Pacific Journal of Mathematics

NONCOMMUTATIVE TENSOR TRIANGULAR GEOMETRY: CLASSIFICATION VIA NOETHERIAN SPECTRA

JAMES ROWE

Volume 330 No. 2 June 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.355

NONCOMMUTATIVE TENSOR TRIANGULAR GEOMETRY: CLASSIFICATION VIA NOETHERIAN SPECTRA

JAMES ROWE

Given a monoidal triangulated category T with Noetherian spectrum, we show that there is an order-preserving bijection between the collection of all Thomason subsets of the noncommutative spectrum of T and the collection of all thick two-sided semiprime ideals of T and that it is universal among all such spaces classifying the ideals in question.

1. Introduction

One application of tensor triangular geometry (tt-geometry) is the classification of various types of subcategories via topological spaces. Following specific classification results in fields such as algebraic geometry (for example, [Thomason 1997]) and topology [Hopkins and Smith 1998], Balmer [2005] proved generic classification theorems within the framework of tt-geometry. Since then, the theory of classification within tt-geometry has expanded to include techniques such as the use of residue functors [Balmer and Favi 2011] and categorical actions [Stevenson 2013], further increasing the range of subcategories for which classification is possible.

This article is concerned with the theory of *monoidal* triangulated categories, or *mt-categories*. Within these categories, the tensor product functor $\otimes$ is no longer required to be symmetric. The general theory of these categories has been of significant interest in recent times, with foundational results established by Nakano, Vashaw, and Yakimov [Nakano et al. 2022a]. These general results include a classification theorem applicable to a wide collection of weak support data [Nakano et al. 2022a, 6.2.1], as well as characterisations of those categories in which the tensor product functor interacts well with support data [Nakano et al. 2022b, 3.1.1].

The classification theorems in both the symmetric and monoidal settings use the connections between a mt-category T and its collection of prime ideals $\mathrm{Spc}(\mathsf{T})$ considered as a topological space. Many of the classifications of subcategories rely on controlling the properties of this space or the prime ideals from which it is

MSC2020: 18G80, 18M05.

Keywords: noncommutative tensor triangular geometry, monoidal triangulated categories, Noetherian spectra.

formed. For example, in the recent work of Mallick and Ray [2023], radical thick tensor ideals of mt-categories are classified using a point-free approach, under the assumption that every prime ideal is completely prime: given a prime ideal $\mathcal{P}$, if there are objects a, b such that $a \otimes b \in \mathcal{P}$, then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

At the level of topological spaces, classification theorems often assume that the spaces used to classify subcategories are *Noetherian* topological spaces. Such spaces are commonly encountered in algebraic geometry: given a commutative Noetherian ring R, the spectrum $\mathrm{Spec}(R)$ is a Noetherian topological space. This is the case in results from both the monoidal setting [Nakano et al. 2022a, 6.2.1], and the symmetric setting (via classifying support data [Balmer 2005, 5.2]), as well as in the formulation of visible points [Balmer and Favi 2011, 7.14].

We follow the original work of Balmer [2005] in the symmetric case and prove analogous statements in the monoidal setting for those mt-categories with Noetherian spectrum, Within this framework we obtain the following classification:

Theorem (Theorem 4.6). *Let* T *be a monoidal triangulated category such that the Balmer spectrum* $\mathrm{Spc}(\mathsf{T})$ *is a Noetherian topological space. Then there is an order-preserving bijection between the collection of all Thomason subsets of the spectrum* $\mathrm{Spc}(\mathsf{T})$ *and the collection of all thick two-sided semiprime ideals of* T.

We also show that if we are given a classification via some other well-behaved support datum (X, σ) then the universal map becomes a homeomorphism.

Theorem (Theorem 5.7). *Suppose* (X, σ) *is a support datum which classifies all thick two-sided semiprime tensor ideals of* T *and the space* X *is Noetherian and* T_0. *Then the universal map* $f_\sigma : X \to \mathrm{Spc}(\mathsf{T})$ *is a homeomorphism.*

In proving these results, we also show that under the Noetherian assumption, the noncommutative spectrum $\mathrm{Spc}(\mathsf{T})$ is a spectral space (Theorem 3.7). These results are all obtained in the *small* setting; we only use thick subcategories and do not require the use of larger structure such as localising subcategories.

2. Preliminaries on mt-categories

Let T be a triangulated category. Many examples in the literature, such as derived categories of commutative rings and the stable homotopy category of spectra, can be equipped with a tensor product functor that interacts well with the triangulated structure of the category. In the case where the tensor product functor is symmetric, these categories are referred to as *tensor triangulated categories* (or tt-categories), while in the general setting where the tensor product need not be symmetric such categories are referred to as *monoidal triangulated categories* (or mt-categories). We will review several of the key definitions and main results from [Nakano et al. 2022a] that will be used throughout this paper.

Definition 2.1. An *essentially small monoidal triangulated category* is a triple of the form $(\mathsf{T}, \otimes, \mathbf{1})$, where T is an essentially small triangulated category and $(\otimes, \mathbf{1})$ is a monoidal structure on T such that $\otimes$ is an exact functor in each variable.

In other words, it is a tensor triangulated category, but the tensor product operation need not be symmetric.

Investigations into both tt-categories and mt-categories often include some specification of the size of the categories to be studied. We may consider a category to be essentially small, and focus only on types of *thick subcategories*, as in [Balmer 2005]. Alternatively, we may consider categories that have a greater level of structure, namely those categories that are rigidly-compactly generated in which the collection of compact objects are also carry the structure of a tt-category or mt-category. This approach via *big* tt-categories is used in work such as [Balmer and Favi 2011] and [Stevenson 2013], and can be used to investigate a variety of different types of subcategory. This is also the approach used in the classification of [Nakano et al. 2022a] in the monoidal setting. We have chosen to use only the essentially small approach, obtaining our results without appealing to this greater level of structure. Although we only obtain a full classification for those cases in which the spectrum is Noetherian, this aligns with the original result of Balmer [2005] that thick subcategories should be controlled by the smaller data of the original category without any higher structure.

With our notion of an mt-category in place, we can now fix the types of subcategory we would like to classify. Aligning with the monoidal classification of [Nakano et al. 2022a] previously discussed, we are aiming to classify certain *semiprime ideals* via the collection of *prime ideals*. Just as when moving from the study of commutative rings to noncommutative rings, the definition of a prime ideal must change as we move from tt-geometry to mt-geometry. We make this precise below:

Definition 2.2 [Nakano et al. 2022a, 1.2]. Let T be an essentially small mt-category.

(1) A *thick two-sided ideal* of T is a thick subcategory closed under left and right tensoring with arbitrary objects of T.

(2) A *prime ideal* of T is a proper thick ideal $\mathcal{P}$ such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ implies $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$ for all thick two-sided ideals $\mathcal{I}$ and $\mathcal{J}$ or T.

(3) A *semiprime* ideal of T is an intersection of prime ideals of T.

(4) A *completely prime* ideal of T is a proper thick ideal $\mathcal{P}$ such that $A \otimes B \in \mathcal{P}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$ for all objects $A, B \in \mathsf{T}$.

In other words, when we talk about prime and semiprime ideals in this setting of mt-geometry, the notion aligns with the usual definitions from noncommutative algebra. While Balmer [2005] classified radical thick tensor ideals (necessarily two-sided) in the symmetric case, we will classify all two-sided thick semiprime ideals. In both cases, the classification is with respect to the collection of prime ideals, considered as a topological space.

Definition 2.3 [Nakano et al. 2022a]. The *noncommutative Balmer spectrum* of an essentially small mt-category T is the set of prime ideals of T. We denote the noncommutative spectrum by $\operatorname{Spc}(\mathsf{T})$. The spectrum can be equipped with a topology generated by the collection of closed subsets

$$V(\mathcal{S}) = \{\mathcal{P} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{P} \cap \mathcal{S} = \varnothing\}$$

for all subsets $\mathcal{S}$ of T.

In the symmetric case the Balmer spectrum is indeed a spectral space (see Definition 3.1) in the sense of Hochster [1969] (see for instance [Balmer 2005] for a direct proof and [Buan et al. 2007, 6.7] for a proof via lattices). The connection to algebraic geometry can be concretely realised via classifications such as the celebrated result of Thomason [1997], which in the framework of tt-geometry results in a homeomorphism

$$X \xrightarrow{\sim} \operatorname{Spc}(\mathrm{D}^{\mathsf{perf}}(X)),$$

where X is a topologically Noetherian scheme and $\mathrm{D}^{\mathsf{perf}}(X)$ is the derived category of perfect complexes over X, equipped with the usual tensor product $\otimes = \otimes^{L}_{\mathcal{O}_X}$ [Balmer 2005, 5.4, 5.6].

The other key element of the classifications in both the symmetric and monoidal settings is the notion of a *support datum*. There is a natural support datum associated to the noncommutative Balmer spectrum:

Definition 2.4 [Nakano et al. 2022a]. For an essentially small mt-category T, the *small noncommutative support* of an object t is given by

$$\operatorname{supp}(t) = \{\mathcal{P} \in \operatorname{Spc}(\mathsf{T}) \mid t \notin \mathcal{P}\}.$$

Note that this is just the restriction to objects of the map V of Definition 2.3.

This noncommutative support carries many desirable properties, interacting with distinguished triangles and sums in the same way as the commutative version of the support. The key distinguishing difference from the symmetric case occurs when taking the intersection of supports, which is investigated further in [Nakano et al. 2022b]. We formally set out the properties below:

Lemma 2.5 [Nakano et al. 2022a, 4.1.2]. *The small noncommutative support satisfies the following properties*:

(1) $\operatorname{supp}(0) = \varnothing$ *and* $\operatorname{supp}(\mathbf{1}) = \operatorname{Spc}(\mathsf{T})$.

(2) $\operatorname{supp}(t \oplus s) = \operatorname{supp}(t) \cup \operatorname{supp}(s)$ *for all* $t, s \in \mathsf{T}$.

(3) $\operatorname{supp}(\Sigma t) = \operatorname{supp}(t)$.

(4) *If* $t \to s \to r \to \Sigma t$ *is a distinguished triangle*, *then* $\mathrm{supp}(t) \subseteq \mathrm{supp}(s) \cup \mathrm{supp}(r)$.

(5) $\bigcup_{r \in \mathsf{T}} \mathrm{supp}(t \otimes r \otimes s) = \mathrm{supp}(t) \cap \mathrm{supp}(s)$ *for all* $t, s \in \mathsf{T}$.

(6) *For all* $t \in \mathsf{T}$ *the subset* $\mathrm{supp}(t)$ *is closed.*

When working with examples, it is not always the case that one can immediately compute a classification directly via the Balmer spectrum. One of the key features of the results in both the symmetric setting [Balmer 2005, 5.2] and the monoidal setting [Nakano et al. 2022a, 6.2.1] is that if a classification can be obtained with a Noetherian topological space equipped with a function that behaves *like* the Balmer support (with potentially some additional hypotheses), then this space must have been homeomorphic to the Balmer spectrum all along. To that end, the general notion of a *support datum* is introduced:

Definition 2.6 [Nakano et al. 2022a, 4.1.1]. A support datum on a mt-category T is a pair (X, σ), where X is a topological space and σ is an assignment $\sigma : \mathsf{T} \to \mathcal{X}$, where $\mathcal{X}$ is the collection of all closed subsets of X, such that σ satisfies the following additional properties:

(1) $\sigma(0) = \varnothing$ and $\sigma(\mathbf{1}) = X$.

(2) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$ for all $a, b \in \mathsf{T}$.

(3) $\sigma(\Sigma a) = \sigma(a)$ for all $a \in \mathsf{T}$.

(4) If $a \to b \to c \to \Sigma a$ is a distinguished triangle in T, then $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$.

(5) $\bigcup_{c \in \mathsf{T}} \sigma(a \otimes c \otimes b) = \sigma(a) \cap \sigma(b)$.

Note that in [Nakano et al. 2022a] it is not required that support data take values in *closed subsets*.

The property on the intersection of pairs of supports can be extended to finite intersections via a simple inductive argument:

Lemma 2.7. *Given a support datum* (X, σ) *and a finite collection of objects* $r_1, r_2, \ldots, r_n$, *we have*

$$\bigcap_{i=1}^{n} \sigma(r_i) = \bigcup_{c_1, c_2, \ldots, c_{n-1} \in \mathsf{T}} \sigma(r_1 \otimes c_1 \otimes r_2 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes r_n).$$

Proof. We proceed by induction, where the base case $\sigma(r_1) \cap \sigma(r_2)$ is satisfied by Definition 2.6. Suppose the result holds for the $n-1$ case. Then

$$\bigcap_{i=1}^{n} \sigma(r_i) = \left(\bigcap_{i=1}^{n-1} \sigma(r_i) \right) \cap \sigma(r_n)$$

$$= \left(\bigcup_{c_1, c_2, \ldots, c_{n-2} \in \mathsf{T}} \sigma(r_1 \otimes c_1 \otimes \cdots \otimes c_{n-2} \otimes r_{n-1}) \right) \cap \sigma(r_n)$$
$$= \bigcup_{c_1, c_2, \ldots, c_{n-2} \in \mathsf{T}} \left(\sigma(r_1 \otimes c_1 \otimes \cdots \otimes c_{n-2} \otimes r_{n-1}) \cap \sigma(r_n) \right)$$
$$= \bigcup_{c_1, c_2, \ldots, c_{n-2} \in \mathsf{T}} \bigcup_{c_{n-1} \in \mathsf{T}} \sigma(r_1 \otimes c_1 \otimes r_2 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes r_n)$$
$$= \bigcup_{c_1, c_2, \ldots, c_{n-1} \in \mathsf{T}} \sigma(r_1 \otimes c_1 \otimes r_2 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes r_n). \qquad \square$$

The homeomorphism alluded to between the spectrum and other spaces that classify ideals is realised by the fact that amongst all support data, the natural support associated to the Balmer spectrum is universal.

Theorem 2.8 [Nakano et al. 2022a, 4.2.2]. *Let (X, σ) be a support datum on T such that $\sigma(t)$ is closed for every object $t \in \mathsf{T}$. Then there is a unique continuous map $f_\sigma : X \to \mathrm{Spc}(\mathsf{T})$ satisfying $\sigma(t) = f_\sigma^{-1}(\mathrm{supp}(t))$ for all $t \in \mathsf{T}$. In other words, $(\mathrm{Spc}(\mathsf{T}), \mathrm{supp})$ is the final support datum among all such support data. The map f_σ is given by*

$$f_\sigma(x) = \{t \in \mathsf{T} \mid x \notin \sigma(t)\}.$$

The definitions and results presented so far contain a mixture of properties defined idealwise (such as prime ideals) and properties that are defined objectwise (such as the properties of support data). Just as in the case of noncommutative rings, prime and semiprime ideals can also be characterised objectwise, allowing convenient translation between the two concepts.

Theorem 2.9 [Nakano et al. 2022a, 1.2.1]. *Let T be an essentially small mt-category. Then the following hold*:

(1) *A proper thick ideal $\mathcal{P}$ of T is prime if and only if, given objects $A, B \in \mathsf{T}$, we have $A \otimes C \otimes B \in \mathcal{P}$ for all $C \in \mathsf{T}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$.*

(2) *A proper thick ideal $\mathcal{P}$ of T is semiprime if and only if, given $A \in \mathsf{T}$, we have $A \otimes C \otimes A \in \mathcal{P}$ for all $C \in \mathsf{T}$ implies $A \in \mathcal{P}$.*

(3) *The noncommutative Balmer spectrum* $\mathrm{Spc}(\mathsf{T})$ *is always nonempty.*

All of the results reviewed so far hold for all mt-categories, irrespective of the properties of their noncommutative Balmer spectrum. Our objective for the remainder of the paper is to demonstrate that for those mt-categories T with Noetherian spectrum $\mathrm{Spc}(\mathsf{T})$, the noncommutative Balmer spectrum is a spectral space classifying all two-sided thick semiprime ideals, and that any Noetherian and T_0 space classifying these ideals via support data is homeomorphic to the Balmer spectrum.

3. Noetherian noncommutative spectra are spectral

Given a commutative ring R, the topological properties of the spectrum $\mathrm{Spec}(R)$ have been the subject of considerable study, particularly in the work of Hochster [1967; 1969], in which the topological spaces that share various properties with ring spectra are classified. To this end, the definition of a spectral space is introduced:

Definition 3.1 [Hochster 1969]. Let X be a topological space and let $K(X)$ denote the set of all quasicompact open subsets of X. The topological space X is *spectral* if it satisfies all of the following conditions:

(1) X is quasicompact and T_0. By T_0 we mean that given any two points $x, y \in X$ there is an open subset of X containing one of these points, but not the other.

(2) $K(X)$ is a basis of open subsets for X.

(3) $K(X)$ is closed under finite intersections.

(4) X is a sober space. That is, every irreducible closed subset of X has a necessarily unique generic point.

Given a commutative ring R, the usual spectrum $\mathrm{Spec}(R)$ is spectral. Moreover, Hochster [1967] proved in his original thesis that given any spectral space X, there exists a commutative ring R such that X is homeomorphic to $\mathrm{Spec}(R)$. The topological properties possessed by spectral spaces are essential to various elements of foundational classification results [Balmer 2005, 5.1, 5.2; Nakano et al. 2022a, 6.2.1].

Although when using general classifying support data we still need to assume the presence of a Noetherian spectral space, in the specific case of the noncommutative Balmer spectrum, we will show that the Noetherian assumption is sufficient. Specifically, we will show that if an mt-category T has Noetherian spectrum $\mathrm{Spc}(\mathsf{T})$, then the spectrum $\mathrm{Spc}(\mathsf{T})$ is a spectral space.

We begin by translating some of the topological properties of the Balmer spectrum investigated in [Balmer 2005] into the monoidal setting.

Let T be an essentially small monoidal triangulated category with spectrum $\mathrm{Spc}(\mathsf{T})$. We aim to show that $\mathrm{Spc}(\mathsf{T})$ is a spectral topological space. As seen in Definition 2.3, a basis of closed sets is given by $\{V(\mathcal{S}) \mid \mathcal{S} \subseteq \mathsf{T}\}$, where

$$V(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{P} \cap \mathcal{S} = \varnothing\}.$$

The corresponding basis of open sets is $\{U(\mathcal{S}) \mid \mathcal{S} \subseteq \mathsf{T}\}$, where

$$U(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{P} \cap \mathcal{S} \neq \varnothing\}.$$

Given an object $s \in \mathsf{T}$, we will simplify the above notation and write $V(s)$ for the basic closed set $V(\{s\})$ and write $U(s)$ for the basic open set $U(\{s\})$.

Immediately we have $V(\mathcal{S}) = \bigcap_{s\in\mathcal{S}} V(s)$ and $U(\mathcal{S}) = \bigcup_{s\in\mathcal{S}} U(s)$. Therefore the sets of the form $V(s) = \operatorname{supp}(s)$ form a basis of closed sets for the topology by Lemma 2.5, while the sets of the form $U(s)$ form a basis of open sets for the topology.

For a collection of objects $\mathcal{E}$ define $\operatorname{supp}(\mathcal{E}) = \bigcup_{s\in\mathcal{E}} \operatorname{supp}(s)$.

Lemma 3.2. *Let $Y \subseteq \operatorname{Spc}(\mathsf{T})$. Then the closure of Y is given by*

$$\bar{Y} = \bigcap_{Y\subseteq \operatorname{supp}(t)} \operatorname{supp}(t).$$

Proof. This follows immediately from the fact that the sets of the form $\operatorname{supp}(s)$ are a basis of closed sets for the topology on $\operatorname{Spc}(\mathsf{T})$. □

Proposition 3.3. *For any point $\mathcal{P} \in \operatorname{Spc}(\mathsf{T})$, the closure of $\mathcal{P}$ is given by*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

In particular, if $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$ then $\mathcal{P}_1 = \mathcal{P}_2$. That is, the space $\operatorname{Spc}(\mathsf{T})$ is T_0.

Proof. The proof is identical to [Balmer 2005, 2.9]. Fix a prime ideal $\mathcal{P}$. Consider the set $\mathcal{S}_0 = \mathsf{T} \setminus \mathcal{P}$ and the associated basic closed subset $V(\mathcal{S}_0) = \{\mathcal{Q} \mid \mathcal{Q} \cap \mathcal{S}_0 = \varnothing\}$. Clearly $\mathcal{P} \in V(\mathcal{S}_0)$. If there is a subset $\mathcal{S} \subseteq \mathsf{T}$ such that $\mathcal{P} \in V(\mathcal{S})$, then $\mathcal{S} \subseteq \mathcal{S}_0$ and so $V(\mathcal{S}_0) \subseteq V(\mathcal{S})$. Therefore $V(\mathcal{S}_0)$ is the smallest closed subset containing $\mathcal{P}$ and is the closure of $\mathcal{P}$. We have

$$\overline{\{\mathcal{P}\}} = V(\mathcal{S}_0) = \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

The fact that $\operatorname{Spc}(\mathsf{T})$ is T_0 follows immediately. □

We will make use of the following theorem from [Nakano et al. 2022a], which is the nonsymmetric version of [Balmer 2005, 2.2].

Theorem 3.4 [Nakano et al. 2022a, 3.2.3]. *Suppose that $\mathcal{M}$ is a multiplicative subset of T and suppose $\mathcal{I}$ is a proper thick two-sided tensor ideal of T such that $\mathcal{I} \cap \mathcal{M} = \varnothing$. The set*

$$X(\mathcal{M}, \mathcal{I}) = \{\mathcal{J} \text{ a thick two-sided tensor ideal of } \mathsf{T} \mid \mathcal{I} \subseteq \mathcal{J} \text{ and } \mathcal{J} \cap \mathcal{M} = \varnothing\}$$

has a maximal element, and moreover this maximal element is prime.

Proposition 3.5. *Nonempty irreducible subsets of $\operatorname{Spc}(\mathsf{T})$ have unique generic points. That is, the noncommutative Balmer spectrum is always a sober space. Indeed for a nonempty closed subset $Z \subseteq \operatorname{Spc}(\mathsf{T})$ the following are equivalent:*

(1) *Z is irreducible.*

(2) *For all $t, s \in \mathsf{T}$, if $U(t \oplus s) \cap Z = \varnothing$, then $U(t) \cap Z = \varnothing$ or $U(s) \cap Z = \varnothing$.*

(3) *The collection $\mathcal{P} = \{t \in \mathsf{T} \mid U(t) \cap Z \neq \varnothing\}$ is a thick prime $\otimes$-ideal.*

Moreover, when these conditions hold, $Z = \overline{\{\mathcal{P}\}}$.

Proof. The proof is very similar to [Balmer 2005, 2.18], although some extra care is needed when proving certain ideals are prime. We have already seen that $\mathrm{Spc}(\mathsf{T})$ is T_0 and so uniqueness of generic points is immediate.

(1) $\implies$ (2): Z irreducible means that for any open subsets $U_1, U_2 \in \mathrm{Spc}(\mathsf{T})$, if $Z \cap U_1 \cap U_2 = \varnothing$ then $Z \cap U_1 = \varnothing$ or $Z \cap U_2 = \varnothing$. This gives (2), since $U(t \oplus s) = U(t) \cap U(s)$.

(2) $\implies$ (3): This will be slightly more involved than the proof in [Balmer 2005]. Condition (2) gives $t, s \in \mathcal{P}$ implies $t \oplus s \in \mathcal{P}$. Using this, we see that if $t, s \in \mathcal{P}$ and $t \to s \to r \to \Sigma t$ is a distinguished triangle, then $r \in \mathrm{thick}^{\otimes}(t \oplus s)$, hence $U(t \oplus s) \subseteq U(r)$ and since $U(t \oplus s) \cap Z \neq \varnothing$, we get $U(r) \cap Z \neq \varnothing$, and so $r \in \mathcal{P}$.

The fact that $\mathcal{P}$ is closed under summands is immediate as $U(t \oplus s) \cap Z \neq \varnothing$ implies $(U(t) \cap U(s)) \cap Z \neq \varnothing$ and therefore $U(t) \cap Z \neq \varnothing$ and $U(s) \cap Z \neq \varnothing$. Therefore, both t and s are objects in $\mathcal{P}$.

It remains to show that $\mathcal{P}$ is a two-sided ideal, and that it is prime. Fix $t \in \mathcal{P}$ and $x \in \mathsf{T}$. We have

$$\begin{aligned} \varnothing &\neq Z \cap U(t) \\ &\subseteq Z \cap (U(t) \cup U(x)) \\ &= Z \cap \Big(\bigcap_{s \in \mathsf{T}} U(t \otimes s \otimes x) \Big) \quad \text{(by Lemma 2.5)} \\ &\subseteq Z \cap U(t \otimes x). \end{aligned}$$

Therefore, $t \otimes x \in \mathcal{P}$. An almost identical argument shows that $x \otimes t \in \mathcal{P}$ and so $\mathcal{P}$ is indeed a two-sided ideal. Now we deal with primeness. Let $\mathcal{I}, \mathcal{J}$ be thick $\otimes$-ideals such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$. In particular, for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$ we have $U(i \otimes j) \cap Z \neq \varnothing$. Now,

$$\bigcup_{i \in \mathcal{I}, j \in \mathcal{J}} \mathrm{supp}(i \otimes j) = \bigcup_{i \in \mathcal{I}} \mathrm{supp}(i) \cap \bigcup_{j \in \mathcal{J}} \mathrm{supp}(j);$$

see for example [Nakano et al. 2022a, 4.4.2]. Therefore,

$$\bigcap_{i \in \mathcal{I}, j \in \mathcal{J}} U(i \otimes j) = \bigcap_{i \in \mathcal{I}} U(i) \cup \bigcap_{j \in \mathcal{J}} U(j).$$

As we assumed $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ we must have $\mathcal{P} \in U(i \otimes j)$ for all i and j. Therefore $\mathcal{P} \in U(i)$ for all i or $\mathcal{P} \in U(j)$ for all j, which is equivalent to asking that $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$, and so we conclude that $\mathcal{P}$ is indeed prime.

(3) $\implies$ (1): We prove that $Z = \overline{\{\mathcal{P}\}}$, which proves (1) and the final statement of the proposition. Let $\mathcal{Q} \in Z$. For $a \in \mathcal{Q}$, we have $\mathcal{Q} \in U(a) \cap Z \neq \varnothing$, and hence $a \in \mathcal{P}$. We have proved $\mathcal{Q} \subseteq \mathcal{P}$, that is, $\mathcal{Q} \in \overline{\{\mathcal{P}\}}$ by Proposition 3.3 for any $\mathcal{Q} \in Z$, and

so $Z \subseteq \overline{\{\mathcal{P}\}}$. Conversely, it suffices to prove $\mathcal{P} \in Z$. To see this, let $s \in \mathsf{T}$ be an object such that $Z \subseteq \operatorname{supp}(s)$. Such objects exist by Lemma 3.2. Then $U(s) \cap Z = \varnothing$, which means $s \notin \mathcal{P}$ or, equivalently, $\mathcal{P} \in \operatorname{supp}(s)$. Therefore, by Lemma 3.2,

$$\mathcal{P} \in \bigcap_{Z \subseteq \operatorname{supp}(s)} \operatorname{supp}(s) = \bar{Z} = Z. \qquad \square$$

So far, all of the topological properties proved hold for the spectrum $\operatorname{Spc}(\mathsf{T})$ of any mt-category T. We will now introduce the Noetherian condition.

Definition 3.6. A topological space X is Noetherian if any of the following equivalent conditions hold:

(1) X satisfies the descending chain condition for closed subsets. That is, for any sequence

$$Y_1 \supseteq Y_2 \supseteq \cdots$$

of closed subsets Y_i of X, there exists an integer m such that for all integers $n \geq m$ we have $Y_m = Y_n$.

(2) Every subspace of X is quasicompact.

(3) Every open subset of X is quasicompact.

By assuming that $\operatorname{Spc}(\mathsf{T})$ is Noetherian, it immediately follows that $\operatorname{Spc}(\mathsf{T})$ is quasicompact, as are all of the basic open subsets $U(\mathcal{S})$, including those of the form $U(t)$ for all objects $t \in \mathsf{T}$.

Theorem 3.7. *If* $\operatorname{Spc}(\mathsf{T})$ *is a Noetherian topological space*, *then it is a spectral space.*

Proof. We verify the conditions required to be spectral.

(1) As $\operatorname{Spc}(\mathsf{T})$ is Noetherian, it is quasicompact. The space is T_0 by Proposition 3.3.

(2) Under the Noetherian assumption, every open subset is quasicompact and so it is immediate that the collection $K(\operatorname{Spc}(\mathsf{T}))$ of all quasicompact open subsets is a basis for $\operatorname{Spc}(\mathsf{T})$.

(3) Given quasicompact basic open sets of the form $U(t)$ and $U(s)$ we have that $U(s) \cap U(t) = U(s \oplus t)$. For a quasicompact basic open set of the form $U(\mathcal{S}) = \bigcup_{s \in \mathcal{S}} U(s)$, by quasicompactness there exists a finite subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $U(\mathcal{S}) = U(\mathcal{S}') = \bigcup_{s \in \mathcal{S}'} U(s)$. Given another such quasicompact basic open set $U(\mathcal{T})$, with finite refinement $\mathcal{T}'$, we obtain

$$U(\mathcal{S}) \cap U(\mathcal{T}) = \bigcup_{s \in \mathcal{S}'} U(s) \cap \bigcup_{t \in \mathcal{T}'} U(t) = \bigcup_{s \in \mathcal{S}'} \bigcup_{t \in \mathcal{T}'} (U(s) \cap U(t)) = \bigcup_{s \in \mathcal{S}'} \bigcup_{t \in \mathcal{T}'} U(s \oplus t).$$

As $\mathcal{S}'$ is a finite set, and $\mathcal{T}'$ is a finite refinement, both unions are finite. As $U(\mathcal{S})$ and $U(\mathcal{T})$ are quasicompact, it follows that the intersection is quasicompact.

(4) By Proposition 3.5, the spectrum $\mathrm{Spc}(\mathsf{T})$ is always a sober space, irrespective of Noetherianity and therefore every nonempty irreducible subset of $\mathrm{Spc}(\mathsf{T})$ has a unique generic point.

Since all of the conditions are satisfied, the spectrum $\mathrm{Spc}(\mathsf{T})$ is a spectral space. □

There are other conditions on the mt-category T which can lead to $\mathrm{Spc}(\mathsf{T})$ being spectral. For example, [Buan et al. 2007, 6.7] proves that the spectrum is spectral under the assumption that the tensor product is symmetric, or that the mt-category T has a generator.

4. Classifying thick two-sided ideals

The objective of this section is to classify all *semiprime* thick tensor ideals of a mt-category T in terms of Thomason subsets of the spectrum $\mathrm{Spc}(\mathsf{T})$, under the assumption that the Balmer spectrum is a Noetherian topological space. If the mt-category T is rigid, then the classification actually covers *all* thick two-sided tensor ideals, providing a monoidal Noetherian analogue to Balmer's original classification [2005, 4.10] in the symmetric case.

As with the previous section, we will first obtain results on tensor ideals in general mt-categories, without assuming that the Balmer spectrum is Noetherian.

Lemma 4.1. *Given a collection of objects $\mathcal{E} \subseteq \mathsf{T}$ there is an equality*

$$\mathrm{supp}(\mathcal{E}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{E} \nsubseteq \mathcal{P}\}.$$

Proof. The proof is identical to [Balmer 2005, 4.6]. We have $\mathcal{P} \in \mathrm{supp}(\mathcal{E})$ if and only if there exists an object $a \in \mathcal{E}$ such that $\mathcal{P} \in \mathrm{supp}(a)$ which means $a \notin \mathcal{P}$, by definition of the support. □

Definition 4.2. Let $\mathcal{J}$ be a thick tensor ideal of T. We denote by $\sqrt{\mathcal{J}}$ the semiprime ideal

$$\sqrt{\mathcal{J}} = \bigcap_{\mathcal{J} \subseteq \mathcal{P} \in \mathrm{Spc}(\mathsf{T})} \mathcal{P}.$$

Definition 4.3. Let $Y \subseteq \mathrm{Spc}(\mathsf{T})$ be a subset. Define the full subcategory T_Y by

$$\mathsf{T}_Y = \{t \in \mathsf{T} \mid \mathrm{supp}(t) \subseteq Y\}.$$

Lemma 4.4. (1) *The subcategory T_Y is a thick two-sided tensor ideal.*

(2) *There is an equality*

$$\mathsf{T}_Y = \bigcap_{\mathcal{P} \notin Y} \mathcal{P} \quad \textit{where } \mathcal{P} \in \mathrm{Spc}(\mathsf{T}).$$

Proof. (1) The statement is similar to [Nakano et al. 2022a, 6.1.1]. The fact that T_Y is a thick subcategory follows immediately from the usual properties of support.

Now let $s \in \mathsf{T}_Y$ and $t \in \mathsf{T}$. Then

$$\operatorname{supp}(s \otimes t) = \operatorname{supp}(s \otimes \mathbf{1} \otimes t) \subseteq \bigcup_{c \in \mathsf{T}} \operatorname{supp}(s \otimes c \otimes t) = \operatorname{supp}(s) \cap \operatorname{supp}(t) \subseteq Y.$$

Therefore, $\operatorname{supp}(s \otimes t) \subseteq Y$ and $s \otimes t \in \mathsf{T}_Y$. That is, T_Y is a right ideal. A similar argument shows that T_Y is a left ideal.

(2) The proof is identical to [Balmer 2005, 4.8]. For an object $t \in \mathsf{T}$, we have $t \in \mathsf{T}_Y$ if and only if $\operatorname{supp}(t) \subseteq Y$. Therefore, for all $\mathcal{P} \in \operatorname{Spc}(\mathsf{T}) \setminus Y$, $t \in \mathsf{T}_Y$ if and only if $\mathcal{P} \notin \operatorname{supp}(t)$ and $t \notin \mathcal{P}$. Hence, $t \in \bigcap_{\mathcal{P} \notin Y} \mathcal{P}$ and the conclusion holds. □

Note that if (X, σ) is a support datum on T then the above lemma can be adjusted to show that the full subcategory $\{t \in \mathsf{T} \mid \sigma(t) \subseteq Y\}$ is a thick two-sided ideal.

Proposition 4.5. *Let $\mathcal{J}$ be a thick tensor ideal of T. Then*

$$\mathsf{T}_{\operatorname{supp}(\mathcal{J})} = \sqrt{\mathcal{J}}.$$

Proof. The proof is identical to [Balmer 2005, 4.9]. By Lemma 4.4, we have

$$\mathsf{T}_{\operatorname{supp}(\mathcal{J})} = \bigcap_{\mathcal{P} \notin \operatorname{supp}(\mathcal{J})} \mathcal{P}.$$

Applying Lemma 4.1, gives $\operatorname{supp}(\mathcal{J}) = \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{J} \not\subseteq \mathcal{Q}\}$ and so

$$\mathsf{T}_{\operatorname{supp}(\mathcal{J})} = \bigcap_{\mathcal{P} \notin \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{J} \not\subseteq \mathcal{Q}\}} \mathcal{P}.$$

The result then immediately follows from the definition of $\sqrt{\mathcal{J}}$. □

We denote by $\mathcal{T}$ the collection of all Thomason subsets of $\operatorname{Spc}(\mathsf{T})$. Recall that a subset $Y \subseteq \operatorname{Spc}(\mathsf{T})$ is Thomason if $Y = \bigcup Y_i$ such that each Y_i is closed and the open complement $\operatorname{Spc}(\mathsf{T}) \setminus Y_i$ is quasicompact. We denote by $\mathcal{S}$ the collection of all semiprime ideals of T.

With the general results in position, we now consider the case in which the Balmer spectrum is Noetherian and obtain the classification result.

Theorem 4.6. *Let T be a monoidal triangulated category such that the Balmer spectrum* $\operatorname{Spc}(\mathsf{T})$ *is a Noetherian topological space. Let $\mathcal{T}$ denote the collection of all Thomason subsets of* $\operatorname{Spc}(\mathsf{T})$ *and let $\mathcal{S}$ denote the collection of all semiprime ideals of T. Then there is an order-preserving bijection $\mathcal{T} \xrightarrow{\sim} \mathcal{S}$ given by*

$$Y \to \mathsf{T}_Y$$

whose inverse is

$$\mathcal{J} \to \operatorname{supp}(\mathcal{J}).$$

Proof. The first map is well defined because T_Y is semiprime by Lemma 4.4. The second map is well defined as the complement of an object's support is quasicompact under the Noetherian assumption. Both maps are clearly inclusion preserving. It remains to show the maps are mutually inverse.

Given a semiprime ideal $\mathcal{J}$ the composite $\mathsf{T}_{\operatorname{supp}(\mathcal{J})}$ is equal to $\sqrt{\mathcal{J}}$ by Proposition 4.5. By assumption, $\mathcal{J}$ is semiprime and so $\mathcal{J} = \sqrt{\mathcal{J}}$, and so the composite

$$\mathcal{J} \to \operatorname{supp}(\mathcal{J}) \to \mathsf{T}_{\operatorname{supp}(\mathcal{J})}$$

is the identity. Now let Y be a Thomason subset of $\operatorname{Spc}(\mathsf{T})$. It remains to show that the composition

$$Y \to \mathsf{T}_Y \to \operatorname{supp}(\mathsf{T}_Y)$$

is the identity. For an object $t \in \mathsf{T}$ we have by definition $t \in \mathsf{T}_Y$ if and only if $\operatorname{supp}(t) \subseteq Y$. Therefore,

$$\operatorname{supp}(\mathsf{T}_Y) = \bigcup_{t \in \mathsf{T}_Y} \operatorname{supp}(t) \subseteq Y.$$

Now we need to show that $Y \subseteq \operatorname{supp}(\mathsf{T}_Y)$. That is, for each prime ideal $\mathcal{P} \in Y$ we must find a compact object x such that $\mathcal{P} \in \operatorname{supp}(x)$ and $\operatorname{supp}(x) \subseteq Y$. As Y is a Thomason subset of $\operatorname{Spc}(\mathsf{T})$, there exist closed subsets Y_i such that $Y = \bigcup Y_i$ and the complement of each Y_i is a quasicompact open subset U_i. Fix a prime $\mathcal{P} \in Y$. Then there exists an index i such that

$$\mathcal{P} \in Y_i = \operatorname{Spc}(\mathsf{T}) \setminus U_i.$$

By assumption, $\operatorname{Spc}(\mathsf{T})$ is Noetherian, so there exists a finite collection of objects $\{r_1, \dots, r_n\}$ such that $U_i = \bigcup_{j=1}^n U(r_j)$. Therefore

$$Y_i = \operatorname{Spc}(\mathsf{T}) \setminus U_i = \operatorname{Spc}(\mathsf{T}) \setminus \left(\bigcup_{j=1}^n U(r_j) \right) = \bigcap_{j=1}^n (\operatorname{Spc}(\mathsf{T}) \setminus U(r_j)) = \bigcap_{j=1}^n \operatorname{supp}(r_j).$$

By Lemma 2.7, there exists compact objects $c_1, \dots, c_{n-1}$ such that for

$$x = r_1 \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes r_n,$$

we have $\mathcal{P} \in \operatorname{supp}(x)$. Moreover,

$$\operatorname{supp}(x) \subseteq \bigcap_{j=1}^n \operatorname{supp}(r_j) = Y_i \subseteq Y$$

and so $x \in \mathsf{T}_Y$, thus completing the proof. □

Proposition 4.7 [Nakano et al. 2022b, 4.1.1]. *Suppose* T *is rigid*, *so that every object is either left or right dualisable. Then every thick two-sided tensor ideal is semiprime.*

Corollary 4.8. *Let* T *be a rigid mt-category with Noetherian spectrum* $\mathrm{Spc}(\mathsf{T})$. *Then the order-preserving bijection of Theorem 4.6 classifies all thick two-sided tensor ideals of* T.

5. Classifying support data and the universal map

We can now investigate the universality of the spectrum with respect to classifying support data. This section provides the monoidal analogue of [Balmer 2005, 5.2], and recovers the general classification result of [Nakano et al. 2022a, 6.2.1]. Note that although the statements of the results are for mt-categories, the majority of the proofs are identical to the arguments in the symmetric setting of Balmer, with only the proofs of Proposition 5.6 and Theorem 5.7 requiring alterations for the monoidal setting.

Definition 5.1. A subset $Y \subseteq X$ of a topological space X is *specialisation closed* if it is the union of closed sets, or equivalently if $y \in Y$ implies $\overline{\{y\}} \subseteq Y$. Given a topological space X we denote by $\mathcal{X}_{\mathrm{sp}}$ the collection of all specialisation closed subsets of X.

Recall that we denote the collection of all thick semiprime ideals of T by $\mathcal{S}$.

Definition 5.2. Let (X, σ) be a support datum on T. We say that (X, σ) is a *classifying support datum* if the following two conditions hold:

(1) The space X is Noetherian and spectral.

(2) We have a bijection $\Theta : \mathcal{X}_{\mathrm{sp}} \to \mathcal{S}$ defined by

$$\Theta(Y) = \{t \in \mathsf{T} \mid \sigma(t) \subseteq Y\}$$

with inverse

$$\Theta^{-1}(\mathcal{J}) = \sigma(\mathcal{J}) = \bigcup_{j \in \mathcal{J}} \sigma(j).$$

Lemma 5.3. *Suppose* (X, σ) *is a classifying support datum on* T. *Then every closed subset* $Z \subseteq X$ *is of the form* $Z = \sigma(t)$ *for some object* $t \in \mathsf{T}$.

Proof. This is the first claim of [Balmer 2005, 5.2]. The proof is identical and included for completeness. Because X is Noetherian, every closed subset has a finite number of irreducible components. Since $\sigma(t_1) \cup \sigma(t_2) \cup \cdots \cup \sigma(t_n) = \sigma(t_1 \oplus t_2 \oplus \cdots \oplus t_n)$ for any finite collection of objects in T, it therefore suffices to prove the lemma for closed sets of the form $Z = \overline{\{x\}}$ for some $x \in X$.

As (X, σ) is classifying, we have

$$\overline{\{x\}} = Z = \Theta^{-1}\Theta(Z) = \bigcup_{t \in \Theta(Z)} \sigma(t).$$

Therefore there exists $t \in \mathsf{T}$ such that $x \in \sigma(t) \subseteq Z$. Hence

$$\overline{\{x\}} \subseteq \sigma(t) \subseteq Z = \overline{\{x\}},$$

proving the lemma. □

Corollary 5.4. *Suppose* Spc(T) *is a Noetherian topological space. Then every open subset of* Spc(T) *is of the form* $U(t)$ *for some object* $t \in \mathsf{T}$.

Proof. For Spc(T) Noetherian, Theorem 4.6 tells us (Spc(T), supp) is a classifying support datum on T. Thus, given an open subset $U \subseteq \operatorname{Spc}(\mathsf{T})$, Lemma 5.3 tells us that $\operatorname{Spc}(\mathsf{T}) \setminus U = \operatorname{supp}(t)$ for some object $t \in \mathsf{T}$. Then $U = \operatorname{Spc}(\mathsf{T}) \setminus \operatorname{supp}(t) = U(t)$. Under the Noetherian assumption, $U(t)$ is quasicompact. □

Proposition 5.5. *If* (X, σ) *is a classifying support datum on* T, *then the universal map* $f_\sigma : X \to \operatorname{Spc}(\mathsf{T})$ *is injective.*

Proof. This is the same as the proof of injectivity in [Balmer 2005, 5.2] and is included for completeness. For $x \in X$ define $Y(x) = \{y \in X \mid x \notin \overline{\{y\}}\}$. Clearly $Y(x)$ is specialisation closed. Fix an object $t \in \mathsf{T}$. We will show that $\sigma(t) \subseteq Y(x)$ if and only if $x \notin \sigma(t)$. Since $x \notin Y(x)$, if $\sigma(t) \subseteq Y(x)$ then $x \notin \sigma(t)$. Conversely, as $\sigma(t)$ is specialisation closed, if $x \notin \sigma(t)$ we have $x \notin \overline{\{y\}}$ for all $y \in \sigma(t)$ and so by definition $\sigma(t) \subseteq Y(x)$. Therefore

$$\Theta(Y(x)) = \{t \in \mathsf{T} \mid \sigma(t) \subseteq Y(x)\} = \{t \in \mathsf{T} \mid x \notin \sigma(t)\} = f_\sigma(x).$$

As (X, σ) is classifying, if $f_\sigma(x_1) = f_\sigma(x_2)$ then $Y(x_1) = Y(x_2)$ and $\overline{\{x_1\}} = \overline{\{x_2\}}$. The space X must be T_0 as (X, σ) is classifying, so $\overline{\{x_1\}} = \overline{\{x_2\}}$ implies $x_1 = x_2$ and the map f_σ is injective. □

Proposition 5.6. *If* (X, σ) *is a classifying support datum on* T, *then the universal map* $f_\sigma : X \to \operatorname{Spc}(\mathsf{T})$ *is surjective.*

Proof. Fix a prime $\mathcal{P} \in \operatorname{Spc}(\mathsf{T})$. As (X, σ) is classifying, there exists a specialisation closed subset $Y \subseteq X$ such that $\mathcal{P} = \Theta(Y)$. As $\mathcal{P}$ is proper, the set $X \setminus Y$ is nonempty. Let $x, y \in X \setminus Y$. By Lemma 5.3, there exist objects $s, t \in \mathsf{T}$ such that $\overline{\{x\}} = \sigma(s)$ and $\overline{\{y\}} = \sigma(t)$. Let $\mathcal{I}$ and $\mathcal{J}$ denote the thick two-sided ideals generated by s and t, respectively. By [Nakano et al. 2022a, 4.3.2],

$$\overline{\{x\}} = \sigma(s) = \sigma(\mathcal{I}) \quad \text{and} \quad \overline{\{y\}} = \sigma(t) = \sigma(\mathcal{J}),$$

and so neither $\mathcal{I}$ nor $\mathcal{J}$ are contained in $\mathcal{P}$. As $\mathcal{P}$ is prime, $\mathcal{I} \otimes \mathcal{J} \nsubseteq \mathcal{P}$ and so $\sigma(\mathcal{I} \otimes \mathcal{J}) \nsubseteq Y$. Therefore, there exists a point $z \in X \setminus Y$ such that

$$z \in \sigma(\mathcal{I} \otimes \mathcal{J}) = \overline{\{x\}} \cap \overline{\{y\}}$$

and hence

$$\overline{\{z\}} \subseteq \overline{\{x\}} \cap \overline{\{y\}}.$$

As in [Balmer 2005, 5.2], as the space X is Noetherian, the nonempty family of sets

$$\{\overline{\{x\}} \mid x \in X \setminus Y\}$$

admits a minimal element which must be the lower bound for inclusion. That is, there exists $x \in X \setminus Y$ such that, for all $y \in X \setminus Y$, we have $x \in \overline{\{y\}}$. Hence,

$$X \setminus Y \subseteq \{y \in X \mid x \in \overline{\{y\}}\}.$$

The reverse inclusion holds because $x \notin Y$, and Y is specialisation closed. Thus,

$$X \setminus Y = \{y \in X \mid x \in \overline{\{y\}}\}$$

and so

$$Y = \{y \in X \mid x \notin \overline{\{y\}}\} = Y(x).$$

Hence,

$$\mathcal{P} = \Theta(Y) = \Theta(Y(x)) = f_\sigma(x),$$

where the final equality is demonstrated in the proof of Proposition 5.5. We conclude that f_σ is surjective. □

Theorem 5.7. *Let* (X, σ) *be a classifying support datum on* T. *Then the universal map* $f_\sigma : X \to \mathrm{Spc}(\mathsf{T})$ *is a homeomorphism.*

Proof. By Proposition 5.5 the universal continuous map f_σ is injective, and by Proposition 5.6 the map is surjective. Therefore, f_σ is bijective. By Theorem 2.8 we have $\sigma(t) = f_\sigma^{-1}(\mathrm{supp}(t))$ for all $t \in \mathsf{T}$. Consequently, $f_\sigma(\sigma(t)) = \mathrm{supp}(t)$ and so f_σ is a closed map as by Lemma 5.3 every closed subset of X is of the form $\sigma(t)$ for some $t \in \mathsf{T}$. We conclude that f_σ is a homeomorphism. □

The requirement that the topological space component of a classifying support datum be Noetherian is satisfied by many examples of interest in both symmetric and monoidal cases. These include the cases of the derived category of perfect complexes over a topologically Noetherian scheme [Thomason 1997] and the stable module category of a finite group scheme [Friedlander and Pevtsova 2007] in the symmetric case and the stable module categories of various quantum groups and Hopf algebras in the monoidal case [Nakano et al. 2022a]. It should be noted that proving that the Noetherian condition is satisfied is often an extensive process (as is the case in all of the stable module category examples mentioned). Moreover, not every mt-category possesses a Noetherian spectrum, including examples in the symmetric case of tt-categories. For example, while the stable homotopy category of finite spectra admits a classification theorem, it has a non-Noetherian spectrum (see [Hopkins and Smith 1998] for the original classification and [Balmer 2010, Section 9] for the tt-geometric context).

However, if the Noetherian condition on the given topological space is satisfied, Theorem 5.7 will guarantee that the space is homeomorphic to the Balmer spectrum.

Acknowledgements

Many thanks to Greg Stevenson and Kent Vashaw for their insightful comments and suggestions. Many thanks also to the referee for their suggested improvements. The author was supported by Engineering and Physical Sciences Research Council (EPSRC) grants EP/N509668/1 and EP/R513222/1.

References

[Balmer 2005] P. Balmer, "The spectrum of prime ideals in tensor triangulated categories", *J. Reine Angew. Math.* **588** (2005), 149–168. MR Zbl

[Balmer 2010] P. Balmer, "Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings", *Algebr. Geom. Topol.* **10**:3 (2010), 1521–1563. MR Zbl

[Balmer and Favi 2011] P. Balmer and G. Favi, "Generalized tensor idempotents and the telescope conjecture", *Proc. Lond. Math. Soc.* (3) **102**:6 (2011), 1161–1185. MR Zbl

[Buan et al. 2007] A. B. Buan, H. Krause, and Ø. Solberg, "Support varieties: an ideal approach", *Homology Homotopy Appl.* **9**:1 (2007), 45–74. MR Zbl

[Friedlander and Pevtsova 2007] E. M. Friedlander and J. Pevtsova, "Π-supports for modules for finite group schemes", *Duke Math. J.* **139**:2 (2007), 317–368. MR Zbl

[Hochster 1967] M. Hochster, *Prime ideal structure in commutative rings*, Ph.D. thesis, Princeton University, 1967, available at https://www.proquest.com/docview/302250870. MR Zbl

[Hochster 1969] M. Hochster, "Prime ideal structure in commutative rings", *Trans. Amer. Math. Soc.* **142** (1969), 43–60. MR Zbl

[Hopkins and Smith 1998] M. J. Hopkins and J. H. Smith, "Nilpotence and stable homotopy theory, II", *Ann. of Math.* (2) **148**:1 (1998), 1–49. MR Zbl

[Mallick and Ray 2023] V. M. Mallick and S. Ray, "Noncommutative tensor triangulated categories and coherent frames", *C. R. Math. Acad. Sci. Paris* **361** (2023), 1415–1427. MR Zbl

[Nakano et al. 2022a] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "Noncommutative tensor triangular geometry", *Amer. J. Math.* **144**:6 (2022), 1681–1724. MR Zbl

[Nakano et al. 2022b] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "Noncommutative tensor triangular geometry and the tensor product property for support maps", *Int. Math. Res. Not.* **2022**:22 (2022), 17766–17796. MR Zbl

[Stevenson 2013] G. Stevenson, "Support theory via actions of tensor triangulated categories", *J. Reine Angew. Math.* **681** (2013), 219–254. MR Zbl

[Thomason 1997] R. W. Thomason, "The classification of triangulated subcategories", *Compositio Math.* **105**:1 (1997), 1–27. MR Zbl

Received September 18, 2023. Revised June 6, 2024.

JAMES ROWE
SCHOOL OF MATHEMATICS & STATISTICS
UNIVERSITY OF GLASGOW
UNITED KINGDOM

james.rowe@glasgow.ac.uk

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 330 No. 2 June 2024
