that the loop curve (m, n) with $m-n=1$, $n > 2$ and $t > 4$ on Σ_K, as a smooth knot in S^3, is never slice. The calculation gets

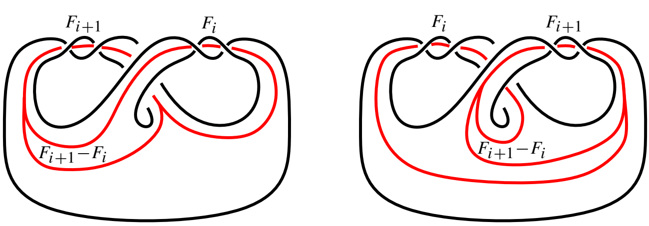


Figure 3. The two infinite families of unknotted curves for the figure-eight knot in S^3. The letters on parts of our curve or in certain locations stands for the number of strands at that particular curve or location. For example, for the (m, n) ∞ curve on the left we will show in Section 3B via explicit isotopies how, starting with the known unknotted (1, 1) ∞ curve, we can recursively obtain the following sequence of unknotted curves: $(1, 1) \sim (3, 2) \sim (8, 5) \sim (21, 13) \sim (55, 34) \sim \cdots$.

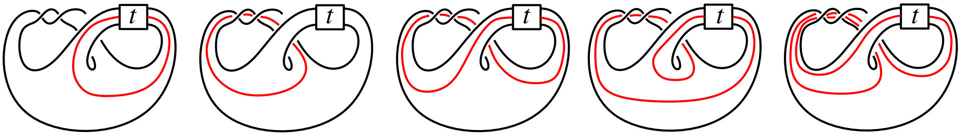


Figure 4. These five curves, from left to right, (0, 1), (1, 0), (1, 1) ∞, (1, 1) loop and (2, 1) ∞, on Σ_K, where $K = K_t$, $t \neq 1, -1$, are unknotted curves in S^3.

quickly complicated once $m - n > 1$, and it stays an open problem if in this range one can find other slice but not unknotted curves.

We can further generalize our study of unknotted essential curves on minimal-genus Seifert surfaces of genus-one knots for the Whitehead doubles of nontrivial knots. We first introduce some notation. Letting P be the twist knot K_t embedded (where $t = 0$ is allowed) in a solid torus $V \subset S^3$ and K denote an arbitrary knot in S^3, we identify a tubular neighborhood of K with V in such a way that the longitude of V is identified with the longitude of K coming from a Seifert surface. The image of P under this identification is a knot, $D^{\pm}(K, t)$, called the positive/negative t-twisted Whitehead double of K. In this situation the knot P is called the pattern for $D^{\pm}(K, t)$, and K is referred to as the companion. Figure 5 depicts the positive -3-twisted Whitehead double of the left-handed trefoil, $D^+(T_{2,-3}, -3)$. If one takes K to be the unknot, then $D^+(K, t)$ is nothing but the twist knot K_t.

Theorem 1.6. *Let K denote a nontrivial knot in S^3. Suppose that Σ_K is a standard genus-one Seifert surface for the Whitehead double of K. Then there is precisely one unknotted homologically essential, simple closed curve in the interior of Σ_K.*

1B. ***From unknotted curves to contractible 4-manifolds.*** The problem of finding unknotted homologically essential curves on a Seifert surface of a genus-one knot is

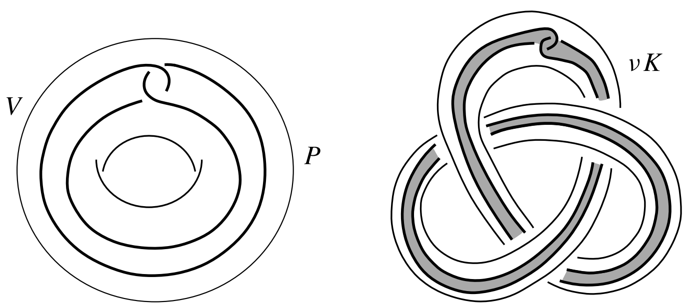


Figure 5. On the left is the solid torus $V \subset S^3$ and the pattern twist knot P (in this case $t = 0$). On the right is the positive -3-twisted Whitehead double of the left-handed trefoil, and its standard genus-one Seifert surface.

interesting on its own, but it is also useful for studying some essential problems in low-dimensional topology. We expand on one of these problems a little more. An important and still open question in low-dimensional topology asks: *which homology* 3-*sphere*[1] *bounds a homology* 4-*ball or contractible* 4-*manifold* (see [Kirby 1978, Problem 4.2])? This problem can be traced back to the famous Whitney embedding theorem and other important subsequent results due to Hirsch [1961], Wall [1965], and Rohlin [1965]. Since then, the research towards understanding this problem has stayed active. It has been shown that many infinite families of homology spheres do bound contractible 4-manifolds [Casson and Harer 1981; Fickle 1984; Stern 1978; Zeeman 1965] and at the same time many powerful techniques and homology cobordism invariants, mainly coming from Floer and gauge theories [Manolescu 2016; Fintushel and Stern 1985; Rohlin 1952], have been used to obtain constraints. See [Şavk 2024] for a detailed recent survey on various constructions and obstructions mentioned above.

In our case, using our main results, we will be able to list some more homology spheres that bound contractible 4-manifolds. This is because of the following theorem of Fickle [1984, Theorem 3.1], which was one of the main motivations for the research in this paper.

Theorem 1.7 (Fickle). *Let K be a knot in S^3 that has a genus-one Seifert surface F with a primitive element $[b] \in H_1(F)$ such that the curve b is unknotted in S^3. If b has self-linking s, then the homology* 3-*sphere obtained by* $1/(s \pm 1)$ *Dehn surgery on K bounds a contractible*[2] 4-*manifold.*

Theorem 1.7 was generalized (along with a somewhat more accessible proof of Fickle's theorem) by Etnyre and Tosun [2023, Theorem 1] to genus-one knots in the boundary of a homology 4-ball W, where the assumption on the curve b is relaxed so that b is slice in W. This will be useful (see Corollary 1.9 below) for applying to the slice but not unknotted curve/knot found in Theorem 1.4.

We also want to take the opportunity to highlight an interesting and still open conjecture [Fickle 1984, page 481, Conjecture] attributed to Fintushel and Stern.

Conjecture 1.8 (Fintushel and Stern). *Let K be a knot in the boundary of a homology* 4-*ball W which has genus-one Seifert surface with a primitive element $[b] \in H_1(F)$ such that b is slice in W. If b has self-linking s, then the homology* 3-*sphere obtained by* $1/k(s \pm 1)$, $k \geq 0$, *Dehn surgery on K bounds a homology* 4-*ball.*

[1] A homology 3-sphere/4-ball is a closed, oriented, smooth 3-/4-manifold having the integral homology groups of S^3/B^4.

[2] Indeed, this contractible manifold is a *Mazur-type* manifold, namely, it is a contractible 4-manifold that has a single handle for each of the indices 0, 1 and 2, where the 2-handle is attached along a knot that links the 1-handle algebraically once. This condition yields a trivial fundamental group.

Corollary 1.9. *Let K_t be a nontrivial twist knot. Then the homology spheres obtained by*

(1) $\pm\frac{1}{2}$ *Dehn surgery on* $K_1 = 4_1$,

(2) $-\frac{1}{2}$ *and* $-\frac{1}{4}$ *Dehn surgeries on* $K_{-1} = T_{2,3}$,

(3) $-\frac{1}{2}$ *and* $1/(t \pm 1)$ *and* $1/((t-2) \pm 1)$ *Dehn surgeries on* K_t *for* $t \neq \pm 1$,

(4) $\frac{1}{2}$ *Dehn surgery on* K_4

bound contractible 4*-manifolds.*

Corollary 1.10. *The homology spheres obtained by* $-\frac{1}{2}$ *Dehn surgery on* $D^+(K, t)$ *each bound a contractible* 4*-manifold.*

Remark 1.11. The 3-manifolds in Corollary 1.9(2) are the Brieskorn spheres $\Sigma(2, 3, 13)$ and $\Sigma(2, 3, 25)$; they were identified by Casson–Harer and Fickle to bound contractible 4-manifolds. Also, it was known already that the result of $\frac{1}{2}$ Dehn surgery on the figure-eight knot bounds a contractible 4-manifold (see [Tosun 2022, Theorem 18 and Figure 6]), and from this we obtain the result in Corollary 1.9(1), as the figure-eight knot is an amphichiral knot. The result in Corollary 1.10 also follows from [Fickle 1984, Theorem 3.6].

Remark 1.12. It is known that the result of $\frac{1}{n}$ Dehn surgery on a slice knot $K \subset S^3$ bounds a contractible 4-manifold. To see this, note that at the 4-manifold level with this surgery operation what we are doing is removing a neighborhood of the slice disk from B^4 (the boundary at this stage is zero surgery on K) and then attaching a 2-handle to a meridian of K with framing $-n$. Now, simple algebraic topology arguments show that this resulting 4-manifold is contractible.

It is a well-known result [Casson and Gordon 1978] that a nontrivial twist knot $K = K_t$ is slice if and only if $K = K_2$ (Stevedore's knot 6_1). So, by arguments above, we already know that result of $\frac{1}{n}$ surgery on K_2 bounds a contractible 4-manifold for any integer n. But, interestingly, we do not recover this by using Theorem 1.3.

Organization. The paper is organized as follows. In Section 2 we set some basic notation and conventions that will be used throughout the paper. Section 3 contains the proofs of Theorems 1.2, 1.3 and 1.4. Our main goal will be to organize, case by case, essential simple closed curves on genus-one Seifert surfaces Σ_K, through sometimes lengthy isotopies, into explicit positive or negative braid closures. Once this is achieved we use a result due to Cromwell that says the Seifert algorithm applied to the closure of a positive/negative braid closure gives a minimal-genus surface. This together with some straightforward calculations will help us to determine the unknotted curves exactly. But sometimes it will not be obvious how or even possible to reduce an essential simple closed curve to a positive or negative braid closure (see Sections 3B, 3C and 3D and Figure 34 in Section 5).

Further analyzing these cases will yield interesting phenomenon listed in Theorems 1.3 and 1.4. Section 4 contains the proof of Theorem 1.6. Finally, Section 5 contains the proofs of Corollaries 1.9 and 1.10 and some final remarks.

2. Preliminaries

In this section, we set some notation and make preparations for the proofs in the next three sections. In Figure 6 we record some basic isotopies/conventions that will be repeatedly used during proofs. Most of these are evident, but for the reader's convenience we explain how the moves in parts (a) and (f) work in Figures 7 and 8. We remind the reader that letters on parts of our curve, as in part (e) of Figure 6, or in a certain location, are used to denote the number of strands that particular curve has.

Recall also an essential, simple closed curve on Σ_K can be represented by a pair of nonnegative integers (m, n), where m is the number of times it runs around the left band and n is the number of times it runs around the right band in Σ_K, and since we are dealing with connected curves we must have that m, n are relatively prime.

We have two cases: $m > n$ or $n > m$. For an (m, n) curve with $m > n$, after the m strands pass under the n strands on the Seifert surface, the curve can be split into two sets of strands. For this case, assume that the top set is made of n strands. They must connect to the n strands going over the right band, leaving the other set to be made of $m - n$ strands. Now, we can split the other side of the set of m

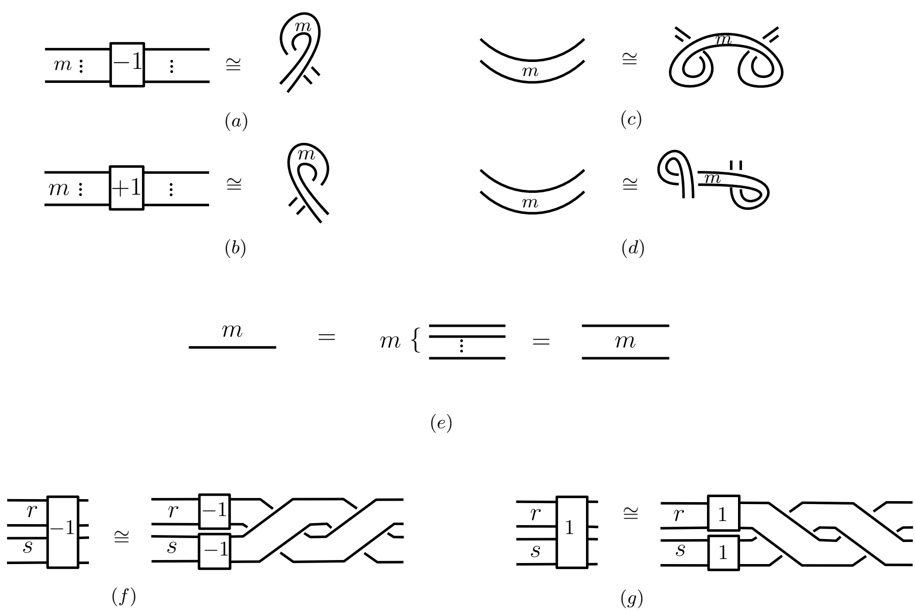


Figure 6. Various isotopies.

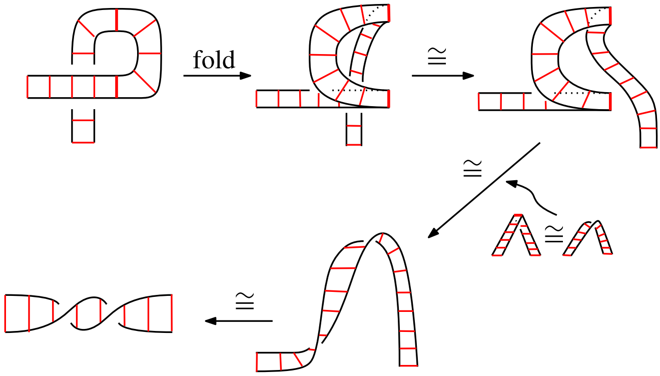


Figure 7. Diagrammatic proof of the move in Figure 6(a). As indicated, the passage from the top right figure to the bottom right is via "smoothing" a creased edge.

strands into two sections. The $m-n$ strands on the right can only go to the bottom of these two sections, because otherwise the curve would have to intersect itself on the surface. This curve is referred to as an (m,n) ∞ curve. See Figure 9(a). The other possibility for an (m,n) curve with $m>n$ has n strands in the bottom set instead, and they loop around to connect with the n strands going over the right band. This leaves the other to have $m-n$ strands. We can split the other side of the set of m strands into two sections. The $m-n$ strands on the right can only go to the top of these two sections, because again otherwise the curve would have to intersect itself on the surface. The remaining subsection must be made of n strands and connect to the n strands going over the right band. This curve is referred to as an (m,n) loop curve. See Figure 9(b). The case of an (m,n) curve with $n>m$ is similar. See Figure 9(c),(d).

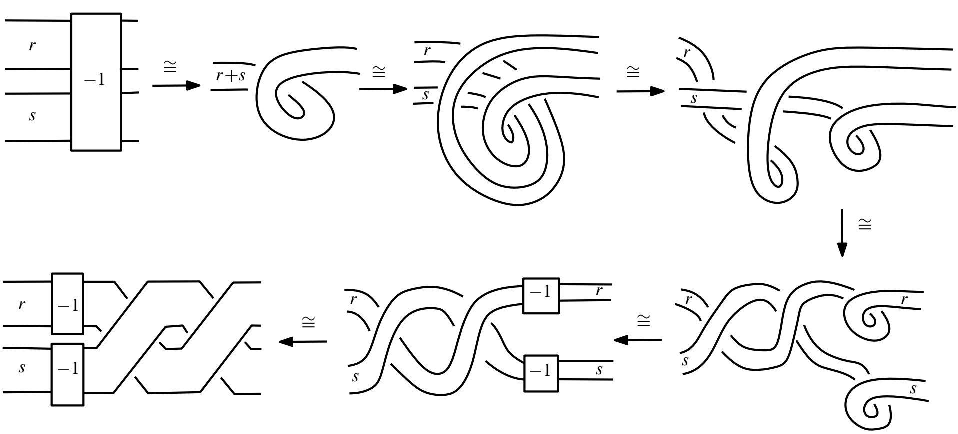
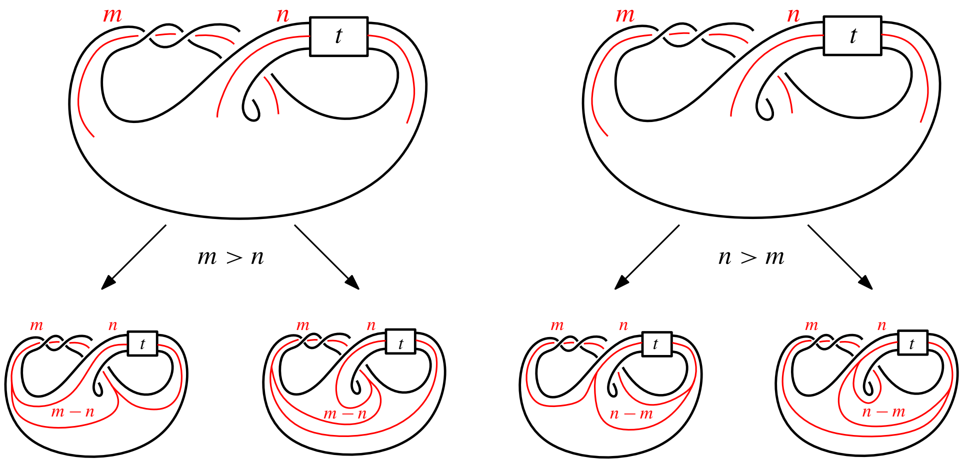


Figure 8. Diagrammatic proof of the move in Figure 6(f).



(a) (m, n) ∞ (b) (m, n) loop (c) (m, n) ∞ (d) (m, n) loop

Figure 9. Possibilities for an essential, simple closed curve (m, n) on Σ_K.

3. Twist knots

In this section we provide the proofs of Theorems 1.2, 1.3 and 1.4. We do this in four parts. Sections 3A and 3B contain all technical details of Theorem 1.2, Section 3C contains details of Theorem 1.3, and Section 3D contains Theorem 1.4 .

3A. ***Twist knot with $t < 0$.*** In this section we consider twist knots $K = K_t$, $t \leq -1$. This in particular includes the right-handed trefoil K_{-1}.

Proposition 3.1. *All essential, simple closed curves on Σ_K can be characterized as the closure of one of the negative braids in Figure 10.*

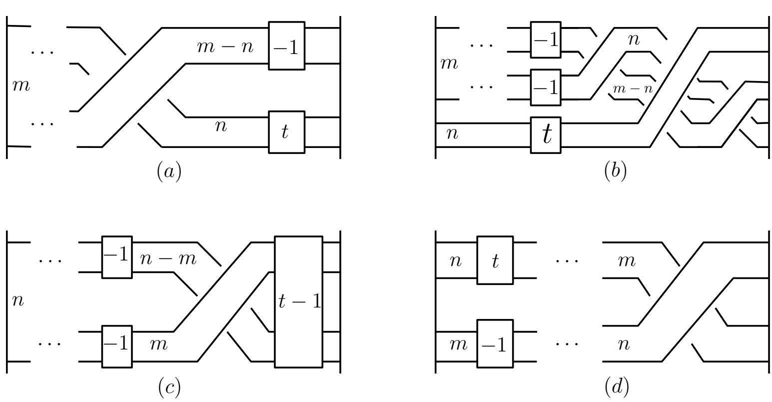


Figure 10

Proof. It suffices to show all possible curves for an arbitrary m and n such that $\gcd(m,n)=1$ are the closures of either braid in Figure 10. As mentioned earlier we will deal with cases where both $m,n \geq 1$ since cases involving 0 are trivial. There are four cases to consider. The arguments for each of these will be quite similar, and so we will explain the first case in detail and refer to the rather self-explanatory drawings/figures for the remaining cases.

Case 1: (m,n) ∞ curve with $m > n > 0$. This case is explained in Figure 11. The picture on top left is the (m,n) curve of interest. The next picture to its right is the (m,n) curve where we ignore the surface it sits on and use the convention from Figure 6(e). The next picture is an isotopy where we push the split between n strands and $m-n$ strands along the dotted blue arc. The next picture is obtained by simple isotopy. The passage from the top right picture to the bottom right is via Figure 6(c). The passage from the bottom right to the figure on its left is obtained by pushing $m-n$ strands around along the green arc. The goal here is to put the curve in a braid closure position. Finally, by applying simple isotopies and Figure 6(a) repeatedly we replace all the loops with full negative twists. Note that we moved the full negative twist on $m-n$ strands clockwise fashion around to bring it in the bottom of the figure. This gives the picture on the bottom left, which is the closure of the negative braid depicted in Figure 10(a).

Case 2: (m,n) loop curve with $m > n > 0$. By a series of isotopies, as indicated in Figure 12, the (m,n) curve in this case can be simplified to the knot depicted on the right of Figure 12, which is the closure of negative braid in Figure 10(b).

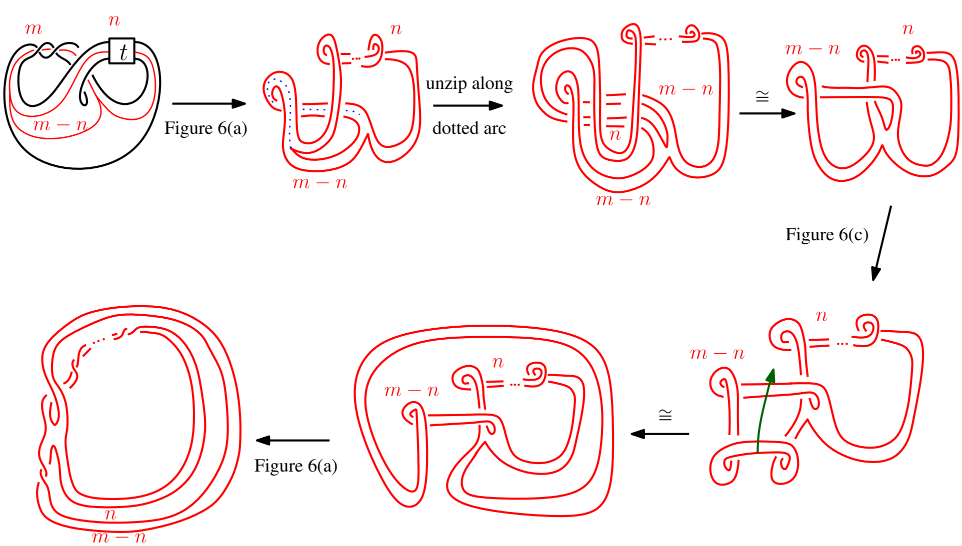


Figure 11

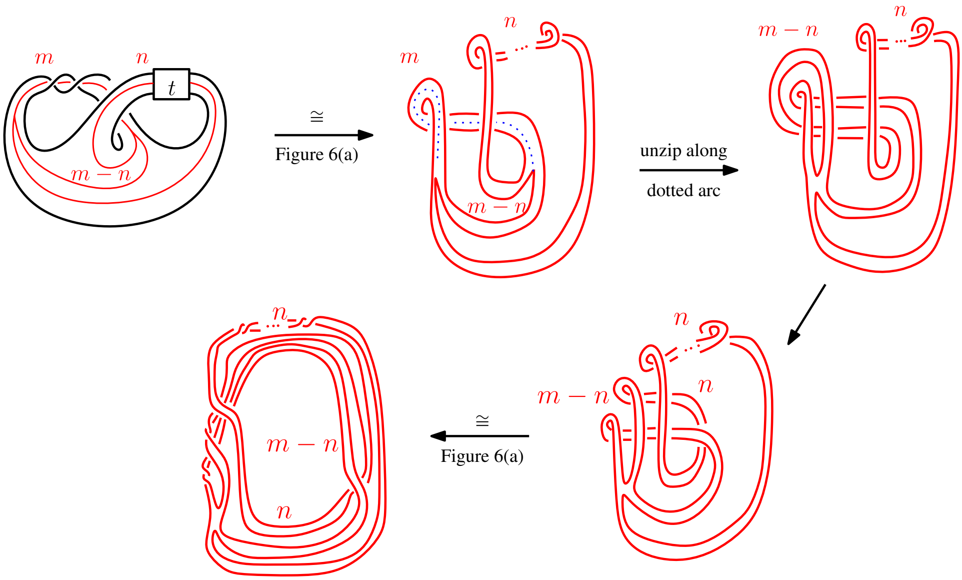


Figure 12

Case 3: (m, n) ∞ curve with $n > m > 0$. By a series of isotopies, as indicated in Figure 13, the (m, n) curve in this case can be simplified to the knot depicted on the bottom left of Figure 13, which is the closure of negative braid in Figure 10(c).

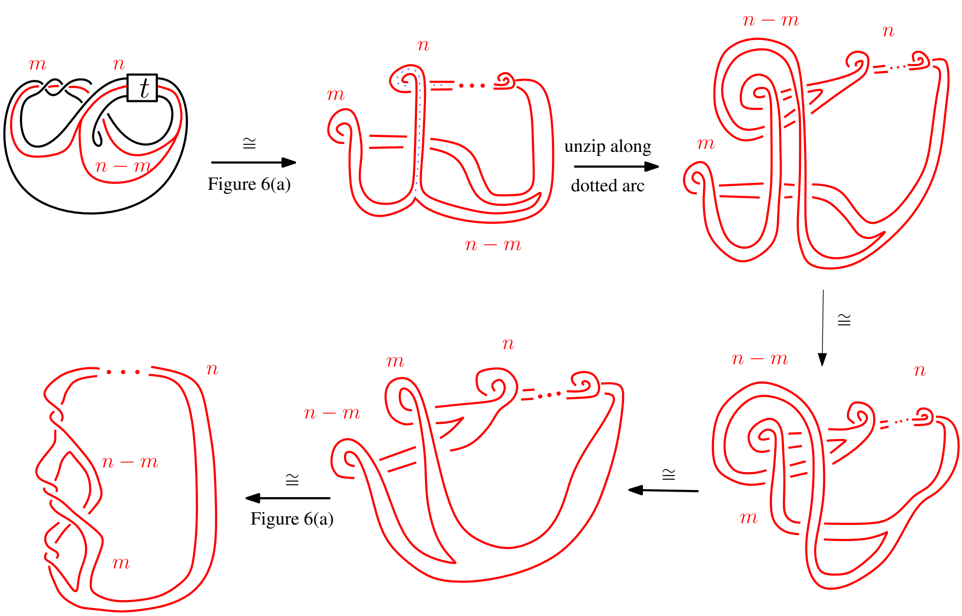


Figure 13

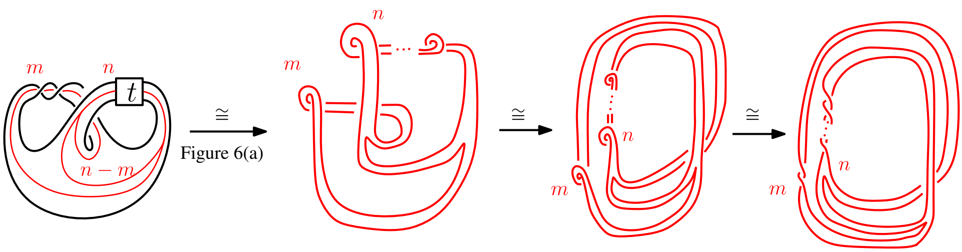


Figure 14

Case 4: (m, n) loop curve with $n > m > 0$. By a series of isotopies, as indicated in Figure 14, the (m, n) curve in this case can be simplified to the knot depicted on the right of Figure 14, which is the closure of negative braid in Figure 10(d). □

Next, we determine which of the curves in Proposition 3.1 are unknotted. It is a classic result due to Cromwell [1989] (see also [Stoimenow 2003, Corollary 4.2]) that the Seifert algorithm applied to the closure of a positive braid gives a minimal-genus surface.

Proposition 3.2. *Let β be a braid as in Figure 10 and $K = \hat{\beta}$ be its closure. Let $s(K)$ be the number of Seifert circles and $l(K)$ be the number of crossings in each braid diagram. Then $(s(K), l(K))$ is equal to*

$$\big(m, |t|n(n-1) + (m-n)(m-n-1) + n(m-n)\big) \qquad \text{for } \beta \text{ as in Figure 10(a)},$$

$$\big(m+n, (|t|+1)n(n-1) + (m-n)(m-n-1) + nm + 2n(m-n)\big) \qquad \text{for } \beta \text{ as in Figure 10(b)},$$

$$\big(n, (|t-1|)n(n-1) + (n-m)(n-m-1) + m(m-1) + m(n-m)\big) \qquad \text{for } \beta \text{ as in Figure 10(c)},$$

$$\big(m+n, |t|n(n-1) + m(m-1) + nm\big) \qquad \text{for } \beta \text{ as in Figure 10(d)}.$$

Proof. Consider the braid β as in Figure 10(a). Clearly, it has m Seifert circles as β has m strands. Next, we will analyze the three locations in which crossings occur. First consider the t negative full twists on n strands. Since each strand crosses over the other $n-1$ strands, we obtain $|t|n(n-1)$ crossings. Second, the negative full twist on $m-n$ strands produces additional $(m-n)(m-n-1)$ crossings. Lastly, notice the part of β where $m-n$ strands overpass the other n strands, and so for each strand in $m-n$ strands we obtain an additional n crossings. Hence, for $K = \hat{\beta}$, we calculate

$$l(\hat{\beta}) = |t|n(n-1) + (m-n)(m-n-1) + n(m-n).$$

The calculations for the other cases are similar. □

We can now prove the first part of Theorem 1.2.

Proof of Theorem 1.2(a). Proposition 3.1 proves the first half of our theorem. To determine there are exactly six unknotted curves when $t=-1$ and five when $t<-1$, let B be the set containing the six and five unknotted curves as in Figures 2 and 4, respectively. It suffices to show an essential, simple closed curve c on Σ_K, where $c \notin B$, cannot be unknotted in S^3. We know by Proposition 3.1 that c is the closure of one of the braids in Figure 10 in S^3, where $m, n \geq 1$ and $\gcd(m,n)=1$. We show, case by case, that the Seifert surface obtained via the Seifert algorithm for curves $c \notin B$ in each case has positive genus, and hence it cannot be unknotted.

- Let $c=(m,n)$ be the closure of the negative braid as in Figure 10(a) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are m Seifert circles, and by Proposition 3.2,

$$l(c)=|t|n(n-1)+(m-n)(m-n-1)+n(m-n).$$

Hence

$$g(\Sigma_c)=\tfrac{1}{2}(1+l-s)=\tfrac{1}{2}\big(m(m-n-2)+n(|t|(n-1)+1)+1\big).$$

If $m=n+1$, then we get $g(\Sigma_c)=\frac{1}{2}|t|n(n-1)$, which is positive as long as $n>1$; note that when $c=(2,1)$ we indeed get an unknotted curve. If $m>n+1$, then $g(\Sigma_c)\geq \frac{1}{2}\big(n(|t|(n-1)+1)+1\big)>0$ as long as $n>0$. So, $c\notin B$ is not an unknotted curve as long as $m>n\geq 1$.

- Let $c=(m,n)$ be the closure of the negative braid as in Figure 10(b) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are $n+m$ Seifert circles, and by Proposition 3.2,

$$l(c)=(|t|+1)n(n-1)+(m-n)(m-n-1)+nm+2n(m-n).$$

Hence

$$g(\Sigma_c)=\tfrac{1}{2}\big(m(m+n-2)+n(|t|(n-1)-1)+1\big).$$

One can easily see that this quantity is always positive as long as $n\geq 1$. So, $c\notin B$ is not an unknotted curve when $m>n\geq 1$.

- Let $c=(m,n)$ be the closure of the negative braid as in Figure 10(c) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are n Seifert circles, and by Proposition 3.2,

$$l(c)=(|t|-1)n(n-1)+(n-m)(n-m-1)+m(m-1)+m(n-m).$$

Hence

$$g(\Sigma_c)=\tfrac{1}{2}\big(n(|t|(n-1)-m-1)+m^2+1\big).$$

This is always positive as long as $m\geq 1$ and $|t|\neq 1$; note that when $c=(1,2)$ and $|t|=1$ we indeed get an unknotted curve. So, $c\notin B$ is not an unknotted curve when $n>m\geq 1$.

- Let $c = (m, n)$ be the closure of the negative braid as in Figure 10(d) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are $n + m$ Seifert circles, and by Proposition 3.2,

$$l(c) = |t|n(n-1) + m(m-1) + nm.$$

Hence

$$g(\Sigma_c) = \tfrac{1}{2}\big(|t|n(n-1) + m(m-2) + n(m-1) + 1\big).$$

One can easily see that this quantity is always positive as long as $m \geq 0$. So, $c \notin B$ is not an unknotted curve when $n > m \geq 1$.

This completes the first part of Theorem 1.2. □

3B. *Figure-eight knot.* The case of figure-eight knot is certainly the most interesting one. It is rather surprising, even to the authors, that there exists a genus-one knot with infinitely many unknotted curves on its genus-one Seifert surface. As we will see, understanding homologically essential curves for the figure-eight knot will be similar to what we did in the previous section. The key difference develops in Cases 2 and 4 below where we show how, under certain conditions, a homologically essential (m, n) ∞ (resp. (m, n) loop) curve can be reduced to the homologically essential $(m-n, 2n-m)$ ∞ (resp. $(2m-n, n-m)$ loop) curve, and how this recursively produces infinitely many distinct homology classes that are represented by the unknot, and we will show that certain Fibonacci numbers can be used to describe these unknotted curves. Finally we will show for the figure-eight knot this is the only way that an unknotted curve can arise. Adapting the notation developed thus far we start characterizing homologically essential simple closed curves on the genus-one Seifert surface Σ_K of the figure-eight knot K.

Proposition 3.3. *All essential, simple closed curves on Σ_K can be characterized as the closure of one of the braids in Figure 15 (note the first and third braids from the left are negative and positive braids, respectively).*

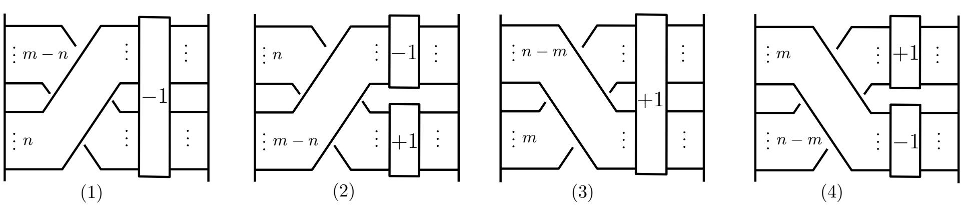


Figure 15. Braid representations of curves on Σ_K, where K is the figure-eight knot. From left to right: (m, n) loop curve with $m > n$; (m, n) ∞ curve with $m > n$; (m, n) ∞ curve with $n > m$; (m, n) loop curve with $n > m$.

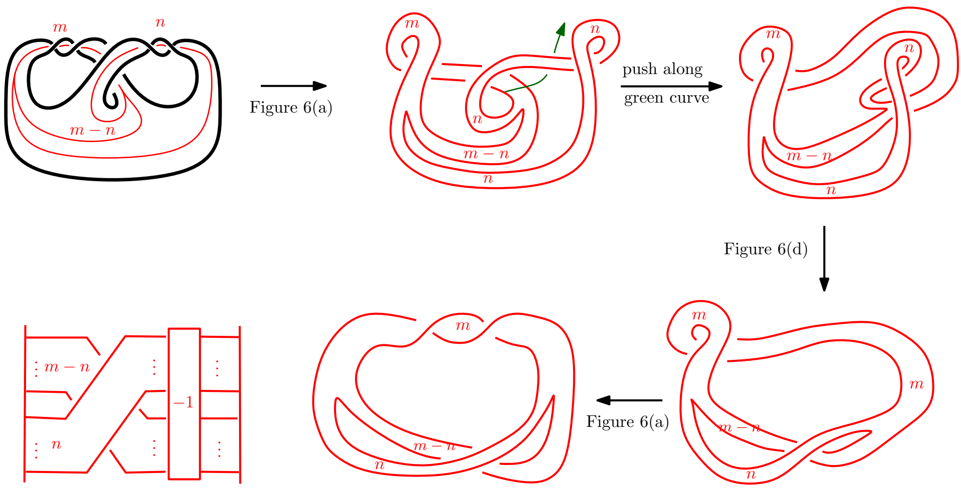


Figure 16

Proof. The curves $(1, 0)$ and $(0, 1)$ are clearly unknots. Moreover, because $\gcd(m, n) = 1$, the only curve with $n = m$ is the $(1, 1)$ curve, which is also the unknot in S^3. For the rest of the arguments below, we will assume $n > m$ or $m > n$. There are four cases to consider:

Case 1: (m, n) loop curve with $m > n > 0$. This curve can be turned into a negative braid following the process in Figure 16. The reader will observe that the process here is very similar to those in the previous section. We mention that the passage from the middle figure on the top to the one on its right is obtained by pushing the m strands along the green curve till it is clear from a positive loop of n strands.

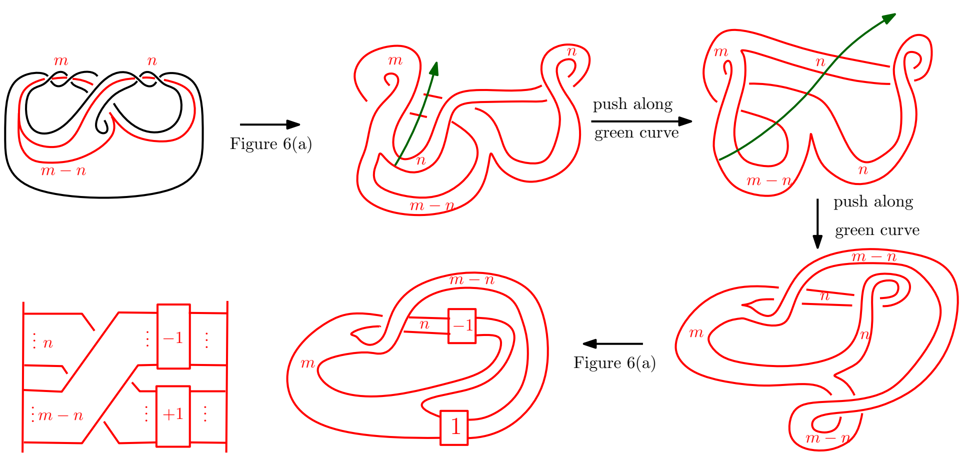


Figure 17

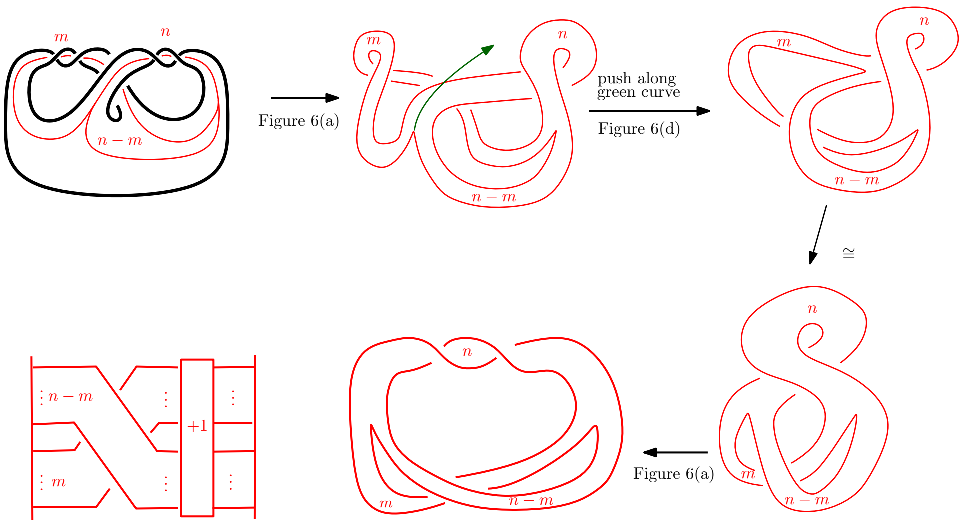


Figure 18

Finally, the middle curve on the bottom is our final curve, which is the closure of the negative braid to its left.

Case 2: (m, n) ∞ curve with $m > n > 0$. As mentioned at the beginning, this case (and Case 4) are much more involved and interesting (in particular the subcases of Cases 2c and 4c; see the proof Proposition 3.4). Following the process as in Figure 17, the curve can be isotoped as in the bottom right of that figure, which is the closure of the braid on its left — that is, the second braid from the left in Figure 15.

Case 3: (m, n) ∞ curve with $n > m > 0$. This curve can be turned into a positive braid following the process in Figure 18.

Case 4: (m, n) loop curve with $n > m > 0$. This curve can be turned into the closure of a braid following the process in Figure 19. □

We next determine which of these curves are unknotted:

Proposition 3.4. *A homologically essential curve c characterized as in Proposition 3.3 is unknotted if and only if it is* (a) *a trivial curve* $(1, 0)$ *or* $(0, 1)$, (b) *an* ∞ *curve in the form of* (F_{i+1}, F_i), *or* (c) *a loop curve in the form of* (F_i, F_{i+1}).

Proof. Let c denote one of these homologically essential curve listed in Proposition 3.3. We will analyze the unknottedness of c in four separate cases.

Case 1: Suppose $c = (m, n)$ is the closure of the negative braid in the bottom left of Figure 16. Note the minimal Seifert surface of c, Σ_c, has $(n)(m-n) + (m)(m-1)$ crossings and m Seifert circles. Hence

$$g(\Sigma_c) = \tfrac{1}{2}(n(m-n) + (m-1)^2).$$

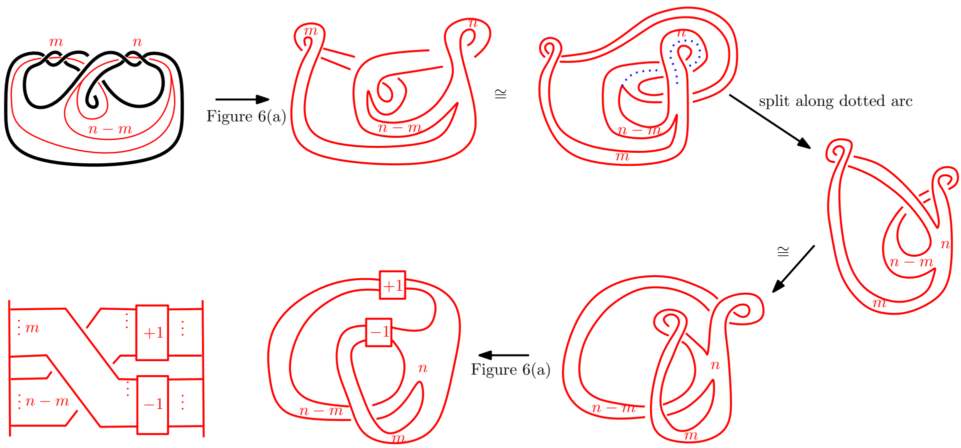


Figure 19

This is a positive integer for all m, n with $m > n$. So c is never unknotted in S^3 as long as long $m > n > 0$.

Case 2: Suppose c is of the form in the bottom right of Figure 17. Since this curve is not a positive or negative braid closure, we cannot directly use Cromwell's result as in Case 1 or the previous section. There are three subcases to consider.

Case 2a: $m - n = n$. Because m and n are relatively prime integers, we must have that $m = 2$ and $n = 1$, and we can easily see that this (2, 1) curve is unknotted.

Case 2b: $m - n > n$. This curve can be turned into a negative braid following the process in Figure 20. More precisely, we start, on the top left of that figure, with the curve appearing on the bottom right of Figure 17. We extend the split along the dotted blue arc and isotope m strands to reach the next figure. We note that this splitting can be done since, by assumption, $m - 2n > 0$. Then using Figure 6(a) and further isotopy we reach the final curve on the bottom right of Figure 20, which is obviously the closure of the negative braid depicted on the bottom left of that picture.

The minimal Seifert surface coming from this negative braid closure contains $m - n$ circles and $(m-2n)n + (m-n)(m-n-1)$ twists. Hence

$$g(\Sigma_c) = \tfrac{1}{2}((m-2n)n + (m-n)(m-n-2) + 1).$$

This a positive integer for all integers m, n with $m - n > n$. So, c is not unknotted in S^3.

Case 2c: $m - n < n$. We organize this curve some more. We start, on the top left of Figure 21, with the curve that is on the bottom left of Figure 17. We extend the split along the dotted blue arc and isotope $m - n$ strands to reach the next figure. After some isotopies we reach the curve on the bottom left of Figure 21. In other

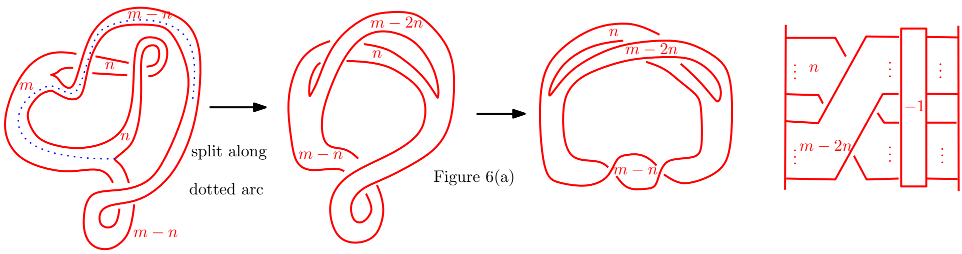


Figure 20

words, this subcase of Case 2c leads to a reduced version of the original picture (top left curve in Figure 17), in the sense that the number of strands over either handle is less than the number of strands in the original picture.

This case can be further subdivided depending on the relationship between $2n-m$ and $m-n$, but this braid (or rather its closure) will turn into a $(m-n, 2n-m)$ ∞ curve when $m-n > 2n-m$:

Case 2c(i): $2n-m = m-n$. This simplifies to $3n = 2m$. Because $\gcd(m, n) = 1$, this will only occur for $m = 3$ and $n = 2$, and the resulting curve is a $(1, 1)$ ∞ curve. In other words here we observed that the $(3, 2)$ curve has been reduced to a $(1, 1)$ curve.

Case 2c(ii): $2n-m > m-n$. This means that we are dealing with a curve under Case 3, and we will see that all curves considered there are positive braid closures.

Case 2c(iii): $2n-m < m-n$. Here, we remain under Case 2. So for $m > n > m-n$, the (m, n) ∞ curve is isotopic to the $(m-n, 2n-m)$ ∞ curve. This isotopy series will be denoted by $(m, n) \sim (m-n, 2n-m)$. Equivalently, there is a series of isotopies such that $(m-n, 2n-m) \sim (m, n)$. If (k, l) denotes a curve at one stage of this isotopy, then $(k, l) \sim ((k+l)+k, k+l)$. So, starting with $k = l = 1$, we

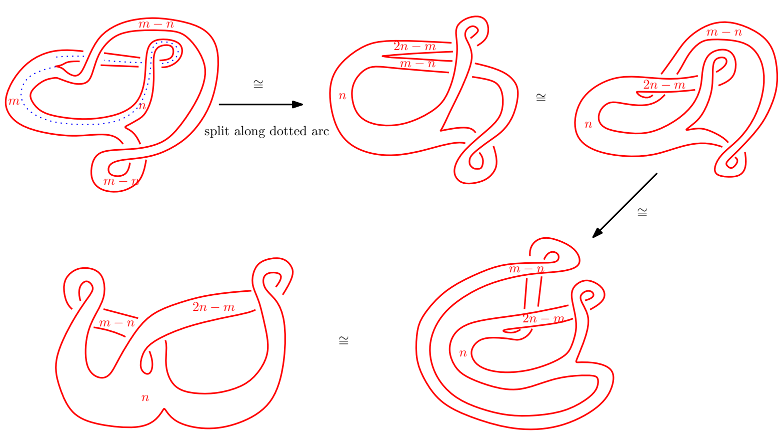


Figure 21

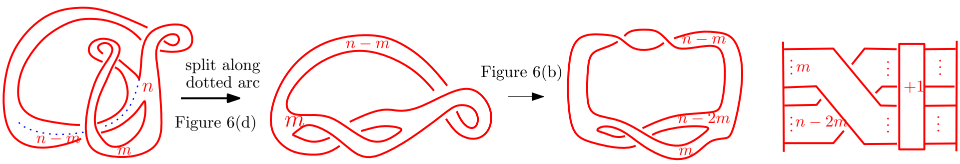


Figure 22

recursively obtain

$$(1,1)\sim(3,2)\sim(8,5)\sim(21,13)\sim(55,34)\sim\cdots.$$

In a similar fashion, if we start with $k=2,\ l=1$, we obtain

$$(2,1)\sim(5,3)\sim(13,8)\sim(34,21)\sim(89,55)\sim\cdots.$$

Notice every curve c above is of the form $c=(F_{i+1},F_i)$, $i\in\mathbb{Z}_{>0}$, where F_i denotes the i-th *Fibonacci number*. We will call these *Fibonacci curves*. We choose (1, 1) and (2, 1) because they are known unknots. As a result, this relation generates an infinite family of homologically distinct simple closed curves on Σ_K that are unknotted in S^3.

Case 3: Suppose a curve, c, is of the form in Figure 15(3), which is the closure of the positive braid depicted in the bottom left of Figure 18. An argument similar to that applied to Case 1 can be used to show c is never unknotted in S^3.

Case 4: Suppose c is of the form as in the bottom right of Figure 19. Similar to Case 2, there are three subcases to consider.

Case 4a: $m=n-m$. Then $2m=n$. Because $\gcd(m,n)=1$, $m=1$ and $n=2$, resulting in the unknot.

Case 4b: $n-m>m$. Then $n-2m>0$, and following the isotopies in Figure 22, the curve can be changed into the closure of the positive braid depicted on the bottom right of that figure.

Identical to Case 2b, the curve c in this case is never unknotted in S^3.

Case 4c: $m>n-m$. Then $2m-n>0$, and we can split the m strands into two: a set of $n-m$ strands and a set of $2m-n$ strands.

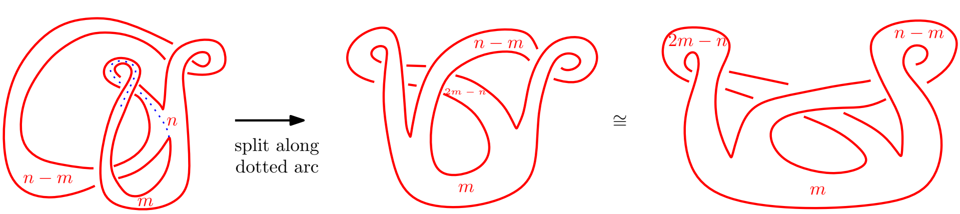


Figure 23

This case can be further subdivided depending on the relationship between $n-m$ and $2m-n$, but this braid will turn into a $(2m-n, n-m)$ loop curve when $n-m > 2m-n$:

Case 4c(i): $2m-n = n-m$. This simplifies to $3m = 2n$. Because $\gcd(m, n) = 1$, this will only occur for $m = 2$ and $n = 3$, and the resulting curve is a $(1, 1)$ loop curve.

Case 4c(ii): $n-m < 2m-n$. This means that we are dealing with a curve under Case 1, and we saw that all curves considered there are negative braid closures.

Case 4c(iii): $n-m > 2m-n$. Here, we remain under Case 4. So for $n > m > n-m$, an (m, n) loop curve has the following isotopy series: $(m, n) \sim (2m-n, n-m)$. If (k, l) denotes a curve at one stage of this isotopy, then the reverse also holds: $(k, l) \sim (k+l, (k+l)+l)$. As a result, much like Case 2c, we can generate two infinite families of unknotted curves in S^3:

$$(1, 1) \sim (2, 3) \sim (5, 8) \sim (13, 21) \sim (34, 55) \sim \cdots,$$
$$(1, 2) \sim (3, 5) \sim (8, 13) \sim (21, 34) \sim (55, 89) \sim \cdots.$$

Notice every curve c is of the form $c = (F_i, F_{i+1})$, $i \in \mathbb{Z}_{>0}$. Finally, we show that this is the only way one can get unknotted curves. That is, we claim:

Lemma 3.5. *If a homologically essential curve c on Σ_K for $K = 4_1$ is unknotted, then it must be a Fibonacci curve.*

Proof. From above, it is clear that if our curve c is Fibonacci, then it is unknotted. So it suffices to show if a curve is not Fibonacci then it is not unknotted. We will demonstrate this for loop curves under Case 4. Let c be a loop curve that is not Fibonacci but is unknotted. Since it is unknotted, it fits into either Case 4a or 4c. But the only unknotted curve from Case 4a is a $(1, 1)$ curve which is a Fibonacci curve, so c must be under Case 4c. By our isotopy relation, $(m, n) \sim (2m-n, n-m)$. So, the curve can be reduced to a minimal form, say (a, b), where $(a, b) \neq (1, 1)$ and $(a, b) \neq (2, 1)$. We will now analyze this reduced curve (a, b):

- If $a = b$, then $(a, b) = (1, 1)$, a contradiction.
- If $a > b$, then (a, b) is under Case 1; none of those are unknotted.
- If $b-a < a < b$, then (a, b) is still under Case 4c, and not in reduced form, a contradiction.
- If $a < b-a < b$, then (a, b) is under Case 4b; none of those are unknotted.
- If $b-a = a < b$, then $(a, b) = (2, 1)$, a contradiction.

So, it has to be that either $(a, b) \sim (1, 1)$ or $(a, b) \sim (2, 1)$. Hence, it must be that $c = (F_i, F_{i+1})$ for some i. The argument for the case where c is an ∞ curve under Case 2 is identical. □

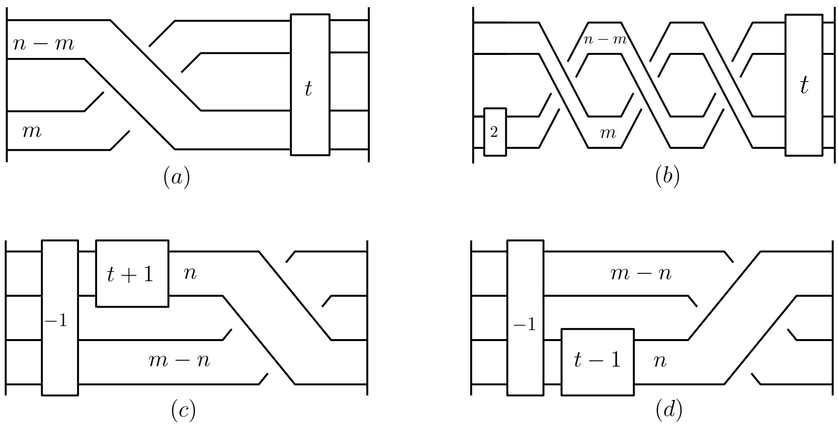


Figure 24

We end this section with a remark which was observed by the authors at the initial stages of the research and was also communicated to the authors by F. Misev.

Remark 3.6. An alternative and perhaps slightly easier way to see the existence of Fibonacci numbers for unknotted curves for the figure-eight knot is as follows: Recall that the figure-eight knot is fibered and its pseudo-Anosov monodromy $\phi : \Sigma \to \Sigma$, where Σ is the genus-one Seifert surface, induces a linear map on the first homology $H_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$ described by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. By applying this matrix repeatedly to the unknotted curves (vectors) (0, 1) and (1, 0) one obtains other unknotted curves that have Fibonacci numbers as their entries exactly as predicted in Proposition 3.4.

We add that this approach cannot capture the full strength of the results about the figure-eight knot: namely, showing that any unknotted curve as in Lemma 3.5 on the genus-one Seifert surface of the figure-eight knot must be a Fibonacci curve or characterizing all homologically essential curves on the Seifert surface of the figure-eight knot as in Proposition 3.3. Moreover our proof technique is by hand and works uniformly for all other twist knots we study in this paper.

3C. ***Twist knot with $t > 1$: part one.*** In this section we consider twist knot $K = K_t$, $t \geq 2$, and give the proof of Theorem 1.3.

Proposition 3.7. *All essential, simple closed curves on Σ_K can be characterized as the closure of one of the braids in Figure 24.*

Proof. It suffices to show all possible curves for an arbitrary m and n such that $\gcd(m, n) = 1$ are the closures of braids in Figure 24. Here too there are four cases to consider but we will analyze these in slightly different order than in the previous two sections.

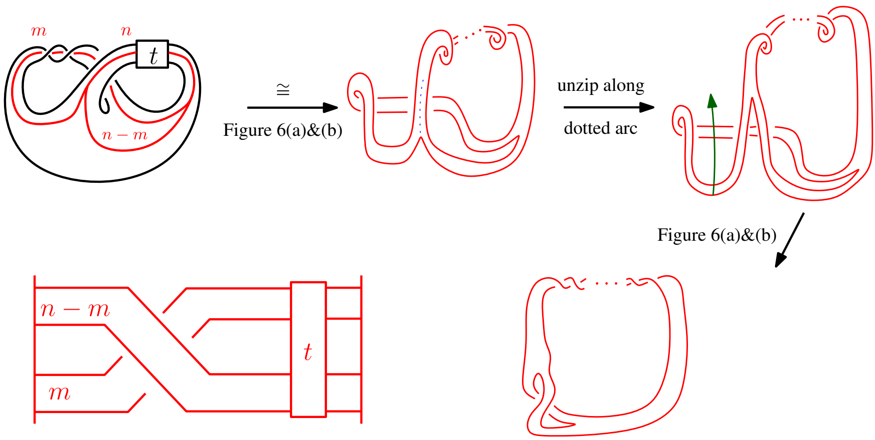


Figure 25

Case 1: (m, n) ∞ curve with $n > m > 0$. In this case the curve is the closure of a positive braid, and this is explained in Figure 25. More precisely, we start with the curve which is drawn in the top left of the figure, and after a sequence of isotopies this becomes the curve in the bottom right of the figure, which is obviously the closure of the braid in the bottom left of the figure. In particular, when $n > m \geq 1$, none of these curves will be unknotted.

Case 2: (m, n) loop curve with $n > m > 0$. In this case too the curve is the closure of a positive braid, and this is explained in Figure 26. In particular, when $n > m > 1$, none of these curves will be unknotted.

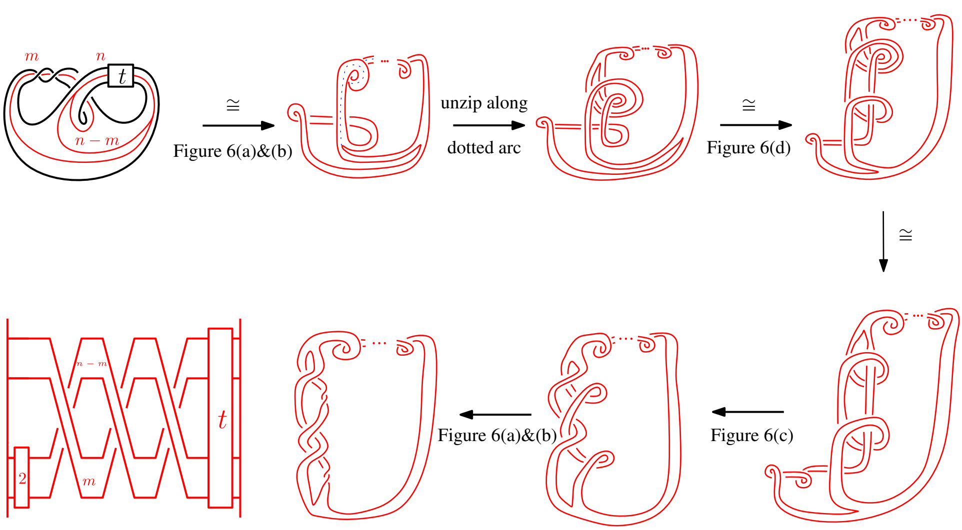


Figure 26

In the remaining two cases we will follow a slightly different way of identifying our curves as braid closures. We will see (as is evident in parts (c) and (d) of Figure 24) that the braids will not be positive or negative braids for general m, n and t values. We will then verify how under the various hypothesis listed in Theorem 1.3 these braids can be reduced to positive or negative braids.

Case 3: (m,n) ∞ curve with $m > n > 0$. We explain in Figure 27 below how the (m,n) ∞ curve with $m > n > 0$ is the closure of the braid in the bottom left of the figure. This braid is not obviously a positive or negative braid.

Case 3a: (m,n) ∞ curve with $m > n > 0$ and $m - tn > 0$. We want to show the braid in the bottom left of Figure 27 under the hypothesis that $m - tn > 0$ can be made a negative braid. We achieve this in Figure 28. More precisely, in part (a) of the figure we see the braid that we are working on. We apply the move in Figure 6(f) and some obvious simplifications to reach the braid in part (d). In part (e) of the figure we reorganize the braid: more precisely, since $m - tn > 0$ and $m - n = m - tn + (t-1)n$, we can split the piece of the braid in part (d) made of $m - n$ strands as the stack of $m - tn$ strands and a set of $t - 1$ many n strands. We then apply the move in Figure 6(f) repeatedly ($t - 1$ times) to obtain the braid in part (f). We note that the block labeled as "all negative crossings" is not important for our purposes to draw explicitly but we emphasize that each time we apply the move in Figure 6(f) it produces a full left-handed twist between a set of n strands and the rest. Next, sliding -1 full twists one by one from n strands over the block of these negative crossings we reach part (g). After further obvious simplifications and organizations in parts (h)–(j) we reach the braid in part (k), which is a negative braid.

Case 3b: (m,n) ∞ curve with $m > n > 0$ and $m - n < n$. We want to show in this case the braid in the bottom left of Figure 27 under the hypothesis that $m - n < n$ can be made a positive braid (regardless of t value). This is achieved in Figure 29.

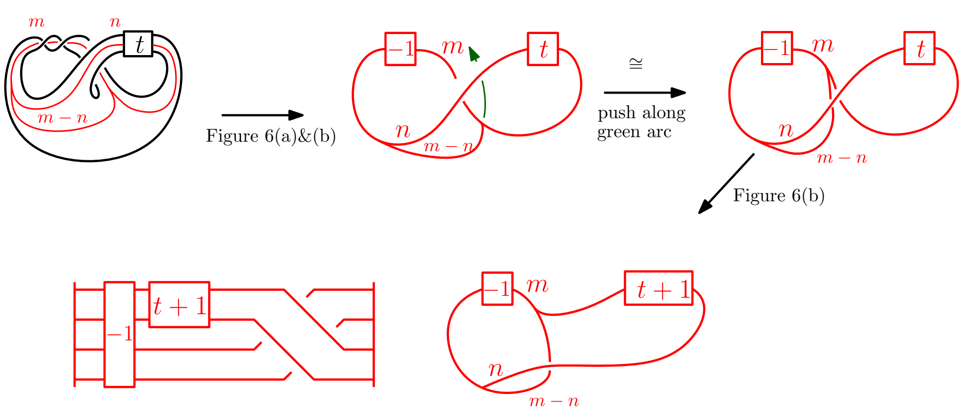


Figure 27

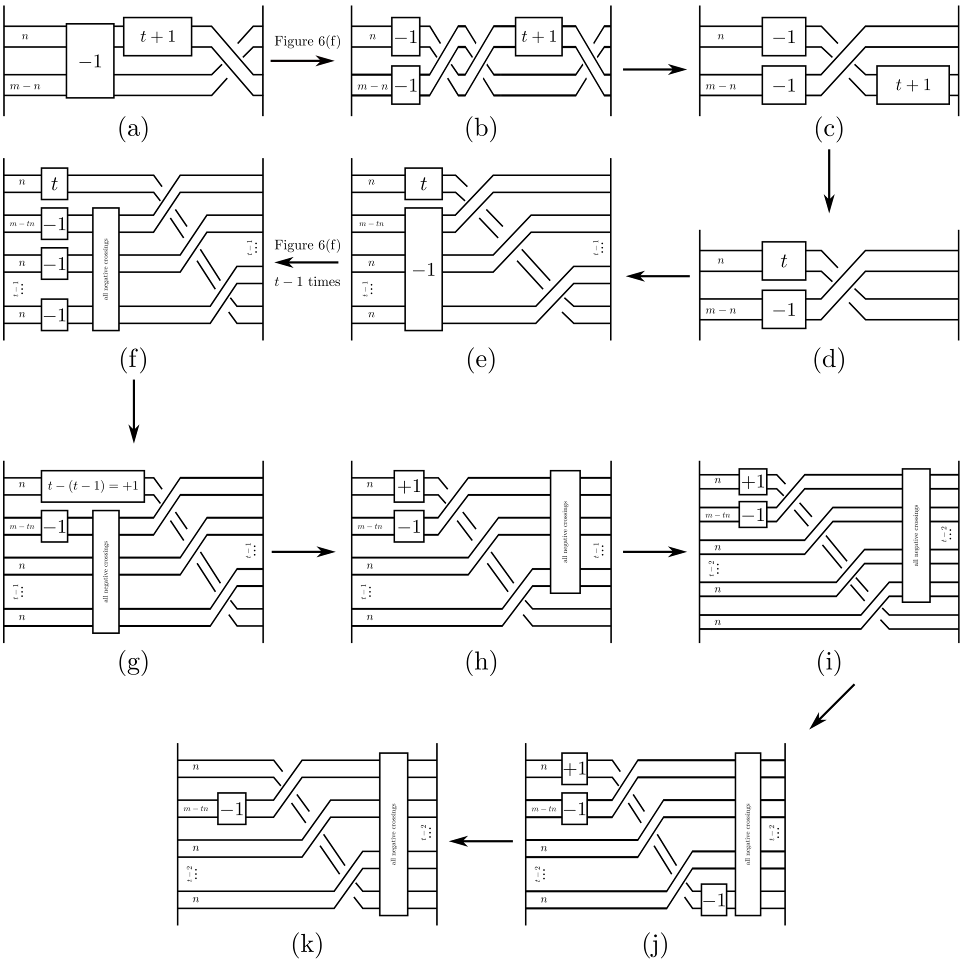


Figure 28

Case 4: (m, n) loop curve with $m > n > 0$. The arguments for this case are identical to those in Cases 3 and 3a above. The (m, n) loop curve with $m > n > 0$ is the closure of the braid that is drawn in the bottom left of Figure 30.

Case 4a: (m, n) loop curve with $m > n > 0$ and $m - tn > 0$. We show the braid, which the (m, n) ∞ curve with $m > n > 0$ is the closure of, can be made a negative braid under the hypothesis $m - tn > 0$. This follows steps very similar to those in Case 3a, which is explained through a series drawings in Figure 31.

Case 4b: (m, n) loop curve with $m > n > 0$ and $m - n < n$. Finally, we consider the (m, n) loop curve with $m > n > 0$ and $m - n < n$. Interestingly, this curve for $t > 2$ does not have to be the closure of a positive or negative braid. This will be further explored in the next section but for now we observe, through Figure 31(a)–(c), that

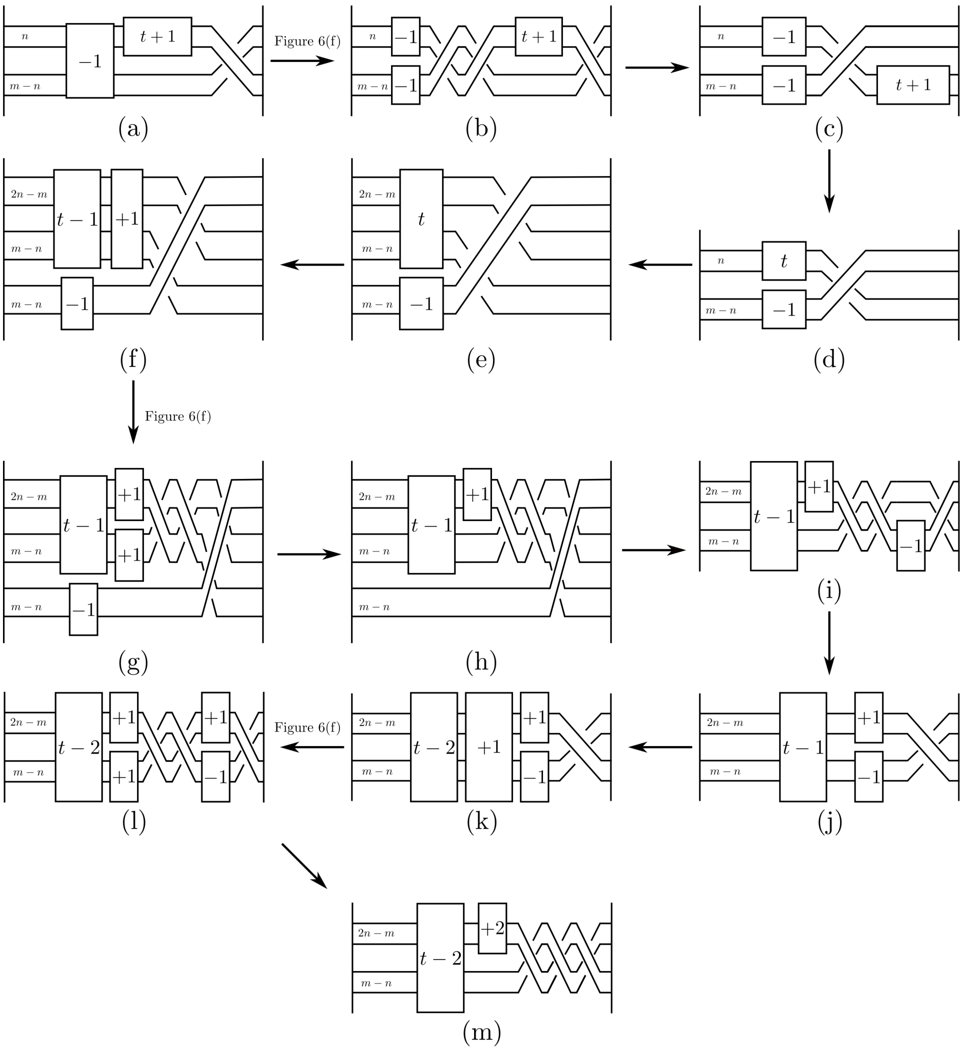


Figure 29

when $t = 2$ the curve is the closure of a negative braid: the braid in (a) in the figure is the braid from Figure 24(d). After applying the move in Figure 6(f) and simple isotopies, we obtain the braid in (c) which is clearly a negative braid when $t = 2$. □

Proof of Theorem 1.3. The proof of part (1) follows from Cases 1 and 2 above. Parts (2)(a)/(b) follows from Cases 3a/b and Case 4a above. As for part (3), observe that when $n > m$ by using Cases 1 and 2 we obtain that all homologically essential curves are the closures of positive braids. When $m > n$, we have either $m - 2n > 0$ or $m - 2n < 0$. In the former case we use Cases 3a and 4a to obtain that all homologically essential curves are the closures of negative braids. In the latter

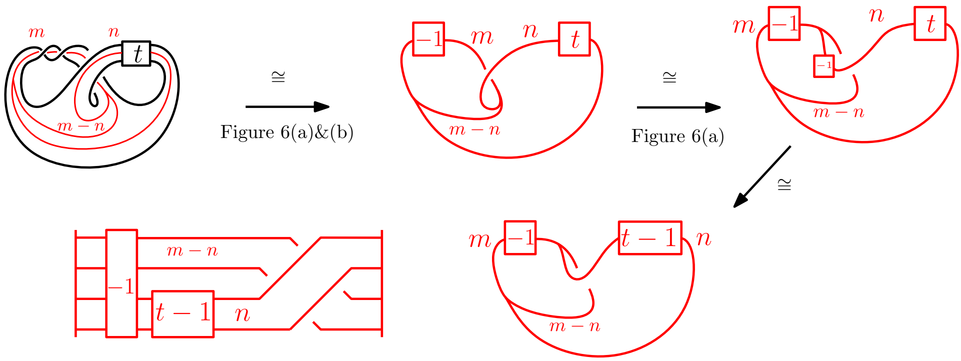


Figure 30

case, first note that $m - 2n < 0$ is equivalent to $m - n < n$, Now by Case 3b all homologically essential ∞ curves are the closures of positive braids, and by Case 4b all homologically essential loop curves are the closures of negative braids. Now by using Cromwell's result and some straightforward genus calculations we deduce that when $m > n > 1$ or $n > m \geq 1$ there are no unknotted curves among

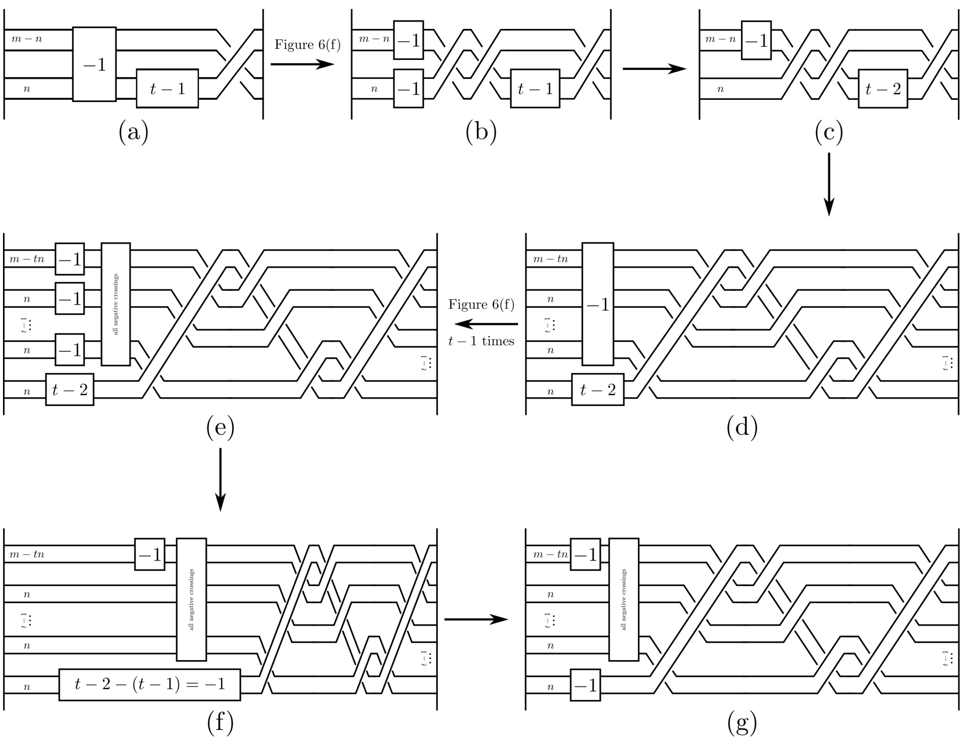


Figure 31

the (positive/negative) braid closures obtained in Cases 1–4 above. Therefore, there are exactly 5 unknotted curves among homologically essential curves on Σ_K for $K = K_t$ in Theorem 1.3. □

3D. *Twist knot with $t > 1$: part two.* In this section we consider the twist knots $K = K_t,\ t \geq 3$, and give the proof of Theorem 1.4.

Proof of Theorem 1.4. We show that the loop curve $(3, 2)$ when $t \geq 3$ is the pretzel knot $P(2t-5, -3, 2)$. This is explained in Figure 32. The braid in (a) is from Figure 24(d) with $m = 3$, $n = 2$, where we moved $(t-2)$ full right-handed twists to the top right end. We take the closure of the braid and cancel the left-handed half-twist on the top left with one of the right-handed half-twists on the top right to reach the knot in (c). In (c)–(g) we implement simple isotopies, and finally reach, in (h), the pretzel knot $P(2t-5, -3, 2)$. This knot has genus $t-1$ [Kim and Lee 2007, Corollary 2.7], and so is never unknotted as long as $t > 1$. This pretzel knot is slice exactly when $2t - 5 + (-3) = 0$, that is, when $t = 4$. The pretzel knot $P(3, -3, 3)$ is also known as 8_{20}. An interesting observation is that although $P(2t-5, -3, 2)$ for $t > 2$ is not a positive braid closure, it is a quasipositive braid closure. □

Proposition 3.8. *The (m, n) loop curve with $m - n = 1$, $n > 3$ and $t > 4$ is never slice.*

Proof. By Rudolph [1993], we have that, for a braid closure $\hat{\beta}$ when $k_+ \neq k_-$,

$$g_4(\hat{\beta}) \geq \tfrac{1}{2}(|k_+ - k_-| - n + 1),$$

where β is a braid in n strands, and $k_\pm$ is the number of positive and negative crossings in β. For quasipositive knots, equality holds, in which case, the Seifert genus is also the same as the four ball (slice) genus.

Now for the loop curve $c = (m, n)$ as in Figure 31(c), we have that

$$k_+ = (t-2)n(n-1) \quad \text{and} \quad k_- = (m-n)(m-n-1) + 3(m-n)n.$$

Hence, when $m - n = 1$, we get that $k_- = 3n$. Notice also that for $n \geq 3$, $t \geq 4$, we have $k_+ > k_-$. Thus, for $n > 3$, $t > 4$, $m - n = 1$, we obtain $c = \hat{\beta}$ is never slice because

$$g_4(\hat{\beta} = c) \geq \tfrac{1}{2}((t-2)n(n-1) - 3n - m + 1) = n((t-2)(n-1) - 4) > 0.$$

One can manually check that the $(4, 3)$ loop curve when $t = 3$ is also not slice. □

Remark 3.9. The inequality in the proof above can also be thought as a generalization to the Seifert genus calculation formula we used for positive/negative braid closures, since for those braids when $|k_+ - k_-|$ is the number of crossings, n, the braid number, is exactly the number of Seifert circles. Thus Rudolph's inequality

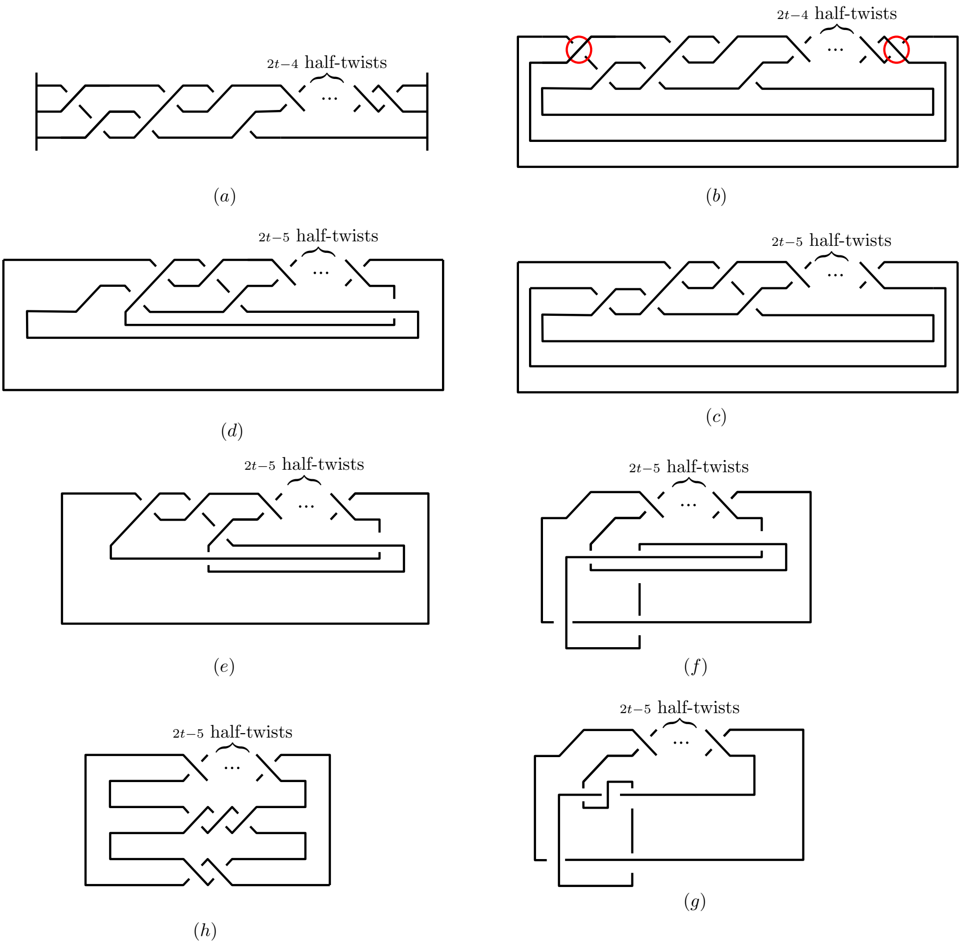


Figure 32

can also be used in the previous cases to show that there are no slice knots in the cases where we found that there are no unknotted curves.

4. Whitehead doubles

Proof of Theorem 1.6. Let $f : S^1 \times D^2 \to S^3$ denote a smooth embedding such that $f(S^1 \times \{0\}) = K$. Set $T = f(S^1 \times D^2)$. Up to isotopy, the collection of essential, simple closed, oriented curves in ∂T is parametrized by

$$\{m\mu + n\lambda \mid m, n \in \mathbb{Z} \text{ and } \gcd(m, n) = 1\},$$

where μ denotes a meridian in ∂T and λ denotes a standard longitude in ∂T coming from a Seifert surface [Rolfsen 1990; Saveliev 1999]. With this parametrization, the only curves that are null-homologous in T are $\pm\mu$ and the only curves that are

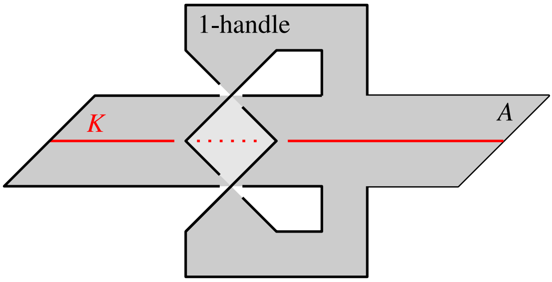


Figure 33. Standard genus-one Seifert surface F for a double of K.

null-homologous in $S^3 \setminus \text{int}(T)$ are $\pm\lambda$. Of course $\pm\mu$ will bound embedded disks in T, but $\pm\lambda$ will not bound embedded disks in $S^3 \setminus \text{int}(T)$ as K is a nontrivial knot. In other words, the only compressing curves for ∂T in S^3 are meridians.

Suppose now that C is a smooth, simple closed curve in the interior of T and there is a smoothly embedded 2-disk, say Δ, in S^3 such that $\partial\Delta = C$. Since C lies in the interior of T, we may assume that Δ meets ∂T transversely in a finite number of circles. Initially observe that if $\Delta \cap \partial T = \varnothing$, then we can use Δ to isotope C in the interior of T so that the result of this isotopy is a curve in the interior of T that misses a meridional disk for T. Now suppose that $\Delta \cap \partial T \neq \varnothing$. We show, in this case too, C can be isotoped to a curve that misses a meridional disk for T. To this end, let σ denote a simple closed curve in $\Delta \cap \partial T$ such that σ is innermost in Δ. That is, σ bounds a subdisk, Δ' say, in Δ, and the interior of Δ' misses ∂T. There are two cases, depending on whether or not σ is essential in ∂T. If σ is essential in ∂T, then, as has already been noted, σ must be a meridian. As such, Δ' will be a meridional disk in T, and C misses Δ'. If σ is not essential in ∂T, then σ bounds an embedded 2-disk, say D, in ∂T. It is possible that Δ meets the interior of D, but we can still cut and paste Δ along a subdisk of D to reduce the number of components in $\Delta \cap \partial T$. Repeating this process yields that if C is smoothly embedded curve in the interior of T and C is unknotted in S^3, then C can be isotoped in the interior of T so as to miss a meridional disk for T. (see [Rolfsen 1990, Theorem 9] and [Jaco 1980, page 13] for a use of similar ideas).

With all this in place, we return to discuss the Whitehead double of K. Suppose that F is a standard, genus-one Seifert surface for a double of K. See Figure 5. The surface F can be viewed as an annulus A with a 1-handle attached to it. Here K is a core circle for A, and the 1-handle is attached to A as depicted in Figure 33

Observe that F can be constructed so that it lives in the interior of T. Now, the curve C that passes once over the 1-handle and zero times around A obviously misses a meridional disk for T, and it obviously is unknotted in S^3. On the other hand, if C is any other essential simple closed curve in the interior of F, then C must go around A some positive number of times. It is not difficult, upon orienting, to see that C can be isotoped so that the strands of C going around A are coherently oriented. As such, C is homologous to some nonzero multiple of K in T. This, in turn, implies that C cannot be isotoped in T so as to miss some meridional disk for T. It follows that C cannot be an unknot in S^3. □

5. Contractible 4-manifolds and final remarks

Proof of Corollaries 1.9 and 1.10. In light of Theorem 1.7, the natural task is to determine the self-linking number s, with respect to the framing induced by the Seifert surface, for the unknotted curves found in Theorems 1.2 and 1.6. For this we use the Seifert matrix given by $S = \left(\begin{smallmatrix} -1 & -1 \\ 0 & t \end{smallmatrix}\right)$ where we use two obvious cycles — both oriented counterclockwise — in Σ_K. Recall that if $c = (m, n)$ is a loop curve, then m and n strands are endowed with the same orientation and hence the same signs. On the other hand, for an ∞ curve they will have opposite orientations and hence opposite signs. Therefore, given t, the self-linking number of a $c = (m, n)$ loop curve is $s = -m^2 - mn + n^2 t$, and the self-linking number of an (m, n) ∞ curve is $s = -m^2 + mn + n^2 t$. A quick calculation shows that the six unknotted curves in Figure 2 for $K_{-1} = T_{2,3}$ share self-linking numbers $s = -1, -3$. As we explained during the proof of Theorem 1.2 the infinitely many unknotted curves for the figure-eight knot $K_1 = 4_1$ reduce (via isotopies) to unknotted curves with $s = -1$ or $s = 1$. The five unknotted curves in Figure 4 for K_t, $t < -1$ or $t > 1$, share self-linking numbers $s = -1$, t and $t - 2$ (see [Cochran and Davis 2015]). Finally, Theorem 1.4 finds a slice but not unknotted curve, which is the curve $(3, 2)$ with $t = 4$. One can calculate from the formula above that this curve has self-linking number $s = 1$. Finally, the unique unknotted curve from Theorem 1.6 has self-linking number $s = -1$. The proofs follow as an obvious consequence of these calculations and Theorem 1.7 and its generalization in [Etnyre and Tosun 2023]. □

Next, we verify through Figure 34 that how not every essential curve on the genus-one Seifert surface of a twist knot must be the closure of a positive (or negative) braid closure. For example, we will show that an $(m, n) = (5, 2)$ ∞ curve on the Seifert surface of the twist knot K_3 as a smooth knot is the twist knot $m(5_2)$, which is known not to be a positive braid closure (e.g., via the KnotInfo database). To this end, we start with the braid as in Figure 34(a), which is the braid in Figure 27 where we substitute $m = 5$, $n = 2$ and $t = 3$. We then apply the move in Figure 6(f) to the full negative twist on 5 strands to obtain the braid in Figure 34(b). After a cancellation between a (-1) twist and a $(+4)$ twist and a small isotopy we get the braid in Figure 34(c). We apply the move in Figure 6(f) again, this time to the full negative twist on 3 strands from the bottom to obtain the braid in Figure 34(d). A small simplification gives the braid in Figure 34(e). Observe that the top strands can be eliminated — here it will be easier to think

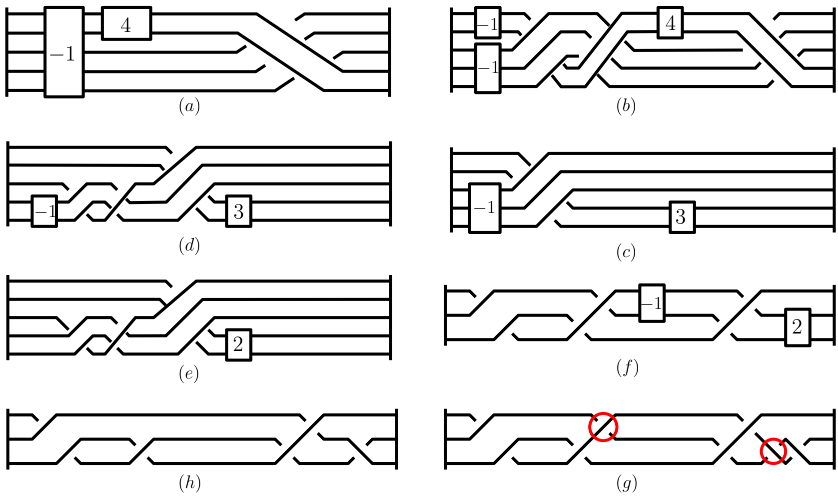


Figure 34. The knot $m(5_2)$ is an essential curve on the genus-one Seifert surface of the twist knot K_3.

of the corresponding braid closure — to get the 3-braid in Figure 34(f). A further simplification gives the braid in Figure 34(g). We can organize and simplify this braid by canceling the encircled half-crossings. This gives the braid in Figure 34(h). We claim that the closure of this braid is the knot $m(5_2)$-mirror of 5_2. One can see this by taking the closure and applying simple plane isotopies. This method is quite easy (and fun) but slightly lengthier. An alternative method is to observe that this braid has braid description $-1, -2, -2, -2, -1, 2$, which we can reorder, via braid isotopy, to be $-2, -2, -2, -1, 2, -1$. Now a quick inspection in the KnotInfo database [Livingston and Moore 2024] shows that the knot 5_2 has braid description $1, 1, 1, 2, -1, 2$. So the closure of the braid in Figure 34 is indeed $m(5_2)$. The KnotInfo database can also be used to verify the knot $m(5_2)$ is not the closure of a positive/negative braid.

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SUBHANKAR DEY
DURHAM UNIVERSITY
DURHAM, ENGLAND
UNITED KINGDOM
subhankar.dey@durham.ac.uk

VERONICA KING
UNIVERSITY OF TEXAS AUSTIN
AUSTIN, TX
UNITED STATES
viking0905@gmail.com

COLBY T. SHAW
SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GA
UNITED STATES
cshaw44@gatech.edu

BÜLENT TOSUN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA
TUSCALOOSA, AL
UNITED STATES

and

SCHOOL OF MATHEMATICS
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ
UNITED STATES
btosun@ua.edu

BRUCE TRACE
DEPARTMENT OF MATHEMATICS
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