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REGULARITY OF MANIFOLDS WITH INTEGRAL SCALAR CURVATURE BOUND AND ENTROPY LOWER BOUND

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We generalize the work of Lee, Naber and Neumayer on regularity of manifolds with lower-bounded scalar curvature and almost Euclidean entropy. We show the same result in the case of integral bounded scalar curvature.

In addition, we also obtain a compactness theorem and an a prior L^p scalar curvature bound estimate for $p < 1$.

1. Introduction

Local regularity is important in the study of manifolds under curvature restrictions. Cheeger [1970] demonstrated that the injectivity radius is uniformly bounded below by a positive constant for compact manifolds with bounded sectional curvature, noncollapsing volume, and bounded diameter. Gromov [1981] further showed a $C^{1,\alpha}$ harmonic radius estimate on such manifolds. These results play a crucial role in the proof of compactness and finiteness theorems (see also [Greene and Wu 1988; Kasue 1989; Peters 1987]). Anderson [1990] extended the $C^{1,\alpha}$ regularity to manifolds with bounded Ricci curvature and bounded injectivity. In the case that the manifold admits only a lower bound on Ricci curvature, Anderson and Cheeger [1992] have given a C^{α} harmonic radius estimate under an additional assumption on the injectivity radius. Moreover, Cheeger and Colding [Colding 1997; Cheeger and Colding 1997] further proved that for manifolds with almost nonnegative Ricci curvature, the unit geodesic ball is Gromov–Hausdorff close to the Euclidean ball if and only if they are close in volume. This is also true for manifolds with integral Ricci curvature lower bound, which is proved by Tian and Zhang [2016].

Regularity of manifolds with bounded scalar curvature would be much more difficult. Recently, Lee, Naber, and Neumayer [Lee et al. 2023] showed that the unit ball of a complete manifold is close to the Euclidean ball in the d_p-distance whenever the scalar curvature, as well as the Perelman ν-functional, is almost nonnegative. This gives a weaker regularity than the usual Gromov–Hausdorff closeness. This paper aims to generalize their result to the integral scalar curvature bound case.

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Recall the Perelman $\mathcal{W}$-functional (see [Perelman 2002] or [Chow et al. 2010]). For a complete manifold (M, g), $\tau > 0$, and $f \in C^\infty(M)$,

$$\mathcal{W}(g, f, \tau) = \int_M \big(\tau(R + |\nabla f|^2) + f - n\big)(4\pi\tau)^{-n/2} e^{-f}\, d\mathrm{vol}_g.$$

Letting $u = (4\pi\tau)^{-n/4} e^{-f/2}$, the Perelman $\mathcal{W}$-function can be reformulated as

$$\mathcal{W}(g, u, \tau) = \int_M \big(\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2\big)\, d\mathrm{vol}_g - \tfrac{1}{2} n \log(4\pi\tau). \tag{1.1}$$

The Perelman entropy $\mu(g, \tau)$ is given by

$$\mu(g, \tau) = \inf\left\{ \mathcal{W}(g, u, \tau) \;\Big|\; \int_M u^2\, d\mathrm{vol}_g = 1 \right\}.$$

Finally, the Perelman ν-functional is given by

$$\nu(g, \tau) = \inf\{\mu(g, \tau') : \tau' \in (0, \tau)\},$$

which satisfies the rescaling invariance $\nu(\alpha g, \alpha\tau) = \nu(g, \tau)$ for all $\alpha > 0$. For any complete Riemannian manifold (M, g) with bounded geometry, the Perelman ν-functional is nonpositive. Furthermore, a rigidity result asserts that for such a manifold, if there exists $\tau > 0$ such that $\nu(g, \tau) = 0$, then (M, g) must be Euclidean space; see [Perelman 2002; Chau et al. 2011; Lee et al. 2023]. When the complete manifold admits only nonnegative scalar curvature, Cheng Liang [Cheng 2022] showed that the manifold must be isometric to Euclidean space whenever the Euclidean isoperimetric inequality holds.

To investigate the stability of the above rigidity result of manifolds with almost nonnegative scalar curvature, Lee, Naber and Neumayer consider the Gromov–Hausdorff convergence under the d_p-distance.

Definition 1.2 (d_p-distance). Given a Riemannian manifold (M^n, g) and a real number $p \in (n, \infty]$, we define the d_p-distance between any $x, y \in M$ by

$$d_p(x, y) = \sup\left\{ |f(x) - f(y)| : \int_M |\nabla f|^p\, d\mathrm{vol}_g \le 1,\ f \in W^{1,p}_{\mathrm{loc}}(M) \cap C^0_{\mathrm{loc}}(M) \right\}.$$

Note that this distance makes sense for any space equipped with a $W^{1,p}$ structure, ensuring the integrability and differentiability of functions. Let $\mathcal{B}_{p,g}(x, r)$ denote the ball centered at x of radius r with respect to d_p, i.e.,

$$\mathcal{B}_{p,g}(x, r) = \{y \in M : d_p(x, y) < r\}.$$

Then the rescaled metric $\tilde{g} = r^{-2} g$ satisfies $\mathcal{B}_{p,\tilde{g}}(x, \rho) = \mathcal{B}_{p,g}(x, \rho r^{1-n/p})$ for any $\rho > 0$.

For a complete Riemannian manifold (M, g), let $B_r(x)$ be the geodesic ball centered at $x \in M$ of radius r, let $R_- = \max\{-R, 0\}$, and let

$$\|R_-\|_{g,q,r} := \sup_{x\in M}\left\{ r^{2-n/q}\left(\int_{B_r(x)} |R_-|^q\right)^{1/q}\right\}.$$

We defined the upper bound of capacity as follows.

Definition 1.3 (capacity). Let (M^n, g) be a Riemannian manifold. For fixed $r > 0$ and $N \in \mathbb{N}^+$, if for any $x \in M$, there exists $\{x_i\}_{i=1}^N \subset B_{2r}(x)$ such that $\{B_r(x_i)\}_{i=1}^N$ forms a covering of $B_{2r}(x)$, then we denote the upper bound of capacity by

$$\mathrm{Cap}_{(M,g)}(r) \le N.$$

The main result of this paper is:

Theorem 1.4 (regularity). *Let (M^n, g) be a complete n-manifold with bounded curvature and fix $\varepsilon, r, N > 0$, $p > n$ and $q > n/2$. There exists $\delta = \delta(n, \varepsilon, N, p, q)$ such that if*

$$\nu(g, 2r^2) \ge -\delta, \quad \|R_-\|_{g,q,r} \le \delta, \quad \mathrm{Cap}_{(M,g)}(r) \le N,$$

then for all $x \in M$ and $0^n \in \mathbb{R}^n$, we have

$$d_{GH}\big((\mathcal{B}_{p,g}(x, r^{1-n/p}), d_{p,g}), (\mathcal{B}_{p,g_{\mathrm{euc}}}(0^n, r^{1-n/p}), d_{p,g_{\mathrm{euc}}})\big) \le \varepsilon r^{1-n/p},$$

and for any $0 < s \le r^{1-n/p}$,

$$1-\varepsilon \le \frac{\mathrm{vol}_g(\mathcal{B}_{p,g}(x,s))}{\mathrm{vol}_{g_{\mathrm{euc}}}(\mathcal{B}_{p,g_{\mathrm{euc}}}(0^n,s))} \le 1+\varepsilon.$$

Remark 1.5. Note that all the statements exhibit scaling invariance, allowing us to assume $r = 1$ in our proofs.

As in the proof of the regularity theorem for manifolds with a pointwise lower bound on scalar curvature [Lee et al. 2023, Theorem 1.7], the main step in our argument is to establish an integral estimate for the Ricci curvature along the Ricci flow; see Lemma 4.1. Once the estimate holds, the proof of Theorem 1.4 is identical with that in [Lee et al. 2023, Sections 5–7] and thus omitted.

Furthermore, we can immediately obtain results analogous to those presented in [Lee et al. 2023].

Theorem 1.6 (compactness). *Fix $\varepsilon, r, N > 0$, $p > n$ and $q > n/2$. There exists $\delta = \delta(n, \varepsilon, N, p, q)$ such that if a sequence of complete pointed Riemannian manifolds $\{(M_i, g_i, x_i)\}$ with bounded curvature satisfies*

$$\nu(g_i, 2r^2) \ge -\delta, \quad \|(R_i)_-\|_{g_i,q,r} \le \delta, \quad \mathrm{Cap}_{(M_i,g_i)}(r) \le N,$$

then there is a subsequence of $\{(M_i, g_i, x_i)\}$ that converges in the pointed d_p sense to (X, g, x), where X is a pointed rectifiable Riemannian space.

See [Lee et al. 2023] for definitions of pointed d_p convergence and pointed rectifiable Riemannian spaces.

For closed manifolds, we establish a prior $L^p(p<1)$ bounds for scalar curvature:

Theorem 1.7. *Fix $n \geq 2$, $\varepsilon > 0$, $p \in (0,1)$ and $q > \max(n/2, 2p)$. Let (M^n, g) be a closed Riemannian n-manifold. There exists $\delta = \delta(n, \varepsilon, p, q) > 0$ such that if*

$$\operatorname{vol}(M)^{2/n} \cdot \left(\fint_M |R_-|^q \right)^{1/q} \leq \delta \quad \textit{and} \quad \nu(g, 2\operatorname{vol}(M)^{2/n}) \geq -\delta,$$

then

$$\operatorname{vol}(M)^{2/n} \cdot \left(\fint_M |R|^p \right)^{1/p} \leq \varepsilon.$$

2. Preliminaries

In this paper, unless specified differently, (M^n, g) will always denote a complete Riemannian manifold of dimension n with bounded curvature.

A Ricci flow $(M^n, g(t))_{t \in [0,T]}$ is a family of smooth metrics $g(t)$ on a smooth manifold M^n satisfying the evolution equation

$$\partial_t g(t) = -2 \operatorname{Ric}_{g(t)} .$$

Along the Ricci flow, the scalar curvature and the volume form evolve by

$$\partial_t R = \Delta_{g(t)} R + 2|\operatorname{Ric}_{g(t)}|^2, \quad \partial_t \, d\mathrm{vol}_{g(t)} = -R_{g(t)} \, d\mathrm{vol}_{g(t)}. \tag{2.1}$$

Consider the heat operator $\partial_t - \Delta_{g(t)}$ coupled to the Ricci flow. Correspondingly, the operator $-\partial_t - \Delta_{g(t)} + R_{g(t)}$ is called the conjugate heat operator. In particular, for $u, v \in C_0^2(M \times [0,T])$, we have

$$\int_M v(\partial_t - \Delta_{g(t)}) u \, d\mathrm{vol}_{g(t)} - \int_M u(-\partial_t - \Delta_{g(t)} + R_{g(t)}) v \, d\mathrm{vol}_{g(t)} = \frac{d}{dt} \int_M uv \, d\mathrm{vol}_{g(t)}.$$

Let $K(\cdot, \cdot; y, s)$ denote the *heat kernel* based at (y, s), i.e.,

$$(\partial_t - \Delta_{x,g(t)}) K(x,t; y,s) = 0, \quad \lim_{t \to s^+} K(\cdot, t; y, s) = \delta_y.$$

The heat kernel exists and is positive; see [Guenther 2002]. For fixed (x,t), the function $K(x,t; \cdot, \cdot)$ is also the *conjugate heat kernel*, i.e.,

$$(-\partial_s - \Delta_{y,g(s)} + R_{g(s)}(y)) K(x,t; y,s) = 0, \quad \lim_{s \to t^-} K(x,t; \cdot, s) = \delta_x.$$

For any $0 \leq s < t < T$ we have

$$\int_M K(x,t; \cdot, s) \, d\mathrm{vol}_{g(s)} = 1. \tag{2.2}$$

Set $\tau(t) = T - t$, and let $u = u(x,t)$ be a solution of the conjugate heat equation along the flow. Chau, Tam and Yu [Chau et al. 2011, Theorem 7.1] show that if $u(\cdot, T) \in C_0^\infty(M)$, then the Perelman $\mathcal{W}$-functional, defined in (1.1), is monotone along the Ricci flow:

$$\mathcal{W}(g(s), u(s), \tau(s)) \le \mathcal{W}(g(t), u(t), \tau(t)) \quad \text{for all } 0 \le s \le t \le T.$$

By taking a compactly supported minimizing sequence for $\mu(g(t), \tau(t))$, we see that $\mu(\tau)$ is also monotone:

$$\mu(g(s), \tau(s)) \le \mu(g(t), \tau(t)) \quad \text{for all } 0 \le s \le t \le T. \tag{2.3}$$

In particular, for Ricci flow $(M, g(t))_{t\in(0,2)}$ and $s \in (0, 1]$, we have $\nu(g(s), 1) \ge \nu(g(0), 2)$.

We restate some basic results of Ricci flow. For the proofs, see [Lee et al. 2023, Section 3].

Proposition 2.4. *Fix $n \ge 2$ and $\lambda > 0$. There exists $\delta = \delta(n, \lambda) > 0$ such that if (M, g) satisfies $\nu(g, 2) \ge -\delta$, then the Ricci flow $(M, g(t))$ with $g(0) = g$ exists for $t \in (0, 1]$ and has the scale-invariant estimate*

$$\sup_{x\in M} |\mathrm{Rm}_{g(t)}(x)| \le \frac{\lambda}{t} \quad \textit{for all } t \in (0, 1]. \tag{2.5}$$

Moreover, for any $x_0 \in M$ and $t \in (0, 1]$, there is a diffeomorphism

$$\phi : B_{g(t)}(x_0, 16t^{\frac{1}{2}}) \to \Omega \subset \mathbb{R}^n$$

such that $\phi(x_0) = 0$ and

$$\tfrac{1}{2}\phi^* g_{\mathrm{euc}}(\phi(x)) \le g(t)(x) \le \tfrac{3}{2}\phi^* g_{\mathrm{euc}}(\phi(x)) \quad \textit{for all } x \in B_{g(t)}(x_0, 16t^{\frac{1}{2}}).$$

In particular, there exists $C = C(n)$ such that

$$C^{-1} r^n \le \mathrm{vol}_{g(t)}(B_{g(t)}(x, r)) \le C r^n \quad \textit{for all } r \in (0, 16t^{\frac{1}{2}}).$$

Combine (2.5) with Shi's estimate [1989], there exists $C = C(k)$ such that $|\nabla^k \mathrm{Rm}_{g(t)}|$ are uniformly bounded by $C/t^{1+k/2}$ for all k. Thus, all Ricci flows are assumed to have bounded curvature throughout the entire paper.

Let $\hat{R}_-(t) = \sup_{x\in M} R_-(x, t)$. It is important to note that the L^1 norm of $K(\cdot, t; y, s)$ has an upper bound:

$$\int_M K(\cdot, t; y, s)\, d\mathrm{vol}_{g(s)} \le \exp\left(\int_s^t \hat{R}_-\right). \tag{2.6}$$

In the compact case, this result can be derived from the following computation:

$$\begin{aligned}\frac{d}{dt}\int_M K(\cdot, t; y, s)\, d\mathrm{vol}_{g(t)} &= \int_M \triangle K - RK\, d\mathrm{vol}_{g(t)} \\ &\le \hat{R}_-(t) \int_M K(\cdot, t; y, s)\, d\mathrm{vol}_{g(t)}.\end{aligned}$$

If M is noncompact, consider an exhaustion of M by smooth domains with compact closure: $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M$. Let $K_{\Omega_i}(x,t;y,s)$ be the corresponding Dirichlet heat kernel on Ω_i. Since Ω_i is compact, we have

$$\int_{\Omega_i} K_{\Omega_i}(\,\cdot\,,t;y,s)\,d\mathrm{vol}_{g(s)} \le e^{\int_s^t \hat{R}_-}.$$

By maximum principle, K_{Ω_i} is an increasing sequence, and K is the limit of K_{Ω_i} as i tend to infinity. By the monotone convergence theorem, we can ascertain that the L^1 norm of $K(\,\cdot\,,t;y,s)$ also satisfies (2.6).

3. Heat kernel estimates for Ricci flow

In this section, we establish the heat kernel's upper and lower bounds. The Gaussian upper bounds for the heat kernel are primarily derived from the heat kernel estimates by Bamler, Cabezas-Rivas, and Wilking in [Bamler et al. 2019, Proposition 3.1]. Incorporating this result with the log Sobolev inequality (see also [Cao and Zhang 2011; Zhang 2011, Theorem 4.2.1]), we achieve a more precise estimation of the heat kernel, specifically about integral scalar curvature, as detailed in the following lemma.

Lemma 3.1. *Fix $n \ge 2$. There exist $\delta = \delta(n) > 0$ and $C = C(n) > 0$ such that if $(M^n, g(t))_{t\in[0,1]}$ satisfies $\nu(g(0),2) \ge -\delta$, then*

$$K(x,t;y,s) \le \frac{C}{(t-s)^{n/2}} \exp\biggl(-\frac{d^2_{g(s)}(x,y)}{C(t-s)} + \int_s^t \hat{R}_-\biggr) \quad \text{for all } 0 \le s < t \le 1.$$

Proof. Up to scaling, we only need to show that if $\nu(g(0),2) \ge -\delta$, then

$$K(x,1;y,0) \le C \exp\biggl(-C^{-1} d^2_{g(0)}(x,y) + \int_0^1 \hat{R}_-\biggr). \tag{3.2}$$

Let $p(s) = 1/(1-s)$ for $s \in [0,1)$, and let $u = u(x,t)$ be a positive solution of the heat equation with Dirichlet boundary condition

$$(\partial_t - \Delta)u = 0 \quad \text{in } \Omega_i \times (0,1), \qquad u = 0 \quad \text{on } \partial\Omega_i \times [0,1].$$

Let $\tau = s(1-s)$ and $v(x,s) = u^{p(s)/2}/\|u^{p(s)/2}\|_2$. By (2.3), we compute

$$\begin{aligned}
\frac{d}{ds}\log\|u\|_{p(s)} &= \frac{p'}{p^2}\int_M (v^2\log v^2) - \frac{p-1}{p^2}\int_M (Rv^2 + 4|\nabla v|^2) - \frac{1}{p^2}\int_M Rv^2 \\
&\le -s(1-s)\int_M (Rv^2 + 4|\nabla v|^2) + \int_M v^2 \log v^2 + \hat{R}_-(s) \\
&= -\bigl(\mathcal{W}(g(s), v(s), \tau) + \tfrac{1}{2} n \log(4\pi\tau) + n\bigr) + \hat{R}_-(s) \\
&\le \delta - \tfrac{1}{2} n \log \tau + \hat{R}_-(s).
\end{aligned}$$

Integrating from $s = 0$ to $s = 1$, we find

$$\|u(\,\cdot\,, 1)\|_\infty \le \exp\left(\delta + n + \int_0^1 \hat{R}_-\right)\|u(\,\cdot\,, 0)\|_1.$$

Since

$$\sup_{x,y} K_{\Omega_i}(x, 1; y, 0) = \sup_{u\neq 0} \frac{\|u(\,\cdot\,, 1)\|_\infty}{\|u(\,\cdot\,, 0)\|_1},$$

letting i tend to infinity, it follows that

$$K(x, 1; y, 0) \le \exp\left(\delta + n + \int_0^1 \hat{R}_-\right). \tag{3.3}$$

This implies the desired bound (3.2) if $d_{g(0)}(x, y)$ is controlled. Therefore, our task reduces to estimating $K(x, 1; \cdot\,, 0)$ when $d_{g(0)}(x, y)$ is large. The proof closely parallels that of [Bamler et al. 2019, Proposition 3.1]. The main deviation lies in the application of formula (2.6) to transform [Bamler et al. 2019, (3.11)] into the form

$$\begin{aligned}\mathcal{I}[B_k] &\le \int_{M/B_{g(0)}(x,r_k)} K(x, 1; \cdot\,, t_k)K(\,\cdot\,, t_k; y, t_{k+1})\, d\mathrm{vol}_{g(t_k)} \\ &\le a_k \exp\left(\int_{t_{k+1}}^{t_k} \hat{R}_-\right).\end{aligned} \tag{3.4}$$

Thus, further details of the proof are omitted here. □

In preparation for the lower bound estimate, we need the following.

Proposition 3.5. *Let $(M, g(t))_{t\in(0,T]}$ be a Ricci flow. Then the following properties hold.*

(1) (*interpolation inequality* [Zhang 2011, Theorom 6.5.1]) *Let u be a positive solution to the heat equation $(\partial_t - \Delta)u = 0$. Then, for $x, y \in M$ and $0 < t \le T$, letting $U = \sup_{M\times[0,T]} u$, we have*

$$u(y, t) \le u(x, t)^{\frac{1}{2}} U^{\frac{1}{2}} \exp\left(\frac{d^2_{g(t)}(x, y)}{t}\right).$$

(2) (*Perelman's differential Harnack inequality* [Perelman 2002, Corollary 9.4]) *Let $\gamma(s)$ be any smooth curve and suppose $w(y, s) = (4\pi(T - s))^{-n/2}e^{-h(y,s)}$ satisfies the conjugate heat equation $(\partial_s + \Delta - R)w = 0$. Then*

$$-\frac{d}{ds}h(\gamma(s), s) \le \frac{1}{2}(R(\gamma(s), s) + |\dot{\gamma}(s)|^2) - \frac{1}{2(T-s)}h(\gamma(s), s).$$

Then, we establish the lower bound estimate on the heat kernel by using the same argument as in the proof of [Zhang 2012, Theorem 1.1].

Lemma 3.6. *Fix $n \geq 2$ and $\lambda > 0$. There exist $\delta = \delta(n, \lambda) > 0$ and $C = C(n) > 0$ such that if $(M, g(t))_{t\in(0,1]}$ satisfies $\nu(g(0), 2) \geq -\delta$, then for any $0 < s < t \leq 1$,*

$$K(x, t; y, s) \geq \frac{C}{(t-s)^{n/2}}\left(\frac{s}{t}\right)^{2\lambda} \exp\left(\frac{-4d^2_{g(t)}(x, y)}{t-s}\right).$$

Proof. By Proposition 2.4, for any $\lambda > 0$, we may choose δ small enough that the Ricci flow enjoys the scale invariant curvature bounds $|\mathrm{Rm}| \leq \lambda/t$. Combining this with Lemma 3.1,

$$K(x, t; y, s) \leq \frac{C}{(t-s)^{n/2}}\left(\frac{t}{s}\right)^{\lambda} \quad \text{for all } 0 < s < t \leq 1. \tag{3.7}$$

Let $K(x, t; y, l) = (4\pi(t-l))^{-n/2}e^{-h(y,l)}$ for $l \in [s, t]$, and let $\gamma(l)$ be the fixed point x. By Proposition 3.5,

$$-\frac{d}{dl}h(x, l) \leq \frac{1}{2}R(x, l) - \frac{1}{2(t-l)}h(x, l).$$

Integrating from s to t, we have $h(x, s) \leq \frac{1}{2}\lambda \log(t/s)$. Consequently,

$$K(x, t; x, s) \geq (4\pi(t-s))^{-n/2}\left(\frac{s}{t}\right)^{\lambda/2}. \tag{3.8}$$

Note that the function $K(y', t'; x, s)$ for $(y', t') \in M \times [(t+s)/2, t]$ is a positive solution to the heat equation. Then Proposition 3.5 implies

$$K(y, t; y, s) \leq K(x, t; y, s)^{\frac{1}{2}} \cdot \left(\sup_{(x',t')\in M\times[(t+s)/2,t]} K(x', t'; y, s)\right)^{\frac{1}{2}} \exp\left(\frac{2d^2_{g(t)}(x, y)}{t-s}\right).$$

Combining this with (3.7) and (3.8), we get the lower bound of $K(x, t; y, s)$. □

For $0 < t \leq 1$ and $x_0 \in M$, let $\varphi : M \times \{t\} \to [0, 1]$ be a cutoff function such that $\varphi(y) \equiv 1$ for $y \in B_{g(t)}(x_0, 8t^{\frac{1}{2}})$ and $\operatorname{supp}\varphi \subset B_{g(t)}(x_0, 16t^{\frac{1}{2}})$. Let $\varphi : M \times [0, t] \to \mathbb{R}$ be the solution of the conjugate heat equation $(\partial_s + \Delta - R)\varphi = 0$ with terminal data $\varphi(y)$.

Applying Lemma 3.6, we can derive the following estimates. For the proof, see [Lee et al. 2023, Proposition 4.4].

Proposition 3.9. *Fix $n \geq 2$ and $\lambda > 0$. There exist $\delta = \delta(n, \lambda) > 0$ and $C = C(n) > 0$ such that if $(M, g(t))_{t\in(0,1]}$ satisfies $\nu(g(0), 2) \geq -\delta$, then for all $(y, s) \in B_{g(t)}(x_0, 4t^{\frac{1}{2}}) \times (0, t)$,*

$$\varphi(y, s) \geq C\left(\frac{s}{t}\right)^{2\lambda}. \tag{3.10}$$

In addition, if the manifold mentioned above also has almost nonnegative integral scalar curvature and an upper capacity bound, we can refine the heat kernel estimates in Lemma 3.1 to obtain lower bounds on scalar curvature along the Ricci flow. Moreover, the lower scalar curvature bounds ensure that the volume of a given set does not expend too much along the Ricci flow.

Lemma 3.11. *Fix $r, N > 0$ $n > 2$ and $q > n/2$. There exist $\delta = \delta(n, N, q)$ and $C = C(n, N, q)$ such that if $(M^n, g(t))_{t\in(0,1/r^2]}$ satisfies*

$$\nu(g(0), 2r^2) \geq -\delta, \quad \|R_-\|_{g(0),q,r} \leq \delta, \quad \mathrm{Cap}_{(M,g(0))}(r) \leq N,$$

then for $0 < t \leq 1/r^2$,

$$R(x, t) \geq -C\delta r^{-(2q-n)/q} t^{-n/(2q)} \quad \text{and} \quad d\mathrm{vol}_{g(t)} \leq e^{C\delta}\, d\mathrm{vol}_{g(s)}.$$

Proof of Lemma 3.11. Up to rescaling, we may assume $r = 1$. We will use the upper bound of capacity to prove that for each $k \in \mathbb{N}^+$ there exists a finite subset $\{x_i\}_{i=1}^{N^k} \subset B_{g(0)}(x, k+1)$ such that

$$B_{g(0)}(x, k+1) \subseteq \bigcup_{i=1}^{N^k} B_{g(0)}(x_i, 1). \tag{3.12}$$

We argue by induction. For $k = 1$ and $y \in M$, by the definition of the capacity, there exists $\{y_j\}_{j=1}^{N}$ such that $B_{g(0)}(y, 2) \subset \bigcup_{j=1}^{N} B_g(y_j, 1)$. For $k > 1$, if there exists $\{x_i\}_{i=1}^{N^{k-1}}$ such that $B_{g(0)}(x, k)$ can be covered by $\{B_{g(0)}(x_i, 1)\}_{i=1}^{N^{k-1}}$, then by the triangle inequality we have $B_{g(0)}(x, k+1) \subset \bigcup_{i=1}^{N^{k-1}} B_{g(0)}(x_i, 2)$. Since for each $B_{g(0)}(x_i, 2)$ there is a finite cover $\{B_{g(0)}(y_{i,j}, 1)\}_{j=1}^{N}$, $\{B_{g(0)}(y_{i,j}, 1)\}_{i,j}$ form a cover for $B_{g(0)}(x, k+1)$. Thus (3.12) follows.

Let w be the solution to the heat equation with the initial data $w(y, 0) = R_-(y, 0)$. By maximum principle, $-R(x, t) \leq w(x, t)$ pointwise. Let $S(t) = \int_0^t \hat{R}_-$ and $A_x(k, k+1) = B_{g(0)}(x, k+1) \setminus B_{g(0)}(x, k)$ for $k \in \mathbb{N}^+$. By Hölder's inequality, (3.12), (2.2) and Lemma 3.1,

$$\begin{aligned} w(x, t) &= \int_M K(x, t; y, 0) R_-(y, 0)\, d\mathrm{vol}_{g(0)}(y) \\ &\leq \sum_{k=0}^{\infty} \|R_-\|_{L^q(B_g(x,k+1))} \Big(\int_{A_x(k,k+1)} K^{q/(q-1)} \Big)^{(q-1)/q} \\ &\leq \sum_{k=0}^{\infty} \sup_{A_x(k,k+1)} K^{1/q} \Big(\int_M K \Big)^{(q-1)/q} \Big(\sum_{i=1}^{N^k} \|R_-\|^q_{L^q(B_g(x_i^k,1))} \Big)^{1/q} \\ &\leq \sum_{k=0}^{\infty} C_0^{1/q} \delta t^{-n/(2q)} \exp\Big(-\frac{k^2}{C_0 q t} + \frac{S(t)}{q} \Big) N^{k/q} \leq C\delta t^{-n/(2q)} e^{S(t)/q}, \end{aligned}$$

where $C_0 = C_0(n)$ and $C = C(n, N, q)$. In particular, we obtain

$$\hat{R}_-(t) \le \max_{x\in M}(w(x,t)) \le C\delta t^{-n/(2q)} e^{S(t)/q}. \tag{3.13}$$

Thus $\frac{d}{dt} q \cdot e^{-S(t)/q} = -e^{-S(t)/q}\hat{R}_-(t) \ge -C\delta t^{-n/(2q)}$. Integrating from 0 to t,

$$S(t) \le -q \log\left(1 - \frac{2C\delta}{2q-n}\right) \quad \text{for all } 0 < t \le 1.$$

If $\delta \le (2q-n)/(4C)$, then $S(t) \le q$. Substituting this in (3.13), $R_-(\cdot, t) \le \hat{R}_-(t) \le 2C\delta t^{-n/(2q)}$. By the evolved equation of volume form (2.1), we have $\partial_\tau \, d\mathrm{vol}_{g(\tau)} \le 2C\delta\tau^{-n/(2q)}\, d\mathrm{vol}_{g(\tau)}$. Integrating τ from s to t completes the proof. □

4. Integral estimate for Ricci curvature under Ricci flow

In this section, we prove an integral estimate for the Ricci curvature, which is scale invariant. The proof of Lemma 4.1 is analogous to [Lee et al. 2023, Theorem 4.1]. In our case, we replace the use of pointed lower bound of initial scalar curvature by the lower bound of $w(x,t)$ along the Ricci flow, as shown in (3.13). For the sake of completeness, we include the proof.

Lemma 4.1 (integral Ricci estimate). *Fix $n > 2$, $\varepsilon, r, N > 0$, $q > n/2$ and $\theta \in \left[0, \frac{1}{2}\right)$. If $(M^n, g(t))_{t\in(0,1/r^2]}$ satisfies*

$$\nu(g(0), 2r^2) \ge -\delta, \quad \|R_-\|_{g(0),q,r} \le \delta, \quad \mathrm{Cap}_{(M,g(0))}(r) \le N,$$

then for any $(x,s) \in M \times (0, 1/r^2]$,

$$\int_0^s \left(\frac{\tau}{s}\right)^{-\theta} \fint_{B_{g(s)}(x,4s^{1/2})} |\mathrm{Ric}_{g(\tau)}| \, d\mathrm{vol}_{g(\tau)} \, d\tau \le \varepsilon^2.$$

As the evolution equation of the scalar curvature contains the term of $|\mathrm{Ric}|^2$, we can combine the scalar curvature estimate and the heat kernel estimate along the Ricci flow to estimate $|\mathrm{Ric}|^2\varphi$. Combining this estimate with Hölder's inequality, we prove Lemma 4.1:

Proof of Lemma 4.1. Up to rescaling the flow, we may assume that $t = 1$. By Proposition 2.4 and the volume comparison in Lemma 3.11, there exists a constant $C_0 = C_0(n, q, N)$ such that for any $(x, s) \in M \times [0, 1]$,

$$\mathrm{vol}_{g(s)}(B_{g(1)}(x, 4)) \ge e^{-C_0\delta}\, \mathrm{vol}_{g(1)}(B_{g(1)}(x, 4)) \ge C_0. \tag{4.2}$$

For fixed $0 < \lambda \le \frac{1}{4} - \frac{1}{2}\theta$, let $\theta_0 = \theta + \lambda$, and then we have $1 - 2\theta_0 \ge \frac{1}{2} - \theta > 0$. Choose δ small enough that Proposition 3.9 holds for this choice of λ. By Hölder's inequality and (4.2) we get

$$\text{(4.3)}\quad \int_0^1 s^{-\theta} \fint_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}|\, d\mathrm{vol}_{g(s)}\, ds \\ \le C_0^{-1}(1-2\theta_0)^{-\frac{1}{2}} \left(\int_0^1 s^{2\lambda} \int_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}|^2\, d\mathrm{vol}_{g(s)}\, ds \right)^{\frac{1}{2}}.$$

Let φ be the cutoff function that evolves by the conjugate heat equation as in Proposition 3.9. Then there exists $C_1 = C_1(n)$ such that

$$\begin{aligned}\int_0^1 s^{2\lambda} \int_{B_{g(1)}(x,4)} &|\mathrm{Ric}_{g(s)}|^2\, d\mathrm{vol}_{g(s)}\, ds \\ &\le C_1 \int_0^1 \int_M |\mathrm{Ric}_{g(s)}|^2 \varphi \\ &= \frac{1}{2} C_1 \int_0^1 \int_M (\partial_s - \triangle) R_{g(s)} \varphi \\ &= \frac{1}{2} C_1 \left(\int_M R_{g(1)} \varphi(\,\cdot\,,1)\, d\mathrm{vol}_{g(1)} - \int_M R_{g(0)} \varphi(\,\cdot\,,0)\, d\mathrm{vol}_{g(0)} \right).\end{aligned}$$

By Proposition 2.4, we find that $\int_M R_{g(1)}\varphi(\,\cdot\,,1)\, d\mathrm{vol}_{g(1)} \le \lambda \,\mathrm{vol}_{g(1)}(B_{g(1)}(x,16)) \le C_1\lambda$. By (3.13), there exists a constant $C_2 = C_2(n,q,N)$ such that

$$\begin{aligned}-\int_M & R_{g(0)}\varphi(\,\cdot\,,0)\, d\mathrm{vol}_{g(0)} \\ &= \int_M R_-(x,0) \int_M K(y,1;x,0)\varphi(y,1)\, d\mathrm{vol}_{g(1)}(y)\, d\mathrm{vol}_{g(0)}(x) \\ &= \int_M \varphi(y,1) \int_M K(y,1;x,0) R_-(x,0)\, d\mathrm{vol}_{g(0)}(x)\, d\mathrm{vol}_{g(1)}(y) \le C_2\delta.\end{aligned}$$

By choosing λ and δ appropriately small, we conclude the proof. □

5. L^p bound for the scalar curvature

For closed manifold, we derive an a prior L^p $(p < 1)$ bound of scalar curvature, Theorem 1.7, which we restate below for convenience. The proof of this theorem closely parallels that of [Lee et al. 2023, Theorem 4.7]. The main difference is that, since the curvature here is only bounded below in an integral sense, we need to estimate the L^p norm of $R + w$ along the Ricci flow, where w evolves by the heat equation with initial data R_-.

Theorem 1.7. *Fix $n \ge 2$, $\varepsilon > 0$, $p \in (0,1)$ and $q > \max(n/2, 2p)$. Let (M^n, g) be a closed Riemannian n-manifold. There exists $\delta = \delta(n,\varepsilon,p,q) > 0$ such that if*

$$\mathrm{vol}(M)^{2/n} \cdot \left(\fint_M |R_-|^q \right)^{1/q} \le \delta \quad \textit{and} \quad \nu(g, 2\,\mathrm{vol}(M)^{2/n}) \ge -\delta,$$

then

$$\mathrm{vol}(M)^{2/n}\cdot\left(\fint_M |R|^p\right)^{1/p}\le\varepsilon.$$

Proof. Up to rescaling, we may assume that $\mathrm{vol}(M)=1$. Choosing $\delta\le\varepsilon/2$, by Hölder's inequality we have $\left(\int_M |R_-|^p\right)^{1/p}\le\left(\int_M |R_-|^q\right)^{1/q}\le\varepsilon/2$, and it is suffices to show that $\int_M |R_+|^p\,d\mathrm{vol}_g\le(\varepsilon/2)^p$, where $R_+=\max(R,0)$. By Proposition 2.4 and Lemma 3.1, for any fixed $\lambda>0$, we may choose δ small enough that the Ricci flow $(M,g(t))$ with $g(0)=g$ exists for $t\in(0,1]$ and there exists a constant $C_0=C_0(n)$ such that

$$|\mathrm{Rm}|\le\lambda/t,\qquad \sup_{x,y\in M}\{K(x,t;y,0)\}\le C_0t^{-n/2}\exp\left(\int_0^t \hat{R}_-\right)\qquad\text{for all }0<t\le 1.$$

Let $w(x,t)$ be the solution of the heat equation with initial data $w(x,0)=R_-(x,0)$ and $f(x,t)=R(x,t)+w(x,t)$, the evolve equation of scalar curvature in (2.1) implies that $(\partial_t-\triangle)f=2|\mathrm{Ric}|^2\ge 0$. By the maximum principle, we have $f(x,t)\ge R_+(x,t)$. Thus, we only need to show $\int_M f^p\,d\mathrm{vol}_{g(0)}\le(\varepsilon/2)^p$.

For any $p\in(0,1)$, we see that f^p is a supersolution of the heat equation:

$$(\partial_t-\triangle)f^p=pf^{p-1}(\partial_t-\triangle)f-p(p-1)f^{p-1}|\nabla f|^2\ge 2pf^{p-1}|\mathrm{Ric}|^2\ge 0.$$

Combining this with Young's inequality, we have

$$\begin{aligned}(5.1)\quad &\int_M f^p\,d\mathrm{vol}_{g(0)}\\ &=\int_M f^p\,d\mathrm{vol}_{g(1)}-\int_0^1\int_M\left((\partial_t-\triangle)f^p-Rf^p\right)d\mathrm{vol}_{g(t)}\,dt\\ &\le\int_M f^p\,d\mathrm{vol}_{g(1)}+\int_0^1\int_M Rf^p\,d\mathrm{vol}_{g(t)}\,dt\\ &\le\int_M f^p\,d\mathrm{vol}_{g(1)}+\int_0^1\int_M (R^{p+1}+Rw^p)\,d\mathrm{vol}_{g(t)}\,dt\\ &\le\int_M f^p\,d\mathrm{vol}_{g(1)}+\frac{p}{q}\int_0^1\int_M w^q\,d\mathrm{vol}_{g(t)}\,dt\\ &\qquad+\int_0^1\int_M\left(R^{p+1}+\frac{q-p}{q}R^{q/(q-p)}\right)d\mathrm{vol}_{g(t)}\,dt.\end{aligned}$$

To bound the right-hand side of (5.1), let $S(t)=\int_0^t \hat{R}_-$. Then Lemma 3.1 implies

$$\begin{aligned}w(x,t)&=\int_M K(x,t;y,0)R_-(y,0)\,d\mathrm{vol}_{g(0)}(y)\\ &\le\left(\int_M |R_-(\cdot,0)|^q\,d\mathrm{vol}_{g(0)}\right)^{1/q}\left(\int_M K(x,t;\cdot,0)^{q/(q-1)}\,d\mathrm{vol}_{g(0)}\right)^{(q-1)/q}\\ &\le\delta\max_{y\in M}K(x,t;y,0)^{1/q}\le C_0^{1/q}\delta t^{-n/(2q)}e^{S(t)/q}.\end{aligned}$$

Similar to the argument in Lemma 3.11, there exists $C_1 = C_1(n,q)$ such that for $x \in M$ and $0 < t \le 1$, we have

$$R_-(x,t) \le w(x,t) \le C_1\delta t^{-n/(2q)}, \quad d\mathrm{vol}_{g(t)} \le e^{C_1\delta}\, d\mathrm{vol}_{g(0)}. \tag{5.2}$$

By (2.6), we find $C_2 = C_2(n,q)$ such that for any $0 \le s < t \le 1$ we have

$$\int_M K(\cdot\,,t;y,s)\,d\mathrm{vol}_{g(t)} \le \exp\biggl(\int_s^t R_-\biggr) \le \exp\biggl(\int_s^t C_1\delta\tau^{-n/(2q)}\biggr)\,d\tau \le C_2.$$

Then we bound each term on the right-hand side of (5.1) separately. For the first term, by (5.2) we see that

$$\int_M f^p\,d\mathrm{vol}_{g(1)} = \int_M (R+w)^p\,d\mathrm{vol}_{g(1)} \le \int_M (\lambda + C_1\delta)^p\,d\mathrm{vol}_{g(1)} \le (\lambda+C_1\delta)^p e^{C_1\delta}.$$

For the second term, by Hölder's inequality, we have

$$\begin{aligned}
&\int_M w^q(\cdot\,,t)\,d\mathrm{vol}_{g(t)} \\
&= \int_M \biggl(\int_M R_-(y,0)K(x,t;y,0)\,d\mathrm{vol}_{g(0)}(y)\biggr)^q d\mathrm{vol}_{g(t)}(x) \\
&\le \int_M \biggl(\int_M R_-^q(y,0)K(x,t;y,0)\,d\mathrm{vol}_{g(0)}(y)\biggr) \\
&\qquad\qquad \cdot \biggl(\int_M K(x,t;y,0)\,d\mathrm{vol}_{g(0)}(y)\biggr)\,d\mathrm{vol}_{g(t)}(x) \\
&= \int_M R_-^q(y,0)\int_M K(x,t;y,0)\,d\mathrm{vol}_{g(t)}(x)\,d\mathrm{vol}_{g(0)}(y) \\
&\le C_2\int_M R_-^q(y,0)\,d\mathrm{vol}_{g(0)}(y) \le C_2\delta^q.
\end{aligned}$$

For the third term, let $\varphi : M \times (0,1) \to \mathbb{R}$ be the solution to the conjugate heat equation with terminal data $\varphi(x,1) = 1$ on $M \times \{1\}$. Using the same proof as [Lee et al. 2023, Proposition 4.4], there exists a constant $C_3 = C_3(n)$ such that for all $y \in M$ and $t \in (0,1]$ we have $\varphi(y,t) \ge C_3 t^{2\lambda}$. Moreover, by using the same argument as in the proof of Lemma 4.1, (5.2) implies that there exists a constant $C_4 = C_4(n,q)$ such that

$$\int_0^1 t^{2\lambda}\int_M |R|^2\,d\mathrm{vol}_{g(t)}\,dt \le C_3^{-1}\int_0^1\int_M |R|^2\varphi(y,t)\,d\mathrm{vol}_{g(t)}\,dt \le C_4(\lambda+\delta). \tag{5.3}$$

For any $0 < \alpha < 2$ and $0 < \lambda < (2-\alpha)/(4\alpha)$, let $\theta = 2\lambda\alpha/(2-\alpha)$. Then, by (5.3),

$$\begin{aligned}
\int_0^1\int_M |R|^\alpha\,d\mathrm{vol}_{g(t)}\,dt &\le \biggl(\int_0^1\int_M t^{-\theta}\,d\mathrm{vol}_s\,ds\biggr)^{\lambda\alpha/\theta}\biggl(\int_0^1 s^{2\lambda}\int_M |R|^2\,d\mathrm{vol}_s\,ds\biggr)^{\alpha/2} \\
&\le 2e^{C_1\delta}C_4^{\alpha/2}(\lambda+\delta)^{\alpha/2}.
\end{aligned}$$

In particular, there exists a constant $C_5 = C_5(n, p, q)$ such that

$$\int_0^1 \int_M R^{p+1} + \frac{q-p}{q} R^{q/(q-p)} \, \mathrm{vol}_{g(t)} \, dt \leq C_5(\lambda+\delta)^{(p+1)/2} + C_5(\lambda+\delta)^{q/(2q-2p)}.$$

Finally, by choosing λ and δ sufficiently small, we conclude the proof. □

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