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DIFFERENTIAL GEOMETRIC APPROACH TO THE DEFORMATION OF A PAIR OF COMPLEX MANIFOLDS AND HIGGS BUNDLES

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Let X be a complex manifold and (E, θ) be a Higgs bundle over X. We study the deformation of the triple (X, E, θ). We introduce the differential graded Lie algebra (DGLA) which governs the deformation. We construct the Kuranishi family of it and prove it contains all the information of small deformations of (X, E, θ).

1. Introduction

Let X be a complex manifold and $(E, \bar{\partial}_E)$ be a holomorphic vector bundle on it. Let $\bar{\partial}_{\mathrm{End}(E)}$ be the natural holomorphic structure on $\mathrm{End}(E)$ induced by E. Let $A^{1,0}(\mathrm{End}(E))$ be the smooth sections of $\mathrm{End}(E) \otimes \Omega^{1,0}$. A Higgs field θ on $(E, \bar{\partial}_E)$ is an additional structure on E such that $\theta \in A^{1,0}(\mathrm{End}(E))$, $\bar{\partial}_{\mathrm{End}(E)}\theta = 0$ and the integrability condition $\theta \wedge \theta = 0$ is satisfied. The Higgs field was introduced in [Hitchin 1987] for the Riemann surfaces case and generalized to the higher dimensional case in [Simpson 1988]. We call a triple (X, E, θ) a *holomorphic-Higgs triple*.

We study the deformation of holomorphic-Higgs triples. Our goal is to derive the differential graded Lie algebra (DGLA) which governs the deformation of a given holomorphic-Higgs triple and construct the Kuranishi family of it. For that sake, we apply the Kodaira–Spencer theory [1958a; 1958b; 1960]. The advantage of studying the deformation in the style of Kodaira–Spencer theory is that we can construct the DGLA differential geometrically. Hence we can use the theory of Kuranishi [1965] to construct the Kuranishi space.

There is a lot of interesting work in studying the deformation of pairs of a complex manifold and a holomorphic bundle over it. Such pairs were studied algebraically in [Huybrechts and Thomas 2010; Li 2008; Martinengo 2009; Sernesi 2006], analytically in [Huang 1995; Siu and Trautmann 1981], and in the style of Kodaira–Spencer theory [Chan and Suen 2016].

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The deformation of holomorphic-Higgs triples was studied algebraically in [Martinengo 2012]. In her work, the DGLA was also obtained. The difference between her work and our work is that she obtained the DGLA purely algebraically while we obtained it differential geometrically.

We first prove the tuple $(L, d_L, [\cdot\,,\cdot]_L)$ is a DGLA. We show that this is the DGLA which governs the deformation afterwards. We prepare some notation. Let (X, E, θ) be a holomorphic-Higgs triple. Let K be a hermitian metric on E and ∂_K be a (1,0)-part of the Chern connection associated to $\bar{\partial}_E$ and K, and for $\phi \in A^{0,i}(TX)$, we define $\{\partial_K, \phi\lrcorner\} := \partial_K(\phi\lrcorner) + (-1)^i \phi\lrcorner\partial_K$. Let $\partial_K^{\mathrm{End}(E)} : A^0(\mathrm{End}(E)) \to A^{1,0}(\mathrm{End}(E))$ be the differential operator induced by ∂_K. Let $[\cdot\,,\cdot]$ be the standard Lie bracket on $A^*(\mathrm{End}(E)) = \bigoplus_i A^i(\mathrm{End}(E))$ and $[\cdot\,,\cdot]_{\mathrm{SN}}$ be the standard Schouten–Nijenhuys bracket on $A^{0,*}(TX) = \bigoplus_i A^{0,i}(TX)$.

Theorem 1.1 (Theorem 3.1). *Let $L^i = \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}(E)) \oplus A^{0,i}(TX)$ and $L := \bigoplus_i L^i$. Let $(A, \phi) \in L^i$ and $(B, \psi) \in L^j$. We set*

$$[(A,\phi),(B,\psi)]_L := \big((-1)^i\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}A - [A,B], [\phi,\psi]_{\mathrm{SN}}\big)$$

We define $B_K \in A^{0,1}(\mathrm{Hom}(TX, \mathrm{End}(E))$ and $\mathbb{C}$-linear map $C_K : A^{0,p}(TX) \to A^{1,p}(\mathrm{End}(E))$ such that they act on $v \in A^{0,p}(TX)$ as

$$B_K(v) := (-1)^p v\lrcorner F_{d_K}, \quad C_K(v) := \{\partial_K^{\mathrm{End}(E)}, v\lrcorner\}\theta.$$

We define $d_L : L \to L$ as,

$$d_L := \begin{pmatrix} \bar{\partial}_{\mathrm{End}(E)} & B_K \\ 0 & \bar{\partial}_{TX} \end{pmatrix} + \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix}.$$

Then $(L, d_L, [\cdot\,,\cdot]_L)$ is a DGLA.

This DGLA is the DGLA which governs the deformation of the holomorphic-Higgs triple. Actually, we have the following.

Theorem 1.2 (see Theorem 3.6 for precise statement). *Let $(A,\phi) \in L^1$. Then (A,ϕ) defines a holomorphic-Higgs triple if and only if (A,ϕ) satisfies the Maurer–Cartan equation*

$$d_L(A,\phi) - \tfrac{1}{2}[(A,\phi),(A,\phi)]_L = 0.$$

Since the governing DGLA is constructed differential geometrically, We can apply the technique of [Kodaira and Spencer 1958a; 1958b; 1960; Kuranishi 1965] to construct the universal family (= *Kuranishi family*) for a triple (X, E, θ).

Let Δ_L be the Laplacian induced by d_L. Since Δ_L is an elliptic operator, $\mathbb{H}^i := \ker(\Delta_L : L^i \to L^i)$ is finite dimensional. Let $\{\eta_1, \dots, \eta_n\}$ be a basis of $\mathbb{H}^1$. Let d_L^* be the formal adjoint of d_L w.r.t. the L^2 metric, $H : L^i \to \mathbb{H}^i$ be the projection and

G be the Green's operator associated to Δ_L. The next result is based on [Kuranishi 1965].

Proposition 1.1 (Propositions 4.1 and 4.2). *Let $t=(t_1,\dots,t_n)\in\mathbb{C}^n$ and $\epsilon_1(t):=\sum_i t_i\eta_i$. For all $|t|\ll 1$ we have a $\epsilon(t)$ such that $\epsilon(t)$ satisfies the equation*

$$\epsilon(t)=\epsilon_1(t)+\tfrac{1}{2}d_L^*G[\epsilon(t),\epsilon(t)]_L.$$

Moreover, $\epsilon(t)$ is holomorphic respect to the variable t and $\epsilon(t)$ satisfies the Maurer–Cartan equation

$$d_L\epsilon(t)-\tfrac{1}{2}[\epsilon(t),\epsilon(t)]_L=0$$

if and only if $H[\epsilon(t),\epsilon(t)]_L=0$.

Let $\Delta\subset\mathbb{C}^n$ be a ball small enough so that $\epsilon(t)$ is holomorphic on Δ. We define $\mathcal{S}\subset\Delta$ as

$$\mathcal{S}:=\{t\in\Delta\mid H[\epsilon(t),\epsilon(t)]_L=0\}.$$

$\mathcal{S}$ might not be smooth, however, it is a complex analytic space. Let $X_{\epsilon(t)}$, $E_{\epsilon(t)}$, $\theta_{\epsilon(t)}$ be the complex manifold, the holomorphic bundle, and the Higgs field which $\epsilon(t)$ defines, respectively. By combining the above results we have a family of holomorphic-Higgs triple $\{(X_{\epsilon(t)},E_{\epsilon(t)},\theta_{\epsilon(t)})\}_{t\in\mathcal{S}}$. We call this family the *Kuranishi family* of (X,E,θ) and $\mathcal{S}$ the *Kuranishi space*.

We recall some properties of the Kuranishi family and space for a compact complex manifold X. Kuranishi [1965] constructed the Kuranishi family and space for arbitrary compact complex manifold and proved the semiuniversality: any other deformation of X is obtained by the pullback of the Kuranishi family. Hence the Kuranishi family contains all the information of small deformations of X. We prove the abbreviated version of the semiuniversality of Kuranishi space. We show that $\{(X_{\epsilon(t)},E_{\epsilon(t)},\theta_{\epsilon(t)})\}_{t\in\mathcal{S}}$ has all the information of small deformations of (X,E,θ).

Let $|\cdot|_k$ be the k-th Sobolev norm on L^1. We assume $k\gg 1$.

Theorem 1.3 (Theorem 4.2). *Let (X,E,θ) be a holomorphic-Higgs triple. Let $\mathcal{S}$ be a Kuranishi family for (X,E,θ). Let $\eta\in L^1$ be a Maurer–Cartan element. If $|\eta|_k$ is small enough, then there is a $t\in\mathcal{S}$ such that $(X_\eta,E_\eta,\theta_\eta)$ is isomorphic to $(X_{\epsilon(t)},E_{\epsilon(t)},\theta_{\epsilon(t)})$ (see Section 4 for the meaning of isomorphic).*

Some applications of Theorem 1.3. Higgs bundles play a core role in the nonabelian Hodge correspondence. Let X be a compact Kähler manifold. The nonabelian Hodge correspondence states there is a one-to-one correspondence in the following objects on X: semisimple representations of the fundamental group of X, flat bundles with a harmonic metric (a.k.a. harmonic bundle), and polystable Higgs bundles with vanishing Chern classes. Here, a harmonic metric is a metric of a flat bundle such that it induces a harmonic map from X to a certain homogenous space.

This correspondence for Riemann surfaces was proved by Hitchin [1987], and the higher dimensional case by Simpson [1988; 1992].

In [Ono 2023], we study the structure of the Kuranishi space for a holomorphic-Higgs triple (X, E, θ) such that X is a compact Kähler manifold and (E, θ) is a polystable Higgs bundle with vanishing Chern classes. We show that the Kuranishi space of such a holomorphic-Higgs triple is isomorphic to the product of the Kuranishi space of X and the Kuranishi space of the Higgs bundle. This predicts that once we construct the moduli space which parametrizes a pair of a Kähler manifold and a polystable Higgs bundle with 0 Chern classes, it locally splits into the moduli space of the Kähler manifold and the moduli space of the Higgs bundle. This phenomenon is interesting since we cannot expect such decomposition globally.

Plan of the paper. In Section 2, we define and study the deformation of holomorphic-Higgs triples. We prove the Newlander–Nirenberg-type theorem in this context (Proposition 2.6). In Section 3, we construct the DGLA which governs the deformation of the holomorphic-Higgs triple. In Section 4, we apply the work of Kuranishi and construct the Kuranishi space for a given holomorphic-Higgs triple and prove its local completeness.

2. Deformation of holomorphic-Higgs triple

For a smooth manifold X, we define $A^p(X)$ to be a space of smooth p-forms on X, and for a smooth vector bundle $E \to X$, we define $A^p(E)$ to be a space of smooth p-forms which take values in E.

Definition 2.1. Let X be a compact complex manifold. Let $\bar{\partial}_{\mathrm{End}(E)}$ be the complex structure on $\mathrm{End}(E)$ induced by E. A Higgs bundle (E, θ) over X is a pair such that

- E is a holomorphic bundle over X,
- θ is a Higgs field such that $\theta \in A^{1,0}(\mathrm{End}(E))$, $\bar{\partial}_{\mathrm{End}(E)}\theta = 0$, and $\theta \wedge \theta = 0$.

We call a triple (X, E, θ) a holomorphic-Higgs triple.

We fix a metric K on E and assume X to be compact throughout this paper.

Definition 2.2. Let (X, E, θ) be a holomorphic-Higgs triple. A family of deformations of holomorphic-Higgs triples $(\mathcal{X}, \mathcal{E}, \Theta)$ over a small ball Δ centered at the origin of $\mathbb{C}^d$ consists of a complex manifold $\mathcal{X}$, a proper submersive holomorphic map

$$\pi : \mathcal{X} \to \Delta$$

and a Higgs bundle $(\mathcal{E}, \Theta)$ over $\mathcal{X}$ such that $\pi^{-1}(0) = X$, $\mathcal{E}|_{\pi^{-1}(0)} = E$, $\Theta|_{\pi^{-1}(0)} = \theta$.

By Ehresmann's theorem and as in [Kodaira 1986, Chapter 7, Lemma 7.1], if we choose Δ small enough, we have maps $F : X \times \Delta \to \mathcal{X}$ and $P : E \times \Delta \to \mathcal{E}$

such that the diagram below commutes, F is a diffeomorphism and P is a smooth bundle isomorphism:

$$\begin{array}{ccc} E\times\Delta & \xrightarrow{P} & \mathcal{E} \\ \downarrow & & \downarrow \\ X\times\Delta & \xrightarrow{F} & \mathcal{X} \\ \downarrow & \swarrow & \\ \Delta & & \end{array}$$

We can induce a complex structure on $X\times\{t\}$ and a Higgs bundle structure on $E\times\{t\}$ using $F|_{X\times\{t\}} : X\times\{t\}\to\mathcal{X}_t=\pi^{-1}(t)$ and $P|_{E\times\{t\}} : E\times\{t\}\to\mathcal{E}|_{\pi^{-1}(t)}$. We denote this family of holomorphic-Higgs triple $\{(X_t, E_t, \theta_t)\}_{t\in\Delta}$.

Since $\{X_t\}_{t\in\Delta}$ is a deformation of the complex manifold X, we have a family of Maurer–Cartan element $\{\phi_t\}_{t\in\Delta}$ such that each ϕ_t determines the complex structure of X_t.

Let $A^{1,0}(X_t) := \{\alpha\in A^1(X) \mid \alpha \text{ is a (1,0)-form of } X_t\}$, $\pi_X^{1,0} : A^1(X)\to A^{1,0}(X)$, and $\pi_X^{0,1} : A^1(X)\to A^{0,1}(X)$ be the natural projection.

Lemma 2.1. *$\alpha\in A^{1,0}(X_t)$ if and only if $\pi_X^{0,1}(\alpha)=\phi_t\lrcorner\pi_X^{1,0}(\alpha)$.*

Proof. It is enough to prove it locally. Let $x\in X$ and U_x be an open neighborhood of x. Let $(\xi_1,\dots,\xi_n)$, $(z_1,\dots,z_n)$ be local coordinates on U_x and $(\xi_1,\dots,\xi_n)$ be a complex coordinate for X_t and $(z_1,\dots,z_n)$ be a complex coordinate for X.

Let $\alpha=\sum_i f_i d\xi_i$. We have

$$\pi_X^{0,1}(\alpha)=\sum_{i,j} f_i\frac{\partial\xi_i}{\partial\bar z_j}d\bar z_j \quad\text{and}\quad \pi_X^{1,0}(\alpha)=\sum_{i,j} f_i\frac{\partial\xi_i}{\partial z_j}dz_j.$$

Recall that $\phi_t=\sum_{i,j}\phi^i_{t,j}\frac{\partial}{\partial z_i}\otimes d\bar z_j$, $(\phi^i_{t,j})=\big(\frac{\partial\xi_i}{\partial z_k}\big)^{-1}\big(\frac{\partial\xi_k}{\partial\bar z_j}\big)$. See [Kodaira 1986] for more details.

Hence

$$\begin{aligned}\phi_t\lrcorner\pi_X^{1,0}(\alpha) &= \Big(\sum_{j,k}\phi^j_{t,k}\frac{\partial}{\partial z_j}\otimes d\bar z_k\Big)\lrcorner\Big(\sum_{i,j} f_i\frac{\partial\xi_i}{\partial z_j}dz_j\Big)\\ &=\sum_{i,j,k} f_i\frac{\partial\xi_i}{\partial z_j}\phi^j_{t,k}d\bar z_k\\ &=\sum_{i,k} f_i\frac{\partial\xi_i}{\partial\bar z_k}d\bar z_k=\pi_X^{0,1}(\alpha).\end{aligned}$$

To prove the converse, we only have to prove that if $\omega\in A^{0,1}(X_t)$ and $\pi_X^{0,1}(\omega)=\phi_t\lrcorner\pi_X^{1,0}(\omega)$ stands then $\omega=0$. Let $\omega=\sum_i h_i d\bar\xi_i$ and assume $\pi_X^{0,1}(\omega)=\phi_t\lrcorner\pi_X^{1,0}(\omega)$.

We have $\pi_X^{0,1}(\omega)=\sum_{i,j} h_i \frac{\partial\bar\xi_i}{\partial\bar z_j} d\bar z_j$ and $\phi_t \lrcorner \pi_X^{1,0}(\omega)=\sum_{i,j,k} h_i \frac{\partial\bar\xi_i}{\partial z_j}\phi^j_{t,k} d\bar z_k$. Since $\pi_X^{0,1}(\omega)=\phi_t\lrcorner\pi_X^{1,0}(\omega)$, we have

$$0=\pi_X^{0,1}(\omega)-\phi_t\lrcorner\pi_X^{1,0}(\omega)=\sum_k\biggl\{\sum_i h_i\biggl(\frac{\partial\bar\xi_i}{\partial\bar z_k}-\sum_j\frac{\partial\bar\xi_i}{\partial z_j}\phi^j_{t,k}\biggr)\biggr\}d\bar z_k.$$

Hence

$$\biggl(\frac{\partial\bar\xi_i}{\partial\bar z_k}-\sum_j\frac{\partial\bar\xi_i}{\partial z_j}\phi^j_{t,k}\biggr)(h_1,\dots,h_n)^T=0.$$

Since ϕ_t defines a near complex structure with respect to the original one

$$\det\biggl(\frac{\partial\bar\xi_i}{\partial\bar z_k}-\sum_j\frac{\partial\bar\xi_i}{\partial z_j}\phi^j_{t,k}\biggr)\neq 0.$$

Hence $(h_1,\dots,h_n)=0$. This implies $\omega=0$. □

Lemma 2.2. *Let $\alpha\in A^{1,0}(X_t)$. α is a holomorphic 1-form on X_t if and only if $(\bar\partial+l_{\psi_t})\pi_X^{1,0}(\alpha)=0$. Here $l_{\psi_t}=\partial(\psi_t\lrcorner)-\psi_t\lrcorner\partial$.*

Proof. As in Lemma 2.1, we only have to prove it locally. We use the notation in the proof of Lemma 2.1.

Let $\alpha=\sum_i f_i d\xi_i$ and let $\alpha^{1,0}=\pi_X^{1,0}(\alpha)$. We first calculate $(\bar\partial+l_{\psi_t})(\alpha^{1,0})$. Since $\alpha^{1,0}=\sum_{i,j} f_i\frac{\partial\xi_i}{\partial z_j}dz_j$, we have

$$\bar\partial\alpha^{1,0}=\bar\partial\biggl(\sum_{i,j} f_i\frac{\partial\xi_i}{\partial z_j}dz_j\biggr)=\sum_{i,j,k}\biggl\{\frac{\partial f_i}{\partial\bar z_k}\frac{\partial\xi_i}{\partial z_j}+f_i\frac{\partial^2\xi_i}{\partial\bar z_k\partial z_j}\biggr\}d\bar z_k\wedge dz_j,$$

$$\begin{aligned}l_{\phi_t}(\alpha^{1,0})&=l_{\phi_t}\biggl(\sum_{i,j} f_i\frac{\partial\xi_i}{\partial z_j}dz_j\biggr)\\&=\partial\biggl(\phi_t\lrcorner\sum_{i,j} f_i\frac{\partial\xi_i}{\partial z_j}dz_j\biggr)-\phi_t\lrcorner\biggl\{\sum_{i,j,k}\biggl(\frac{\partial f_i}{\partial z_k}\frac{\partial\xi_i}{\partial z_j}+f_i\frac{\partial^2\xi_i}{\partial z_k\partial z_j}\biggr)dz_k\wedge dz_j\biggr\}\\&=\partial\biggl(\sum_{i,j,k} f_i\frac{\partial\xi_i}{\partial z_j}\phi^j_{t,k}d\bar z_k\biggr)-\phi_t\lrcorner\biggl\{\sum_{i,j,k}\biggl(\frac{\partial f_i}{\partial z_k}\frac{\partial\xi_i}{\partial z_j}\biggr)dz_k\wedge dz_j\biggr\},\end{aligned}$$

and

$$\begin{aligned}\partial\biggl(\sum_{i,j,k} f_i\frac{\partial\xi_i}{\partial z_j}\phi^j_{t,k}d\bar z_k\biggr)&=\partial\biggl(\sum_{i,k} f_i\frac{\partial\xi_i}{\partial\bar z_k}d\bar z_k\biggr)\\&=\sum_{i,j,k}\frac{\partial f_i}{\partial z_j}\frac{\partial\xi_i}{\partial\bar z_k}dz_j\wedge d\bar z_k+\sum_{i,j,k} f_i\frac{\partial^2\xi_i}{\partial\bar z_k\partial z_j}dz_j\wedge d\bar z_k\\&=\sum_{i,j,k}\frac{\partial f_i}{\partial z_k}\frac{\partial\xi_i}{\partial z_j}\sum_l\phi^k_{t,l}d\bar z_l\wedge dz_j-\sum_{i,j,k}\frac{\partial f_i}{\partial z_k}\frac{\partial\xi_i}{\partial z_j}\sum_l\phi^j_{t,l}d\bar z_l\wedge dz_k\\&=\sum_{i,j,k,l}\frac{\partial f_i}{\partial z_k}\frac{\partial\xi_i}{\partial z_j}\phi^k_{t,l}d\bar z_l\wedge dz_j-\sum_{i,k,l}\frac{\partial f_i}{\partial z_k}\frac{\partial\xi_i}{\partial\bar z_l}d\bar z_l\wedge dz_k.\end{aligned}$$

Hence

$$(1)\quad (\bar\partial + l_{\phi_t})(\alpha^{1,0}) = \sum_{i,j,k} \frac{\partial f_i}{\partial \bar z_k}\frac{\partial \xi_i}{\partial z_j} d\bar z_k \wedge dz_j + \sum_{i,j,k} f_i \frac{\partial^2 \xi_i}{\partial \bar z_k \partial z_j} d\bar z_k \wedge dz_j$$
$$+ \sum_{i,j,k} \frac{\partial f_i}{\partial z_j}\frac{\partial \xi_i}{\partial \bar z_k} dz_j \wedge d\bar z_k + \sum_{i,j,k} f_i \frac{\partial^2 \xi_i}{\partial \bar z_k \partial z_j} dz_j \wedge d\bar z_k$$
$$- \sum_{i,j,k,l} \frac{\partial f_i}{\partial z_k}\frac{\partial \xi_i}{\partial z_j}\phi^k_{t,l} d\bar z_l \wedge dz_j + \sum_{i,k,l} \frac{\partial f_i}{\partial z_k}\frac{\partial \xi_i}{\partial \bar z_l} d\bar z_l \wedge dz_k$$
$$= \sum_{i,j,l} \frac{\partial f_i}{\partial \bar z_l}\frac{\partial \xi_i}{\partial z_j} d\bar z_l \wedge dz_j - \sum_{i,j,k,l} \frac{\partial f_i}{\partial z_k}\frac{\partial \xi_i}{\partial z_j}\phi^k_{t,l} d\bar z_l \wedge dz_j$$
$$= \sum_{j,l}\sum_i \frac{\partial \xi_i}{\partial z_j}\left(\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t.l}\right) d\bar z_l \wedge dz_j.$$

If we assume α to be a holomorphic 1-form on X_t, this implies that $\{f_i\}_i$ are holomorphic functions on X_t. Hence we have

$$\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t,l} = 0.$$

Hence by (1), when α is a holomorphic 1-form on X_t, $(\bar\partial + l_{\phi_t})(\alpha^{1,0}) = 0$.

Conversely, if we assume $(\bar\partial + l_{\phi_t})(\alpha^{1,0}) = 0$, by (1) we have

$$\frac{\partial \xi_i}{\partial z_j}\left(\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t,l}\right) = 0.$$

Since ϕ_t defines a near complex structure to X, we have $\det\left(\frac{\partial \xi_i}{\partial z_j}\right) \neq 0$. Hence $\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t,l} = 0$. This shows that $\{f_i\}$ are holomorphic function on X_t and α is a holomorphic 1-form on X_t. □

By Lemma 2.1, θ_t can be decomposed as $\theta_t = \omega_t + \phi_t \lrcorner \omega_t$, where $\omega_t = \pi^{1,0}_X(\theta_t)$. We define an operator $\bar D_t : A^0(E) \to A^1(E)$ as

$$\bar D_t(s) = \bar D_t(s^k e_k) := (\partial + l_{\phi_t}) s^k \otimes e_k + \omega_t \wedge s, \quad s \in A^0(E).$$

Here, $\{e_k\}$ is a local holomorphic frame of E_t and we used the Einstein summation rule.

Proposition 2.1. *$\bar D_t$ is a well defined operator, that is, $\bar D_t$ is independent of the holomorphic frame of E_t. Also $\bar D_t$ satisfies the Leibniz rule:*

$$\bar D_t(\alpha \wedge s) = (\bar\partial + l_{\phi_t})\alpha \otimes s + (-1)^p \alpha \wedge \bar D_t(s)$$

for every $\alpha \in A^p(X)$ and $s \in A^0(E)$.

Proof. To prove well-definedness, we need to show that $\overline{D}_t$ is independent of the choice of a local holomorphic frame $\{e_k\}$ of E_t. Take another local holomorphic frame $\{f_j\}$ of E_t. Let h^k_j be a holomorphic function of X_t such that $f_j = h^k_j e_j$. Then for local section $s \in A(E)$, $s = \tilde{s}^j f_j = s^k e_k$, we have $\tilde{s}_j h^k_j = s_k$, thus we have

$$\begin{aligned}\overline{D}_t(\tilde{s}^j f_j) &= (\bar\partial + l_{\phi_t})\tilde{s}^j \otimes f_j + \omega_t \wedge (\tilde{s}^j f_j)\\ &= (\bar\partial + l_{\phi_t})\tilde{s}^j \otimes h^k_j e_k + \omega_t \wedge s\\ &= (\bar\partial + l_{\phi_t})(\tilde{s}^j h^k_j) \otimes e_k + \omega_t \wedge (s^k e_k)\\ &= (\bar\partial + l_{\phi_t})(s^k) \otimes e_k + \omega_t (s^k e_k)\\ &= \overline{D}_t(s^k e_k).\end{aligned}$$

Hence $\overline{D}_t$ is well defined.

The Leibniz rule for $\overline{D}_t$ follows from the fact that $\alpha \in A^p(X)$, $\beta \in A^q(X)$, $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ stands and $\bar\partial + l_{\phi_t}$ satisfies the Leibniz rule:

$$(\bar\partial + l_{\phi_t})(\alpha \wedge \beta) = (\bar\partial + l_{\phi_t})(\alpha) \wedge \beta + (-1)^p \alpha \wedge (\bar\partial + l_{\phi_t})(\beta). \qquad \square$$

Proposition 2.2. $\overline{D}_t^2 = 0$.

Proof. We calculate $\overline{D}_t^2$ locally and show $\overline{D}_t^2 = 0$. Since $\overline{D}_t$ satisfies the Leibniz rule, we only have to prove $(\bar\partial + l_{\phi_t})^2 = 0$ and $\overline{D}_t^2(s) = 0$ for $s \in A^0(E)$.

First we prove $(\bar\partial + l_{\phi_t})^2 = 0$. According to [Martinengo 2012], we have

$$(\bar\partial + l_{\phi_t})^2 = l_{\bar\partial_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t]}. \tag{2}$$

Since ϕ_t is a Maurer–Cartan element, we have $\bar\partial_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t] = 0$. Hence $(\bar\partial + l_{\phi_t})^2 = 0$.

Next we prove $\overline{D}_t^2(s) = 0$ for $s \in A^0(E)$. Let $\{e_k\}$ be a holomorphic frame for E_t. Assume that s and ω_t has a trivialization as $s = s^k e_k$ and $\omega_t = g_i dz_i$, $g_i = (a^s_{i,t})$ respect to the frame $\{e_k\}$. Here s^k, $a^s_{i,t} \in A^0(X)$ and $g_i \in A^0(\mathrm{End}(E))$. Since $\omega_t = \pi_X^{1,0}(\theta_t)$ and θ_t is a Higgs field we have $\omega_t \wedge \omega_t = 0$. Applying Lemma 2.2 and the fact that $\overline{D}_t$ satisfies the Leibniz rule, we have

$$\begin{aligned}\overline{D}_t^2(s) &= \overline{D}_t^2(s^k \otimes e_k)\\ &= \overline{D}_t((\bar\partial + l_{\phi_t})s^k \otimes e_k + \omega_t \wedge s)\\ &= (\bar\partial + l_{\phi_t})^2 s^k \otimes e_k + \omega_t \wedge (\bar\partial + l_{\phi_t})s^k \otimes e_k + (\bar\partial + l_{\phi_t})(a^s_{i,k} s^k dz_i) \otimes e_s \omega_t \wedge \omega_t \wedge s\\ &= \omega_t \wedge (\bar\partial + l_{\phi_t})s^k \otimes e_k + (\bar\partial + l_{\phi_t})(a^s_{i,k} dz_i) \wedge s^k \otimes e_s\\ &\qquad - \omega_t \wedge (\bar\partial + l_{\phi_t})s^k \otimes e_k + \omega_t \wedge \omega_t \wedge s\\ &= 0.\end{aligned}$$

Since $s \in A(E)$ is an arbitrary smooth section, this proves the claim. $\square$

Proposition 2.3. *We define $A_t := \bar{D}_t - \bar{\partial}_E - \{\partial_K, \phi_t\} - \theta$. Then $A_t \in A^1(\mathrm{End}(E))$. Here ∂_K is a (1,0)-part of the Chern connection which is uniquely determined by $\bar{\partial}_E$ and the hermitian metric K. $\{\partial_K, \phi_t \lrcorner\}$ is the operator such that $\{\partial_K, \phi_t \lrcorner\} = \partial_K(\phi_t \lrcorner) - \phi_t \lrcorner \partial_K$.*

Proof. Let $f \in A^0(X)$ and $s \in A^0(E)$. Using the Leibniz rule and the fact that the contraction is only taken in the (1,0)-part, we have

$$\begin{aligned} A_t(fs) &= (\bar{\partial} + l_{\phi_t}) f \otimes s + f \bar{D}_t(s) - \bar{\partial} f \otimes s - f \bar{\partial}_E s + \phi_t \lrcorner \partial_K(fs) - \theta \wedge (fs) \\ &= (\bar{\partial} - \phi_t \lrcorner \partial) f \otimes s + f \bar{D}_t(s) - \bar{\partial} f \otimes s - f \bar{\partial}_E s + \phi_t \lrcorner (\partial f \otimes s + f \partial_K s) - f\theta \wedge s \\ &= f(\bar{D}_t - \bar{\partial}_E - \{\partial_K, \phi_t\} - \theta)s \\ &= f A_t(s). \end{aligned}$$

This shows that $A_t \in A^1(\mathrm{End}(E))$. □

We summarize the results so far.

Proposition 2.4. *Let (X, E, θ) be a holomorphic-Higgs triple. Let $(\mathcal{X}, \mathcal{E}, \Theta)$ be a deformation family of (X, E, θ) over Δ and $\{(X_t, E_t, \theta_t)\}_{t \in \Delta}$ be the family obtained from $(\mathcal{X}, \mathcal{E}, \Theta)$. Combining ϕ_t and θ_t, we can construct a well-defined differential operator $\bar{D}_t$ such that $(\bar{D}_t)^2 = 0$. Let $A_t := \bar{D}_t - \bar{\partial}_E - \{\partial_K, \phi_t \lrcorner\} - \theta$. Then $A_t \in A^1(\mathrm{End}(E))$.*

We want the converse of the above proposition. Suppose we have a given smooth family $A_t \in A^{0,1}(\mathrm{End}E)$, $B_t \in A^{1,0}(\mathrm{End}E)$ and $\phi_t \in A^{0,1}(TX)$ parametrized by $t \in \Delta$.

We define the operator $\bar{D}_t : A^0(E) \to A^1(E)$ as

$$\bar{D}_t := \bar{\partial}_E + \{\partial_K, \phi_t \lrcorner\} + A_t + \theta + B_t.$$

We extend $\bar{D}_t$ to $A^p(E)$ in an obvious way so that it satisfies the Leibniz rule:

$$\bar{D}_t(\alpha \otimes s) = (\bar{\partial} + l_{\phi_t})\alpha \otimes s + (-1)^p \alpha \wedge \bar{D}_t(s).$$

We want to show that if $\bar{D}_t^2 = 0$, (A_t, B_t, ϕ_t) defines a holomorphic-Higgs triple (X_t, E_t, θ_t). First of all, we have:

Proposition 2.5. *If $\bar{D}_t^2 = 0$, ϕ_t defines a holomorphic structure on X. We denote this complex manifold by X_t.*

Proof. Let $f \in A^0(X)$ and $s \in A^0(E)$. Since $\bar{D}_t^2 = 0$, it satisfies the Leibniz rule:

$$0 = \bar{D}_t^2(f \otimes s) = (\bar{\partial} + l_{\phi_t})^2 f \otimes s.$$

Since f and s are arbitrary function and section, we have $(\bar{\partial} + l_{\phi_t})^2 = 0$. By (2), we have

$$0 = (\bar{\partial} + l_{\phi_t})^2 = l_{\bar{\partial}_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t]}.$$

Hence $\bar{\partial}_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t] = 0$. Hence ϕ_t defines a integrable complex structure on X. □

Next, we show that E admits a holomorphic structure over X_t and we can induce a Higgs field on it. Let us define $\bar{D}'_t : A^0(E) \to A^{0,1}(E)$ as $\bar{D}'_t := \bar{\partial}_E + \{\partial_K, \phi_t \lrcorner\} + A_t$. We remark that $\bar{D}_t = \bar{D}'_t + \theta + B_t$. The next claim was proved in [Moroianu 2007].

Lemma 2.3. $\ker(\bar{D}'_t)$ *generates* $A^0(E)$ *locally.*

Proof. See the proof of [Chan and Suen 2016, Lemma 3.11.]. □

The above lemma tells us that for every $x \in X$ we have an open neighborhood U of x and a frame $\{e_k\}$ on U such that $\{e_k\} \subset \ker(\bar{D}'_t)$. Let $\{e_k\}$ be a local frame of E such that $\{e_k\} \subset \ker(\bar{D}'_t)$. Let $\bar{\partial}_t$ be the Dolbeault operator of X_t. We can then define $\bar{\partial}_{E_t}$ by

$$\bar{\partial}_{E_t}(s^k e_k) := \bar{\partial}_t s^k \otimes e_k.$$

Let $\{f_j\} \subset \ker(\bar{D}'_t)$ be an another local frame of E, then there exist (h^k_j) such that $f_j = h^k_j e_k$. Applying $\bar{D}'_t$, we have

$$\bar{D}'_t(f_j) = \bar{D}'_t(h^k_j e_j) = (\bar{\partial} - \phi_t \lrcorner \partial) h^k_j \otimes e_k.$$

Since e_k is a local frame, we have $(\bar{\partial} - \phi_t \lrcorner \partial) h^k_j = 0$, which is equivalent to $\bar{\partial}_t h^k_j = 0$. We can now check $\bar{\partial}_{E_t}$ is well defined. Let $s \in A^0(E)$ and assume s has local trivialization as $s = \tilde{s}_j f_j = s_k e_k$. Applying $\bar{\partial}_{E_t}$ we have

$$\bar{\partial}_{E_t}(s^k e_k) = \bar{\partial}_t s^k \otimes e_k = \bar{\partial}_t(\tilde{s}_j h^k_j) \otimes e_k = \bar{\partial}_t \tilde{s}_j \otimes h^k_j e_k = \bar{\partial}_t \tilde{s}_j \otimes f_j = \bar{\partial}_{E_t}(\tilde{s}_j f_j).$$

This proves the well-definedness. By definition, $\bar{\partial}_E$ satisfies the Leibniz rule:

$$\bar{\partial}_{E_t}(\alpha \otimes s) = \bar{\partial}_t \alpha \otimes s + (-1)^p \alpha \wedge \bar{\partial}_{E_t} s$$

and $\bar{\partial}^2_{E_t} = 0$ since ϕ_t defines an integral almost complex structure on X. Hence, by the linearized version of the Newlander–Nirenberg theorem, $E_t = (E, \bar{\partial}_{E_t})$ is a holomorphic bundle over X_t.

We want to show next that $\theta_t = \theta + B_t + \phi_t \lrcorner(\theta + B_t)$ is a Higgs field for E_t under the above assertion. By Lemma 2.1, θ_t is a (1,0)-form of X_t which takes value in $\mathrm{End}(E)$.

Let $e_k \subset \ker(\bar{D}'_t)$ be a local frame of E and assume $\theta + B_t$ is written as $\theta + B_t = \sum_i g_i dz_i$ ($g_i \in A^0(\mathrm{End}(E))$) respect to this frame. By Lemma 2.2, to show θ_t is a Higgs field on E_t, it is enough to show $(\bar{\partial} + l_{\phi_t}) g_i dz_i = 0$ and $(\theta + B_t) \wedge (\theta + B_t) = 0$.

Since $\bar{D}_t$ satisfies the Leibniz rule

$$\begin{aligned} 0 = \bar{D}^2_t(e_k) &= \bar{D}_t(\bar{D}_t(e_k)) = \bar{D}_t((\theta + B_t)(e_k)) = \bar{D}_t(g_i dz_i(e_k)) \\ &= (\bar{\partial} + l_{\phi_t})(g_i dz_i) e_k - g_i dz_i \wedge \bar{D}_t(e_k) \\ &= (\bar{\partial} + l_{\phi_t})(g_i dz_i) e_k - (\theta + B_t) \wedge (\theta + B_t)(e_k). \end{aligned}$$

Hence θ_t is a Higgs field for E_t and (X_t, E_t, θ_t) is a holomorphic-Higgs triple. In summary, we have proved the following,

Proposition 2.6. *Suppose we have a given smooth family $A_t \in A^{0,1}(\mathrm{End}(E))$, $B_t \in A^{1,0}(\mathrm{End}(E))$, $\phi_t \in A^{0,1}(TX)$ parametrized by t. If the induced differential operator $\overline{D}_t : A^p(E) \to A^{p+1}(E)$ satisfies $\overline{D}_t^2 = 0$ and the Leibniz rule*

$$\overline{D}_t(\alpha \wedge s) = (\bar{\partial} + l_{\phi_t})\alpha \otimes s + (-1)^p \alpha \wedge \overline{D}_t(s),$$

then E admits a holomorphic structure over the complex manifold X_t, which we denote by E_t, and a Higgs field θ_t such that (X_t, E_t, θ_t) is a holomorphic-Higgs triple.

3. DGLA and the Maurer–Cartan equation

Let us recall the definition of DGLA.

Definition 3.1. A differential graded Lie algebra (DGLA) $(V, [\cdot,\cdot], d)$ is the date of $\mathbb{Z}$-graded vector space $L = \bigoplus_{i\in\mathbb{Z}} L^i$ with a bilinear bracket $[\cdot,\cdot] : V \times V \to V$ and a linear map d such that:

1. $[a, b] + (-1)^{ij}[b, a] = 0$ for $a \in V^i$, $b \in V^j$.
2. The graded Jacobi identity holds:
$$[a, [b, c]] = [[a, b], c] + (-1)^{ij}[b, [a, c]], \quad a \in V^i,\ b \in V^j,\ c \in V^k.$$
3. $d(V^i) \subset V^{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^i [a, db]$ for $a \in V^i$, $b \in V^j$. The map d is called the differential of V.

We recall the definition of the Maurer–Cartan equation of a DGLA.

Definition 3.2. The Maurer–Cartan equation of a DGLA V is

$$da - \tfrac{1}{2}[a, a] = 0, \quad a \in V^1.$$

The solutions of the Maurer–Cartan equation are called the Maurer–Cartan elements of the DGLA L.

We derive the Maurer–Cartan equation and DGLA which governs the deformation of the holomorphic-Higgs triple. The next proposition is important to construct the DGLA. Before we state it, we introduce some notation. Let $\partial_K^{\mathrm{End}(E)} : A^0(\mathrm{End}(E)) \to A^{1,0}(\mathrm{End}(E))$ be the differential operator induced by ∂_K. Let F_{d_K} be the curvature of the Chern connection. Let the bracket $[\cdot,\cdot]$ be the canonical Lie bracket defined on $A^*(\mathrm{End}(E))$ and $[\cdot,\cdot]_{\mathrm{SH}}$ be the standard Schouten–Nijenhuys bracket defined on $A^{0,*}(TX)$.

Proposition 3.1. *Suppose we have a $A \in A^{0,1}(\mathrm{End}(E))$, $B \in A^{1,0}(\mathrm{End}(E))$ and $\phi \in A^{0,1}(TX)$. Let $\overline{D}$ be the differential operator defined as*

$$\overline{D} := \bar{\partial}_E + \{\partial_K, \phi\lrcorner\} + \theta + A + B.$$

$\overline{D}^2 = 0$ *holds if and only if the following two equations hold*:

$$\bar\partial_{\mathrm{End}(E)}(A+B) - \phi\lrcorner F_{d_K} + [\theta, A+B] \\ + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\theta + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(A+B) + \tfrac{1}{2}[A+B, A+B] = 0,$$

$$\bar\partial_{TX}\phi - \tfrac{1}{2}[\phi, \phi] = 0.$$

From now on we denote $[\,\cdot\,,\cdot\,]_{\mathrm{SH}}$ as $[\,\cdot\,,\cdot\,]$ if there is no confusion. The proof of the above proposition will be given at the end of the section.

Let us define some notation. Let L^i be $L^i := \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}(E)) \oplus A^{0,i}(TX))$. Let for $\phi \in A^{0,i}(TX)$, $\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\} := \partial_K^{\mathrm{End}(E)}(\phi\lrcorner) + (-1)^i \phi\lrcorner\partial_K^{\mathrm{End}(E)}$. Define the bracket $[\,\cdot\,,\cdot\,]_L : L^i \times L^j \to L^{i+j}$ by

$$[(A,\phi),(B,\psi)]_L := \\ \big((-1)^i\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}A - (-1)^{(i+1)j}\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}B - [A,B], [\phi,\psi]\big).$$

We define $B_K \in A^{0,1}(\mathrm{Hom}(TX, \mathrm{End}(E))$ and the $\mathbb{C}$-linear map $C_K : A^{0,p}(TX) \to A^{1,p}(\mathrm{End}(E))$ such that they act on $v \in A^{0,p}(TX)$ as

$$B_K(v) := (-1)^p v\lrcorner F_{d_K}, \quad C_K(v) := \{\partial_K^{\mathrm{End}(E)}, v\lrcorner\}\theta.$$

We define the linear operator $d_L : L \to L$ as

$$d_L := \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} + \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix}$$

Theorem 3.1. $\big(L = \bigoplus_i L_i, [\,\cdot\,,\cdot\,]_L, d_L\big)$ *is a DGLA.*

We separate the proof of the theorem into the two propositions below. Before going to the proof, we introduce some formulas which are useful for the proof.

Lemma 3.1 [Martinengo 2012, Lemma 3.1]. *Let* $i_\xi(\omega) = \xi\lrcorner\omega$ *for all* $\omega \in A^*(X)$. *For every* $\xi, \eta \in A^{0,*}(TX)$,

$$i_{[\xi,\eta]} = [i_\xi, [\partial, i_\eta]], \quad [i_\xi, i_\eta] = 0. \tag{3}$$

We slightly modify Lemma 3.1 so that we can use it in our proof.

Lemma 3.2. *Let* X *be a complex manifold and* E *be a holomorphic bundle over* X. *Let* K *be a hermitian metric on* E *and* ∂_K *be a* (1,0)*-part of the Chern connection. By considering the degree of the differential form of* (3), *for any* $\omega \in A^*(E)$ *and any* $\phi \in A^{0,j}(TX)$ *and* $\psi \in A^{0,k}(TX)$, *we have*

$$[\phi,\psi]\lrcorner\omega = \phi\lrcorner\partial_K(\psi\lrcorner\omega) \\ -(-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner\omega)) - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner\omega) - (-1)^{jk+k}\psi\lrcorner\phi\lrcorner\partial_K\omega.$$

We obtain the corollaries below by applying Lemma 3.2.

Corollary 3.2. *Let* $A \in A^i(\mathrm{End}(E))$, $\phi \in A^{0,j}(TX)$ *and* $\psi \in A^{0,k}(TX)$. *Then we have*

$$\begin{aligned}&\{\partial_K^{\mathrm{End}(E)}, [\phi,\psi]\lrcorner\}A\\ &\qquad = \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}A - (-1)^{jk}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}A.\end{aligned}$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

Applying Lemma 3.2 to the left-hand side of the equation we have

$$\begin{aligned}&\{\partial_K, [\phi,\psi]\lrcorner\}A\\
&= \partial_K([\phi,\psi]\lrcorner A) + (-1)^{j+k}[\phi,\psi]\lrcorner\partial_K A\\
&= \partial_K\big\{\phi\lrcorner\partial_K(\psi\lrcorner A) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner A))\\
&\qquad - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner A) - (-1)^{jk+k}\psi\lrcorner\phi\lrcorner\partial_K A\big\}\\
&\qquad + (-1)^{j+k}\{\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner\partial_K A))\\
&\qquad - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\}\\
&= \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A)) - (-1)^{jk}\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A)) - (-1)^{jk+k}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A)\\
&\qquad + (-1)^{j+k}\big\{\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner\partial_K A))\\
&\qquad - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\big\}\\
&= \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A)) - (-1)^{jk}\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A)) - (-1)^{jk+k}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A)\\
&\qquad + (-1)^{j+k}\big\{\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) - (-1)^{jk+j}\partial_K(\psi\lrcorner(\phi\lrcorner\partial_K A))\\
&\qquad - (-1)^{jk+j+k}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\big\}.\end{aligned}$$

We apply (3) for the computation of the right-hand side of the equation.

$$\begin{aligned}&\{\partial_K,\phi\lrcorner\}\{\partial_K,\psi\lrcorner\}A-(-1)^{jk}\{\partial_K,\psi\lrcorner\}\{\partial_K,\phi\lrcorner\}A\\
&= \{\partial_K,\phi\lrcorner\}(\partial_K\psi\lrcorner A+(-1)^k\psi\lrcorner\partial_K)A-(-1)^{jk}\{\partial_K,\psi\lrcorner\}(\partial_K\phi\lrcorner A+(-1)^j\phi\lrcorner\partial_K A)\\
&= \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A))+(-1)^k\partial_K(\phi\lrcorner\psi\lrcorner\partial_K A)+(-1)^{j+k}\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A)\\
&\quad -(-1)^{jk}\{\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A))+(-1)^j\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A)+(-1)^{j+k}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\}\\
&= \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A))-(-1)^{jk+k}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A)+(-1)^{j+k}\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A)\\
&\quad -(-1)^{jk}\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A))-(-1)^{jk+j}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A)\\
&\quad -(-1)^{jk+j+k}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A).\end{aligned}$$

Hence we have equality holds. □

Corollary 3.3. *Let F_{d_K} be the curvature of the Chern connection. Let $\phi \in A^{0,i}(TX)$ and $\psi \in A^{0,j}(TX)$. Then we have*

$$[\phi, \psi]\lrcorner F_{d_K} = (-1)^i \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\psi\lrcorner F_{d_K} - (-1)^{ij+j}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}\phi\lrcorner F_{d_K}.$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

Recall that F_{d_K} is a (1, 1)-form which takes values in $\mathrm{End}(E)$.

Applying Lemma 3.2 to the left-hand side of the equation and by the Bianchi identity we have

$$\begin{aligned}[\phi, \psi]\lrcorner F_{d_K} &= \phi\lrcorner\partial_K(\psi\lrcorner F_{d_K}) \\ &\quad - (-1)^{ij+j}\partial_K(\psi\lrcorner(\phi\lrcorner F_{d_K})) - (-1)^{ij}\psi\lrcorner\partial_K(\phi\lrcorner F_{d_K}) \\ &\quad - (-1)^{ij+j}\psi\lrcorner\phi\lrcorner\partial_K F_{d_K} \\ &= \phi\lrcorner\partial_K(\psi\lrcorner F_{d_K}) - (-1)^{ij}\psi\lrcorner\partial_K(\phi\lrcorner F_{d_K}).\end{aligned}$$

By direct computation for the right-hand side of the equation, we have

$$\begin{aligned}&(-1)^i\{\partial_K, \phi\lrcorner\}\psi\lrcorner F_{d_K} - (-1)^{ij+j}\{\partial_K, \psi\lrcorner\}\phi\lrcorner F_{d_K} \\ &\quad = (-1)^i\partial_K(\phi\lrcorner\psi\lrcorner F_{d_K}) + \phi\lrcorner\partial_K\psi\lrcorner F_{d_K} \\ &\qquad - (-1)^{ij+j}\partial_K(\psi\lrcorner\phi\lrcorner F_{d_K}) - (-1)^{ij}\psi\lrcorner\partial_K\phi\lrcorner F_{d_K} \\ &\quad = \phi\lrcorner\partial_K\psi\lrcorner F_{d_K} - (-1)^{ij}\psi\lrcorner\partial_K\phi\lrcorner F_{d_K}.\end{aligned}$$

Hence we have the desired equality. □

By direct computation, we obtain some corollaries.

Corollary 3.4. *Let $A \in A^i(\mathrm{End}(E))$, $B \in A^j(\mathrm{End}(E))$ and $\phi \in A^{0,k}(TX)$. Then*

$$\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}[A, B] = [\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}A, B] + (-1)^{ik}[A, \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}B]. \tag{4}$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

By using local trivialization we have

$$\begin{aligned}&\{\partial_K, \phi\lrcorner\}[A, B] \\ &= \partial_K(\phi\lrcorner[A, B]) + (-1)^k\phi\lrcorner\partial_K[A, B] \\ &= \partial(\phi\lrcorner[A, B]) + [K^{-1}\partial K, \phi\lrcorner[A, B]] + (-1)^k\phi\lrcorner(\partial[A, B]) + [K^{-1}\partial K, [A, B]] \\ &= [\partial(\phi\lrcorner A), B] + (-1)^{i+k-1}[\phi\lrcorner A, \partial B] + (-1)^{i+ik}[\partial A, \phi\lrcorner B] + (-1)^{ik}[A, \partial(\phi\lrcorner B)] \\ &\quad + (-1)^k[\phi\lrcorner\partial A, B] + (-1)^{ik+i+1}[\partial A, \phi\lrcorner B] + (-1)^{k+i}[\phi\lrcorner A, \partial B] \\ &\quad + (-1)^{k+ki}[A, \phi\lrcorner\partial B] + (-1)^k[\phi\lrcorner K^{-1}\partial K, [A, B]] \\ &= [\partial(\phi\lrcorner A), B] + (-1)^{ik}[A, \partial(\phi\lrcorner B)] + (-1)^k[\phi\lrcorner\partial A, B] + (-1)^{ik+k}[A, \phi\lrcorner\partial B] \\ &\quad + (-1)^k[[\phi\lrcorner K^{-1}\partial K, A], B] + (-1)^{ki+k}[A, [\phi\lrcorner K^{-1}\partial K, B]] \\ &= [\{\partial_K, \phi\lrcorner\}A, B] + (-1)^{ik}[A, \{\partial_K, \phi\lrcorner\}B].\end{aligned}$$

Hence we have the desired equality. □

Corollary 3.5. *Let* $A \in A^i(\mathrm{End}(E))$ *and* $\phi \in A^{0,j}(TX)$. *Then*

$$\bar\partial_{\mathrm{End}(E)}\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}A = (-1)^j\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\bar\partial_{\mathrm{End}(E)}A - \{\partial_K^{\mathrm{End}(E)}, \bar\partial_{TX}\phi\lrcorner\}A - [\phi\lrcorner F_{d_K}, A].$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

We prove the above equality by using local trivialization:

$$\begin{aligned}
\bar\partial_{\mathrm{End}(E)}&\{\partial_K, \phi\lrcorner\}A \\
&= \bar\partial_{\mathrm{End}(E)}\{\partial_K(\phi\lrcorner) + (-1)^j\phi\lrcorner\partial_K A\} \\
&= \bar\partial_{\mathrm{End}(E)}\{\partial(\phi\lrcorner A) + (-1)^j[\phi\lrcorner K^{-1}\partial K, A] + (-1)^j\phi\lrcorner\partial A\} \\
&= -\partial(\bar\partial_{TX}\phi\lrcorner A) + (-1)^j\partial(\phi\lrcorner\bar\partial_{\mathrm{End}(E)}A) + (-1)^j[\bar\partial_{EndE}(\phi\lrcorner K^{-1}\partial K), A] \\
&\qquad + [\phi\lrcorner K^{-1}\partial K, \bar\partial_{\mathrm{End}(E)}A] + (-1)^j\bar\partial_{TX}\phi\lrcorner\partial A - \phi\lrcorner\bar\partial_{\mathrm{End}(E)}\partial A \\
&= (-1)^j\{\partial_K, \phi\lrcorner\}\bar\partial_{\mathrm{End}(E)}A - [\phi\lrcorner F_{d_K}, A] - \partial(\bar\partial_{TX})\phi\lrcorner A \\
&\qquad + (-1)^j[(\bar\partial_{TX}\phi)\lrcorner K^{-1}\partial K, A] + (-1)^j\bar\partial_{TX}\phi\lrcorner\partial A \\
&= (-1)^j\{\partial_K, \phi\lrcorner\}\bar\partial_{\mathrm{End}(E)}A - \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}A - [\phi\lrcorner F_{d_K}, A].
\end{aligned}$$

Hence we have the desired equality. □

Proposition 3.2. *The bracket* $[\,\cdot\,,\cdot\,]_L : L \times L \to L$ *satisfies the following*:

1. *For every* $(A, \phi) \in L^i$, $(B, \psi) \in L^j\,(i, j \in \mathbb{Z})$,

$$[(A, \phi), (B, \psi)]_L + (-1)^{pq}[(B, \psi), (A, \phi)]_L = 0.$$

2. *The graded Jacobi identity holds*: *for every* $(A, \phi) \in L^i$, $(B, \psi) \in L^j$, $(C, \tau) \in L^k$, *and* i, j, k,

$$[(A, \phi), [(B, \psi), (C, \tau)]_L]_L = [[(A, \phi), (B, \psi)]_L, (C, \tau)]_L + (-1)^{ij}[(B, \psi), [(A, \phi), (C, \tau)]_L]_L$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

1. is obvious from the definition. We prove 2.

We first calculate each component. First we have

$$\begin{aligned}
(5)\quad &[(A, \phi), [(B, \psi), (C, \tau)]_L]_L \\
&= \big[(A, \phi), ((-1)^j\{\partial_K, \psi\lrcorner\}C - (-1)^{(j+1)k}\{\partial_K, \tau\lrcorner\}B - [B, C], [\psi, \tau])\big]_L \\
&= \begin{pmatrix} \alpha \\ [\phi, [\psi, \tau]] \end{pmatrix}
\end{aligned}$$

where

$$\alpha = (-1)^i\{\partial_K, \phi\lrcorner\}\{(-1)^j\{\partial_K, \psi\lrcorner\}C - (-1)^{(j+1)k}\{\partial_K, \tau\lrcorner\}B - [B, C]\}$$
$$- (-1)^{(i+1)(j+k)}\{\partial_K, [\psi, \tau]\lrcorner\}A$$
$$- [A, (-1)^j\{\partial_K, \psi\lrcorner\}C - (-1)^{(j+1)k}\{\partial_K, \tau\lrcorner\}B - [B, C]]$$

Next we have

$$\big[[(A, \phi), (B, \psi)]_L, (C, \tau)\big]_L = \begin{pmatrix} \beta \\ [[\phi, \psi], \tau] \end{pmatrix} \tag{6}$$

where

$$\beta = (-1)^{i+j}\{\partial, [\phi, \psi]\lrcorner\}C$$
$$- (-1)^{(i+j+1)k}\{\partial_K, \tau\lrcorner\}\{(-1)^i\{\partial_K, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K, \psi\lrcorner\}A - [A, B]\}$$
$$- [(-1)^i\{\partial_K, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K, \psi\lrcorner\}A - [A, B], C].$$

We also have

$$(-1)^{ij}\big[(B, \psi), [(A, \phi), (C, \tau)]_L\big]_L = \begin{pmatrix} \gamma \\ [\psi, [\phi, \tau]] \end{pmatrix} \tag{7}$$

where

$$\gamma = (-1)^{ij}((-1)^i\{\partial_K, \psi\lrcorner\}\{(-1)^i\{\partial_K, \phi\lrcorner\}C - (-1)^{(i+1)k}\{\partial_K, \phi\lrcorner\}A - [A, C]\}$$
$$- (-1)^{(j+1)(i+k)}\{\partial_K, [\phi, \tau]\lrcorner\}B$$
$$- [B, (-1)^i\{\partial_K, \psi\lrcorner\}(-1)^i\{\partial_K, \phi\lrcorner\}C - (-1)^{(i+1)k}\{\partial_K, \phi\lrcorner\}A - [A, C]].$$

Hence by (5), (6), and (7) we only have to prove the equations

$$\{\partial_K, [\phi, \psi]\lrcorner\}A = \{\partial_K, \phi\lrcorner\}\{\partial_K, \psi\lrcorner\}A - (-1)^{jk}\{\partial_K, \psi\lrcorner\}\{\partial_K, \phi\lrcorner\}A,$$
$$\{\partial_K, \phi\lrcorner\}[A, B] = [\{\partial_K, \phi\lrcorner\}A, B] + (-1)^{ik}[A, \{\partial_K, \phi\lrcorner\}B],$$
$$[A, [B, C]] = [[A, B], C] + (-1)^{ij}[B, [A, C]],$$
$$[\phi, [\psi, \tau]] = [[\phi, \psi], \tau] + (-1)^{ij}[\psi, [\phi, \tau]].$$

The above equations follow from Corollaries 3.2 and 3.4 and the fact that the Schouten–Nijenhuis bracket satisfies the Jacobi identity. Hence we proved that $[\cdot, \cdot]_L$ satisfies the Jacobi identity. □

Proposition 3.3. *d_L is a differential with respect to the bracket $[\cdot, \cdot]_L$, that is,*

1. $d_L(L^i) \subset L^{i+1}$,
2. $d_L \circ d_L = 0$,
3. *for every $(A, \phi) \in L^i$, $(B, \psi) \in L^j$ and i, j,*

$$d_L[(A, \phi), (B, \psi)]_L = [d_L(A, \phi), (B, \psi)]_L + (-1)^i[(A, \phi), d_L(B, \psi)]_L$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

1. is obvious from the definition of d_L.

We prove 2. for $d_L \circ d_L : L^1 \to L^3$. Let $(A, \phi) \in L^1$. We calculate $d_L \circ d_L(A, \phi)$:

$$d_L(A,\phi) = \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{T(X)} \end{pmatrix} + \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}$$

$$\cdot\, d_L\left(\begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{T(X)}\phi \end{pmatrix} + \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{T(X)} \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{T(X)}\phi \end{pmatrix} \tag{8}$$

$$+ \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{TX}\phi \end{pmatrix} + \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix} \tag{9}$$

$$+ \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}. \tag{10}$$

Let us show (8) = (9) = (10) = 0:

$$(8) = \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{TX}\phi \end{pmatrix}$$

$$= \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} \circ \bar\partial_{\mathrm{End}(E)} A + \bar\partial_{\mathrm{End}(E)}(\phi \lrcorner F_{d_K}) + B_K(\bar\partial_{T(X)}\phi) \\ \bar\partial_{TX} \circ \bar\partial_{TX}\phi \end{pmatrix}$$

$$= \begin{pmatrix} \bar\partial_{TX}\phi \lrcorner F_{d_K} + \phi \lrcorner \bar\partial_{\mathrm{End}(E)} F_{d_k} - \bar\partial_{TX}\phi \lrcorner F_{d_K} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \phi \lrcorner \bar\partial_{\mathrm{End}(E)} F_{d_k} \\ 0 \end{pmatrix} = 0.$$

The last equation comes from the Bianchi identity. Next, we show (9) = 0:

$$(9) = \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{TX}\phi \end{pmatrix} + \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} [\theta, \bar\partial_{\mathrm{End}(E)} A] - [\theta, \phi \lrcorner F_{d_K}] + \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}\theta \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \bar\partial_{\mathrm{End}(E)}[\theta, A] + \bar\partial_{\mathrm{End}(E)}(\{\partial_K, \phi\lrcorner\}\theta) \\ 0 \end{pmatrix}.$$

Since θ is a Higgs field, $\bar\partial_{\mathrm{End}(E)}[\theta, A] = -[\theta, \bar\partial_{\mathrm{End}(E)} A]$. Hence we have

$$(9) = \begin{pmatrix} -[\theta, \phi \lrcorner F_{d_K}] + \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}\theta + \bar\partial_{\mathrm{End}(E)}(\{\partial_K, \phi\lrcorner\}\theta) \\ 0 \end{pmatrix}. \tag{11}$$

By direct computation using the local realization we have

$$\begin{aligned}\{\partial_K, \bar{\partial}_{TX}\phi\lrcorner\}\theta &= \partial_K(\bar{\partial}_{TX}\phi\lrcorner\theta) + \bar{\partial}_{TX}\phi\lrcorner(\partial_K\theta)\\ &= \partial(\bar{\partial}_{TX}\phi\lrcorner\theta) + [K^{-1}\partial K, \bar{\partial}_{TX}\phi\lrcorner\theta] + \bar{\partial}_{TX}\phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta]).\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}_{\mathrm{End}(E)}(\{\partial_K, \phi\lrcorner\}\theta) &= \bar{\partial}_{\mathrm{End}(E)}\{\partial(\phi\lrcorner\theta) + [K^{-1}\partial K, \phi\lrcorner\theta] - \phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta])\}\\ &= -\partial\bar{\partial}(\phi\lrcorner\theta) + [F_{d_K}, \phi\lrcorner\theta] - [K^{-1}\partial K, \bar{\partial}_{\mathrm{End}(E)}(\phi\lrcorner\theta)]\\ &\qquad - \bar{\partial}_{TX}\phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta]) - \phi\lrcorner[F_{d_K}, \theta]\\ &= -\partial(\bar{\partial}_{TX}\phi\lrcorner\theta) - [\phi\lrcorner F_{d_K}, \theta] - [K^{-1}\partial K, \bar{\partial}_{TX}\phi\lrcorner\theta]\\ &\qquad - \bar{\partial}_{TX}\phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta]).\end{aligned}$$

Hence by (11) and the above two displays, we obtain that (9) $= 0$.

Next, we show (10) $= 0$:

$$\text{(12)}\qquad (10) = \begin{pmatrix}\theta & C_K\\ 0 & 0\end{pmatrix}\begin{pmatrix}[\theta, A] + \{\partial_K, \phi\lrcorner\}\theta\\ 0\end{pmatrix} = \begin{pmatrix}[\theta, \{\partial_K, \phi\lrcorner\}\theta]\\ 0\end{pmatrix}.$$

By direct computation using the local realization we have

$$\begin{aligned}&[\theta, \{\partial_K, \phi\lrcorner\}\theta]\\ &= \theta\wedge\{\partial_K, \phi\lrcorner\}\theta - \{\partial_K, \phi\lrcorner\}\theta\wedge\theta\\ &= \theta\wedge\{\partial(\phi\lrcorner\theta) + [K^{-1}\partial K, \phi\lrcorner\theta] - \phi\lrcorner\partial\theta - \phi\lrcorner[K^{-1}\partial K, \theta]\}\\ &\qquad - \{\partial(\phi\lrcorner\theta) + [K^{-1}\partial K, \phi\lrcorner]\theta - \phi\lrcorner\partial\theta - \phi\lrcorner[K^{-1}\partial K, \theta]\}\wedge\theta\\ &= \theta\wedge\{\partial(\phi\lrcorner\theta) - \phi\lrcorner\partial\theta - [\phi\lrcorner K^{-1}\partial K, \theta]\} - \{\partial(\phi\lrcorner\theta) - \phi\lrcorner\partial\theta - [\phi\lrcorner K^{-1}\partial K, \theta]\}\wedge\theta\\ &= \theta\wedge\partial(\phi\lrcorner\theta) - \theta\wedge\phi\lrcorner\partial\theta - \partial(\phi\lrcorner\theta)\wedge\theta + (\phi\lrcorner\partial\theta)\wedge\theta.\end{aligned}$$

Since $\theta\wedge\theta = 0$, we have

$$\begin{aligned}0 &= \partial(\phi\lrcorner(\theta\wedge\theta)) - \phi(\partial(\theta\wedge\theta))\\ &= \partial(\phi\lrcorner\theta)\wedge\theta - (\phi\lrcorner\theta)\wedge\partial\theta + \partial\theta\wedge\phi\lrcorner\theta - \theta\wedge\partial(\phi\lrcorner\theta)\\ &\qquad - (\phi\lrcorner\partial\theta)\wedge\theta - \partial\theta\wedge(\phi\lrcorner\theta) + (\phi\lrcorner\theta)\wedge\partial\theta + \theta\wedge(\phi\lrcorner\partial\theta)\\ &= \partial(\phi\lrcorner\theta)\wedge\theta - \theta\wedge\partial(\phi\lrcorner\theta) - (\phi\lrcorner\partial\theta)\wedge\theta + \theta\wedge\phi\lrcorner\partial\theta.\end{aligned}$$

Hence by (12) and the above two displays, we obtain that (10) $= 0$. This completes the proof of 2.

Next we prove 3.

Let $(A, \phi) \in L^i$ and $(B, \psi) \in L^j$. We first calculate each component of 3.

First we have

$$d_L[(A, \phi), (B, \psi)]_L = \begin{pmatrix}\alpha\\ \bar{\partial}_{TX}[\phi, \psi]\end{pmatrix}$$

where

$$\alpha = (\bar\partial_{\mathrm{End}(E)} + \theta)\big((-1)^i\{\{\partial_K, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K, \psi\lrcorner\}A - [A, B]\}\big) \\ + (-1)^{i+j}[\phi, \psi]\lrcorner F_{d_K} + \{\partial_K, [\phi, \psi]\lrcorner\}\theta.$$

Next we have

$$[d_L(A,\phi), (B,\psi)]_L = \begin{pmatrix} \beta \\ [\bar\partial_{TX}\phi, \psi]) \end{pmatrix},$$

where

$$\beta = (-1)^{i+1}\{\partial_K, \bar\partial_{TX}\phi\lrcorner\}B \\ - (-1)^{(i+2)j}\{\partial_K, \psi\lrcorner\}(\bar\partial_{\mathrm{End}(E)}A + (-1)^i\phi\lrcorner F_{d_K} + [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta) \\ - [\bar\partial_{\mathrm{End}(E)}A + (-1)^i\phi\lrcorner F_{d_K} + [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta, B].$$

We also have

$$(-1)^i[(A,\phi), d_L(B,\psi)]_L = \begin{pmatrix} \gamma \\ (-1)^i[\phi, \bar\partial_{TX}\psi] \end{pmatrix},$$

where

$$\gamma = \{\partial_K, \phi\lrcorner\}(\bar\partial_{\mathrm{End}(E)}B + (-1)^j\psi\lrcorner F_{d_K} + [\theta, B] + \{\partial_K, \phi\lrcorner\}\theta) \\ - (-1)^{(i+1)(j+1)+i}\{\partial_K, \bar\partial_{TX}\psi\lrcorner\}A \\ - (-1)^i[A, \bar\partial_{\mathrm{End}(E)}B + (-1)^j\psi\lrcorner F_{d_K} + [\theta, B] + \{\partial_K, \psi\lrcorner\}\theta].$$

Hence by the above equations, we have to prove

$$\begin{aligned} \bar\partial_{\mathrm{End}(E)}\{\partial_K, \phi\lrcorner\}A &= (-1)^j\{\partial_K, \phi\}\bar\partial_{\mathrm{End}(E)}A - \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}A - [\phi\lrcorner F_{d_K}, A], \\ \{\partial_K, \phi\lrcorner\}[\theta, A] &= [\{\partial_K, \phi\lrcorner\}\theta, A] + (-1)^i[\theta, \{\partial_K, \phi\lrcorner\}A], \\ [\phi, \psi]\lrcorner F_{d_K} &= (-1)^i\{\partial_K, \phi\lrcorner\}\psi\lrcorner F_{d_K} - (-1)^{ij+j}\{\partial_K, \psi\lrcorner\}\phi\lrcorner F_{d_K}, \\ [\theta, [A, B]] &= [[\theta, A], B] + (-1)^i[A, [\theta, B]], \\ \bar\partial_{\mathrm{End}(E)}[A, B] &= [\bar\partial_{\mathrm{End}(E)}A, B] + (-1)^i[A, \bar\partial_{\mathrm{End}(E)}B], \\ \bar\partial_{TX}[\phi, \psi] &= [\bar\partial_{TX}\phi, \psi] + (-1)^i[\phi, \bar\partial_{TX}\psi]. \end{aligned}$$

These equations follow from Corollaries 3.2–3.5 and the fact that $\bar\partial_{\mathrm{End}(E)}$ and $\bar\partial_{TX}$ satisfy the Leibniz rule and the canonical bracket satisfies the Jacobi identity. □

Propositions 3.2 and 3.3 show us that $(L, [\cdot,\cdot]_L, d_L)$ is a DGLA. Hence we proved Theorem 3.1. Combining Propositions 2.6 and 3.1 with Theorem 3.1, we have:

Theorem 3.6. *Given a holomorphic-Higgs triple (X, E, θ) and a smooth family of elements $\{A_t, B_t, \phi_t\}_{t\in\Delta} \subset A^{0,1}(\mathrm{End}(E)) \oplus A^{1,0}(\mathrm{End}(E)) \oplus A^{0,1}(TX)$. Then,*

(A_t, B_t, ϕ_t) defines a holomorphic-Higgs triple if and only if (A_t, B_t, ϕ_t) satisfies the Maurer–Cartan equation

$$d_L(A_t + B_t, \phi_t) - \tfrac{1}{2}[(A_t + B_t, \phi_t), (A_t + B_t, \phi_t)] = 0. \tag{13}$$

We now give the proof of Proposition 3.1.

Proof of Proposition 3.1. We calculate $\bar{D}^2$. Using Corollaries 3.2 and 3.5, we have

$$\begin{aligned}
\bar{D}^2 &= (\bar{\partial}_E + \{\partial_K, \phi\lrcorner\} + A + \theta + B)^2 \\
&= \bar{\partial}_E\bar{\partial}_E + \bar{\partial}_E\{\partial_K, \phi\lrcorner\} + \{\partial_K, \phi\lrcorner\}\bar{\partial}_E + \{\partial_K, \phi\lrcorner\}\{\partial_K, \phi\lrcorner\} \\
&\qquad + \{\partial_K, \phi\lrcorner\}(A + \theta + B) + (A + \theta + B)\{\partial_K, \phi\lrcorner\} \\
&\qquad\quad + \bar{\partial}_E(A + \theta + B) + (A + \theta + B)\bar{\partial}_E + [\theta, A + B] + \tfrac{1}{2}[A + B, A + B] \\
&= -\{\partial_K, \bar{\partial}_{TX}\phi\lrcorner\} - \phi\lrcorner F_{d_K} + \tfrac{1}{2}\{\partial_K, [\phi, \phi]\lrcorner\} + \bar{\partial}_{\mathrm{End}(E)}(\theta + A + B) \\
&\qquad + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(\theta + A + B) + [\theta, A + B] + \tfrac{1}{2}[A + B, A + B] \\
&= -\bigl\{\partial_K, \bigl(\bar{\partial}_{TX}\phi - \tfrac{1}{2}[\phi, \phi]\bigr)\lrcorner\bigr\} \\
&\qquad + \bar{\partial}_{\mathrm{End}(E)}(A + B) - \phi\lrcorner F_{d_K} + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\theta + [\theta, A + B] \\
&\qquad\quad + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(A + B) + \tfrac{1}{2}[A + B, A + B].
\end{aligned}$$

Hence by the above calculation, $\bar{D}^2 = 0$ is equivalent to

$$\begin{aligned}
&\bar{\partial}_{\mathrm{End}(E)}(A + B) - \phi\lrcorner F_{d_K} + [\theta, A + B] \\
&\qquad + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\theta + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(A + B) + \tfrac{1}{2}[A + B, A + B] = 0, \\
&\qquad\qquad \bar{\partial}_{TX}\phi - \tfrac{1}{2}[\phi, \phi] = 0.
\end{aligned}$$

Hence we have the proof. □

4. Kuranishi family

4A. ***Construction of Kuranishi family.*** Kuranishi [1965] constructed a universal family for any complex manifold X over a possible singular analytic space. We want to construct a family of holomorphic-Higgs triples over a certain singular space which becomes a universal family in this context.

Here we recall some differential operators and inequalities we need. These are commonly used in classical Hodge theory. We choose a hermitian metric g on X and a hermitian metric K on E. Using these two metrics, we can define an inner product $(\cdot, \cdot)$ on $L = \bigoplus_i L^i$. We remark that L^i and L^j are orthogonal with respect to this inner product. We first define the formal adjoint of d_L with respect to $(\cdot, \cdot)$ by

$$(d_L\alpha, \beta) = (\alpha, d_L^*\beta).$$

Then the Laplacian Δ_L is defined by

$$\Delta_L = d_L \circ d_L^* + d_L^* \circ d_L.$$

This is an elliptic self-adjoint operator. Hence by Theorem 4.12 in [Wells 1980, Chapter 4], it has a finite dimensional kernel $\mathbb{H}^i$. We call the elements of $\mathbb{H}^i$ a *harmonic form*. Let $\widetilde{L}^i$ be a completion of L^i with respect to the inner product $(\cdot\,,\cdot)$, and let $H : L^i \to \mathbb{H}^i$ be the harmonic projection. The Green's operator $G : L^i \to L^i$ is defined by

$$I = H + \Delta_L \circ G = H + G \circ \Delta_L,$$

where I is the identity for L^i. H and G can be extended to the bounded operator $H, G : \widetilde{L}^i \to \widetilde{L}^i$. G commutes with d_L and d_L^*.

Now let $\{\eta_1, \dots, \eta_n\} \subset \mathbb{H}^1$ be a basis and $\epsilon_1(t) := \sum_{j=1}^n t_j \eta_j \in \mathbb{H}^1$. Consider

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

We define the Hölder norm $\|\cdot\|_{k,\alpha}$ as in [Morrow and Kodaira 2006]. We have the inequalities

$$\|d_L^*\epsilon\|_{k,\alpha} \le C_1 \|\epsilon\|_{k+1.\alpha},$$
$$\|[\epsilon, \delta]\|_{k,\alpha} \le C_2 \|\epsilon\|_{k+1,\alpha} \|\delta\|_{k+1,\alpha}.$$

Douglis and Nirenberg [1955] proved the a priori estimate

$$\|\epsilon\|_{k,\alpha} \le C_3(\|\Delta_L \epsilon\|_{k-2,\alpha} + \|\epsilon\|_{0,\alpha}).$$

Applying these and following the proof of Proposition 2.3 in [Morrow and Kodaira 2006, Chapter 4] one can deduce an estimate for the Green's operator G:

$$\|G\epsilon\|_{k,\alpha} \le C_4 \|\epsilon\|_{k-2,\alpha},$$

where all C_i's are positive constants which depend only on k and α.

Then by applying the proof of Proposition 2.4 in [Morrow and Kodaira 2006, Chapter 4] or using the implicit function theorem for Banach spaces as in [Kuranishi 1965], we obtain a unique solution $\epsilon(t)$ which satisfies

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L,$$

which is analytic in the variable in t. The solution $\epsilon(t)$ is also a smooth section for L^1. By applying the Laplacian to the above equation, we get

$$\Delta_L \epsilon(t) - \tfrac{1}{2} d_L^* [\epsilon(t), \epsilon(t)]_L = 0.$$

Since $\epsilon(t)$ is holomorphic in t, we have

$$\sum_{i,j} \frac{\partial^2}{\partial t_i \partial \bar{t}_j} \epsilon(t) = 0.$$

Hence we have

$$\left(\Delta_L + \sum_{i,j} \frac{\partial^2}{\partial t_i \partial \bar{t}_j}\right)\epsilon(t) - \tfrac{1}{2} d_L^*[\epsilon(t), \epsilon(t)]_L = 0.$$

Since the operator

$$\Delta_L + \sum_{i,j} \frac{\partial^2}{\partial t_i \partial \bar{t}_j}$$

is elliptic, we can say that $\epsilon(t)$ is smooth by elliptic regularity.

From the discussion so far, we have:

Proposition 4.1. *Let $\{\eta_1, \dots, \eta_n\} \subset \mathbb{H}^1$ be a basis. Let $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ and $\epsilon_1(t) := \sum_i t_i \eta_i$. For all $|t| \ll 1$ we have a $\epsilon(t)$ such that $\epsilon(t)$ satisfies*

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

Moreover, $\epsilon(t)$ is holomorphic with respect to the variable t.

Following Kuranishi [1965] we have:

Proposition 4.2. *If we take $|t|$ small enough, the solution $\epsilon(t)$ that satisfies*

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L$$

solves the Maurer–Cartan equation if and only if $H[\epsilon(t), \epsilon(t)]_L = 0$. Here H is the harmonic projection.

Proof. Suppose the Maurer–Cartan equation holds. Then

$$d_L \epsilon(t) - \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L = 0.$$

Hence we have

$$H[\epsilon(t), \epsilon(t)]_L = 2H d_L \epsilon(t) = 0.$$

Conversely, suppose that $H[\epsilon(t), \epsilon(t)]_L = 0$. We have to show

$$\delta(t) := d_L \epsilon(t) - \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L = 0.$$

Since $\epsilon(t)$ is a solution to

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L$$

and $\epsilon_1(t)$ is d_L-closed, applying d_L we get

$$d_L \epsilon(t) = \tfrac{1}{2} d_L d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

Hence

$$2\delta(t) = d_L d_L^* G[\epsilon(t), \epsilon(t)]_L - [\epsilon(t), \epsilon(t)]_L.$$

By the Hodge decomposition, we can write

$$[\epsilon(t), \epsilon(t)]_L = H[\epsilon(t), \epsilon(t)]_L + \Delta_L G[\epsilon(t), \epsilon(t)]_L = \Delta_L G[\epsilon(t), \epsilon(t)]_L.$$

Therefore

$$\begin{aligned} 2\delta(t) &= d_L d_L^* G[\epsilon(t), \epsilon(t)]_L - \Delta_L G[\epsilon(t), \epsilon(t)]_L \\ &= -d_L^* d_L G[\epsilon(t), \epsilon(t)]_L \\ &= -d_L^* G d_L[\epsilon(t), \epsilon(t)]_L \\ &= -2 d_L^* G[d_L \epsilon(t), \epsilon(t)]_L. \end{aligned}$$

Hence we get

$$\begin{aligned} \delta(t) &= -d_L^* G[d_L \epsilon(t), \epsilon(t)]_L \\ &= -d_L^* G\big[\delta(t) + \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L, \epsilon(t)\big]_L \\ &= -d_L^* G[\delta(t), \epsilon(t)]_L. \end{aligned}$$

We used the Jacobi identity in the last equality. Using the estimate

$$\|[\xi, \eta]\|_{k,\alpha} \le C_{k,\alpha} \|\xi\|_{k+1,\alpha} \|\eta\|_{k+1,\alpha},$$

we get

$$\|\delta(t)\|_{k,\alpha} \le C_{k,\alpha} \|\delta(t)\|_{k,\alpha} \|\epsilon(t)\|_{k,\alpha}.$$

If we take $|t|$ small enough such that $C_{k,\alpha}\|\epsilon(t)\|_{k,\alpha} < 1$, we obtain $\delta(t) = 0$. This stands for all $|t|$ small enough. This finishes the proof. □

In the case when $H[\epsilon(t), \epsilon(t)]_L = 0$ for all t or $\mathbb{H}^2 = 0$, we have:

Corollary 4.1. *Let n be the dimension of $\mathbb{H}^1$. If $H[\epsilon(t), \epsilon(t)]_L = 0$ for all t, we have a family of deformation of holomorphic-Higgs triples over a small ball Δ centered at the origin of $\mathbb{C}^n$.*

Proof. If $H[\epsilon(t), \epsilon(t)] = 0$ for all t, $\epsilon(t) = (A_t + B_t, \phi_t)$ satisfies the Maurer–Cartan equation and so we obtain family of holomorphic-Higgs triple (X_t, E_t, θ_t). Since ϕ_t is holomorphic for variable t, applying the Newlander–Nirenberg theorem, we can define a complex structure on $\mathcal{X} := X \times \Delta$ such that $X_t = \mathcal{X}|_{X\times\{t\}}$. Let $\mathcal{E} := E \times \Delta$. By applying the linearized Newlander–Nirenberg theorem as in [Moroianu 2007], we have a local frame $\{e(x, t)\}$ of $\mathcal{E}$ on $\mathcal{X}$ such that for each t, $\{e(x, t)\} \subset \ker(\overline{D}'_t) = \ker(\bar\partial_{E_t})$ and is holomorphic respect to variable t. Let $\sigma : \mathcal{X} \to \mathcal{E}$ be a smooth section and locally trivialized as $\sigma(x, t) = \sum_k s^k(x, t) e_k(x, t)$ where s^k are smooth function on $\mathcal{X}$. We define $\bar\partial_{\mathcal{E}} : A(\mathcal{E}) \to A^{0,1}_{\mathcal{X}}(\mathcal{E})$ as

$$\bar\partial_{\mathcal{E}}(\sigma(x, t)) := \sum_k \bar\partial_{\mathcal{X}} s^k(x, t) \otimes e_k(x, t).$$

Note that $\bar{\partial}_{\mathcal{E}}$ is well defined and $\bar{\partial}_{\mathcal{E}}|_{E\times t} = \bar{\partial}_{E_t}$. It is clear that $\bar{\partial}_{\mathcal{E}}^2 = 0$ so that $\mathcal{E}$ is a holomorphic bundle over $\mathcal{X}$.

Let $\Theta = \theta + B_t + \phi_t \lrcorner (\theta + B_t)$. Since ϕ_t, B_t is holomorphic respect to the variable t and $\theta + B_t + \phi_t \lrcorner (\theta + B_t)$ is a Higgs field for (X_t, E_t), we have $\bar{\partial}_{\mathrm{End}\mathcal{E}}\Theta = 0$, $\Theta \wedge \Theta = 0$. Hence Θ is a Higgs field for $(\mathcal{X}, \mathcal{E})$.

Let $\pi : \mathcal{X} = X \times \Delta \to \Delta$ be a natural projection, this is a holomorphic submersion. Also $\pi^{-1}(0) = X$, $\mathcal{E}|_{\pi^{-1}(0)} = E$ and $\Theta|_{\pi^{-1}(0)} = \theta$ stands. Hence we have a family of deformation of a holomorphic-Higgs triple over Δ. □

In general, the condition $\mathbb{H}^2 = 0$ may not be satisfied. However, we can define a possible singular analytic space

$$\mathcal{S} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)]_L = 0\}.$$

Let $X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)}$ be the complex manifold, the holomorphic bundle, and the Higgs field defined by $\epsilon(t)$. By the above results, we have a family of holomorphic-Higgs triples $\{(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})\}_{t\in\mathcal{S}}$. We call this family the *Kuranishi family* of (X, E, θ) and $\mathcal{S}$ the *Kuranishi space*.

4B. ***Local completeness of Kuranishi family.*** We give a proof of the local completeness of the Kuranishi family for the deformation of the triple (X, E, θ). Here we follow Kuranishi's method.

Recall that in Section 4A we proved that for a given $\epsilon_1(t) = \sum_i t_i \eta_i \in \mathbb{H}^1 = \ker(\Delta_L : L^1 \to L^1)$ the existence of solutions $\epsilon(t)$ to

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L$$

and proved that $\epsilon(t)$ satisfies the Maurer–Cartan equation if and only if

$$H[\epsilon(t), \epsilon(t)]_L = 0.$$

Hence we obtain a family of holomorphic-Higgs triples over

$$\mathcal{S} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)]_L = 0\}.$$

Before we state the main theorem of this paper, we introduce the Sobolev norm for L and collect some estimates.

First, let us recall the Sobolev norm on Euclidean space. Let U be an open subset of $\mathbb{R}^n$ and f and g be a complex-valued smooth function on $\overline{U}$. Here, $\overline{U}$ is a closure of U. We set

$$(f, g)_k := \sum_{|\alpha|<k} \int_U D^\alpha f \cdot \overline{D^\alpha g}\, dx,$$

where we use the multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$, $|\alpha| = \sum_i \alpha_i$ and $D^\alpha = \bigl(\frac{\partial}{\partial x_1}\bigr)^{\alpha_1} \cdots \bigl(\frac{\partial}{\partial x_n}\bigr)^{\alpha_n}$.

Then we define k-th Sobolev norm $|\cdot|_k$ as

$$|f|_k = |f|_k^U := \sqrt{(f, f)_k}. \tag{14}$$

Let V be a relatively compact open subset of U. By [Morrow and Kodaira 2006, Chapter 4, Lemma 3.1], we have an estimate such that

$$|fg|_k^V \leq c|f|_k^U \cdot |g|_k^U, \quad k \geq n+2, \tag{15}$$

where c is a constant.

By using a partition of unity and the metric of E and X, we can define k-th Sobolev $|\eta|_k$ for any $\eta \in L^i = \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}(E)) \oplus A^{0,i}(TX)$. We list some estimates that we need. Let c_k be a constant. Then the following estimates hold (see [Morrow and Kodaira 2006] for more details):

$$\begin{aligned} &|[\phi, \psi]_k| \leq c_k |\phi|_{k+1} |\psi|_{k+1}, \quad k \geq 2n+2, \ \dim_{\mathbb{C}} X = n, \\ &|H\phi| \leq c_k |\phi|_k, \\ &|d_L^* G\phi|_k \leq c_k |\phi|_{k-1}. \end{aligned} \tag{16}$$

From now on, we choose a k large enough such that the above estimates hold.

Let $\eta := (A+B, \phi) \in L^1$ be a Maurer–Cartan element and assume $|\eta|_k$ is small enough so that η can define a holomorphic-Higgs triple. Let $X_\eta, E_\eta, \theta_\eta$ be the complex manifold, the holomorphic bundle, and the Higgs field which η defines, respectively. We denote this holomorphic-Higgs triple $(X_\eta, E_\eta, \theta_\eta)$. Let $\eta' \in L^1$ be another Maurer–Cartan element and assume that η' also defines a holomorphic-Higgs triple $(X_{\eta'}, E_{\eta'}, \theta_{\eta'})$. We denote as $(X_\eta, E_\eta, \theta_\eta) \cong (X_{\eta'}, E_{\eta'}, \theta_{\eta'})$ when there is a biholomorphic map $F : X_\eta \to X'_\eta$, a holomorphic bundle isomorphism $\Phi : E_\eta \to E'_\eta$ which is compatible with F and $\theta_\eta = \widehat{\Phi}^{-1} \circ F^*(\theta'_\eta) \circ \widehat{\Phi}$ holds. Here $\widehat{\Phi} : E_\eta \to F^*(E_{\eta'})$ is the holomorphic bundle isomorphism induced by Φ. $F^*(E_{\eta'})$ is the pull back of the bundle $E_{\eta'}$ by F.

Now we state the main theorem of this paper.

Theorem 4.2. *Let $\eta := (A+B, \phi) \in L^1$ be a Maurer–Cartan element. If $|\eta|_k$ is small enough, then there exists some $t \in \mathcal{S}$ such that $(X_\eta, E_\eta, \theta_\eta) \cong (X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})$.*

Proposition 4.3. *Let $\epsilon_1(t) \in \mathbb{H}^1$, $t \in S$. Assume that ϵ solves the equation*

$$\epsilon = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon, \epsilon]_L.$$

If $|\epsilon|_k$ is small enough, then the solution is unique.

Proof. Suppose ϵ is another solution. Let $\delta = \epsilon - \epsilon(t)$. Then

$$\begin{aligned} \delta &= \tfrac{1}{2} d_L^* G\big([\epsilon, \epsilon]_L - [\epsilon(t), \epsilon(t)]_L\big) \\ &= \tfrac{1}{2} d_L^* G\big([\delta, \epsilon(t)]_L + [\epsilon(t), \delta]_L + [\delta, \delta]_L\big) \\ &= \tfrac{1}{2} d_L^* G\big(2[\delta, \epsilon(t)]_L + [\delta, \delta]_L\big). \end{aligned}$$

Estimating $|\delta|_k$ gives

$$|\delta|_k \le D_k\big(|\delta|_k|\epsilon(t)|_k + |\delta|_k^2\big) \le D_k|\delta|_k\big(|\epsilon(t)|_k + |\delta|_k\big).$$

If $|\epsilon(t)|_k$ is small enough, the above estimate holds if and only if $|\delta|_k = 0$. This proves the proposition. □

Proposition 4.4. *Suppose $\eta \in L^1$ satisfies the Maurer–Cartan equation (13). If $d_L^*\eta = 0$ and $|\eta|_k$ is small enough, then $\eta = \epsilon(t)$ for some $t \in S$.*

Proof. Since η satisfies the Maurer–Cartan equation, we have

$$d_L\eta - \tfrac{1}{2}[\eta, \eta]_L = 0.$$

Since $d_L^*\eta = 0$, we have

$$\Delta_L\eta = d_L^* d_L\eta + d_L d_L^*\eta = \tfrac{1}{2}d_L^*[\eta, \eta]_L.$$

Hence

$$\eta - H\eta = G\Delta_L\eta = \tfrac{1}{2}Gd_L^*[\eta, \eta]_L.$$

Let $\psi := H\eta$. Then $\eta = \psi + \frac{1}{2}Gd_L^*[\eta, \eta]_L$. By the assumption such that $|\eta|_k$ is small, $|\psi|_k$ is small by (16). Hence $\psi = \epsilon_1(t)$ for $|t|$ small enough. Hence by Proposition 4.3, $\eta = \epsilon(t)$ for some $t \in S$. □

In general $d_L^*\eta \neq 0$ so we must try something else. We follow the idea of [Kuranishi 1965]. Let us recall how we solved this problem in the complex manifold setting. The idea is that for a given Maurer–Cartan element $\phi \in A^{0,1}(TX)$, we deform ϕ along a diffeomorphism $f : X \to X$.

Let X_ϕ be a complex manifold such that the complex structure comes from ϕ. Let $f : X \to X$ be a diffeomorphism. We can induce a complex structure on X by f. We denote the corresponding Maurer–Cartan element as $\phi \circ f$. Note that $f : X_{\phi\circ f} \to X_\phi$ is a biholomorphic map.

Kuranishi showed that for every Maurer–Cartan elements ϕ with $|\phi|_k$ small, there is a diffeomorphism f such that $\bar\partial_{TX}(\phi \circ f) = 0$. We recall how we obtain such f.

Let $g = (g_{i\bar j})$ be a fixed hermitian metric on X. Let $\xi = \sum_i \xi_i(z)\frac{\partial}{\partial z_i} \in A^0(TX)$ and $\bar\xi$ be the conjugate. Let $z_0 \in X$. Let $c(t) = c(t, z_0, \xi) = (c_1(t), \dots, c_n(t))$ be the geodesic curve starting from z_0 with initial velocity $\xi + \bar\xi$. Let $f_\xi(z_0) := c(1, z_0, \xi)$. Since X is compact, f_ξ is a diffeomorphism. By using Taylor expansion for f_ξ, we obtain

$$\phi \circ f_\xi = \phi + \bar\partial_{TX}\xi + R(\phi, \xi) \tag{17}$$

where $R(t\phi, t\xi) = t^2 R_1(\phi, \xi, t)$ if t is a real number and both R, R_1 are smooth map on X. In [Kuranishi 1965], it was shown that there is a $\xi \in A^0(TX)$ such that $\bar{\partial}_{TX}(\phi \circ f_\xi) = 0$ for any ϕ with $|\phi|_k$ small by the implicit function theorem between Banach spaces.

Let $\eta = (A + B, \phi) \in L^1$ be a Maurer–Cartan elements and assume $|\eta|_k$ is small enough so that η can define a holomorphic-Higgs triple $(X_\eta, E_\eta, \theta_\eta)$. By Kuranishi's work we have a $\xi \in A^0(TX)$ such that $\bar{\partial}_{TX}(\phi \circ f_\xi) = 0$.

Let $P_\xi : E \to E$ be the parallel transport of the Chern connection along f_ξ. Let $\upsilon \in A^0(\mathrm{End}(E))$ and $\exp(\upsilon) := \sum_{n=0}^{\infty} \frac{\upsilon^n}{n!} \in A^0(\mathrm{End}(E))$. $\exp(\upsilon) : E \to E$ is an automorphism and the inverse is given as $\exp(-\upsilon)$. Note that $(\upsilon, \xi) \in L^0$.

Let $\Phi := P_\xi \circ \exp(\upsilon)$. Since P_ξ is an isomorphism and compatible with f_ξ, Φ also is. Hence there is a smooth bundle isomorphism $\widehat{\Phi} : E \to f_\xi^* E_\eta$ which is induced by Φ. Hence we can induce a holomorphic-Higgs triple structure on (X, E, θ) via Φ and f_ξ. This holomorphic-Higgs triple is isomorphic to $(X_\eta, E_\eta, \theta_\eta)$. We denote the corresponding Maurer–Cartan element as $\eta_\gamma := ((A+B) \star \Phi, \phi \circ f_\xi)$. We show the existence of $\gamma := (\upsilon, \xi) \in L^0$ such that $d_L^* \eta_\gamma = 0$.

We first prove the next proposition.

Proposition 4.5. *Let $\eta_\gamma = ((A + B) \star \Phi, \phi \circ f_\xi)$, $\eta = (A + B, \phi)$ and $\gamma = (\upsilon, \xi)$ be as above. Then we have*

$$((A + B) \star \Phi, \phi \circ f_\xi) = (A + B, \phi) + d_L(\upsilon, \xi) + R((A, B, \phi), (\upsilon, \xi)). \tag{18}$$

The error term R is of order t^2 in the sense that

$$R(t(A, B, \phi), t(\upsilon, \xi)) = t^2 R_1((A, B, \phi), (\upsilon, \xi), t).$$

where t is a real number and R_1 is a smooth map.

Proof. Before going to the proof, we prepare some terminologies. Let $A \in A^{0,1}(\mathrm{End}(E))$, $B \in A^{1,0}(\mathrm{End}(E))$, $\upsilon \in A(\mathrm{End}(E))$, $\phi \in A^{0,1}(TX)$ and $\xi \in A^0(TX)$. The map $R((A, B, \phi), (\upsilon, \xi))$ is a smooth map on X such that R depends on A, B, υ, ϕ and ξ and R is of order t^2 in the sense that

$$R(t(A, B, \phi), t(\upsilon, \xi)) = t^2 R_1((A, B, \phi), (\upsilon, \xi), t),$$

where t is a real number and $R_1((A, B, \phi), (\upsilon, \xi), t)$ is a smooth map defined on X. We assume that the same property holds for $R((A, \phi), (\upsilon, \xi))$, $R((B, \phi), (\upsilon, \xi))$.

The map $R'((A, B, \phi), (\upsilon, \xi))$ is a smooth map defined on some open set of X such that R' depends on A, B, υ, ϕ, and ξ and R' is of order t^2 in the sense that

$$R'(t(A, B, \phi), t(\upsilon, \xi)) = t^2 R'_1((A, B, \phi), (\upsilon, \xi), t),$$

where t is a real number and $R'_1((A,B,\phi),(\upsilon,\xi),t)$ is a smooth map defined on some open set of X. We assume that the same property holds for $R'((A,\phi),(\upsilon,\xi))$, $R'((B,\phi),(\upsilon,\xi))$, $R'(\upsilon,\xi)$ and $R'(\phi,\xi)$.

By (17), we only have to prove

$$(A+B)\star\Phi = A+B+\bar{\partial}_{\mathrm{End}(E)}\upsilon+\xi\lrcorner F_{d_K}+[\theta,\upsilon]+\{\partial_K^{\mathrm{End}(E)},\xi\lrcorner\}\theta+R((A,B,\phi),(\upsilon,\xi)). \tag{19}$$

First we prove

$$A\star\Phi = A+\bar{\partial}_{\mathrm{End}(E)}\upsilon+\xi\lrcorner F_{d_K}+R((A,\phi),(\upsilon,\xi)). \tag{20}$$

Let U' and U be open sets of X such that $U'\subset U$ and $f_\xi(U')\subset U$. We calculate $(A\star\Phi-A)(z)$ for $z\in U'$. Let $\{e_k\}$ be a holomorphic frame on U for $E_{\eta'}$. Since $E_{\eta'}$'s complex structure is induced by Φ, $\Phi : E_{\eta'}\to E_\eta$ is a holomorphic bundle isomorphism. Hence $\Phi(e_k)$ is a holomorphic section for E_η. Hence we have

$$\bar{\partial}_E e_k+\{\partial_K,(\phi\circ f_\xi)\lrcorner\}e_k+(A\star\Phi)e_k=0, \tag{21}$$

$$\Phi^{-1}\circ(\bar{\partial}_E+\{\partial_k,\phi\lrcorner\}+A)\circ\Phi(e_k)=0. \tag{22}$$

By (17), (21) is equivalent to

$$\bar{\partial}_E e_k+\{\partial_K,\phi\lrcorner\}e_k+\{\partial_K,\bar{\partial}_{TX}\xi\lrcorner\}e_k+(A\star\Phi)e_k+R'(\phi,\xi)(e_k)=0. \tag{23}$$

Let P'_ξ be the first order of P_ξ. Since $\Phi=P_\xi\circ\exp(\upsilon)$, we have an expansion for $\Phi(e_k)$ such that

$$\Phi(e_k)=P_\xi\circ\exp(\upsilon)(e_k)=e_k+P'_\xi(e_k)+\upsilon(e_k)+R'(\upsilon,\xi)(e_k).$$

Hence (22) is equivalent to

$$\bar{\partial}_E e_k+\{\partial_K,\phi\lrcorner\}e_k+Ae_k+\bar{\partial}_E(P'_\xi e_k)-P'_\xi\bar{\partial}_E e_k \\ +\bar{\partial}_E\upsilon e_k-\upsilon\bar{\partial}_E(e_k)+R'((A,\phi),(\upsilon,\xi))(e_k)=0.$$

Since $\bar{\partial}_E(\upsilon e_k)-\upsilon\bar{\partial}_E e_k=(\bar{\partial}_{\mathrm{End}(E)}\upsilon)e_k$, we have

$$\bar{\partial}_E e_k+\{\partial_K,\phi\lrcorner\}e_k+Ae_k+\bar{\partial}_E(P'_\xi e_k)-P'_\xi\bar{\partial}_E e_k \\ +(\bar{\partial}_{\mathrm{End}(E)}\upsilon)e_k+R'((A,\phi),(\upsilon,\xi))(e_k)=0. \tag{24}$$

Hence by (23), (24),

$$(A\star\Phi)e_k-Ae_k+\{\partial_K,\bar{\partial}_{TX}\xi\lrcorner\}e_k-\bar{\partial}_E(P'_\xi e_k)+P'_\xi\bar{\partial}_E e_k \\ -(\bar{\partial}_{\mathrm{End}(E)}\upsilon)e_k+R'((A,\phi),(\upsilon,\xi))(e_k)=0. \tag{25}$$

We have to prove $\{\partial_K,\bar{\partial}_{TX}\xi\lrcorner\}-\bar{\partial}_E\circ P'_\xi+P'_\xi\circ\bar{\partial}_E=-\xi\lrcorner F_{d_K}$. We prove this for a holomorphic frame $\{e'_k\}$ for E on U. Since P_ξ is the parallel transport along f_ξ

respect to the Chern connection, we have $P'_\xi(e'_k) = -\xi \lrcorner K^{-1}\partial K(e'_k)$. (See [Spivak 1999] for more details). Hence we have

$$\begin{aligned}\{\partial_K, \bar\partial_{TX}\xi\lrcorner\}e'_k - \bar\partial_E \circ P'_\xi e'_k + P'_\xi \circ \bar\partial_E e'_k &= -\bar\partial_{TX}\xi\lrcorner\partial_K e'_k + \bar\partial_E(\xi\lrcorner K^{-1}\partial K e'_k)\\ &= -\bar\partial_{TX}\xi\lrcorner\partial_K e'_k + \bar\partial_{TX}\xi\lrcorner K^{-1}\partial K e'_k - \xi\lrcorner(\bar\partial_{\mathrm{End}(E)}(K^{-1}\partial K))e'_k\\ &= (-\xi\lrcorner F_{d_K})e'_k.\end{aligned}$$

Hence by (25), we have

$$(A\star\Phi)e_k = Ae_k + (\bar\partial_{\mathrm{End}(E)}\upsilon)e_k + \xi\lrcorner F_{d_K}(e_k) + R'((A,\phi),(\upsilon,\xi))(e_k).$$

Since $\{e_k\}$ is an arbitrary holomorphic frame on $E_{\eta'}$ and $R'((A,\phi),(\upsilon,\xi))(e_k)$ is a local expression of $A\star\Phi - A - \bar\partial_{\mathrm{End}(E)}\upsilon - \xi\lrcorner F_{d_K}$ we proved (20).

Next, we prove that

$$B\star\Phi = B + [\theta,\upsilon] + \{\partial_K^{\mathrm{End}(E)}, \xi\lrcorner\}\theta + R((B,\phi),(\upsilon,\xi)). \tag{26}$$

We recall that $\widehat\Phi : E_{\eta'} \to f_\xi^*(E_\eta)$ is a holomorphic bundle isomorphism and $\theta_{\eta'} = \widehat\Phi^{-1}\circ f_\xi^*(\theta_\eta)\circ\widehat\Phi$. Let $\theta_{\eta'}^{1,0}$ is the $(1,0)$-part of $\theta_{\eta'}$ respect to the original complex structure, then we have $B\star\Phi = \theta_{\eta'}^{1,0} - \theta$. We calculate $B\star\Phi$ locally.

Let (U,z) be a local coordinate and $U'\subset U$. We assume $f_\xi(U')\subset U$ and $\xi = \sum_i \xi_i(z)\big(\frac{\partial}{\partial z_i}\big)$. By the definition of f_ξ, for $z\in U'$, we have

$$f_\xi(z) = \big(z_1 + \xi_1(z) + O(|\xi|^2), \dots, z_n + \xi_n(z) + O(|\xi|^2)\big).$$

Let $\{e_k\}$ be a holomorphic frame on U for E. Let $g_i, B_i \in A^0(\mathrm{End}(E))$. Assume that θ is locally expressed as $\sum_i g_i(z)dz_i$ and B as $\sum_i B_i(z)dz_i$ respect to this frame.

Let the bracket $[\,\cdot\,,\cdot\,]$ be the canonical Lie bracket defined on $A^*(\mathrm{End}(E))$. Since $\theta_\eta = \theta + B + \phi\lrcorner(\theta+B) = (g_i+B_i)dz_i + (g_i+B_i)\phi^i_j d\bar z_j$ and $\widehat\Phi$ is induced by Φ, we have

$$\begin{aligned}\theta_{\eta'}(e_k) &= \widehat\Phi^{-1}\circ f_\xi^*(\theta_\eta)\circ\widehat\Phi(e_k)\\ &= \widehat\Phi^{-1}\circ\{(g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j}\}\circ\widehat\Phi(e_k)\\ &= ((g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j})(e_k)\\ &\quad + [(g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j}, P'_\xi](e_k)\\ &\quad + [(g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j}, \upsilon](e_k)\\ &\quad + R'((B,\phi),(\upsilon,\xi))(e_k).\end{aligned}$$

Hence

$$\begin{aligned}(27)\quad \theta^{1,0}_{\eta'}(e_k) &= \left((g_i+B_i)(f_\xi(z))dz_i+(g_i+B_i)(f_\xi(z))\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k)\\ &\quad+\left[((g_i+B_i)(f_\xi(z))dz_i+(g_i+B_i)(f_\xi(z))\frac{\partial \xi_i}{\partial z_j}dz_j,\, P'_\xi\right](e_k)\\ &\quad+\left[((g_i+B_i)(f_\xi(z))dz_i+(g_i+B_i)(f_\xi(z))\frac{\partial \xi_i}{\partial z_j}dz_j,\, \upsilon\right](e_k)\\ &\quad+R'((B,\phi),(\upsilon,\xi))(e_k)\\ &= \left((g_i+B_i)(f_\xi(z))dz_i+g_i(f_\xi(z))\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k)\\ &\quad+\big[g_i(f_\xi(z))dz_i,\, P'_\xi\big](e_k)+\big[g_i(f_\xi(z))dz_i,\, \upsilon\big](e_k)\\ &\quad+R'((B,\phi),(\upsilon,\xi))(e_k).\end{aligned}$$

Since $f_\xi(z)=c(z,\xi,1)$, we have the Taylor expansion at $t=0$ for $g_i(f_\xi(z))$ and $B_i(f_\xi(z))$:

$$g_i(f_\xi(z)) = g_i(z)+\xi_j(z)\frac{\partial g_i}{\partial z_j}(z)+O(|\xi|^2),$$

$$B_i(f_\xi(z)) = B_i(z)+\xi_j(z)\frac{\partial B_i}{\partial z_j}(z)+O(|\xi|^2).$$

Hence by (27), $\theta^{1,0}_{\eta'}(e_k)$ becomes

$$\begin{aligned}\theta^{1,0}_{\eta'}(e_k) &= \left(g_i(z)dz_i+\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+B_i(z)dz_i+g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k)\\ &\quad+[g_i(z)dz_i,\, P'_\xi](e_k)+[g_i(z)dz_i,\, \upsilon](e_k)+R'((B,\phi),(\upsilon,\xi))(e_k)\\ &= (\theta+B)(e_k)+\left(\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k)\\ &\quad+[\theta,\, P'_\xi](e_k)+[\theta,\upsilon](e_k)+R'((B,\phi),(\upsilon,\xi))(e_k).\end{aligned}$$

Hence

$$\begin{aligned}(28)\qquad B\star\Phi(e_k) &= \theta^{1,0}_{\eta}(e_k)-\theta(e_k)\\ &= B(e_k)+\left(\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k)\\ &\quad+[\theta,\, P'_\xi](e_k)+[\theta,\upsilon](e_k)+R'((B,\phi),(\upsilon,\xi))(e_k).\end{aligned}$$

Hence the only thing we have to prove is

$$\left(\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k)+[\theta,\, P'_\xi](e_k)=(\{\partial_K^{\mathrm{End}(E)},\xi\lrcorner\}\theta)(e_k).$$

Since $\{e_k\}$ is a local holomorphic frame for E, we have

$$\begin{aligned}
&\left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k) + [\theta, P'_\xi](e_k)\\
&= \left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k) + \theta(-\xi \lrcorner K^{-1}\partial K)(e_k) + \xi \lrcorner K^{-1}\partial K(\theta(e_k))\\
&= \left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k) + [\xi \lrcorner K^{-1}\partial K, \theta](e_k)
\end{aligned}$$

and

$$\begin{aligned}
&\{\partial_K, \xi \lrcorner\}\theta\\
&= \partial_K(\xi \lrcorner \theta) + \xi \lrcorner \partial_K(\theta)\\
&= \partial(\xi_i(z)g_i) + [K^{-1}\partial K, \xi \lrcorner \theta] + \xi_j(z)\frac{\partial g_i}{\partial z_j}dz_i - \xi_i(z)\frac{\partial g_i}{\partial z_j}dz_j + \xi \lrcorner [K^{-1}\partial K, \theta]\\
&= \frac{\partial}{\partial z_j}(\xi_i(z)g_i(z))dz_j + \xi_j(z)\frac{\partial g_i}{\partial z_j}dz_i - \xi_i(z)\frac{\partial g_i}{\partial z_j}dz_j + [\xi \lrcorner K^{-1}\partial K, \theta]\\
&= \left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right) + [\xi \lrcorner K^{-1}\partial K, \theta].
\end{aligned}$$

Hence we have the desired equality. Hence by (28) we have

$$B \star \Phi(e_k) = B(e_k) + [\theta, \upsilon](e_k) + (\{\partial_K^{\mathrm{End}(E)}, \xi \lrcorner\}\theta)(e_k) + R'((B, \phi), (\upsilon, \xi))(e_k).$$

Since $\{e_k\}$ is an arbitrary local holomorphic frame on E and $R'((B, \phi), (\upsilon, \xi))(e_k)$ is a local expression of $B \star \Phi - B - [\theta, \upsilon] - (\{\partial_K^{\mathrm{End}(E)}, \xi \lrcorner\}\theta)$, we proved (26).

Hence by (20) and (26) we proved (19). This completes the proof. □

Recall that $\mathbb{H}^0 = \ker(\Delta_L) \subset L^0$. Let F^0 be the orthogonal complement of $\mathbb{H}^0$ w.r.t. the inner product $(\cdot, \cdot)$. Note that $\ker(H) = F^0$. H is the harmonic projection. Then, for $\gamma \in F^0$,

$$\eta = G\Delta_L\gamma + H\gamma = G\Delta_L\gamma.$$

Since d_L^* is zero on L^0, $d_L^*(\gamma) = 0$. Hence

$$\Delta_L\gamma = d_L^* d_L\gamma.$$

This yields,

$$\gamma = G d_L^* d_L \gamma. \tag{29}$$

From now on, we think of L^1, F^0 as normed by the k-th Sobolev norm and by the $(k-1)$-th Sobolev norm. Let L^1_{k-1}, L^1_k, F^0_k be the completion of L^1, F^0 with respect to the corresponding norms.

Proposition 4.6. *Let $\eta(\gamma) := \eta + d_L\gamma + R(\eta, \gamma)$. There are neighborhoods of the origin U and V in L^1 and F^0 such that for any $\eta \in U$ there is a $\gamma \in V$ such that*

$$d_L^*(\eta(\gamma)) = 0. \tag{30}$$

Proof. Let $\gamma := (\upsilon, \xi) \in F^0$. By the definition of $\eta(\gamma)$, (30) is equivalent to

$$0 = d_L^*(\eta(\gamma)) = d_L^*\eta + d_L^* d_L\gamma + d_L^* R(\eta, \gamma).$$

By (29),

$$\gamma = G d_L^* d_L \gamma = -G d_L^* \eta - G d_L^* R(\eta, \gamma).$$

Thus (30) is equivalent to

$$\gamma + G d_L^* \eta + G d_L^* R(\eta, \gamma) = 0.$$

Let U_1 and V_1 are neighborhoods of the origin of L_k^1 and F_k^0. By the local property of $R(\eta, \gamma)$ which we observed in Proposition 4.5, we can define C^1 map $h : U_1 \times V_1 \to L_{k-1}^1$ by

$$h(\eta, \gamma) = \gamma + G d_L^* \eta + G d_L^* R(\eta, \gamma).$$

By the order condition on the error term R, the identity map is the derivative of h concerning γ at $(0, 0)$. Hence by the implicit function theorem for Banach spaces, there exists an open neighborhood U_0 of $0 \in L_k^1$ and a continuous map $g : U_0 \to V_1$ such that $g(0) = 0$ and such that $h(\eta, \gamma) = 0$ if and only if $\gamma = g(\eta)$ for all $\eta \in U_0$ (see [Lang 1993] for details).

Let $U := U_0 \cap L^0$ and $V := g(U_0) \cap F^0$. Let $\eta \in U$ and $\gamma := g(\eta)$. By the previous section, we have $h(\eta, \gamma) = 0$. If we take U_0 small enough, $\Delta_L + d_L^* R(\eta, \cdot) + d_L^* \eta$ is a quasilinear elliptic operator. By elliptic regularity, γ is smooth. Hence $\gamma \in V$. Hence this completes the proof. □

We can now give the proof of Theorem 4.2.

Proof of Theorem 4.2. Let $\eta \in L^1$ be a Maurer–Cartan element and $|\eta|_k \ll 1$. By Proposition 4.4, we only have to prove the theorem for $d_{T(E)}^* \eta \neq 0$. By Proposition 4.6, we have a $\gamma = (\upsilon, \xi) \in L^0$ such that

$$d_L^* \eta + d_L^* d_L \gamma + d_L^* R(\eta, \gamma) = 0.$$

Let $\Phi := P_\xi \circ \exp(\upsilon)$. We can induce a structure of holomorphic-Higgs triple on (X, E, θ) that is isomorphic to $(X_\eta, E_\eta, \theta_\eta)$ by Φ and f_ξ. We denote the corresponding Maurer–Cartan element as η_γ. By Proposition 4.5, we have

$$\eta_\gamma = \eta + d_L\gamma + R(\eta, \gamma).$$

We can easily see that $d_L^* \eta_\gamma = 0$. Hence by Proposition 4.4, we have some $t \in \mathcal{S}$ such that $(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)}) \cong (X_\eta, E_\eta, \theta_\eta)$. This completes the proof. □

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