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**THE RESTRICTION OF EFFICIENT GEODESICS  
TO THE NONSEPARATING COMPLEX OF CURVES**

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# THE RESTRICTION OF EFFICIENT GEODESICS TO THE NONSEPARATING COMPLEX OF CURVES

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In the complex of curves of a closed orientable surface of genus  $g$ ,  $\mathcal{C}(S_g)$ , a preferred finite set of geodesics between any two vertices, called *efficient geodesics*, was introduced by Birman, Margalit, and Menasco (2016). The main tool they used to establish the existence of efficient geodesics was a *dot graph*, which records the intersection pattern of a reference arc with the simple closed curves associated with a geodesic path. The idea behind the construction was that a geodesic that is not initially efficient contains shapes in its corresponding dot graph. These shapes then correspond to surgeries that reduce the intersection with the reference arc. We show that the efficient geodesic algorithm can be restricted to the nonseparating curve complex; the proof of this will involve analysis of the dot graph and its corresponding surgeries. Moreover, we demonstrate that given any geodesic in the complex of curves we may obtain an efficient geodesic whose vertices, with the possible exception of the endpoints, are all nonseparating curves.

## 1. Introduction

*The complex of curves and geodesics.* The complex of curves  $\mathcal{C}(S)$  for a compact surface  $S$  is a simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves in  $S$  and whose edges connect vertices with disjoint representatives. We can endow the 0-skeleton of  $\mathcal{C}(S)$  with a metric by defining the distance between two vertices  $u$  and  $v$  to be the minimal number of edges among paths between them. In this paper, as in [Birman et al. 2016], we assume that the surfaces we are considering are closed and have genus at least two. It is a fundamental result that in this case  $\mathcal{C}(S)$  is connected. Thus, the distance is defined for all pairs of vertices in  $\mathcal{C}(S)$ . The trouble is that the complex of curves is, in fact, too connected. It turns out that  $\mathcal{C}(S)$  is locally infinite (for any vertex  $v$  there are infinitely many adjacent vertices  $w$ ) and there are infinitely many geodesics between many pairs of vertices. Thus, it is useful to have a preferred finite subset of geodesics to choose from. This is the idea behind the introduction of *tight geodesics*

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in [Masur and Minsky 2000]. Birman, Margalit, and Menasco [Birman et al. 2016] introduced an alternate preferred finite set of geodesics, called *efficient geodesics*. The novel feature of this particular set of geodesics is the algorithm used to generate them. The algorithm can (and has been, in [Glenn et al. 2017]) implemented on a computer to find efficient geodesics for small distances.

We let  $\mathcal{N}(S)$  denote the subcomplex of  $\mathcal{C}(S)$  spanned by vertices corresponding to nonseparating simple closed curves. This complex is called the *complex of nonseparating curves*. Again, it is a standard result that if the genus of  $S$  is at least two,  $\mathcal{N}(S)$  is connected. The complex of nonseparating curves has attracted interest in the literature. See [Rasmussen 2020; Hamenstädt 2014; Wright 2023].

We first show that given a geodesic in the complex of curves we can always find a geodesic whose interior is contained in the subcomplex of nonseparating curves. It is then natural to wonder whether *efficient geodesics* exist that are contained in this subcomplex. Our main result is that they in fact do.

**Theorem 1.1.** *Let  $g \geq 2$ . If  $v$  and  $w$  are vertices of  $\mathcal{N}(S_g)$  with  $d(v, w) \geq 3$ , then there exists an efficient geodesic from  $v$  to  $w$  in  $\mathcal{N}(S_g)$ . Additionally, there are finitely many efficient geodesics from  $v$  to  $w$ .*

**Efficient geodesics.** The idea behind obtaining an efficient geodesic  $v_0, \dots, v_n$  in  $\mathcal{C}(S)$  is to iteratively decrease intersections with an arc as we move along the path. We explain further below. The following construction and results were first introduced in [Birman et al. 2016]; the reader familiar with these results is invited to skip to Section 2.

Suppose that  $\gamma$  is an arc in  $S$  and  $\alpha$  is a simple closed curve in  $S$ . Then we say that  $\gamma$  and  $\alpha$  are in *minimal position* if  $\alpha$  is disjoint from the endpoints of  $\gamma$  and the number of points of intersection of  $\alpha$  and  $\gamma$  is smallest over all simple closed curves homotopic to  $\alpha$  through homotopies that do not pass through the endpoints of  $\gamma$ .

Let  $v_0, \dots, v_n$  be a geodesic of length at least three in the complex of curves, and let  $\alpha_0, \alpha_1$ , and  $\alpha_n$  be representatives of  $v_0, v_1$ , and  $v_n$  that are pairwise in minimal position. A *reference arc* for the triple  $\alpha_0, \alpha_1, \alpha_n$  is an arc  $\gamma$  that is in minimal position with  $\alpha_1$  and whose interior is disjoint from  $\alpha_0 \cup \alpha_n$ .

We say that the oriented geodesic  $v_0, \dots, v_n$  is *initially efficient* if

$$|\alpha_1 \cap \gamma| \leq n - 1$$

for all choices of reference arcs  $\gamma$ . Finally, we say that  $v = v_0, \dots, v_n = w$  is *efficient* if the oriented geodesic  $v_k, \dots, v_n$  is initially efficient for each  $0 \leq k \leq n - 3$  and the oriented geodesic  $v_n, v_{n-1}, v_{n-2}, v_{n-3}$  is also initially efficient. Thus, to test the efficiency of a geodesic we look at all the triples  $v_k, v_{k+1}, v_n$  and count the intersection of  $v_{k+1}$  with any reference arc. While it may seem impossible to check intersections with *all* reference arcs, it turns out that there are finitely many of

them. Moreover, in special cases it is sufficient to check the intersections of  $\alpha_i \cap \alpha_n$  for  $1 \leq i \leq n - 1$  [Birman et al. 2016].

Given a vertex path  $v_0, \dots, v_n$  in  $\mathcal{C}(S)$  with representative curves  $\alpha_0, \dots, \alpha_n$  and an oriented reference arc  $\gamma$  for  $\alpha_0, \alpha_1$  and  $\alpha_n$ , we may traverse  $\gamma$  in the direction of its orientation and record the order in which the curves  $\alpha_0, \dots, \alpha_n$  intersect  $\gamma$ . The result is a sequence of natural numbers  $\sigma \in \{1, \dots, n - 1\}^N$ , where  $N$  is the minimal cardinality of  $\gamma \cap (\alpha_1 \cup \dots \cup \alpha_{n-1})$ . The sequence  $\sigma$  is called the *intersection sequence* of the  $\alpha_i$  along  $\gamma$ .

The *complexity* of an oriented path  $v_0, \dots, v_n \in \mathcal{C}(S)$  is defined to be

$$\sum_{k=1}^{n-1} (i(v_0, v_k) + i(v_k, v_n)).$$

We say that a sequence  $\sigma$  of natural numbers is *reducible* under the following circumstances: whenever  $\sigma$  arises as an intersection sequence for a path  $v_0, \dots, v_n$  in  $\mathcal{C}(S)$  there is another path  $v_0, v'_1, \dots, v'_{n-1}, v_n$  with smaller complexity. The following proposition demonstrates that paths of minimal complexity must be initially efficient.

**Proposition 1.1** [Birman et al. 2016, Proposition 3.1]. *Suppose  $\sigma$  is a sequence of elements of  $\{1, \dots, n - 1\}$ . If  $\sigma$  has more than  $n - 1$  entries equal to 1, then  $\sigma$  is reducible.*

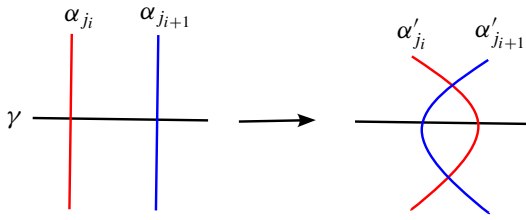
From the above proposition Birman, Margalit, and Menasco deduce the existence of initially efficient geodesics.

**Proposition 1.2** [Birman et al. 2016, Proposition 3.2]. *Let  $g \geq 2$ . If  $v$  and  $w$  are vertices of  $\mathcal{C}(S)$ , with  $d(v, w) \geq 3$ , then there exists an initially efficient geodesic from  $v$  to  $w$ .*

We note that the definition of complexity of a path works when restricted to the complex of nonseparating curves. It is interesting to consider how this measure may change when restricted to the nonseparating curve complex. See Questions 2 and 3 in Section 3.

**Sawtooth form and the dot graph.** The proof of Proposition 1.1 was carried out in three stages. First, the intersection sequence was put into a normal form. This is called *sawtooth form*. Then, associated to the sawtooth form for the sequence is a diagram called the *dot graph*. The reducibility of an intersection sequence then corresponds to certain geometric features in the dot graph. We review these now.

We may exchange the order of intersection of two curves that are adjacent in the intersection sequence by performing a *commutation* as described in [Birman et al. 2016, Lemma 3.3]; see Figure 1.



**Figure 1.** A commutation.

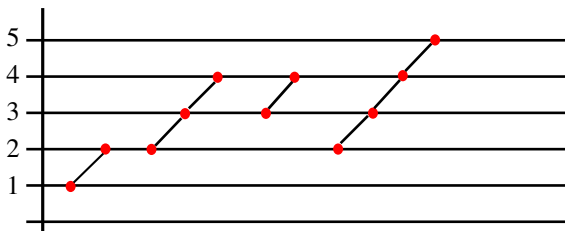
The result is a sequence that is in *sawtooth form*. That is, we say a sequence  $(j_1, j_2, \dots, j_k)$  of natural numbers is in *sawtooth form* if

$$j_i < j_{i+1} \implies j_{i+1} = j_i + 1.$$

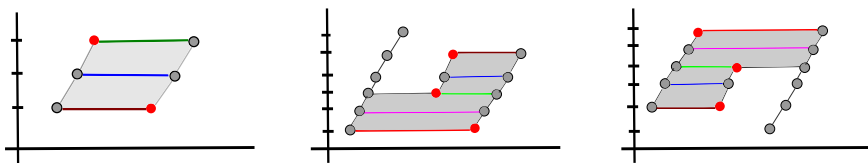
An example of a sequence in sawtooth form is  $(1, 2, 2, 3, 4, 3, 4, 2, 3, 4, 5)$ . Given a sequence of natural numbers in sawtooth form, we also consider its *ascending sequences*, which are the maximal subsequences of the form  $(k, k+1, \dots, k+m)$ . In the above example the ascending sequences are  $(1, 2)$ ,  $(2, 3, 4)$ ,  $(3, 4)$ ,  $(2, 3, 4, 5)$ . It is clear that if we have an intersection sequence and we perform a finite number of commutations we may get the intersection sequence into sawtooth form while keeping the number of intersections of the  $\alpha_i$ 's with  $\gamma$  constant.

Next, given an intersection sequence  $\sigma$  in sawtooth form, we may regard it as a function  $1, \dots, N \rightarrow \mathbb{N}$  and plot it in  $\mathbb{R}_{\geq 0}^2$ . The points of the graph of a sequence will be called *dots*. We decorate the graph by connecting the dots that lie on a given line of slope 1; these line segments are called *ascending segments*. The resulting decorated graph is called the *dot graph* of  $\sigma$  and is denoted by  $G(\sigma)$ . See [Figure 2](#). Again, the idea behind this construction is that given a geodesic that is not efficient we can see shapes in its corresponding dot graph that correspond to surgeries that reduce the intersection with the reference arc.

**Dot graph polygons and surgery.** This section is a summary of [\[Birman et al. 2016, Section 3.3\]](#). We first review the surgeries described, and then in the next section we discuss the results of these surgeries when restricted to  $\mathcal{N}(S)$ .



**Figure 2.** The dot graph of  $(1, 2, 2, 3, 4, 3, 4, 2, 3, 4, 5)$ .

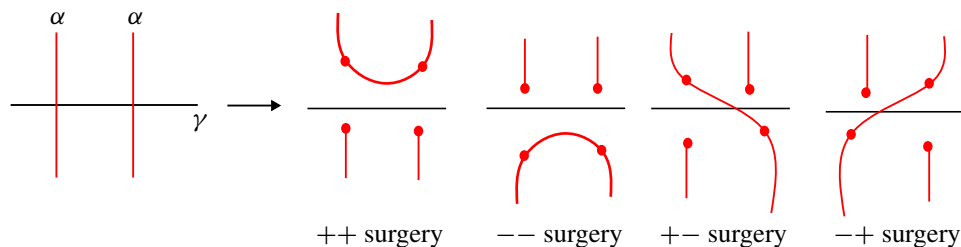


**Figure 3.** Dot graph polygons (box, hexagon type 1, hexagon type 2).

After putting an intersection sequence into sawtooth form and constructing its dot graph, it was shown that if the dot graph contained certain geometric shapes, it corresponded to a sequence that was reducible. These shapes were called *dot graph polygons*. In particular, the existence of a *box*, *hexagon of type I* or *hexagon of type II* in the dot graph (see Figure 3) implied that the sequence  $\sigma$  was reducible. To remove these shapes from the dot graph, *surgeries* on the curves in the intersection pattern corresponding to these shapes were introduced.

We do surgery on a curve  $\alpha$  that intersects our intersection arc  $\gamma$  in at least two points. We draw a neighborhood of  $\gamma$  so that it is horizontal and oriented to the right. We then remove from  $\alpha$  small neighborhoods of its points of intersection with  $\gamma$ . This results in a pair of curves. We will then join two of the endpoints back together forming a new simple closed curve, and discard the other curve. Depending on how we join pairs of endpoints, we say that  $\alpha'$  is obtained from  $\alpha$  by  $++$ ,  $+-$ ,  $-+$ , or  $--$  surgery along  $\gamma$ . The first symbol is  $+$  or  $-$  depending on whether the first endpoint of  $\alpha$  lies to the left or right of  $\gamma$ , respectively. Similarly for the second symbol. When we are considering an arbitrary simple closed curve, exactly two of the four possible surgeries result in a simple closed curve. If we give  $\alpha$  an orientation then two intersection points of  $\alpha$  and  $\gamma$  can either agree or disagree in orientation. If they agree, then the  $+-$  and  $-+$  surgeries, which are called *odd*, each result in a simple closed curve. Otherwise, the  $++$  and  $--$  surgeries, called *even* surgeries, result in simple closed curves.

Suppose that we have a geodesic in  $\mathcal{C}(S)$  containing  $\alpha$  as a representative for some vertex  $v_i$ . If we are to perform surgery on  $\alpha$  then it must intersect our intersection arc  $\gamma$  at least twice (otherwise it stays in the geodesic and is not replaced). Thus,



**Figure 4.** Surgery on a curve.

either  $\alpha$  intersects  $\gamma$  consecutively, or between these intersections  $\gamma$  intersects with at least one other curve  $\beta$ . In the complex of curves, we can immediately get rid of the first case, where  $\alpha$  intersects  $\gamma$  consecutively, by performing the surgeries described above. This does not follow so easily in the subcomplex of nonseparating curves (see Proposition 2.5). However, if  $\alpha$  does not intersect  $\gamma$  consecutively, as is the case when we see boxes, and hexagons in the dot graph, we will see that we have the same choice of surgeries as before.

Using the above surgeries, it was shown in [Birman et al. 2016] that a dot graph with an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2 corresponds to a sequence that is reducible. This was done by prescribing a sequence of surgeries that replaced the  $\alpha_i$  curves with new  $\alpha'_i$  that resulted in a path with smaller complexity. Below we state the required sequence of surgeries corresponding to the type of dot graph polygon.

Suppose the dot graph  $G(\sigma)$  has an empty, unpierced box  $P$ . Then the corresponding sequence of intersections along  $\gamma$  has the form

$$\alpha_k, \dots, \alpha_{k+m}, \alpha_k, \dots, \alpha_{k+m},$$

where  $1 \leq k \leq k+m \leq n-1$ . For the vertices not in  $\{k, \dots, k+m\}$ , they remain unchanged. We define  $\alpha'_k, \dots, \alpha'_{k+m}$  inductively: for  $i = k, \dots, k+m$  the curve  $\alpha'_i$  is obtained by performing surgery along  $\gamma$  between the two points of  $\alpha_i \cap \gamma$  corresponding to dots of  $P$ ; the surgeries are chosen so that they form a path in the directed graph in Figure 5. The vertices of the graph correspond to the four types of surgeries described above: the rule is that the second sign of the origin of a directed edge is opposite of the first sign of the terminus. It is clear from the graph that the desired sequence of surgeries exists. We demonstrate this procedure in Figure 5, where we perform  $-+$  surgery on  $\alpha_3$ , then  $--$  surgery on  $\alpha_4$ , and finally  $+-$  surgery on  $\alpha_5$ . It is an easy check that replacing the curves  $\alpha_i$  with these new ones results in a reduced intersection sequence  $\sigma$ .

Suppose the dot graph  $G(\sigma)$  has an empty, unpierced hexagon  $P$  of type 1. The case of a type 2 hexagon is nearly identical so it will be omitted. By definition of sawtooth form and of a type 1 hexagon, there are no ascending segments of  $G(\sigma)$

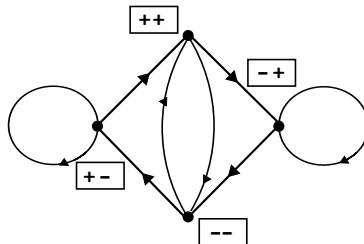


Figure 5

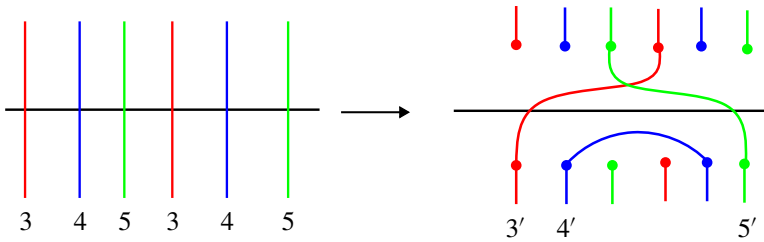


Figure 6

in the vertical strip between the leftmost and middle ascending edges of  $P$  and any ascending segments of  $G(\sigma)$  that lie in the vertical strip between the middle and rightmost ascending segments have their highest point strictly below the lower-right horizontal edge of  $P$ . See Figure 7.

It follows that the dots of  $P$  correspond to a sequence of intersections along  $\gamma$  of the form

$$\alpha_k, \dots, \alpha_{k+m}, \alpha_k, \dots, \alpha_{k+l}, \alpha_{j_1}, \dots, \alpha_{j_p}, \alpha_{k+l}, \dots, \alpha_{k+m},$$

where  $1 \leq k \leq k+l \leq k+m \leq n-1$ ,  $p \geq 0$ , and each  $j_i < \alpha_{k+l}$ . See Figure 7, where  $k = 3$ ,  $l = 2$ ,  $m = 4$ , and  $p = 0$ .

As in the case with the box, whenever we have  $\alpha_i$  with  $i \notin \{k, \dots, k+m\}$  we set  $\alpha'_i = \alpha_i$ . Each of the remaining  $\alpha_i$  correspond to two dots in  $P$  except for  $\alpha_{k+l}$ , which corresponds to three. Let  $\alpha'_{k+l}$  be the curve obtained from  $\alpha_{k+l}$  via surgery along  $\gamma$  between the first two (leftmost) points of  $\alpha_{k+l} \cap \gamma$  corresponding to dots of  $P$  and satisfying the following property:  $\alpha'_{k+l}$  does not contain the arc of  $\alpha_{k+l}$  containing the third (rightmost) point of  $\alpha_{k+l} \cap \gamma$  corresponding to a dot on  $P$ . We then define  $\alpha'_{k+l-1}, \dots, \alpha'_k$  inductively as before using the directed graph in Figure 5, and finally, we define  $\alpha'_{k+l+1}, \dots, \alpha'_{k+m}$  inductively as before. It is readily verified that this procedure reduces  $\sigma$ .

Using these surgeries to remove from the dot graph the above polygons results in an initially efficient geodesic. The last step is to inductively produce initially

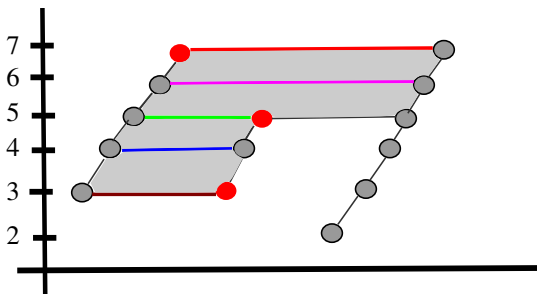
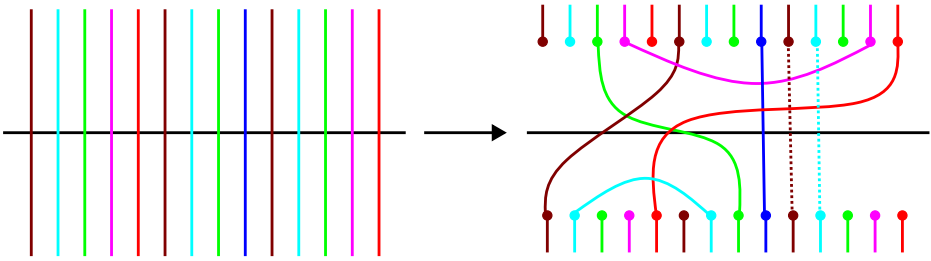


Figure 7. Type 1 hexagon.





**Figure 8.** The surgery described above on a type 1 hexagon.

efficient geodesics for the triples  $v_k, \dots, v_n$  for each  $0 \leq k \leq n-3$  and the oriented geodesic  $v_n, v_{n-1}, v_{n-2}, v_{n-3}$ . This was done in [Birman et al. 2016, Section 3.5]. The exact inductive argument works when restricting to  $\mathcal{N}(S)$ , so our work lies solely in showing the construction of initially efficient geodesics restricts to  $\mathcal{N}(S)$ .

## 2. Existence of efficient geodesics in complex of nonseparating curves

Our main result is that efficient geodesics exist in the complex of nonseparating curves. Since this is a subcomplex of the complex of curves, the fact that there are finitely many of them follows from [Birman et al. 2016, Theorem 1.1]. We restate our main theorem:

**Theorem 1.1.** *Let  $g \geq 2$ . If  $v$  and  $w$  are vertices of  $\mathcal{N}(S_g)$  with  $d(v, w) \geq 3$ , then there exists an efficient geodesic from  $v$  to  $w$  in  $\mathcal{N}(S_g)$ . Additionally, there are finitely many efficient geodesics from  $v$  to  $w$ .*

**Proof of Theorem 1.1.** The result lies in following exactly the proof setup for the complex of curves in [Birman et al. 2016]. We prove the existence of initially efficient geodesics in  $\mathcal{N}(S)$  (see Proposition 2.2). Then the additional inductive step will follow exactly as outlined in [Birman et al. 2016, Section 3.5]. The key observation of this paper is that Lemma 2.1 holds when restricting to  $\mathcal{N}(S)$ :

**Lemma 2.1.** *Suppose that  $\sigma$  is a sequence of natural numbers in sawtooth form and that  $G(\sigma)$  has an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2. Then  $\sigma$  is reducible (in  $\mathcal{N}(S)$ ).*

This will allow us to prove:

**Proposition 2.2.** *Let  $g \geq 2$ . If  $v$  and  $w$  are vertices of  $\mathcal{N}(S)$  with  $d(v, w) \geq 3$ , then there exists an initially efficient geodesic from  $v$  to  $w$  in  $\mathcal{N}(S)$ .*

In [Birman et al. 2016] the existence of the above polygons in the dot graph came with surgeries on the curves representing vertices in the given geodesic that removed these shapes in the dot graph. To prove Lemma 2.1, we need only to show that it is possible to do the surgery constructions outlined in Section 1.4, so that

each new curve is nonseparating. For most of the surgeries this is straightforward, while a few need to be redone carefully.

We begin with the well-known fact that a separating curve must intersect any other curve an even number of times.

**Proposition 2.3.** *Let  $\alpha$  be a simple closed curve in  $S$ . If  $\alpha$  separates  $S$  into two components, then for any simple closed curve  $\beta$  we have the geometric intersection number,  $i(\alpha, \beta)$ , is even.*

*Proof.* Let  $S_\alpha$  be the surface that results from splitting  $S$  along  $\alpha$ . In  $S_\alpha$ , the curve  $\beta$  is either unchanged (all of  $\beta$  is in one of the connected components of  $S_\alpha$ ) or  $\beta$  is a collection of arcs with endpoints along  $\alpha$ . Each of these arcs has two endpoints. Thus intersections, should they exist, come in pairs. In either case this gives an even number of intersections with  $\alpha$ .  $\square$

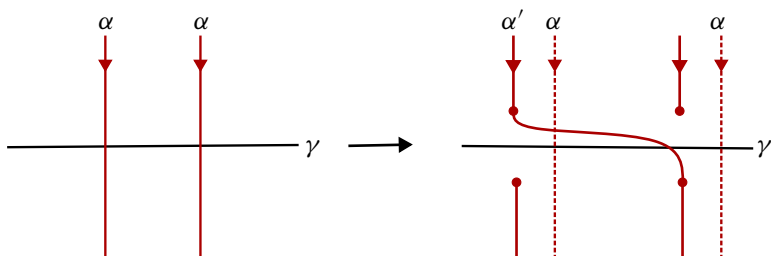
The quick test we will use when deciding if a curve made via surgery is nonseparating is to see if  $i(\alpha', \beta) = 1$  for some curve  $\beta$ . Next, we must show that given a geodesic in the complex of curves, we may take all the vertices in the path to be nonseparating curves. This allows us to start with geodesic in  $\mathcal{N}(S)$  and attempt to make it efficient. The following proposition was also observed in [Hamenstädt 2014, Corollary 3.3].

**Proposition 2.4.** *Given a geodesic in  $\mathcal{C}(S)$  with endpoints  $v, w \in \mathcal{N}(S)$  there exists a geodesic from  $v$  to  $w$  with each vertex a nonseparating curve.*

*Proof.* Let  $v = v_0, \dots, v_n = w$  be a geodesic in  $\mathcal{C}(S)$ . If all the vertices in this geodesic are nonseparating curves then we are done. Assume that  $v_i$  is a separating curve in the above geodesic with lowest index  $i$ . Then consider the subpath  $v_{i-1}, v_i, v_{i+1}$  because  $v_i$  is separating it divides the surface  $S$  into two components. Both  $v_{i-1}$  and  $v_{i+1}$  are disjoint from  $v_i$  however since there is not an edge between them they intersect each other. Therefore, they are both in one of the connected components of  $S_v$ . In the other connected component choose a nonseparating simple closed curve  $v'_i$ . Such a curve exists, as otherwise  $v_i$  would be inessential. This curve is disjoint from both  $v_{i-1}$  and  $v_{i+1}$  and may replace  $v_i$  in the geodesic. Continuing in this way gives a geodesic in the subcomplex of nonseparating curves.  $\square$

Notice that the connectivity of  $\mathcal{N}(S)$  follows immediately from Proposition 2.4.

**The trivial surgeries.** The above proposition allows us to start with a geodesic path in  $\mathcal{N}(S)$ ; now we wish to do simplifying surgeries on it. We begin with the case where our reference arc  $\gamma$  sees a curve  $\alpha$  consecutively. In  $\mathcal{C}(S)$ , we performed an even or odd surgery depending on the orientation of  $\alpha$  and were guaranteed that each resulted in a simple closed curve. However, when we restrict to the



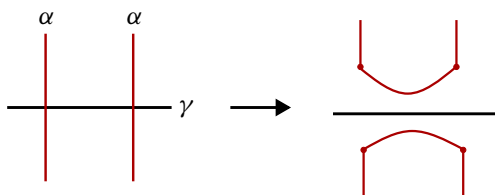
**Figure 9.** Odd surgery always results in a nonseparating curve.

subcomplex of nonseparating curves we are no longer guaranteed that both curves are nonseparating. For instance, take a separating curve and connect sum it with a nonseparating curve. Then performing surgery along an arc separates the curve into a separating curve and a nonseparating one. Proposition 2.5 shows, however, that performing surgery on a nonseparating curve that meets a reference arc  $\gamma$  consecutively will always yield at least one nonseparating curve.

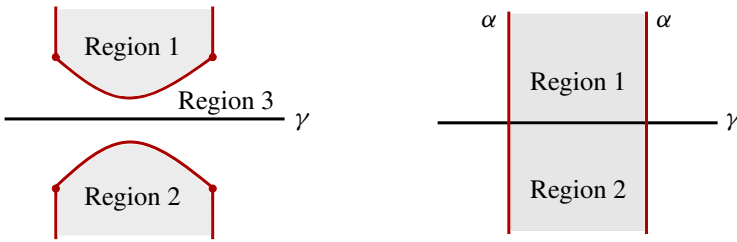
**Proposition 2.5.** *Let  $\gamma$  be a reference arc, and  $\alpha$  be a nonseparating simple closed curve. Suppose that  $i(\alpha, \gamma) \geq 2$ . Then there exists a simple closed curve  $\alpha'$  obtained from the nonseparating simple closed curve  $\alpha$  via some surgery along  $\gamma$  that is still nonseparating.*

*Proof.* Orient  $\gamma$  and  $\alpha$ . Consider two points of intersection that are consecutive along  $\gamma$ . The orientation of  $\gamma$  and  $\alpha$  allow us to assign an index to each intersection either  $+1$  or  $-1$ . If two points of intersection have the same index, we perform an odd surgery. The resulting curve  $\alpha'$  crosses  $\gamma$  one time and intersects  $\alpha$  exactly once. (See Figure 9 for  $+ -$  surgery.) We emphasize that we construct  $\alpha'$  so that outside the local picture in Figure 9,  $\alpha'$  lies just to the right of  $\alpha$ . Since  $i(\alpha, \alpha') = 1$ ,  $\alpha'$  is nonseparating.

If the two intersection points have opposite indices we perform an even surgery. In this case, we may need to make a choice of curve to replace  $\alpha$  with, since one of the surgeries may give a nonseparating curve. One of the curves remains above  $\gamma$  and the other remains below  $\gamma$ . We argue that at least one of these curves is nonseparating.



**Figure 10.**  $++$  and  $--$  surgery on  $\alpha$ .



**Figure 11.** Figure for the proof of [Proposition 2.5](#).

Assume for contradiction that both curves were separating this would divide  $S$  into three connected components. Then joining the curves back together would give back our original curve  $\alpha$  however  $\alpha$  would still separate our surface.

Thus, performing an even surgery always results in at least one nonseparating curve.  $\square$

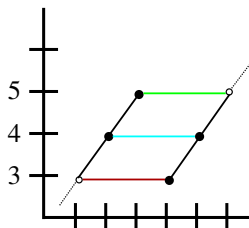
**The nontrivial surgeries.** We use the dot graph exactly as in [\[Birman et al. 2016\]](#) to determine how to reduce our intersections sequence. When restricting to the nonseparating curve complex we need to show that each dot graph polygon has a surgery that results in the removal of the polygon and whose new curves are all nonseparating. This will prove [Lemma 2.1](#).

Throughout we assume that  $\sigma$  is an intersection sequence of nonseparating curves in sawtooth form.

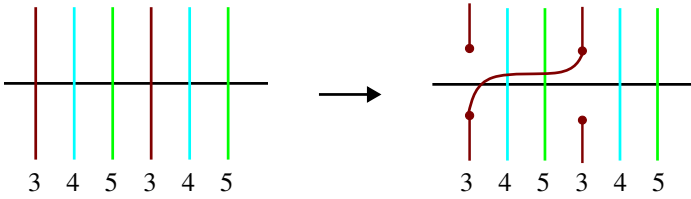
Case 1: Suppose that  $\sigma$  is a sequence of natural numbers in sawtooth form and that  $G(\sigma)$  has an empty, unpierced box. Then  $\sigma$  is reducible.

This is the easy case. Carry out the surgeries exactly as in  $\mathcal{C}(S)$ . Performing the surgeries one at a time, notice that regardless of the type of surgery, the resulting curve will intersect a curve adjacent to it exactly once. The figures below demonstrate this when the box has three curves involved. The general case follows exactly the same. See [Figures 13](#) and [14](#) for the first two steps in the surgery sequence of a box containing the curves 3, 4, and 5.

Continuing in this way, removes the box from the dot graph and replaces all curves with other nonseparating curves.



**Figure 12.** An empty, unpierced box.



**Figure 13.** The intersection sequence corresponding to the above box in the dot graph. Odd surgery on curve 3 intersects curve 4 exactly once, and thus it is nonseparating. Clearly, an even surgery would do the same, so there are two surgery options for curve 3 exactly as before.

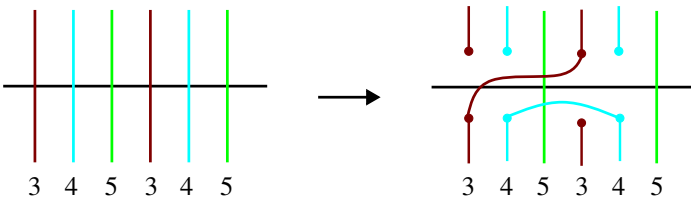
Case 2: Suppose that  $\sigma$  is a sequence of natural numbers in sawtooth form and that  $G(\sigma)$  has an empty, unpierced hexagon of type 1 or 2. Then  $\sigma$  is reducible.

We will treat the case of an empty, unpierced hexagon of type 1. The other case follows exactly the same procedure. The surgery instructions for this case are similar to the instructions for  $\mathcal{C}(S)$  but one new idea is needed. We introduce some new terminology to simplify the discussion. Given a empty, unpierced hexagon of type 1 in the dot graph, its vertices have the form

$$\alpha_k, \dots, \alpha_{k+m}, \alpha_k, \dots, \alpha_{k+l}, \alpha_{j_1}, \dots, \alpha_{j_p}, \alpha_{k+l}, \dots, \alpha_{k+m},$$

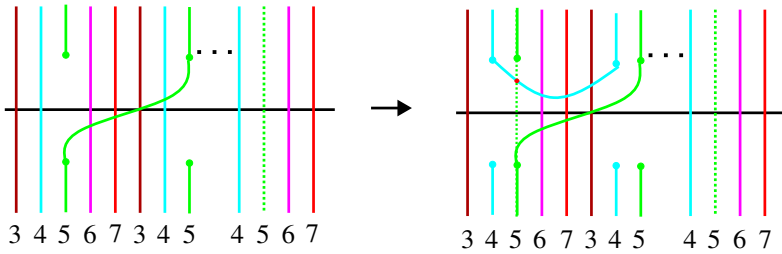
where  $1 \leq k \leq k+l \leq k+m \leq n-1$ ,  $p \geq 0$ , and each  $j_i < \alpha_{k+l}$ . We will call the integer  $l$  the *step length* of the hexagon, the number of vertices in  $\alpha_{j_1}, \dots, \alpha_{j_p}$  the *tail length* of the hexagon, and the integer  $m$  the *total length* of the hexagon. From the dot graph it is easy to see these values. For instance the hexagon in Figure 7 has a step length 2, and tail length 4. We will call the curve  $\alpha_{k+l}$  the *curve at step length  $l$* .

In the discussion below we will also refer to vertices on the dot graph as being “above” or “below” each other. This is referring to the actual placement of the vertices on the dot graph. Along the intersection arc, a curve being “above” another means that its index is higher than the other.



**Figure 14.** Now surgery on curve 4 is performed. This time an even surgery is demonstrated. As in Figure 13, this curve intersects curve 5 exactly one time, so it is nonseparating. An odd surgery would do the same.





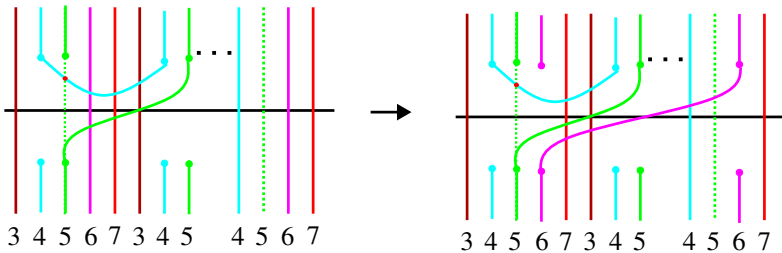
**Figure 17.** Now surgery on curve 4 is performed. This time an even surgery is demonstrated. As above this curve intersects the old curve 5 exactly one time, so it is nonseparating. An odd surgery would do the same. The same follows for curve 3.

the curve at step length  $l$  (above it in the dot graph) may cause issues. We break this into two cases:

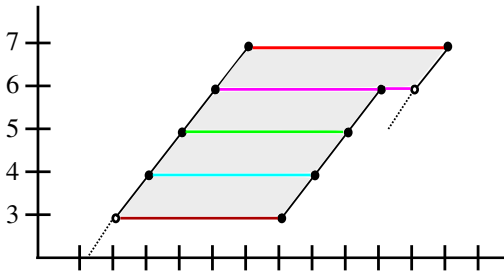
Subcase 2.1: Let  $m$  denote the total length of the hexagon, and let  $l$  denote the step length. If  $m > l + 1$ , then the surgeries follow exactly as in  $\mathcal{C}(S)$ . This is clear since all the curves obtained from surgery above the curve at step length  $l$  intersect an adjacent curve (above or below in the dot graph) exactly once. Notice that for the curve directly above the curve at step length  $l$ ,  $\alpha_{k+l+1}$ , we use the intersection with the curve above it  $\alpha_{k+l+2}$  to show it is nonseparating. For curves above  $\alpha_{k+l+1}$  in the dot graph, say  $\alpha_j$ , we look at the intersection with  $\alpha_{j-1}$  to show it is nonseparating. See [Figure 18](#).

Subcase 2.2: Let  $m$  denote the total length of the hexagon, and let  $l$  denote the step length. If  $m = l + 1$ , then we introduce a new surgery on the curve  $\alpha_{k+l+1}$ .

All the curves below the curve at step length  $l$  have the surgeries performed on them as before, each is nonseparating. We now need to perform a surgery on the curve  $\alpha_{k+l+1}$  and argue that is nonseparating.



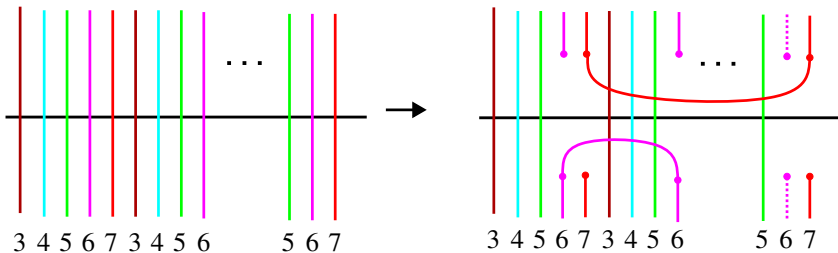
**Figure 18.** Surgery on curve 6 is performed. This curve intersects the curve 7 exactly one time, so it is nonseparating. An odd surgery would do the same. Notice the surgery would intersect the old curve 5 twice, so it is not possible to argue that it is nonseparating with curve 5.



**Figure 19.** Surgery on the curves 3 through 5 will be just fine, since either type of surgery will intersect the curve directly above it exactly once. But surgery on 7 will intersect 6 twice.

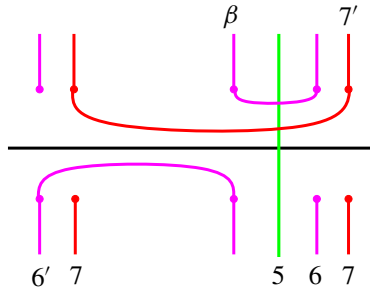
If the curve  $\alpha_{k+l+1}$  requires odd surgery, then it is nonseparating because the surgered curve  $\alpha'_{k+l+1}$  intersects  $\alpha_{k+l+1}$  exactly once as in Proposition 2.5. In the case where  $\alpha_{k+l+1}$  requires even surgery and the even surgery required forces the new curve  $\alpha'_{k+l+1}$  to be separating, will we consider the part of  $\alpha_{k+l}$  (the curve at step length  $l$ ) that is “inside” of the curve  $\alpha'_{k+l+1}$  (We call the “inside” of  $\alpha'_{k+l+1}$  the part of the surface disjoint from the curve  $\alpha_{k+l}$ ). Since we are assuming our hexagon has a tail, we will also see the curve  $\alpha_{k+l-1}$  intersect  $\gamma$  one time inside the region along gamma between the second occurrence of  $\alpha_{k+l}$  and the third occurrence of  $\alpha_{k+1}$ . If we were able to perform even surgery on these two parts of  $\alpha_{k+l}$ , the resulting curve is nonseparating since it intersects  $\alpha_{k+l-1}$  exactly once. By the assumption that  $\alpha'_{k+l+1}$  is separating, the orientation of  $\alpha_{k+l}$  inside of  $\alpha'_{k+l+1}$  must be consistent with an even surgery, otherwise  $\alpha_{k+l}$  would intersect  $\alpha_{k+l+1}$ . See Figure 20.

Let the new nonseparating curve that is obtained by joining the ends of curve  $\alpha_{k+l}$  inside of  $\alpha'_{k+l+1}$  be denoted by  $\beta$ . We want  $\beta$  to be a replacement curve for curve  $\alpha_{k+l+1}$  so it must be disjoint from  $\alpha'_{k+l}$  and  $\alpha_{k+l+2}$ . Clearly,  $\beta$  is disjoint



**Figure 20.** After performing surgery on curve 6, and doing the required even surgery on curve 7 (in the figure it is ++), notice the two parts of curve 6 bounded by the new curve 7'. If we assume that 7' is separating then the two parts of 6 inside 7' must be oriented to allow for ++ surgery.





**Figure 21.** The curve  $\beta$  is disjoint from curve  $6'$  since  $7'$  is disjoint from  $6'$  and  $\beta$ , and  $\beta$  is on the other side of  $7'$  than  $6'$ .

from  $\alpha'_{k+l}$  since  $\alpha'_{k+l+1}$  is disjoint from  $\alpha'_{k+l}$ ,  $\beta$  is disjoint from  $\alpha'_{k+l+1}$  and  $\beta$  is on the other side of  $\alpha'_{k+l+1}$  than  $\alpha'_{k+l}$ . This is demonstrated in [Figure 21](#).

Now we need to argue that the curve  $\beta$  is disjoint from the curve  $\alpha_{k+l+2}$ . Observe that  $\alpha'_{k+l+1}$  must be disjoint from  $\alpha_{k+l+2}$ . Indeed, it is obtained by surgery on the curve  $\alpha_{k+l+1}$  with an arc of  $\gamma$ , both of which are disjoint from  $\alpha_{k+l+2}$ , because  $\alpha_{k+l+2}$  does not enter the hexagon. Thus, if  $\alpha_{k+l+2}$  intersects  $\beta$  it is contained entirely inside  $\alpha'_{k+l+1}$  and therefore disjoint from  $\alpha'_{k+l}$ . This is a contradiction. Since then the curve  $\alpha_{k+l+1}$  is not required in a geodesic path. We would have a path  $\alpha_0, \alpha'_1, \dots, \alpha'_{k+l}, \alpha_{k+l+2}, \dots, \alpha_n$  of length  $n - 1$  contradicting the assumption that the distance between  $v_0$  and  $v_n$  is exactly  $n$ . So,  $\beta$  is a nonseparating curve disjoint from  $\alpha_{k+l+2}$  and  $\alpha'_{k+l}$ . Therefore,  $\beta$  is a suitable replacement for  $\alpha_{k+l+1}$ .

The case for a hexagon of type 2 follows the exact same argument. This covers all the cases, thus proving [Lemma 2.1](#).

### 3. Conclusion

We have demonstrated that efficient geodesics exist in the nonseparating curve complex. Moreover, we demonstrated that given any geodesic in the complex of curves we may obtain an efficient geodesic whose vertices, with the possible exception of the endpoints, are all nonseparating curves.

**Question 1.** Birman, Margalit, and Menasco [[2016, Theorem 1.1](#)] prove that, given vertices  $v_0$  and  $v_n$  at distance  $n$  in  $\mathcal{C}(S)$ , there are at most  $n^{6g-6}$  vertices that may appear as the first curve in an initially efficient geodesic. Can this bound be reduced when we restrict to  $\mathcal{N}(S)$ ?

**Question 2.** Does replacing a separating curve with a nonseparating curve in a path ever increase the complexity measure?

**Question 3.** Find an example of an efficient geodesic that contains a separating curve. Are there any restrictions on the number of separating curves in a efficient geodesics?

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## References

- [Birman et al. 2016] J. Birman, D. Margalit, and W. Menasco, “Efficient geodesics and an effective algorithm for distance in the complex of curves”, *Math. Ann.* **366**:3-4 (2016), 1253–1279. [MR](#) [Zbl](#)
- [Glenn et al. 2017] P. Glenn, W. W. Menasco, K. Morrell, and M. J. Morse, “MICC: a tool for computing short distances in the curve complex”, *J. Symbolic Comput.* **78** (2017), 115–132. [MR](#) [Zbl](#)
- [Hamenstädt 2014] U. Hamenstädt, “Hyperbolicity of the graph of nonseparating multicurves”, *Algebr. Geom. Topol.* **14**:3 (2014), 1759–1778. [MR](#) [Zbl](#)
- [Masur and Minsky 2000] H. A. Masur and Y. N. Minsky, “Geometry of the complex of curves, II: Hierarchical structure”, *Geom. Funct. Anal.* **10**:4 (2000), 902–974. [MR](#) [Zbl](#)
- [Rasmussen 2020] A. J. Rasmussen, “Uniform hyperbolicity of the graphs of nonseparating curves via bicorn curves”, *Proc. Amer. Math. Soc.* **148**:6 (2020), 2345–2357. [MR](#) [Zbl](#)
- [Wright 2023] A. Wright, “Spheres in the curve graph and linear connectivity of the Gromov boundary”, preprint, 2023. [arXiv 2304.03004](#)

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# PACIFIC JOURNAL OF MATHEMATICS

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Integrality relations for polygonal dissections	199
AARON ABRAMS and JAMES POMMERSHEIM	
The $h$ -principle for maps transverse to bracket-generating distributions	207
ARITRA BHOWMICK	
The restriction of efficient geodesics to the nonseparating complex of curves	233
SETH HOVLAND and GREG VINAL	
On multiplicity-free weight modules over quantum affine algebras	251
XINGPENG LIU	
Differential geometric approach to the deformation of a pair of complex manifolds and Higgs bundles	283
TAKASHI ONO	
Uniform extension of definable $C^{m,\omega}$ -Whitney jets	317
ADAM PARUSIŃSKI and ARMIN RAINER	
Noncommutative tensor triangular geometry: classification via Noetherian spectra	355
JAMES ROWE	
Regularity of manifolds with integral scalar curvature bound and entropy lower bound	373
SHANGZHI ZOU	
Correction to the article A Hecke algebra isomorphism over close local fields	389
RADHIKA GANAPATHY	