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THE h-PRINCIPLE FOR MAPS TRANSVERSE TO BRACKET-GENERATING DISTRIBUTIONS

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Given a smooth bracket-generating distribution $\mathcal{D}$ of constant growth on a manifold M, we prove that maps from an arbitrary manifold Σ to M, which are transverse to $\mathcal{D}$, satisfy the complete h-principle. This partially settles a question posed by M. Gromov (1986).

1. Introduction

A *distribution* $\mathcal{D}$ on a manifold M is a (smooth) subbundle of the tangent bundle TM. Given such a $\mathcal{D}$, we can consider the sheaf $\Gamma\mathcal{D}$ of local sections to $\mathcal{D}$, i.e, local vector fields on M taking values in $\mathcal{D}$. The distribution $\mathcal{D}$ is called *bracket-generating* if at each point $x \in M$, the tangent space $T_x M$ is spanned by the vector fields obtained by taking finitely many successive Lie brackets of vector fields in $\Gamma\mathcal{D}$. We say $\mathcal{D}$ is $(r-1)$*-step bracket-generating* at $x \in M$, if there exists some integer $r = r(x)$ such that

$$T_x M = \operatorname{Span}\big\{[X_1, \dots, [X_{k-1}, X_k] \dots]_x \mid X_1, \dots, X_k \in \Gamma\mathcal{D},\ 1 \le k \le r\big\}.$$

Note that what we call a $(r-1)$-step bracket-generating is usually called an r-step bracket-generating distribution elsewhere in the literature. Bracket-generating distributions are the stepping stone for the field of sub-Riemannian geometry [Gromov 1996; Montgomery 2002].

One possible way to study a given distribution $\mathcal{D}$ is via smooth maps $u : \Sigma \to M$ from an arbitrary manifold Σ and looking at how the image of the differential $du : T\Sigma \to TM$ intersects $\mathcal{D}$. The map u is said to be *transverse* to $\mathcal{D}$ if we have that $du(T_\sigma \Sigma) + \mathcal{D}_{u(\sigma)} = T_{u(\sigma)} M$ holds for every $\sigma \in \Sigma$. Gromov [1986] asked the reader to prove the following.

Theorem. *Given a bracket-generating distribution $\mathcal{D}$ on a manifold M, maps $\Sigma \to M$ transverse to $\mathcal{D}$ satisfy the h-principle.*

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The idea of the proof as indicated in [Gromov 1986, p. 84] contained an error which was later acknowledged in [Gromov 1996, p. 254]. Eliashberg and Mishachev [2002, p. 136] showed that the proof indeed goes through if the distribution $\mathcal{D}$ is contact. In fact, their argument remains valid for any strongly bracket-generating distribution (or fat distribution, see [Montgomery 2002] for a definition). Moreover, they also planned out a strategy that could work for an arbitrary bracket-generating distribution as well. Del Pino and Shin [2020] carried out the ideas of Eliashberg and Mishachev [2002] and proved the h-principle for smooth maps transverse to *real analytic* bracket-generating distributions on a real analytic manifold. Their argument heavily depends on estimating the codimension of certain semianalytic sets in the jet bundle. It was also conjectured in the same article that Gromov's original statement should hold for a *smooth* bracket-generating distribution if certain higher-order jet calculations are performed.

The main goal of this article is to identify a suitable higher jet "regularity" condition (Definition 3.1) so that the sheaf of $\mathcal{D}$-horizontal maps $\mathbb{R} \to M$ satisfying this regularity is microflexible. The difficulty lies in proving the local h-principle for this class of maps provided the distribution is equiregularly bracket-generating (Definition 2.3), which is proved in Theorem 3.7. Then, applying Gromov's analytic and sheaf-theoretic techniques, the h-principles for transverse maps (Theorem 4.1) and for transverse immersions (Theorem 4.3), follow by a standard argument. We refer to [Gromov 1986; Eliashberg and Mishachev 2002] for the details of this theory.

The article is organized as follows: In Section 2, we recall some basic notions about bracket-generating distributions. In Section 3, we obtain the regularity criterion for maps $\mathbb{R} \to M$ horizontal to a bracket-generating distribution and prove the local h-principle for such maps. In Section 4 we prove the main h-principles.

2. Bracket-generating distributions

Definition 2.1. A *distribution* of rank n (and corank p) on a manifold M is a smooth vector subbundle of rank n (and corank p) of the tangent bundle TM.

Given any distribution $\mathcal{D} \subset TM$, we have the sheaf of local sections $\Gamma\mathcal{D}$, which is a sheaf of local vector fields on M. By the notation $X \in \mathcal{D}$ we shall mean a local section $X \in \Gamma\mathcal{D}$ defined on some unspecified open set of M. Given two arbitrary sheaves $\mathcal{E}, \mathcal{F}$ of vector fields on M (not necessarily given as sheaves of sections of some distribution), we can define the sheaf

$$[\mathcal{E}, \mathcal{F}] = \operatorname{Span}\big\{[X, Y] \mid X \in \mathcal{E},\ Y \in \mathcal{F}\big\},$$

where the span is taken over $C^\infty(M)$. Inductively, we then define

$$\mathcal{D}^0 = 0, \qquad \mathcal{D}^1 = \mathcal{D}, \qquad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i], \quad i \geq 1.$$

Definition 2.2. A distribution $\mathcal{D}$ is said to be *bracket-generating* if for each $x \in M$ we have $T_x M = \mathcal{D}_x^{r+1}$ for some $r = r(x)$. $\mathcal{D}$ is said to have type $\mathfrak{m} = \mathfrak{m}(x)$ at x for the $(r+2)$-tuple $\mathfrak{m} = (0 = m_0 \leq \cdots \leq m_{r+1} = \dim M)$, where $m_i = \operatorname{rk} \mathcal{D}_x^i$ for $0 \leq i \leq r+1$.

For a *generic* bracket-generating distribution $\mathcal{D}$ on M, the number of steps it takes to bracket-generate $T_x M$ is nonconstant and the sheaves $\mathcal{D}^i$ may fail to be of constant rank.

Definition 2.3. A bracket-generating distribution $\mathcal{D}$ on M is said to be r-step bracket-generating if $T_x M = \mathcal{D}_x^{r+1}$ for all $x \in M$. Furthermore, $\mathcal{D}$ is said to be *equiregular* (or, *of constant growth*) of type $\mathfrak{m} = (m_0 < \cdots < m_{r+1})$ if $\mathcal{D}$ has the type $\mathfrak{m}$ at every $x \in M$.

Throughout this article, we shall mostly restrict ourselves to equiregular distributions $\mathcal{D}$ of some fixed type $\mathfrak{m}$. In particular, each $\mathcal{D}^s$ will be a distribution, and we get a flag

$$0 = \mathcal{D}^0 \subset \mathcal{D} = \mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^{r+1} = TM.$$

It should be noted that in general, equiregularity is a nongeneric condition on the germs of distributions of a given rank, although most of the interesting distributions appearing in the literature possess this property.

Example 2.4. Contact and Engel distributions are well-studied examples of bracket-generating distributions that bracket-generates the tangent bundle in 1 and 2 steps respectively. More generally, we have Goursat structures which are certain rank 2, r-step bracket-generating distributions on manifolds of dimension $r+2$. Note that all of these distributions are equiregular as well. On the other hand, the Martinet distribution, given as the kernel $\ker(dz - y^2 dx)$ on $\mathbb{R}^3$ is *not* equiregular. We refer to [Montgomery 2002] for many more examples.

We shall need the following lemma in the next section.

Lemma 2.5. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution on M, of type $\mathfrak{m} = (m_0 < \cdots < m_{r+1})$. Set $p_s = \operatorname{rk}(\mathcal{D}^{s+1}/\mathcal{D}^s) = m_{s+1} - m_s > 0$. Then, for any $x \in M$ and for $1 \leq j \leq p_s$, $1 \leq s \leq r$, there exists a collection of vectors*

$$\tau^{s,j} \in \mathcal{D}_x, \quad \eta^{s,j} \in \mathcal{D}_x^s \setminus \mathcal{D}_x^{s-1}, \quad \zeta^{s,j} \in \mathcal{D}_x^{s+1} \setminus \mathcal{D}_x^s,$$

and 1-forms $\lambda^{s,j}$ defined near x, such that:

- *For each $1 \leq s \leq r$, $\mathcal{D}_x^{s+1} = \mathcal{D}_x^s + \operatorname{Span}\langle \zeta_x^{s,1}, \ldots, \zeta_x^{s,p_s} \rangle$.*
- *The 1-forms $\{\lambda^{s,j}\}$ are dual to $\{\zeta^{s,j}\}$ at x. Also, $\{\lambda^{s,j} \mid 1 \leq j \leq p_s, 1 \leq s \leq r\}$ is a frame for the annihilator bundle of $\mathcal{D}$ near x.*

- *For each* $1 \le s \le s' \le r$ *and* $1 \le j \le p_s,\ 1 \le j' \le p_{s'}$, *we have*

$$d\lambda^{s',j'}|_x(\tau^{s,j}, \eta^{s,j}) = \begin{cases} \delta_{j,j'}, & s' = s, \\ 0, & s' > s, \end{cases}$$

where $\delta_{j,j'}$ *is the Kronecker's delta function.*

Proof. Since $\mathcal{D}^{s+1} = \mathcal{D}^s + [\mathcal{D}, \mathcal{D}^s]$, we have the well-defined sheaf homomorphism

$$\begin{aligned} \Omega^s : \mathcal{D} \otimes \mathcal{D}^s/\mathcal{D}^{s-1} &\to \mathcal{D}^{s+1}/\mathcal{D}^s, \\ X \otimes (Y \bmod \mathcal{D}^{s-1}) &\mapsto -[X, Y] \mod \mathcal{D}^s. \end{aligned}$$

Furthermore, Ω^s is $C^\infty(M)$-linear and hence, for vectors $X \in \mathcal{D}_x$, $Y \in \mathcal{D}^s_x$ we have the linear maps

$$\Omega^s_x(X, Y \bmod \mathcal{D}^{s-1}_x) = -[\tilde{X}, \tilde{Y}]_x \mod \mathcal{D}^s_x,$$

where $\tilde{X} \in \mathcal{D}$, $\tilde{Y} \in \mathcal{D}^s$ are arbitrary extensions of X, Y respectively. Thus, we have that $\Omega^s : \mathcal{D} \otimes \mathcal{D}^s/\mathcal{D}^{s-1} \to \mathcal{D}^{s+1}/\mathcal{D}^s$ is a bundle map, which is surjective since $\mathcal{D}$ is bracket-generating, for $1 \le s \le r$.

Choose vectors $\tau^{s,j} \in \mathcal{D}_x,\ \eta^{s,j} \in \mathcal{D}^s_x \setminus \mathcal{D}^{s-1}_x$ so that $\{\Omega^s_x(\tau^{s,j}, \eta^{s,j}) \mid 1 \le j \le p_s\}$ forms a frame of $\mathcal{D}^{s+1}_x/\mathcal{D}^s_x$. Let us consider some arbitrary extensions $\tilde{\tau}^{s,j} \in \mathcal{D}$, $\tilde{\eta}^{s,j} \in \mathcal{D}^s \setminus \mathcal{D}^{s-1}$ of $\tau^{s,j}, \eta^{s,j}$ respectively, and denote $\tilde{\zeta}^{s,j} = -[\tilde{\tau}^{s,j}, \tilde{\eta}^{s,j}] \in \mathcal{D}^{s+1}$. Note that $\Omega^s(\tilde{\tau}^{s,j}, \tilde{\eta}^{s,j}) = \tilde{\zeta}^{s,j} \bmod \mathcal{D}^s$. Since $TM = \oplus_{s=0}^r \mathcal{D}^{s+1}/\mathcal{D}^s$, we have a local framing

$$TM \underset{\text{loc.}}{=} \mathcal{D} \oplus \operatorname{Span}\langle \tilde{\zeta}^{s,j},\ 1 \le j \le p_s,\ 1 \le s \le r \rangle$$

near x. Next, choose independent local 1-forms $\lambda^{s,j}$ near x, which are in the annihilator bundle $\operatorname{Ann}\mathcal{D}$ (i.e., $\lambda^{s,j}|_{\mathcal{D}} = 0$), and $\{\lambda^{s,j}\}$ is dual to $\{\tilde{\zeta}^{s,j}\}$. Note that

$$\mathcal{D}^s \underset{\text{loc.}}{=} \bigcap_{s' \ge s} \bigcap_{j=1}^{p^{s'}} \ker \lambda^{s',j}, \quad 1 \le s \le r.$$

Hence, for $s' \ge s$, we have

$$\begin{aligned} d\lambda^{s',j'}(\tau^{s,j}, \eta^{s,j}) &= \Big[\tilde{\tau}^{s,j}\underbrace{(\lambda^{s',j'}(\tilde{\eta}^{s,j}))}_{0} - \tilde{\eta}^{s,j}\underbrace{(\lambda^{s',j'}(\tilde{\tau}^{s,j}))}_{0} - \lambda^{s',j'}\underbrace{([\tilde{\tau}^{s,j}, \tilde{\eta}^{s,j}])}_{-\tilde{\zeta}^{s,j}}\Big]_x \\ &= \lambda^{s',j'}(\zeta^{s,j}) = \begin{cases} \delta_{j',j}, & s' = s, \\ 0, & s' > s. \end{cases} \end{aligned}$$

This concludes the proof of Lemma 2.5. □

3. Regularity of horizontal curves

Let us fix an arbitrary distribution $\mathcal{D}$ that has rank n and corank p on M with $\dim M = N = n + p$. Given a manifold Σ, a map $u : \Sigma \to M$ is called *$\mathcal{D}$-horizontal* if $du(T_\sigma \Sigma) \subset \mathcal{D}_{u(\sigma)}$ for each $\sigma \in \Sigma$. For simplicity, let us assume that $\mathcal{D}$ is given as the kernel of 1-forms $\lambda^1, \dots, \lambda^p$ on M. Then, $\mathcal{D}$-horizontal maps $\Sigma \to M$ are precisely the solutions of the following *nonlinear* differential operator:

$$\begin{aligned} \mathfrak{D} : C^\infty(\Sigma, M) &\to \Omega^1(\Sigma, \mathbb{R}^p) = \Gamma \hom(T\Sigma, \mathbb{R}^p), \\ u &\mapsto (u^*\lambda^1, \dots, u^*\lambda^p). \end{aligned} \tag{1}$$

To find solutions of $\mathfrak{D}$, we appeal to the Nash–Gromov implicit function theorem. As a first step, linearizing $\mathfrak{D}$ at some $u : \Sigma \to M$ we get the *linear* differential operator:

$$\begin{aligned} \mathfrak{L}_u : \Gamma u^*TM &\to \Omega^1(\Sigma, \mathbb{R}^p), \\ \xi &\mapsto \big[X \mapsto \big(d\lambda^s(\xi, u_* X) + X(\lambda^s \circ \xi)\big)_{s=1}^p\big], \end{aligned} \tag{2}$$

which restricts to the *bundle map* on $\Gamma u^*\mathcal{D}$:

$$\mathcal{L}_u := \mathfrak{L}_u|_{\Gamma u^*\mathcal{D}} : \xi \mapsto [X \mapsto (d\lambda^s(\xi, u_* X))].$$

An immersion $u : \Sigma \to M$ is called *$(d\lambda^s)$-regular* if the bundle map $\mathcal{L}_u$ is surjective. In general, $(d\lambda^s)$-regularity depends on our choice of 1-forms λ^s, whereas $(d\lambda^s)$-regularity of a $\mathcal{D}$-horizontal map is independent of any such choice. A $(d\lambda^s)$-regular horizontal immersion is also called *Ω-regular*, where $\Omega : \Lambda^2\mathcal{D} \to TM/\mathcal{D}$ is the associated curvature 2-form. It follows that the sheaf of Ω-regular horizontal maps $\Sigma \to M$ is microflexible [Gromov 1986, p. 339]. Note that $(d\lambda^s)$-regularity is a first-order condition on the class of maps. For the existence of a $(d\lambda^s)$-regular horizontal map, even when $\dim \Sigma = 1$, $\mathcal{D}$ must be 1-step bracket-generating. We now identify a suitable higher-order regularity for maps $\mathbb{R} \to M$ horizontal to an arbitrary distribution.

$\mathcal{W}$-regular horizontal curves. The arguments presented in this section follow the general scheme of algebraically solving (underdetermined) linear partial differential operators as in [Gromov 1986, p. 155]. Briefly, the idea is as follows. In order to solve the linear operator $\mathfrak{L}_u$ for some u, ideally we need to find out some linear operator $\mathfrak{M}_u : \Omega^1(\Sigma, \mathbb{R}^p) \to \Gamma u^*TM$ so that $\mathfrak{L}_u \circ \mathfrak{M}_u = \mathrm{Id}$ holds. This involves solving partial differential equations in the coefficients of $\mathfrak{M}_u$, which is in general hard to do. Instead, we try to look for a linear operator $\mathfrak{S}_u : \Gamma u^*TM \to \Omega^1(\Sigma, \mathbb{R}^p)$ satisfying $\mathfrak{S}_u \circ \overline{\mathfrak{L}_u} = \mathrm{Id}$, where $\overline{\mathfrak{L}_u} : \Omega^1(\Sigma, \mathbb{R}^p) \to \Gamma u^*TM$ is the formal adjoint of $\mathfrak{L}_u$. Note that this system is *algebraic* in the coefficients of $\mathfrak{S}_u$, and thus are considerably easier to solve. Once we get a smooth solution $\mathfrak{S}_u$, we can take the formal adjoint of the whole equation, and obtain $\mathfrak{L}_u \circ \overline{\mathfrak{S}_u} = \mathrm{Id}$, since $\overline{\overline{\mathfrak{L}_u}} = \mathfrak{L}_u$. Taking $\mathfrak{M}_u = \overline{\mathfrak{S}_u}$ we then have the desired solution to the original problem. This observation was used by Gromov to prove the fact that a generic *underdetermined* linear operator admits (universal) a right inverse [Gromov 1986, p. 156]. Although

$\mathfrak{L}_u$ is underdetermined, we are not able to appeal to this theorem directly, since we do not know whether the operator is sufficiently generic in the sense of Gromov. Instead, we explicitly identify a class of maps u for which the algebraic system always admits a (smooth) solution. We would like to note that a similar approach was also successfully used in [De Leo 2019], where the author proved the existence of nonfree isometric immersions.

Without loss of generality, we assume $M = \mathbb{R}^N$ and fix some coordinates $y^1, \dots, y^N$ on M. Let us write the 1-forms λ^s as

$$\lambda^s = \lambda^s_\mu \, dy^\mu, \quad 1 \le s \le p.$$

We also fix the (global) coordinate t on $\mathbb{R}$. For a function $u : \mathbb{R} \to \mathbb{R}^N$, the linearization operator $\mathfrak{L}_u$ (see (2)) is then given as

$$\begin{aligned}\mathfrak{L}_u : C^\infty(\mathbb{R}, \mathbb{R}^N) &\to C^\infty(\mathbb{R}, \mathbb{R}^p), \\ \xi = (\xi^\mu) &\to \big((\partial_\mu \lambda^s_\nu \circ u)\, \partial_t u^\nu \, \xi^\mu + (\lambda^s_\mu \circ u)\, \partial_t \, \xi^\mu\big)^p_{s=1}.\end{aligned} \tag{3}$$

Written in a matrix form we have $\mathfrak{L}_u(\xi) = \mathfrak{L}^0_u \xi + \mathfrak{L}^1_u \partial_t \xi$ where the $p \times N$ matrices $\mathfrak{L}^i_u$ are given as

$$\mathfrak{L}^0_u = \big((\partial_\mu \lambda^s_\nu \circ u)\, \partial_t u^\nu\big)_{p\times N}, \quad \mathfrak{L}^1_u = (\lambda^s_\mu \circ u)_{p\times N}. \tag{4}$$

Taking the formal adjoint of $\mathfrak{L}_u$, we get the first-order operator

$$\mathfrak{R}_u : C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$$

which can be written as

$$\mathfrak{R}_u = \mathfrak{R}^0_u + \mathfrak{R}^1_u \partial_t = (\mathfrak{L}^0_u - \partial_t \mathfrak{L}^1_u)^\dagger - (\mathfrak{L}^1_u)^\dagger \partial_t. \tag{5}$$

Observe that

$$\begin{aligned}\mathfrak{L}^0_u - \partial_t \mathfrak{L}^1_u &= \big((\partial_\mu \lambda^s_\nu \circ u)\, \partial_t u^\nu - \partial_t (\lambda^s_\mu \circ u)\big)_{p\times N} \\ &= \big((\partial_\mu \lambda^s_\nu \circ u - \partial_\nu \lambda^s_\mu \circ u)\, \partial_t u^\nu\big)_{p\times N} = \big(-(\iota_{u_* \partial_t} d\lambda^s)(\partial_\mu)\big)_{p\times N}.\end{aligned} \tag{6}$$

Let us now consider the equation

$$\mathfrak{S} \circ \mathfrak{R}_u = \mathrm{Id} \tag{7}$$

for an arbitrary order q linear operator $\mathfrak{S} : C^\infty(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$ given as

$$\mathfrak{S} := \mathfrak{S}^0 + \mathfrak{S}^1 \partial_t + \cdots + \mathfrak{S}^q \partial^q_t,$$

where $\mathfrak{S}^i$ are $p \times N$ matrices of functions. Note that (7) is *algebraic* in the entries of the matrices $\mathfrak{S}^i$. In fact, equation (7) represents a total of $p^2(q+2)$ many equations in $pN(q+1)$ many variables, namely, $\{\mathfrak{S}^i_{\alpha\beta} \mid 1 \le \alpha \le p,\ 1 \le \beta \le N,\ 0 \le i \le q\}$. This system is *underdetermined* if and only if

$$p^2(q+2) < pN(q+1) \quad \Leftrightarrow \quad nq > p-n. \tag{8}$$

Expanding out (7) we have

$$\begin{aligned}
\mathrm{Id} = \mathfrak{S} \circ \mathfrak{R}_u &= (\mathfrak{S}^0 + \mathfrak{S}^1 \partial_t + \cdots + \mathfrak{S}^q \partial_t^q) \circ (\mathfrak{R}^0_u + \mathfrak{R}^1_u \partial_t) \\
&= (\mathfrak{S}^0 \mathfrak{R}^0_u + \mathfrak{S}^1 \partial_t \mathfrak{R}^0_u + \cdots + \mathfrak{S}^q \partial_t^q \mathfrak{R}^0_u) \\
&\quad + \big(\mathfrak{S}^0 \mathfrak{R}^1_u + \mathfrak{S}^1(\mathfrak{R}^0_u + \partial_t \mathfrak{R}^1_u) + \cdots + \mathfrak{S}^q(q\,\partial_t^{q-1}\mathfrak{R}^0_u + \partial_t^q \mathfrak{R}^1_u)\big)\,\partial_t \\
&\quad + \cdots \\
&\quad + \big(\mathfrak{S}^{q-1}\mathfrak{R}^1_u + \mathfrak{S}^q(\mathfrak{R}^0_u + q\,\partial_t \mathfrak{R}^1_u)\big)\,\partial_t^q \\
&\quad + \mathfrak{S}^q \mathfrak{R}^1_u \partial_t^{q+1}.
\end{aligned}$$

Comparing both sides, we get the block-matrix system:

$$\begin{aligned}
&\begin{pmatrix} \mathfrak{S}^0 & \mathfrak{S}^1 & \ldots & \mathfrak{S}^{q-1} & \mathfrak{S}^q \end{pmatrix} \\
&\qquad \times \begin{pmatrix}
\mathfrak{R}^0_u & \mathfrak{R}^1_u & \ldots & 0 & 0 \\
\partial_t \mathfrak{R}^0_u & \mathfrak{R}^0_u + \partial_t \mathfrak{R}^1_u & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
\partial_t^{q-1}\mathfrak{R}^0_u & (q-1)\,\partial_t^{q-2}\mathfrak{R}^0_u + \partial_t^{q-1}\mathfrak{R}^1_u & \ldots & \mathfrak{R}^1_u & 0 \\
\partial_t^q \mathfrak{R}^0_u & q\,\partial_t^{q-1}\mathfrak{R}^0_u + \partial_t^q \mathfrak{R}^1_u & \ldots & \mathfrak{R}^0_u + q\,\partial_t \mathfrak{R}^1_u & \mathfrak{R}^1_u
\end{pmatrix} \\
&= \begin{pmatrix} \mathrm{Id}_{p\times p} & 0_{p\times p} & \ldots & 0_{p\times p} & 0_{p\times p} \end{pmatrix}.
\end{aligned} \tag{9}$$

Let us denote

$$R_u := -(\mathfrak{L}^0_u - \partial_t \mathfrak{L}^1_u) = (\iota_{u_*\partial_t}\, d\lambda^s)_{p\times N}, \qquad \Lambda := \mathfrak{L}^1_u = (\lambda^s_\mu \circ u)_{p\times N}, \tag{10}$$

so that from (5) we have

$$\mathfrak{R}^0_u = -(R_u)^\dagger \quad \text{and} \quad \mathfrak{R}^1_u = -\Lambda^\dagger. \tag{11}$$

Taking the adjoint of the coefficient matrix in (9) and multiplying by -1 we then get the following matrix:

$$A := \begin{pmatrix}
R_u & \partial_t R_u & \ldots & \partial_t^{q-1} R_u & \partial_t^q R_u \\
\Lambda & R_u + \partial_t \Lambda & \ldots & (q-1)\,\partial_t^{q-1} R_u + \partial_t^{q-1}\Lambda & q\,\partial_t^{q-1} R_u + \partial_t^q \Lambda \\
0 & \Lambda & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \Lambda & R_u + q\,\partial_t \Lambda \\
0 & 0 & \ldots & 0 & \Lambda
\end{pmatrix}. \tag{12}$$

A is a matrix of size $p(q+2) \times N(q+1)$, which depends on the $(q+1)$-th jet of the map u. The rows of this matrix can be linearly independent only if $nq \geq p-n$ holds (see (8)). Under the full rank condition, one can always solve for $\mathfrak{S}^i$ smoothly in (9), and thus solving (7). It should be noted that there is no unique solution, but instead, we have an affine space of them. The full rank of A will enable us to choose a solution that varies smoothly depending on $j_u^{q+1}(x)$.

Definition 3.1. For some fixed q satisfying $nq \geq p-n$, let us define the relation $\mathcal{W} \subset J^{q+1}(\mathbb{R}, \mathbb{R}^N)$ as

$$\mathcal{W} = \left\{ j_u^{q+1}(x) \mid du_x \text{ is injective and } A = A(j_u^{q+1}(x)) \text{ has full (row) rank} \right\}.$$

A smooth solution of $\mathcal{W}$ is called a *$\mathcal{W}$-regular* or *weakly $(d\lambda^s)$-regular* map. We denote by $\operatorname{Sol}\mathcal{W}$ the space of all $\mathcal{W}$-regular maps.

Lemma 3.2. *$(d\lambda^s)$-regular maps are $\mathcal{W}$-regular.*

Proof. If an immersion $u : \mathbb{R} \to \mathbb{R}^N$ is $(d\lambda^s)$-regular, then the block $\binom{R_u}{\Lambda}_{2p\times N}$ in the top left corner of A has full (row) rank, which makes the first two row-blocks of A full rank. On the other hand, the Λ blocks on the "off-diagonal" are always full rank, since the rows of Λ consist of linearly independent 1-forms $\{\lambda^s\}$. Note that there is no overlap between the $\binom{R_u}{\Lambda}_{2p\times N}$ block and the rest of the diagonal Λ blocks. Hence, the rows of A are linearly independent whenever u is $(d\lambda^s)$-regular, i.e., u is then $\mathcal{W}$-regular. □

In light of the above lemma, one observes that the first row-block of A is the one where the rank of A might drop, and one may consider $\mathcal{W}$-regularity as the natural higher-order analog of $d\lambda^s$-regularity. This observation shall become more clear in the proof of Theorem 3.7. Let us now show that $\mathfrak{L}_u$ admits a universal right inverse over $\mathcal{W}$-regular maps, i.e., one can solve $\mathfrak{L}_u \circ \mathfrak{M}_u = \mathrm{Id}$ for any $\mathcal{W}$-regular map u, such that $\mathfrak{M}_u$ depends smoothly on u.

Proposition 3.3. *Fix q satisfying $nq \geq p-n$ and the relation $\mathcal{W} \subset J^{q+1}(\mathbb{R}, \mathbb{R}^N)$. Then, for $\mathcal{W}$-regular maps $u : \mathbb{R} \to \mathbb{R}^N$, there exists a linear partial differential operator $\mathfrak{M}_u : C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$ of order q, satisfying $\mathfrak{L}_u \circ \mathfrak{M}_u = \mathrm{Id}$. Furthermore, the assignment*

$$\operatorname{Sol}\mathcal{W} \times C^\infty(\mathbb{R}, \mathbb{R}^p) \ni (u, P) \mapsto \mathfrak{M}(u, P) := \mathfrak{M}_u(P) \in C^\infty(\mathbb{R}, \mathbb{R}^N)$$

is a differential operator, nonlinear of order $2q+1$ in the first variable.

Proof. Fix a jet $\sigma = j_u^{q+1}(t) \in \mathcal{W}|_t$, represented by some map $u : \mathcal{O}p(t) \to \mathbb{R}^N$. The first-order operator $\mathfrak{R}_u$ defined on $\mathcal{O}p(t)$ gives rise to the (linear) symbol map

$$\Delta_{\mathfrak{R}_\sigma} : J^1(\mathbb{R}, \mathbb{R}^p)|_t \to J^0(\mathbb{R}, \mathbb{R}^N)|_t = C^\infty(\mathbb{R}, \mathbb{R}^N)|_t.$$

For any jet $j^1_P(t) \in J^1(\mathbb{R}, \mathbb{R}^p)|_t$ represented by some $P : \mathcal{O}\mathrm{p}(t) \to \mathbb{R}^p$, we then have $\Delta_{\mathfrak{R}_\sigma}(j^1_P(t)) = \mathfrak{R}_u(P)(t)$. We define $\Delta^{(q)}_{\mathfrak{R}_\sigma} : J^{q+1}(\mathbb{R}, \mathbb{R}^p)|_t \to J^q(\mathbb{R}, \mathbb{R}^N)|_t$ by

$$\Delta^{(q)}_{\mathfrak{R}_\sigma}(j^{q+1}_P(t)) = j^q_{\mathfrak{R}_u(P)}(t).$$

Since $\sigma \in \mathcal{W}$, the matrix $A_\sigma = A(j^{q+1}_u(t))$ has full row rank. We can then readily solve for $\mathfrak{S}^j = \mathfrak{S}^j(\sigma)$ in (9), in terms of rational polynomials of the terms of A_σ. Indeed, we get a (linear) map $\Delta_{\mathfrak{S}_\sigma} : J^q(\mathbb{R}, \mathbb{R}^N)|_t \to C^\infty(\mathbb{R}, \mathbb{R}^p)|_t$ satisfying the commutative diagram

$$\begin{array}{ccc} J^{q+1}(\mathbb{R}, \mathbb{R}^p)|_t & \xrightarrow{\ \Delta^{(q)}_{\mathfrak{R}_\sigma}\ } & J^q(\mathbb{R}, \mathbb{R}^N)|_t \\ & {\scriptstyle p^{q+1}_0} \searrow \quad \swarrow {\scriptstyle \Delta_{\mathfrak{S}_\sigma}} & \\ & J^0(\mathbb{R}, \mathbb{R}^p)|_t & \end{array}$$

Consider an open neighborhood $U(\sigma) \subset \mathcal{W}$ of σ so that the denominators of all the rational polynomials in $\Delta_{\mathfrak{S}_\sigma}$ remain nonzero for all jets $\tau \in U(\sigma)$. Shrinking $U(\sigma)$ if necessary, assume that $U(\sigma)$ projects down to an open neighborhood $V(\sigma) \subset \mathbb{R}$ of t. We then have a smooth map

$$\Delta_\sigma : U(\sigma) \times J^q(V(\sigma), \mathbb{R}^N) \to C^\infty(V(\sigma), \mathbb{R}^p),$$

so that for $\tau \in U(\sigma)|_s$ with $s \in V(\sigma)$ and for any $j^{q+1}_P(s)$, the following holds:

$$\Delta_\sigma\big(\tau, \Delta^{(q)}_{\mathfrak{R}_\tau}(j^{q+1}_P(s))\big) = p^{q+1}_0(j^{q+1}_P(s)) = P(s).$$

Note that Δ_σ is nonlinear in the first term, whereas it is linear in the second term.

We now have an open cover $\mathfrak{U} = \{U(\sigma)\}_{\sigma \in \mathcal{W}}$ of $\mathcal{W}$. Fix a partition of unity $\{\rho_\alpha\}_{\alpha \in \Lambda}$ on $\mathcal{W}$ subordinate to $\mathfrak{U}$, so that $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for some $U_\alpha \in \mathfrak{U}$. We denote the corresponding open set $V_\alpha \subset \mathbb{R}$ and the map

$$\Delta_\alpha : U_\alpha \times J^q(V_\alpha, \mathbb{R}^N) \to C^\infty(V_\alpha, \mathbb{R}^p).$$

Define the bundle map

$$\Delta_{\mathfrak{S}} : \mathcal{W} \times J^q(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$$

via the formula

$$\Delta_{\mathfrak{S}}(\tau, \eta) := \sum_\alpha \Delta_\alpha(\tau, \rho_\alpha(\tau)\eta).$$

Since each Δ_α is linear in the second argument, the map $\Delta_{\mathfrak{S}}$ is well-defined and smooth. Now, for jets $\tau = j^{q+1}_u(s) \in \mathcal{W}$ and $\eta = j^{q+1}_P(s) \in J^{q+1}(\mathbb{R}, \mathbb{R}^p)$, we have

$$\begin{aligned}
\Delta_{\mathfrak{S}}(\tau, \Delta^{(q)}_{\mathfrak{R}_\tau}(\eta)) &= \sum_\alpha \Delta_\alpha\big(\tau, \rho_\alpha(\tau)\Delta^{(q)}_{\mathfrak{R}_\tau}(\eta)\big) \\
&= \sum_\alpha \Delta_\alpha\big(\tau, \Delta^{(q)}_{\mathfrak{R}_\tau}(\rho_\alpha(\tau)\eta)\big), \quad \text{as } \Delta^{(q)}_{\mathfrak{R}_\tau} \text{ is linear} \\
&= \sum_\alpha p_0^{q+1}(\rho_\alpha(\tau)\eta), \qquad \text{by the construction of } \Delta_\alpha \\
&= p_0^{q+1}\Big(\sum_\alpha \rho_\alpha(\tau)\eta\Big) \\
&= p_0^{q+1}(\eta) = P(s).
\end{aligned}$$

Define $\mathfrak{S} : \operatorname{Sol}\mathcal{W} \times C^\infty(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$ via the formula

$$\mathfrak{S}(u, \xi) = \Delta_{\mathfrak{S}}(j_u^{q+1}, j_\xi^q).$$

The operator $\mathfrak{S}$ is nonlinear of order $q+1$ in the first component and linear of order q in the second component. We have $\mathfrak{S}(u, \mathfrak{R}_u(P)) = P$ for any $u \in \operatorname{Sol}\mathcal{W}$ and $P \in C^\infty(\mathbb{R}, \mathbb{R}^p)$.

Lastly, define the operator $\mathfrak{M} : \operatorname{Sol}\mathcal{W} \times C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$ by

$$\mathfrak{M}(u, P) = \mathfrak{M}_u(P) = \overline{\mathfrak{S}_u}(P),$$

where $\overline{\mathfrak{S}_u} : C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$ is the formal adjoint to the operator $\mathfrak{S}_u : \xi \mapsto \mathfrak{S}(u, \xi)$. We have

$$\mathfrak{L}_u \circ \mathfrak{M}_u = \overline{\mathfrak{R}_u} \circ \overline{\mathfrak{S}_u} = \overline{\mathfrak{S}_u \circ \mathfrak{R}_u} = \overline{\mathrm{Id}} = \mathrm{Id} \quad \text{for any } u \in \operatorname{Sol}\mathcal{W}.$$

Clearly $\mathfrak{M}$ is a differential operator, which is linear of order q in the second component. Since taking adjoint of the q-th order operator $\mathfrak{S}_u$ itself has order q, we have $\mathfrak{M}$ is nonlinear of order $2q+1$ in the first component. □

Following Gromov's terminology [Gromov 1986, pp. 115–116], Proposition 3.3 implies that for any q satisfying $nq \geq p - n$, the first-order operator

$$\mathfrak{D} : C^\infty(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$$

is infinitesimally invertible over $\mathcal{W}$-regular maps, with defect $2q+1$ and order of inversion q. For $\alpha \geq 0$, denote the relation of α-infinitesimal solutions of $\mathfrak{D} = 0$ as

(13) $$\mathcal{R}^\alpha_{\mathrm{tang}} = \{ j_u^{\alpha+1}(x) \mid j^\alpha_{\mathfrak{D}(u)}(x) = 0 \} \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N), \quad \alpha \geq 0.$$

Next, for $\alpha \geq (2q+1) - 1 = 2q$ denote the relation of $\mathcal{W}$-regular α-infinitesimal solutions of $\mathfrak{D} = 0$ by

(14) $$\mathcal{W}_\alpha = (p^{\alpha+1}_{q+1})^{-1}(\mathcal{W}) \cap \mathcal{R}^\alpha_{\mathrm{tang}} \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N).$$

Each $\mathcal{W}_\alpha$ has the same C^∞-solutions for $\alpha \geq 2q$, namely, the $\mathcal{W}$-regular $\mathcal{D}$-horizontal curves. Denote the sheaves

$$\Phi^{\mathcal{W}} = \operatorname{Sol} \mathcal{W}_\alpha \quad \text{and} \quad \Psi_\alpha^{\mathcal{W}} = \Gamma \mathcal{W}_\alpha \qquad \text{for } \alpha \geq 2q. \tag{15}$$

A direct application of the results in [Gromov 1986, pp. 118–120] then gives us:

Theorem 3.4. *Fix q satisfying $nq \geq p - n$, where $p = \operatorname{cork} \mathcal{D}$, $n = \operatorname{rk} \mathcal{D}$. Then*:

- *$\Phi^{\mathcal{W}}$ is microflexible.*
- *For any*
$$\alpha \geq \max\{2q + 1 + q,\ 2.1 + 2q\} = 3q + 1,$$
the jet map $j^{\alpha+1} : \Phi^{\mathcal{W}} \to \Psi_\alpha^{\mathcal{W}}$ is a local weak homotopy equivalence.

Remark 3.5. It should be noted that the $\mathcal{W}$-regularity of $\mathcal{D}$-horizontal maps $\mathbb{R} \to M$ is independent of any choice of coordinates on M or choice of defining 1-forms λ^s. Indeed, these are precisely the class of maps $\mathbb{R} \to M$ over which the operator $\mathfrak{D}$ (see (1)) is infinitesimally invertible. Since the solution space $\mathfrak{D} = 0$ is independent of any choice, so is the regularity of such maps.

Local h-principle for $\mathcal{W}$-regular horizontal curves. To keep the notation light, throughout the rest of this section, we shall treat any higher jet $j_u^q(x)$ formally as variables. That is, $j_u^q(x)$ really represents the tuple of *formal* maps

$$\left(F^i : \odot^i T_x \mathbb{R} \to T_{u(x)} \mathbb{R}^N, \quad 1 \leq i \leq q\right)$$

in the jet space $J^q_{(x,u(x))}(\mathbb{R}, \mathbb{R}^N)$, and each component $\partial_t^i u(x) \equiv F^i(\partial_t^i) \in \mathbb{R}^N$ are independent variables. For any 1-form λ defining $\mathcal{D}$ near $y = u(x)$, the components of the higher jets $j_\lambda^q(y)$ will be treated as known scalar values.

Now, consider the first-order relation

$$\mathcal{R}_{\text{imm-tang}} = \left\{ j_u^1(x) \in J^1(\mathbb{R}, \mathbb{R}^N) \mid du_x \text{ is injective and } \operatorname{Im} du_x \subset \mathcal{D}_{u(x)} \right\}. \tag{16}$$

The solution sheaf $\operatorname{Sol} \mathcal{R}_{\text{imm-tang}}$ consists of all the $\mathcal{D}$-horizontal immersed curves. It follows from (13) that $\mathcal{R}_{\text{imm-tang}} \subset \mathcal{R}^0_{\text{tang}}$. For any $\alpha \geq q$, where q satisfies $nq \geq p - n$, we have from (14) that the jet projection map $p_1^{\alpha+1} : J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N) \to J^1(\mathbb{R}, \mathbb{R}^N)$ restricts to a map

$$p_1^{\alpha+1}|_{\mathcal{W}_\alpha} : \mathcal{W}_\alpha \to \mathcal{R}_{\text{imm-tang}}.$$

In fact, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{W}_\alpha & \hookrightarrow & \mathcal{R}^\alpha_{\text{tang}} & \hookrightarrow & J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N) \\
\downarrow {\scriptstyle p_1^{\alpha+1}|_{\mathcal{W}_\alpha}} & & \downarrow {\scriptstyle p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}}} & & \downarrow {\scriptstyle p_1^{\alpha+1}} \\
\mathcal{R}_{\text{imm-tang}} & \hookrightarrow & \mathcal{R}^0_{\text{tang}} & \hookrightarrow & J^1(\mathbb{R}, \mathbb{R}^N)
\end{array}$$

Note that $p_1^{\alpha+1}$ is an affine bundle. Let us now make the following easy observation.

Lemma 3.6. *For any $\sigma \in \mathcal{R}^0_{\text{tang}}$, the fiber of $p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}}$ over σ is contractible.*

Proof. A jet $j_u^{\alpha+1}(x) \in \mathcal{R}^\alpha_{\text{tang}} \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N)$ is defined by the equation $j^\alpha_{\mathfrak{D}(u)}(x) = 0$ (see (13)), which expands to the following system:

$$\left.\begin{aligned} 0 &= \partial_t^k (u^* \lambda^s(\partial_t))|_x \\ &= \partial_t^k ((\lambda^s_\mu \circ u)\, \partial_t u^\mu)|_x \\ &\Rightarrow (\lambda^s_\mu \circ u)\, \partial_t^{k+1} u^\mu|_x + \text{terms involving } j_u^k(x) = 0 \end{aligned}\right\}, \tag{17}$$

where $1 \le s \le p$, $0 \le k \le \alpha$. Recall the matrix $\Lambda = (\lambda^s_\mu \circ u(x))_{p \times N}$ from (10) and denote by $\partial_t^k u$ the column vector $\partial_t^k u = (\partial_t^k u^\mu(x))_{N \times 1}$. Then, equation (17) can be expressed as the following affine system:

$$\left.\begin{aligned} \Lambda \partial_t u &= 0 \\ \Lambda \partial_t^2 u &= \big(-(\partial_\nu \lambda^s_\mu \circ u)\, \partial_t u^\nu\, \partial_t u^\mu\big)_{p\times 1}\big|_x \\ &\;\;\vdots \\ \Lambda \partial_t^{\alpha+1} u &= p \times 1 \text{ vector involving } j_u^\alpha(x) \end{aligned}\right\}. \tag{18}$$

Note that we have $\mathcal{R}^0_{\text{tang}}|_{(x,u(x))} = \ker \Lambda$. Since Λ has full rank, given any value of $\sigma = j_u^1(x) \equiv \partial_t u \in \mathcal{R}^0_{\text{tang}}$, the above system can always be solved in a triangular way. Clearly, at each step the solution space is affine. It follows that the fiber of $p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}}$ over σ is contractible. □

The discussion so far had no extra assumption on the distribution $\mathcal{D}$. From this point onwards, we shall only consider equiregular, bracket-generating distributions. The main goal of this section is to prove the following.

Theorem 3.7. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution of rank n and corank p on $\mathbb{R}^{N=n+p}$, with type $\mathfrak{m}$. Let q_0 satisfy $nq_0 \ge p - n$. Fix a jet $\sigma = j_u^1(x) \in \mathcal{R}_{\text{imm-tang}}$. Then for each $K \ge 1$, there exists some $q(\mathfrak{m}, K) \ge q_0$, such that for any $\alpha \ge q(\mathfrak{m}, K)$ the complement of the fiber $\mathcal{W}_\alpha|_\sigma = (p_1^{\alpha+1}|_{\mathcal{W}_\alpha})^{-1}(\sigma)$ has codimension at least K in $\mathcal{R}^\alpha_{\text{tang}}|_\sigma = (p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}})^{-1}(\sigma)$.*

As a corollary, we get the local h-principle for $\mathcal{W}$-regular horizontal maps.

Corollary 3.8. *The sheaf map $\Phi^{\mathcal{W}} \to \Gamma\mathcal{R}_{\text{imm-tang}}$ induced by the differential map is a local weak homotopy equivalence.*

Proof. Fix some jet $\sigma \in \mathcal{R}_{\text{imm-tang}}$. It follows from Theorem 3.7 and Lemma 3.6 that the fiber of $p_1^{\alpha+1}|_{\mathcal{W}_\alpha}$ over σ is K-connected for α sufficiently large. Hence, passing to the infinity jet, we get that the fiber is weakly contractible. By an argument of Gromov [1986, p. 77], the sheaf map $p_1^\infty : \Psi^{\mathcal{W}}_\infty = \Gamma\mathcal{W}_\infty \to \Gamma\mathcal{R}_{\text{imm-tang}}$ is then a local weak homotopy equivalence. Also, from Theorem 3.4 we have the sheaf map $j^\infty : \Phi^{\mathcal{W}} \to \Psi^{\mathcal{W}}_\infty$ is a local weak homotopy equivalence. But then the composition

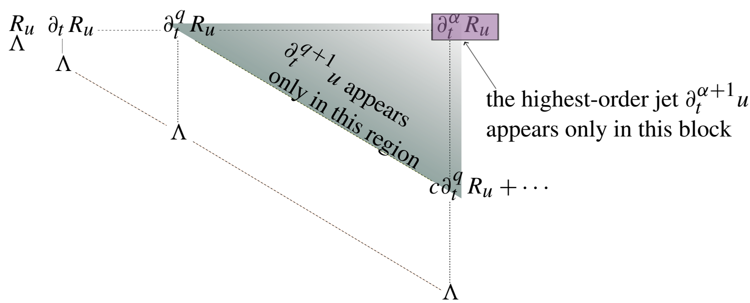

Figure 1. Highest-order jet of u in A.

$p_1^\infty \circ j^\infty : \Phi^{\mathcal{W}} \to \Gamma\mathcal{R}_{\text{imm-tang}}$ is a local weak homotopy equivalence as well. Note that the composition map is nothing but the differential map $u \mapsto du$. □

To prove Theorem 3.7, we need to understand the equations involved in defining the relation $\mathcal{W}_\alpha \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N)$ as in (14). We have already seen in Lemma 3.6 that given a jet $\sigma \in \mathcal{R}_{\text{imm-tang}}$, the fiber of $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$ is the solution space of a triangular affine system (see (18)). But a jet $j_u^{\alpha+1}(x) \in \mathcal{W}_\alpha|_\sigma \subset \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ must satisfy $\mathcal{W}$-regularity as well, i.e., the matrix $A = A(j_u^{\alpha+1}(x))$ as given in (12) must have independent rows. We note the following features of the matrix that will become useful later in the proof.

Firstly, the Λ blocks in the off-diagonal of A have full rank. Thus, the rank can only drop at the first row-block. In any block above the Λ-diagonal, the highest-order jet term $\partial_t^{q+1} u$ is contributed by the $\partial_t^q R_u$ factor, and it appears linearly. In fact, from (10) we have

$$\partial_t^q R_u = \left(d\lambda^s(\partial_t^{q+1} u, \partial_\mu)\right)_{p\times N} + \text{a } p \times N \text{ matrix in } j_u^q(x). \tag{19}$$

Furthermore, no component of $\partial_t^{q+1} u$ appears anywhere below the diagonal passing through this block (see Figure 1). In particular, in each column-block, the highest-order derivative of u occurs in the first row-block only.

Secondly, each column-block of A has N many columns, which can be labeled by the framing $\{\partial_1, \ldots, \partial_N\}$ of $T\mathbb{R}^N$, as it is clear from (19). For any arbitrary choice of frame $\{W_1, \ldots, W_N\}$, we can always perform some (invertible) column operations on A so that the columns in the target column-block, say the $(q+1)$-th column-block, are now labeled by $\{W_1, \ldots, W_N\}$. Indeed, if we write $W_i = W_i^j \partial_j$, then one can consider the invertible matrix $W = (W_1^j \ \ldots \ W_N^j)_{N\times N}$, so that multiplying the $\partial_t^q R_u$ block from the right by W transforms it into the following block:

$$\left(d\lambda^s(\partial_t^{q+1} u, W_\mu)\right)_{p\times N} + \text{a } p \times N \text{ matrix in } j_u^q(x). \tag{20}$$

We extend this W to a block-diagonal-matrix $\widetilde{W}$ of size $N(\alpha+1) \times N(\alpha+1)$ by putting W as the $(q+1)$-th diagonal block, and Id_N in all the other diagonal positions. Now, if we multiply the matrix A by $\widetilde{W}$ from the right, it will perform column manipulations precisely at the $(q+1)$-th column-block. In particular, the top row-block in this column is now $(\partial_t^q R_u)W$, and thus by looking at (20), we may label this column-block by $W_1, \dots, W_N$. Note that this process does not change the rank of the matrix, since the column manipulation is invertible by construction.

In the proof, for each column-block of A, we shall only prescribe a subframe of $T\mathbb{R}^N$ (obtained by using Lemma 2.5), which will then be arbitrarily extended to a full frame. Performing the column manipulation as described above will make sure that the target column-block is labeled first by the prescribed subframe and then by the arbitrary choice of extension. As we shall see, we are not interested in the columns which are labeled arbitrarily during this process. If the matrix has full (row) rank after discarding a few columns, then the original matrix will also have full rank. Thus, given a subframe, say, $(W_1, \dots, W_t)$ for the $(q+1)$-th column-block, we shall say that *the* $(q+1)$*-th column-block* is relabeled by the subframe, while discarding the arbitrarily extended part.

Let us now proceed with the proof of Theorem 3.7. We refer to page 226, where the major steps of the proof are carried out in detail for an example case.

Proof of Theorem 3.7. Let $\sigma = j_u^1(x) \in \mathcal{R}_{\text{imm-tang}}$ be a given jet, and $y = u(x)$. Suppose $\mathcal{D}$ has the type $\mathfrak{m} = (0 = m_0 < \cdots < m_{r+1} = N = n+p)$, where $m_s = \dim \mathcal{D}_y^s$ for $0 \le s \le r+1$. We denote $p_s = \dim(\mathcal{D}_y^{s+1}/\mathcal{D}_y^s) = m_{s+1} - m_s$ for $1 \le s \le r$ and set $p_0 = 0$. Using Lemma 2.5, we get the vectors

$$\tau^{s,j} \in \mathcal{D}_y, \quad \eta^{s,j} \in \mathcal{D}_y^s \setminus \mathcal{D}_y^{s-1}, \quad \zeta^{s,j} \in \mathcal{D}_y^{s+1} \setminus \mathcal{D}_y^s, \qquad 1 \le j \le p_s,\ 1 \le s \le r$$

at y, and the 1-forms $\lambda^{s,j}$ near y. We write the matrix A (see (12)) in terms of these $\lambda^{s,j}$'s. As observed in Remark 3.5, this does not change the relations $\mathcal{W}_\alpha$.

Notations: We label the row- and column-blocks of A starting from 0, so that the $(0,q)$-th block is $\partial_t^q R_u$ and the $(q+1,q)$-th block is the Λ block. We will use $\zeta^{s,\bullet}$ to mean the tuple of vectors $(\zeta^{s,1}, \dots, \zeta^{s,p_s})$ and similarly $\tau^{s,\bullet}$, $\eta^{s,\bullet}$, $\lambda^{s,\bullet}$ etc. In particular, the matrix Λ is then given by $\Lambda^\dagger = (\lambda^{1,\bullet} \ \dots \ \lambda^{p,\bullet})_{N\times p}$. We shall also use the notation $\zeta^\bullet$ for the tuple $(\zeta^{1,\bullet}, \dots, \zeta^{r,\bullet})$ of size p, and $\hat{\zeta}^{s,\bullet}$ for $1 \le s \le r$ for the tuple of size $p - p_s$, obtained by dropping the tuple $\zeta^{s,\bullet}$ from $\zeta^\bullet$. For notational convenience, we set $\hat{\zeta}^{0,\bullet} = \zeta^\bullet$.

Let us first assume $K = 1$. We need to show that the complement of $\mathcal{W}_\alpha|_\sigma$ in $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$ has codimension 1 for some α large enough. To achieve this, we shall find a polynomial P in the jet variables $j_u^\alpha(x)$ so that P being nonzero at some $\tilde{\sigma} \in \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ implies that the matrix $A = A(\tilde{\sigma})$ has full (row) rank. Let us briefly discuss the proof strategy.

Step 1: For each column-block of A, we shall prescribe some subframe of $T_y\mathbb{R}^N$, consisting of some suitable vectors as obtained by Lemma 2.5 (and thus using the fact that $\mathcal{D}$ is bracket-generating). This step (Algorithm 1) is recursive, and it will determine the value of $q(\mathfrak{m}, 1)$. Extending each subframe arbitrarily to a full frame and then relabeling the column-blocks, we get a new matrix, say, A_1. Since these operations are invertible, A has full rank if and only if A_1 has full rank.

Step 2: We shall consider the submatrix B of A_1, with columns labeled by the prescribed subframes as above and all the rows of A_1. B will be a square matrix and $P = \det B$ will be our candidate polynomial in the higher jet variables $j_u^{\alpha+1}(x)$. Since it is a minor of A_1, $\det B \neq 0$ at some higher jet $\tilde{\sigma} \in \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ implies that $A(\tilde{\sigma})$ has full rank, i.e, $\tilde{\sigma} \in \mathcal{W}_\alpha|_\sigma$. We shall keep using the notation Λ and $\partial_t^q R_u$ to denote the respective blocks in B obtained after the column transformations and curtailing of A. Performing some more (invertible) row and column operations on B, we shall produce a new matrix B_1, so that $\det B = \det B_1$. Next, we shall extract a square submatrix C of B_1 and observe that $\det B = \det B_1 = \pm \det C$.

Step 3: It follows from (18) that if $j_u^q(x)$ is solved, then the solution space for $\partial_t^{q+1}u$ is given as the *affine* space $V_q + \ker\Lambda = V_q + \mathcal{D}_{u(x)}$, where the $N \times 1$ vector $V_q = V_q(j_u^q(x))$ is obtained using some fixed choice of right inverse Λ^{-1}. In particular, we can write $\partial_t^{q+1}u = X_q\tau^q + V_q(j_u^q(x))$ for $1 \le q \le \alpha$, where X_q is some indeterminate and τ^q is a vector suitably set in Algorithm 1 to either $0 \in \mathcal{D}_y$ or to one of the vectors $\tau^{s,j} \in \mathcal{D}_y$ chosen earlier. Inductively, we then have

$$(21)\qquad \partial_t^{q+1}u = X_q\tau^q + \text{terms in } X_1, \dots, X_{q-1},\ \tau^1, \dots, \tau^{q-1} \text{ and } \sigma = j_u^1(x)$$

for $1 \le q \le \alpha$. Arbitrary values of $X_1, \dots, X_\alpha$ will produce $j_u^{\alpha+1}(x) \in \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ from (21). We will replace the values of $\partial_t^{q+1}u$ in the matrix C. Treating $\det C$ as a polynomial in the indeterminates X_q we shall show that $\det C$ is nonvanishing for suitably large values of X_q. Thus, $P = \det B$ is nonvanishing when restricted to $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$. This will conclude the proof for $K = 1$.

The crux of the proof lies in suitably choosing the subframes for each of the column-blocks of A as done in Step 1. We produce a schematic diagram to explain the process (Figure 2). As discussed earlier, the rank can only drop in the first row-block consisting of $\partial_t^q R_u$, which are represented by the red boxes in Figure 2, whereas the blue boxes represent the Λ blocks. Suppose we have dealt with the rows corresponding to $d\lambda^{1,1}, \dots d\lambda^{s',j'}$ of the first row-block in a manner that, ignoring the rest of the rows from the first row-blcok, the matrix A up to, say, the q-th column-block has full rank. We could keep choosing $\zeta^\bullet$ for the subsequent column-blocks, which will transform all the Λ blocks appearing in those column-blocks into Id_p, and thus make sure that the matrix A, after *ignoring* the rest of the rows from the first row-block, is indeed full rank.

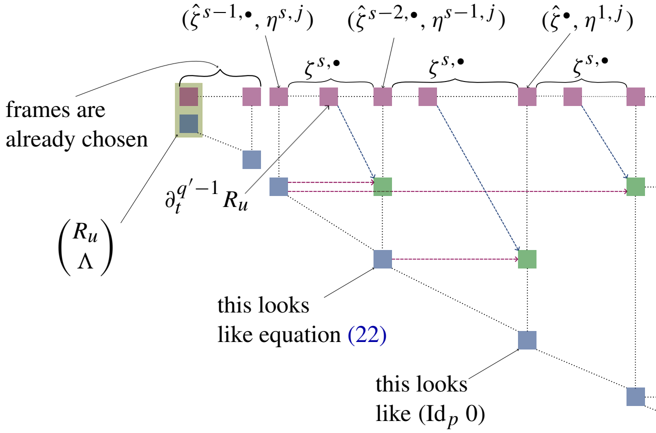

Figure 2. How Algorithm 1 chooses the labeling subframes.

Now, suppose the next row in the first row-block corresponds to $d\lambda^{s,j}$. We label the $(q+1)$-th column-block by the frame $(\hat{\zeta}^{s-1,\bullet}, \eta^{s,j})$. It follows from (20) that the last column of the $\partial_t^q R_u$ block transforms into

$$d\lambda^{s,j}(\partial_t^{q+1} u, \eta^{s,j}) + \text{a } p \times 1 \text{ vector in } j_u^q(x).$$

Since $\partial_t^{q+1} u$ is a jet that did not appear earlier in the matrix (Figure 1), we can prescribe its value arbitrarily to make sure that (at least) the row corresponding to $d\lambda^{s,j}$ in $\partial_t^q R_u$ must be linearly independent. Indeed, it follows from Lemma 3.6, that given $j_u^q(x)$, we may choose $\partial_t^{q+1} u$ from an *affine* space, which is parallel to the distribution $\mathcal{D} = \ker \Lambda$, and so, given a particular solution of $\partial_t^{q+1} u$, we can always add a vector proportional to $\tau^q = \tau^{s,j} \in \mathcal{D}^s$. This is done in Step 3, by adding $X_q \tau^q$ for large value of X_q. It follows from Lemma 3.6 that this will introduce an X_q variable (with coefficient 1) at the $d\lambda^{s,j}$ row, which does not appear anywhere below this row. Note that the rows appearing above might have some instances of X_q, but these rows will be taken care of by some $X_{q'}$ variable appearing earlier. In other words, each row gets assigned a unique X_q.

But choosing this subframe $(\hat{\zeta}^{s-1,j}, \eta^{s,j})$ also introduces a column vector of unknown scalars in the corresponding Λ-block at the $\lambda^{s-1,\bullet}$-rows (see (22)), and in turn (possibly) reduces the rank for the corresponding row-block. To compensate for this, we choose a sufficiently high jet of u that we have not used (recall that we have utilized the jet $\partial_t^{q+1} u$ so far). Indeed, we may consider $\partial_t^{q'} u$ for some $q' \gg q$, and look at the block to the right of this Λ block where this jet appears for the first time in this row-block. This is represented by a red dashed arrow in Figure 2, the arrowhead pointing to the block (the green box in the figure) which is used to compensate for the drop of rank in the Λ block. It follows from (12) that this block must look like $c\partial_t^{q'-1} R_u + \text{a } p \times N$ matrix in $j_u^{q'-1}(x)$, for some positive integer c. We now perform the same process as above: we choose another subframe, say,

Input: q, s, j, d
Output: q

function ChooseSubFrame(q, s, j, d)
  if $d = 0$ **then** ▷ We are in row-block 0
    $\tau^q \leftarrow \tau^{s,j}$
  else ▷ We are in row-block d
    $\tau^q \leftarrow 0$
    $\tau^{q-d} \leftarrow \tau^{s,j}$
  Pick subframe $(\hat{\zeta}^{s-1,\bullet}, \eta^{s,j})$ for column-block q
  $q_0 \leftarrow q$
  ▷ If $s = 1$, then $p_{s-1} = 0$ and we do not enter the following loop
  for $1 \le a \le p_{s-1}$ **do**
    for $1 \le b \le q_0 + 1$ **do**
      Pick subframe $\zeta^\bullet$ for column-block $q + b$
      $\tau^q \leftarrow 0 \in \mathcal{D}_y$
    $q \leftarrow$ ChooseSubFrame($q + q_0 + 2, s - 1, a, q_0 + 1$)
  return q

Algorithm 1. Algorithm for choosing subframes and τ^q's.

$(\hat{\zeta}^{s-2,j'}, \eta^{s-1,j'})$ for the corresponding column-block, choose a value of $\partial_t^{q'-1} u$, suitably add some vector proportional to $\tau^{s-1,j'}$, and thus make sure that at least one of the rows of this row-block is now independent. But now we need to keep doing this recursively, as this will again drop the rank of some Λ block down the line. The recursion ends when we choose the frame $(\zeta^\bullet, \eta^{1,j})$, since this frame will transform the corresponding Λ-block into $(\mathrm{Id}_p \;\; 0_{p\times 1})$, which already has full rank.

We give a recursive algorithm (Algorithm 1) to choose the appropriate subframes for some $1 \le s \le r$ and some $1 \le j \le p_s$ starting from some column-block q, while at the same time suitably fixing the vectors $\tau^{q'} \in \mathcal{D}_y$ for $q' \ge q$ that needs to be used later in Step 3. The inputs of the algorithm correspond to the situation described above: we have dealt with all the rows appearing before the $d\lambda^{s,j}$-row in the first row-block, using up till the $(q-1)$-th column-block. The integer d corresponds to the depth of the recursion; we begin at $d = 0$, and then d increases as we compensate for subsequent the Λ blocks, as discussed above.

Algorithm 1 outputs the last column-block q for which the subframe and the vector τ^q have been chosen. Note that for $s \ge 2$, the algorithm is recursive. Whereas for $s = 1$, we choose the frame $(\zeta^{0,\bullet}, \eta^{1,j}) = (\zeta^\bullet, \eta^{1,j})$ and do not enter the for-loop since $p_{s-1} = 0$ for $s = 1$. We can now find all the subframes, starting from some column-block q by Algorithm 2.

As before, Algorithm 2 outputs the last column-block q for which the subframe and the vector τ^q have been chosen. We are now in a position to get a suitable value of $q(\mathfrak{m}, K)$ for $K = 1$. First, we choose the frame $\zeta^\bullet$ for each of the column-block

Input: q
Output: q

```
function ChooseAllSubFrames(q)
    for 1 ≤ s ≤ r do
        for 1 ≤ j ≤ p_s do
            q ← ChooseSubFrame(q, s, j, 0) + 1
    return q − 1
```

Algorithm 2. Algorithm for choosing all subframes.

$0, \dots, q_0$, where $q_0 \geq 0$ satisfies $nq_0 \geq p - n$ and set $\tau^0 = \cdots = \tau^{q_0} = 0$. Then, let $q(\mathfrak{m}, 1) = \texttt{ChooseAllSubFrames}(q_0 + 1)$. Let us denote the transformed matrix by $A_1 = A_1(j_u^{q(\mathfrak{m},1)+1}(x))$. This concludes Step 1.

Next, choose the submatrix B of A_1 which are labeled by the prescribed columns chosen as in Step 1. Firstly, note that whenever we are choosing the subframe $\zeta^\bullet$ for column-block q, the Λ-block in that column-block of B becomes Id_p, since $\lambda^\bullet$ is dual to $\zeta^\bullet$. On the other hand, choosing the frame $(\hat{\zeta}^{s-1,\bullet}, \eta^{s,j})$ for some (s, j) with $s > 1$ makes the Λ block of the form

$$\begin{pmatrix} \mathrm{Id}_{p_1+\cdots+p_{s-2}} & 0 & 0 \\ 0 & 0 & *_{p_{s-1}\times 1} \\ 0 & \mathrm{Id}_{p_s+\cdots+p_r} & 0 \end{pmatrix}_{p\times(p-p_{s-1}+1)}, \tag{22}$$

where $*$ represents the column vector $(\lambda^{s,\bullet}(\eta^{s,j}))$. Lastly, choosing $(\zeta^\bullet, \eta^{1,j})$ transforms the Λ block into $(\mathrm{Id}_p \;\; 0_{p\times 1})_{p\times(p+1)}$. It follows that B is a square matrix of size $p(q(\mathfrak{m}, 1) + 2) \times p(q(\mathfrak{m}, 1) + 2)$.

Let us now perform some invertible row and column operations on B, keeping its rank fixed. Starting from the bottom right corner of B and then going towards the top left corner, we consider each Id block in the off-diagonal Λ block. Next using these Id blocks in order, we make everything zero first along the columns and then along the rows. Denote the new matrix by B_1 and observe that $\det B = \det B_1$. In any nonzero block of B_1 above the Λ diagonal, the highest-order derivative of u, say $\partial_t^{q+1} u$, is still contributed by the $\partial_t^q R_u$ factor of this block. In fact, any of these blocks look like

$$\left(0_{p\times(p'-1)} \;\; c\, d\lambda^\bullet(\partial_t^{q+1} u, \eta^{s,j}) + \text{terms in } j_u^q(x)\right)_{p\times p'}$$

for some integer $1 \leq c \leq q(\mathfrak{m}, 1)$ and some $p' \in \{p+1,\ p - p_s + 1,\ 1 \leq s \leq r\}$. Note that this integer c is the integer coefficient of the respective $\partial_t^q R_u$ factor as in (12). Let C be the square submatrix of B_1 obtained by removing the rows and columns corresponding to the Id blocks so that we have $\det B = \det B_1 = \pm \det C$, concluding Step 2.

The columns of C are precisely those columns corresponding to some $\eta^{s,j}$ chosen earlier via Algorithm 2, whereas the rows of C correspond to each row from the first row-block of B_1, and all the rows starting with some $\lambda^{s,j'}(\eta^{s,j})$ from the other row-blocks. Let us now show that $\det C \neq 0$. We replace $\partial_t^{q+1}u = X_q\tau^q + \dots$ in C by using (21). By (i) the construction of C, (ii) the choice of τ^q's in Algorithm 1, and (iii) from Lemma 2.5, it follows that in each row of C there exists a unique column so that the element in this position satisfies the following:

- The element looks like, $cX_q +$ terms in $X_1, \dots, X_{q-1}$, for some q and some integer $1 \leq c \leq q(\mathfrak{m}, 1)$.
- X_q does not appear anywhere in C to the left of this column.
- X_q does not appear anywhere in this column below this row (but may appear above this row).

Note that, not every variable X_q appears in C, since we have set many $\tau^q = 0$. In fact, there are precisely $\bar{q}$ many variables appearing, where C has the size $\bar{q} \times \bar{q}$. Hence, for notational convenience, let us rename the appearing X_q's to $\{Y_1, \dots, Y_{\bar{q}}\}$ in the increasing order. By expanding $\det C$, we then have the recursive formula:

$$\left.\begin{aligned} \det C &= Y_{\bar{q}} f_{\bar{q}-1}(\sigma, Y_1, \dots, Y_{\bar{q}-1}) + g_{\bar{q}-1}(\sigma, Y_1, \dots, Y_{\bar{q}-1}) \\ f_{\bar{q}-1}(\sigma, Y_1, \dots, Y_{\bar{q}-1}) &= Y_{\bar{q}-1} f_{\bar{q}-2}(\sigma, Y_1, \dots, Y_{\bar{q}-2}) + g_{\bar{q}-2}(\sigma, Y_1, \dots, Y_{\bar{q}-2}) \\ &\vdots \\ f_2(\sigma, Y_1, Y_2) &= Y_2 f_1(\sigma, Y_1) + g_1(\sigma, Y_1) \\ f_1(\sigma, Y_1) &= Y_1 \det\tilde{C} + g_0(\sigma) \end{aligned}\right\}. \tag{23}$$

Above, f_i, g_i are polynomial functions, where g_0 depends only on the choice of the first jet $\sigma = j^1_u(x)$. The matrix $\tilde{C}$ has the following property: for each row of $\tilde{C}$, there exists a unique column, so that the element in that position is an integer (corresponding to the coefficient of some X_q) and everything below this integer in this column is 0. Indeed, this column corresponds to the unique column of C with the first occurrence of X_q as discussed above. Then, some column permutation puts the matrix $\tilde{C}$ in an upper triangular form, with nonzero integers in the diagonal. In particular, $\det \tilde{C} \neq 0$. Hence, choosing $Y_1, Y_2, \dots, Y_{\bar{q}}$ successively and sufficiently large, we can see from (23) that $\det C \neq 0$. In other words, we have shown that the polynomial $\det B$ is nonvanishing when restricted to $\mathcal{R}^{q(\mathfrak{m},1)}_{\text{tang}}|_\sigma$, which concludes Step 3.

To finish the proof for $K = 1$, let $\alpha > q(\mathfrak{m}, 1)$. First perform `ChooseAllSub-Frames`$(q_0 + 1)$ as above. Next, for $q(\mathfrak{m}, 1) + 1 \leq q \leq \alpha$, choose the subframe $\zeta^\bullet$ for the column-block q and set $\tau^q = 0$. We can then continue with the rest of the argument as above without any change. In particular, for any $\alpha \geq q(\mathfrak{m}, 1)$, we have the codimension of the complement of $\mathcal{W}_\alpha|_\sigma$ is at least 1 in $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$.

Let us now induct over K. Suppose for some $K \geq 1$, we have obtained a suitable $q(\mathfrak{m}, K)$ and some polynomials $P_1, \ldots, P_K$ in the jets $j_u^{q(\mathfrak{m},K)}(x)$ so that

$$(\mathcal{W}_{q(\mathfrak{m},K)})^c \subset \{P_1 = \cdots = P_K = 0\}, \quad \operatorname{codim}\{P_1 = \cdots = P_K = 0\} \geq K \text{ in } \mathcal{R}_{\text{tang}}^{q(\mathfrak{m},K)}|_\sigma.$$

We set $q(\mathfrak{m}, K+1) = \texttt{ChooseAllSubFrames}(q(\mathfrak{m}, K)+1)$. This will produce a new polynomial P_{K+1} involving jets that were not involved in any of the $P_1, \ldots, P_K$. But then $\operatorname{codim}\{P_1 = \cdots = P_{K+1} = 0\} \geq K+1$ in $\mathcal{R}_{\text{tang}}^{q(\mathfrak{m},K+1)}|_\sigma$. Also by construction, $(\mathcal{W}_{q(\mathfrak{m},K+1)})^c \subset \{P_1 = \cdots = P_{K+1} = 0\}$. Proceeding similarly as in the $K = 1$ case, we can finish the inductive step.

This concludes the proof of Theorem 3.7. □

A toy case. Let us consider a toy case to illustrate the major steps in the proof of Theorem 3.7. We consider a bracket-generating distribution $\mathcal{D}$, of rank $n = 10$ and corank $p = 4$, on a manifold M with $\dim M = 14$. Suppose $\mathcal{D}^3 = TM$ and $\operatorname{rk}(\mathcal{D}^2/\mathcal{D}) = 2 = \operatorname{rk}(\mathcal{D}^3/\mathcal{D}^2)$. Thus, $\mathcal{D}$ has the type

$$\mathfrak{m} = (0 = m_0 < m_1 = 10 < m_2 = 12 < m_3 = 14),$$

and we have $p_1 = 2 = p_2$. For some $x \in M$, using Lemma 2.5, we choose the necessary vectors $\tau^{s,j} \in \mathcal{D}_x$, $\eta^{s,j} \in \mathcal{D}^s$, $\zeta^{s,j} \in \mathcal{D}_x^{s+1}$ for $1 \leq j \leq p_s$ and $1 \leq s \leq 2$ and the 1-forms $\lambda^{s,j}$. Clearly, for $q_0 = 0$ we have $nq_0 > p - n = -6$. Let us now determine $q(\mathfrak{m}, 1)$ and see how Algorithm 2 produces the subframes for different column-blocks of the matrix A:

pick $\zeta^\bullet$ for column-block 0, pick $(\zeta^\bullet, \eta^{1,1})$ for column-block 1,
pick $(\zeta^\bullet, \eta^{1,2})$ for column-block 2, pick $(\hat{\zeta}^{1,\bullet}, \eta^{2,1})$ for column-block 3,
pick $\zeta^\bullet$ for column-blocks 4 to 7, pick $(\zeta^\bullet, \eta^{1,1})$ for column-block 8,
pick $\zeta^\bullet$ for column-blocks 9 to 12, pick $(\zeta^\bullet, \eta^{1,2})$ for column-block 13,
pick $(\hat{\zeta}^{1,\bullet}, \eta^{2,2})$ for column-block 14, pick $\zeta^\bullet$ for column-blocks 15 to 29,
pick $(\zeta^\bullet, \eta^{1,1})$ for column-block 30, pick $\zeta^\bullet$ for column-blocks 31 to 45,
pick $(\zeta^\bullet, \eta^{1,2})$ for column-block 46.

Thus, we may take $q(\mathfrak{m}, 1) = 46$. The submatrix B is a square matrix of size $4 \times (46 + 2) = 192$. We also choose the vectors τ^q as

$$\tau^1 = \tau^{1,1}, \quad \tau^2 = \tau^{1,2}, \quad \tau^3 = \eta^{2,1}, \quad \tau^4 = \tau^{1,1},$$
$$\tau^9 = \tau^{1,2}, \quad \tau^{14} = \tau^{2,2}, \quad \tau^{15} = \tau^{1,1}, \quad \tau^{31} = \tau^{1,2}$$

and set all other $\tau^q = 0$ for $0 \leq q \leq 46$.

Removing all the rows and columns corresponding to some Id block in the off-diagonal, we get the matrix C of size 8×8 from B. Let us also replace

$\partial_t^{q+1} u = X_q \tau^q + \dots$ in C. Then C looks like

$$
\begin{array}{cccccccc}
\eta^{1,1} & \eta^{1,2} & \eta^{2,1} & \eta^{1,1} & \eta^{1,2} & \eta^{2,1} & \eta^{1,1} & \eta^{1,2}
\end{array}
$$

$$
C = \begin{pmatrix}
X_1+* & ?X_2+* & ?X_3+* & ?X_4+* & ?X_9+* & ?X_{14}+* & ?X_{15}+* & ?X_{31}+* \\
0X_1+* & X_2+* & ?X_3+* & ?X_4+* & ?X_9+* & ?X_{14}+* & ?X_{15}+* & ?X_{31}+* \\
0X_1+* & 0X_2+* & X_3+* & ?X_4+* & ?X_9+* & ?X_{14}+* & ?X_{15}+* & ?X_{31}+* \\
0X_1+* & 0X_2+* & 0X_3+* & ?X_4+* & ?X_9+* & X_{14}+* & ?X_{15}+* & ?X_{31}+* \\
0 & 0 & \lambda^{2,1}(\eta^{2,1}) & X_4+* & ?X_9+* & 0X_{14}+* & ?X_{15}+* & ?X_{31}+* \\
0 & 0 & \lambda^{2,2}(\eta^{2,1}) & 0X_4+* & X_9+* & 0X_{14}+* & ?X_{15}+* & ?X_{31}+* \\
0 & 0 & 0 & 0 & 0 & \lambda^{2,1}(\eta^{2,1}) & X_{15}+* & ?X_{31}+* \\
0 & 0 & 0 & 0 & 0 & \lambda^{2,2}(\eta^{2,1}) & 0X_{15}+* & X_{31}+*
\end{pmatrix}_{8\times 8}
$$

Above, $*$ in some $?X_i + *$ denotes terms involving $X_1, \dots, X_{i-1}$. Also, to keep the calculation simple, we have set the (nonzero) integer coefficients to 1 for the highest-order X_q uniquely associated with each row. We can find a recursive formula for $\det C$ as in (23). In particular, the matrix $\tilde{C}$ is given by

$$
\tilde{C} = \begin{pmatrix}
1 & ? & ? & ? & ? & ? & ? & ? \\
0 & 1 & ? & ? & ? & ? & ? & ? \\
0 & 0 & 1 & ? & ? & ? & ? & ? \\
0 & 0 & 0 & ? & ? & 1 & ? & ? \\
0 & 0 & 0 & 1 & ? & 0 & ? & ? \\
0 & 0 & 0 & 0 & 1 & 0 & ? & ? \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & ? \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}_{8\times 8}
$$

which obviously satisfies $\det \tilde{C} \neq 0$.

4. The h-principle for transverse maps

Given a distribution $\mathcal{D}$, a map $u : \Sigma \to M$ is said to be *transverse* to $\mathcal{D}$ if the composition map $T\Sigma \xrightarrow{du} u^*TM \xrightarrow{\lambda} u^*TM/\mathcal{D}$ is surjective, where $\lambda : TM \to TM/\mathcal{D}$ is the quotient map. We have the following theorem.

Theorem 4.1. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution on a manifold M. Then, maps transverse to $\mathcal{D}$ satisfy the C^0-dense parametric h-principle.*

Once we have the microflexibility (Theorem 3.4) and the local h-principle (Corollary 3.8) for $\mathcal{W}$-regular horizontal maps $\mathbb{R} \to M$, the proof essentially follows from the steps outlined as in [Gromov 1986, p. 84]. The details were worked out in [Eliashberg and Mishachev 2002, Theorem 14.2.1] when the distribution is contact, and in [Del Pino and Presas 2019, Theorem 4] for the case of Engel distributions. We include the sketch of the proof for the general case.

Proof of Theorem 4.1. Let $\mathcal{R}_{\text{tran}} = \{ j_u^1(x) \mid \lambda \circ du_x \text{ is surjective} \} \subset J^1(\Sigma, M)$ be the relation of $\mathcal{D}$-transverse maps $\Sigma \to M$. Since $\mathcal{R}_{\text{tran}}$ is open, we have

$\operatorname{Sol}\mathcal{R}_{\mathrm{tran}}$ is microflexible and furthermore, $j^1 : \operatorname{Sol}\mathcal{R}_{\mathrm{tran}} \to \Gamma\mathcal{R}_{\mathrm{tran}}$ is a local weak homotopy equivalence. To prove the h-principle, we need to find some suitable (local) microextensions (in the sense of [Bhowmick and Datta 2023, Definition 5.10]) to maps $\widetilde{\Sigma} \to M$, where $\widetilde{\Sigma} = \Sigma \times \mathbb{R}$.

Let us consider the following class of maps:

$$\tilde{\Phi}^{\mathcal{W}\text{-tran}} = \left\{ u : \widetilde{\Sigma} \to M \;\middle|\; \begin{array}{c} u \text{ is transverse to } \mathcal{D}, \text{ and for each } \sigma \in \Sigma \\ u|_{\sigma\times\mathbb{R}} \text{ is a } \mathcal{W}\text{-regular, } \mathcal{D}\text{-horizontal, immersed curve} \end{array} \right\}.$$

We prove the microflexibility and local h-principle for $\tilde{\Phi}^{\mathcal{W}\text{-tran}}$. Assuming that $\mathcal{D} = \bigcap_{s=1}^{p} \ker \lambda^s$, let us consider the differential operator

$$\begin{aligned} \widetilde{\mathfrak{D}} : C^\infty(\widetilde{\Sigma}, M) &\to \hom(T\mathbb{R}, \mathbb{R}^p) = C^\infty(\widetilde{\Sigma}, M), \\ u &\mapsto (u^*\lambda^s|_{T\mathbb{R}}) = (u^*\lambda^s(\partial_t)). \end{aligned}$$

Clearly, $u : \widetilde{\Sigma} \to M$ is a solution of $\widetilde{\mathfrak{D}} = 0$ precisely when $u|_{\sigma\times\mathbb{R}}$ is $\mathcal{D}$-horizontal for each $\sigma \in \Sigma$. The linearization of $\widetilde{\mathfrak{D}}$ at some u is then given by a formula identical to (3). Since t is a global coordinate on $\widetilde{\Sigma} = \Sigma \times \mathbb{R}$, we have a splitting of the jet spaces. Denote by $j_u^{q+1,\perp}(x)$ the higher derivatives purely along the t direction. We then see that the matrix $A = A(j_u^{q+1})$ as in (12), in fact depends only on $j_u^{q+1,\perp}$. Let $\widetilde{\mathcal{R}}_{\mathrm{tran}} \subset J^1(\widetilde{\Sigma}, M)$ be the relation of $\mathcal{D}$-transverse maps $\widetilde{\Sigma} \to M$ and define $\mathcal{W}^{\mathrm{tran}} \subset J^{q+1}(\widetilde{\Sigma}, M)$ by

$$\mathcal{W}^{\mathrm{tran}} = \left\{ j_u^{q+1}(x) \mid j_u^1(x) \in \widetilde{\mathcal{R}}_{\mathrm{tran}},\ \partial_t u(x) \neq 0,\ A(j_u^{q+1,\perp}(x)) \text{ has full rank} \right\}.$$

By similar arguments as in Section 3, the operator $\widetilde{\mathfrak{D}}$ is infinitesimally invertible on $\mathcal{W}^{\mathrm{tran}}$-regular maps.

Let $\widetilde{\mathcal{R}}^{\alpha}_{\mathrm{tang}} = \{ j_u^{\alpha+1}(x) \mid j^{\alpha}_{\widetilde{\mathfrak{D}}(u)}(x) = 0 \}$ for $\alpha \geq 0$ and then for $\alpha \geq 2q$ define

$$\mathcal{W}^{\mathrm{tran}}_{\alpha} = (p^{\alpha+1}_{q+1})^{-1}(\mathcal{W}^{\mathrm{tran}}) \cap \widetilde{\mathcal{R}}^{\alpha}_{\mathrm{tang}} \subset J^{\alpha+1}(\widetilde{\Sigma}, M).$$

It is then immediate that the smooth solutions of $\mathcal{W}^{\mathrm{tran}}_{\alpha}$ are all same and in fact $\tilde{\Phi}^{\mathcal{W}\text{-tran}} = \operatorname{Sol}\mathcal{W}^{\mathrm{tran}}_{\alpha}$. Just as in Theorem 3.4, we have the microflexibility for $\tilde{\Phi}^{\mathcal{W}\text{-tran}}$, and $j^{\alpha+1} : \tilde{\Phi}^{\mathcal{W}\text{-tran}} \to \Gamma\mathcal{W}^{\mathrm{tran}}_{\alpha}$ is a local weak homotopy equivalence for α large enough. Since $\tilde{\Phi}^{\mathcal{W}\text{-tran}}$ is invariant under the pseudogroup of fiber-preserving local diffeomorphisms of $\widetilde{\Sigma}$, we have the flexibility of the restricted sheaf $\tilde{\Phi}^{\mathcal{W}\text{-tran}}|_{\Sigma\times 0}$ [Gromov 1986, p. 78].

Define the relation

$$\widetilde{\mathcal{R}}_{\text{tang-tran}} = \left\{ j_u^1(x) \in \widetilde{\mathcal{R}}_{\mathrm{tran}} \mid 0 \neq \partial_t u(x) \in \mathcal{D}_{u(x)} \right\} \subset J^1(\widetilde{\Sigma}, M).$$

We have the jet projection map $J^{\alpha+1}(\widetilde{\Sigma}, M) \to J^1(\widetilde{\Sigma}, M)$ which restricts to a map $\mathcal{W}^{\mathrm{tran}}_{\alpha} \to \widetilde{\mathcal{R}}_{\text{tang-tran}}$. Since transversality is a first-order regularity condition, we deduce analogous to Corollary 3.8 that the map $\tilde{\Phi}^{\mathcal{W}\text{-tran}} \to \Gamma\widetilde{\mathcal{R}}_{\text{tang-tran}}$ induced by the differential, is a local weak homotopy equivalence. Now, we have a map

$ev : \Gamma\widetilde{\mathcal{R}}_{\text{tang-tran}} \to \Gamma\mathcal{R}_{\text{tran}}$ induced by the restriction $u \mapsto u|_{\Sigma\times 0}$. By choosing some arbitrary nonvanishing local sections of $\mathcal{D}$, we can easily get local (parametric) formal extensions along this ev map on contractible open sets of Σ. Thus, $\widetilde{\mathcal{R}}_{\text{tang-tran}}$ is a microextension of $\mathcal{R}_{\text{tran}}$. We can now finish the proof of the h-principle as in [Bhowmick and Datta 2023, Theorem 2.18]. □

We would like to note that Martínez-Aguinaga and Del Pino [2022] recently have proved the above h-principle (among many other strong results) under similar constant growth assumption on the distribution, albeit using a completely different rather geometric technique. We believe that the technique utilized in the present article can be adapted to a broader class of problems, including the existence of horizontal immersions of submanifolds. Indeed, this method seems promising to address the following conjecture by Gromov.

Conjecture [Gromov 1996, p. 259]. Given a distribution $\mathcal{D}$ on M, Ω-regular (i.e., $(d\lambda^s)$-regular) $\mathcal{D}$-horizontal immersions $\Sigma \to M$ satisfy the complete h-principle, provided $\dim M \geq (\dim \Sigma + 1) \operatorname{codim} \mathcal{D}$.

Immersions transverse to a distribution. Whenever $\dim \Sigma \leq \dim M$, an immersion $u : \Sigma \to M$ is said to be *transverse to* $\mathcal{D}$ if u is an immersion and the composition map $T\Sigma \to u^*(TM/\mathcal{D})$ is of full rank. The following h-principle is well known.

Theorem 4.2 [Gromov 1986, p. 87; Eliashberg and Mishachev 2002, p. 71]. *Let $\mathcal{D}$ be an arbitrary distribution on M. If $\dim \Sigma < \operatorname{cork} \mathcal{D}$, then immersions $\Sigma \to M$ transverse to $\mathcal{D}$ satisfy all forms of h-principles.*

The critical dimension $\dim \Sigma = \operatorname{cork} \mathcal{D}$ is not covered by the above theorem. Although, for the special case of $\mathcal{D}$ being either a contact [Eliashberg and Mishachev 2002, Theorem 14.2.2] or an Engel distribution [Del Pino and Presas 2019], the h-principle holds for all transverse immersions. The h-principle for smooth immersions transverse to *real analytic* bracket-generating distributions was proved in [Del Pino and Shin 2020]. We have the following.

Theorem 4.3. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution. Then, C^0-dense, parametric h-principle holds for immersions $\Sigma \to M$ transverse to $\mathcal{D}$, provided* $\dim \Sigma < \dim M$.

Proof. The proof is identical to that of Theorem 4.1. Denote by $\mathcal{R}_{\text{imm}} \subset J^1(\Sigma, M)$ the relation of immersions $\Sigma \to M$, and similarly $\widetilde{\mathcal{R}}_{\text{imm}} \subset J^1(\widetilde{\Sigma}, M)$ where we have $\widetilde{\Sigma} = \Sigma \times \mathbb{R}$. Then, $\mathcal{R}_{\text{imm-tran}} = \mathcal{R}_{\text{tran}} \cap \mathcal{R}_{\text{imm}}$ is the relation of immersions $\Sigma \to M$ which are transverse to $\mathcal{D}$. The microextension is provided by the relation

$$\widetilde{\mathcal{R}}_{\text{imm-tang-tran}} = \widetilde{\mathcal{R}}_{\text{tang-tran}} \cap \widetilde{\mathcal{R}}_{\text{imm}},$$

where $\widetilde{\mathcal{R}}_{\text{tang-tran}}$ is as in Theorem 4.1. We have the map

$$ev : \Gamma\widetilde{\mathcal{R}}_{\text{imm-tang-tran}} \to \Gamma\mathcal{R}_{\text{imm-tran}}$$

induced by $u \mapsto u|_{\Sigma\times 0}$. Since we have that $\dim \Sigma < \dim M$, we can always find *nonvanishing* (local) extensions along the ev map. The h-principle then follows. □

In particular, we have the h-principle for $\mathcal{D}$-transverse maps $\Sigma \to M$ for $\dim \Sigma = \operatorname{cork} \mathcal{D}$, provided $\mathcal{D}$ is bracket-generating. Also, taking $\mathcal{D} = TM$, Theorem 4.3 reduces to Hirsch's h-principle for immersions $\Sigma \to M$ [Hirsch 1959].

Remark 4.4. When $\dim \Sigma \geq \operatorname{cork} \mathcal{D}$, one can also treat immersions $\Sigma \to (M, \mathcal{D})$ transverse to $\mathcal{D}$ as partially horizontal immersions [Gromov 1996, p. 256]. It turns out that all such maps are $\Omega_\bullet$-regular in the sense of Gromov. One can then get a stronger version of Theorem 4.3, where the distribution can be taken to be arbitrary, and thus extending Theorem 4.2.

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