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ON MULTIPLICITY-FREE WEIGHT MODULES OVER QUANTUM AFFINE ALGEBRAS

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Our goal is to construct and study the multiplicity-free weight modules of quantum affine algebras. For this, we introduce the notion of shiftability condition with respect to a symmetrizable generalized Cartan matrix, and investigate its applications on the study of quantum affine algebra structures and the realizations of the infinite-dimensional multiplicity-free weight modules. We also compute the highest ℓ-weights of the infinite-dimensional multiplicity-free weight modules as highest ℓ-weight modules.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantum affine algebra (without derivation) associated to an affine Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ in which q is not a root of unity. In this note, we are concerned with infinite-dimensional multiplicity-free weight representations, i.e., those with all of their weight subspaces one-dimensional, over $U_q(\mathfrak{g})$. As we shall see, these representations are the basic representations toward the infinite-dimensional modules of quantum affine algebras.

In the classical cases, the multiplicity-free weight representations over finite-dimensional simple Lie algebras, or more generally, the bounded weight representations have been extensively studied in [Benkart et al. 1997; Britten et al. 1994; Grantcharov and Serganova 2006; 2010]. These representations play a crucial role in the classification of simple weight modules of finite-dimensional simple Lie algebras (see [Mathieu 2000]). For the quantum groups of finite type, Futorny, Hartwig, and Wilson [Futorny et al. 2015] gave a classification of all infinite-dimensional irreducible multiplicity-free weight representations of type A_n. Recently, the infinite-dimensional multiplicity-free weight representations of the quantum groups of types A_n, B_n and C_n were constructed in [Chen et al. 2024].

As an important class of multiplicity-free weight modules, the q-oscillator representations over $U_q(\mathfrak{g})$ of types $A_n^{(1)}$, $C_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$ have been obtained in the works of T. Hayashi [1990] and A. Kuniba and M. Okado [2018; 2013; 2015]. Our goal is to construct infinite-dimensional multiplicity-free weight representations

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of $U_q(\mathfrak{g})$ in a general way. For this, associated to each symmetrizable generalized Cartan matrix, we introduce a system of equations in a Laurent polynomial ring $\mathcal{A}$ (essentially, the Cartan part of $U_q(\mathfrak{g})$) by the shift operators. We say that the corresponding generalized Cartan matrix satisfies the *shiftability condition* if the system of equations has solutions (see Section 4A). One result of this note is that an affine Cartan matrix satisfies the shiftability condition if and only if the relevant Dynkin diagram is one of the types mentioned above (see Theorem 4.2). The solutions allow us to define $U_q(\mathfrak{g})$-module structures on $\mathcal{A}$, and to relate the quantum affine algebra structures with the n-fold quantized oscillator algebra. Our method for the construction is parallel with the earlier work concerning U^0-free modules [Chen et al. 2024]. Namely, we can get the multiplicity-free weight modules of $U_q(\mathfrak{g})$ by applying the "weighting" procedure to the above modules on $\mathcal{A}$. In particular, the q-oscillator representations can also be reconstructed.

For the study of weight representations of quantum affine algebras, the concepts of ℓ-weights and ℓ-weight vectors have proven especially useful, allowing one to refine the spectral data properly in weight representations. For example, we have the classification of irreducible finite-dimensional representations (see [Chari and Pressley 1991; 1998]) and infinite-dimensional weight representations of quantum affine algebras [Hernandez 2005; Mukhin and Young 2014] by highest ℓ-weights (their highest ℓ-weights are determined by *Drinfeld polynomials* and *rational functions*, respectively). In this note, we shall compute explicitly the highest ℓ-weight of the q-oscillator representations. For the type $A_n^{(1)}$, the highest ℓ-weights of q-oscillator representations also were discussed in [Boos et al. 2016; 2017; Kwon and Lee 2023].

The paper is organized as follows. In Section 2, we give some necessary notation, and review two presentations of quantum affine algebras. In Section 3, we recall the definition of highest ℓ-weight representations. Then we obtain the classification of highest ℓ-modules with finite weight multiplicities in general. In Section 4, we introduce the notion of shiftability condition, and present the solutions to the corresponding system of equations, which allow us to study the compatible structures of quantum affine algebras with the n-fold quantized oscillator algebra. In Section 5, the infinite-dimensional multiplicity-free weight modules are constructed. In Section 6, we compute the highest ℓ-weight of the q-oscillator representations.

Conventions. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of integers, real numbers and complex numbers respectively. Denote $\mathbb{C}\setminus\{0\}$ by $\mathbb{C}^\times$ and the set of nonnegative integers by $\mathbb{Z}_{\geq 0}$. Finally, δ_{ij} is the Kronecker symbol.

2. Preliminaries and notation

First, let us recall some necessary notation and two presentations of quantum affine algebras based on [Beck and Nakajima 2004; Drinfeld 1987; Kac 1990].

2A. *Affine Kac–Moody algebras.* Let $\mathfrak{g} = \mathfrak{g}(X_N^{(r)})$ be an affine Kac–Moody algebra with respect to the generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ of type $X_N^{(r)}$, where $I = \{0, 1, \dots, n\}$ is an indexed set and $X_N^{(r)}$ is a Dynkin diagram from [Kac 1990, Table Aff r], except in the case of $X_N^{(r)} = A_{2n}^{(2)}$ ($n \geq 1$), where we reverse the numbering of the simple roots.

Let $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ (resp. $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$) denote the set of simple roots (resp. simple coroots) such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$. Let $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the *root lattice* of $\mathfrak{g}$. Set $Q_+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. Assume that $\delta = \sum a_i\alpha_i$ and $c = \sum a_i^\vee \alpha_i^\vee$ are the smallest positive imaginary root and a central element of $\mathfrak{g}$, where a_i and $a_i^\vee$ are the numerical labels of the Dynkin diagrams of $X_N^{(r)}$ and its dual, respectively. Let $\{\omega_i\}_{i\in I}$ denote the *fundamental weights* of $\mathfrak{g}$, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in I$.

Let W be the *affine Weyl group* of $\mathfrak{g}$ (which is a subgroup of the general linear group of $\mathfrak{h}^*$) generated by the *simple reflections* $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, $\lambda \in \mathfrak{h}^*$, $i \in I$. Note that $w(\delta) = \delta$ for all $w \in W$. Set $I_0 = I \setminus \{0\}$. Denote by $\mathring{W}$ the subgroup of W generated by the simple reflections s_i for $i \in I_0$. It is a finite group.

Take the nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ invariant under the action of W, which is normalized uniquely by $(\lambda, \delta) = \langle \lambda, c \rangle$ for $\lambda \in \mathfrak{h}^*$. Define D as the diagonal matrix $\mathrm{diag}(d_0, \dots, d_n)$ with $d_i = a_i^{-1} a_i^\vee$. Then $(\alpha_i, \alpha_j) = d_i a_{ij}$ for all $i, j \in I$. Let $\triangle$ be the *root system* of $\mathfrak{g}$, $\triangle^{\pm} = \triangle \cap (\pm Q_+)$, and let $\triangle^{\mathrm{re}} = \triangle \setminus \mathbb{Z}\delta$ be the set of *real roots*. For each $\alpha \in \triangle^{\mathrm{re}}$ we set $\widetilde{d}_\alpha = \max\bigl(1, \frac{1}{2}(\alpha, \alpha)\bigr)$. In particular, write $\widetilde{d}_i$ simply for $\widetilde{d}_{\alpha_i}$. Then

$$\widetilde{d}_i = \begin{cases} 1 & \text{if } r = 1 \text{ or } X_N^{(r)} = A_{2n}^{(2)}, \\ d_i & \text{otherwise.} \end{cases}$$

Denote by $\mathring{A} = (a_{ij})_{i,j\in I_0}$ the Cartan matrix of finite type, and let $\mathring{\mathfrak{g}}$ be the associated simple finite-dimensional Lie algebra. Then $\{\alpha_i\}_{i\in I_0}$ is a set of simple roots for $\mathring{\mathfrak{g}}$. Let $\mathring{Q} = \bigoplus_{i\in I_0} \mathbb{Z}\alpha_i$ be the root lattice for $\mathring{\mathfrak{g}}$, and let $\widetilde{P}$ be the weight lattice of the euclidean space $\mathbb{R} \otimes_{\mathbb{Z}} \mathring{Q} \subset \mathfrak{h}^*$ defined as $\widetilde{P} = \bigoplus_{i\in I_0} \mathbb{Z}\widetilde{\omega}_i$, where $(\widetilde{\omega}_i, \alpha_j) = \delta_{ij}\widetilde{d}_i$. Then $\mathring{Q}$ can be naturally embedded into $\widetilde{P}$, which provides a W-invariant action on $\mathfrak{h}^*$ by $x(\lambda) = \lambda - (x, \lambda)\delta$ for $x \in \widetilde{P}, \lambda \in \mathfrak{h}^*$.

Define the *extended Weyl group* by $\widetilde{W} = \mathring{W} \ltimes \widetilde{P}$. We also have $\widetilde{W} = W \ltimes \mathscr{T}$, where $\mathscr{T} = \{w \in \widetilde{W} \mid w(\triangle^+) \subset \triangle^+\}$, which is a subgroup of the group of the Dynkin diagram automorphisms. An expression for $w \in \widetilde{W}$ is called *reduced* if $w = \tau s_{i_1} \cdots s_{i_l}$, where $\tau \in \mathscr{T}$ and l is minimal. We call the minimal integer l the *length* of w, and denote it by $l(w)$.

2B. *Quantum affine algebras.* The *quantum affine algebra* $U_q(\mathfrak{g})$ in the Drinfeld–Jimbo realization [Drinfeld 1985; Jimbo 1985] is the unital associative algebra over $\mathbb{C}$ generated by X_i^+, X_i^-, $K_i^{\pm 1}$, $i \in I$, with the following relations:

(2-1) $$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$$

$$K_i X_j^{\pm} K_i^{-1} = q_i^{\pm a_{ij}} X_j^{\pm}, \tag{2-2}$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2-3}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0 \quad \text{for } i \neq j, \tag{2-4}$$

where $q \in \mathbb{C}^{\times}$ is not a root of unity and $q_i = q^{d_i}$. We have used the standard notation:

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q^! = [m]_q [m-1]_q \cdots [1]_q, \quad \begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q^!}{[r]_q^! [m-r]_q^!}.$$

In particular, we denote $[m]_{q_i}$ by $[m]_i$ for simplicity.

Let U^0 be the commutative subalgebra of $U := U_q(\mathfrak{g})$ generated by K_i, K_i^{-1}, $i \in I$. It is clear that each element in U^0 is a linear combination of the monomials $K_\beta := K_0^{b_0} K_1^{b_1} \cdots K_n^{b_n}$ for $\beta = \sum_{i \in I} b_i \alpha_i \in Q$. In particular, K_δ is a central element in U. Let U^+ (resp. U^-) denote the span of monomials in X_i^+ (resp. X_i^-). Recall that U has a canonical triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$. For later use, we note that U^+ is graded by Q_+ in the usual way: $U^+ = \bigoplus_{\beta \in Q_+} U_\beta^+$.

Let us recall the Hopf algebra structure of U with the coproduct Δ, the antipode S, and the counit ϵ defined as follows:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(X_i^+) = X_i^+ \otimes 1 + K_i \otimes X_i^+,$$
$$\Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-,$$
$$S(X_i^+) = -K_i^{-1} X_i^+, \quad S(X_i^-) = -X_i^- K_i, \quad S(K_i) = K_i^{-1},$$
$$\epsilon(X_i^+) = 0 = \epsilon(X_i^-), \quad \epsilon(K_i) = 1.$$

There exists another presentation of U due to Drinfeld [1987]. Just like the realizations of the affine Kac–Moody algebras $\mathfrak{g}$ as (twisted) loop algebras, this presentation of U is generated by the Drinfeld's "loop-like" generators.

Consider the root datum (X_N, σ) with σ a diagram automorphism of X_N of order r. Let $\bar{A} = (\bar{a}_{ij})_{1 \le i,j \le N}$ be the Cartan matrix of the type X_N, and let ω be a fixed primitive r-th root of unity. Note that if $r = 1$ (i.e., σ is an identity) we have $N = n$ and $\bar{A} = \mathring{A}$; if $r > 1$, then X_N is one of the simply laced types: A_N $(N \ge 2)$, D_{n+1} $(n \ge 2)$, or E_6. We use $\bar{i} \in I_0$ to stand for one representative of the σ-orbit of i on $\{1, 2, \ldots, N\}$ such that $\bar{i} \le \sigma^s(i)$ for any s. Take the set of simple roots $\{\bar{\alpha}_i\}_{1 \le i \le N}$ and the normalized bilinear form (,) (by abuse of notation) such that $(\bar{\alpha}_i, \bar{\alpha}_j) = d_i a_{ij}$ if $r = 1$ and otherwise $(\bar{\alpha}_i, \bar{\alpha}_j) = \bar{a}_{ij}$ for $1 \le i, j \le N$.

The quantum affine algebra U (add the central elements $K_\delta^{\pm 1/2}$) is isomorphic to the algebra generated by $x_{i,k}^{\pm}$ $(1 \le i \le N,\ k \in \mathbb{Z})$, $h_{i,k}$ $(1 \le i \le N,\ k \in \mathbb{Z} \setminus \{0\})$, $k_i^{\pm 1}$ $(1 \le i \le N)$, and the central elements $C^{\pm 1/2}$, subject to the following relations:

$$x^{\pm}_{\sigma(i),k} = \omega^k x^{\pm}_{i,k}, \quad h^{\pm}_{\sigma(i),k} = \omega^k h^{\pm}_{i,k}, \quad k^{\pm 1}_{\sigma(i)} = k^{\pm 1}_i,$$
$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \quad k_i h_{j,l} = h_{j,l} k_i,$$
$$k_i x^{\pm}_{j,k} = q_{\bar{i}}^{\pm a_{\bar{i}\bar{j}}} x^{\pm}_{j,k} k_i,$$

$$[h_{i,k}, h_{j,l}] = \delta_{k,-l} \frac{1}{k} \left(\sum_{s=1}^{r} \left[\frac{k(\overline{\alpha}_i, \overline{\alpha}_{\sigma^s(j)})}{d_{\bar{i}}} \right]_{\bar{i}} \omega^{ks} \right) \frac{C^k - C^{-k}}{q_{\bar{i}} - q_{\bar{i}}^{-1}}, \tag{2-5}$$

$$[h_{i,k}, x^{\pm}_{j,l}] = \pm \frac{1}{k} \left(\sum_{s=1}^{r} \left[\frac{k(\overline{\alpha}_i, \overline{\alpha}_{\sigma^s(j)})}{d_{\bar{i}}} \right]_{\bar{i}} \omega^{ks} \right) C^{\mp |k|/2} x^{\pm}_{j,k+l},$$

$$[x^+_{i,k}, x^-_{j,l}] = \left(\sum_{s=1}^{r} \frac{\delta_{\sigma^s(i) j} \omega^{sl}}{\widetilde{d}_{\bar{i}}} \right) \frac{C^{(k-l)/2} \psi^+_{i,k+l} - C^{-(k-l)/2} \psi^-_{i,k+l}}{q_{\bar{i}} - q_{\bar{i}}^{-1}},$$

where the $\psi^{\pm}_{i,k}$ are the elements determined by the following identity of the formal power series in z:

$$\sum_{k=0}^{\infty} \psi^{\pm}_{i,\pm k} z^{\pm k} = k_i^{\pm 1} \exp\left(\pm (q_{\bar{i}} - q_{\bar{i}}^{-1}) \sum_{l=1}^{\infty} h_{i,\pm l} z^{\pm l} \right), \tag{2-6}$$

together with the *quantum Serre–Drinfeld relations*, whose explicit forms will not be used in this paper. One can refer to [Drinfeld 1987] for more details and to [Beck 1994; Jing 1998] and [Damiani 2000; 2012; 2015][1] for a proof.

Under the isomorphism, we have $X_i^{\pm} = x^{\pm}_{i,0}$, $K_i^{\pm 1} = k_i^{\pm 1}$ for $i \in I_0$, and $K_\delta = C$. Note that $\psi^+_{i,-k} = \psi^-_{i,k} = 0$ for any positive integers k, and $\psi^{\pm}_{i,0} = k_i^{\pm 1}$ from the identity (2-6).

From the relations in the Drinfeld presentation, U is essentially generated by $x^{\pm}_{i,\widetilde{d}_i k}$ ($i \in I_0, k \in \mathbb{Z}$), $h_{i,\widetilde{d}_i k}$ ($i \in I_0, k \in \mathbb{Z} \setminus \{0\}$), $k_i^{\pm 1}$ ($i \in I_0$), and the central elements $C^{\pm 1/2}$ (see [Damiani 2012, Proposition 4.25]). Moreover, the quantum affine algebra U has a triangular decomposition [Chari and Pressley 1994; 1998]:

$$U \cong U(\leqslant) \otimes U(0) \otimes U(\geqslant), \tag{2-7}$$

where $U(\geqslant)$ (resp. $U(\leqslant)$) is the subalgebra generated by $x^+_{i,\widetilde{d}_i k}$ (resp. $x^-_{i,\widetilde{d}_i k}$), $i \in I_0$, $k \in \mathbb{Z}$, and $U(0)$ is the subalgebra generated by $C^{\pm 1/2}$ and $k_i^{\pm}$, $h_{i,k}$, $i \in I_0$, $k \in \mathbb{Z} \setminus \{0\}$.

3. Highest ℓ-weight representations with finite weight multiplicities

In this section, we recall basic notation of representations over quantum affine algebras: weight modules, ℓ-weights, and highest ℓ-weight modules. Most of the

[1] The author used the notation $\widetilde{H}^{\pm}_{i,l}$, $H_{i,l}$, which is related with the notation $\psi^{\pm}_{i,l}$, $h_{i,l}$ by $\widetilde{H}^{\pm}_{i,l} = C^{l/2} k_i^{\mp 1} \psi^{\pm}_{i,l}$ and $H_{i,l} = C^{l/2} h_{i,l}$.

definitions and results in this section are well-known; one can refer to [Chari and Pressley 1991; Mukhin and Young 2014].

3A. ***Highest ℓ-weight modules.*** We begin with the notion of highest ℓ-weight modules. Thanks to the Hopf algebra structure of U^0 (inherited from U), the set of all *algebra characters* of U^0, i.e., all algebra homomorphisms from U^0 to $\mathbb{C}$, has an abelian group structure. The addition and the inverse are given by

$$(\lambda+\mu)(u) = (\lambda\otimes\mu)\circ\Delta(u), \quad (-\lambda)(u) = \lambda\circ S(u)$$

for any algebra characters λ, μ, and $u \in U^0$. Denote this group simply by $(\mathscr{X}, +)$. Any $\beta \in \mathfrak{h}^*$ induces a character in $\mathscr{X}$ by assigning K_i to $q^{(\beta,\alpha_i)}$ for $i \in I$, which is unique up to a constant multiple of δ, so we still denote it by $\beta \in \mathscr{X}$.

For a U-module V and $\lambda \in \mathscr{X}$, define

$$V_\lambda = \{v \in V \mid u.v = \lambda(u)v \text{ for all } u \in U^0\}.$$

By the defining relations (2-2) we have $X_i^{\pm}.V_\lambda \subset V_{\lambda\pm\alpha_i}$. If V_λ is nonzero, then we say λ is a *weight* of V, and V_λ is a *weight space* of *weight* λ. A nonzero vector $v \in V_\lambda$ is called a *weight vector* of weight λ. If the weight space V_λ is finite-dimensional, then $\dim V_\lambda$ is called the *multiplicity* of the weight λ. Call V a *weight module* if $V = \bigoplus_\lambda V_\lambda$. Moreover, a weight module V is said to be *multiplicity-free* if $\dim V_\lambda \le 1$ for all $\lambda \in \mathscr{X}$.

Throughout this note, we assume that the central element C acts trivially on a U-module. So any weight λ of a U-module is *level-zero*, that is, $\lambda(K_\delta) = 1$.

Note that the actions of the $\psi_{i,k}^{\pm}$ on a U-module commute with each other by (2-5) and (2-6). For a weight λ of V with finite multiplicity, we may refine the weight space V_λ as

$$V_\lambda = \bigoplus_{\boldsymbol{\gamma}:\mathrm{wt}(\boldsymbol{\gamma})=\lambda} V_{\boldsymbol{\gamma}},$$

$$V_{\boldsymbol{\gamma}} = \{v \in V_\lambda \mid \forall 1 \le i \le N, k \ge 0, \exists m \in \mathbb{Z}_{>0}, (\psi_{i,\pm k}^{\pm} - \gamma_{i,\pm k}^{\pm})^m.v = 0\},$$

where $\boldsymbol{\gamma} = (\gamma_{i,\pm k}^{\pm})_{1\le i\le N, k\in\mathbb{Z}_{\ge 0}}$ is any N-tuple of sequences of complex numbers satisfying that $\gamma_{i,0}^{+}\gamma_{i,0}^{-} = 1$ and $\gamma_{\sigma(i),\pm k}^{\pm} = \omega^{\pm k}\gamma_{i,\pm k}^{\pm}$ for all $1 \le i \le N$, and we associate $\boldsymbol{\gamma}$ with a level-zero weight $\mathrm{wt}(\boldsymbol{\gamma}) \in \mathscr{X}$ by setting $\mathrm{wt}(\boldsymbol{\gamma})(K_i) = \gamma_{i,0}^{+}$ for all $i \in I_0$. Call such a sequence $\boldsymbol{\gamma}$ an *ℓ-weight* and $V_{\boldsymbol{\gamma}}$ the *ℓ-weight space* of $\boldsymbol{\gamma}$ if $V_{\boldsymbol{\gamma}}$ is not zero.

Given an ℓ-weight $\boldsymbol{\gamma}$, the defining relations in the Drinfeld presentation imply that $\boldsymbol{\gamma}$ is completely determined by the tuple of complex numbers $(\gamma_{i,\pm\tilde{d}_i k}^{\pm})_{i\in I_0, k\in\mathbb{Z}_{\ge 0}}$. Note that the $\gamma_{i,k}^{\pm}$ for $\tilde{d}_i \nmid k$ are zero. Hence we may write $\boldsymbol{\gamma} \equiv (\gamma_{i,\pm\tilde{d}_i k}^{\pm})_{i\in I_0, k\in\mathbb{Z}_{\ge 0}}$ directly without any ambiguity.

Now we can define the highest ℓ-weight modules.

Definition 3.1. We say V is a *highest ℓ-weight modules* of highest ℓ-weight $\boldsymbol{\gamma}$ if $V = U.v$ for some nonzero vector $v \in V$ such that $x^+_{i,k}.v = 0$ for $1 \le i \le N$, $k \in \mathbb{Z}$, and $\psi^\pm_{i,\pm k}.v = \gamma^\pm_{i,\pm k} v$ for $1 \le i \le N$, $k \in \mathbb{Z}_{\ge 0}$. By (2-7), $\dim V_{\boldsymbol{\gamma}} = 1$, so v is unique up to a scalar; we call it the *highest ℓ-weight vector* of V.

3B. ***The classification theorem: rationality.*** In this subsection we give the classification of simple highest ℓ-weight modules with finite weight multiplicity, which appeared in [Mukhin and Young 2014] for untwisted cases.

We say an ℓ-weight $\boldsymbol{f} = (f^\pm_{i,\pm\tilde{d}_i k})_{i\in I_0, k\in\mathbb{Z}_{\ge 0}}$ is *rational* if there is a tuple of complex-valued rational functions $(f_i(z))_{i\in I_0}$ in a formal variable z such that for each $i \in I_0$, $f_i(z)$ is regular at 0 and ∞, $f_i(0) f_i(\infty) = 1$, and

$$\sum_{k=0}^{\infty} f^+_{i,\tilde{d}_i k} z^k = f_i(z) = \sum_{k=0}^{\infty} f^-_{i,-\tilde{d}_i k} z^{-k}$$

in the sense that the left- and right-hand sides are the Laurent expansions of $f_i(z)$ at 0 and ∞, respectively.

Let $\mathcal{R}$ be the set of all rational ℓ-weights. Then $\mathcal{R}$ forms an abelian group with the group operation $(\boldsymbol{f}, \boldsymbol{g}) \mapsto \boldsymbol{f}\boldsymbol{g}$ being given by componentwise multiplication of the corresponding tuples of rational functions. We will not always distinguish between a rational ℓ-weight $\boldsymbol{f}$ and the corresponding tuple $(f_i(z))_{i\in I_0}$ of rational functions.

Recall from [Chari and Pressley 1991; 1998] that simple finite-dimensional modules of U are highest ℓ-weight modules, and their highest ℓ-weights $\boldsymbol{f}$ are parametrized by the tuples of the *Drinfeld polynomials*. More precisely, there exists a tuple of polynomials $(P_i(z))_{i\in I_0}$ with all $P_i(z)$ having constant coefficient 1 such that $\boldsymbol{f}$ satisfies that for $i \in I_0$,

$$f_i(z) = \begin{cases} q_n^{2\deg P_n}(P_n(q_n^{-4}z)/P_n(z)) & \text{if } (X_N^{(r)}, i) = (A_{2n}^{(2)}, n), \\ q_i^{\deg P_i}(P_i(q_i^{-2}z)/P_i(z)) & \text{otherwise.} \end{cases}$$

Therefore, the highest ℓ-weight of any simple finite-dimensional module is rational.

In general, we have the following theorem.

Theorem 3.2. *Let V be an irreducible highest ℓ-weight module. Then all weight spaces of V are finite-dimensional if and only if its highest ℓ-weight $\boldsymbol{f}$ belongs to $\mathcal{R}$.*

Proof. For the nontwisted cases, one can refer to [Mukhin and Young 2014, Theorem 3.7]. The proof of the twisted cases is essentially parallel to that of the untwisted cases thanks to the triangular decomposition (2-7) of the Drinfeld realization. □

4. Shiftability conditions and algebra homomorphisms

In this section, the notion of the shiftability condition with respect to a generalized Cartan matrix will be introduced, and the compatible structures of the quantum affine algebras with the n-fold q-oscillator algebras are given from the q-shiftability condition.

4A. ***Shiftability conditions.*** Let $A = (a_{ij})_{i,j\in I}$ be any symmetrizable generalized Cartan matrix. Let $\mathcal{A}$ be the Laurent polynomial ring over $\mathbb{C}$ in the variables x_i, $i \in I$, i.e., $\mathcal{A} = \mathbb{C}[x_i^{\pm 1}, i \in I]$. For each $i \in I$, consider the algebra automorphism $\zeta_i : \mathcal{A} \to \mathcal{A}$ given by $\zeta_i(x_j) = q_i^{-\delta_{ij}} x_j$ for $j \in I$. For any distinct $i, j \in I$, we say a pair of Laurent polynomials (f, g) in $\mathcal{A}$ is (i, j)-*shiftable* if f, g satisfy

$$fg = \zeta_j^{-1}(f)\zeta_i^{-1}(g).$$

Set $\{x\}_i := (x - x^{-1})/(q_i - q_i^{-1})$ for any unit x in $\mathcal{A}$, and for simplicity, write $\{x\} = (x - x^{-1})/(q - q^{-1})$. Define the elements $y_i, y_i^{-1} \in \mathcal{A}$ by

$$y_i^{\pm 1} = \prod_{j\in I} x_j^{\pm a_{ji}}.$$

Consider the following system of equations with respect to the variables ϕ_i, $i \in I$, in $\mathcal{A}$:

$$\begin{cases} \zeta_i(\phi_i) - \phi_i = \{y_i\}_i, \\ \phi_i\phi_j = \zeta_j^{-1}(\phi_i)\zeta_i^{-1}(\phi_j), \end{cases} \qquad i, j \in I,\ i \neq j. \tag{4-1}$$

In general, this system of equations does not always have a solution. It depends on the choice of the generalized Cartan matrix A. Therefore, we can say A admits the q-*shiftability condition* when the corresponding system (4-1) has a solution.

By a quick computation, we obtain a family of solutions to (4-1) for A of types A_2 and $A_1^{(1)}$.

Example 4.1. (i) For the type A_2, a pair of Laurent polynomials (ϕ_1, ϕ_2) where $\phi_1 = \{qbx_1\}\{bx_1^{-1}x_2\}$ and $\phi_2 = \{qbx_1^{-1}x_2\}\{bx_2^{-1}\}$ for each scalar $b \in \mathbb{C}^\times$ is a solution.

(ii) For the type $A_1^{(1)}$, consider the Laurent polynomials $\phi_0 = \{qbx_0x_1^{-1}\}\{bx_0^{-1}x_1\}$ and $\phi_1 = \{qbx_0^{-1}x_1\}\{bx_0x_1^{-1}\}$ for any scalar $b \in \mathbb{C}^\times$. It is easy to check that (ϕ_0, ϕ_1) is a solution.

In what follows, the q-shiftability condition for the generalized Cartan matrices of affine types will be investigated. Now assume that A is an affine Cartan matrix as in Section 2. Then we have the first main result in this section.

Theorem 4.2. *There exists an $(n+1)$-tuple of Laurent polynomials in $\mathcal{A}$ satisfying the system (4-1) if and only if A is of the type $A_n^{(1)}$ $(n \geq 1)$, $C_n^{(1)}$ $(n \geq 2)$, $A_{2n}^{(2)}$ $(n \geq 1)$ or $D_{n+1}^{(2)}$ $(n \geq 2)$.*

The proof of Theorem 4.2 will be given in the Appendix. Here we list all tuples of Laurent polynomials $(\phi_i)_{i\in I}$ satisfying (4-1) for each affine Cartan matrix A in the theorem above.

For the type $A_n^{(1)}$ $(n \geq 1)$:

$$\big(\{qb_A z_0\}\{b_A z_1\},\ \{qb_A z_1\}\{b_A z_2\},\ \ldots,\ \{qb_A z_n\}\{b_A z_0\}\big).$$

For the type $C_n^{(1)}$ $(n \geq 2)$:

$$\big(\{q_0 b_C z_1^{-1}\}_0\{b_C z_1\}_0,\ \{q_1 b_C z_1\}_1\{b_C z_2\}_1,\ \ldots,\ \{q_{n-1} b_C z_{n-1}\}_{n-1}\{b_C z_n\}_{n-1},\ \{q_n b_C z_n\}_n\{b_C z_n^{-1}\}_n\big).$$

For the type $A_{2n}^{(2)}$ $(n \geq 1)$:

$$\left(\{\iota q^{-\frac{3}{2}} z_1\}_0\{\iota q^{-\frac{1}{2}} z_1\}_0,\ \{\iota q^{\frac{1}{2}} z_1\}_1\{\iota q^{-\frac{1}{2}} z_2\}_1,\ \ldots,\ \{\iota q^{\frac{1}{2}} z_{n-1}\}_{n-1}\{\iota q^{-\frac{1}{2}} z_n\}_{n-1},\ \frac{\iota}{q_n - q_n^{-1}}\{\iota q^{\frac{1}{2}} z_n\}_n\right).$$

For the type $D_{n+1}^{(2)}$ $(n \geq 2)$:

$$\left(\frac{\iota}{q_0 - q_0^{-1}}\{\iota q^{-1} z_1\}_0,\ \{\iota q z_1\}_1\{\iota q^{-1} z_2\}_1,\ \ldots,\ \{\iota q z_{n-1}\}_{n-1}\{\iota q^{-1} z_n\}_{n-1},\ \frac{\iota}{q_n - q_n^{-1}}\{\iota q z_n\}_n\right).$$

Here, $\iota = \sqrt{-1}$. The elements $z_i \in \mathcal{A}$ involved in the above solutions and the relations in our notation are given as follows for each type:

For $A_n^{(1)}$,

$$z_i = x_{i-1}^{-1} x_i, \quad z_0 = (z_1 \cdots z_n)^{-1}, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n z_0^{-1}, \quad b_{A_n^{(1)}} \in \mathbb{C}^\times.$$

For $C_n^{(1)}$,

$$z_i = x_{i-1}^{-1} x_i, \quad y_0 = z_1^{-2}, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n^2, \quad b_{C_n^{(1)}} = q^{-1/4} \text{ or } \iota q^{-1/4}.$$

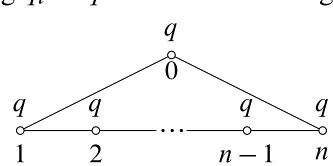
For $A_{2n}^{(2)}$,

$$z_i = x_{i-1}^{-1} x_i, \quad z_n = x_{n-1}^{-1} x_n^2, \quad y_0 = z_1^{-2}, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n, \quad b_{A_{2n}^{(2)}} = \iota q^{-\frac{1}{2}}.$$

For $D_{n+1}^{(2)}$,

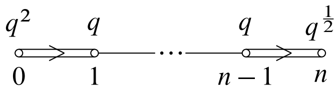
$$z_1 = x_0^{-2} x_1, \quad z_i = x_{i-1}^{-1} x_i, \quad z_n = x_{n-1}^{-1} x_n^2, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n, \quad y_0 = z_1^{-1}, \quad b_{D_{n+1}^{(2)}} = \iota q^{-1}.$$

By our convention, the Dynkin diagrams of the above four types and the corresponding $q_i = q^{d_i}$ are the following:

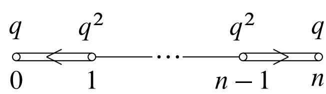


$(A_n^{(1)})$

$(C_n^{(1)})$



$(A_{2n}^{(2)})$



$(D_{n+1}^{(2)})$

Remark 4.3. One can also consider the shiftability condition for a generalized Cartan matrix A in the classical sense. More precisely, consider the polynomial ring $\mathcal{A}^+ = \mathbb{C}[x_i, i \in I]$, and the algebra automorphisms $\zeta_i : \mathcal{A}^+ \to \mathcal{A}^+$ defined by $\zeta_i(x_j) = x_j - \delta_{ij}$ for all $j \in I$. Write $y_i = \sum_{j \in I} a_{ij} x_j$. Then a similar system of equations in $\mathcal{A}^+$ (replace $\{y_i\}_i$ in (4-1) by y_i) can be obtained.

4B. ***Quantized oscillator algebra and algebra homomorphisms.*** One interesting application of the q-shiftability condition is to study the compatible structures of quantum affine algebras of types $X_N^{(r)}$ with the n-fold quantized oscillator algebra.

Fix $\nu \in \mathbb{C}^\times$. The (*symmetric*) *quantized oscillator algebra* $\mathscr{B}_\nu$ is the unital associative algebra over $\mathbb{C}$ generated by four elements $\boldsymbol{a}^+$, $\boldsymbol{a}$, $\boldsymbol{k}^{\pm 1}$ subject to the relations:

$$[\boldsymbol{a}, \boldsymbol{a}^+]_\nu = \boldsymbol{k}, \quad [\boldsymbol{a}, \boldsymbol{a}^+]_{\nu^{-1}} = \boldsymbol{k}^{-1}, \quad \boldsymbol{k}\boldsymbol{k}^{-1} = \boldsymbol{k}^{-1}\boldsymbol{k} = 1,$$
$$\boldsymbol{k}\boldsymbol{a}\boldsymbol{k}^{-1} = \nu^{-1}\boldsymbol{a}, \quad \boldsymbol{k}\boldsymbol{a}^+\boldsymbol{k}^{-1} = \nu\boldsymbol{a}^+,$$

where $[x, y]_\nu := xy - \nu^{-1} yx$. Then $\boldsymbol{a}^+\boldsymbol{a} = \{\boldsymbol{k}\}$, $\boldsymbol{a}\boldsymbol{a}^+ = \{\nu\boldsymbol{k}\}$ and $\{\boldsymbol{k}\}\boldsymbol{a}^+ = \boldsymbol{a}^+\{\nu\boldsymbol{k}\}$, $\boldsymbol{a}\{\boldsymbol{k}\} = \{\nu\boldsymbol{k}\}\boldsymbol{a}$ in $\mathscr{B}_\nu$. Here we define $\{\boldsymbol{x}\} = \{\boldsymbol{x}\}_\nu = (\boldsymbol{x} - \boldsymbol{x}^{-1})/(\nu - \nu^{-1})$ for $\boldsymbol{x} = \boldsymbol{k}$ or $\nu\boldsymbol{k}$.

One can easily check the following results.

Lemma 4.4. (i) *There exists a unique* $\mathbb{C}$*-algebra automorphism* (*an involution*) $\vartheta : \mathscr{B}_\nu \to \mathscr{B}_\nu$ *such that* $\vartheta(\boldsymbol{a}^+) = -\boldsymbol{a}$, $\vartheta(\boldsymbol{a}) = \boldsymbol{a}^+$ *and* $\vartheta(\boldsymbol{k}) = \nu^{-1}\boldsymbol{k}^{-1}$.

(ii) *For any* $b \in \mathbb{C}^\times$ *and* $m \in \mathbb{Z}$, *there exists a family of* $\mathbb{C}$*-algebra automorphisms* $\theta_{b,m} : \mathscr{B}_\nu \to \mathscr{B}_\nu$ *such that* $\theta_{b,m}(\boldsymbol{a}) = b\boldsymbol{k}^m\boldsymbol{a}$, $\theta_{b,m}(\boldsymbol{a}^+) = b^{-1}\boldsymbol{a}^+\boldsymbol{k}^{-m}$ *and* $\theta_{b,m}(\boldsymbol{k}^{\pm 1}) = \boldsymbol{k}^{\pm 1}$.

Consider the algebra $\mathscr{B}_\nu^{\otimes n}$ of the n-fold tensor product of $\mathscr{B}_\nu$. Denote the generators of its i-th component by $\boldsymbol{a}_i^+$, $\boldsymbol{a}_i$ and $\boldsymbol{k}_i^{\pm 1}$, which satisfy the above relations. Let $U_q(X_N^{(r)})$ be the quantum affine algebra U of the type $X_N^{(r)}$ in Theorem 4.2. For convenience, if X is of the type A then we shall deal with $A_{n-1}^{(1)} (n \geq 2)$ instead of $A_n^{(1)} (n \geq 1)$ from now on.

Fix a solution $(\phi_i)_{i \in I}$ in Section 4A. We define the algebra homomorphism $\pi_{X_N^{(r)}} : U_q(X_N^{(r)}) \to \mathscr{B}_\nu^{\otimes n}$ as follows: regard ϕ_i and $\zeta_i(\phi_i)$ as the images of $X_i^- X_i^+$ and $X_i^+ X_i^-$ respectively under $\pi_{X_N^{(r)}}$ by setting $\boldsymbol{k}_i = q^{-1} b_A^{-1} z_i^{-1}$ for the type $A_{n-1}^{(1)}$, and $\boldsymbol{k}_i = \imath \nu^{-1/2} z_i^{-1}$ otherwise (here we consider the solution with $b_C = \imath q^{-1/4}$ for the type $C_n^{(1)}$), where ν is defined as in the following proposition for each type. Then the relations (2-3) for $i = j$ hold under $\pi_{X_N^{(r)}}$ since ϕ_i satisfies $\zeta_i(\phi_i) - \phi_i = \{y_i\}_i$. In this sense, $\mathcal{A}_0 := \mathbb{C}[\boldsymbol{k}_1^{\pm 1}, \dots, \boldsymbol{k}_n^{\pm 1}]$ is a subalgebra of $\mathcal{A}$, and $\pi_{X_N^{(r)}}(K_i) = y_i$ for $i \in I$. On the other hand, we choose $\pi_{X_N^{(r)}}(X_i^{\pm}) \in \mathscr{B}_\nu^{\otimes n}$ satisfying

$$\pi_{X_N^{(r)}}(X_i^{\pm}) f = \zeta_i^{\pm 1}(f) \pi_{X_N^{(r)}}(X_i^{\pm})$$

for any $f \in \mathcal{A}_0$. The above choice yields that the relations (2-1)–(2-4) hold. Then we get the following algebra homomorphisms, which were obtained in [Hayashi 1990; Kuniba et al. 2015].

Proposition 4.5 [Hayashi 1990; Kuniba et al. 2015]. *For a parameter z, there exist algebra homomorphisms from $U_q(X_N^{(r)})$ to $\mathscr{B}_\nu^{\otimes n}[z, z^{-1}]$ defined as follows.*

For the type $A_{n-1}^{(1)}$ and $\nu = q$:

$$\pi_{A_{n-1}^{(1)},z} : U_q(A_{n-1}^{(1)}) \to \mathscr{B}_\nu^{\otimes n}[z, z^{-1}],$$

$$X_i^+ \mapsto z^{\delta_{i,0}} \boldsymbol{a}_i \boldsymbol{a}_{i+1}^+, \quad X_i^- \mapsto z^{-\delta_{i,0}} \boldsymbol{a}_i^+ \boldsymbol{a}_{i+1}, \quad K_i \mapsto \boldsymbol{k}_i^{-1} \boldsymbol{k}_{i+1}.$$

For this type, we always read the index i as i modulo n.

For the type $C_n^{(1)}$ and $\nu = q^{\frac{1}{2}}$:

$$\pi_{C_n^{(1)},z} : U_q(C_n^{(1)}) \to \mathscr{B}_\nu^{\otimes n}[z, z^{-1}],$$

$$\begin{aligned}
&X_0^+ \mapsto z(\boldsymbol{a}_1^+)^2/[2]_\nu, && X_0^- \mapsto z^{-1}\boldsymbol{a}_1^2/[2]_\nu, && K_0 \mapsto -\nu \boldsymbol{k}_1^2,\\
&X_i^+ \mapsto \boldsymbol{a}_i \boldsymbol{a}_{i+1}^+, && X_i^- \mapsto \boldsymbol{a}_i^+ \boldsymbol{a}_{i+1}, && K_i \mapsto \boldsymbol{k}_i^{-1} \boldsymbol{k}_{i+1},\\
&X_n^+ \mapsto \boldsymbol{a}_n^2/[2]_\nu, && X_n^- \mapsto (\boldsymbol{a}_n^+)^2/[2]_\nu, && K_n \mapsto -\nu^{-1} \boldsymbol{k}_n^{-2}.
\end{aligned}$$

For the type $A_{2n}^{(2)}$ and $\nu = q$:

$$\pi_{A_{2n}^{(2)},z} : U_q(A_{2n}^{(2)}) \to \mathscr{B}_\nu^{\otimes n}[z, z^{-1}],$$

$$\begin{aligned}
&X_0^+ \mapsto z(\boldsymbol{a}_1^+)^2/[2]_\nu, && X_0^- \mapsto z^{-1}\boldsymbol{a}_1^2/[2]_\nu, && K_0 \mapsto -\nu \boldsymbol{k}_1^2,\\
&X_i^+ \mapsto \boldsymbol{a}_i \boldsymbol{a}_{i+1}^+, && X_i^- \mapsto \boldsymbol{a}_i^+ \boldsymbol{a}_{i+1}, && K_i \mapsto \boldsymbol{k}_i^{-1} \boldsymbol{k}_{i+1},\\
&X_n^+ \mapsto \imath \tau_\nu \boldsymbol{a}_n, && X_n^- \mapsto \boldsymbol{a}_n^+, && K_n \mapsto \imath \nu^{-\frac{1}{2}} \boldsymbol{k}_n^{-1},
\end{aligned}$$

For the type $D_{n+1}^{(2)}$ and $\nu = q^2$:

$$\pi_{D_{n+1}^{(2)},z} : U_q(D_{n+1}^{(2)}) \to \mathscr{B}_\nu^{\otimes n}[z, z^{-1}],$$

$$\begin{aligned}
&X_0^+ \mapsto z\boldsymbol{a}_1^+, && X_0^- \mapsto \imath \tau_\nu z^{-1}\boldsymbol{a}_1, && K_0 \mapsto -\imath \nu^{\frac{1}{2}} \boldsymbol{k}_1,\\
&X_i^+ \mapsto \boldsymbol{a}_i \boldsymbol{a}_{i+1}^+, && X_i^- \mapsto \boldsymbol{a}_i^+ \boldsymbol{a}_{i+1}, && K_i \mapsto \boldsymbol{k}_i^{-1} \boldsymbol{k}_{i+1},\\
&X_n^+ \mapsto \imath \tau_\nu \boldsymbol{a}_n, && X_n^- \mapsto \boldsymbol{a}_n^+, && K_n \mapsto \imath \nu^{-\frac{1}{2}} \boldsymbol{k}_n^{-1}.
\end{aligned}$$

Here, $\tau_\nu = (\nu + 1)/(\nu - 1)$. □

5. Multiplicity-free weight modules

In this section, we construct the multiplicity-free weight representations over U from the solutions and the algebra homomorphisms in the previous section. Throughout, we assume that U is the quantum affine algebra of type $X_N^{(r)}$ in Proposition 4.5.

5A. ***Module structures on $\mathcal{A}_0$.*** In order to construct the multiplicity-free weight representations, we first consider the auxiliary U-module structures on $\mathcal{A}_0 = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}]$.

Let us fix some notation here. Note that α_0 and α_n are long roots in the type $C_n^{(1)}$, while both of them are short roots in $D_{n+1}^{(2)}$. In addition, by our assumption, α_0 is long and α_n is short in $A_{2n}^{(2)}$. We define a pair $\kappa := (\kappa_1, \kappa_2)$ of signs such that κ_1, κ_2 are equal to 0 or 1, depending on the length of the roots α_0 and α_n for each type, that is,

$$(\kappa_1, \kappa_2) = (1, 1) \quad \text{for } C_n^{(1)},$$
$$(\kappa_1, \kappa_2) = (1, 0) \quad \text{for } A_{2n}^{(2)},$$
$$(\kappa_1, \kappa_2) = (0, 0) \quad \text{for } D_{n+1}^{(2)}.$$

Fix a solution $(\phi_i)_{i\in I}$ of (4-1), and recall the units z_i for each type, and the shift operators ζ_i defined in Section 4A. Put $b = b_{X_N^{(r)}}$. Then we have:

Theorem 5.1. *Let z be a parameter valued in $\mathbb{C}^\times$. For an n-tuple $\boldsymbol{f} = (f_i)_{1\le i\le n}$ such that f_i is equal either to 1 or to $bz_i - b^{-1}z_i^{-1}$ for $1 \le i \le n$, there exists a U-module structure on the algebra $\mathcal{A}_0$ for each type defined in the following*:

For the type $A_{n-1}^{(1)}$,

$$X_i^+.u = z^{\delta_{i,0}} f_i \zeta_i\left(\frac{\{bz_{i+1}\}}{f_{i+1}}\right)\zeta_i(u), \quad X_i^-.u = z^{-\delta_{i,0}}\zeta_i^{-1}\left(\frac{\{bz_i\}}{f_i}\right) f_{i+1}\zeta_i^{-1}(u),$$

and $K_i^{\pm 1}.u = y_i^{\pm 1}u$, for any $u \in \mathcal{A}_0$.

For other types,

$$X_0^+.u = z\frac{\zeta_0(\phi_0)\zeta_0(u)}{\zeta_1^{-1}(f_1)^{\kappa_1}\zeta_0(f_1)}, \qquad X_0^-.u = z^{-1} f_1\zeta_1(f_1)^{\kappa_1}\zeta_0^{-1}(u),$$
$$X_i^+.u = f_i\zeta_i\left(\frac{\{bz_{i+1}\}_i}{f_{i+1}}\right)\zeta_i(u), \quad X_i^-.u = \zeta_i^{-1}\left(\frac{\{bz_i\}_i}{f_i}\right) f_{i+1}\zeta_i^{-1}(u),$$
$$X_n^+.u = f_n\zeta_{n-1}^{-1}(f_n)^{\kappa_2}\zeta_n(u), \qquad X_n^-.u = \frac{\phi_n\zeta_n^{-1}(u)}{\zeta_n^{-1}(f_n)\zeta_{n-1}(f_n)^{\kappa_2}},$$

and $K_i^{\pm 1}.u = y_i^{\pm 1}u$, for any $u \in \mathcal{A}_0$.

Proof. Taking $u = 1$ in the above construction we have the precise expressions of the actions $X_i^\pm.1$. In addition, for any $u \in \mathcal{A}_0$ we have $X_i^\pm.u = \zeta_i^{\pm 1}(u)X_i^\pm.1$. We can check each defining relation directly. For the relations (2-2), we have

$$(K_i X_j^{\pm} K_i^{-1}).u = y_i (X_j^{\pm}.(y_i^{-1}u)) \\ = y_i \zeta_j^{\pm 1}(y_i^{-1}u) X_j^{\pm}.1 = y_i \zeta_j^{\pm 1}(y_i^{-1}) X_j^{\pm}.u = q_i^{\pm a_{ij}} X_j^{\pm}.u.$$

For the relations (2-3), we consider three cases:

(1) If $i = j$, then we have

$$(X_i^+ X_i^- - X_i^- X_i^+).u = u\big(\zeta_i(X_i^-.1)X_i^+.1 - \zeta_i^{-1}(X_i^+.1)X_i^-.1\big) = u(\zeta_i(\phi_i) - \phi_i).$$

Here, $\phi_i := \zeta_i^{-1}(X_i^+.1)X_i^-.1,\ i \in I$, is just a solution to the system (4-1), by construction, which implies (2-3) for $i = j$, as desired.

(2) If $|i - j| > 1$, then we have $\zeta_i(f_k) = f_k$ and $\zeta_i(\{bz_k\}_k) = \{bz_k\}_k$ for $k = j, j+1$. Similarly, $\zeta_j(f_k) = f_k$ and $\zeta_j(\{bz_k\}_k) = \{bz_k\}_k$ for $k = i, i+1$. Therefore

$$X_i^+ X_j^-.u = \zeta_i \zeta_j^{-1}(u)\zeta_i(X_j^-.1)X_i^+.1 = \zeta_i \zeta_j^{-1}(u)(X_j^-.1)(X_i^+.1) = X_j^- X_i^+.u.$$

(3) If $|i - j| = 1$, we need to do more detailed calculations for each type. First assume $a_{ij}a_{ji} = 1$. Then we have to show

$$\zeta_i \zeta_j^{-1}\left(\frac{\{bz_j\}_j}{f_j}\right)\zeta_i(f_{j+1}) f_i \zeta_i\left(\frac{\{bz_{i+1}\}_i}{f_{i+1}}\right) \\ = \zeta_j^{-1}\left(\frac{\{bz_j\}_j}{f_j}\right) f_{j+1}\zeta_j^{-1}(f_i)\zeta_i\zeta_j^{-1}\left(\frac{\{bz_{i+1}\}_i}{f_{i+1}}\right).$$

If $j = i + 1$, then $\zeta_i(f_{j+1}) = f_{j+1},\ \zeta_j^{-1}(f_i) = f_i$, and $\zeta_i(f_j) = \zeta_j^{-1}(f_j)$, while for $i = j+1$, we have $\zeta_j^{-1}(f_{i+1}) = f_{i+1},\ \zeta_i(f_j) = f_j$, and $\zeta_i(f_i) = \zeta_j^{-1}(f_i)$. Both cases imply the above equality holds. When $X_N^{(r)} \neq A_{n-1}^{(1)}$, a direct computation yields the following equalities:

$$\{bz_1\}_1\zeta_1\phi_0 = \phi_0\zeta_0^{-1}\{bz_1\}_1, \qquad f_1^{\kappa_1}\zeta_0^{\pm 1}\zeta_1^{\pm 1}(f_1) = f_1\zeta_1^{\mp 1}(f_1)^{\kappa_1},$$
$$\{bz_n\}_{n-1}\phi_n = \zeta_n^{-1}(\{bz_n\}_{n-1})\zeta_{n-1}^{-1}(\phi_n), \quad f_n^{\kappa_2}\zeta_{n-1}^{\pm 1}\zeta_n^{\pm 1}(f_n) = f_n\zeta_{n-1}^{\mp 1}(f_n)^{\kappa_2}.$$

Then similar arguments for the case that $a_{ij}a_{ji} = 2$ are true. Any tuple $(\phi_i)_i$ satisfying (4-1) and the choice of $(f_i)_i$ also guarantee that these actions hold under the quantum Serre relations (2-4). □

Denote by $S_z(\boldsymbol{f})$ the U-module on $\mathcal{A}_0$ related to z and $\boldsymbol{f}$ given by Theorem 5.1. Recall $\pi_{X_N^{(r)}}(K_i) = y_i$ in the construction in Section 4B. Then

$$\pi_{X_N^{(r)}}(K_\delta) = \prod_{i\in I} y_i^{a_i} = \prod_{i\in I}\prod_{j\in I} x_j^{a_{ji}a_i} = \prod_{j\in I} x_j^{\sum_{i\in I} a_{ji}a_i} = 1. \tag{5-1}$$

In particular, K_δ acts trivially on $S_z(\boldsymbol{f})$. Therefore, $S_z(\boldsymbol{f})$ is finitely U^0-generated instead of U^0-diagonalizable when restricted as a U^0-module.

Now let us explain the "weighting" procedure mentioned in the introduction. That is, to a U-module $S_z(\boldsymbol{f})$, we associate a weight module $\mathcal{W}(S_z(\boldsymbol{f}))$ in the following way. Consider the algebra homomorphism from U^0 to $\mathcal{A}_0$ that assigns K_i to y_i for $i \in I_0$, which induces a natural group homomorphism from the group of characters of $\mathcal{A}_0$ to $\mathscr{X}$. For any character φ of $\mathcal{A}_0$, denote by $\mathfrak{m}_\varphi$ ($:= \ker\varphi$) the corresponding maximal ideal of $\mathcal{A}_0$. Extending $\alpha_j \in \mathscr{X}$ to a character of $\mathcal{A}_0$ by setting $\alpha_j(y_i) = q_j^{a_{ji}}$ (still denoting it by α_j), we have

$$K_i.\mathfrak{m}_\varphi S_z(\boldsymbol{f}) \subset \mathfrak{m}_\varphi S_z(\boldsymbol{f}), \quad X_i^{\pm}.\mathfrak{m}_\varphi S_z(\boldsymbol{f}) \subset \mathfrak{m}_{\varphi\pm\alpha_i} S_z(\boldsymbol{f}).$$

Define

$$\mathcal{W}(S_z(\boldsymbol{f})) = \bigoplus_{\varphi} S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f}),$$

where φ is taken over all characters of $\mathcal{A}_0$.

Corollary 5.2. *For any U-module $S_z(\boldsymbol{f})$, we have that $\mathcal{W}(S_z(\boldsymbol{f}))$ is a weight module, and all its simple subquotients are multiplicity-free.*

Proof. It is clear that $S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f})$ is 1-dimensional and K_i acts diagonally. In particular, K_δ acts by 1. The first assertion follows from the previous statements, and the λ-weight space $\mathcal{W}(S_z(\boldsymbol{f}))_\lambda = \bigoplus_{\overline{\varphi}=\lambda} S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f})$, where $\overline{\varphi}$ means the image of φ in $\mathscr{X}$. Since we have

$$X_i^{\pm}.S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f}) \subset S_z(\boldsymbol{f})/\mathfrak{m}_{\varphi\pm\alpha_i} S_z(\boldsymbol{f}),$$

the second assertion follows. □

Remark 5.3. In fact, the U-module $\mathcal{W}(S_z(\boldsymbol{f}))$ is a q-analog of the *coherent family* in the sense of [Nilsson 2016]. This "weighting" procedure was first suggested by O. Mathieu [2000].

Now let us study the possible highest weights of $\mathcal{W}(S_z(\boldsymbol{f}))$ when restricted as a $U_q(\mathring{\mathfrak{g}})$-module. Assume that the weight vector $1 + \mathfrak{m}_\varphi S_z(\boldsymbol{f})$ of $\mathcal{W}(S_z(\boldsymbol{f}))$ is a highest weight vector for some φ. Then we have $X_i^{+}.(1 + \mathfrak{m}_\varphi S_z(\boldsymbol{f})) = 0$ for $i \in I_0$, which implies that

$$(\varphi + \alpha_i)(\zeta_i(\phi_i)) = 0 \quad \text{for } i \in I_0. \tag{5-2}$$

The weight $\lambda = \overline{\varphi}$ is level-zero and is determined uniquely by the values $\lambda(K_i)$, $i \in I_0$. Therefore, all level-zero weights can be seen automatically as weights over $U_q(\mathring{\mathfrak{g}})$. As a result, we can obtain the following result.

Corollary 5.4. *Let $\lambda \in \mathscr{X}$ be a weight of $\mathcal{W}(S_z(\boldsymbol{f}))$ for some $\boldsymbol{f}$. If λ is a highest $U_q(\mathring{\mathfrak{g}})$-weight, then up to twistings by the automorphisms of $U_q(\mathring{\mathfrak{g}})$, we have*

(1) *for the type $A_{n-1}^{(1)}$, the weight λ is of the form $\omega_0 + a\omega_s - (a+1)\omega_{s+1}$ for some $a \in \mathbb{C}$ and $s \in I$ up to a constant multiple of δ;*

(2) *for the type* $C_n^{(1)}$, *the weight* λ *has the form* $\frac{1}{2}\omega_0+\omega_{n-1}-\frac{3}{2}\omega_n$ *or* $\frac{1}{2}(\omega_0-\omega_n)$, *up to a constant multiple of* δ;

(3) *for the type* $A_{2n}^{(2)}$ (*resp.* $D_{n+1}^{(2)}$), *the weight* $\lambda=(\lambda(K_0),\dots,\lambda(K_n))$ *is defined as*

$$(-q,1,\dots,1,\imath q^{-1/2})\quad (\textit{resp. } (-\imath q,1,\dots,1,\imath q^{-1})).$$

Proof. The result can be deduced directly from (5-2). For example, in the type $A_{n-1}^{(1)}$, the equations (5-2) become

$$\begin{cases} m_0 m_1\cdots m_{n-1}=1, \\ \{qbm_i\}\{bm_{i+1}\}=0, \quad 1\le i\le n-1, \end{cases} \tag{5-3}$$

where we denote $\varphi(z_i)$ by m_i for $i\in I$. Let $\lambda=\overline{\varphi}$. Then $\lambda(K_i)=m_im_{i+1}^{-1}$. To solve the equations (5-3), we consider two cases: if $\{qbm_1\}$ is not zero, then $\{bm_i\}=0$ for $i\ge 2$; if $\{qbm_1\}$ is zero, then we assume that s is the maximal index such that $\{qbm_s\}$ is zero, and then $\{qbm_i\}=0$ for $i\le s$ and $\{bm_j\}=0$ for $j\ge s+2$. In the first case, the possible solutions are $m_1=\pm b^{n-1}$ and $m_i=\pm b^{-1}$ for $i\ne 1$. Then, up to twistings by sign automorphisms of $U_q(\mathring{\mathfrak{g}})$, the weight λ is given by

$$\lambda(K_0)=b^{-n},\qquad \lambda(K_1)=b^n,\qquad \lambda(K_i)=1\quad\text{for } i\ge 2,$$

which is of the form $(a+1)\omega_0-(a+1)\omega_1$ for some $a\in\mathbb{C}$. In the second case, $m_i=\pm q^{-1}b^{-1}$ for $0\le i\le s$, $m_{s+1}=\pm q^{s+1}b^{n-1}$, and $m_j=\pm b^{-1}$ otherwise. Then, up to signs, the weight λ is the following:

$$\lambda(K_s)=q^{-s-2}b^{-n},\qquad \lambda(K_{s+1})=q^{s+1}b^n,\qquad \lambda(K_i)=1\quad\text{for } i\ne s,s+1,$$

which is exactly of the form in (1). So assertion (1) follows. □

All simple subquotients obtained in Corollary 5.4 can be realized as q-oscillator representations by using the Fock space representations of $\mathscr{B}_\nu$ and the algebra homomorphisms in Proposition 4.5 (see, e.g., [Kuniba 2018]). In the following subsection, we shall recall the q-oscillator representations.

5B. ***Realization of multiplicity-free weight modules.*** Let $F=\bigoplus_{m\in\mathbb{Z}_{\ge 0}}\mathbb{C}|m\rangle$ be the Fock space representation of $\mathscr{B}_\nu$ on which the generators $\boldsymbol{a}^+$ and $\boldsymbol{a}$ act as the creation and annihilation operators, respectively, and the element $\boldsymbol{a}^+\boldsymbol{a}=\{\boldsymbol{k}\}_\nu$ corresponds to the number operator; more precisely, for any $m\in\mathbb{Z}_{\ge 0}$,

$$\boldsymbol{a}^+.|m\rangle=|m+1\rangle,\quad \boldsymbol{a}.|m\rangle=[m]_\nu|m-1\rangle,\quad \boldsymbol{k}^{\pm 1}.|m\rangle=\nu^{\pm m}|m\rangle.$$

In particular, $\boldsymbol{a}.|0\rangle=0$.

Denote this representation by $\rho : \mathscr{B}_\nu \to \mathrm{End}_{\mathbb{C}}(F)$. For any $b \in \mathbb{C}^\times$ and $\varepsilon \in \{0, 1\}$, we denote by $\rho_{\varepsilon,b}$ the composition $\rho \circ \vartheta^\varepsilon \circ \theta_{b,0}$ (see Lemma 4.4). Then F has a new $\mathscr{B}_\nu$-module structure via $\rho_{\varepsilon,b}$.

Definition 5.5. Let z be a parameter valued in $\mathbb{C}^\times$. We define the representation $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ of U on the space $F^{\otimes n}$ via the composition of the algebra homomorphisms $\pi_z := \pi_{X_N^{(r)},z}$ defined in Proposition 4.5 and

$$\mathscr{B}_\nu^{\otimes n}[z, z^{-1}] \xrightarrow{\rho_{\varepsilon_1,b_1}\otimes\cdots\otimes\rho_{\varepsilon_n,b_n}} \mathrm{End}_{\mathbb{C}}(F^{\otimes n}),$$

where $\boldsymbol{\varepsilon} = (\varepsilon_i)_i \in \{0, 1\}^n$, and $\boldsymbol{b} = (b_i)_i \in (\mathbb{C}^\times)^n$.

For any n-tuple $(m_i)_i \in (\mathbb{Z}_{\geq 0})^n$, we use $|\boldsymbol{m}\rangle := |m_1\rangle \otimes \cdots \otimes |m_n\rangle$ for the basis vector of $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$. Let $\boldsymbol{e}_j$ be the j-th standard vector in $\mathbb{Z}^n$ with 1 at the j-th term and 0 otherwise for $1 \leq j \leq n$. Moreover, set $\mathbf{0}$ for $(0, \ldots, 0) \in \mathbb{Z}^n$.

For $n \geq 2$, note that U-module actions of $X_i^{\pm 1}$, K_i for $1 \leq i \leq n-1$ on $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$, by Definition 5.5, can be written down uniformly as follows:

$$\begin{aligned} &X_i^+.|\boldsymbol{m}\rangle \\ &\quad = (-1)^{\varepsilon_i} b_i b_{i+1}^{-1} [m_i/m_i^{\varepsilon_i}]_\nu [m_{i+1}^{\varepsilon_{i+1}}]_\nu |\boldsymbol{m} - (-1)^{\varepsilon_i}\boldsymbol{e}_i + (-1)^{\varepsilon_{i+1}}\boldsymbol{e}_{i+1}\rangle, \end{aligned} \tag{5-4}$$

$$\begin{aligned} &X_i^-.|\boldsymbol{m}\rangle \\ &\quad = (-1)^{\varepsilon_{i+1}} b_i^{-1} b_{i+1} [m_i^{\varepsilon_i}]_\nu [m_{i+1}/m_{i+1}^{\varepsilon_{i+1}}]_\nu |\boldsymbol{m} + (-1)^{\varepsilon_i}\boldsymbol{e}_i - (-1)^{\varepsilon_{i+1}}\boldsymbol{e}_{i+1}\rangle, \end{aligned} \tag{5-5}$$

$$K_i.|\boldsymbol{m}\rangle = \nu^{-(-1)^{\varepsilon_i}(m_i+\varepsilon_i)+(-1)^{\varepsilon_{i+1}}(m_{i+1}+\varepsilon_{i+1})}|\boldsymbol{m}\rangle, \tag{5-6}$$

for $\boldsymbol{m} \in (\mathbb{Z}_{\geq 0})^n$, where ν is defined in Proposition 4.5 for each type, and $|\boldsymbol{m}\rangle$ for $\boldsymbol{m} \notin (\mathbb{Z}_{\geq 0})^n$ can be read as 0. Here we remark that the $U_q(A_{n-1}^{(1)})$-module actions of $X_0^\pm$, K_0 on $|\boldsymbol{m}\rangle$ also have the above forms, where we understand the indices $i, i+1$ as $n, 1 \pmod n$, respectively.

Regard $U_q(A_{n-1})$ as the subalgebra of $U_q(\mathfrak{g})$ $(n \geq 2)$ via forgetting the actions of the Drinfeld–Jimbo generators indexed by 0 and n. One can check that $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ as a $U_q(A_{n-1})$-module is a multiplicity-free weight module. In fact, $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ has the following direct sum decomposition:

$$\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{l=-\infty}^{\infty} \mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}, \quad \mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}}=l} \mathbb{C}|\boldsymbol{m}\rangle.$$

For each $\boldsymbol{m} \in \mathbb{Z}^n$, write $|\boldsymbol{m}|_{\boldsymbol{\varepsilon}} = \sum_i (-1)^{\varepsilon_i} m_i$. Each $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is an irreducible, multiplicity-free weight $U_q(A_{n-1})$-module by formulae (5-4)–(5-6) as q (or ν) is not a root of unity.

Fix $1 \leq i \leq n$. Define the algebra homomorphism $\delta_i : \mathcal{A}_0 \to \mathbb{C}$ by $\boldsymbol{k}_j \mapsto \nu^{\delta_{ij}}$ for $1 \leq j \leq n$. It induces an algebra character $\widetilde{\delta}_i \in \mathscr{X}$ by

$$\widetilde{\delta}_i : U^0 \xrightarrow{\pi|_{U^0}} \mathcal{A}_0 \xrightarrow{\delta_i} \mathbb{C}.$$

Then we have:

Proposition 5.6. *For $\boldsymbol{\varepsilon} \in \{0,1\}^n$ and $\boldsymbol{b} \in (\mathbb{C}^\times)^n$, we have*:

(i) *$\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is a weight module with $\dim(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda \le 1$ for any $\lambda \in \mathscr{X}$.*

(ii) *If $\dim(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda = 1$, then there exists $\boldsymbol{m} \in (\mathbb{Z}_{\ge 0})^n$ such that $(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda = \mathbb{C}|\boldsymbol{m}\rangle$ with*

$$\lambda = \sum_{i=1}^{n} (-1)^{\varepsilon_i}(m_i + \varepsilon_i)\widetilde{\delta}_i. \tag{5-7}$$

Proof. Clearly, $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is a weight module. By (5-6) and the actions of K_0 and K_n, which are defined for $X_N^{(r)} \ne A_{n-1}^{(1)}$ by

$$K_0.|\boldsymbol{m}\rangle = -\iota^{1-\kappa_1}\nu^{(-1)^{\varepsilon_1}(\kappa_1+1)(m_1+1/2)}|\boldsymbol{m}\rangle, \tag{5-8}$$

$$K_n.|\boldsymbol{m}\rangle = \iota^{1+\kappa_2}\nu^{-(-1)^{\varepsilon_n}(\kappa_2+1)(m_n+1/2)}|\boldsymbol{m}\rangle, \tag{5-9}$$

where κ_1 and κ_2 are defined in Section 5A, the relative weight of $|\boldsymbol{m}\rangle$ is given by the right-hand side of the equality (5-7). By the above statement, $\dim(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda \le 1$ for any $\lambda \in \mathscr{X}$. □

Consider the following decomposition of $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$:

$$\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \mathcal{F}^{z,+}_{\boldsymbol{\varepsilon},\boldsymbol{b}} \oplus \mathcal{F}^{z,-}_{\boldsymbol{\varepsilon},\boldsymbol{b}}, \quad \mathcal{F}^{z,+}_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}} \equiv 0 \ (\mathrm{mod}\ 2)} \mathbb{C}|\boldsymbol{m}\rangle, \quad \mathcal{F}^{z,-}_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}} \equiv 1 \ (\mathrm{mod}\ 2)} \mathbb{C}|\boldsymbol{m}\rangle.$$

For $0 \le s \le n$, let $\boldsymbol{\varepsilon}_{>s} \in \{0,1\}^n$ satisfy

$$\varepsilon_1 = \cdots = \varepsilon_s = 0, \quad \varepsilon_{s+1} = \cdots = \varepsilon_n = 1.$$

For example, $\boldsymbol{\varepsilon}_{>0} = (1,\dots,1)$ and $\boldsymbol{\varepsilon}_{>n} = (0,\dots,0)$.

Then we have:

Proposition 5.7. *For any $\boldsymbol{\varepsilon} \in \{0,1\}^n$ and $\boldsymbol{b} \in (\mathbb{C}^\times)^n$, we have*:

(i) *As a $U_q(A_{n-1}^{(1)})$-module, $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is irreducible for any admissible $l \in \mathbb{Z}$ (defined in (5-10)); it is a highest ℓ-weight module with a highest ℓ-weight vector $v_{l,s}$ if and only if $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>s}$ for some $0 \le s \le n$, where*

$$v_{l,s} = \begin{cases} |l\boldsymbol{e}_s\rangle & \text{if } l \ge 0 \text{ and } 0 < s \le n, \\ |-l\boldsymbol{e}_{s+1}\rangle & \text{if } l < 0 \text{ and } 0 \le s < n. \end{cases} \tag{5-10}$$

(ii) *As $U_q(C_n^{(1)})$-modules, $\mathcal{F}^{z,+}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ and $\mathcal{F}^{z,-}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ are irreducible; they are highest ℓ-weight modules with highest ℓ-weight vector $v^+ = |\boldsymbol{0}\rangle$ and $v^- = |\boldsymbol{e}_n\rangle$ respectively whenever $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$.*

(iii) *As a $U_q(A_{2n}^{(2)})$ or $U_q(D_{n+1}^{(2)})$-module, $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is irreducible; it is a highest ℓ-weight module with a highest ℓ-weight vector $v = |\boldsymbol{0}\rangle$ whenever $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$.*

Proof. Note K_δ acts trivially on $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ by (5-1). The defining relations (2-5) imply that the actions of $h_{i,k}$, $1 \le i \le N$, $k \in \mathbb{Z} \setminus \{0\}$ on $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ commute pairwise. Hence $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is an ℓ-weight U-module. It is clear that $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is closed under the action of $U_q(A^{(1)}_{n-1})$. The irreducibility of $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ can be checked by the actions (5-4)–(5-6). Note that for $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>0}$ (resp. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$), $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is finite-dimensional for $l < 0$ (resp. $l \ge 0$). As a $U_q(A_{n-1})$-module (ignore the actions of $X^{\pm}_0$, K_0), $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is a highest weight module if and only if $\varepsilon_1 \le \cdots \le \varepsilon_n$, and the corresponding highest weight vector can be chosen as (5-10), which is also a highest ℓ-weight vector by weight consideration.

For $X^{(r)}_N \ne A^{(1)}_{n-1}$, the actions of X^+_0 and X^+_n are given by

$$X^+_0.|\boldsymbol{m}\rangle = z\frac{b_1^{-\kappa_1-1}}{[\kappa_1+1]_\nu}\prod_{j=0}^{\kappa_1}[(m_1-j)^{\varepsilon_1}]_\nu|\boldsymbol{m}+(-1)^{\varepsilon_1}(\kappa_1+1)\boldsymbol{e}_1\rangle,$$

and $X^+_n.|\boldsymbol{m}\rangle = x|\boldsymbol{m}-(-1)^{\varepsilon_n}(\kappa_2+1)\boldsymbol{e}_n\rangle$, where $a \in \mathbb{C}^\times$ is defined by

$$x = (-1)^{\varepsilon_n(1-\kappa_2)}b_n^{\kappa_2+1}\frac{\iota^{1-\kappa_2}}{[\kappa_2+1]_\nu}\tau_{\nu,\kappa_2}\prod_{j=0}^{\kappa_2}[(m_n-j)/(m_n-j)^{\varepsilon_n}]_\nu,$$

and $\tau_{\nu,\kappa_2} = (\nu-\kappa_2+1)/(\nu+\kappa_2-1)$. Similarly, we can obtain the actions of X^-_0 and X^-_n. Assertions (ii) and (iii) can be deduced directly from the above actions. □

6. Highest ℓ-weights

In this section, we focus on the irreducible highest ℓ-weight representations constructed in the previous section, and compute their highest ℓ-weights explicitly.

6A. ***Multiplicity-free highest ℓ-weight modules.*** Fix $0 < s < n$. Let $\mathcal{W}_s := F^{\otimes n}$ be the $U_q(A^{(1)}_{n-1})$-module defined as follows (see also [Kwon and Lee 2023]):

$$X^+_0.|\boldsymbol{m}\rangle = |\boldsymbol{m}+\boldsymbol{e}_1+\boldsymbol{e}_n\rangle, \qquad X^-_0.|\boldsymbol{m}\rangle = -[m_1][m_n]|\boldsymbol{m}-\boldsymbol{e}_1-\boldsymbol{e}_n\rangle,$$
$$X^+_s.|\boldsymbol{m}\rangle = -[m_s][m_{s+1}]|\boldsymbol{m}-\boldsymbol{e}_s-\boldsymbol{e}_{s+1}\rangle, \qquad X^-_s.|\boldsymbol{m}\rangle = |\boldsymbol{m}+\boldsymbol{e}_s+\boldsymbol{e}_{s+1}\rangle,$$
$$X^+_i.|\boldsymbol{m}\rangle = [m_i]|\boldsymbol{m}-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}\rangle, \qquad X^-_i.|\boldsymbol{m}\rangle = [m_{i+1}]|\boldsymbol{m}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}\rangle,$$
$$X^+_j.|\boldsymbol{m}\rangle = [m_{j+1}]|\boldsymbol{m}+\boldsymbol{e}_j-\boldsymbol{e}_{j+1}\rangle, \qquad X^-_j.|\boldsymbol{m}\rangle = [m_j]|\boldsymbol{m}-\boldsymbol{e}_j+\boldsymbol{e}_{j+1}\rangle,$$

and

$$K_0.|\boldsymbol{m}\rangle = q^{m_1+m_n+1}|\boldsymbol{m}\rangle, \quad K_s.|\boldsymbol{m}\rangle = q^{-m_s-m_{s+1}-1}|\boldsymbol{m}\rangle,$$
$$K_i.|\boldsymbol{m}\rangle = q^{m_{i+1}-m_i}|\boldsymbol{m}\rangle, \quad K_j.|\boldsymbol{m}\rangle = q^{m_j-m_{j+1}}|\boldsymbol{m}\rangle,$$

where $1 \le i < s < j \le n-1$ and $\boldsymbol{m} \in (\mathbb{Z}_{\ge 0})^n$. From the actions (5-4)–(5-6), $\mathcal{W}_s$ is just the twisting of the module $\mathcal{F}^1_{\boldsymbol{\varepsilon},\boldsymbol{b}}$, where $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>s}$ and $\boldsymbol{b} = (1,\ldots,1)$, by the automorphism of $U_q(A_{n-1}^{(1)})$ sending $X_k^{\pm}$ to $-X_k^{\pm}$ for $s \le k \le n$ along with other Drinfeld–Jimbo generators fixed. Let $\mathcal{W}_s^{(l)}$ denote the l-th irreducible component of $\mathcal{W}_s$, i.e., $\mathcal{W}_s^{(l)} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}}=l} \mathbb{C}|\boldsymbol{m}\rangle$.

Let $(X_N^{(r)}, \nu)$ be one of the types in Proposition 4.5 except $(A_{n-1}^{(1)}, q)$. Let $\mathcal{W} = F^{\otimes n}$ be the $U_q(X_N^{(r)})$-module defined as (see [Kuniba and Okado 2015]):

$$
\begin{aligned}
X_0^+.|\boldsymbol{m}\rangle &= \frac{1}{[\kappa_1+1]_\nu}|\boldsymbol{m}+(\kappa_1+1)\boldsymbol{e}_1\rangle,\\
X_0^-.|\boldsymbol{m}\rangle &= -(-1)^{|\kappa|}\frac{\iota^{1-\kappa_1}}{[\kappa_1+1]_\nu}\tau_{\nu,\kappa_1}\prod_{j=0}^{\kappa_1}[m_1-j]_\nu|\boldsymbol{m}-(\kappa_1+1)\boldsymbol{e}_1\rangle,\\
K_0.|\boldsymbol{m}\rangle &= (-1)^{|\kappa|}\iota^{1-\kappa_1}\nu^{(\kappa_1+1)(m_1+1/2)}|\boldsymbol{m}\rangle,\\
X_i^+.|\boldsymbol{m}\rangle &= [m_i]_\nu|\boldsymbol{m}-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}\rangle && (1\le i\le n-1),\\
X_i^-.|\boldsymbol{m}\rangle &= [m_{i+1}]_\nu|\boldsymbol{m}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}\rangle && (1\le i\le n-1),\\
K_i.|\boldsymbol{m}\rangle &= \nu^{-m_i+m_{i+1}}|\boldsymbol{m}\rangle && (1\le i\le n-1),\\
X_n^+.|\boldsymbol{m}\rangle &= \frac{\iota^{1+\kappa_2}}{[\kappa_2+1]_\nu}\tau_{\nu,\kappa_2}\prod_{j=0}^{\kappa_2}[m_n-j]_\nu|\boldsymbol{m}-(\kappa_2+1)\boldsymbol{e}_n\rangle,\\
X_n^-.|\boldsymbol{m}\rangle &= \frac{1}{[\kappa_2+1]_\nu}|\boldsymbol{m}+(\kappa_2+1)\boldsymbol{e}_n\rangle,\\
K_n.|\boldsymbol{m}\rangle &= \iota^{1-\kappa_2}\nu^{-(\kappa_2+1)(m_n+1/2)}|\boldsymbol{m}\rangle,
\end{aligned}
$$

where $\boldsymbol{m} \in (\mathbb{Z}_{\ge 0})^n$, $|\kappa| = \kappa_1+\kappa_2$, and $\tau_{\nu,\kappa_i} = (\nu-\kappa_i+1)/(\nu+\kappa_i-1)$. Here the κ_i are defined in Section 5A. This module can be obtained from $\mathcal{F}^1_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ with $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$ and $\boldsymbol{b} = (1,\dots,1)$ by the automorphism of $U_q(X_N^{(r)})$ defined by

$$
\begin{aligned}
X_0^- &\mapsto (-1)^{|\kappa|+1}X_0^-, & K_0 &\mapsto (-1)^{|\kappa|+1}K_0,\\
X_n^+ &\mapsto (-1)^{\kappa_2}X_n^+, & K_n &\mapsto (-1)^{\kappa_2}K_n,
\end{aligned}
$$

with other generators fixed. For the type $C_n^{(1)}$, denote the irreducible components $\mathcal{F}^{1,\pm}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ of the $U_q(C_n^{(1)})$-module $\mathcal{W}$ by $\mathcal{W}^{\pm}$, for convenience.

Lemma 6.1. *Let $L(\boldsymbol{f})$ be an irreducible highest ℓ-weight U-module with $\boldsymbol{f} = (f_i(z))_{i\in I_0} \in \mathcal{R}$. If $\dim L(\boldsymbol{f})_{\mathrm{wt}(\boldsymbol{f})-\alpha_i} = 1$ for some $i \in I_0$ then $f_i(z)$ satisfies*

$$
f_i(z) = f_{i,0}^{+}\frac{1-(a-b)z}{1-az}, \tag{6-1}
$$

where $a, b \in \mathbb{C}$ satisfy $f^+_{i,2\tilde{d}_i} = a f^+_{i,\tilde{d}_i}$ and $f^+_{i,\tilde{d}_i} = b f^+_{i,0}$.

Proof. Suppose that $v \in L(\boldsymbol{f})$ is a nonzero ℓ-weight vector of $\boldsymbol{f}$. Note that $\{x^-_{i,k}.v, k \in \mathbb{Z}\}$ spans the weight space $L(\boldsymbol{f})_{\mathrm{wt}(\boldsymbol{f})-\alpha_i}$. If $\dim(L(\boldsymbol{f})_{\mathrm{wt}(\boldsymbol{f})-\alpha_i}) = 1$, then there exist $j \in \mathbb{Z}$ such that $x^-_{i,\tilde{d}_i j}.v$ is nonzero, and $a \in \mathbb{C}$ such that

$$x^-_{i,\tilde{d}_i (j+1)}.v = a x^-_{i,\tilde{d}_i j}.v. \tag{6-2}$$

Consider the actions of $x^+_{i,\tilde{d}_i k}$, $k \in \mathbb{Z}$, on (6-2). The defining relations (2-5) imply

$$f^+_{i,\tilde{d}_i (k+j+1)} - f^-_{i,\tilde{d}_i (k+j+1)} = a(f^+_{i,\tilde{d}_i (k+j)} - f^-_{i,\tilde{d}_i (k+j)})$$

for any $k \in \mathbb{Z}$. Because $f^-_{i,k} = f^+_{i,-k} = 0$ for $k > 0$, we have $f^+_{i,\tilde{d}_i (k+1)} = a f^+_{i,\tilde{d}_i k}$ for any $k > 0$. Taking the series $f_i(z) = \sum_{k=0}^{\infty} z^k f^+_{i,\tilde{d}_i k}$, we have

$$\begin{aligned} f_i(z)(1-az) &= \sum_{k=0}^{\infty} z^k f^+_{i,\tilde{d}_i k} - a \sum_{k=0}^{\infty} z^{k+1} f^+_{i,\tilde{d}_i k} \\ [-3pt] &= f^+_{i,0} + \sum_{k=1}^{\infty} z^k (f^+_{i,\tilde{d}_i k} - a f^+_{i,\tilde{d}_i (k-1)}) \\ &= f^+_{i,0} + (f^+_{i,\tilde{d}_i} - a f^+_{i,0}) z. \end{aligned}$$

Hence $f_i(z)$ has the rational form (6-1). □

6B. *Highest ℓ-weights.* Let us first study some properties of the Weyl group and the description of the root vectors of quantum affine algebras, which will enable us to compute the highest ℓ-weight explicitly.

Lemma 6.2. *Let $i, j \in I$, and $i \neq j$.*

1. *If $a_{ij} a_{ji} = 1$, then $s_j s_i \alpha_j = \alpha_i$.*
2. *If $a_{ij} a_{ji} = 2$, then $s_i s_j s_i \alpha_j = \alpha_j$.*

Proof. Both (1) and (2) are easy to deduce from $s_j s_i \alpha_j = (a_{ij} a_{ji} - 1)\alpha_j - a_{ij}\alpha_i$ and

$$s_i s_j s_i \alpha_j = (a_{ij} a_{ji} - 1)\alpha_j + (2 - a_{ij} a_{ji}) a_{ij} \alpha_i,$$

respectively. □

Recall the braid group operators associated to $\widetilde{W}$ introduced by Lusztig [1990]. For each simple reflection s_i, there is an algebra automorphism $T_i = T_{s_i}$ of U defined by

$$T_i X_i^+ = -X_i^- K_i, \quad T_i X_i^- = -K_i^{-1} X_i^+, \quad T_i K_\beta = K_{s_i \beta},$$

$$T_i X_j^+ = \sum_{k=0}^{-a_{ij}} (-1)^{k-a_{ij}} q_i^{-k} (X_i^+)^{(-a_{ij}-k)} X_j^+ (X_i^+)^{(k)} \quad (i \neq j),$$

$$T_i X_j^- = \sum_{k=0}^{-a_{ij}} (-1)^{k-a_{ij}} q_i^{k} (X_i^-)^{(k)} X_j^- (X_i^-)^{(-a_{ij}-k)} \quad (i \neq j),$$

where $\beta \in Q$ and $(X_i^\pm)^{(k)} = (X_i^\pm)^k / [k]_i^!$. Then $\Phi T_i = T_i^{-1} \Phi$, where Φ is the

$\mathbb{C}$-linear anti-automorphism of U sending $X_i^{\pm}$ to $X_i^{\pm}$, K_i to K_i^{-1} for $i \in I$. For any $\tau \in \mathscr{T}$, define T_τ by $T_\tau(X_i^{\pm}) = X_{\tau(i)}^{\pm}$ and $T_\tau(K_i) = K_{\tau(i)}$.

For later use, we list some well-known properties of braid group operators (see [Lusztig 1993; Beck 1994]). Choose one element $w \in \widetilde{W}$. If $\tau s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression of w, then the automorphism $T_w = T_\tau T_{i_1} T_{i_2} \cdots T_{i_m}$ of U is independent on the choice of the reduced expression of w. In particular, one reduced expression can be transformed to another by a finite sequence of braid relations. If $w(\alpha_i) = \alpha_j$ then $T_w(X_i^+) = X_j^+$. Moreover, if $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression and $l(ws_i) = l(w) + 1$, then we have $T_w X_i^+ \in U^+$.

Remark 6.3. For $i \le j$, put $\boldsymbol{s}_{(i,j)} = s_i s_{i+1} \cdots s_j$.

(i) In the type $A_{n-1}^{(1)}$, a reduced expression of $\widetilde{\omega}_i$ for $1 \le i \le n-1$ can be chosen as

$$\widetilde{\omega}_i = \tau^i \boldsymbol{s}_{(1,n-i)}^{-1} \boldsymbol{s}_{(2,n-i+1)}^{-1} \cdots \boldsymbol{s}_{(i,n-1)}^{-1},$$

where τ is the diagram automorphism of $A_{n-1}^{(1)}$ sending j to $j+1 \pmod n$ for $j \in I$ (see [Jang et al. 2023, Section 3.3]).

(ii) In the type $A_{2n}^{(2)}$, the reduced expression of $\widetilde{\omega}_n$ can be chosen (see [Damiani 2000, Corollary 4.2.4]) as

$$\widetilde{\omega}_n = (s_0 s_1 \cdots s_n)^n.$$

(iii) In the type $C_n^{(1)}$ (resp. $D_{n+1}^{(2)}$), the reduced expressions of $\widetilde{\omega}_{n-1}$ and $\widetilde{\omega}_n$ can be chosen as

$$\widetilde{\omega}_{n-1} = (\boldsymbol{s}_{(0,n)} s_{n-1})^{n-1} \quad \text{and} \quad \widetilde{\omega}_n = \tau s_n \boldsymbol{s}_{(n-1,n)} \boldsymbol{s}_{(n-2,n)} \cdots \boldsymbol{s}_{(1,n)},$$

respectively, where τ is the diagram automorphism of $C_n^{(1)}$ (resp. $D_{n+1}^{(2)}$) sending i to $n-i$ for $i \in I$.

Now, let us define the root vectors in U. We refer the reader to [Beck and Nakajima 2004] for the construction of root vectors X_β^+, $\beta \in \triangle$ (i.e., the E_β defined therein). In particular, the real root vectors $X_{k\widetilde{d}_i\delta \pm \alpha_i}^+$ are described explicitly by

$$X_{k\widetilde{d}_i\delta + \alpha_i}^+ = T_{\widetilde{\omega}_i}^{-k} X_i^+ \quad (k \ge 0), \qquad X_{k\widetilde{d}_i\delta - \alpha_i}^+ = T_{\widetilde{\omega}_i}^{k} T_i^{-1} X_i^+ \quad (k > 0).$$

Then $X_{k\widetilde{d}_i\delta \pm \alpha_i}^+ \in U^+$. The imaginary root vectors are defined by

$$\widetilde{\psi}_{i,k\widetilde{d}_i} = X_{k\widetilde{d}_i\delta - \alpha_i}^+ X_i^+ - q_i^{-2} X_i^+ X_{k\widetilde{d}_i\delta - \alpha_i}^+ \quad (k > 0), \tag{6-3}$$

and define the elements $X_{i,k\widetilde{d}_i\delta}^+$ by the following formal series in z:

$$\exp\left((q_i - q_i^{-1}) \sum_{k \ge 1} X_{i,k\widetilde{d}_i\delta}^+ z^k \right) = 1 + \sum_{k \ge 1} (q_i - q_i^{-1}) \widetilde{\psi}_{i,k\widetilde{d}_i} z^k. \tag{6-4}$$

Under the isomorphism of two presentations of $U_q(\mathfrak{g})$, the generators $\psi^+_{i,k\widetilde{d}_i}$ and the imaginary root vectors $\widetilde{\psi}_{i,k\widetilde{d}_i}$ are related (see [Beck 1994; Damiani 2012]); more precisely, for $k > 0$ and $i \in I_0$, we have

$$\psi^+_{i,k\widetilde{d}_i} = o(i)^k (q_i - q_i^{-1}) C^{-k\widetilde{d}_i/2} k_i \widetilde{\psi}_{i,k\widetilde{d}_i}, \tag{6-5}$$

where $o : I_0 \to \{\pm 1\}$ is a map such that $o(i) = -o(j)$ whenever

(i) $a_{ij} \le 0$ implies that $o(i)o(j) = -1$,

(ii) in the twisted cases different from $A^{(2)}_{2n}$, if $a_{ij} = -2$ then $o(i) = 1$.

Note that $o(n) = 1$ in the type $D^{(2)}_{n+1}$ as $a_{n,n-1} = -2$. Thus, we can deduce that the map $o : I_0 \to \{\pm 1\}$ is uniquely determined in the type $D^{(2)}_{n+1}$.

In Lemma 6.1, the scalars a and b can be described by the root vectors according to the above relations, which will become more computable in our case. Let $v \in L(\boldsymbol{f})$ be a nonzero ℓ-weight vector of $\boldsymbol{f}$. Since $C^{1/2}$ acts trivially on $L(\boldsymbol{f})$ and k_i commutes with $\widetilde{\psi}_{i,k\widetilde{d}_i}$, this implies that

$$X^+_{2\widetilde{d}_i\delta-\alpha_i}.v = o(i) a X^+_{\widetilde{d}_i\delta-\alpha_i}.v \quad \text{and} \quad \widetilde{\psi}_{i,\widetilde{d}_i}.v = o(i)\frac{b}{q_i - q_i^{-1}} v. \tag{6-6}$$

Lemma 6.4. *For any $i \in I_0$ and $k \in \mathbb{Z}_{>0}$, the root vectors $X^+_{k\widetilde{d}_i\delta-\alpha_i}$ in $U_q(X^{(r)}_N)$ have the following relations*:

$$X^+_{(k+1)\widetilde{d}_i\delta-\alpha_i} = \begin{cases} \frac{1}{[3]^!_n}[X^+_{k\delta-\alpha_n}, [X^+_{\delta-\alpha_n}, X^+_n]_q] & \text{if } (X^{(r)}_N, i) = (A^{(2)}_{2n}, n), \\ \frac{1}{[2]_i}[X^+_{k\widetilde{d}_i\delta-\alpha_i}, [X^+_{\widetilde{d}_i\delta-\alpha_i}, X^+_i]_{q_i^2}] & \text{otherwise}. \end{cases}$$

Proof. We may use the following relations [Damiani 2000, Proposition 2.2.4, Corollary 3.2.4] (see [Beck 1994]): for $k \in \mathbb{Z}_{>0}$,

$$\begin{cases} [X^+_{k\delta-\alpha_n}, X^+_{n,\delta}] = [3]^!_n X^+_{(k+1)\delta-\alpha_n} & \text{if } (X^{(r)}_N, i) = (A^{(2)}_{2n}, n), \\ [X^+_{k\widetilde{d}_i\delta-\alpha_i}, X^+_{i,\widetilde{d}_i\delta}] = [2]_i X^+_{(k+1)\widetilde{d}_i\delta-\alpha_i} & \text{otherwise}. \end{cases} \tag{6-7}$$

Note that $X^+_{i,\widetilde{d}_i\delta} \in U$ is defined by the formal series (6-4). Because, in (6-3), $\widetilde{\psi}_{i,k\widetilde{d}_i} = [X^+_{k\widetilde{d}_i\delta-\alpha_i}, X^+_i]_{q_i^2}$, by comparing the coefficients of z in (6-4), we can get

$$X^+_{i,\widetilde{d}_i\delta} = \widetilde{\psi}_{i,\widetilde{d}_i} = [X^+_{\widetilde{d}_i\delta-\alpha_i}, X^+_i]_{q_i^2}, \tag{6-8}$$

which implies the lemma by (6-7) and (6-8). □

Let $\alpha \in Q_+$. We introduce the *height* $\operatorname{ht}\alpha$ of α as $\operatorname{ht}\alpha = \sum_{i\in I} m_i$ if $\alpha = \sum_{i\in I} m_i\alpha_i$. Define a subset $Q_+(\alpha)$ of Q_+ as follows:

$$Q_+(\alpha) = \{\beta \in Q_+ \mid \operatorname{ht}\alpha - \operatorname{ht}\beta = 1, \alpha - \beta \ne \alpha_0\}.$$

Let $U^+(\alpha)$ be the subspace of U^+_α defined as $U^+(\alpha) = \sum_{\beta\in Q_+(\alpha)} U^+_\beta X^+_{\alpha-\beta}$.

Lemma 6.5. (1) *For $i \in I_0$, the root vector $X^+_{\delta-\alpha_i}$ in $U_q(A^{(1)}_{n-1})$ has the following form*:

$$X^+_{\delta-\alpha_i} \equiv (-q^{-1})^{n-2}(X^+_{i+1}\cdots X^+_{n-1})(X^+_{i-1}\cdots X^+_2 X^+_1)X^+_0 \pmod{U^+(\delta-\alpha_i)}.$$

(2) *In $U_q(C^{(1)}_n)$*,

$$X^+_{\delta-\alpha_{n-1}} \equiv q^{-n}(X^+_n X^+_{n-2}\cdots X^+_1)(X^+_{n-1}X^+_{n-2}\cdots X^+_1)X^+_0 \pmod{U^+(\delta-\alpha_{n-1})},$$

$$X^+_{\delta-\alpha_n} \equiv \left(\frac{q^{-1}}{[2]_1}\right)^{n-1}(X^+_{n-1})^2(X^+_{n-2})^2\cdots(X^+_1)^2X^+_0 \pmod{U^+(\delta-\alpha_n)}.$$

(3) *In $U_q(A^{(2)}_{2n})$*,

$$X^+_{\delta-\alpha_n} \equiv q^{-2n}(X^+_{n-1}X^+_{n-2}\cdots X^+_1)(X^+_n X^+_{n-1}\cdots X^+_1)X^+_0 \pmod{U^+(\delta-\alpha_n)}.$$

(4) *In $U_q(D^{(2)}_{n+1})$*,

$$X^+_{\delta-\alpha_n} \equiv q^{-2n+2}X^+_{n-1}X^+_{n-2}\cdots X^+_1X^+_0 \pmod{U^+(\delta-\alpha_n)}.$$

Proof. Thanks to the reduced expressions of $\widetilde{\omega}_i$ in Remark 6.3, the lemma can be deduced directly by the definition. One can refer to [Jang et al. 2023, Lemma 4.7] for assertion (1). To see the remaining assertions we define the operators $\mathcal{D}^{(1)}_i$ and $\mathcal{D}^{(2)}_i$ of U for $i \in I_0$ by $\mathcal{D}^{(1)}_i(X) = [X, X^+_i]_{q_i}$ and $\mathcal{D}^{(2)}_i(X) = [[X, X^+_i], X^+_i]_q$ for any $X \in U$, respectively. In the type $C^{(1)}_n$, we note that $T_\tau\mathcal{D}^{(s)}_i = \mathcal{D}^{(s)}_{n-i}T_\tau$ and $T_j\mathcal{D}^{(s)}_i = \mathcal{D}^{(s)}_iT_j$ for $|i-j|>1$ and $s=1,2$. Write $T_{\boldsymbol{s}_{(i,j)}} = T_{(i,j)}$ for simplicity. Then

$$T_{(0,n)}T_{n-1}\mathcal{D}^{(s)}_n = \mathcal{D}^{(s)}_nT_{(0,n)}T_{n-1},$$

since $\boldsymbol{s}_{(0,n)}s_{n-1}\alpha_n = \alpha_n$, and for any $0<i<n-1$, we have $(\boldsymbol{s}_{(0,n)}s_{n-1})^{i-1}\alpha_1 = \alpha_i$ and $\boldsymbol{s}_{(i+1,n-1)}\boldsymbol{s}_{(i,n-2)}\alpha_{n-2} = \alpha_i$ by using Lemma 6.2, so we get

$$\begin{aligned}(T_{(0,n)}T_{n-1})^iT_{(0,n-2)}X^+_{n-1} &= (T_{(0,n)}T_{n-1})^{i-1}T_{(0,n)}T_{(0,n-3)}X^+_{n-2}\\ &= (T_{(0,n)}T_{n-1})^{i-1}T_{(0,n-2)}T_{(0,n-3)}T_{n-1}X^+_{n-2}\\ &= -(T_{(0,n)}T_{n-1})^{i-1}[T_{(0,n-2)}X^+_{n-1}, X^+_1]_{q_{n-1}}\\ &= -\mathcal{D}^{(1)}_i(T_{(0,n)}T_{n-1})^{i-1}T_{(0,n-2)}X^+_{n-1},\end{aligned}$$

and

$$\begin{aligned}T_{(i+1,n)}T_{(i,n-1)}X^+_n &= T_{(i+1,n-1)}T_{(i,n-2)}T_nT_{n-1}X^+_n\\ &= T_{(i+1,n-1)}T_{(i,n-2)}T^{-1}_{n-1}X^+_n\\ &= \frac{1}{[2]_{n-1}}[[T_{(i+1,n-1)}X^+_n, X^+_i], X^+_i]_q\\ &= \frac{1}{[2]_1}\mathcal{D}^{(2)}_iT_{(i+1,n-1)}X^+_n.\end{aligned}$$

Finally, the definition of the root vectors and Remark 6.3 imply that

$$\begin{aligned}
X^+_{\delta-\alpha_{n-1}} &= (T_{(0,n)}T_{n-1})^{n-2}T_{(0,n)}X^+_{n-1} \\
&= -(T_{(0,n)}T_{n-1})^{n-2}T_{(0,n-2)}\mathcal{D}^{(1)}_n X^+_{n-1} \\
&= -\mathcal{D}^{(1)}_n (T_{(0,n)}T_{n-1})^{n-2}T_{(0,n-2)}X^+_{n-1} \\
&= \mathcal{D}^{(1)}_n \mathcal{D}^{(1)}_{n-2}(T_{(0,n)}T_{n-1})^{n-3}T_{(0,n-2)}X^+_{n-1} \\
&\;\;\vdots \\
&= (-1)^{n-1}\mathcal{D}^{(1)}_n \mathcal{D}^{(1)}_{n-2}\cdots\mathcal{D}^{(1)}_1 T_{(0,n-2)}X^+_{n-1} \\
&= \mathcal{D}^{(1)}_n \mathcal{D}^{(1)}_{n-2}\cdots\mathcal{D}^{(1)}_1 \mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_{1,q}X^+_0,
\end{aligned}$$

where $\mathcal{D}^{(1)}_{i,q}(X) = [X, X^+_i]_q$ for $X \in U$, and

$$\begin{aligned}
X^+_{\delta-\alpha_n} &= T_\tau T_n T_{(n-1,n)}\cdots T_{(2,n)}T_{(1,n-1)}X^+_n \\
&= \frac{1}{[2]_1}T_\tau T_n T_{(n-1,n)}\cdots T_{(3,n)}\mathcal{D}^{(2)}_1 T_{(2,n-1)}X^+_n \\
&= \frac{1}{[2]_1}\mathcal{D}^{(2)}_{n-1}T_\tau T_n T_{(n-1,n)}\cdots T_{(3,n)}T_{(2,n-1)}X^+_n \\
&\;\;\vdots \\
&= \Big(\frac{1}{[2]_1}\Big)^{n-1}\mathcal{D}^{(2)}_{n-1}\mathcal{D}^{(2)}_{n-2}\cdots\mathcal{D}^{(2)}_1 X^+_0,
\end{aligned}$$

which implies assertion (2). Similarly, we can prove that

$$X^+_{\delta-\alpha_n} = -\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_1\mathcal{D}^{(1)}_{n,q_1}\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_2\mathcal{D}^{(1)}_{1,q_0}(X^+_0) \quad \text{in } U_q(A^{(2)}_{2n}), \tag{6-9}$$

$$X^+_{\delta-\alpha_n} = (-1)^{n-1}\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_1(X^+_0) \quad \text{in } U_q(D^{(2)}_{n+1}), \tag{6-10}$$

which imply (3) and (4). □

Remark 6.6. In order to simplify computations in the following theorem for the type $A^{(2)}_{2n}$ ($n \geq 2$), we actually only need two terms of $X^+_{\delta-\alpha_n}$,

$$\begin{aligned}
&q^{-2n}(X^+_{n-1}X^+_{n-2}\cdots X^+_1)(X^+_n X^+_{n-1}\cdots X^+_1)X^+_0 \\
&\qquad\qquad - q^{-2n+1}(X^+_{n-1}\cdots X^+_1)^2 X^+_0 X^+_n,
\end{aligned} \tag{6-11}$$

which can be deduced directly from formula (6-9).

Now we compute the highest ℓ-weights of the q-oscillator representations defined in Section 6A.

Theorem 6.7. (1) *Fix $0<s<n$ and $l\in\mathbb{Z}$. The $U_q(A_{n-1}^{(1)})$-module $\mathcal{W}_s^{(l)}$ has highest ℓ-weight $\boldsymbol{f}=(f_i(z))_{i\in I_0}$ given by*

$$f_i(z)=\frac{c_{i,l}+u}{1+c_{i,l}u}\quad \textit{with}\;\; c_{i,l}=\begin{cases} q^{\delta_{i,s-1}l-\delta_{i,s}(l+1)} & \textit{if } l\geq 0,\\ q^{\delta_{i,s}(l-1)-\delta_{i,s+1}l} & \textit{if } l<0\end{cases}$$

for $1\leq i\leq n-1$, where $u=o(s)(-q^{-1})^n z$.

(2) *The highest ℓ-weight of the $U_q(C_n^{(1)})$-module $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) is given by*

$$\left(1,\dots,1,\frac{q^{-1/2}+u}{1+q^{-1/2}u}\right)\quad\left(\textit{resp.}\;\left(1,\dots,1,\frac{q^{1/2}+u}{1+q^{1/2}u},\frac{q^{-3/2}+u}{1+q^{-3/2}u}\right)\right),$$

where $u=o(n)q^{-n-1}z$.

(3) *The highest ℓ-weight of the $U_q(A_{2n}^{(2)})$-module (resp. $U_q(D_{n+1}^{(2)})$-module) $\mathcal{W}$ is given by*

$$\left(1,\dots,1,\frac{\imath q_n^{-1}+u}{1+\imath q_n^{-1}u}\right),$$

where $u=o(n)\imath\tau_q q^{-2n-1}z$ (resp. $u=q^{-2n}z$).

Proof. The proof of the first assertion can be found in [Kwon and Lee 2023, Theorem 4.10]. For (2), we have verified in Proposition 5.7 that $v^+=|\mathbf{0}\rangle$ and $v^-=|\boldsymbol{e}_n\rangle$ are highest ℓ-weight vectors of $U_q(C_n^{(1)})$-modules $\mathcal{W}^+$ and $\mathcal{W}^-$ respectively. Therefore, it follows from Lemma 6.1 and formulae (6-6) that we only need to compute the actions of $X^+_{2\widetilde{d}_i\delta-\alpha_i}$ and $\widetilde{\psi}_{i,\widetilde{d}_i}$ on $v^\pm$.

Note that $X_j^+.v^\pm=0$ for all $j\in I_0$. By using Lemma 6.5 we have

$$X^+_{\delta-\alpha_n}.v^+=\frac{q^{-n+1}}{[2]_1}|2\boldsymbol{e}_n\rangle,\quad X^+_{\delta-\alpha_n}.v^-=\frac{q^{-n+1}}{[2]_1}|3\boldsymbol{e}_n\rangle,$$

and then

$$\widetilde{\psi}_{n,1}.v^+=\frac{q^{-n-1}}{[2]_1}v^+,\quad \widetilde{\psi}_{n,1}.v^-=\frac{q^{-n-1}}{[2]_1}[3]_1v^-.$$

Therefore, we have

$$\begin{aligned} X^+_{2\delta-\alpha_n}.v^+ &=\frac{1}{[2]}[X^+_{\delta-\alpha_n},\widetilde{\psi}_{n,1}].v^+\\ &=\frac{1}{[2]}\left(\frac{q^{-n-1}}{[2]_1}X^+_{\delta-\alpha_n}.v^+-\frac{q^{-n+1}}{[2]_1}\widetilde{\psi}_{n,1}.|2\boldsymbol{e}_n\rangle\right)\\ &=\frac{q^{-n+1}}{[2][2]_1}\left(q^{-2}+1-q^{-2}\frac{[4]_1[3]_1}{[2]_1}\right)X^+_{\delta-\alpha_n}.v^+\\ &=-q^{-n-3/2}X^+_{\delta-\alpha_n}.v^+,\end{aligned}$$

and

$$X^+_{2\delta-\alpha_n}.v^- = \frac{1}{[2]}[X^+_{\delta-\alpha_n}, \widetilde{\psi}_{n,1}].v^- = \frac{q^{-n+1}}{[2][2]_1}(q^{-2}[3]_1 X^+_{\delta-\alpha_n}.v^- - \widetilde{\psi}_{n,1}.|3\boldsymbol{e}_n\rangle)$$
$$= \frac{q^{-n+1}}{[2][2]_1}\left(q^{-2}[3]_1 + [3]_1 - q^{-2}\frac{[5]_1[4]_1}{[2]_1}\right)X^+_{\delta-\alpha_n}.v^-$$
$$= -q^{-n-5/2}X^+_{\delta-\alpha_n}.v^-.$$

On the other hand,

$$X^+_{\delta-\alpha_{n-1}}.v^- = -q^{-n}|\boldsymbol{e}_{n-1}\rangle, \quad \widetilde{\psi}_{n-1,1}.v^- = q^{-n-1}v^-,$$

and then

$$X^+_{2\delta-\alpha_{n-1}}.v^- = \frac{1}{[2]_{n-1}}[X^+_{\delta-\alpha_{n-1}}, \widetilde{\psi}_{n-1,1}].v^-$$
$$= \frac{q^{-n}}{[2]_{n-1}}(q^{-1}X^+_{\delta-\alpha_{n-1}}.v^- + \widetilde{\psi}_{n-1,1}.|\boldsymbol{e}_{n-1}\rangle)$$
$$= \frac{q^{-n}}{[2]_{n-1}}(q^{-1}+1)X^+_{\delta-\alpha_{n-1}}.v^-$$
$$= q^{-n-1/2}X^+_{\delta-\alpha_{n-1}}.v^-.$$

In other cases, we can check that $X^+_{\delta-\alpha_i}.v = 0$, so $\widetilde{\psi}_{i,1}.v = 0$ and $X^+_{2\delta-\alpha_i}.v = 0$. Thus, we get (2) as desired.

To get (3), let $v := |0\rangle$. We first focus on the type $D^{(2)}_{n+1}$. By Lemma 6.5(4),

$$X^+_{\delta-\alpha_n}.v = q^{-2n+2}|2\boldsymbol{e}_n\rangle, \quad \widetilde{\psi}_{n,1}.v = -q^{-2}X^+_n X^+_{\delta-\alpha_n}.v = -\imath\tau_\nu q^{-2n}v,$$

and then

$$X^+_{2\delta-\alpha_n}.v = \frac{1}{[2]}[X^+_{\delta-\alpha_n}, \widetilde{\psi}_{n,1}].v = \frac{1}{[2]}(-\imath\tau_\nu q^{-2n}X^+_{\delta-\alpha_n}.v - q^{-2n+2}\widetilde{\psi}_{n,1}.|\boldsymbol{e}_n\rangle),$$
$$= \frac{q^{-2n+2}}{[2]}(-\imath\tau_\nu q^{-2} - \imath\tau_\nu + \imath\tau_\nu[2]_\nu q^{-2})X^+_{\delta-\alpha_n}.v$$
$$= -\imath q^{-2n-1}X^+_{\delta-\alpha_n}.v.$$

For $i \neq n$, we have $X^+_{\widetilde{d}_i\delta-\alpha_i}.v = 0$, and then $\widetilde{\psi}_{i,\widetilde{d}_i}.v = 0$ and $X^+_{2\widetilde{d}_i\delta-\alpha_i}.v = 0$. This proves assertion (3) for the type $D^{(2)}_{n+1}$. For the type $A^{(2)}_{2n}$, Lemma 6.5(3) yields

$$X^+_{\delta-\alpha_n}.v = q^{-2n}\imath\tau_q|\boldsymbol{e}_n\rangle, \quad \widetilde{\psi}_{n,1}.v = -q^{-1}X^+_n X^+_{\delta-\alpha_n}.v = \tau_q^2 q^{-2n-1}v.$$

Note that all terms in the expression of $X^+_{\delta-\alpha_n}$ vanish on the vector $|\boldsymbol{e}_n\rangle$ except for the two terms in (6-11). We can compute the following action by using (6-11):

$$X^+_{\delta-\alpha_n}.|\boldsymbol{e}_n\rangle = \imath\tau_q q^{-2n-1}|2\boldsymbol{e}_n\rangle.$$

Therefore,

$$\begin{aligned}X^+_{2\delta-\alpha_n}.v &= \frac{1}{[3]^!_n}[X^+_{\delta-\alpha_n}, \widetilde{\psi}_{n,1}].v \\ &= \frac{1}{[3]^!_n}(\tau_q^2 q^{-2n-1} X^+_{\delta-\alpha_n}.v - \imath\tau_q q^{-2n}\widetilde{\psi}_{n,1}.|e_n\rangle), \\ &= \frac{q^{-2n}\tau_q}{[3]^!_n}(q^{-1}+1-q^{-2}[2])X^+_{\delta-\alpha_n}.v \\ &= q^{-2n-3/2}\tau_q X^+_{\delta-\alpha_n}.v.\end{aligned}$$

Then $a = o(n)\tau_q q^{-2n-3/2}$ and $b = o(n)\tau_q q^{-2n}(q^{-1/2}+q^{-3/2})$. Assertion (3) for the type $A^{(2)}_{2n}$ follows from Lemma 6.1 and Corollary 5.4(3). □

Appendix: Proof of Theorem 4.2

Proof of Theorem 4.2. For generalized Cartan matrices of finite types, the corresponding system of equations (4-1) has been solved in [Chen et al. 2024]. The method followed in this appendix is parallel with the one there.

Given an affine Cartan matrix A. Fix one $i \in I$ and denote by $\mathcal{A}_{x_i}$ the subalgebra of $\mathcal{A}$ generated by all $x_j^{\pm 1}$ for $j \neq i$. Then there is a natural isomorphism $\mathcal{A} \cong \mathcal{A}_{x_i}[x_i^{\pm 1}]$. Suppose that $(\phi_i)_{i\in I}$ is any solution to the system (4-1).

We have the following two crucial lemmas.

Lemma A.1. *Any $\phi \in \mathcal{A}$ satisfying $\zeta_i(\phi) - \phi = \{y_i\}_i$ has the form*

$$\phi = \beta_i^+ y_i + \phi_0 + \beta_i^- y_i^{-1},$$

where $\phi_0 \in \mathcal{A}_{x_i}$, $\beta_i^+ = -q_i(q_i - q_i^{-1})^{-2}$ and $\beta_i^- = -q_i^{-1}(q_i - q_i^{-1})^{-2}$.

Proof. Let $\phi = \sum_k \phi_k x_i^k$ with $\phi_k \in \mathcal{A}_{x_i}$. Then $\zeta_i(\phi) - \phi = \{y_i\}_i$ implies that

$$\sum_k (q_i^{-k}-1)\phi_k x_i^k = \frac{1}{q_i - q_i^{-1}} x_i^2 \prod_{j\neq i} x_j^{a_{ji}} - \frac{1}{q_i - q_i^{-1}} x_i^{-2} \prod_{j\neq i} x_j^{-a_{ji}}.$$

Hence ϕ_k is zero unless $k = 0, \pm 2$ and

$$\phi_2 = \beta_i^+ \prod_{j\neq i} x_j^{a_{ji}}, \quad \phi_{-2} = \beta_i^- \prod_{j\neq i} x_j^{-a_{ji}}.$$

So the lemma is proved. □

Therefore, we may always assume that ϕ_i in the system (4-1) satisfies $\phi_i = \beta_i^+ y_i + \phi_{i,0} + \beta_i^- y_i^{-1}$, where $\phi_{i,0} \in \mathcal{A}_{x_i}$.

Note that any pair (ϕ_i, ϕ_j) is (i,j)-shiftable. This condition can further restrict the choices of $\phi_{i,0}$ and $\phi_{j,0}$ when the nodes i and j are not connected in the Dynkin diagram of A, namely, $a_{ij} = 0$. More precisely, we have:

Lemma A.2. *If $a_{ij}=0$, then both $\phi_{i,0}$ and $\phi_{j,0}$ lie in $\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}$.*

Proof. Since $a_{ij}=0$, we have $a_{ji}=0$, and $y_i\in\mathcal{A}_{x_j}$ and $y_j\in\mathcal{A}_{x_i}$. If $\phi_{i,0}=0$, there is nothing to do. Assume that $\phi_{i,0}$ is not zero. We rewrite $\phi_{i,0}$ uniquely as the Laurent polynomial in x_j, i.e., a unique form in $\mathcal{A}_{x_j}[x_j^{\pm1}]$. Take the nonzero term in this form of $\phi_{i,0}$ such that x_j has the highest (resp. lowest) power, denoted by $\phi_{i,\max}$ (resp. $\phi_{i,\min}$). Then the shiftability of (ϕ_i,ϕ_j) implies that

$$q_j^m\beta_j^+\phi_{i,\max}y_j=\beta_j^+\phi_{i,\max}y_j,\quad q_j^l\beta_j^-\phi_{i,\min}y_j^{-1}=\beta_j^-\phi_{i,\min}y_j^{-1},$$

where $m=\deg_{x_j}\phi_{i,\max}$ and $l=\deg_{x_j}\phi_{i,\max}$. Hence $m=0=l$. Then we can conclude that $\phi_{i,0}\in\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}$ as desired. Similarly, we have $\phi_{j,0}\in\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}$. □

Let us first focus on the rank-two cases. Fix $i\neq j$ in I and $J=\{i,j\}$. Due to Lemma A.2 we may assume that the nodes i and j are connected. Without loss of generality, we set $\lambda=a_{ij}$, $\mu=a_{ji}$ and $|\lambda|\geq|\mu|$. Then

$$A_J=\begin{pmatrix}2&\lambda\\ \mu&2\end{pmatrix}\quad\text{where } 1\leq\lambda\mu\leq4.$$

Then $y_i=x_i^2x_j^\mu$ and $y_j=x_i^\lambda x_j^2$ in this case. Assume that φ_i and φ_j have the forms as in Lemma A.1, i.e.,

$$\varphi_l=\beta_l^+y_l+\phi_{l,0}+\beta_l^-y_l^{-1},$$

where $\varphi_{l,0}\in\mathcal{A}_{x_l}$, $l\in J$, and let φ_i and φ_j satisfy the equality

$$\varphi_i\varphi_j=\zeta_j^{-1}(\varphi_i)\zeta_i^{-1}(\varphi_j).\tag{$*$}$$

Record the (x_i,x_j)-degrees of a monomial u in $\mathcal{A}$ by a *degree vector*

$$\begin{pmatrix}\deg_{x_i}u\\ \deg_{x_j}u\end{pmatrix}.$$

Then a Laurent polynomial f corresponds to a matrix with each column vector representing for the (x_i,x_j)-degrees of certain term of f. Moreover, if $\varphi_{i,0}$ is not zero (resp. $\varphi_{j,0}$ is not zero), then we use one vector with a parameter s (resp. t),

$$\begin{pmatrix}0\\ s\end{pmatrix}\quad\left(\text{resp. }\begin{pmatrix}t\\ 0\end{pmatrix}\right),$$

to stand for the possible (x_i,x_j)-degrees of $\varphi_{i,0}$ (resp. $\varphi_{j,0}$). For example, by Lemma A.2, if $a_{ij}=0$, then s and t always equal 0. Therefore, we obtain the following matrix with possible (x_i,x_j)-degrees of terms of $\varphi_i\varphi_j$:

$$\begin{pmatrix}q_i^\lambda q_j^\mu & q_i^tq_j^\mu & 1 & q_i^\lambda q_j^s & q_i^tq_j^s & q_i^{-\lambda}q_j^s & 1 & q_i^tq_j^{-\mu} & q_i^{-\lambda}q_j^{-\mu}\\ 2+\lambda & 2+t & 2-\lambda & \lambda & t & -\lambda & \lambda-2 & t-2 & -\lambda-2\\ \mu+2 & \mu & \mu-2 & s+2 & s & s-2 & 2-\mu & -\mu & -2-\mu\end{pmatrix},$$

where the first row contains the corresponding *shifted coefficients* in $\zeta_i^{-1}(\varphi_j)\zeta_j^{-1}(\varphi_i)$. The terms with shifted coefficient 1 can be canceled on the left- and right-hand sides of the equality $(*)$, so we may omit such terms. Therefore, we have the matrix

$$\begin{pmatrix} q_j^{2\mu} & q_i^t q_j^\mu & q_i^\lambda q_j^s & q_i^t q_j^s & q_i^{-\lambda} q_j^s & q_i^t q_j^{-\mu} & q_j^{-2\mu} \\ 2+\lambda & 2+t & \lambda & t & -\lambda & t-2 & -\lambda-2 \\ \mu+2 & \mu & s+2 & s & s-2 & -\mu & -2-\mu \end{pmatrix}. \tag{M1}$$

One useful statement is that if a shifted coefficient is not 1, then the corresponding degree vector has to be equal to another one in the matrix (M1) by the equality $(*)$. Therefore, we can determine all possible (x_i, x_j)-degrees of $\varphi_{i,0}$ and $\varphi_{j,0}$ as follows:

types	(λ, μ)	possible values of s and t
A_2	$(-1,-1)$	$s, t \in \{1, -1\}$
$B_2\ (= C_2)$	$(-2,-1)$	$\varphi_{i,0} = 0,\ t = \pm 2$ or $s = \pm 1,\ t = 0$
G_2	$(-3,-1)$	none
$A_1^{(1)}$	$(-2,-2)$	$s = 0 = t$
$A_2^{(2)}$	$(-4,-1)$	$\varphi_{i,0} = 0,\ t = 0$

Note that there is no (x_i, x_j)-degree vector for the type G_2 satisfying the above statement, so neither for the types $G_2^{(1)}$ and $D_4^{(3)}$. We have obtained all solutions for the type $A_1^{(1)}$ in Example 4.1. For type $A_2^{(2)}$, we substitute the reduced forms of $\varphi_{i,0}$ and $\varphi_{j,0}$, i.e., $\varphi_{i,0} = 0$, $\varphi_{j,0} \in \mathbb{C}^\times$, into (4-1), and then get

$$\varphi_i = \frac{\imath}{q_i - q_i^{-1}}\{\imath q^{\frac{1}{2}} x_i^2 x_j^{-1}\}_i, \quad \varphi_j = \{\imath q^{-\frac{1}{2}} x_i^2 x_j^{-1}\}_j \{\imath q^{\frac{3}{2}} x_i^{-2} x_j\}_j.$$

By our assumption in Section 2, we have $i = 1$, $j = 0$ and $\phi_0 = \varphi_j$, $\phi_1 = \varphi_i$ for the type $A_2^{(2)}$.

Let us turn to the higher-rank cases. The next result tells us how to "glue" the rank-two cases together.

Lemma A.3. *Let $j \in I$ be a node which connects to the other two distinct nodes i and l in the Dynkin diagram. Assume that $\phi_{j,0} \neq 0$ and the pair of integers (m, t) is the (x_i, x_l)-degree of any nonzero (monomial) term of $\phi_{j,0}$. Then we have $mt \leq 0$.*

Proof. Otherwise, assume that $mt > 0$ and the corresponding nonzero term of $\phi_{j,0}$ is $\phi_{j,0}^{(1)}$. Without loss of generality, we may let $m > 0$ and $t > 0$. Consider the term $\beta_i^- y_i^{-1} \phi_{j,0}^{(1)}$ of $\phi_i \phi_j$ which has the factor $x_i^{m-2} x_j^{-a_{ji}} x_l^{t-a_{li}}$. So we have that the shifted coefficient $q_i^m q_j^{-a_{ji}} = q_i^{m-a_{ij}}$ in $\zeta_j^{-1}\phi_i \zeta_i^{-1}\phi_j$ is not 1. However, there is no other term in $\phi_i\phi_j$ whose (x_i, x_j, x_l)-degree vector equals $(m-2, -a_{ji}, t-a_{li})$. This is a contradiction. Hence $mt \leq 0$. ☐

Lemma A.3 implies that there is no solution to the system (4-1) for A whose Dynkin diagram contains D_4 or F_4 as a subdiagram.

Up to this point, we have ruled out all affine Cartan matrices except those of types $A_n^{(1)}$ $(n \geq 1)$, $C_n^{(1)}$ $(n \geq 2)$, $A_{2n}^{(2)}$ $(n \geq 1)$ or $D_{n+1}^{(2)}$ $(n \geq 2)$. Now we can substitute the reduced forms of the $\phi_{i,0}$ into the system (4-1) to determine the coefficients of the possible terms. Then we obtain all solutions as listed below Theorem 4.2. Therefore, Theorem 4.2 is proved, as desired.

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References

[Beck 1994] J. Beck, "Braid group action and quantum affine algebras", *Comm. Math. Phys.* **165**:3 (1994), 555–568. MR Zbl

[Beck and Nakajima 2004] J. Beck and H. Nakajima, "Crystal bases and two-sided cells of quantum affine algebras", *Duke Math. J.* **123**:2 (2004), 335–402. MR Zbl

[Benkart et al. 1997] G. Benkart, D. Britten, and F. Lemire, "Modules with bounded weight multiplicities for simple Lie algebras", *Math. Z.* **225**:2 (1997), 333–353. MR Zbl

[Boos et al. 2016] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, "Oscillator versus prefundamental representations", *J. Math. Phys.* **57**:11 (2016), art. id. 111702. MR Zbl

[Boos et al. 2017] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, "Oscillator versus prefundamental representations, II: Arbitrary higher ranks", *Journal of Mathematical Physics* **58**:9 (2017), art. id. 093504. MR Zbl

[Britten et al. 1994] D. J. Britten, J. Hooper, and F. W. Lemire, "Simple C_n modules with multiplicities 1 and applications", *Canad. J. Phys.* **72**:7-8 (1994), 326–335. MR Zbl

[Chari and Pressley 1991] V. Chari and A. Pressley, "Quantum affine algebras", *Comm. Math. Phys.* **142**:2 (1991), 261–283. MR Zbl

[Chari and Pressley 1994] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge Univ. Press, 1994. MR Zbl

[Chari and Pressley 1998] V. Chari and A. Pressley, "Twisted quantum affine algebras", *Comm. Math. Phys.* **196**:2 (1998), 461–476. MR Zbl

[Chen et al. 2024] H. Chen, Y. Gao, X. Liu, and L. Wang, "U^0-free quantum group representations", *J. Algebra* **642** (2024), 330–366. MR Zbl

[Damiani 2000] I. Damiani, "The R-matrix for (twisted) affine quantum algebras", pp. 89–144 in *Representations and quantizations* (Shanghai, 1998), edited by J. Wang and Z. Lin, China High. Educ. Press, Beijing, 2000. MR Zbl

[Damiani 2012] I. Damiani, "Drinfeld realization of affine quantum algebras: the relations", *Publ. Res. Inst. Math. Sci.* **48**:3 (2012), 661–733. MR Zbl

[Damiani 2015] I. Damiani, "From the Drinfeld realization to the Drinfeld–Jimbo presentation of affine quantum algebras: injectivity", *Publ. Res. Inst. Math. Sci.* **51**:1 (2015), 131–171. MR Zbl

[Drinfeld 1985] V. G. Drinfeld, "Hopf algebras and the quantum Yang–Baxter equation", *Dokl. Akad. Nauk SSSR* **283**:5 (1985), 1060–1064. In Russian. MR Zbl

[Drinfeld 1987] V. G. Drinfeld, "A new realization of Yangians and of quantum affine algebras", *Dokl. Akad. Nauk SSSR* **296**:1 (1987), 13–17. In Russian. MR Zbl

[Futorny et al. 2015] V. Futorny, J. Hartwig, and E. Wilson, "Irreducible completely pointed modules of quantum groups of type A", *J. Algebra* **432** (2015), 252–279. MR Zbl

[Grantcharov and Serganova 2006] D. Grantcharov and V. Serganova, "Category of $\mathfrak{sp}(2n)$-modules with bounded weight multiplicities", *Mosc. Math. J.* **6**:1 (2006), 119–134, 222. MR Zbl

[Grantcharov and Serganova 2010] D. Grantcharov and V. Serganova, "Cuspidal representations of $\mathfrak{sl}(n+1)$", *Adv. Math.* **224**:4 (2010), 1517–1547. MR Zbl

[Hayashi 1990] T. Hayashi, "q-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras", *Comm. Math. Phys.* **127**:1 (1990), 129–144. MR Zbl

[Hernandez 2005] D. Hernandez, "Representations of quantum affinizations and fusion product", *Transform. Groups* **10**:2 (2005), 163–200. MR Zbl

[Jang et al. 2023] I.-S. Jang, J.-H. Kwon, and E. Park, "Unipotent quantum coordinate ring and prefundamental representations for types $A_n^{(1)}$ and $D_n^{(1)}$", *Int. Math. Res. Not.* **2023**:2 (2023), 1119–1172. MR Zbl

[Jimbo 1985] M. Jimbo, "A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation", *Lett. Math. Phys.* **10**:1 (1985), 63–69. MR Zbl

[Jing 1998] N. Jing, "On Drinfeld realization of quantum affine algebras", pp. 195–206 in *The Monster and Lie algebras* (Columbus, OH, 1996), edited by J. Ferrar and K. Harada, Ohio State Univ. Math. Res. Inst. Publ. **7**, de Gruyter, Berlin, 1998. MR Zbl

[Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge Univ. Press, 1990. MR Zbl

[Kuniba 2018] A. Kuniba, "Tetrahedron equation and quantum R matrices for q-oscillator representations mixing particles and holes", *Symm. Integr. Geom. Methods Appl.* **14** (2018), art. id. 067. MR Zbl

[Kuniba and Okado 2013] A. Kuniba and M. Okado, "Tetrahedron equation and quantum R matrices for infinite-dimensional modules of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$", *J. Phys. A* **46**:48 (2013), art. id. 485203. MR Zbl

[Kuniba and Okado 2015] A. Kuniba and M. Okado, "Tetrahedron equation and quantum R matrices for q-oscillator representations of $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$", *Comm. Math. Phys.* **334**:3 (2015), 1219–1244. MR Zbl

[Kuniba et al. 2015] A. Kuniba, M. Okado, and S. Sergeev, "Tetrahedron equation and quantum R matrices for modular double of $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$", *Lett. Math. Phys.* **105**:3 (2015), 447–461. MR Zbl

[Kwon and Lee 2023] J.-H. Kwon and S.-M. Lee, "Affinization of q-oscillator representations of $U_q(\mathfrak{gl}_n)$", *Lett. Math. Phys.* **113**:3 (2023), art. id. 58. MR Zbl

[Lusztig 1990] G. Lusztig, "Quantum groups at roots of 1", *Geom. Dedicata* **35**:1-3 (1990), 89–113. MR Zbl

[Lusztig 1993] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics **110**, Birkhäuser, Boston, 1993. MR Zbl

[Mathieu 2000] O. Mathieu, "Classification of irreducible weight modules", *Ann. Inst. Fourier* (*Grenoble*) **50**:2 (2000), 537–592. MR Zbl

[Mukhin and Young 2014] E. Mukhin and C. A. S. Young, "Affinization of category $\mathcal{O}$ for quantum groups", *Trans. Amer. Math. Soc.* **366**:9 (2014), 4815–4847. MR Zbl

[Nilsson 2016] J. Nilsson, "$\mathcal{U}(\mathfrak{h})$-free modules and coherent families", *J. Pure Appl. Algebra* **220**:4 (2016), 1475–1488. MR Zbl

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