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CORRECTION TO THE ARTICLE A HECKE ALGEBRA ISOMORPHISM OVER CLOSE LOCAL FIELDS

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The proof of Lemma 2.5 of the author's article "A Hecke algebra isomorphism over close local fields" (*Pacific J. Math.* 319:2 (2022), 307–332) is incorrect. We use a slight variant of the original approach to correct the proof. This leads to some modifications to some parts of Section 3 of the original article, and these are given in Section 2 of this note. With these modifications, Theorem 4.1 of the original article holds.

We retain the notation in [Ganapathy 2022, Section 2]. Let T be a torus over F. Then T is determined by the Γ_F-module $X_*(T)$. Let $\mathscr{T}^{\mathrm{ft}}$ be the Néron–Raynaud model of T and $\mathscr{T}$ its identity component. Let $m \geq 1$ be such that T splits over an at most m-ramified Galois extension of F. Then the action of Γ_F on $X_*(T)$ factors through Γ_F/I_F^m. For any field F' that is at least m-close to F, we obtain a torus T' over F' via the action of $\Gamma_{F'} \to \Gamma_{F'}/I_{F'}^m \xrightarrow[\cong]{\mathrm{Del}_m^{-1}} \Gamma_F/I_F^m$ on $X_*(T)$. This torus splits over an at most m-ramified extension of F'. Let $\mathscr{T}'^{\mathrm{ft}}$ be the Néron–Raynaud model of T' and $\mathscr{T}'$ its identity component.

Theorem 0.1 [Chai and Yu 2001, Section 9]. *Let $m \geq 1$ and let h be as in* [Chai and Yu 2001, Section 8]. *Assume $e \geq m+3h$. Then for any nonarchimedean local field F' that is e-close to F, the group schemes $\mathscr{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ and $\mathscr{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ are isomorphic. In particular,*

$$\mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong \mathscr{T}'^{\mathrm{ft}}(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m)$$

as groups. This isomorphism continues to hold when we replace $\mathscr{T}^{\mathrm{ft}}$ by $\mathscr{T}$.

In [Ganapathy 2022, Section 2C], we had constructed a group-theoretic section of the Kottwitz homomorphism $\kappa_{T,F} : T(F) \to X_*(T)_{I_F}^{\sigma}$ and had used Theorem 0.1 for the neutral component $\mathscr{T}$ to give a proof of Lemma 2.5 in the same article. If T splits over an unramified extension of F or is an induced torus over F, the results in [Ganapathy 2022, Section 2] go through. However, the Kottwitz homomorphism for

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a general torus need not admit a group-theoretic section, as the following example illustrates.

Example 0.2. Let $\breve{F}$ be the completion of the maximal unramified subextension of $\mathbb{Q}_2$. Let $L = \breve{F}\big(\sqrt{-1}\big)$. Then L is a wildly ramified quadratic extension of $\breve{F}$. Let $T = \mathrm{Nm}^1_{L/\breve{F}}\mathbb{G}_m$ denote the norm-1 torus. Let γ be the nontrivial element of $\mathrm{Gal}(L/\breve{F})$. Then $X_*(T)_{I_F} \cong \mathbb{Z}/2\mathbb{Z}$. Note that $\kappa_{T,\breve{F}}$ has a group-theoretic section if and only if $-1 \in T(\breve{F})$ does not lie in $T(\breve{F})_1$. Note that

$$T(\breve{F})_1 = \{y \in L^\times \mid x\gamma(x)^{-1} = y \text{ for some } x \in L^\times\}.$$

Since $-1 = \big(\sqrt{-1}\big)\gamma\big(\sqrt{-1}\big)^{-1}$, -1 indeed lies in $T(\breve{F})_1$. We conclude that $\kappa_{T,\breve{F}}$ does not admit a group-theoretic section.

The error in [Ganapathy 2022, Section 2] is that Lemma 2.3 is false in general (the $n_{\tilde{\lambda}}$ defined in the line above Lemma 2.3 may not be well-defined). Consequently, Lemma 2.4 cannot be salvaged to yield a well-defined set of representatives for the torsion elements of $X_*(T)_{I_F}$ that forms a group and is σ-stable.

1. Proof of [Ganapathy 2022, Lemma 2.5]

Let T be a torus over F and let $\widetilde{F}$ be the splitting extension of $T_{\breve{F}}$ in the completion of F_s. Fix a uniformizer $\varpi_{\widetilde{F}}$ of $\widetilde{F}$. Consider the Kottwitz homomorphism $\kappa_{T,\breve{F}} : T(\breve{F}) \to X^*(T)_{I_F}$. Let $X_*(T)_{I_F}/\mathrm{tor}$ denote the quotient of $X_*(T)_{I_F}$ by its torsion subgroup. Note that $X_*(T)_{I_F}/\mathrm{tor}$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(X^*(T)^{I_F}, \mathbb{Z})$. This leads to the valuation homomorphism $\omega_{T,\breve{F}} : T(\breve{F}) \to \mathrm{Hom}_{\mathbb{Z}}(X^*(T)^{I_F}, \mathbb{Z})$. Note that $\mathrm{Ker}(\omega_{T,\breve{F}}) = T(\breve{F})_b = \mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}})$ is the maximal bounded subgroup of $T(\breve{F})$ and it contains $T(\breve{F})_1$. We will construct a group-theoretic section of the valuation homomorphism. We will then use Theorem 0.1 for $\mathscr{T}^{\mathrm{ft}}$ to prove [Ganapathy 2022, Lemma 2.5] over $\breve{F}$. We will show that this isomorphism over $\breve{F}$ is σ-equivariant to obtain the required isomorphism over F (see Lemmas 1.2 and 1.3).

1A. ***A group-theoretic section of the valuation homomorphism and its consequences.*** Let $\breve{\lambda}_1, \dots, \breve{\lambda}_n \in X_*(T)_{I_F}$ be such that their images $\breve{\lambda}^t_1, \dots, \breve{\lambda}^t_n$ form a basis of $X_*(T)_{I_F}/\mathrm{tor}$. Fix $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in X_*(T)$ such that $\mathrm{pr}(\tilde{\lambda}_i) = \breve{\lambda}_i$, where $\mathrm{pr} : X_*(T) \to X_*(T)_{I_F}$ is the natural surjection. Define $n_{\tilde{\lambda}_i} = \tilde{\lambda}_i(\varpi_{\widetilde{F}})$. Define $n_{\breve{\lambda}^t_i} = n_{\breve{\lambda}_i} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, n_{\tilde{\lambda}_i}$. For $\breve{\lambda}^t \in X_*(T)_{I_F}/\mathrm{tor}$, write $\breve{\lambda}^t = \sum_i c_i \breve{\lambda}^t_i$ and define $n_{\breve{\lambda}^t} = \prod_i n^{c_i}_{\breve{\lambda}^t_i}$. Note that $n_0 = 1$ by construction.

Lemma 1.1. *The set $\mathscr{S} := \{n_{\breve{\lambda}^t} \mid \breve{\lambda}^t \in X_*(T)_{I_F}/\mathrm{tor}\}$ is a subgroup of $T(\breve{F})$. The map $\nabla_{T,\breve{F}} : X_*(T)_{I_F}/\mathrm{tor} \to \mathscr{S}$, $\breve{\lambda}^t \mapsto n_{\breve{\lambda}^t}$, is a group isomorphism.*

Proof. It is clear that $\mathscr{S}$ is a subgroup of $T(\breve{F})$. It is also clear that $\nabla_{T,\breve{F}}$ is a surjective group homomorphism. We just need to see that it is injective. Suppose $n_{\breve{\lambda}^t} = 1$.

We need to show that $\breve{\lambda}^t = 0$. Write $\breve{\lambda}^t = \sum_i c_i \breve{\lambda}^t_i$. The natural pairing between $X_*(T)$ and $X^*(T)$ induces a perfect pairing $\langle\cdot,\cdot\rangle : X_*(T)_{I_F}/\mathrm{tor} \times X^*(T)^{I_F} \to \mathbb{Z}$. Let $\breve{\chi}_1, \ldots, \breve{\chi}_n \in X^*(T)^{I_F}$ be such that $\langle \tilde{\lambda}_j, \breve{\chi}_k\rangle = \delta_{j,k}$, $1 \le j,k \le n$. Now $n_{\breve{\lambda}^t} = \prod_i \mathrm{Nm}_{\widetilde{F}/\breve{F}} n_{\tilde{\lambda}_i}^{c_i} = 1$. This implies that $1 = \breve{\chi}_j(n_{\breve{\lambda}^t}) = \mathrm{Nm}_{\widetilde{F}/\breve{F}} \breve{\chi}_j(n_{\tilde{\lambda}_j})^{c_j} = (\mathrm{Nm}_{\widetilde{F}/\breve{F}} \varpi_{\widetilde{F}}^{c_j})$. This forces $c_j = 0$. Since j was arbitrary, this shows that $\breve{\lambda}^t = 0$. □

Lemma 1.2. *Let T be a torus over F. Let $\mathscr{T}^{\mathrm{ft}}$ be as above and for $m \ge 1$, let $\breve{T}_m = \mathrm{Ker}(\mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}}) \to \mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}}/\mathfrak{p}^m_{\breve{F}}))$. Let $e \ge m + 4h$. If $\breve{F}$ and $\breve{F}'$ are e-close, we have an isomorphism*

$$\breve{\mathscr{T}}_m : T(\breve{F})/\breve{T}_m \to T'(\breve{F}')/\breve{T}'_m.$$

Proof. Since $\mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}}) = T(\breve{F})_b$, we have by Theorem 0.1 (which also holds over $\breve{F}$; see [Chai and Yu 2001]) an isomorphism

$$T(\breve{F})_b/\breve{T}_m \to T'(\breve{F}')_b/\breve{T}'_m. \tag{1-1}$$

Since T splits over an at most m-ramified extension of F, the action of Γ_F on $X_*(T)$ factors through Γ_F/I_F^m. Since the action of Γ_F/I_F^m on $X_*(T)$ is Del_m-equivariant, we have $X_*(T)_{I_F} \cong X_*(T)_{I_{F'}}$ and $X_*(T)_{I_F}/\mathrm{tor} \cong X_*(T)_{I_{F'}}/\mathrm{tor}$ via Del_m. We identify these groups via these isomorphisms. Let $\varpi_{\widetilde{F}'}$ be a uniformizer of $\widetilde{F}'$ such that $\varpi_{\widetilde{F}} \bmod \mathfrak{p}^{rm}_{\widetilde{F}} \mapsto \varpi_{\widetilde{F}'} \bmod \mathfrak{p}'^{rm}_{\widetilde{F}}$ where $r = [\widetilde{F} : \breve{F}]$. For $1 \le i \le n$, define $n'_{\tilde{\lambda}_i} = \tilde{\lambda}_i(\varpi_{\widetilde{F}'})$, $n'_{\breve{\lambda}^t_i} = \mathrm{Nm}_{\widetilde{F}'/\breve{F}'} n_{\tilde{\lambda}_i}$. Form the subgroup $\mathscr{S}' \subset T(\breve{F}')$ as before. Since $\nabla_{T,\breve{F}}$, $\nabla_{T',\breve{F}'}$ are group isomorphisms, we get

$$T(\breve{F})/\breve{T}_m \cong X_*(T)_{I_F}/\mathrm{tor} \times T(\breve{F})_b/\breve{T}_m,$$

and similarly over $\breve{F}'$. These observations, combined with (1-1), finish the proof of the lemma. □

Lemma 1.3. *The isomorphism $\breve{\mathscr{T}}_m : T(\breve{F})/\breve{T}_m \to T'(\breve{F}')/\breve{T}'_m$ of Lemma 1.2 is σ-equivariant. It induces a group isomorphism $\mathscr{T}_m : T(F)/T_m \to T'(F')/T'_m$.*

Proof. We know that the isomorphism in (1-1) is σ-equivariant. We need to see that for $\breve{\lambda}^t \in \mathscr{S}$, $\sigma(n_{\breve{\lambda}}) \bmod \breve{T}_m \mapsto \sigma'(n'_{\breve{\lambda}}) \bmod \breve{T}'_m$. It suffices to see this for $\breve{\lambda}^t_i$, $1 \le i \le n$. Fix i and let $\breve{\lambda}^t = \breve{\lambda}^t_i$. Write

$$\sigma(\breve{\lambda}^t) = \sum_j c_j \breve{\lambda}^t_j. \tag{1-2}$$

Let $\tilde{\sigma}$ be any lift of σ to Γ_F/I_F^m and we denote its action on $X_*(T)$ as $\tilde{\sigma}$. We know

$$\sigma(n_{\breve{\lambda}^t}) = \mathrm{Nm}_{\widetilde{F}/\breve{F}} \tilde{\sigma}(n_{\tilde{\lambda}}) = \mathrm{Nm}_{\widetilde{F}/\breve{F}} \tilde{\sigma}(\tilde{\lambda})(\tilde{\sigma}(\varpi_{\widetilde{F}}))$$

and

$$n_{\sigma(\breve{\lambda}^t)} = \prod_j \mathrm{Nm}_{\widetilde{F}/\breve{F}} n_{\tilde{\lambda}_j}^{c_j} = \prod_j \mathrm{Nm}_{\widetilde{F}/\breve{F}} \tilde{\lambda}_j(\varpi_{\widetilde{F}})^{c_j}.$$

Equation (1-2) implies that $\tilde{\sigma}(\tilde{\lambda}) - \sum_j c_j \tilde{\lambda}_j \in X_*(T)(I_F)$, so

$$\tilde{\sigma}(\tilde{\lambda}) - \sum_j c_j \tilde{\lambda}_j = \sum_k d_k(\gamma_k(\tilde{\mu}_k) - \tilde{\mu}_k),$$

for suitable $\gamma_k \in I_F/I_F^m$ and $\tilde{\mu}_k \in X_*(T)$. Now,

$$\tilde{\sigma}(\tilde{\lambda})(\tilde{\sigma}(\varpi_{\widetilde{F}})) = \prod_j \tilde{\lambda}_j(\tilde{\sigma}(\varpi_{\widetilde{F}})^{c_j}) \cdot \prod_k (\gamma_k(\mu_k) - \mu_k)(\tilde{\sigma}(\varpi_{\widetilde{F}})^{d_k})$$

Define

$$u_{\tilde{\lambda},\tilde{\sigma}} = \prod_j \tilde{\lambda}_j((\tilde{\sigma}(\varpi_{\widetilde{F}})\varpi_{\widetilde{F}}^{-1})^{c_j}) \prod_k \mu_k\big(\gamma_k^{-1}(\tilde{\sigma}(\varpi_{\widetilde{F}}))(\tilde{\sigma}(\varpi_{\widetilde{F}})^{-1})^{d_k}\big)$$

and define $u_{\breve{\lambda},\sigma} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, u_{\tilde{\lambda},\tilde{\sigma}}$. Then we have $\sigma(n_{\breve{\lambda}^t}) = u_{\breve{\lambda},\sigma} \cdot n_{\sigma(\breve{\lambda}^t)}$.

By construction of $\breve{\mathscr{T}}_m$, we have $n_{\sigma(\breve{\lambda}^t)} \bmod \breve{T}_m \mapsto n'_{\sigma(\breve{\lambda}^t)} \bmod \breve{T}'_m$. Further $u_{\tilde{\lambda},\tilde{\sigma}} \in T(\widetilde{F})_1$. Recall that $r = [\widetilde{F} : \breve{F}]$. With $\varpi_{\widetilde{F}}$ and $\varpi_{\widetilde{F}'}$ as above, the map $X_*(T) \to T(\widetilde{F})$, $\tilde{\lambda} \mapsto \tilde{\lambda}(\varpi_{\widetilde{F}})$, is a group-theoretic section of the Kottwitz homomorphism over $\widetilde{F}$, and using the Chai–Yu isomorphism $T(\tilde{F})_1/\tilde{T}_{rm} \cong T'(\widetilde{F}')_1/\tilde{T}'_{rm}$ we obtain that

$$\tilde{\mathscr{T}}_{rm} : T(\widetilde{F})/\tilde{T}_{rm} \cong T(\widetilde{F}')/\tilde{T}'_{rm}$$

as groups. Since under the isomorphism $\mathfrak{O}_{\widetilde{F}}/\mathfrak{p}_{\widetilde{F}}^{rm} \cong \mathfrak{O}_{\widetilde{F}'}/\mathfrak{p}_{\widetilde{F}'}^{rm}$, we have

$$\tilde{\sigma}(\varpi_{\widetilde{F}})\varpi_{\widetilde{F}}^{-1} \bmod \mathfrak{p}_{\widetilde{F}}^{rm} \mapsto \tilde{\sigma}'(\varpi'_{\widetilde{F}})\varpi'^{-1}_{\widetilde{F}} \bmod \mathfrak{p}_{\widetilde{F}'}^{rm},$$
$$\gamma_k^{-1}(\tilde{\sigma}(\varpi_{\widetilde{F}}))(\tilde{\sigma}(\varpi_{\widetilde{F}}))^{-1} \bmod \mathfrak{p}_{\widetilde{F}}^{rm} \mapsto \gamma'^{-1}_k(\tilde{\sigma}'(\varpi'_{\widetilde{F}}))(\tilde{\sigma}'(\varpi'_{\widetilde{F}}))^{-1} \bmod \mathfrak{p}_{\widetilde{F}'}^{rm}$$

we have that $u_{\tilde{\lambda},\tilde{\sigma}} \bmod \tilde{T}_{rm} \mapsto u_{\tilde{\lambda}',\tilde{\sigma}'} \bmod \tilde{T}'_{rm}$ via $\tilde{\mathscr{T}}_{rm}$. By the functoriality of the Chai–Yu isomorphism [2001, Section 9.2], we have the commutative diagram

$$\begin{array}{ccc} T(\widetilde{F})_1/\tilde{T}_{rm} & \xrightarrow{\mathrm{Nm}} & T(\breve{F})_1/\breve{T}_m \\ \cong\downarrow & & \downarrow\cong \\ T'(\widetilde{F}')_1/\tilde{T}'_{rm} & \xrightarrow{\mathrm{Nm}} & T'(\breve{F}')_1/\breve{T}'_m \end{array}$$

It follows that $u_{\breve{\lambda},\sigma} \bmod \breve{T}_m \mapsto u'_{\breve{\lambda},\sigma'} \bmod \breve{T}'_m$. We have proved that $\sigma(n_{\breve{\lambda}^t}) \bmod \breve{T}_m \mapsto \sigma'(n'_{\breve{\lambda}^t}) \bmod \breve{T}'_m$ for all $\breve{\lambda}^t = \breve{\lambda}_i^t$, $1 \le i \le n$. Hence this same claim holds for all $\breve{\lambda}^t \in X_*(T)_{I_F}/\mathrm{tor}$. This implies that $\breve{\mathscr{T}}_m$ is σ-equivariant. The claim that $\breve{\mathscr{T}}_m$ restricts to an isomorphism $\mathscr{T}_m : T(F)/T_m \to T'(F')/T'_m$ follows from the fact that $H^1(\sigma, \breve{T}_m) = 1$ (see [Serre 1979, Chapter XII, §3, Lemma 3]). □

1B. ***Some remarks.*** Assume $e \ge m + 4h$. We have σ-equivariant isomorphisms $\breve{\mathscr{T}}_m$ and $\breve{\mathscr{T}}_{m+h}$ constructed above (we also have $\tilde{\mathscr{T}}_{rm}$ and $\tilde{\mathscr{T}}_{r(m+h)}$). Let $t \in T(\breve{F})_b$ with

$\kappa_{T,\breve{F}}(t)=\breve{\mu}$. Write $t=\mathrm{Nm}_{\widetilde{F}/\breve{F}}\ \tilde{t}$, with $\tilde{t}\in T(\widetilde{F})$. By functoriality of the Chai–Yu isomorphism (for $T_{\breve{F}}\hookrightarrow R_{\breve{F}}=\mathrm{Res}_{\widetilde{F}/\breve{F}}T_{\widetilde{F}}$), we have the commutative diagram

$$\begin{array}{ccc} T(\breve{F})_b/\breve{T}_{m+h} & \xrightarrow{\ i\ } & T(\widetilde{F})_b/\tilde{T}_{r(m+h)} \\ {\scriptstyle \breve{\mathscr{T}}_{m+h}}\downarrow & & \downarrow{\scriptstyle \tilde{\mathscr{T}}_{r(m+h)}} \\ T'(\breve{F}')_b/\breve{T}'_{m+h} & \xrightarrow{\ i'\ } & T'(\widetilde{F}')_b/\tilde{T}'_{r(m+h)} \end{array}$$

As explained in [Aubert and Varma 2024, Theorem 2.5.3], it follows from the arguments in [Chai and Yu 2001, Section 8] that $T(\breve{F})_b\cap\tilde{T}_{r(m+h)}\subset\breve{T}_m$. Let $\tilde{t}'\in T(\widetilde{F}')$ be such that $\tilde{\mathscr{T}}_{r(m+h)}(\tilde{t}\bmod\tilde{T}_{r(m+h)})=\tilde{t}'\bmod\tilde{T}'_{r(m+h)}$. Using the Galois equivariance of $\tilde{\mathscr{T}}_{r(m+h)}$ and the commutativity of the above diagram, we have $\breve{\mathscr{T}}_{m+h}(t\bmod T(\breve{F})_b\cap\tilde{T}_{r(m+h)})=t'\bmod T'(\breve{F}')_b\cap\tilde{T}'_{r(m+h)}$ where $t'=\mathrm{Nm}_{\widetilde{F}'/\breve{F}'}\ \tilde{t}'$. Hence $\breve{\mathscr{T}}_m(t\bmod\breve{T}_m)=\breve{\mathscr{T}}_{m+h}(t\bmod\breve{T}_m)=t'\bmod\breve{T}'_m$. By Diagram (7.3.1) in [Kottwitz 1997], $\kappa_{T',\breve{F}'}(t')=\breve{\mu}$.

Now, let $t\in T(\breve{F})$. Write $t=t_1n_{\breve{\mu}^t}$ for suitable $t_1\in T(\breve{F})_b$ and $\breve{\mu}^t\in X_*(T)_{I_F}/\mathrm{tor}$. Then $\kappa_{T,\breve{F}}(t)=\kappa_{T,\breve{F}}(t_1)+\breve{\mu}$. Also $t\bmod\breve{T}_m\mapsto(t_1'\bmod\breve{T}'_m)(n'_{\breve{\mu}^t}\bmod\breve{T}'_m)$ for a suitable $t_1'\in T'(\breve{F}')_b$. Then $\kappa_{T',\breve{F}'}(t_1'n'_{\breve{\mu}^t})=\kappa_{T',\breve{F}'}(t_1')+\breve{\mu}$. By the preceding paragraph, we see that $\kappa_{T,\breve{F}}(t_1)=\kappa_{T',\breve{F}'}(t_1')$. Hence $\breve{\mathscr{T}}_m$ is compatible with the Kottwitz homomorphism $\kappa_{T,\breve{F}}$. Also $\mathscr{T}_m$ is compatible with $\kappa_{T,F}$.

2. Modifications to [Ganapathy 2022, Section 3]

2A. ***Modifications to*** [Ganapathy 2022, Section 3A]**.** The correction given in Section 1 leads to some corrections in [Ganapathy 2022, Section 3]. One important modification is that we need to replace the set of representatives $\{n_{\breve{\lambda}}\mid\breve{\lambda}\in X_*(T)_{I_F}\}$ and $\{n_{\breve{\lambda}_{\mathrm{ad}}}\mid\breve{\lambda}\in X_*(T_{\mathrm{ad}})_{I_F}\}$ used in the proofs in [Ganapathy 2022, Section 3A] with the set of representatives given in Lemma 2.1. Let M, M^*, A, S, T, B and σ be as in [Ganapathy 2022, Section 3]. So M^* is an inner form of a quasisplit connected, reductive group M with $M_{\mathrm{ad}}\cong\mathrm{Res}_{L/F}\,\mathrm{PGL}_n$ for a finite separable extension L/F. Let $\widetilde{F}\supset L\breve{F}$ be the splitting extension of $T_{\breve{F}}$. Let $e=[L:L\cap\breve{F}]$ and $f=[L\cap\breve{F}:F]$. Fix a uniformizer $\varpi_{\widetilde{F}}$ of $\widetilde{F}$.

Lemma 2.1. *Let* $\omega_{T,\breve{F}}:T(\breve{F})\to X_*(T)_{I_F}/\mathrm{tor}$ *and* $\omega_{T_{\mathrm{ad}},\breve{F}}=\kappa_{T_{\mathrm{ad}},\breve{F}}:T_{\mathrm{ad}}(\breve{F})\to X_*(T_{\mathrm{ad}})_{I_F}$ *be the valuation homomorphisms on* T *and* T_{ad}, *respectively. There exist group-theoretic sections* $\nabla_{T,\breve{F}}:X_*(T)_{I_F}/\mathrm{tor}\to T(\breve{F})$ *and* $\nabla_{T_{\mathrm{ad}},\breve{F}}:X_*(T_{\mathrm{ad}})_{I_F}\to T_{\mathrm{ad}}(\breve{F})$ *of* $\omega_{T,\breve{F}}$ *and* $\omega_{T_{\mathrm{ad}},\breve{F}}$, *respectively, such that* $\nabla_{T,\breve{F}}$ *and* $\nabla_{T_{\mathrm{ad}},\breve{F}}$ *agree on the subset* $X_*(T_{\mathrm{sc}})_{I_F}$.

Proof. Let us begin by noting that $X_*(T_{\mathrm{ad}})$ has a $\mathbb{Z}$-basis permuted by Γ_F and $X_*(T_{\mathrm{ad}})_{I_F}$ is torsion-free and admits a $\mathbb{Z}$-basis permuted by σ. Note that $M_{\mathrm{ad},\widetilde{F}}=\prod_{1\le i\le e,1\le j\le f}M^{(i,j)}_{\mathrm{ad},\widetilde{F}}$ where each $M^{(i,j)}_{\mathrm{ad},\widetilde{F}}\cong\mathrm{PGL}_n/\widetilde{F}$. Following the notation of

[Bourbaki 2002], for $1 \le i \le e$, $1 \le j \le f$, let

$$\tilde{\lambda}^{(i,j)}_{\mathrm{ad},n-1} = \epsilon^{(i,j)}_1 - \frac{1}{n}(\epsilon^{(i,j)}_1 + \epsilon^{(i,j)}_2 + \cdots + \epsilon^{(i,j)}_n),$$

and, for $1 \le k \le n-2$,

$$\tilde{\lambda}^{(i,j)}_{\mathrm{ad},k} = \epsilon^{(i,j)}_k - \epsilon^{(i,j)}_{k+1}.$$

The set

$$\{\tilde{\lambda}^{(i,j)}_{\mathrm{ad},k} \mid 1 \le k \le n-1,\ 1 \le i \le e,\ 1 \le j \le f\}$$

yields a $\mathbb{Z}$-basis of $X_*(T_{\mathrm{ad}})$. Let $\mathrm{pr} : X_*(T_{\mathrm{ad}}) \to X_*(T_{\mathrm{ad}})_{I_F}$ be the natural projection. For $1 \le k \le n-1$ and $1 \le j \le f$, let $\breve{\lambda}^{(j)}_{\mathrm{ad},k} = \mathrm{pr}(\tilde{\lambda}^{(1,j)}_{\mathrm{ad},k})$. Then the set

$$\{\breve{\lambda}^{(j)}_{\mathrm{ad},k} \mid 1 \le k \le n-1,\ 1 \le j \le f\}$$

yields a $\mathbb{Z}$-basis of $X_*(T_{\mathrm{ad}})_{I_F}$. Let

$$n_{\breve{\lambda}^{(j)}_{\mathrm{ad},k}} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\lambda}^{(1,j)}_{\mathrm{ad},k}(\varpi_{\widetilde{F}}),\quad 1 \le k \le n-2, \quad \text{and} \quad n_{\breve{\lambda}^{(j)}_{\mathrm{ad},n-1}} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\lambda}^{(1,j)}_{\mathrm{ad},n-1}(\varpi_{\widetilde{F}}).$$

The elements $n_{\breve{\lambda}^{(j)}_{\mathrm{ad},k}}$, $1 \le k \le n-1$, $1 \le j \le f$, are used to obtain a set of representatives

$$\{n_{\breve{\lambda}_{\mathrm{ad}}} \mid \breve{\lambda}_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})_{I_F}\}$$

that form a group; see Lemma 1.1. Let $\nabla_{T_{\mathrm{ad}},\breve{F}} : X_*(T_{\mathrm{ad}})_{I_F} \to T_{\mathrm{ad}}(\breve{F})$, $\breve{\lambda}_{\mathrm{ad}} \mapsto n_{\breve{\lambda}_{\mathrm{ad}}}$, denote this group-theoretic section of $\omega_{T_{\mathrm{ad}},\breve{F}}$.

Next note that $X_*(T_{\mathrm{sc}})_{I_F} \subset X_*(T)_{I_F}/\mathrm{tor}$. Hence the elements $\breve{\lambda}^{(j)}_{\mathrm{ad},k}$, $1 \le k \le n-2$, $1 \le j \le f$, lie in $X_*(T)_{I_F}/\mathrm{tor}$. Also, $j(X_*(T)_{I_F}/\mathrm{tor})$ is of finite index in $X_*(T_{\mathrm{ad}})_{I_F}$, so there exists a nonnegative integer r, which we may choose as small as possible, such that for each $1 \le j \le f$, $r \cdot \breve{\lambda}^{(j)}_{\mathrm{ad},n-1} = j(\breve{\lambda}^{(j)}_{n-1})$ for a $\breve{\lambda}^{(j)}_{n-1} \in X_*(T)_{I_F}/\mathrm{tor}$. For the same r, there exists $\tilde{\lambda}^{(1,j)}_{n-1} \in X_*(T)$ such that $j(\tilde{\lambda}^{(1,j)}_{n-1}) = r \cdot \tilde{\lambda}^{(1,j)}_{\mathrm{ad},n-1}$ and $\mathrm{pr}(\tilde{\lambda}^{(1,j)}_{n-1}) = \breve{\lambda}^{(j)}_{n-1}$. For $1 \le k \le n-2$,

$$\tilde{\lambda}^{(1,1)}_{\mathrm{ad},k} \in X_*(T), \quad \mathrm{pr}(\tilde{\lambda}^{(1,1)}_{\mathrm{ad},k}) = \breve{\lambda}^{(1)}_{\mathrm{ad},k} \quad \text{and} \quad j(\breve{\lambda}^{(1)}_{\mathrm{ad},k}) = \breve{\lambda}^{(1)}_{\mathrm{ad},k}.$$

Set

$$n_{\breve{\lambda}^{(j)}_{\mathrm{ad},k}} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\lambda}^{(1,j)}_{\mathrm{ad},k}(\varpi_{\widetilde{F}}),\quad 1 \le k \le n-2, \quad \text{and} \quad n_{\breve{\lambda}^{(j)}_{n-1}} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\lambda}^{(1,j)}_{n-1}(\varpi_{\widetilde{F}}).$$

Now, the set $\{\breve{\lambda}^{(j)}_{\mathrm{ad},k} \mid 1 \le k \le n-2,\ 1 \le j \le f\} \cup \{\breve{\lambda}^{(j)}_{n-1} \mid 1 \le j \le f\}$ is $\mathbb{Z}$-linearly independent. Further, it may be extended to a basis of $X_*(T)_{I_F}/\mathrm{tor}$. For the remaining basis elements of $X_*(T)_{I_F}/\mathrm{tor}$, we choose representatives as in Section 1A. This then yields a set of representatives $\{n_{\breve{\lambda}} \mid \breve{\lambda} \in X_*(T)_{I_F}/\mathrm{tor}\}$ that forms a group. Let $\nabla_{T,\breve{F}} : X_*(T)_{I_F}/\mathrm{tor} \to T(\breve{F})$, $\breve{\lambda} \to n_{\breve{\lambda}}$ denote this group-theoretic section of $\omega_{T,\breve{F}}$. By construction, we have $\nabla_{T,\breve{F}}$ and $\nabla_{T_{\mathrm{ad}},\breve{F}}$ agree on $X_*(T_{\mathrm{sc}})_{I_F}$. This finishes the proof of the lemma. □

Lemmas 3.1 and 3.2 in [Ganapathy 2022] are not affected.

Let $\Omega_{\breve{M}}$ and $\Omega_{\breve{M},\mathrm{ad}}$ be as in [Ganapathy 2022, Section 3A]. We fix a σ-stable alcove $\breve{a}$ in $\mathscr{A}(S, \breve{F})$ and identify $\Omega_{\breve{M}}$ with $\Omega_{\breve{a}}$ and $\Omega_{\breve{M}_{\mathrm{ad}}}$ with $\Omega_{\breve{a},\mathrm{ad}}$. Let $\breve{\nu}_{\mathrm{ad}} = t_{\breve{\eta}_{\mathrm{ad}}}\breve{z}$ be as in [Ganapathy 2022, Section 3A]. With notation as in Lemma 2.1, $\breve{\eta}_{\mathrm{ad}} = \breve{\lambda}^{(1)}_{\mathrm{ad},n-1}$. Let $\breve{z} = \breve{z}^{(1)} = s_1^{(1)} \cdots s_{n-1}^{(1)}$. Let $n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}} \in T_{\mathrm{ad}}(\breve{F})$ be as in Lemma 2.1. We fix a system of pinnings $\{x_{\breve{a}} \mid \breve{a} \in \breve{\Phi}(M, S)\}$ that is σ-stable as in [Ganapathy 2022, Section 3A]. Let $n_{\breve{z}^{(1)}} = n_{s_1^{(1)}} \cdots n_{s_{n-1}^{(1)}}$. Let $\sigma^* = \mathrm{Ad}(n_{\breve{\nu}_{\mathrm{ad}}}) \circ \sigma$ where $n_{\breve{\nu}_{\mathrm{ad}}} = n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}} n_{\breve{z}^{(1)}}$, and let $M^* = M_{\breve{F}}^{\sigma^*}$. Let $\Omega_M = \Omega_{\breve{M}}^{\sigma}$ and $\Omega_{M^*} = \Omega_{\breve{M}}^{\sigma^*}$. Similarly define $\Omega_{M,\mathrm{ad}}$ and $\Omega_{M^*,\mathrm{ad}}$. By [Ganapathy 2022, Lemma 3.2] we have $\Omega_M = \Omega_{M^*}$ and $\Omega_{M,\mathrm{ad}} = \Omega_{M^*,\mathrm{ad}} \cong \mathbb{Z}/n\mathbb{Z}$. The group $j(\Omega_M) \subset \Omega_{M,\mathrm{ad}}$ is cyclic. Assume $[\Omega_{M,\mathrm{ad}} : j(\Omega_M)] = r$ and that $j(\Omega_M) \neq 0$. Let $\breve{\tau}_0 \in \Omega_M \subset \Omega_{\breve{M}}^{\sigma}$ be such that $j(\breve{\tau}_0)$ is a generator of $j(\Omega_M)$. Then $j(\breve{\tau}_0) = \breve{\nu}_{\mathrm{ad}}^r \sigma(\breve{\nu}_{\mathrm{ad}})^r \cdots \sigma^{k-1}(\breve{\nu}_{\mathrm{ad}})^r$. Write $\breve{\tau}_0 = t_{\breve{\lambda}_0}\breve{y}_0$, where $\breve{\lambda}_0 \in X_*(T)_{I_F}^{\sigma}$ and $\breve{y}_0 \in W(M, S)$. Note that $\breve{y}_0 = (\breve{z}^{(1)})^r \sigma(\breve{z}^{(1)})^r \cdots \sigma^{f-1}(\breve{z}^{(1)})^r$. We may and do assume that $\breve{\lambda}_0 \in (X_*(T)_{I_F}/\mathrm{tor})^{\sigma}$. Let $n_{\breve{\lambda}_0} \in T(\breve{F})$ be as in Lemma 2.1. Note that $n_{\breve{\lambda}_0}$ may not be fixed by σ. Let $n_{\breve{y}_0} := n^r_{\breve{z}^{(1)}} \sigma(n^r_{\breve{z}^{(1)}}) \cdots \sigma^{f-1}(n^r_{\breve{z}^{(1)}})$.

Lemma 2.2. *Let $\breve{\tau}_0$ be as in the preceding paragraph. There exists $v \in T(\breve{F})_1$ such that $n_{\breve{\tau}_0} = v n_{\breve{\lambda}_0} n_{\breve{y}_0} \in M^*(F)$ and $\kappa_{M^*,F}(n_{\breve{\tau}_0}) = \breve{\tau}_0$.*

Proof. Recall that we have fixed representatives $\{n_{\breve{\lambda}} \mid \breve{\lambda} \in X_*(T)_{I_F}/\mathrm{tor}\}$ that forms a group. Note that $\sigma(\breve{\lambda}_0) = \breve{\lambda}_0$ and $\sigma(\breve{y}_0) = \breve{y}_0$. Let us compute $\sigma^*(n_{\breve{\lambda}_0} n_{\breve{y}_0})$. Using the definition of $n_{\breve{y}_0}$, we have $\sigma(n_{\breve{y}_0}) = n_{\breve{y}_0}$. Using [Ganapathy 2022, Lemma 3.1(b)], we have

$$\sigma^*(n_{\breve{\lambda}_0} n_{\breve{y}_0}) = \sigma^*(n_{\breve{\lambda}_0}) n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - \breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})} n_{\breve{y}_0}.$$

Now,

$$u = \sigma^*(n_{\breve{\lambda}_0}) n^{-1}_{\sigma^*(\breve{\lambda}_0)} \in T(\breve{F})_1$$

since its image under $\kappa_{T,\breve{F}}$ is 0. Since $H^1(\sigma^*, T(\breve{F})_1) = 1$, there exists $v \in T(\breve{F})_1$ such that $\sigma^*(v)v^{-1} = u^{-1}$. Now $\sigma^*(v n_{\breve{\lambda}_0}) = v u^{-1} \sigma^*(n_{\breve{\lambda}_0}) = v n_{\sigma^*(\breve{\lambda}_0)}$. Then

$$\sigma^*(v n_{\breve{\lambda}_0} n_{\breve{y}_0}) = v n_{\sigma^*(\breve{\lambda}_0)} n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - \breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})} n_{\breve{y}_0} = v n_{\sigma^*(\breve{\lambda}_0) + \breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - \breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})} n_{\breve{y}_0} = v n_{\breve{\lambda}_0} n_{\breve{y}_0}.$$

The second equality follows from Lemma 2.1 and that $\breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - \breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}) \in X_*(T_{\mathrm{sc}})_{I_F} \subset X_*(T)_{I_F}/\mathrm{tor}$. To get the third equality, note that from the proof of [Ganapathy 2022, Lemma 3.2], $\sigma^*(\breve{\lambda}_0) - \sigma(\breve{\lambda}_0) = \breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - (\mathrm{Ad}(z^{(1)})(\breve{y}_0))(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})$ but $\sigma(\breve{\lambda}_0) = \breve{\lambda}_0$ and $\mathrm{Ad}(z^{(1)})(\breve{y}_0) = \breve{y}_0$. This finishes the proof of the lemma. □

Now, given $\breve{\tau} = t_{\breve{\lambda}}\breve{w} \in \Omega_M$ with $t_{\breve{\lambda}} \in X_*(T)_{I_F}$ and $\breve{w} \in W(M, S)$, we have $j(\breve{\tau}) = sj(\breve{\tau}_0)$ for a unique nonnegative integer s with $0 \leq s < n/r$. Let $\breve{\mu} = \breve{\tau} - s\breve{\tau}_0$. Write $\breve{\mu} = t_{\breve{\mu}_0} \cdot \breve{w}_0 \in \Omega_M$. Then $j(\breve{\tau}) = sj(\breve{\tau}_0)$ implies that $\breve{w} = \breve{y}_0^s$, so $\breve{w}_0 = 1$ and the element $\breve{\mu}$ is just given by the translation $t_{\breve{\mu}_0} \in X_*(T)_{I_F}$. We identify $\breve{\mu}$ and $\breve{\mu}_0$. Since σ fixes $\breve{\tau}$ and $\breve{\tau}_0$, we have $\sigma(\breve{\mu}) = \breve{\mu}$. We claim that $\sigma^*(\breve{\mu}) = \breve{\mu}$. To see this, note that since $j(\breve{\mu})=0$, we have that $j(\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})-\breve{\mu})=0$, but since $\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})-\breve{\mu}\in X_*(T_{\mathrm{sc}})_{I_F}$, and since j acts as identity on $X_*(T_{\mathrm{sc}})_{I_F}$, it follows that $\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})-\breve{\mu}=0$. This then implies that $\sigma^*(\breve{\mu})=\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})=\breve{\mu}$. So $\breve{\mu}\in X_*(T)_{I_F}^{\sigma^*}$. Set $n_{\breve{\tau}}=n_{\breve{\mu}}n_{\breve{\tau}_0}^s$ with $n_{\breve{\mu}}\in T^*(F)$ satisfies $\kappa_{T^*,F}(n_{\breve{\mu}})=\breve{\mu}$.

Proposition 2.3. *Let $\breve{\tau}\in\Omega_{M^*}=\Omega_M$. Then $\sigma^*(n_{\breve{\tau}})=n_{\breve{\tau}}$. In particular, $n_{\breve{\tau}}\in M^*(F)$ and $\widetilde{p}:\Omega_{M^*}\to M^*(F)$, $\breve{\tau}\mapsto n_{\breve{\tau}}$, is a (set-theoretic) section of $\kappa_{M^*,F}$.*

Proof. It suffices to prove that $\sigma^*(n_{\breve{\tau}_0})=n_{\breve{\tau}_0}$, but this is Lemma 2.2. □

2B. ***Modifications to*** **[Ganapathy 2022, Section 3B].** Via Del_m, we have isomorphisms $X_*(T)\cong X_*(T')$ and $X_*(T_{\mathrm{ad}})\cong X_*(T'_{\mathrm{ad}})$ that are Γ_F/I_F^m-equivariant, and $\Omega_{\breve{M}}\cong\Omega_{\breve{M}'}$ and $\Omega_{\breve{M}_{\mathrm{ad}}}\cong\Omega_{\breve{M}'_{\mathrm{ad}}}$. We identify these groups via these isomorphisms. We construct $\nabla_{T',\breve{F}'}:X_*(T)_{I_F}/\mathrm{tor}\to T'(\breve{F}')$, $\breve{\lambda}\mapsto n'_{\breve{\lambda}}$, and $\nabla_{T'_{\mathrm{ad}},\breve{F}'}:X_*(T_{\mathrm{ad}})_{I_F}\to T'_{\mathrm{ad}}(\breve{F}')$, $\breve{\lambda}\mapsto n_{\breve{\lambda}}$, exactly as in Lemma 2.1, but with $\varpi_{\widetilde{F}}$ replaced with $\varpi_{\widetilde{F}'}$ where $\varpi_{\widetilde{F}}\bmod\mathfrak{p}_{\widetilde{F}}^{rm}\mapsto\varpi_{\widetilde{F}'}\bmod\mathfrak{p}_{\widetilde{F}}'^{rm}$ as in Lemma 1.2. Let $\breve{\tau}_0$ be as in Lemma 2.2. Let $n'_{\breve{\lambda}_0},n'_{\sigma^*(\breve{\lambda}_0)}\in T'(\breve{F}')$ be such that under $\breve{\mathscr{T}}_m$, $n_{\breve{\lambda}_0}\bmod\breve{T}_m\mapsto n'_{\breve{\lambda}_0}\bmod\breve{T}'_m$, and similarly for $n'_{\sigma^*(\breve{\lambda}_0)}$. Then, since $\breve{\mathscr{T}}_m$ is σ^*-equivariant, we have $u\bmod\breve{T}_m\mapsto u'\bmod\breve{T}'_m$, where $u'=\sigma'^*(n'_{\breve{\lambda}_0})n'^{-1}_{\sigma'^*(\breve{\lambda}_0)}$. By the proof of the fact that $H^1(\sigma^*,T(\breve{F})_1)=1$ [Serre 1979, Chapter XII, §3, Lemma 3], it follows that we may choose $v'\in T(\breve{F})_1$ such that $\sigma^*(v')v'^{-1}=u'^{-1}$ and such that $v\bmod\breve{T}_m\mapsto v'\bmod\breve{T}'_m$. Let $n'_{\breve{y}_0}=n'^r_{\breve{z}^{(1)}}\sigma'(n'_{\breve{z}^{(1)}})^r\cdots\sigma^{f-1}(n'_{\breve{z}^{(1)}})^r$. Set $n'_{\breve{\tau}_0}=v'n'_{\breve{\lambda}_0}n'_{\breve{y}_0}$. Given $\breve{\tau}\in\Omega_M$, we may write $\breve{\tau}=\breve{\mu}+s\breve{\tau}_0$ for a unique $0\le s<n/r$ as in the paragraph preceding Proposition 2.3. Set $n'_{\breve{\tau}}=n'_{\breve{\mu}}n'^s_{\breve{\tau}_0}$ where $n'_{\breve{\mu}}\in T'^*(F')$ with $\mathscr{T}_m^*(n_{\breve{\mu}}\bmod T_m^*)\mapsto n'_{\breve{\mu}}\bmod T'^*_m$. Note that $\kappa_{T'^*,F'}(n'_{\breve{\mu}})=\breve{\mu}$ by Section 1B. By Proposition 2.3, $n'_{\breve{\tau}}\in M'^*(F')$.

Proposition 2.4 [Ganapathy 2022, Proposition 3.4]. *Let $m\ge 1$ and let $e\ge m+4h$. If the fields F and F' are e-close, then we have an isomorphism $M^*(F)/M_m^*\cong M'^*(F')/M'^*_m$.*

Proof. The proof given in [Ganapathy 2022, Proposition 3.4] works with straightforward modifications.

Consider the set theoretic section $\widetilde{p}:\Omega_{M^*}\to M^*(F)$ in Proposition 2.3 and let p be its composition with the natural projection $M^*(F)\to M^*(F)/M_m^*$. Similarly, we get $\tilde{p}':\Omega_{M^*}\xrightarrow{\widetilde{p}}M'^*(F')$ and p'.

It suffices to prove that the sections p and p' satisfy (a) and (b) of [Ganapathy 2022, Proposition 3.4].

To see (a), it suffices to prove that

$$\begin{array}{ccc} M^*(F)_1/M_m^* & \xrightarrow{\cong} & M'^*(F')_1/M'^*_m \\ \Big\downarrow{\scriptstyle \mathrm{Inn}(n_{\breve{\tau}})} & & \Big\downarrow{\scriptstyle \mathrm{Inn}(n'_{\breve{\tau}})} \\ M^*(F)_1/M_m^* & \xrightarrow{\cong} & M'^*(F')_1/M'^*_m \end{array}$$

is commutative for $\breve{\tau} \in \Omega_{M^*}$. Let $\breve{P}$ be the Iwahori subgroup of $M(\breve{F})$ $(= M^*(\breve{F}))$ attached to the σ-stable alcove $\breve{a}$ and let $\breve{P}'$ be the corresponding Iwahori subgroup of $M'(\breve{F}')$. Then by [Ganapathy 2019, Theorem 4.5], we have that $\breve{P}/\breve{P}_m \cong \breve{P}'/\breve{P}'_m$. Since $\breve{\nu}_{\mathrm{ad}} \in \Omega_{\breve{a},\mathrm{ad}}$, the alcove $\breve{a}$ is also σ^*-stable. By Propositions 4.10 and 6.2 in [Ganapathy 2019], the isomorphism $\breve{P}/\breve{P}_m \cong \breve{P}'/\breve{P}'_m$ is σ- and σ^*-equivariant. This implies that $\breve{P} \cap M^*(F) = M^*(F)_1$, $\breve{P}_m \cap M^*(F) = M^*_m$ and similarly that $\breve{P}' \cap M'^*(F') = M'^*(F')_1$, $\breve{P}'_m \cap M'^*(F') = M'^*_m$. Since $\breve{\tau} \in \Omega_{M^*} = \Omega_{\breve{a}}^{\sigma^*} \subset \Omega_{\breve{a}}$, we see that $n_{\breve{\tau}}$ normalizes $\breve{P}$ and $\breve{P}_m$. To finish the proof of (a), it suffices to observe that the following diagram is commutative:

$$\begin{array}{ccc} \breve{P}/\breve{P}_m & \xrightarrow{\cong} & \breve{P}'/\breve{P}'_m \\ \big\downarrow{\scriptstyle \mathrm{Inn}(n_{\breve{\tau}})} & & \big\downarrow{\scriptstyle \mathrm{Inn}(n'_{\breve{\tau}})} \\ \breve{P}/\breve{P}_m & \xrightarrow{\cong} & \breve{P}'/\breve{P}'_m \end{array}$$

This follows by arguing as in the proof of [Ganapathy 2019, Proposition 6.2].

Let us prove (b). The element $n_{\breve{y}_0}^{n/r}$ equals $\breve{a}^\vee(-1) \in M^*(F)_1$ for a suitable $\breve{a} \in \breve{\Phi}(M, S)$.

Let $\breve{\tau}_1, \breve{\tau}_2 \in \Omega_{M^*}$. As in the proof of Proposition 2.3, write $\breve{\tau}_i = \breve{\mu}_i + s_i \breve{\tau}_0$, and $\breve{\tau}_1 + \breve{\tau}_2 = \breve{\mu} + s\breve{\tau}_0$. Note that $s \bmod (n/r) \equiv s_1 + s_2 \bmod (n/r)$.

Recall that $n_{\breve{\mu}}, n_{\breve{\mu}_1}, n_{\breve{\mu}_2} \in T^*(F)$ and $n_{\breve{\mu}} \bmod T^*_m \mapsto n'_{\breve{\mu}} \bmod T'^*_m$ and for $i = 1, 2$, $n_{\breve{\mu}_i} \bmod T^*_m \mapsto n'_{\breve{\mu}_i} \bmod T'^*_m$. Write $n^s_{\breve{\tau}_0} = t_s n^s_{\breve{y}_0}$ where $t_s \in T(\breve{F})$. Similarly write $n'^s_{\breve{\tau}_0} = t'_s n'^s_{\breve{y}_0}$. Then it is straightforward to see that $t_s \bmod \breve{T}_m \mapsto t'_s \bmod \breve{T}'_m$ via $\breve{\mathscr{T}}_m$. The same claim holds for t_{s_i}, $i = 1, 2$. Also, $\breve{a}^\vee(-1) \bmod \breve{T}_m \to \breve{a}'^\vee(-1) \bmod \breve{T}_m$. Finally, we note that $n_{\breve{\tau}_1+\breve{\tau}_2} n_{\breve{\tau}_1}^{-1} n_{\breve{\tau}_2}^{-1} \in M^*(F)_1 \cap T(\breve{F})$ and by [Ganapathy 2019, Proof of Proposition 6.2 and Corollary 6.3], we see that on the subgroup $M^*(F)_1 \cap T(\breve{F})$ the isomorphism of [Ganapathy 2019, Corollary 6.3] restricts to $\mathscr{T}^*_m$. Hence the sections p, p' satisfy (b). □

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