Pacific Journal of Mathematics

Volume 330 No. 2 June 2024

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.199

INTEGRALITY RELATIONS FOR POLYGONAL DISSECTIONS

AARON ABRAMS AND JAMES POMMERSHEIM

Given a parallelogram dissected into triangles, the area of any one of the triangles of the dissection is integral over the ring generated by the areas of the other triangles. Given a trapezoid dissected into triangles, the area of any triangle determined by either diagonal of the trapezoid is integral over the ring generated by the areas of the triangles in the dissection. In both cases, the integrality relations are invariant under deformation of the dissection.

The trapezoid theorem implies and provides a new context for Monsky's equidissection theorem that a square cannot be dissected into an odd number of triangles of equal area. A corollary of these results is that the area polynomials for parallelograms we introduced and studied in previous work (2014; 2022; 2023) have all leading coefficients equal to ±1.

1. Introduction

We establish several new results about the geometry of dissections of certain Euclidean plane polygons. A *dissection* of such a polygon T into triangles is a collection of triangles in the plane whose union is T and whose interiors are disjoint.

Theorem 1. *Let T be a trapezoid in the Euclidean plane with vertices $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}$, in counterclockwise order. Suppose that T is dissected into n triangles of areas $a_1, \ldots, a_n$. Then the area of the triangle $\boldsymbol{pqs}$ is integral over $\mathbb{Z}[a_1, \ldots, a_n]$.*

Theorem 2. *Let T be a parallelogram in the Euclidean plane with a dissection into n triangles of areas $a_1, \ldots, a_n$. Then a_n is integral over $\mathbb{Z}[a_1, \ldots, a_{n-1}]$.*

Theorem 1 immediately implies Monsky's theorem [1970] that a parallelogram cannot be dissected into an odd number of triangles of equal area, since $\frac{1}{2}$ is not integral over $\mathbb{Z}[1/n]$ when n is odd. Thus Theorem 1 generalizes and provides a new context for Monsky's theorem. However, this cannot be considered a new proof of Monsky's theorem, since our proof proceeds along the same lines as the original, using valuations to 3-color points of a certain affine plane and appealing to Sperner's lemma. See [Monsky 1970; Jepsen and Monsky 2008].

MSC2020: primary 52B45; secondary 51M25.

Keywords: integrality relation, dissection, Monsky polynomial.

We also show that in a certain sense, the integrality relations arising in these theorems are invariant under deformation; that is, the integrality relations actually hold for the quadratic polynomials that express the areas of the triangles, and not just for the numerical areas a_i. See Theorems 1+ and 3 below.

Theorem 2 goes hand in hand with a result about the *area polynomial* p_T that was introduced in [Abrams and Pommersheim 2014] and further studied in [Abrams and Pommersheim 2022; 2023]. For any combinatorial triangulation T of a quadrilateral, there is a unique (up to sign) nonzero homogeneous irreducible integer polynomial p_T with one variable A_i for each triangle such that $p_T(a_1, \ldots, a_n) = 0$ whenever T is drawn in the plane with a parallelogram boundary and triangles of areas $a_1, \ldots, a_n$. Here by *combinatorial triangulation* we mean a simplicial complex homeomorphic to a disk, with four vertices on the boundary. (The connection with dissections is that every dissection of a planar trapezoid can be viewed as the image of a combinatorial triangulation under a piecewise linear map to the plane which may collapse some triangles; see, e.g, [AP 2022, Propositions 2 and 5].) The mod 2 structure of p_T is completely specified by [AP 2022, Theorem 9.1], which implies in particular that the coefficients of the leading terms are odd integers. Further, in [AP 2023, Theorem 6.2] it is shown that these leading terms must all be equal up to sign.

Theorem 3. *For any combinatorial triangulation T, the area polynomial p_T is monic. That is, for any i the coefficient of A_i^d is ± 1, where $d = \deg p_T$.*

This is a special case of the positivity conjecture from [AP 2022, Conjecture 4].

Remark. Monsky's equidissection theorem applies to arbitrary dissections, as do our Theorems 1 and 2, whereas the combinatorial triangulations of Theorem 3 are by definition simplicial complexes. It is easy to see that Theorem 3 also holds for any dissection that has an area polynomial (the "hyper" case in the language of [AP 2022, Defintion 26]). However it is not known whether every dissection of a parallelogram has this property; this question is discussed in [AP 2022, Section 8].

We also note that integrality conditions have previously appeared in theorems about equidissections of trapezoids. For example, [Jepsen and Monsky 2008, Theorem 1.1] (see also [Kasimatis and Stein 1990]) gives a necessary condition for the existence of an equidissection of a trapezoid of a given shape into a given number of triangles. Theorem 1 strengthens that result.

The theorems are proved by combining ideas originally due to Monsky [1970] with some technical machinery developed in [AP 2014; 2022; 2023]. Some familiarity with those works may be helpful for the reader; in order to focus on the results, we have not attempted to make the arguments here entirely self-contained.

2. Integrality for trapezoids

In this section, we prove Theorem 1 by establishing an integrality relation for the triangle $\boldsymbol{pqs}$ of a dissected trapezoid. In fact we prove a stronger version of this theorem (Theorem 1+) that allows deformations of the trapezoid.

Let T be a combinatorial triangulation of a quadrilateral $\boldsymbol{pqrs}$. For each vertex v other than $\boldsymbol{r}$, we introduce two variables x_v and y_v. We treat $v = \boldsymbol{r}$ differently so that our ring will reflect the geometric condition that $\boldsymbol{pqrs}$ be a trapezoid rather than an arbitrary quadrilateral. For this final vertex, we introduce a variable t which represents the ratio of the lengths of side $\boldsymbol{sr}$ to side $\boldsymbol{pq}$. Thus we work in the polynomial ring

$$\mathcal{R} = \mathbb{C}\big[\big\{x_v, y_v \mid v \in \text{Vertices}(T) \setminus \{\boldsymbol{r}\}\big\}, t\big].$$

In $\mathcal{R}$, we use the abbreviations $x_{\boldsymbol{r}} = x_{\boldsymbol{s}} + t(x_{\boldsymbol{q}} - x_{\boldsymbol{p}})$ and $y_{\boldsymbol{r}} = y_{\boldsymbol{s}} + t(y_{\boldsymbol{q}} - y_{\boldsymbol{p}})$. In $\mathcal{R}$, it is natural to consider the variables x_v and y_v as having degree 1, while t has degree 0.

Orienting the boundary in the direction $\boldsymbol{pqrs}$ endows each triangle Δ_i of T with an orientation. For each Δ_i, we introduce a quadratic polynomial $W_i \in \mathcal{R}$ which expresses twice the area of the oriented triangle Δ_i. For convenience, we prefer to work with doubled areas throughout. This makes little difference, as all the relations we obtain will be homogeneous. We use $W_U \in \mathcal{R}$ to denote the quadratic polynomial representing twice the area of triangle $\boldsymbol{psq}$; this choice of orientation is consistent with the other triangles. We sometimes abuse language and refer to the W_i and W_U as the areas.

Theorem 4 (**Theorem 1+**). *Let T be a combinatorial triangulation of a quadrilateral $\boldsymbol{pqrs}$ into n triangles. Let $W_1, \dots, W_n \in \mathcal{R}$ denote the polynomials expressing the areas of the triangles of T, and let $W_U \in \mathcal{R}$ denote the polynomial expressing the area of the triangle $\boldsymbol{psq}$. Then W_U is integral over $\mathbb{Z}[W_1, \dots, W_n]$.*

Proof. We use many of the ideas from the proof of Theorem 7.2 (Monsky+) from [AP 2022]. To show that W_U is integral over $\mathcal{S} = \mathbb{Z}[W_1, \dots, W_n]$, it is enough to show that if ν is a valuation on the fraction field of $\mathbb{Z}[W_U, W_1, \dots, W_n]$ such that $\nu(W_i) \geq 0$ for all i, then $\nu(W_U) \geq 0$ (see, e.g, [Atiyah and Macdonald 1969, 5.22]). Given such a ν, extend it to the fraction field $\mathcal{F} = \text{Frac}(\mathcal{R})$ and, following Monsky [1970], use ν to color each point of $\mathcal{F} \times \mathcal{F}$ one of three colors A, B, C as in the proof of [AP 2022, Theorem 7.2].

Let $M : \mathcal{F} \times \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ be the unique affine transformation taking $(x_{\boldsymbol{p}}, y_{\boldsymbol{p}})$ to $(0, 0)$, $(x_{\boldsymbol{q}}, y_{\boldsymbol{q}})$ to $(1, 0)$, and $(x_{\boldsymbol{s}}, y_{\boldsymbol{s}})$ to $(0, 1)$. Note that

$$\det M = \begin{vmatrix} x_{\boldsymbol{q}} - x_{\boldsymbol{p}} & x_{\boldsymbol{s}} - x_{\boldsymbol{p}} \\ y_{\boldsymbol{q}} - y_{\boldsymbol{p}} & y_{\boldsymbol{s}} - y_{\boldsymbol{p}} \end{vmatrix}^{-1} = -W_U^{-1}.$$

We now color the vertices of T by using M to pull back the coloring of $\mathcal{F} \times \mathcal{F}$. That is, if v is a vertex of T, then we color v with the color of the point $M(x_v, y_v)$.

This assigns $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{s}$ the colors C, A, B, respectively. As for $\boldsymbol{r}$, one sees that $M(x_{\boldsymbol{r}}, y_{\boldsymbol{r}}) = (t, 1)$, so $\boldsymbol{r}$ has color A or B. The boundary of T is thus colored $CAAB$ or $CABB$, and in either case we may apply Sperner's lemma to conclude that T has an ABC triangle Δ_j. For such a triangle we have $\nu(\text{Area}(M\Delta_j)) \leq 0$, which means $\nu(-W_U^{-1} W_j) \leq 0$. Hence $\nu(W_j) \leq \nu(W_U)$, which implies $\nu(W_U) \geq 0$. □

We now show that Theorem 1+ implies Theorem 1.

Proof. Let $\Delta = \boldsymbol{pqrs}$ be a trapezoid in the plane with a dissection into n triangles of areas $a_1, \dots, a_n$, and let u denote the area of triangle $\boldsymbol{psq}$. As in [AP 2022, Propositions 2.6, 3.2], there exists a combinatorial triangulation T with $m \geq n$ triangles obtained by poofing the dissection, and a drawing ρ of T that has the same set of nondegenerate triangles as the original dissection along with $m - n$ degenerate triangles of area 0. By Theorem 1+, there is an integral equation $g_T(W_U, W_1, \dots, W_m) = 0$, where we may take g_T to be homogeneous in its $m+1$ variables. If $u = 0$, then we are done. Otherwise, $\rho(\boldsymbol{p}) \neq \rho(\boldsymbol{q})$, and we may solve for t and substitute this value along with the given values of x_i and y_i into g_T. After this substitution the W_j corresponding to degenerate triangles vanish. As the W_i and W_U stand for twice the areas, we now divide by $2^{\deg g_T}$ to get the desired integral equation for u over $a_1, \dots, a_n$. □

We conclude this section with a consequence for parallelograms which generalizes a theorem of Monsky.

Corollary 5. *Let $T = \boldsymbol{pqrs}$ be a parallelogram in the Euclidean plane with a dissection into n triangles of areas $a_1, \dots, a_n$. Let σ denote the area of T. Then $\frac{1}{2}\sigma$ is integral over $\mathbb{Z}[a_1, \dots, a_n]$.*

This corollary implies the fact due to Monsky [1970] that if a square of area 1 in the Euclidean plane is dissected into n triangles of areas $a_1, \dots, a_n$, then there is a polynomial f with integer coefficients such that $2f(a_1, \dots, a_n) = 1$. (To see this, take the integral equation for $\frac{1}{2}\sigma = \frac{1}{2}$ and multiply by a power of 2 to clear denominators.) Likewise, Theorem 17 of [AP 2022], which extends Monsky's theorem to handle deformations, can be derived from Theorem 1+.

3. The area map for trapezoids

Theorem 1+ tells us that W_U is integral over $\mathbb{Z}[W_1, \dots, W_n]$, i.e., there exists a polynomial $g = g_T \in \mathbb{Z}[U, B_1, \dots, B_n]$, monic in U, such that $g(W_U, W_1, \dots, W_n) = 0$ in $\mathcal{R}$. Assuming that g has been chosen with minimal degree, we will now show that almost all points in the zero set of g are realized as areas of triangles in an actual trapezoidal drawing of T. For this purpose, we introduce a *drawing space* $\text{Trap}(T)$ and an *area map* for this situation.

Let T be a combinatorial triangulation of a quadrilateral with corners $\boldsymbol{pqrs}$. A *drawing* of T is a map $\rho : \text{Vertices}(T) \to \mathbb{C}^2$ that takes $\boldsymbol{pqrs}$ to a trapezoid; this means that the vectors $\boldsymbol{q} - \boldsymbol{p}$ and $\boldsymbol{r} - \boldsymbol{s}$ are linearly dependent. Let $\text{Trap} = \text{Trap}(T)$ be the space of drawings of T. An open dense subset of Trap is parameterized by the affine space $X = X(T)$ with coordinates x_v, y_v for all vertices v except $\boldsymbol{r}$ and an additional coordinate t. We will keep track of the areas of the triangles of T as well as the area U of the triangle formed by the images of $\boldsymbol{p}$, $\boldsymbol{s}$, and $\boldsymbol{q}$ (even though these vertices probably do not form a triangle of the triangulation); thus let $Y = Y(T)$ denote the projective space with one coordinate for each triangle of T and one additional coordinate U. Now let $\text{Area} : X \dashrightarrow Y$ be the (rational) area map that records the areas of the triangles in the corresponding coordinates and the area of the triangle $\rho(\boldsymbol{p})$, $\rho(\boldsymbol{s})$, $\rho(\boldsymbol{q})$ in the U coordinate.

Let $V = V(T)$ denote the closure of the image of the map Area. Thus $V \subset Y$ is a rational variety.

Theorem 6. *For any T, the variety $V(T)$ is an irreducible hypersurface in Y defined by a homogeneous polynomial $z_T(U, B_1, \dots, B_n)$ that is monic in U.*

Proof. The parameter space X is irreducible, so $V(T)$ is also irreducible. To show $V(T)$ is a hypersurface, we appeal to the argument from [AP 2014, Theorem 5] that Area is generically locally injective after modding out by affine transformations. A dimension count then shows that the image of Area has codimension 1 in Y.

Let z_T be the defining equation of $V(T)$, scaled to have integer coefficients. We wish to show that z_T is monic in U. By Theorem 1+, there exists $g \in \mathbb{Z}[U, B_1, \dots, B_n]$ which is monic in U and such that $g(W_U, W_1, \dots, W_n) = 0$ in $\mathcal{R}$. We assume that we have chosen such a g with minimal degree. Note that $g = g(U, B_1, \dots, B_n)$ vanishes on the image of Area, so z_T divides g.

We now argue that in fact $g = \pm z_T$. The W_i are algebraically independent over $\mathbb{C}$, because if there were a dependence $r(W_1, \dots, W_n) = 0$, we would have z_T divides r, which implies that z_T does not contain the variable U. But then g, which is a multiple of z_T, would not be monic in U, a contradiction. We conclude that $\mathbb{Z}[W_1, \dots, W_n]$ is isomorphic to a polynomial ring, which is a UFD. By Gauss's lemma, the integral equation g may be chosen to be irreducible as a polynomial in $\mathbb{Q}(W_1, \dots, W_n)[U]$. It follows that $g(U, B_1, \dots, B_n)$ is irreducible in $\mathbb{Q}[U, B_1, \dots, B_n]$. From this we see that $g = \pm z_T$, and so z_T is monic in U, as desired. $\square$

4. Integrality for parallelograms

In this section we prove Theorems 2 and 3. The proofs of these integrality theorems for parallelograms rely on our integrality theorem for trapezoids.

The polynomial p_T for parallelograms, studied in [AP 2014; 2022; 2023], can be linked to the polynomial z_T for trapezoids using a simple geometric observation: a trapezoid $T = \boldsymbol{pqrs}$ is a parallelogram if and only if its area is twice the area of triangle $\boldsymbol{pqs}$. For a triangulated trapezoid, this condition is represented by the equation $-2U = S$, where S denotes $\sum_{i=1}^{n} B_i$. This observation implies the relation

$$p_T(B_1, \dots, B_n) \mid z_T(-S, 2B_1, \dots, 2B_n),$$

from which we will tease out the monicity of p_T.

To do this, one further fact about z_T is required.

Proposition 7. *For any T, we have $z_T(U, B, 0, \dots, 0) = \pm U^e (U + B)^f$ for non-negative integers e and f.*

Proving this requires understanding points of V that are not in the image of the area map. The paper [AP 2023] studies this question in a nearly identical context, namely the area map for a triangulated parallelogram. One main conclusion there is that if w is a point of V then either w is in the image of Area or else there is a subset of the coordinates that sums nontrivially to 0. This conclusion is also valid for the trapezoid area map.

Lemma 8. *Suppose $w = [u : b_1 : \cdots : b_n] \in V \setminus \operatorname{Im} \mathrm{Area}$. Let $b_0 = u$. Then there is a subset Z of $\{0, \dots, n\}$ such that $\sum_{i \in Z} b_i = 0$, but $b_i \neq 0$ for some $i \in Z$.*

Proof. We view Area as the area map associated to the complex $\hat{T} = T \cup U$ which is a triangulation of the triangle $\boldsymbol{qrs}$. The proof is nearly identical to the parallelogram case [AP 2023, Main Theorem 3]. Here are the main points of the argument. We use the language of generating paths and bubbles introduced in [AP 2023, Section 3].

Suppose $w \in V \setminus \operatorname{Im} \mathrm{Area}$. Then there is a generating path for w, which is a path $\gamma(s)$ of drawings in Trap converging to a limiting $\rho \in$ Trap as $s \to 0$ and such that $\mathrm{Area}(\gamma(s)) \to w$.

If ρ maps the boundary $\boldsymbol{qrs}$ to a single point, then ρ contains a bubble. Otherwise there are two adjacent points of the boundary V_1 and V_2 such that $\rho(V_1) \neq \rho(V_2)$. Using an invertible affine transformation we may assume $\rho(V_1) = (0, 0)$ and $\rho(V_2) = (1, 0)$, and a further affine transformation that converges to the identity as $s \to 0$ fixes $\gamma(s)(V_1) = (0, 0)$ and $\gamma(s)(V_2) = (1, 0)$. We then rescale vertically so that some vertex is not converging to the x-axis. This produces a new generating path, with a limiting drawing that we still call ρ. By the elastic lemma of [AP 2023], ρ must have a bubble.

We conclude that there exists a generating path for w with a bubble. The bubble corollary of [AP 2023] then asserts that the coordinates inside this bubble sum to zero but are not all zero. □

We now prove the proposition.

Proof. From Theorem 1+, z_T is monic in U and hence also $z_T(U, B, 0, \dots, 0)$ is monic in U. Thus it suffices to show that the only zeros of $z_T(U, B, 0, \dots, 0)$ have $U = 0$ or $U = -B$.

Note that $[1 : 0 : \cdots]$ is not in V, again since z_T is monic in U. So we may assume $B \neq 0$, and suppose $w = [U : 1 : 0 : \cdots] \in V$. We will show that $U = 0$ or $U = -1$.

If $U = -1$, we are done. Otherwise, by Lemma 8, we have $w \in \operatorname{Im}\mathrm{Area}$. Thus, there is a drawing with $B = 1$ and the areas of all other triangles of T equal to 0. It follows from [AP 2022, Corollary 5.6(1)] that the boundary of T must be drawn as a degenerate trapezoid. But the vertices of the boundary cannot all be collinear, since then the B_i would sum to 0. Thus the image of the boundary is a nondegenerate triangle of area 1, and the four points $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$ map onto the three corners of this triangle. Thus we see that either the U triangle, $\boldsymbol{psq}$, or the U' triangle, $\boldsymbol{qsr}$, is degenerate. However U and U' add up to $-\sum B_i$, which equals -1. Hence $U = 0$ or $U = -1$. ☐

We now prove Theorems 3 and 2, in that order.

Proof. We first consider the coefficient α of B_1^d in the polynomial

$$\tilde{z}(B_1, \dots, B_n) = z(-S, 2B_1, \dots, 2B_n).$$

This coefficient α is the same as the coefficient of B_1^d in $z(-B_1, 2B_1, 0, \dots, 0)$, which equals $\pm(-B_1)^e B_1^f$ by the proposition. Thus $\alpha = \pm 1$. Since p is a factor of $\tilde{z}$, it follows from Gauss's lemma that $B_1^{d'}$ has coefficient ± 1 in p, where d' is the degree of p. This proves Theorem 3.

To prove Theorem 2 for triangulations, we view the polynomial $p_T(B_1, \dots, B_n)$ as a polynomial in B_n with coefficients in $\mathbb{Z}[B_1, \dots, B_{n-1}]$. We have just established that the leading coefficient is ± 1. Thus p_T provides the required integral equation for B_n over $\mathbb{Z}[B_1, \dots, B_{n-1}]$.

To prove Theorem 2 for dissections, apply the poofing argument used in Theorem 1 to produce a combinatorial triangulation to which the previous paragraph applies. ☐

Example. The triangulation T_n with vertices $\boldsymbol{p} = p_0, p_1, \dots, p_{n+1} = \boldsymbol{r}$, $\boldsymbol{q}$, $\boldsymbol{s}$ and triangles $A_i = \boldsymbol{s}\, p_{i-1} p_i$ and $B_i = \boldsymbol{q}\, p_i p_{i-1}$ (for $1 \le i \le n+1$), called the *diagonal case* in [AP 2014], has

$$z_{T_n} = \left(\prod_{k=0}^{n+1} \ell_k\right)\left(\frac{1}{\ell_0} - \sum_{k=0}^{n} \frac{A_{k+1}}{\ell_k \ell_{k+1}}\right)$$

where ℓ_k stands for the linear form $A_1 + \cdots + A_k + B_1 + \cdots + B_k + U$. Its degree is $n+1$. For example $z_{T_1} = U^2 + 2UB_1 + UB_2 + UB_4 + B_1^2 + B_1B_2 + B_1B_3 + B_1B_4$. We then have $z_{T_n}(-S, 2A_i, 2B_i) = S \cdot p_{T_n}$, where p_{T_n} is computed in [AP 2014].

Acknowledgements

The authors thank Paul Monsky for numerous insightful communications and inspirations. In addition Abrams gratefully acknowledges the support of the MTA Distinguished Guest Scientist Fellowship Programme 2022. Pommersheim thanks the Fulbright US Scholar Program for their support. We also thank Dezső Miklós and András Stipsicz for their roles in making this work possible.

References

[Abrams and Pommersheim 2014] A. Abrams and J. Pommersheim, "Spaces of polygonal triangulations and Monsky polynomials", *Discrete Comput. Geom.* **51**:1 (2014), 132–160. MR Zbl

[Abrams and Pommersheim 2022] A. Abrams and J. Pommersheim, "Generalized dissections and Monsky's theorem", *Discrete Comput. Geom.* **67**:3 (2022), 947–983. MR Zbl

[Abrams and Pommersheim 2023] A. Abrams and J. Pommersheim, "An illustrated encyclopedia of area relations", *Eur. J. Math.* **9**:3 (2023), art. id. 49. MR Zbl

[Atiyah and Macdonald 1969] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969. MR Zbl

[Jepsen and Monsky 2008] C. H. Jepsen and P. Monsky, "Constructing equidissections for certain classes of trapezoids", *Discrete Math.* **308**:23 (2008), 5672–5681. MR Zbl

[Kasimatis and Stein 1990] E. A. Kasimatis and S. K. Stein, "Equidissections of polygons", *Discrete Math.* **85**:3 (1990), 281–294. MR Zbl

[Monsky 1970] P. Monsky, "On dividing a square into triangles", *Amer. Math. Monthly* **77** (1970), 161–164. MR Zbl

Received June 16, 2023. Revised January 30, 2024.

AARON ABRAMS
WASHINGTON AND LEE UNIVERSITY
LEXINGTON, VA
UNITED STATES
Current address:
SCHOOL OF DATA SCIENCE
UNIVERSITY OF VIRGINIA
CHARLOTTESVILLE, VA
UNITED STATES

abrams.aaron@gmail.com

JAMES POMMERSHEIM
DEPARTMENT OF MATHEMATICS
REED COLLEGE
PORTLAND, OR
UNITED STATES

jamie@reed.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.207

THE h-PRINCIPLE FOR MAPS TRANSVERSE TO BRACKET-GENERATING DISTRIBUTIONS

ARITRA BHOWMICK

Given a smooth bracket-generating distribution $\mathcal{D}$ of constant growth on a manifold M, we prove that maps from an arbitrary manifold Σ to M, which are transverse to $\mathcal{D}$, satisfy the complete h-principle. This partially settles a question posed by M. Gromov (1986).

1. Introduction

A *distribution* $\mathcal{D}$ on a manifold M is a (smooth) subbundle of the tangent bundle TM. Given such a $\mathcal{D}$, we can consider the sheaf $\Gamma\mathcal{D}$ of local sections to $\mathcal{D}$, i.e, local vector fields on M taking values in $\mathcal{D}$. The distribution $\mathcal{D}$ is called *bracket-generating* if at each point $x \in M$, the tangent space $T_x M$ is spanned by the vector fields obtained by taking finitely many successive Lie brackets of vector fields in $\Gamma\mathcal{D}$. We say $\mathcal{D}$ is $(r-1)$-*step bracket-generating* at $x \in M$, if there exists some integer $r = r(x)$ such that

$$T_x M = \operatorname{Span}\big\{[X_1, \ldots, [X_{k-1}, X_k] \ldots]_x \mid X_1, \ldots, X_k \in \Gamma\mathcal{D},\ 1 \le k \le r\big\}.$$

Note that what we call a $(r-1)$-step bracket-generating is usually called an r-step bracket-generating distribution elsewhere in the literature. Bracket-generating distributions are the stepping stone for the field of sub-Riemannian geometry [Gromov 1996; Montgomery 2002].

One possible way to study a given distribution $\mathcal{D}$ is via smooth maps $u : \Sigma \to M$ from an arbitrary manifold Σ and looking at how the image of the differential $du : T\Sigma \to TM$ intersects $\mathcal{D}$. The map u is said to be *transverse* to $\mathcal{D}$ if we have that $du(T_\sigma \Sigma) + \mathcal{D}_{u(\sigma)} = T_{u(\sigma)} M$ holds for every $\sigma \in \Sigma$. Gromov [1986] asked the reader to prove the following.

Theorem. *Given a bracket-generating distribution $\mathcal{D}$ on a manifold M, maps $\Sigma \to M$ transverse to $\mathcal{D}$ satisfy the h-principle.*

MSC2020: primary 58A20, 58A30; secondary 57R42, 58A17.
Keywords: transverse maps, bracket-generating distributions, h-principle.

The idea of the proof as indicated in [Gromov 1986, p. 84] contained an error which was later acknowledged in [Gromov 1996, p. 254]. Eliashberg and Mishachev [2002, p. 136] showed that the proof indeed goes through if the distribution $\mathcal{D}$ is contact. In fact, their argument remains valid for any strongly bracket-generating distribution (or fat distribution, see [Montgomery 2002] for a definition). Moreover, they also planned out a strategy that could work for an arbitrary bracket-generating distribution as well. Del Pino and Shin [2020] carried out the ideas of Eliashberg and Mishachev [2002] and proved the h-principle for smooth maps transverse to *real analytic* bracket-generating distributions on a real analytic manifold. Their argument heavily depends on estimating the codimension of certain semianalytic sets in the jet bundle. It was also conjectured in the same article that Gromov's original statement should hold for a *smooth* bracket-generating distribution if certain higher-order jet calculations are performed.

The main goal of this article is to identify a suitable higher jet "regularity" condition (Definition 3.1) so that the sheaf of $\mathcal{D}$-horizontal maps $\mathbb{R} \to M$ satisfying this regularity is microflexible. The difficulty lies in proving the local h-principle for this class of maps provided the distribution is equiregularly bracket-generating (Definition 2.3), which is proved in Theorem 3.7. Then, applying Gromov's analytic and sheaf-theoretic techniques, the h-principles for transverse maps (Theorem 4.1) and for transverse immersions (Theorem 4.3), follow by a standard argument. We refer to [Gromov 1986; Eliashberg and Mishachev 2002] for the details of this theory.

The article is organized as follows: In Section 2, we recall some basic notions about bracket-generating distributions. In Section 3, we obtain the regularity criterion for maps $\mathbb{R} \to M$ horizontal to a bracket-generating distribution and prove the local h-principle for such maps. In Section 4 we prove the main h-principles.

2. Bracket-generating distributions

Definition 2.1. A *distribution* of rank n (and corank p) on a manifold M is a smooth vector subbundle of rank n (and corank p) of the tangent bundle TM.

Given any distribution $\mathcal{D} \subset TM$, we have the sheaf of local sections $\Gamma\mathcal{D}$, which is a sheaf of local vector fields on M. By the notation $X \in \mathcal{D}$ we shall mean a local section $X \in \Gamma\mathcal{D}$ defined on some unspecified open set of M. Given two arbitrary sheaves $\mathcal{E}, \mathcal{F}$ of vector fields on M (not necessarily given as sheaves of sections of some distribution), we can define the sheaf

$$[\mathcal{E}, \mathcal{F}] = \operatorname{Span}\big\{[X, Y] \mid X \in \mathcal{E},\ Y \in \mathcal{F}\big\},$$

where the span is taken over $C^\infty(M)$. Inductively, we then define

$$\mathcal{D}^0 = 0, \qquad \mathcal{D}^1 = \mathcal{D}, \qquad \mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i], \quad i \geq 1.$$

Definition 2.2. A distribution $\mathcal{D}$ is said to be *bracket-generating* if for each $x \in M$ we have $T_x M = \mathcal{D}_x^{r+1}$ for some $r = r(x)$. $\mathcal{D}$ is said to have type $\mathfrak{m} = \mathfrak{m}(x)$ at x for the $(r+2)$-tuple $\mathfrak{m} = (0 = m_0 \leq \cdots \leq m_{r+1} = \dim M)$, where $m_i = \operatorname{rk} \mathcal{D}_x^i$ for $0 \leq i \leq r+1$.

For a *generic* bracket-generating distribution $\mathcal{D}$ on M, the number of steps it takes to bracket-generate $T_x M$ is nonconstant and the sheaves $\mathcal{D}^i$ may fail to be of constant rank.

Definition 2.3. A bracket-generating distribution $\mathcal{D}$ on M is said to be r-step bracket-generating if $T_x M = \mathcal{D}_x^{r+1}$ for all $x \in M$. Furthermore, $\mathcal{D}$ is said to be *equiregular* (or, *of constant growth*) of type $\mathfrak{m} = (m_0 < \cdots < m_{r+1})$ if $\mathcal{D}$ has the type $\mathfrak{m}$ at every $x \in M$.

Throughout this article, we shall mostly restrict ourselves to equiregular distributions $\mathcal{D}$ of some fixed type $\mathfrak{m}$. In particular, each $\mathcal{D}^s$ will be a distribution, and we get a flag

$$0 = \mathcal{D}^0 \subset \mathcal{D} = \mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^{r+1} = TM.$$

It should be noted that in general, equiregularity is a nongeneric condition on the germs of distributions of a given rank, although most of the interesting distributions appearing in the literature possess this property.

Example 2.4. Contact and Engel distributions are well-studied examples of bracket-generating distributions that bracket-generates the tangent bundle in 1 and 2 steps respectively. More generally, we have Goursat structures which are certain rank 2, r-step bracket-generating distributions on manifolds of dimension $r+2$. Note that all of these distributions are equiregular as well. On the other hand, the Martinet distribution, given as the kernel $\ker(dz - y^2 dx)$ on $\mathbb{R}^3$ is *not* equiregular. We refer to [Montgomery 2002] for many more examples.

We shall need the following lemma in the next section.

Lemma 2.5. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution on M, of type $\mathfrak{m} = (m_0 < \cdots < m_{r+1})$. Set $p_s = \operatorname{rk}\big(\mathcal{D}^{s+1}/\mathcal{D}^s\big) = m_{s+1} - m_s > 0$. Then, for any $x \in M$ and for $1 \leq j \leq p_s$, $1 \leq s \leq r$, there exists a collection of vectors*

$$\tau^{s,j} \in \mathcal{D}_x, \quad \eta^{s,j} \in \mathcal{D}_x^s \setminus \mathcal{D}_x^{s-1}, \quad \zeta^{s,j} \in \mathcal{D}_x^{s+1} \setminus \mathcal{D}_x^s,$$

and 1-forms $\lambda^{s,j}$ defined near x, such that:

- *For each $1 \leq s \leq r$, $\mathcal{D}_x^{s+1} = \mathcal{D}_x^s + \operatorname{Span}\langle \zeta_x^{s,1}, \ldots, \zeta_x^{s,p_s} \rangle$.*
- *The 1-forms $\{\lambda^{s,j}\}$ are dual to $\{\zeta^{s,j}\}$ at x. Also, $\{\lambda^{s,j} \mid 1 \leq j \leq p_s, 1 \leq s \leq r\}$ is a frame for the annihilator bundle of $\mathcal{D}$ near x.*

- *For each* $1 \le s \le s' \le r$ *and* $1 \le j \le p_s,\ 1 \le j' \le p_{s'}$, *we have*
$$d\lambda^{s',j'}|_x(\tau^{s,j}, \eta^{s,j}) = \begin{cases} \delta_{j,j'}, & s' = s, \\ 0, & s' > s, \end{cases}$$
where $\delta_{j,j'}$ *is the Kronecker's delta function.*

Proof. Since $\mathcal{D}^{s+1} = \mathcal{D}^s + [\mathcal{D}, \mathcal{D}^s]$, we have the well-defined sheaf homomorphism
$$\begin{aligned} \Omega^s : \mathcal{D} \otimes \mathcal{D}^s/\mathcal{D}^{s-1} &\to \mathcal{D}^{s+1}/\mathcal{D}^s, \\ X \otimes (Y \bmod \mathcal{D}^{s-1}) &\mapsto -[X, Y] \mod \mathcal{D}^s. \end{aligned}$$
Furthermore, Ω^s is $C^\infty(M)$-linear and hence, for vectors $X \in \mathcal{D}_x$, $Y \in \mathcal{D}^s_x$ we have the linear maps
$$\Omega^s_x(X, Y \bmod \mathcal{D}^{s-1}_x) = -[\tilde{X}, \tilde{Y}]_x \mod \mathcal{D}^s_x,$$
where $\tilde{X} \in \mathcal{D}$, $\tilde{Y} \in \mathcal{D}^s$ are arbitrary extensions of X, Y respectively. Thus, we have that $\Omega^s : \mathcal{D} \otimes \mathcal{D}^s/\mathcal{D}^{s-1} \to \mathcal{D}^{s+1}/\mathcal{D}^s$ is a bundle map, which is surjective since $\mathcal{D}$ is bracket-generating, for $1 \le s \le r$.

Choose vectors $\tau^{s,j} \in \mathcal{D}_x,\ \eta^{s,j} \in \mathcal{D}^s_x \setminus \mathcal{D}^{s-1}_x$ so that $\{\Omega^s_x(\tau^{s,j}, \eta^{s,j}) \mid 1 \le j \le p_s\}$ forms a frame of $\mathcal{D}^{s+1}_x/\mathcal{D}^s_x$. Let us consider some arbitrary extensions $\tilde{\tau}^{s,j} \in \mathcal{D}$, $\tilde{\eta}^{s,j} \in \mathcal{D}^s \setminus \mathcal{D}^{s-1}$ of $\tau^{s,j}, \eta^{s,j}$ respectively, and denote $\tilde{\zeta}^{s,j} = -[\tilde{\tau}^{s,j}, \tilde{\eta}^{s,j}] \in \mathcal{D}^{s+1}$. Note that $\Omega^s(\tilde{\tau}^{s,j}, \tilde{\eta}^{s,j}) = \tilde{\zeta}^{s,j} \bmod \mathcal{D}^s$. Since $TM = \oplus_{s=0}^r \mathcal{D}^{s+1}/\mathcal{D}^s$, we have a local framing
$$TM \underset{\text{loc.}}{=} \mathcal{D} \oplus \operatorname{Span}\langle \tilde{\zeta}^{s,j},\ 1 \le j \le p_s,\ 1 \le s \le r \rangle$$
near x. Next, choose independent local 1-forms $\lambda^{s,j}$ near x, which are in the annihilator bundle $\operatorname{Ann} \mathcal{D}$ (i.e., $\lambda^{s,j}|_{\mathcal{D}} = 0$), and $\{\lambda^{s,j}\}$ is dual to $\{\tilde{\zeta}^{s,j}\}$. Note that
$$\mathcal{D}^s \underset{\text{loc.}}{=} \bigcap_{s' \ge s} \bigcap_{j=1}^{p^{s'}} \ker \lambda^{s',j}, \quad 1 \le s \le r.$$
Hence, for $s' \ge s$, we have
$$\begin{aligned} d\lambda^{s',j'}(\tau^{s,j}, \eta^{s,j}) &= \Big[\tilde{\tau}^{s,j} \underbrace{(\lambda^{s',j'}(\tilde{\eta}^{s,j}))}_{0} - \tilde{\eta}^{s,j} \underbrace{(\lambda^{s',j'}(\tilde{\tau}^{s,j}))}_{0} - \lambda^{s',j'} (\underbrace{[\tilde{\tau}^{s,j}, \tilde{\eta}^{s,j}]}_{-\tilde{\zeta}^{s,j}}) \Big]_x \\ &= \lambda^{s',j'}(\zeta^{s,j}) = \begin{cases} \delta_{j',j}, & s' = s, \\ 0, & s' > s. \end{cases} \end{aligned}$$
This concludes the proof of Lemma 2.5. □

3. Regularity of horizontal curves

Let us fix an arbitrary distribution $\mathcal{D}$ that has rank n and corank p on M with $\dim M = N = n + p$. Given a manifold Σ, a map $u : \Sigma \to M$ is called $\mathcal{D}$*-horizontal* if $du(T_\sigma \Sigma) \subset \mathcal{D}_{u(\sigma)}$ for each $\sigma \in \Sigma$. For simplicity, let us assume that $\mathcal{D}$ is given as the kernel of 1-forms $\lambda^1, \ldots, \lambda^p$ on M. Then, $\mathcal{D}$-horizontal maps $\Sigma \to M$ are precisely the solutions of the following *nonlinear* differential operator:

$$\begin{aligned} \mathfrak{D} : C^\infty(\Sigma, M) &\to \Omega^1(\Sigma, \mathbb{R}^p) = \Gamma \hom(T\Sigma, \mathbb{R}^p), \\ u &\mapsto (u^*\lambda^1, \ldots, u^*\lambda^p). \end{aligned} \tag{1}$$

To find solutions of $\mathfrak{D}$, we appeal to the Nash–Gromov implicit function theorem. As a first step, linearizing $\mathfrak{D}$ at some $u : \Sigma \to M$ we get the *linear* differential operator:

$$\begin{aligned} \mathfrak{L}_u : \Gamma u^*TM &\to \Omega^1(\Sigma, \mathbb{R}^p), \\ \xi &\mapsto \big[X \mapsto \big(d\lambda^s(\xi, u_*X) + X(\lambda^s \circ \xi)\big)_{s=1}^p\big], \end{aligned} \tag{2}$$

which restricts to the *bundle map* on $\Gamma u^*\mathcal{D}$:

$$\mathcal{L}_u := \mathfrak{L}_u|_{\Gamma u^*\mathcal{D}} : \xi \mapsto [X \mapsto (d\lambda^s(\xi, u_*X))].$$

An immersion $u : \Sigma \to M$ is called $(d\lambda^s)$*-regular* if the bundle map $\mathcal{L}_u$ is surjective. In general, $(d\lambda^s)$-regularity depends on our choice of 1-forms λ^s, whereas $(d\lambda^s)$-regularity of a $\mathcal{D}$-horizontal map is independent of any such choice. A $(d\lambda^s)$-regular horizontal immersion is also called Ω*-regular*, where $\Omega : \Lambda^2\mathcal{D} \to TM/\mathcal{D}$ is the associated curvature 2-form. It follows that the sheaf of Ω-regular horizontal maps $\Sigma \to M$ is microflexible [Gromov 1986, p. 339]. Note that $(d\lambda^s)$-regularity is a first-order condition on the class of maps. For the existence of a $(d\lambda^s)$-regular horizontal map, even when $\dim \Sigma = 1$, $\mathcal{D}$ must be 1-step bracket-generating. We now identify a suitable higher-order regularity for maps $\mathbb{R} \to M$ horizontal to an arbitrary distribution.

$\mathcal{W}$-regular horizontal curves. The arguments presented in this section follow the general scheme of algebraically solving (underdetermined) linear partial differential operators as in [Gromov 1986, p. 155]. Briefly, the idea is as follows. In order to solve the linear operator $\mathfrak{L}_u$ for some u, ideally we need to find out some linear operator $\mathfrak{M}_u : \Omega^1(\Sigma, \mathbb{R}^p) \to \Gamma u^*TM$ so that $\mathfrak{L}_u \circ \mathfrak{M}_u = \mathrm{Id}$ holds. This involves solving partial differential equations in the coefficients of $\mathfrak{M}_u$, which is in general hard to do. Instead, we try to look for a linear operator $\mathfrak{S}_u : \Gamma u^*TM \to \Omega^1(\Sigma, \mathbb{R}^p)$ satisfying $\mathfrak{S}_u \circ \overline{\mathfrak{L}_u} = \mathrm{Id}$, where $\overline{\mathfrak{L}_u} : \Omega^1(\Sigma, \mathbb{R}^p) \to \Gamma u^*TM$ is the formal adjoint of $\mathfrak{L}_u$. Note that this system is *algebraic* in the coefficients of $\mathfrak{S}_u$, and thus are considerably easier to solve. Once we get a smooth solution $\mathfrak{S}_u$, we can take the formal adjoint of the whole equation, and obtain $\mathfrak{L}_u \circ \overline{\mathfrak{S}_u} = \mathrm{Id}$, since $\overline{\overline{\mathfrak{L}_u}} = \mathfrak{L}_u$. Taking $\mathfrak{M}_u = \overline{\mathfrak{S}_u}$ we then have the desired solution to the original problem. This observation was used by Gromov to prove the fact that a generic *underdetermined* linear operator admits (universal) a right inverse [Gromov 1986, p. 156]. Although

$\mathfrak{L}_u$ is underdetermined, we are not able to appeal to this theorem directly, since we do not know whether the operator is sufficiently generic in the sense of Gromov. Instead, we explicitly identify a class of maps u for which the algebraic system always admits a (smooth) solution. We would like to note that a similar approach was also successfully used in [De Leo 2019], where the author proved the existence of nonfree isometric immersions.

Without loss of generality, we assume $M = \mathbb{R}^N$ and fix some coordinates $y^1, \ldots, y^N$ on M. Let us write the 1-forms λ^s as

$$\lambda^s = \lambda^s_\mu \, dy^\mu, \quad 1 \le s \le p.$$

We also fix the (global) coordinate t on $\mathbb{R}$. For a function $u : \mathbb{R} \to \mathbb{R}^N$, the linearization operator $\mathfrak{L}_u$ (see (2)) is then given as

$$\begin{aligned}\mathfrak{L}_u : C^\infty(\mathbb{R}, \mathbb{R}^N) &\to C^\infty(\mathbb{R}, \mathbb{R}^p),\\ \xi = (\xi^\mu) &\to \big((\partial_\mu \lambda^s_\nu \circ u)\, \partial_t u^\nu \xi^\mu + (\lambda^s_\mu \circ u)\, \partial_t \xi^\mu\big)_{s=1}^p.\end{aligned} \tag{3}$$

Written in a matrix form we have $\mathfrak{L}_u(\xi) = \mathfrak{L}^0_u \xi + \mathfrak{L}^1_u \partial_t \xi$ where the $p \times N$ matrices $\mathfrak{L}^i_u$ are given as

$$\mathfrak{L}^0_u = \big((\partial_\mu \lambda^s_\nu \circ u)\, \partial_t u^\nu\big)_{p\times N}, \quad \mathfrak{L}^1_u = (\lambda^s_\mu \circ u)_{p\times N}. \tag{4}$$

Taking the formal adjoint of $\mathfrak{L}_u$, we get the first-order operator

$$\mathfrak{R}_u : C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$$

which can be written as

$$\mathfrak{R}_u = \mathfrak{R}^0_u + \mathfrak{R}^1_u \partial_t = (\mathfrak{L}^0_u - \partial_t \mathfrak{L}^1_u)^\dagger - (\mathfrak{L}^1_u)^\dagger \partial_t. \tag{5}$$

Observe that

$$\begin{aligned}\mathfrak{L}^0_u - \partial_t \mathfrak{L}^1_u &= \big((\partial_\mu \lambda^s_\nu \circ u)\, \partial_t u^\nu - \partial_t(\lambda^s_\mu \circ u)\big)_{p\times N}\\ &= \big((\partial_\mu \lambda^s_\nu \circ u - \partial_\nu \lambda^s_\mu \circ u)\, \partial_t u^\nu\big)_{p\times N} = \big(-(\iota_{u_*\partial_t} d\lambda^s)(\partial_\mu)\big)_{p\times N}.\end{aligned} \tag{6}$$

Let us now consider the equation

$$\mathfrak{S} \circ \mathfrak{R}_u = \mathrm{Id} \tag{7}$$

for an arbitrary order q linear operator $\mathfrak{S} : C^\infty(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$ given as

$$\mathfrak{S} := \mathfrak{S}^0 + \mathfrak{S}^1 \partial_t + \cdots + \mathfrak{S}^q \partial_t^q,$$

where $\mathfrak{S}^i$ are $p \times N$ matrices of functions. Note that (7) is *algebraic* in the entries of the matrices $\mathfrak{S}^i$. In fact, equation (7) represents a total of $p^2(q+2)$ many equations in $pN(q+1)$ many variables, namely, $\{\mathfrak{S}^i_{\alpha\beta} \mid 1 \le \alpha \le p,\ 1 \le \beta \le N,\ 0 \le i \le q\}$. This system is *underdetermined* if and only if

$$p^2(q+2) < pN(q+1) \quad \Leftrightarrow \quad nq > p-n. \tag{8}$$

Expanding out (7) we have

$$\begin{aligned}
\mathrm{Id} = \mathfrak{S} \circ \mathfrak{R}_u &= (\mathfrak{S}^0 + \mathfrak{S}^1 \partial_t + \cdots + \mathfrak{S}^q \partial_t^q) \circ (\mathfrak{R}_u^0 + \mathfrak{R}_u^1 \partial_t) \\
&= (\mathfrak{S}^0 \mathfrak{R}_u^0 + \mathfrak{S}^1 \partial_t \mathfrak{R}_u^0 + \cdots + \mathfrak{S}^q \partial_t^q \mathfrak{R}_u^0) \\
&\quad + \big(\mathfrak{S}^0 \mathfrak{R}_u^1 + \mathfrak{S}^1 (\mathfrak{R}_u^0 + \partial_t \mathfrak{R}_u^1) + \cdots + \mathfrak{S}^q (q\,\partial_t^{q-1} \mathfrak{R}_u^0 + \partial_t^q \mathfrak{R}_u^1)\big)\, \partial_t \\
&\quad + \cdots \\
&\quad + \big(\mathfrak{S}^{q-1} \mathfrak{R}_u^1 + \mathfrak{S}^q (\mathfrak{R}_u^0 + q\,\partial_t \mathfrak{R}_u^1)\big)\, \partial_t^q \\
&\quad + \mathfrak{S}^q \mathfrak{R}_u^1 \partial_t^{q+1}.
\end{aligned}$$

Comparing both sides, we get the block-matrix system:

$$\begin{aligned}
&\begin{pmatrix} \mathfrak{S}^0 & \mathfrak{S}^1 & \dots & \mathfrak{S}^{q-1} & \mathfrak{S}^q \end{pmatrix} \\
&\qquad \times \begin{pmatrix}
\mathfrak{R}_u^0 & \mathfrak{R}_u^1 & \dots & 0 & 0 \\
\partial_t \mathfrak{R}_u^0 & \mathfrak{R}_u^0 + \partial_t \mathfrak{R}_u^1 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
\partial_t^{q-1} \mathfrak{R}_u^0 & (q-1)\,\partial_t^{q-2} \mathfrak{R}_u^0 + \partial_t^{q-1} \mathfrak{R}_u^1 & \dots & \mathfrak{R}_u^1 & 0 \\
\partial_t^q \mathfrak{R}_u^0 & q\,\partial_t^{q-1} \mathfrak{R}_u^0 + \partial_t^q \mathfrak{R}_u^1 & \dots & \mathfrak{R}_u^0 + q\,\partial_t \mathfrak{R}_u^1 & \mathfrak{R}_u^1
\end{pmatrix} \\
&= \begin{pmatrix} \mathrm{Id}_{p\times p} & 0_{p\times p} & \dots & 0_{p\times p} & 0_{p\times p} \end{pmatrix}.
\end{aligned} \tag{9}$$

Let us denote

$$R_u := -(\mathfrak{L}_u^0 - \partial_t \mathfrak{L}_u^1) = (\iota_{u_* \partial_t} d\lambda^s)_{p\times N}, \qquad \Lambda := \mathfrak{L}_u^1 = (\lambda_\mu^s \circ u)_{p\times N}, \tag{10}$$

so that from (5) we have

$$\mathfrak{R}_u^0 = -(R_u)^\dagger \quad \text{and} \quad \mathfrak{R}_u^1 = -\Lambda^\dagger. \tag{11}$$

Taking the adjoint of the coefficient matrix in (9) and multiplying by -1 we then get the following matrix:

$$A := \begin{pmatrix}
R_u & \partial_t R_u & \dots & \partial_t^{q-1} R_u & \partial_t^q R_u \\
\Lambda & R_u + \partial_t \Lambda & \dots & (q-1)\,\partial_t^{q-1} R_u + \partial_t^{q-1} \Lambda & q\,\partial_t^{q-1} R_u + \partial_t^q \Lambda \\
0 & \Lambda & \dots & \dots & \dots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & \Lambda & R_u + q\,\partial_t \Lambda \\
0 & 0 & \dots & 0 & \Lambda
\end{pmatrix}. \tag{12}$$

A is a matrix of size $p(q+2) \times N(q+1)$, which depends on the $(q+1)$-th jet of the map u. The rows of this matrix can be linearly independent only if $nq \geq p-n$ holds (see (8)). Under the full rank condition, one can always solve for $\mathfrak{S}^i$ smoothly in (9), and thus solving (7). It should be noted that there is no unique solution, but instead, we have an affine space of them. The full rank of A will enable us to choose a solution that varies smoothly depending on $j_u^{q+1}(x)$.

Definition 3.1. For some fixed q satisfying $nq \geq p-n$, let us define the relation $\mathcal{W} \subset J^{q+1}(\mathbb{R}, \mathbb{R}^N)$ as

$$\mathcal{W} = \left\{ j_u^{q+1}(x) \mid du_x \text{ is injective and } A = A(j_u^{q+1}(x)) \text{ has full (row) rank} \right\}.$$

A smooth solution of $\mathcal{W}$ is called a *$\mathcal{W}$-regular* or *weakly $(d\lambda^s)$-regular* map. We denote by $\operatorname{Sol}\mathcal{W}$ the space of all $\mathcal{W}$-regular maps.

Lemma 3.2. *$(d\lambda^s)$-regular maps are $\mathcal{W}$-regular.*

Proof. If an immersion $u : \mathbb{R} \to \mathbb{R}^N$ is $(d\lambda^s)$-regular, then the block $\binom{R_u}{\Lambda}_{2p\times N}$ in the top left corner of A has full (row) rank, which makes the first two row-blocks of A full rank. On the other hand, the Λ blocks on the "off-diagonal" are always full rank, since the rows of Λ consist of linearly independent 1-forms $\{\lambda^s\}$. Note that there is no overlap between the $\binom{R_u}{\Lambda}_{2p\times N}$ block and the rest of the diagonal Λ blocks. Hence, the rows of A are linearly independent whenever u is $(d\lambda^s)$-regular, i.e., u is then $\mathcal{W}$-regular. □

In light of the above lemma, one observes that the first row-block of A is the one where the rank of A might drop, and one may consider $\mathcal{W}$-regularity as the natural higher-order analog of $d\lambda^s$-regularity. This observation shall become more clear in the proof of Theorem 3.7. Let us now show that $\mathfrak{L}_u$ admits a universal right inverse over $\mathcal{W}$-regular maps, i.e., one can solve $\mathfrak{L}_u \circ \mathfrak{M}_u = \mathrm{Id}$ for any $\mathcal{W}$-regular map u, such that $\mathfrak{M}_u$ depends smoothly on u.

Proposition 3.3. *Fix q satisfying $nq \geq p-n$ and the relation $\mathcal{W} \subset J^{q+1}(\mathbb{R}, \mathbb{R}^N)$. Then, for $\mathcal{W}$-regular maps $u : \mathbb{R} \to \mathbb{R}^N$, there exists a linear partial differential operator $\mathfrak{M}_u : C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$ of order q, satisfying $\mathfrak{L}_u \circ \mathfrak{M}_u = \mathrm{Id}$. Furthermore, the assignment*

$$\operatorname{Sol}\mathcal{W} \times C^\infty(\mathbb{R}, \mathbb{R}^p) \ni (u, P) \mapsto \mathfrak{M}(u, P) := \mathfrak{M}_u(P) \in C^\infty(\mathbb{R}, \mathbb{R}^N)$$

is a differential operator, nonlinear of order $2q+1$ in the first variable.

Proof. Fix a jet $\sigma = j_u^{q+1}(t) \in \mathcal{W}|_t$, represented by some map $u : \mathcal{O}p(t) \to \mathbb{R}^N$. The first-order operator $\mathfrak{R}_u$ defined on $\mathcal{O}p(t)$ gives rise to the (linear) symbol map

$$\Delta_{\mathfrak{R}_\sigma} : J^1(\mathbb{R}, \mathbb{R}^p)|_t \to J^0(\mathbb{R}, \mathbb{R}^N)|_t = C^\infty(\mathbb{R}, \mathbb{R}^N)|_t.$$

For any jet $j^1_P(t) \in J^1(\mathbb{R}, \mathbb{R}^p)|_t$ represented by some $P : \mathcal{O}p(t) \to \mathbb{R}^p$, we then have $\Delta_{\mathfrak{R}_\sigma}(j^1_P(t)) = \mathfrak{R}_u(P)(t)$. We define $\Delta^{(q)}_{\mathfrak{R}_\sigma} : J^{q+1}(\mathbb{R}, \mathbb{R}^p)|_t \to J^q(\mathbb{R}, \mathbb{R}^N)|_t$ by

$$\Delta^{(q)}_{\mathfrak{R}_\sigma}(j^{q+1}_P(t)) = j^q_{\mathfrak{R}_u(P)}(t).$$

Since $\sigma \in \mathcal{W}$, the matrix $A_\sigma = A(j^{q+1}_u(t))$ has full row rank. We can then readily solve for $\mathfrak{S}^j = \mathfrak{S}^j(\sigma)$ in (9), in terms of rational polynomials of the terms of A_σ. Indeed, we get a (linear) map $\Delta_{\mathfrak{S}_\sigma} : J^q(\mathbb{R}, \mathbb{R}^N)|_t \to C^\infty(\mathbb{R}, \mathbb{R}^p)|_t$ satisfying the commutative diagram

$$\begin{tikzcd}
J^{q+1}(\mathbb{R}, \mathbb{R}^p)|_t \arrow[rr, "\Delta^{(q)}_{\mathfrak{R}_\sigma}"] \arrow[dr, "p^{q+1}_0"'] & & J^q(\mathbb{R}, \mathbb{R}^N)|_t \arrow[dl, "\Delta_{\mathfrak{S}_\sigma}"] \\
& J^0(\mathbb{R}, \mathbb{R}^p)|_t &
\end{tikzcd}$$

Consider an open neighborhood $U(\sigma) \subset \mathcal{W}$ of σ so that the denominators of all the rational polynomials in $\Delta_{\mathfrak{S}_\sigma}$ remain nonzero for all jets $\tau \in U(\sigma)$. Shrinking $U(\sigma)$ if necessary, assume that $U(\sigma)$ projects down to an open neighborhood $V(\sigma) \subset \mathbb{R}$ of t. We then have a smooth map

$$\Delta_\sigma : U(\sigma) \times J^q(V(\sigma), \mathbb{R}^N) \to C^\infty(V(\sigma), \mathbb{R}^p),$$

so that for $\tau \in U(\sigma)|_s$ with $s \in V(\sigma)$ and for any $j^{q+1}_P(s)$, the following holds:

$$\Delta_\sigma\big(\tau, \Delta^{(q)}_{\mathfrak{R}_\tau}(j^{q+1}_P(s))\big) = p^{q+1}_0(j^{q+1}_P(s)) = P(s).$$

Note that Δ_σ is nonlinear in the first term, whereas it is linear in the second term.

We now have an open cover $\mathfrak{U} = \{U(\sigma)\}_{\sigma \in \mathcal{W}}$ of $\mathcal{W}$. Fix a partition of unity $\{\rho_\alpha\}_{\alpha \in \Lambda}$ on $\mathcal{W}$ subordinate to $\mathfrak{U}$, so that $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for some $U_\alpha \in \mathfrak{U}$. We denote the corresponding open set $V_\alpha \subset \mathbb{R}$ and the map

$$\Delta_\alpha : U_\alpha \times J^q(V_\alpha, \mathbb{R}^N) \to C^\infty(V_\alpha, \mathbb{R}^p).$$

Define the bundle map

$$\Delta_{\mathfrak{S}} : \mathcal{W} \times J^q(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$$

via the formula

$$\Delta_{\mathfrak{S}}(\tau, \eta) := \sum_\alpha \Delta_\alpha(\tau, \rho_\alpha(\tau)\eta).$$

Since each Δ_α is linear in the second argument, the map $\Delta_{\mathfrak{S}}$ is well-defined and smooth. Now, for jets $\tau = j^{q+1}_u(s) \in \mathcal{W}$ and $\eta = j^{q+1}_P(s) \in J^{q+1}(\mathbb{R}, \mathbb{R}^p)$, we have

$$\begin{aligned}
\Delta_{\mathfrak{S}}(\tau, \Delta^{(q)}_{\mathfrak{R}_\tau}(\eta)) &= \sum_\alpha \Delta_\alpha\big(\tau, \rho_\alpha(\tau)\Delta^{(q)}_{\mathfrak{R}_\tau}(\eta)\big) \\
&= \sum_\alpha \Delta_\alpha\big(\tau, \Delta^{(q)}_{\mathfrak{R}_\tau}(\rho_\alpha(\tau)\eta)\big), \quad \text{as } \Delta^{(q)}_{\mathfrak{R}_\tau} \text{ is linear} \\
&= \sum_\alpha p_0^{q+1}(\rho_\alpha(\tau)\eta), \qquad \text{by the construction of } \Delta_\alpha \\
&= p_0^{q+1}\Big(\sum_\alpha \rho_\alpha(\tau)\eta\Big) \\
&= p_0^{q+1}(\eta) = P(s).
\end{aligned}$$

Define $\mathfrak{S} : \mathrm{Sol}\,\mathcal{W} \times C^\infty(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$ via the formula

$$\mathfrak{S}(u, \xi) = \Delta_{\mathfrak{S}}(j_u^{q+1}, j_\xi^q).$$

The operator $\mathfrak{S}$ is nonlinear of order $q+1$ in the first component and linear of order q in the second component. We have $\mathfrak{S}(u, \mathfrak{R}_u(P)) = P$ for any $u \in \mathrm{Sol}\,\mathcal{W}$ and $P \in C^\infty(\mathbb{R}, \mathbb{R}^p)$.

Lastly, define the operator $\mathfrak{M} : \mathrm{Sol}\,\mathcal{W} \times C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$ by

$$\mathfrak{M}(u, P) = \mathfrak{M}_u(P) = \overline{\mathfrak{S}_u}(P),$$

where $\overline{\mathfrak{S}_u} : C^\infty(\mathbb{R}, \mathbb{R}^p) \to C^\infty(\mathbb{R}, \mathbb{R}^N)$ is the formal adjoint to the operator $\mathfrak{S}_u : \xi \mapsto \mathfrak{S}(u, \xi)$. We have

$$\mathfrak{L}_u \circ \mathfrak{M}_u = \overline{\mathfrak{R}_u} \circ \overline{\mathfrak{S}_u} = \overline{\mathfrak{S}_u \circ \mathfrak{R}_u} = \overline{\mathrm{Id}} = \mathrm{Id} \quad \text{for any } u \in \mathrm{Sol}\,\mathcal{W}.$$

Clearly $\mathfrak{M}$ is a differential operator, which is linear of order q in the second component. Since taking adjoint of the q-th order operator $\mathfrak{S}_u$ itself has order q, we have $\mathfrak{M}$ is nonlinear of order $2q+1$ in the first component. □

Following Gromov's terminology [Gromov 1986, pp. 115–116], Proposition 3.3 implies that for any q satisfying $nq \geq p - n$, the first-order operator

$$\mathfrak{D} : C^\infty(\mathbb{R}, \mathbb{R}^N) \to C^\infty(\mathbb{R}, \mathbb{R}^p)$$

is infinitesimally invertible over $\mathcal{W}$-regular maps, with defect $2q+1$ and order of inversion q. For $\alpha \geq 0$, denote the relation of α-infinitesimal solutions of $\mathfrak{D} = 0$ as

(13) $$\mathcal{R}^\alpha_{\mathrm{tang}} = \{ j_u^{\alpha+1}(x) \mid j^\alpha_{\mathfrak{D}(u)}(x) = 0 \} \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N), \quad \alpha \geq 0.$$

Next, for $\alpha \geq (2q+1) - 1 = 2q$ denote the relation of $\mathcal{W}$-regular α-infinitesimal solutions of $\mathfrak{D} = 0$ by

(14) $$\mathcal{W}_\alpha = (p^{\alpha+1}_{q+1})^{-1}(\mathcal{W}) \cap \mathcal{R}^\alpha_{\mathrm{tang}} \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N).$$

Each $\mathcal{W}_\alpha$ has the same C^∞-solutions for $\alpha \geq 2q$, namely, the $\mathcal{W}$-regular $\mathcal{D}$-horizontal curves. Denote the sheaves

$$(15) \qquad \Phi^{\mathcal{W}} = \mathrm{Sol}\,\mathcal{W}_\alpha \quad \text{and} \quad \Psi_\alpha^{\mathcal{W}} = \Gamma\mathcal{W}_\alpha \qquad \text{for } \alpha \geq 2q.$$

A direct application of the results in [Gromov 1986, pp. 118–120] then gives us:

Theorem 3.4. *Fix q satisfying $nq \geq p-n$, where $p = \operatorname{cork}\mathcal{D}$, $n = \operatorname{rk}\mathcal{D}$. Then*:

- *$\Phi^{\mathcal{W}}$ is microflexible.*
- *For any*
$$\alpha \geq \max\{2q+1+q,\ 2.1+2q\} = 3q+1,$$
the jet map $j^{\alpha+1} : \Phi^{\mathcal{W}} \to \Psi_\alpha^{\mathcal{W}}$ is a local weak homotopy equivalence.

Remark 3.5. It should be noted that the $\mathcal{W}$-regularity of $\mathcal{D}$-horizontal maps $\mathbb{R} \to M$ is independent of any choice of coordinates on M or choice of defining 1-forms λ^s. Indeed, these are precisely the class of maps $\mathbb{R} \to M$ over which the operator $\mathfrak{D}$ (see (1)) is infinitesimally invertible. Since the solution space $\mathfrak{D} = 0$ is independent of any choice, so is the regularity of such maps.

Local h-principle for $\mathcal{W}$-regular horizontal curves. To keep the notation light, throughout the rest of this section, we shall treat any higher jet $j_u^q(x)$ formally as variables. That is, $j_u^q(x)$ really represents the tuple of *formal* maps

$$\left(F^i : \odot^i T_x\mathbb{R} \to T_{u(x)}\mathbb{R}^N, \quad 1 \leq i \leq q\right)$$

in the jet space $J^q_{(x,u(x))}(\mathbb{R}, \mathbb{R}^N)$, and each component $\partial_t^i u(x) \equiv F^i(\partial_t^i) \in \mathbb{R}^N$ are independent variables. For any 1-form λ defining $\mathcal{D}$ near $y = u(x)$, the components of the higher jets $j_\lambda^q(y)$ will be treated as known scalar values.

Now, consider the first-order relation

$$(16) \quad \mathcal{R}_{\text{imm-tang}} = \left\{ j_u^1(x) \in J^1(\mathbb{R}, \mathbb{R}^N) \mid du_x \text{ is injective and } \operatorname{Im} du_x \subset \mathcal{D}_{u(x)} \right\}.$$

The solution sheaf $\mathrm{Sol}\,\mathcal{R}_{\text{imm-tang}}$ consists of all the $\mathcal{D}$-horizontal immersed curves. It follows from (13) that $\mathcal{R}_{\text{imm-tang}} \subset \mathcal{R}^0_{\text{tang}}$. For any $\alpha \geq q$, where q satisfies $nq \geq p-n$, we have from (14) that the jet projection map $p_1^{\alpha+1} : J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N) \to J^1(\mathbb{R}, \mathbb{R}^N)$ restricts to a map

$$p_1^{\alpha+1}|_{\mathcal{W}_\alpha} : \mathcal{W}_\alpha \to \mathcal{R}_{\text{imm-tang}}.$$

In fact, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{W}_\alpha \arrow[r, hook] \arrow[d, "p_1^{\alpha+1}|_{\mathcal{W}_\alpha}"'] & \mathcal{R}^\alpha_{\text{tang}} \arrow[r, hook] \arrow[d, "p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}}"] & J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N) \arrow[d, "p_1^{\alpha+1}"] \\
\mathcal{R}_{\text{imm-tang}} \arrow[r, hook] & \mathcal{R}^0_{\text{tang}} \arrow[r, hook] & J^1(\mathbb{R}, \mathbb{R}^N)
\end{tikzcd}$$

Note that $p_1^{\alpha+1}$ is an affine bundle. Let us now make the following easy observation.

Lemma 3.6. *For any $\sigma \in \mathcal{R}^0_{\text{tang}}$, the fiber of $p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}}$ over σ is contractible.*

Proof. A jet $j_u^{\alpha+1}(x) \in \mathcal{R}^\alpha_{\text{tang}} \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N)$ is defined by the equation $j^\alpha_{\mathfrak{D}(u)}(x) = 0$ (see (13)), which expands to the following system:

$$
\left.\begin{aligned}
0 &= \partial_t^k (u^* \lambda^s(\partial_t))|_x \\
&= \partial_t^k ((\lambda^s_\mu \circ u)\, \partial_t u^\mu)|_x \\
&\Rightarrow (\lambda^s_\mu \circ u)\, \partial_t^{k+1} u^\mu|_x + \text{terms involving } j_u^k(x) = 0
\end{aligned}\right\}, \tag{17}
$$

where $1 \le s \le p$, $0 \le k \le \alpha$. Recall the matrix $\Lambda = (\lambda^s_\mu \circ u(x))_{p \times N}$ from (10) and denote by $\partial_t^k u$ the column vector $\partial_t^k u = (\partial_t^k u^\mu(x))_{N \times 1}$. Then, equation (17) can be expressed as the following affine system:

$$
\left.\begin{aligned}
\Lambda \partial_t u &= 0 \\
\Lambda \partial_t^2 u &= \left(-(\partial_\nu \lambda^s_\mu \circ u)\, \partial_t u^\nu\, \partial_t u^\mu\right)_{p \times 1}\Big|_x \\
&\;\;\vdots \\
\Lambda \partial_t^{\alpha+1} u &= p \times 1 \text{ vector involving } j_u^\alpha(x)
\end{aligned}\right\}. \tag{18}
$$

Note that we have $\mathcal{R}^0_{\text{tang}}|_{(x,u(x))} = \ker \Lambda$. Since Λ has full rank, given any value of $\sigma = j_u^1(x) \equiv \partial_t u \in \mathcal{R}^0_{\text{tang}}$, the above system can always be solved in a triangular way. Clearly, at each step the solution space is affine. It follows that the fiber of $p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}}$ over σ is contractible. □

The discussion so far had no extra assumption on the distribution $\mathcal{D}$. From this point onwards, we shall only consider equiregular, bracket-generating distributions. The main goal of this section is to prove the following.

Theorem 3.7. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution of rank n and corank p on $\mathbb{R}^{N=n+p}$, with type $\mathfrak{m}$. Let q_0 satisfy $nq_0 \ge p - n$. Fix a jet $\sigma = j_u^1(x) \in \mathcal{R}_{\text{imm-tang}}$. Then for each $K \ge 1$, there exists some $q(\mathfrak{m}, K) \ge q_0$, such that for any $\alpha \ge q(\mathfrak{m}, K)$ the complement of the fiber $\mathcal{W}_\alpha|_\sigma = (p_1^{\alpha+1}|_{\mathcal{W}_\alpha})^{-1}(\sigma)$ has codimension at least K in $\mathcal{R}^\alpha_{\text{tang}}|_\sigma = (p_1^{\alpha+1}|_{\mathcal{R}^\alpha_{\text{tang}}})^{-1}(\sigma)$.*

As a corollary, we get the local h-principle for $\mathcal{W}$-regular horizontal maps.

Corollary 3.8. *The sheaf map $\Phi^{\mathcal{W}} \to \Gamma \mathcal{R}_{\text{imm-tang}}$ induced by the differential map is a local weak homotopy equivalence.*

Proof. Fix some jet $\sigma \in \mathcal{R}_{\text{imm-tang}}$. It follows from Theorem 3.7 and Lemma 3.6 that the fiber of $p_1^{\alpha+1}|_{\mathcal{W}_\alpha}$ over σ is K-connected for α sufficiently large. Hence, passing to the infinity jet, we get that the fiber is weakly contractible. By an argument of Gromov [1986, p. 77], the sheaf map $p_1^\infty : \Psi^{\mathcal{W}}_\infty = \Gamma \mathcal{W}_\infty \to \Gamma \mathcal{R}_{\text{imm-tang}}$ is then a local weak homotopy equivalence. Also, from Theorem 3.4 we have the sheaf map $j^\infty : \Phi^{\mathcal{W}} \to \Psi^{\mathcal{W}}_\infty$ is a local weak homotopy equivalence. But then the composition

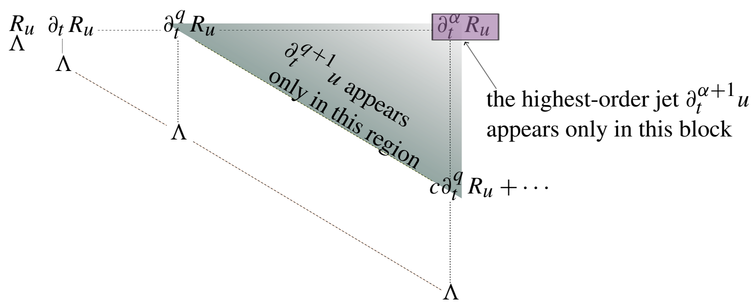


Figure 1. Highest-order jet of u in A.

$p_1^\infty \circ j^\infty : \Phi^{\mathcal{W}} \to \Gamma\mathcal{R}_{\text{imm-tang}}$ is a local weak homotopy equivalence as well. Note that the composition map is nothing but the differential map $u \mapsto du$. □

To prove Theorem 3.7, we need to understand the equations involved in defining the relation $\mathcal{W}_\alpha \subset J^{\alpha+1}(\mathbb{R}, \mathbb{R}^N)$ as in (14). We have already seen in Lemma 3.6 that given a jet $\sigma \in \mathcal{R}_{\text{imm-tang}}$, the fiber of $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$ is the solution space of a triangular affine system (see (18)). But a jet $j_u^{\alpha+1}(x) \in \mathcal{W}_\alpha|_\sigma \subset \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ must satisfy $\mathcal{W}$-regularity as well, i.e., the matrix $A = A(j_u^{\alpha+1}(x))$ as given in (12) must have independent rows. We note the following features of the matrix that will become useful later in the proof.

Firstly, the Λ blocks in the off-diagonal of A have full rank. Thus, the rank can only drop at the first row-block. In any block above the Λ-diagonal, the highest-order jet term $\partial_t^{q+1} u$ is contributed by the $\partial_t^q R_u$ factor, and it appears linearly. In fact, from (10) we have

$$\partial_t^q R_u = \big(d\lambda^s(\partial_t^{q+1} u, \partial_\mu)\big)_{p\times N} + \text{a } p \times N \text{ matrix in } j_u^q(x). \tag{19}$$

Furthermore, no component of $\partial_t^{q+1} u$ appears anywhere below the diagonal passing through this block (see Figure 1). In particular, in each column-block, the highest-order derivative of u occurs in the first row-block only.

Secondly, each column-block of A has N many columns, which can be labeled by the framing $\{\partial_1, \ldots, \partial_N\}$ of $T\mathbb{R}^N$, as it is clear from (19). For any arbitrary choice of frame $\{W_1, \ldots, W_N\}$, we can always perform some (invertible) column operations on A so that the columns in the target column-block, say the $(q+1)$-th column-block, are now labeled by $\{W_1, \ldots, W_N\}$. Indeed, if we write $W_i = W_i^j \partial_j$, then one can consider the invertible matrix $W = (W_1^j \ \ \ldots \ \ W_N^j)_{N\times N}$, so that multiplying the $\partial_t^q R_u$ block from the right by W transforms it into the following block:

$$\big(d\lambda^s(\partial_t^{q+1} u, W_\mu)\big)_{p\times N} + \text{a } p \times N \text{ matrix in } j_u^q(x). \tag{20}$$

We extend this W to a block-diagonal-matrix $\widetilde{W}$ of size $N(\alpha+1)\times N(\alpha+1)$ by putting W as the $(q+1)$-th diagonal block, and Id_N in all the other diagonal positions. Now, if we multiply the matrix A by $\widetilde{W}$ from the right, it will perform column manipulations precisely at the $(q+1)$-th column-block. In particular, the top row-block in this column is now $(\partial_t^q R_u)W$, and thus by looking at (20), we may label this column-block by $W_1, \ldots, W_N$. Note that this process does not change the rank of the matrix, since the column manipulation is invertible by construction.

In the proof, for each column-block of A, we shall only prescribe a subframe of $T\mathbb{R}^N$ (obtained by using Lemma 2.5), which will then be arbitrarily extended to a full frame. Performing the column manipulation as described above will make sure that the target column-block is labeled first by the prescribed subframe and then by the arbitrary choice of extension. As we shall see, we are not interested in the columns which are labeled arbitrarily during this process. If the matrix has full (row) rank after discarding a few columns, then the original matrix will also have full rank. Thus, given a subframe, say, $(W_1, \ldots, W_t)$ for the $(q+1)$-th column-block, we shall say that *the* $(q+1)$*-th column-block* is relabeled by the subframe, while discarding the arbitrarily extended part.

Let us now proceed with the proof of Theorem 3.7. We refer to page 226, where the major steps of the proof are carried out in detail for an example case.

Proof of Theorem 3.7. Let $\sigma = j_u^1(x) \in \mathcal{R}_{\text{imm-tang}}$ be a given jet, and $y = u(x)$. Suppose $\mathcal{D}$ has the type $\mathfrak{m} = (0 = m_0 < \cdots < m_{r+1} = N = n+p)$, where $m_s = \dim \mathcal{D}_y^s$ for $0 \le s \le r+1$. We denote $p_s = \dim(\mathcal{D}_y^{s+1}/\mathcal{D}_y^s) = m_{s+1} - m_s$ for $1 \le s \le r$ and set $p_0 = 0$. Using Lemma 2.5, we get the vectors

$$\tau^{s,j} \in \mathcal{D}_y, \quad \eta^{s,j} \in \mathcal{D}_y^s \setminus \mathcal{D}_y^{s-1}, \quad \zeta^{s,j} \in \mathcal{D}_y^{s+1} \setminus \mathcal{D}_y^s, \qquad 1 \le j \le p_s,\ 1 \le s \le r$$

at y, and the 1-forms $\lambda^{s,j}$ near y. We write the matrix A (see (12)) in terms of these $\lambda^{s,j}$'s. As observed in Remark 3.5, this does not change the relations $\mathcal{W}_\alpha$.

Notations: We label the row- and column-blocks of A starting from 0, so that the $(0, q)$-th block is $\partial_t^q R_u$ and the $(q+1, q)$-th block is the Λ block. We will use $\zeta^{s,\bullet}$ to mean the tuple of vectors $(\zeta^{s,1}, \ldots, \zeta^{s,p_s})$ and similarly $\tau^{s,\bullet}$, $\eta^{s,\bullet}$, $\lambda^{s,\bullet}$ etc. In particular, the matrix Λ is then given by $\Lambda^\dagger = (\lambda^{1,\bullet}\ \ldots\ \lambda^{p,\bullet})_{N\times p}$. We shall also use the notation $\zeta^\bullet$ for the tuple $(\zeta^{1,\bullet}, \ldots, \zeta^{r,\bullet})$ of size p, and $\hat{\zeta}^{s,\bullet}$ for $1 \le s \le r$ for the tuple of size $p - p_s$, obtained by dropping the tuple $\zeta^{s,\bullet}$ from $\zeta^\bullet$. For notational convenience, we set $\hat{\zeta}^{0,\bullet} = \zeta^\bullet$.

Let us first assume $K = 1$. We need to show that the complement of $\mathcal{W}_\alpha|_\sigma$ in $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$ has codimension 1 for some α large enough. To achieve this, we shall find a polynomial P in the jet variables $j_u^\alpha(x)$ so that P being nonzero at some $\tilde{\sigma} \in \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ implies that the matrix $A = A(\tilde{\sigma})$ has full (row) rank. Let us briefly discuss the proof strategy.

Step 1: For each column-block of A, we shall prescribe some subframe of $T_y\mathbb{R}^N$, consisting of some suitable vectors as obtained by Lemma 2.5 (and thus using the fact that $\mathcal{D}$ is bracket-generating). This step (Algorithm 1) is recursive, and it will determine the value of $q(\mathfrak{m}, 1)$. Extending each subframe arbitrarily to a full frame and then relabeling the column-blocks, we get a new matrix, say, A_1. Since these operations are invertible, A has full rank if and only if A_1 has full rank.

Step 2: We shall consider the submatrix B of A_1, with columns labeled by the prescribed subframes as above and all the rows of A_1. B will be a square matrix and $P = \det B$ will be our candidate polynomial in the higher jet variables $j_u^{\alpha+1}(x)$. Since it is a minor of A_1, $\det B \neq 0$ at some higher jet $\tilde{\sigma} \in \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ implies that $A(\tilde{\sigma})$ has full rank, i.e, $\tilde{\sigma} \in \mathcal{W}_\alpha|_\sigma$. We shall keep using the notation Λ and $\partial_t^q R_u$ to denote the respective blocks in B obtained after the column transformations and curtailing of A. Performing some more (invertible) row and column operations on B, we shall produce a new matrix B_1, so that $\det B = \det B_1$. Next, we shall extract a square submatrix C of B_1 and observe that $\det B = \det B_1 = \pm \det C$.

Step 3: It follows from (18) that if $j_u^q(x)$ is solved, then the solution space for $\partial_t^{q+1}u$ is given as the *affine* space $V_q + \ker\Lambda = V_q + \mathcal{D}_{u(x)}$, where the $N \times 1$ vector $V_q = V_q(j_u^q(x))$ is obtained using some fixed choice of right inverse Λ^{-1}. In particular, we can write $\partial_t^{q+1}u = X_q\tau^q + V_q(j_u^q(x))$ for $1 \le q \le \alpha$, where X_q is some indeterminate and τ^q is a vector suitably set in Algorithm 1 to either $0 \in \mathcal{D}_y$ or to one of the vectors $\tau^{s,j} \in \mathcal{D}_y$ chosen earlier. Inductively, we then have

$$(21) \qquad \partial_t^{q+1}u = X_q\tau^q + \text{terms in } X_1, \dots, X_{q-1},\ \tau^1, \dots, \tau^{q-1} \text{ and } \sigma = j_u^1(x)$$

for $1 \le q \le \alpha$. Arbitrary values of $X_1, \dots, X_\alpha$ will produce $j_u^{\alpha+1}(x) \in \mathcal{R}^\alpha_{\text{tang}}|_\sigma$ from (21). We will replace the values of $\partial_t^{q+1}u$ in the matrix C. Treating $\det C$ as a polynomial in the indeterminates X_q we shall show that $\det C$ is nonvanishing for suitably large values of X_q. Thus, $P = \det B$ is nonvanishing when restricted to $\mathcal{R}^\alpha_{\text{tang}}|_\sigma$. This will conclude the proof for $K = 1$.

The crux of the proof lies in suitably choosing the subframes for each of the column-blocks of A as done in Step 1. We produce a schematic diagram to explain the process (Figure 2). As discussed earlier, the rank can only drop in the first row-block consisting of $\partial_t^q R_u$, which are represented by the red boxes in Figure 2, whereas the blue boxes represent the Λ blocks. Suppose we have dealt with the rows corresponding to $d\lambda^{1,1}, \dots d\lambda^{s',j'}$ of the first row-block in a manner that, ignoring the rest of the rows from the first row-blcok, the matrix A up to, say, the q-th column-block has full rank. We could keep choosing $\zeta^\bullet$ for the subsequent column-blocks, which will transform all the Λ blocks appearing in those column-blocks into Id_p, and thus make sure that the matrix A, after *ignoring* the rest of the rows from the first row-block, is indeed full rank.

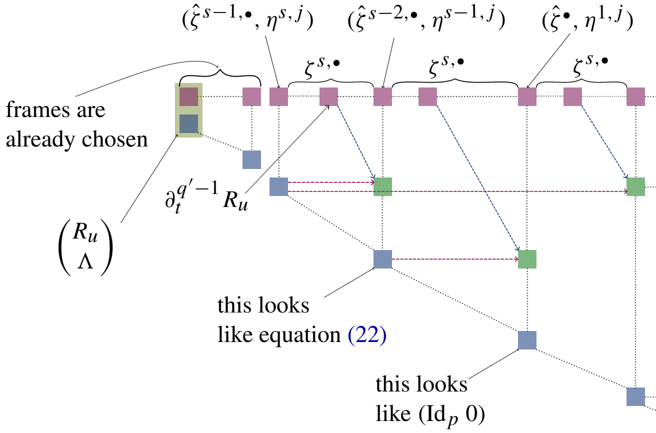


Figure 2. How Algorithm 1 chooses the labeling subframes.

Now, suppose the next row in the first row-block corresponds to $d\lambda^{s,j}$. We label the $(q+1)$-th column-block by the frame $(\hat{\zeta}^{s-1,\bullet}, \eta^{s,j})$. It follows from (20) that the last column of the $\partial_t^q R_u$ block transforms into

$$d\lambda^{s,j}(\partial_t^{q+1} u, \eta^{s,j}) + \text{a } p \times 1 \text{ vector in } j_u^q(x).$$

Since $\partial_t^{q+1} u$ is a jet that did not appear earlier in the matrix (Figure 1), we can prescribe its value arbitrarily to make sure that (at least) the row corresponding to $d\lambda^{s,j}$ in $\partial_t^q R_u$ must be linearly independent. Indeed, it follows from Lemma 3.6, that given $j_u^q(x)$, we may choose $\partial_t^{q+1} u$ from an *affine* space, which is parallel to the distribution $\mathcal{D} = \ker \Lambda$, and so, given a particular solution of $\partial_t^{q+1} u$, we can always add a vector proportional to $\tau^q = \tau^{s,j} \in \mathcal{D}^s$. This is done in Step 3, by adding $X_q \tau^q$ for large value of X_q. It follows from Lemma 3.6 that this will introduce an X_q variable (with coefficient 1) at the $d\lambda^{s,j}$ row, which does not appear anywhere below this row. Note that the rows appearing above might have some instances of X_q, but these rows will be taken care of by some $X_{q'}$ variable appearing earlier. In other words, each row gets assigned a unique X_q.

But choosing this subframe $(\hat{\zeta}^{s-1,j}, \eta^{s,j})$ also introduces a column vector of unknown scalars in the corresponding Λ-block at the $\lambda^{s-1,\bullet}$-rows (see (22)), and in turn (possibly) reduces the rank for the corresponding row-block. To compensate for this, we choose a sufficiently high jet of u that we have not used (recall that we have utilized the jet $\partial_t^{q+1} u$ so far). Indeed, we may consider $\partial_t^{q'} u$ for some $q' \gg q$, and look at the block to the right of this Λ block where this jet appears for the first time in this row-block. This is represented by a red dashed arrow in Figure 2, the arrowhead pointing to the block (the green box in the figure) which is used to compensate for the drop of rank in the Λ block. It follows from (12) that this block must look like $c\partial_t^{q'-1} R_u + \text{a } p \times N$ matrix in $j_u^{q'-1}(x)$, for some positive integer c. We now perform the same process as above: we choose another subframe, say,

Input: q, s, j, d
Output: q

function `ChooseSubFrame`(q, s, j, d)
  if $d = 0$ **then** ▷ We are in row-block 0
    $\tau^q \leftarrow \tau^{s,j}$
  else ▷ We are in row-block d
    $\tau^q \leftarrow 0$
    $\tau^{q-d} \leftarrow \tau^{s,j}$
  Pick subframe $(\hat{\zeta}^{s-1,\bullet}, \eta^{s,j})$ for column-block q
  $q_0 \leftarrow q$
  ▷ If $s = 1$, then $p_{s-1} = 0$ and we do not enter the following loop
  for $1 \leq a \leq p_{s-1}$ **do**
    for $1 \leq b \leq q_0 + 1$ **do**
      Pick subframe $\zeta^\bullet$ for column-block $q + b$
      $\tau^q \leftarrow 0 \in \mathcal{D}_y$
    $q \leftarrow$ `ChooseSubFrame`$(q + q_0 + 2, s - 1, a, q_0 + 1)$
  return q

Algorithm 1. Algorithm for choosing subframes and τ^q's.

$(\hat{\zeta}^{s-2,j'}, \eta^{s-1,j'})$ for the corresponding column-block, choose a value of $\partial_t^{q'-1} u$, suitably add some vector proportional to $\tau^{s-1,j'}$, and thus make sure that at least one of the rows of this row-block is now independent. But now we need to keep doing this recursively, as this will again drop the rank of some Λ block down the line. The recursion ends when we choose the frame $(\zeta^\bullet, \eta^{1,j})$, since this frame will transform the corresponding Λ-block into $(\mathrm{Id}_p \ \ 0_{p\times 1})$, which already has full rank.

We give a recursive algorithm (Algorithm 1) to choose the appropriate subframes for some $1 \leq s \leq r$ and some $1 \leq j \leq p_s$ starting from some column-block q, while at the same time suitably fixing the vectors $\tau^{q'} \in \mathcal{D}_y$ for $q' \geq q$ that needs to be used later in Step 3. The inputs of the algorithm correspond to the situation described above: we have dealt with all the rows appearing before the $d\lambda^{s,j}$-row in the first row-block, using up till the $(q-1)$-th column-block. The integer d corresponds to the depth of the recursion; we begin at $d = 0$, and then d increases as we compensate for subsequent the Λ blocks, as discussed above.

Algorithm 1 outputs the last column-block q for which the subframe and the vector τ^q have been chosen. Note that for $s \geq 2$, the algorithm is recursive. Whereas for $s = 1$, we choose the frame $(\zeta^{0,\bullet}, \eta^{1,j}) = (\zeta^\bullet, \eta^{1,j})$ and do not enter the for-loop since $p_{s-1} = 0$ for $s = 1$. We can now find all the subframes, starting from some column-block q by Algorithm 2.

As before, Algorithm 2 outputs the last column-block q for which the subframe and the vector τ^q have been chosen. We are now in a position to get a suitable value of $q(\mathfrak{m}, K)$ for $K = 1$. First, we choose the frame $\zeta^\bullet$ for each of the column-block

Input: q
Output: q

```
1: function ChooseAllSubFrames(q)
2:     for 1 ≤ s ≤ r do
3:         for 1 ≤ j ≤ p_s do
4:             q ← ChooseSubFrame(q, s, j, 0) + 1
       return q − 1
```

Algorithm 2. Algorithm for choosing all subframes.

$0, \dots, q_0$, where $q_0 \geq 0$ satisfies $nq_0 \geq p-n$ and set $\tau^0 = \cdots = \tau^{q_0} = 0$. Then, let $q(\mathfrak{m}, 1) = \texttt{ChooseAllSubFrames}(q_0 + 1)$. Let us denote the transformed matrix by $A_1 = A_1(j_u^{q(\mathfrak{m},1)+1}(x))$. This concludes Step 1.

Next, choose the submatrix B of A_1 which are labeled by the prescribed columns chosen as in Step 1. Firstly, note that whenever we are choosing the subframe $\zeta^\bullet$ for column-block q, the Λ-block in that column-block of B becomes Id_p, since $\lambda^\bullet$ is dual to $\zeta^\bullet$. On the other hand, choosing the frame $(\hat{\zeta}^{s-1,\bullet}, \eta^{s,j})$ for some (s, j) with $s > 1$ makes the Λ block of the form

$$\begin{pmatrix} \mathrm{Id}_{p_1+\cdots+p_{s-2}} & 0 & 0 \\ 0 & 0 & *_{p_{s-1}\times 1} \\ 0 & \mathrm{Id}_{p_s+\cdots+p_r} & 0 \end{pmatrix}_{p\times(p-p_{s-1}+1)}, \tag{22}$$

where $*$ represents the column vector $(\lambda^{s,\bullet}(\eta^{s,j}))$. Lastly, choosing $(\zeta^\bullet, \eta^{1,j})$ transforms the Λ block into $(\mathrm{Id}_p \;\; 0_{p\times 1})_{p\times(p+1)}$. It follows that B is a square matrix of size $p(q(\mathfrak{m}, 1)+2) \times p(q(\mathfrak{m}, 1)+2)$.

Let us now perform some invertible row and column operations on B, keeping its rank fixed. Starting from the bottom right corner of B and then going towards the top left corner, we consider each Id block in the off-diagonal Λ block. Next using these Id blocks in order, we make everything zero first along the columns and then along the rows. Denote the new matrix by B_1 and observe that $\det B = \det B_1$. In any nonzero block of B_1 above the Λ diagonal, the highest-order derivative of u, say $\partial_t^{q+1} u$, is still contributed by the $\partial_t^q R_u$ factor of this block. In fact, any of these blocks look like

$$\left(0_{p\times(p'-1)} \;\; c\, d\lambda^\bullet(\partial_t^{q+1} u, \eta^{s,j}) + \text{ terms in } j_u^q(x)\right)_{p\times p'}$$

for some integer $1 \leq c \leq q(\mathfrak{m}, 1)$ and some $p' \in \{p+1,\; p-p_s+1,\; 1 \leq s \leq r\}$. Note that this integer c is the integer coefficient of the respective $\partial_t^q R_u$ factor as in (12). Let C be the square submatrix of B_1 obtained by removing the rows and columns corresponding to the Id blocks so that we have $\det B = \det B_1 = \pm \det C$, concluding Step 2.

The columns of C are precisely those columns corresponding to some $\eta^{s,j}$ chosen earlier via Algorithm 2, whereas the rows of C correspond to each row from the first row-block of B_1, and all the rows starting with some $\lambda^{s,j'}(\eta^{s,j})$ from the other row-blocks. Let us now show that $\det C \neq 0$. We replace $\partial_t^{q+1} u = X_q \tau^q + \dots$ in C by using (21). By (i) the construction of C, (ii) the choice of τ^q's in Algorithm 1, and (iii) from Lemma 2.5, it follows that in each row of C there exists a unique column so that the element in this position satisfies the following:

- The element looks like, $cX_q +$ terms in $X_1, \dots, X_{q-1}$, for some q and some integer $1 \leq c \leq q(\mathfrak{m}, 1)$.
- X_q does not appear anywhere in C to the left of this column.
- X_q does not appear anywhere in this column below this row (but may appear above this row).

Note that, not every variable X_q appears in C, since we have set many $\tau^q = 0$. In fact, there are precisely $\bar{q}$ many variables appearing, where C has the size $\bar{q} \times \bar{q}$. Hence, for notational convenience, let us rename the appearing X_q's to $\{Y_1, \dots, Y_{\bar{q}}\}$ in the increasing order. By expanding $\det C$, we then have the recursive formula:

$$
\left.\begin{aligned}
\det C &= Y_{\bar{q}} f_{\bar{q}-1}(\sigma, Y_1, \dots, Y_{\bar{q}-1}) + g_{\bar{q}-1}(\sigma, Y_1, \dots, Y_{\bar{q}-1}) \\
f_{\bar{q}-1}(\sigma, Y_1, \dots, Y_{\bar{q}-1}) &= Y_{\bar{q}-1} f_{\bar{q}-2}(\sigma, Y_1, \dots, Y_{\bar{q}-2}) + g_{\bar{q}-2}(\sigma, Y_1, \dots, Y_{\bar{q}-2}) \\
&\ \ \vdots \\
f_2(\sigma, Y_1, Y_2) &= Y_2 f_1(\sigma, Y_1) + g_1(\sigma, Y_1) \\
f_1(\sigma, Y_1) &= Y_1 \det \tilde{C} + g_0(\sigma)
\end{aligned}\right\} . \tag{23}
$$

Above, f_i, g_i are polynomial functions, where g_0 depends only on the choice of the first jet $\sigma = j_u^1(x)$. The matrix $\tilde{C}$ has the following property: for each row of $\tilde{C}$, there exists a unique column, so that the element in that position is an integer (corresponding to the coefficient of some X_q) and everything below this integer in this column is 0. Indeed, this column corresponds to the unique column of C with the first occurrence of X_q as discussed above. Then, some column permutation puts the matrix $\tilde{C}$ in an upper triangular form, with nonzero integers in the diagonal. In particular, $\det \tilde{C} \neq 0$. Hence, choosing $Y_1, Y_2, \dots, Y_{\bar{q}}$ successively and sufficiently large, we can see from (23) that $\det C \neq 0$. In other words, we have shown that the polynomial $\det B$ is nonvanishing when restricted to $\mathcal{R}_{\text{tang}}^{q(\mathfrak{m},1)}|_\sigma$, which concludes Step 3.

To finish the proof for $K = 1$, let $\alpha > q(\mathfrak{m}, 1)$. First perform `ChooseAllSubFrames`$(q_0 + 1)$ as above. Next, for $q(\mathfrak{m}, 1) + 1 \leq q \leq \alpha$, choose the subframe $\zeta^\bullet$ for the column-block q and set $\tau^q = 0$. We can then continue with the rest of the argument as above without any change. In particular, for any $\alpha \geq q(\mathfrak{m}, 1)$, we have the codimension of the complement of $\mathcal{W}_\alpha|_\sigma$ is at least 1 in $\mathcal{R}_{\text{tang}}^\alpha|_\sigma$.

Let us now induct over K. Suppose for some $K \geq 1$, we have obtained a suitable $q(\mathfrak{m}, K)$ and some polynomials $P_1, \ldots, P_K$ in the jets $j_u^{q(\mathfrak{m},K)}(x)$ so that

$$(\mathcal{W}_{q(\mathfrak{m},K)})^c \subset \{P_1 = \cdots = P_K = 0\}, \quad \operatorname{codim}\{P_1 = \cdots = P_K = 0\} \geq K \text{ in } \mathcal{R}_{\text{tang}}^{q(\mathfrak{m},K)}|_\sigma.$$

We set $q(\mathfrak{m}, K+1) = \texttt{ChooseAllSubFrames}(q(\mathfrak{m}, K)+1)$. This will produce a new polynomial P_{K+1} involving jets that were not involved in any of the $P_1, \ldots, P_K$. But then $\operatorname{codim}\{P_1 = \cdots = P_{K+1} = 0\} \geq K+1$ in $\mathcal{R}_{\text{tang}}^{q(\mathfrak{m},K+1)}|_\sigma$. Also by construction, $(\mathcal{W}_{q(\mathfrak{m},K+1)})^c \subset \{P_1 = \cdots = P_{K+1} = 0\}$. Proceeding similarly as in the $K = 1$ case, we can finish the inductive step.

This concludes the proof of Theorem 3.7. □

A toy case. Let us consider a toy case to illustrate the major steps in the proof of Theorem 3.7. We consider a bracket-generating distribution $\mathcal{D}$, of rank $n = 10$ and corank $p = 4$, on a manifold M with $\dim M = 14$. Suppose $\mathcal{D}^3 = TM$ and $\operatorname{rk}(\mathcal{D}^2/\mathcal{D}) = 2 = \operatorname{rk}(\mathcal{D}^3/\mathcal{D}^2)$. Thus, $\mathcal{D}$ has the type

$$\mathfrak{m} = (0 = m_0 < m_1 = 10 < m_2 = 12 < m_3 = 14),$$

and we have $p_1 = 2 = p_2$. For some $x \in M$, using Lemma 2.5, we choose the necessary vectors $\tau^{s,j} \in \mathcal{D}_x$, $\eta^{s,j} \in \mathcal{D}^s$, $\zeta^{s,j} \in \mathcal{D}_x^{s+1}$ for $1 \leq j \leq p_s$ and $1 \leq s \leq 2$ and the 1-forms $\lambda^{s,j}$. Clearly, for $q_0 = 0$ we have $nq_0 > p - n = -6$. Let us now determine $q(\mathfrak{m}, 1)$ and see how Algorithm 2 produces the subframes for different column-blocks of the matrix A:

pick $\zeta^\bullet$ for column-block 0,	pick $(\zeta^\bullet, \eta^{1,1})$ for column-block 1,
pick $(\zeta^\bullet, \eta^{1,2})$ for column-block 2,	pick $(\hat{\zeta}^{1,\bullet}, \eta^{2,1})$ for column-block 3,
pick $\zeta^\bullet$ for column-blocks 4 to 7,	pick $(\zeta^\bullet, \eta^{1,1})$ for column-block 8,
pick $\zeta^\bullet$ for column-blocks 9 to 12,	pick $(\zeta^\bullet, \eta^{1,2})$ for column-block 13,
pick $(\hat{\zeta}^{1,\bullet}, \eta^{2,2})$ for column-block 14,	pick $\zeta^\bullet$ for column-blocks 15 to 29,
pick $(\zeta^\bullet, \eta^{1,1})$ for column-block 30,	pick $\zeta^\bullet$ for column-blocks 31 to 45,
pick $(\zeta^\bullet, \eta^{1,2})$ for column-block 46.	

Thus, we may take $q(\mathfrak{m}, 1) = 46$. The submatrix B is a square matrix of size $4 \times (46+2) = 192$. We also choose the vectors τ^q as

$$\tau^1 = \tau^{1,1}, \quad \tau^2 = \tau^{1,2}, \quad \tau^3 = \eta^{2,1}, \quad \tau^4 = \tau^{1,1},$$
$$\tau^9 = \tau^{1,2}, \quad \tau^{14} = \tau^{2,2}, \quad \tau^{15} = \tau^{1,1}, \quad \tau^{31} = \tau^{1,2}$$

and set all other $\tau^q = 0$ for $0 \leq q \leq 46$.

Removing all the rows and columns corresponding to some Id block in the off-diagonal, we get the matrix C of size 8×8 from B. Let us also replace

$\partial_t^{q+1} u = X_q \tau^q + \dots$ in C. Then C looks like

$$C = \begin{array}{c} \begin{array}{cccccccc} \eta^{1,1} & \eta^{1,2} & \eta^{2,1} & \eta^{1,1} & \eta^{1,2} & \eta^{2,1} & \eta^{1,1} & \eta^{1,2} \end{array} \\ \begin{pmatrix} X_1+* & ?X_2+* & ?X_3+* & ?X_4+* & ?X_9+* & ?X_{14}+* & ?X_{15}+* & ?X_{31}+* \\ 0X_1+* & X_2+* & ?X_3+* & ?X_4+* & ?X_9+* & ?X_{14}+* & ?X_{15}+* & ?X_{31}+* \\ 0X_1+* & 0X_2+* & X_3+* & ?X_4+* & ?X_9+* & ?X_{14}+* & ?X_{15}+* & ?X_{31}+* \\ 0X_1+* & 0X_2+* & 0X_3+* & ?X_4+* & ?X_9+* & X_{14}+* & ?X_{15}+* & ?X_{31}+* \\ 0 & 0 & \lambda^{2,1}(\eta^{2,1}) & X_4+* & ?X_9+* & 0X_{14}+* & ?X_{15}+* & ?X_{31}+* \\ 0 & 0 & \lambda^{2,2}(\eta^{2,1}) & 0X_4+* & X_9+* & 0X_{14}+* & ?X_{15}+* & ?X_{31}+* \\ 0 & 0 & 0 & 0 & 0 & \lambda^{2,1}(\eta^{2,1}) & X_{15}+* & ?X_{31}+* \\ 0 & 0 & 0 & 0 & 0 & \lambda^{2,2}(\eta^{2,1}) & 0X_{15}+* & X_{31}+* \end{pmatrix}_{8\times 8} \end{array}$$

Above, $*$ in some $?X_i + *$ denotes terms involving $X_1, \dots, X_{i-1}$. Also, to keep the calculation simple, we have set the (nonzero) integer coefficients to 1 for the highest-order X_q uniquely associated with each row. We can find a recursive formula for $\det C$ as in (23). In particular, the matrix $\tilde{C}$ is given by

$$\tilde{C} = \begin{pmatrix} 1 & ? & ? & ? & ? & ? & ? & ? \\ 0 & 1 & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 1 & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & ? & ? & 1 & ? & ? \\ 0 & 0 & 0 & 1 & ? & 0 & ? & ? \\ 0 & 0 & 0 & 0 & 1 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{8\times 8}$$

which obviously satisfies $\det \tilde{C} \neq 0$.

4. The h-principle for transverse maps

Given a distribution $\mathcal{D}$, a map $u : \Sigma \to M$ is said to be *transverse* to $\mathcal{D}$ if the composition map $T\Sigma \xrightarrow{du} u^*TM \xrightarrow{\lambda} u^*TM/\mathcal{D}$ is surjective, where $\lambda : TM \to TM/\mathcal{D}$ is the quotient map. We have the following theorem.

Theorem 4.1. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution on a manifold M. Then, maps transverse to $\mathcal{D}$ satisfy the C^0-dense parametric h-principle.*

Once we have the microflexibility (Theorem 3.4) and the local h-principle (Corollary 3.8) for $\mathcal{W}$-regular horizontal maps $\mathbb{R} \to M$, the proof essentially follows from the steps outlined as in [Gromov 1986, p. 84]. The details were worked out in [Eliashberg and Mishachev 2002, Theorem 14.2.1] when the distribution is contact, and in [Del Pino and Presas 2019, Theorem 4] for the case of Engel distributions. We include the sketch of the proof for the general case.

Proof of Theorem 4.1. Let $\mathcal{R}_{\text{tran}} = \{ j_u^1(x) \mid \lambda \circ du_x \text{ is surjective} \} \subset J^1(\Sigma, M)$ be the relation of $\mathcal{D}$-transverse maps $\Sigma \to M$. Since $\mathcal{R}_{\text{tran}}$ is open, we have

$\operatorname{Sol}\mathcal{R}_{\text{tran}}$ is microflexible and furthermore, $j^1 : \operatorname{Sol}\mathcal{R}_{\text{tran}} \to \Gamma\mathcal{R}_{\text{tran}}$ is a local weak homotopy equivalence. To prove the h-principle, we need to find some suitable (local) microextensions (in the sense of [Bhowmick and Datta 2023, Definition 5.10]) to maps $\widetilde{\Sigma} \to M$, where $\widetilde{\Sigma} = \Sigma \times \mathbb{R}$.

Let us consider the following class of maps:

$$\tilde{\Phi}^{\mathcal{W}\text{-tran}} = \left\{ u : \widetilde{\Sigma} \to M \;\middle|\; \begin{array}{c} u \text{ is transverse to } \mathcal{D}, \text{ and for each } \sigma \in \Sigma \\ u|_{\sigma\times\mathbb{R}} \text{ is a } \mathcal{W}\text{-regular, } \mathcal{D}\text{-horizontal, immersed curve} \end{array} \right\}.$$

We prove the microflexibility and local h-principle for $\tilde{\Phi}^{\mathcal{W}\text{-tran}}$. Assuming that $\mathcal{D} = \bigcap_{s=1}^{p} \ker \lambda^s$, let us consider the differential operator

$$\begin{aligned} \widetilde{\mathfrak{D}} : C^\infty(\widetilde{\Sigma}, M) &\to \hom(T\mathbb{R}, \mathbb{R}^p) = C^\infty(\widetilde{\Sigma}, M), \\ u &\mapsto (u^*\lambda^s|_{T\mathbb{R}}) = (u^*\lambda^s(\partial_t)). \end{aligned}$$

Clearly, $u : \widetilde{\Sigma} \to M$ is a solution of $\widetilde{\mathfrak{D}} = 0$ precisely when $u|_{\sigma\times\mathbb{R}}$ is $\mathcal{D}$-horizontal for each $\sigma \in \Sigma$. The linearization of $\widetilde{\mathfrak{D}}$ at some u is then given by a formula identical to (3). Since t is a global coordinate on $\widetilde{\Sigma} = \Sigma \times \mathbb{R}$, we have a splitting of the jet spaces. Denote by $j_u^{q+1,\perp}(x)$ the higher derivatives purely along the t direction. We then see that the matrix $A = A(j_u^{q+1})$ as in (12), in fact depends only on $j_u^{q+1,\perp}$. Let $\widetilde{\mathcal{R}}_{\text{tran}} \subset J^1(\widetilde{\Sigma}, M)$ be the relation of $\mathcal{D}$-transverse maps $\widetilde{\Sigma} \to M$ and define $\mathcal{W}^{\text{tran}} \subset J^{q+1}(\widetilde{\Sigma}, M)$ by

$$\mathcal{W}^{\text{tran}} = \left\{ j_u^{q+1}(x) \mid j_u^1(x) \in \widetilde{\mathcal{R}}_{\text{tran}},\ \partial_t u(x) \neq 0,\ A(j_u^{q+1,\perp}(x)) \text{ has full rank} \right\}.$$

By similar arguments as in Section 3, the operator $\widetilde{\mathfrak{D}}$ is infinitesimally invertible on $\mathcal{W}^{\text{tran}}$-regular maps.

Let $\widetilde{\mathcal{R}}^{\alpha}_{\text{tang}} = \{ j_u^{\alpha+1}(x) \mid j^{\alpha}_{\widetilde{\mathfrak{D}}(u)}(x) = 0 \}$ for $\alpha \geq 0$ and then for $\alpha \geq 2q$ define

$$\mathcal{W}^{\text{tran}}_{\alpha} = (p^{\alpha+1}_{q+1})^{-1}(\mathcal{W}^{\text{tran}}) \cap \widetilde{\mathcal{R}}^{\alpha}_{\text{tang}} \subset J^{\alpha+1}(\widetilde{\Sigma}, M).$$

It is then immediate that the smooth solutions of $\mathcal{W}^{\text{tran}}_{\alpha}$ are all same and in fact $\tilde{\Phi}^{\mathcal{W}\text{-tran}} = \operatorname{Sol}\mathcal{W}^{\text{tran}}_{\alpha}$. Just as in Theorem 3.4, we have the microflexibility for $\tilde{\Phi}^{\mathcal{W}\text{-tran}}$, and $j^{\alpha+1} : \tilde{\Phi}^{\mathcal{W}\text{-tran}} \to \Gamma\mathcal{W}^{\text{tran}}_{\alpha}$ is a local weak homotopy equivalence for α large enough. Since $\tilde{\Phi}^{\mathcal{W}\text{-tran}}$ is invariant under the pseudogroup of fiber-preserving local diffeomorphisms of $\widetilde{\Sigma}$, we have the flexibility of the restricted sheaf $\tilde{\Phi}^{\mathcal{W}\text{-tran}}|_{\Sigma\times 0}$ [Gromov 1986, p. 78].

Define the relation

$$\widetilde{\mathcal{R}}_{\text{tang-tran}} = \left\{ j_u^1(x) \in \widetilde{\mathcal{R}}_{\text{tran}} \mid 0 \neq \partial_t u(x) \in \mathcal{D}_{u(x)} \right\} \subset J^1(\widetilde{\Sigma}, M).$$

We have the jet projection map $J^{\alpha+1}(\widetilde{\Sigma}, M) \to J^1(\widetilde{\Sigma}, M)$ which restricts to a map $\mathcal{W}^{\text{tran}}_{\alpha} \to \widetilde{\mathcal{R}}_{\text{tang-tran}}$. Since transversality is a first-order regularity condition, we deduce analogous to Corollary 3.8 that the map $\tilde{\Phi}^{\mathcal{W}\text{-tran}} \to \Gamma\widetilde{\mathcal{R}}_{\text{tang-tran}}$ induced by the differential, is a local weak homotopy equivalence. Now, we have a map

$ev : \Gamma\widetilde{\mathcal{R}}_{\text{tang-tran}} \to \Gamma\mathcal{R}_{\text{tran}}$ induced by the restriction $u \mapsto u|_{\Sigma\times 0}$. By choosing some arbitrary nonvanishing local sections of $\mathcal{D}$, we can easily get local (parametric) formal extensions along this ev map on contractible open sets of Σ. Thus, $\widetilde{\mathcal{R}}_{\text{tang-tran}}$ is a microextension of $\mathcal{R}_{\text{tran}}$. We can now finish the proof of the h-principle as in [Bhowmick and Datta 2023, Theorem 2.18]. □

We would like to note that Martínez-Aguinaga and Del Pino [2022] recently have proved the above h-principle (among many other strong results) under similar constant growth assumption on the distribution, albeit using a completely different rather geometric technique. We believe that the technique utilized in the present article can be adapted to a broader class of problems, including the existence of horizontal immersions of submanifolds. Indeed, this method seems promising to address the following conjecture by Gromov.

Conjecture [Gromov 1996, p. 259]. Given a distribution $\mathcal{D}$ on M, Ω-regular (i.e., $(d\lambda^s)$-regular) $\mathcal{D}$-horizontal immersions $\Sigma \to M$ satisfy the complete h-principle, provided $\dim M \geq (\dim\Sigma + 1)\operatorname{codim}\mathcal{D}$.

Immersions transverse to a distribution. Whenever $\dim\Sigma \leq \dim M$, an immersion $u : \Sigma \to M$ is said to be *transverse to* $\mathcal{D}$ if u is an immersion and the composition map $T\Sigma \to u^*(TM/\mathcal{D})$ is of full rank. The following h-principle is well known.

Theorem 4.2 [Gromov 1986, p. 87; Eliashberg and Mishachev 2002, p. 71]. *Let $\mathcal{D}$ be an arbitrary distribution on M. If* $\dim\Sigma < \operatorname{cork}\mathcal{D}$, *then immersions $\Sigma \to M$ transverse to $\mathcal{D}$ satisfy all forms of h-principles.*

The critical dimension $\dim\Sigma = \operatorname{cork}\mathcal{D}$ is not covered by the above theorem. Although, for the special case of $\mathcal{D}$ being either a contact [Eliashberg and Mishachev 2002, Theorem 14.2.2] or an Engel distribution [Del Pino and Presas 2019], the h-principle holds for all transverse immersions. The h-principle for smooth immersions transverse to *real analytic* bracket-generating distributions was proved in [Del Pino and Shin 2020]. We have the following.

Theorem 4.3. *Let $\mathcal{D}$ be an equiregular bracket-generating distribution. Then, C^0-dense, parametric h-principle holds for immersions $\Sigma \to M$ transverse to $\mathcal{D}$, provided* $\dim\Sigma < \dim M$.

Proof. The proof is identical to that of Theorem 4.1. Denote by $\mathcal{R}_{\text{imm}} \subset J^1(\Sigma, M)$ the relation of immersions $\Sigma \to M$, and similarly $\widetilde{\mathcal{R}}_{\text{imm}} \subset J^1(\widetilde{\Sigma}, M)$ where we have $\widetilde{\Sigma} = \Sigma \times \mathbb{R}$. Then, $\mathcal{R}_{\text{imm-tran}} = \mathcal{R}_{\text{tran}} \cap \mathcal{R}_{\text{imm}}$ is the relation of immersions $\Sigma \to M$ which are transverse to $\mathcal{D}$. The microextension is provided by the relation

$$\widetilde{\mathcal{R}}_{\text{imm-tang-tran}} = \widetilde{\mathcal{R}}_{\text{tang-tran}} \cap \widetilde{\mathcal{R}}_{\text{imm}},$$

where $\widetilde{\mathcal{R}}_{\text{tang-tran}}$ is as in Theorem 4.1. We have the map

$$ev : \Gamma\widetilde{\mathcal{R}}_{\text{imm-tang-tran}} \to \Gamma\mathcal{R}_{\text{imm-tran}}$$

induced by $u \mapsto u|_{\Sigma\times 0}$. Since we have that $\dim\Sigma < \dim M$, we can always find *nonvanishing* (local) extensions along the ev map. The h-principle then follows. □

In particular, we have the h-principle for $\mathcal{D}$-transverse maps $\Sigma \to M$ for $\dim\Sigma = \operatorname{cork}\mathcal{D}$, provided $\mathcal{D}$ is bracket-generating. Also, taking $\mathcal{D} = TM$, Theorem 4.3 reduces to Hirsch's h-principle for immersions $\Sigma \to M$ [Hirsch 1959].

Remark 4.4. When $\dim\Sigma \geq \operatorname{cork}\mathcal{D}$, one can also treat immersions $\Sigma \to (M, \mathcal{D})$ transverse to $\mathcal{D}$ as partially horizontal immersions [Gromov 1996, p. 256]. It turns out that all such maps are $\Omega_\bullet$-regular in the sense of Gromov. One can then get a stronger version of Theorem 4.3, where the distribution can be taken to be arbitrary, and thus extending Theorem 4.2.

Acknowledgements

The author would like to thank M. Datta for many valuable comments and insightful discussions, and S. Prasad for their help with the presentation of the article. The author would like to thank the referee for pointing out a mistake in Lemma 2.5 which was originally stated without the equiregularity assumption, and for other suggestions to improve this article. This work was supported by the IISER Kolkata funded postdoctoral fellowship (IISER-K/FA/PDF/DMS/2021/628).

References

[Bhowmick and Datta 2023] A. Bhowmick and M. Datta, "Existence of horizontal immersions in fat distributions", *Internat. J. Math.* **34**:10 (2023), art. id. 2350056. MR Zbl

[De Leo 2019] R. De Leo, "Proof of a Gromov conjecture on the infinitesimal invertibility of the metric-inducing operators", *Asian J. Math.* **23**:6 (2019), 919–932. MR Zbl

[Del Pino and Presas 2019] Á. Del Pino and F. Presas, "Flexibility for tangent and transverse immersions in Engel manifolds", *Rev. Mat. Complut.* **32**:1 (2019), 215–238. MR Zbl

[Del Pino and Shin 2020] Á. Del Pino and T. Shin, "Microflexiblity and local integrability of horizontal curves", preprint, 2020. arXiv 2009.14518

[Eliashberg and Mishachev 2002] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics **48**, American Mathematical Society, Providence, RI, 2002. MR Zbl

[Gromov 1986] M. Gromov, *Partial differential relations*, Results in Mathematics and Related Areas **3**, Springer, Berlin, 1986. MR Zbl

[Gromov 1996] M. Gromov, "Carnot–Carathéodory spaces seen from within", pp. 79–323 in *Sub-Riemannian geometry*, Progr. Math. **144**, Birkhäuser, Basel, 1996. MR Zbl

[Hirsch 1959] M. W. Hirsch, "Immersions of manifolds", *Trans. Amer. Math. Soc.* **93** (1959), 242–276. MR Zbl

[Martínez-Aguinaga and Del Pino 2022] J. Martínez-Aguinaga and Á. Del Pino, "Classification of tangent and transverse knots in bracket-generating distributions", preprint, 2022. arXiv 2210.00582

[Montgomery 2002] R. Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs **91**, American Mathematical Society, Providence, RI, 2002. MR Zbl

Received May 24, 2022. Revised July 24, 2023.

ARITRA BHOWMICK
DEPARTMENT OF MATHEMATICS AND STATISTICS
IISER KOLKATA
WEST BENGAL
INDIA
aritrabh.pdf@iiserkol.ac.in

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.233

THE RESTRICTION OF EFFICIENT GEODESICS TO THE NONSEPARATING COMPLEX OF CURVES

SETH HOVLAND AND GREG VINAL

In the complex of curves of a closed orientable surface of genus g, $\mathcal{C}(S_g)$, a preferred finite set of geodesics between any two vertices, called *efficient geodesics*, was introduced by Birman, Margalit, and Menasco (2016). The main tool they used to establish the existence of efficient geodesics was a *dot graph*, which records the intersection pattern of a reference arc with the simple closed curves associated with a geodesic path. The idea behind the construction was that a geodesic that is not initially efficient contains shapes in its corresponding dot graph. These shapes then correspond to surgeries that reduce the intersection with the reference arc. We show that the efficient geodesic algorithm can be restricted to the nonseparating curve complex; the proof of this will involve analysis of the dot graph and its corresponding surgeries. Moreover, we demonstrate that given any geodesic in the complex of curves we may obtain an efficient geodesic whose vertices, with the possible exception of the endpoints, are all nonseparating curves.

1. Introduction

The complex of curves and geodesics. The complex of curves $\mathcal{C}(S)$ for a compact surface S is a simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves in S and whose edges connect vertices with disjoint representatives. We can endow the 0-skeleton of $\mathcal{C}(S)$ with a metric by defining the distance between two vertices u and v to be the minimal number of edges among paths between them. In this paper, as in [Birman et al. 2016], we assume that the surfaces we are considering are closed and have genus at least two. It is a fundamental result that in this case $\mathcal{C}(S)$ is connected. Thus, the distance is defined for all pairs of vertices in $\mathcal{C}(S)$. The trouble is that the complex of curves is, in fact, too connected. It turns out that $\mathcal{C}(S)$ is locally infinite (for any vertex v there are infinitely many adjacent vertices w) and there are infinitely many geodesics between many pairs of vertices. Thus, it is useful to have a preferred finite subset of geodesics to choose from. This is the idea behind the introduction of *tight geodesics*

MSC2020: 57K20.
Keywords: curve complex, nonseparating curves, efficient geodesics.

in [Masur and Minsky 2000]. Birman, Margalit, and Menasco [Birman et al. 2016] introduced an alternate preferred finite set of geodesics, called *efficient geodesics*. The novel feature of this particular set of geodesics is the algorithm used to generate them. The algorithm can (and has been, in [Glenn et al. 2017]) implemented on a computer to find efficient geodesics for small distances.

We let $\mathcal{N}(S)$ denote the subcomplex of $\mathcal{C}(S)$ spanned by vertices corresponding to nonseparating simple closed curves. This complex is called the *complex of nonseparating curves*. Again, it is a standard result that if the genus of S is at least two, $\mathcal{N}(S)$ is connected. The complex of nonseparating curves has attracted interest in the literature. See [Rasmussen 2020; Hamenstädt 2014; Wright 2023].

We first show that given a geodesic in the complex of curves we can always find a geodesic whose interior is contained in the subcomplex of nonseparating curves. It is then natural to wonder whether *efficient geodesics* exist that are contained in this subcomplex. Our main result is that they in fact do.

Theorem 1.1. *Let $g \geq 2$. If v and w are vertices of $\mathcal{N}(S_g)$ with $d(v, w) \geq 3$, then there exists an efficient geodesic from v to w in $\mathcal{N}(S_g)$. Additionally, there are finitely many efficient geodesics from v to w.*

Efficient geodesics. The idea behind obtaining an efficient geodesic $v_0, \ldots, v_n$ in $\mathcal{C}(S)$ is to iteratively decrease intersections with an arc as we move along the path. We explain further below. The following construction and results were first introduced in [Birman et al. 2016]; the reader familiar with these results is invited to skip to Section 2.

Suppose that γ is an arc in S and α is a simple closed curve in S. Then we say that γ and α are in *minimal position* if α is disjoint from the endpoints of γ and the number of points of intersection of α and γ is smallest over all simple closed curves homotopic to α through homotopies that do not pass through the endpoints of γ.

Let $v_0, \ldots, v_n$ be a geodesic of length at least three in the complex of curves, and let α_0, α_1, and α_n be representatives of v_0, v_1, and v_n that are pairwise in minimal position. A *reference arc* for the triple $\alpha_0, \alpha_1, \alpha_n$ is an arc γ that is in minimal position with α_1 and whose interior is disjoint from $\alpha_0 \cup \alpha_n$.

We say that the oriented geodesic $v_0, \ldots, v_n$ is *initially efficient* if

$$|\alpha_1 \cap \gamma| \leq n - 1$$

for all choices of reference arcs γ. Finally, we say that $v = v_0, \ldots, v_n = w$ is *efficient* if the oriented geodesic $v_k, \ldots, v_n$ is initially efficient for each $0 \leq k \leq n - 3$ and the oriented geodesic $v_n, v_{n-1}, v_{n-2}, v_{n-3}$ is also initially efficient. Thus, to test the efficiency of a geodesic we look at all the triples v_k, v_{k+1}, v_n and count the intersection of v_{k+1} with any reference arc. While it may seem impossible to check intersections with *all* reference arcs, it turns out that there are finitely many of them. Moreover, in special cases it is sufficient to check the intersections of $\alpha_i \cap \alpha_n$ for $1 \le i \le n-1$ [Birman et al. 2016].

Given a vertex path $v_0, \ldots, v_n$ in $\mathcal{C}(S)$ with representative curves $\alpha_0, \ldots, \alpha_n$ and an oriented reference arc γ for α_0, α_1 and α_n, we may traverse γ in the direction of its orientation and record the order in which the curves $\alpha_0, \ldots, \alpha_n$ intersect γ. The result is a sequence of natural numbers $\sigma \in \{1, \ldots, n-1\}^N$, where N is the minimal cardinality of $\gamma \cap (\alpha_1 \cup \cdots \cup \alpha_{n-1})$. The sequence σ is called the *intersection sequence* of the α_i along γ.

The *complexity* of an oriented path $v_0, \ldots, v_n \in \mathcal{C}(S)$ is defined to be

$$\sum_{k=1}^{n-1} (i(v_0, v_k) + i(v_k, v_n)).$$

We say that a sequence σ of natural numbers is *reducible* under the following circumstances: whenever σ arises as an intersection sequence for a path $v_0, \ldots, v_n$ in $\mathcal{C}(S)$ there is another path $v_0, v_1', \ldots, v_{n-1}', v_n$ with smaller complexity. The following proposition demonstrates that paths of minimal complexity must be initially efficient.

Proposition 1.1 [Birman et al. 2016, Proposition 3.1]. *Suppose σ is a sequence of elements of $\{1, \ldots, n-1\}$. If σ has more than $n-1$ entries equal to 1, then σ is reducible.*

From the above proposition Birman, Margalit, and Menasco deduce the existence of initially efficient geodesics.

Proposition 1.2 [Birman et al. 2016, Proposition 3.2]. *Let $g \ge 2$. If v and w are vertices of $\mathcal{C}(S)$, with $d(v, w) \ge 3$, then there exists an initially efficient geodesic from v to w.*

We note that the definition of complexity of a path works when restricted to the complex of nonseparating curves. It is interesting to consider how this measure may change when restricted to the nonseparating curve complex. See Questions 2 and 3 in Section 3.

Sawtooth form and the dot graph. The proof of Proposition 1.1 was carried out in three stages. First, the intersection sequence was put into a normal form. This is called *sawtooth form*. Then, associated to the sawtooth form for the sequence is a diagram called the *dot graph*. The reducibility of an intersection sequence then corresponds to certain geometric features in the dot graph. We review these now.

We may exchange the order of intersection of two curves that are adjacent in the intersection sequence by performing a *commutation* as described in [Birman et al. 2016, Lemma 3.3]; see Figure 1.

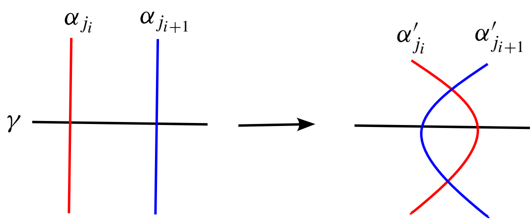


Figure 1. A commutation.

The result is a sequence that is in *sawtooth form*. That is, we say a sequence $(j_1, j_2, \ldots, j_k)$ of natural numbers is in *sawtooth form* if

$$j_i < j_{i+1} \implies j_{i+1} = j_i + 1.$$

An example of a sequence in sawtooth form is $(1, 2, 2, 3, 4, 3, 4, 2, 3, 4, 5)$. Given a sequence of natural numbers in sawtooth form, we also consider its *ascending sequences*, which are the maximal subsequences of the form $(k, k+1, \ldots, k+m)$. In the above example the ascending sequences are $(1, 2)$, $(2, 3, 4)$, $(3, 4)$, $(2, 3, 4, 5)$. It is clear that if we have an intersection sequence and we perform a finite number of commutations we may get the intersection sequence into sawtooth form while keeping the number of intersections of the α_i's with γ constant.

Next, given an intersection sequence σ in sawtooth form, we may regard it as a function $1, \ldots, N \to \mathbb{N}$ and plot it in $\mathbb{R}^2_{\geq 0}$. The points of the graph of a sequence will be called *dots*. We decorate the graph by connecting the dots that lie on a given line of slope 1; these line segments are called *ascending segments*. The resulting decorated graph is called the *dot graph* of σ and is denoted by $G(\sigma)$. See Figure 2. Again, the idea behind this construction is that given a geodesic that is not efficient we can see shapes in its corresponding dot graph that correspond to surgeries that reduce the intersection with the reference arc.

Dot graph polygons and surgery. This section is a summary of [Birman et al. 2016, Section 3.3]. We first review the surgeries described, and then in the next section we discuss the results of these surgeries when restricted to $\mathcal{N}(S)$.

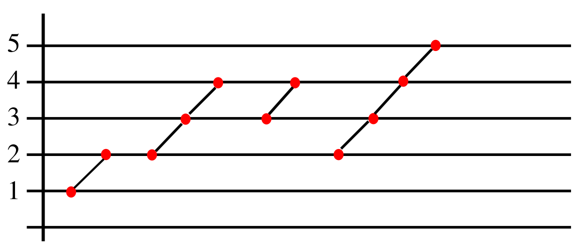


Figure 2. The dot graph of $(1, 2, 2, 3, 4, 3, 4, 2, 3, 4, 5)$.

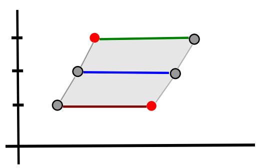


Figure 3. Dot graph polygons (box, hexagon type 1, hexagon type 2).

After putting an intersection sequence into sawtooth form and constructing its dot graph, it was shown that if the dot graph contained certain geometric shapes, it corresponded to a sequence that was reducible. These shapes were called *dot graph polygons*. In particular, the existence of a *box*, *hexagon of type I* or *hexagon of type II* in the dot graph (see Figure 3) implied that the sequence σ was reducible. To remove these shapes from the dot graph, *surgeries* on the curves in the intersection pattern corresponding to these shapes were introduced.

We do surgery on a curve α that intersects our intersection arc γ in at least two points. We draw a neighborhood of γ so that it is horizontal and oriented to the right. We then remove from α small neighborhoods of its points of intersection with γ. This results in a pair of curves. We will then join two of the endpoints back together forming a new simple closed curve, and discard the other curve. Depending on how we join pairs of endpoints, we say that α' is obtained from α by $++$, $+-$, $-+$, or $--$ surgery along γ. The first symbol is $+$ or $-$ depending on whether the first endpoint of α lies to the left or right of γ, respectively. Similarly for the second symbol. When we are considering an arbitrary simple closed curve, exactly two of the four possible surgeries result in a simple closed curve. If we give α an orientation then two intersection points of α and γ can either agree or disagree in orientation. If they agree, then the $+-$ and $-+$ surgeries, which are called *odd*, each result in a simple closed curve. Otherwise, the $++$ and $--$ surgeries, called *even* surgeries, result in simple closed curves.

Suppose that we have a geodesic in $\mathcal{C}(S)$ containing α as a representative for some vertex v_i. If we are to perform surgery on α then it must intersect our intersection arc γ at least twice (otherwise it stays in the geodesic and is not replaced). Thus,

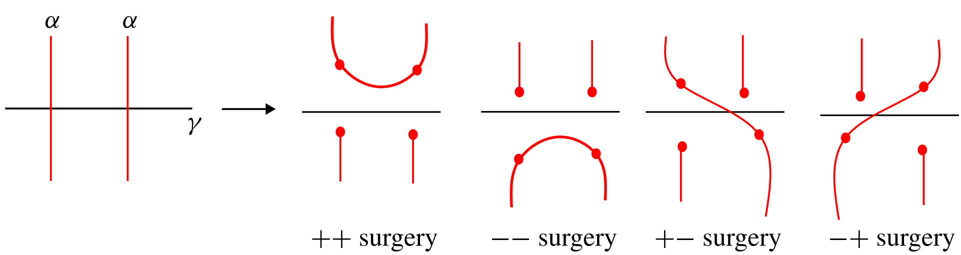


Figure 4. Surgery on a curve.

either α intersects γ consecutively, or between these intersections γ intersects with at least one other curve β. In the complex of curves, we can immediately get rid of the first case, where α intersects γ consecutively, by performing the surgeries described above. This does not follow so easily in the subcomplex of nonseparating curves (see Proposition 2.5). However, if α does not intersect γ consecutively, as is the case when we see boxes, and hexagons in the dot graph, we will see that we have the same choice of surgeries as before.

Using the above surgeries, it was shown in [Birman et al. 2016] that a dot graph with an empty, unpierced box or an empty, unpierced hexagon of type 1 or 2 corresponds to a sequence that is reducible. This was done by prescribing a sequence of surgeries that replaced the α_i curves with new α_i' that resulted in a path with smaller complexity. Below we state the required sequence of surgeries corresponding to the type of dot graph polygon.

Suppose the dot graph $G(\sigma)$ has an empty, unpierced box P. Then the corresponding sequence of intersections along γ has the form

$$\alpha_k, \ldots, \alpha_{k+m}, \alpha_k, \ldots, \alpha_{k+m},$$

where $1 \le k \le k+m \le n-1$. For the vertices not in $\{k, \ldots, k+m\}$, they remain unchanged. We define $\alpha_k', \ldots, \alpha_{k+m}'$ inductively: for $i = k, \ldots, k+m$ the curve α_i' is obtained by performing surgery along γ between the two points of $\alpha_i \cap \gamma$ corresponding to dots of P; the surgeries are chosen so that they form a path in the directed graph in Figure 5. The vertices of the graph correspond to the four types of surgeries described above: the rule is that the second sign of the origin of a directed edge is opposite of the first sign of the terminus. It is clear from the graph that the desired sequence of surgeries exists. We demonstrate this procedure in Figure 5, where we perform $-+$ surgery on α_3, then $--$ surgery on α_4, and finally $+-$ surgery on α_5. It is an easy check that replacing the curves α_i with these new ones results in a reduced intersection sequence σ.

Suppose the dot graph $G(\sigma)$ has an empty, unpierced hexagon P of type 1. The case of a type 2 hexagon is nearly identical so it will be omitted. By definition of sawtooth form and of a type 1 hexagon, there are no ascending segments of $G(\sigma)$

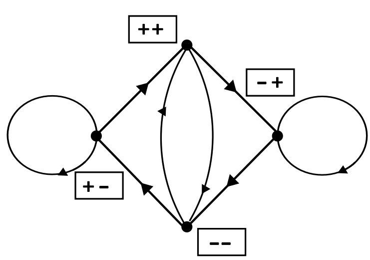


Figure 5

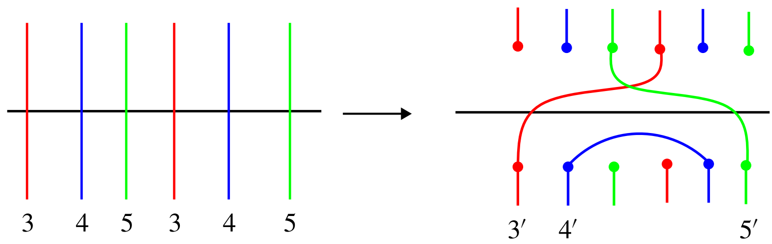


Figure 6

in the vertical strip between the leftmost and middle ascending edges of P and any ascending segments of $G(\sigma)$ that lie in the vertical strip between the middle and rightmost ascending segments have their highest point strictly below the lower-right horizontal edge of P. See Figure 7.

It follows that the dots of P correspond to a sequence of intersections along γ of the form

$$\alpha_k, \ldots, \alpha_{k+m}, \alpha_k, \ldots, \alpha_{k+l}, \alpha_{j_1}, \ldots, \alpha_{j_p}, \alpha_{k+l}, \ldots, \alpha_{k+m},$$

where $1 \leq k \leq k+l \leq k+m \leq n-1$, $p \geq 0$, and each $j_i < \alpha_{k+l}$. See Figure 7, where $k=3$, $l=2$, $m=4$, and $p=0$.

As in the case with the box, whenever we have α_i with $i \notin \{k, \ldots, k+m\}$ we set $\alpha_i' = \alpha_i$. Each of the remaining α_i correspond to two dots in P except for α_{k+l}, which corresponds to three. Let α_{k+l}' be the curve obtained from α_{k+l} via surgery along γ between the first two (leftmost) points of $\alpha_{k+l} \cap \gamma$ corresponding to dots of P and satisfying the following property: α_{k+l}' does not contain the arc of α_{k+l} containing the third (rightmost) point of $\alpha_{k+l} \cap \gamma$ corresponding to a dot on P. We then define $\alpha_{k+l-1}', \ldots, \alpha_k'$ inductively as before using the directed graph in Figure 5, and finally, we define $\alpha_{k+l+1}', \ldots, \alpha_{k+m}'$ inductively as before. It is readily verified that this procedure reduces σ.

Using these surgeries to remove from the dot graph the above polygons results in an initially efficient geodesic. The last step is to inductively produce initially

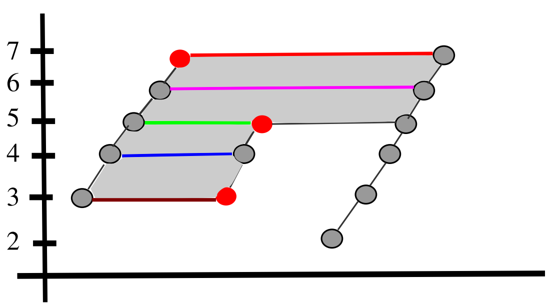


Figure 7. Type 1 hexagon.

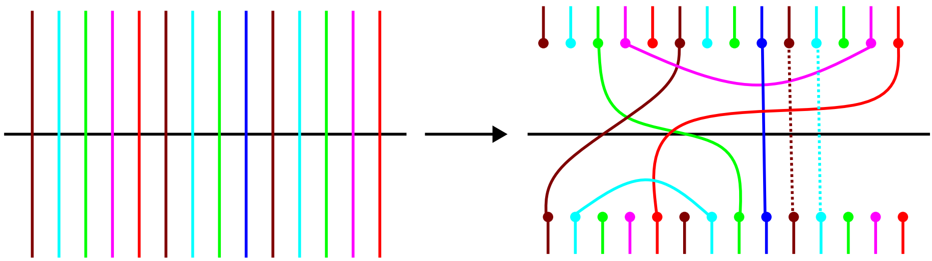


Figure 8. The surgery described above on a type 1 hexagon.

efficient geodesics for the triples $v_k, \ldots, v_n$ for each $0 \le k \le n-3$ and the oriented geodesic $v_n, v_{n-1}, v_{n-2}, v_{n-3}$. This was done in [Birman et al. 2016, Section 3.5]. The exact inductive argument works when restricting to $\mathcal{N}(S)$, so our work lies solely in showing the construction of initially efficient geodesics restricts to $\mathcal{N}(S)$.

2. Existence of efficient geodesics in complex of nonseparating curves

Our main result is that efficient geodesics exist in the complex of nonseparating curves. Since this is a subcomplex of the complex of curves, the fact that there are finitely many of them follows from [Birman et al. 2016, Theorem 1.1]. We restate our main theorem:

Theorem 1.1. *Let $g \ge 2$. If v and w are vertices of $\mathcal{N}(S_g)$ with $d(v, w) \ge 3$, then there exists an efficient geodesic from v to w in $\mathcal{N}(S_g)$. Additionally, there are finitely many efficient geodesics from v to w.*

Proof of Theorem 1.1. The result lies in following exactly the proof setup for the complex of curves in [Birman et al. 2016]. We prove the existence of initially efficient geodesics in $\mathcal{N}(S)$ (see Proposition 2.2). Then the additional inductive step will follow exactly as outlined in [Birman et al. 2016, Section 3.5]. The key observation of this paper is that Lemma 2.1 holds when restricting to $\mathcal{N}(S)$:

Lemma 2.1. *Suppose that σ is a sequence of natural numbers in sawtooth form and that $G(\sigma)$ has an empty, unpierced box or an empty, unpierced hexagon of type* 1 *or* 2*. Then σ is reducible* (*in* $\mathcal{N}(S)$).

This will allow us to prove:

Proposition 2.2. *Let $g \ge 2$. If v and w are vertices of $\mathcal{N}(S)$ with $d(v, w) \ge 3$, then there exists an initially efficient geodesic from v to w in $\mathcal{N}(S)$.*

In [Birman et al. 2016] the existence of the above polygons in the dot graph came with surgeries on the curves representing vertices in the given geodesic that removed these shapes in the dot graph. To prove Lemma 2.1, we need only to show that it is possible to do the surgery constructions outlined in Section 1.4, so that each new curve is nonseparating. For most of the surgeries this is straightforward, while a few need to be redone carefully.

We begin with the well-known fact that a separating curve must intersect any other curve an even number of times.

Proposition 2.3. *Let α be a simple closed curve in S. If α separates S into two components, then for any simple closed curve β we have the geometric intersection number, $i(\alpha, \beta)$, is even.*

Proof. Let S_α be the surface that results from splitting S along α. In S_α, the curve β is either unchanged (all of β is in one of the connected components of S_α) or β is a collection of arcs with endpoints along α. Each of these arcs has two endpoints. Thus intersections, should they exist, come in pairs. In either case this gives an even number of intersections with α. □

The quick test we will use when deciding if a curve made via surgery is nonseparating is to see if $i(\alpha', \beta) = 1$ for some curve β. Next, we must show that given a geodesic in the complex of curves, we may take all the vertices in the path to be nonseparating curves. This allows us to start with geodesic in $\mathcal{N}(S)$ and attempt to make it efficient. The following proposition was also observed in [Hamenstädt 2014, Corollary 3.3].

Proposition 2.4. *Given a geodesic in $\mathcal{C}(S)$ with endpoints $v, w \in \mathcal{N}(S)$ there exists a geodesic from v to w with each vertex a nonseparating curve.*

Proof. Let $v = v_0, \ldots, v_n = w$ be a geodesic in $\mathcal{C}(S)$. If all the vertices in this geodesic are nonseparating curves then we are done. Assume that v_i is a separating curve in the above geodesic with lowest index i. Then consider the subpath v_{i-1}, v_i, v_{i+1} because v_i is separating it divides the surface S into two components. Both v_{i-1} and v_{i+1} are disjoint from v_i however since there is not an edge between them they intersect each other. Therefore, they are both in one of the connected components of S_v. In the other connected component choose a nonseparating simple closed curve v_i'. Such a curve exists, as otherwise v_i would be inessential. This curve is disjoint from both v_{i-1} and v_{i+1} and may replace v_i in the geodesic. Continuing in this way gives a geodesic in the subcomplex of nonseparating curves. □

Notice that the connectivity of $\mathcal{N}(S)$ follows immediately from Proposition 2.4.

The trivial surgeries. The above proposition allows us to start with a geodesic path in $\mathcal{N}(S)$; now we wish to do simplifying surgeries on it. We begin with the case where our reference arc γ sees a curve α consecutively. In $\mathcal{C}(S)$, we performed an even or odd surgery depending on the orientation of α and were guaranteed that each resulted in a simple closed curve. However, when we restrict to the

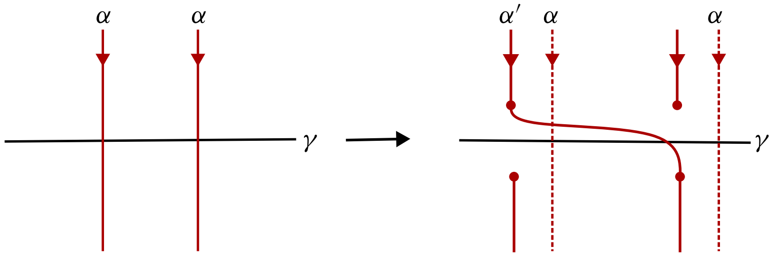


Figure 9. Odd surgery always results in a nonseparating curve.

subcomplex of nonseparating curves we are no longer guaranteed that both curves are nonseparating. For instance, take a separating curve and connect sum it with a nonseparating curve. Then performing surgery along an arc separates the curve into a separating curve and an nonseparating one. Proposition 2.5 shows, however, that performing surgery on a nonseparating curve that meets a reference arc γ consecutively will always yield at least one nonseparating curve.

Proposition 2.5. *Let γ be a reference arc, and α be a nonseparating simple closed curve. Suppose that $i(\alpha, \gamma) \geq 2$. Then there exists a simple closed curve α' obtained from the nonseparating simple closed curve α via some surgery along γ that is still nonseparating.*

Proof. Orient γ and α. Consider two points of intersection that are consecutive along γ. The orientation of γ and α allow us to assign an index to each intersection either $+1$ or -1. If two points of intersection have the same index, we preform an odd surgery. The resulting curve α' crosses γ one time and intersects α exactly once. (See Figure 9 for $+-$ surgery.) We emphasize that we construct α' so that outside the local picture in Figure 9, α' lies just to the right of α. Since $i(\alpha, \alpha') = 1$, α' is nonseparating.

If the two intersection points have opposite indices we perform an even surgery. In this case, we may need to make a choice of curve to replace α with, since one of the surgeries may give a nonseparating curve. One of the curves remains above γ and the other remains below γ. We argue that at least one of these curves is nonseparating.

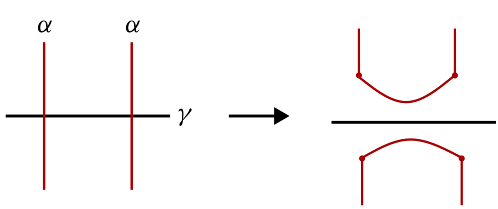


Figure 10. $++$ and $--$ surgery on α.

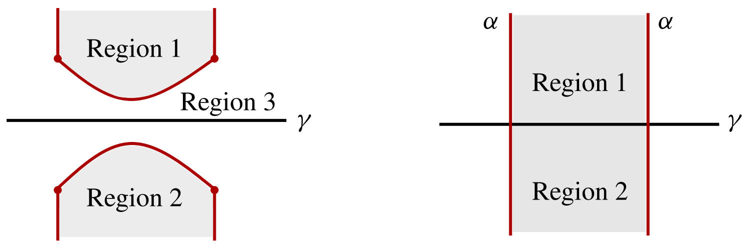


Figure 11. Figure for the proof of Proposition 2.5.

Assume for contradiction that both curves were separating this would divide S into three connected components. Then joining the curves back together would give back our original curve α however α would still separate our surface.

Thus, performing an even surgery always results in at least one nonseparating curve. □

The nontrivial surgeries. We use the dot graph exactly as in [Birman et al. 2016] to determine how to reduce our intersections sequence. When restricting to the nonseparating curve complex we need to show that each dot graph polygon has a surgery that results in the removal of the polygon and whose new curves are all nonseparating. This will prove Lemma 2.1.

Throughout we assume that σ is an intersection sequence of nonseparating curves in sawtooth form.

Case 1: Suppose that σ is a sequence of natural numbers in sawtooth form and that $G(\sigma)$ has an empty, unpierced box. Then σ is reducible.

This is the easy case. Carry out the surgeries exactly as in $\mathcal{C}(S)$. Performing the surgeries one at a time, notice that regardless of the type of surgery, the resulting curve will intersect a curve adjacent to it exactly once. The figures below demonstrate this when the box has three curves involved. The general case follows exactly the same. See Figures 13 and 14 for the first two steps in the surgery sequence of a box containing the curves 3, 4, and 5.

Continuing in this way, removes the box from the dot graph and replaces all curves with other nonseparating curves.

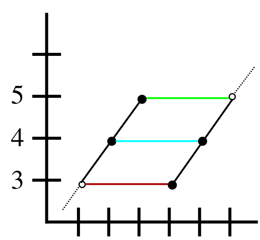


Figure 12. An empty, unpierced box.

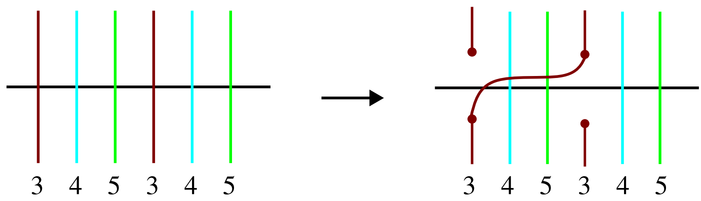


Figure 13. The intersection sequence corresponding to the above box in the dot graph. Odd surgery on curve 3 intersects curve 4 exactly once, and thus it is nonseparating. Clearly, an even surgery would do the same, so there are two surgery options for curve 3 exactly as before.

Case 2: Suppose that σ is a sequence of natural numbers in sawtooth form and that $G(\sigma)$ has an empty, unpierced hexagon of type 1 or 2. Then σ is reducible.

We will treat the case of an empty, unpierced hexagon of type 1. The other case follows exactly the same procedure. The surgery instructions for this case are similar to the instructions for $\mathcal{C}(S)$ but one new idea is needed. We introduce some new terminology to simplify the discussion. Given a empty, unpierced hexagon of type 1 in the dot graph, its vertices have the form

$$\alpha_k, \ldots, \alpha_{k+m}, \alpha_k, \ldots, \alpha_{k+l}, \alpha_{j_1}, \ldots, \alpha_{j_p}, \alpha_{k+l}, \ldots, \alpha_{k+m},$$

where $1 \leq k \leq k+l \leq k+m \leq n-1$, $p \geq 0$, and each $j_i < \alpha_{k+l}$. We will call the integer l the *step length* of the hexagon, the number of vertices in $\alpha_{j_1}, \ldots, \alpha_{j_p}$ the *tail length* of the hexagon, and the integer m the *total length* of the hexagon. From the dot graph it is easy to see these values. For instance the hexagon in Figure 7 has a step length 2, and tail length 4. We will call the curve α_{k+l} *the curve at step length l*.

In the discussion below we will also refer to vertices on the dot graph as being "above" or "below" each other. This is referring to the actual placement of the vertices on the dot graph. Along the intersection arc, a curve being "above" another means that its index is higher than the other.

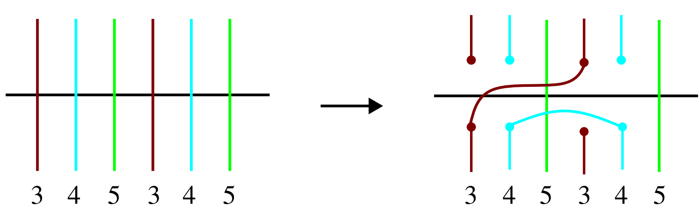


Figure 14. Now surgery on curve 4 is performed. This time an even surgery is demonstrated. As in Figure 13, this curve intersects curve 5 exactly one time, so it is nonseparating. An odd surgery would do the same.

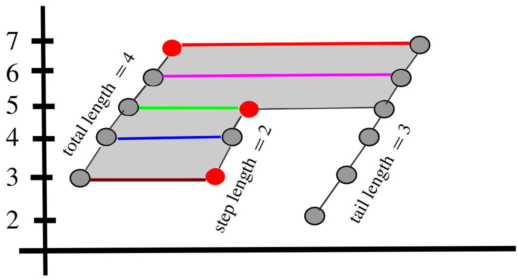


Figure 15. These three numbers, along with the starting vertex, completely determine a type 1 hexagon.

Notice that we may always assume that the tail length of a hexagon is nonzero. If it were zero, the curve at step length l would occur consecutively in the dot graph, and a trivial surgery on this curve would remove the hexagon from the dot graph. We begin the hexagon surgery the same. Consider the curve at step length l, this is the only curve that occurs three times in the hexagon. There exists a surgery on the first two intersection points that removes the third intersection point. This surgery is determined by the orientations of the first two intersection points. However, whatever surgery is required intersects the curves directly adjacent (above and below in the dot graph) to it exactly one time, thus the result is nonseparating. See Figure 16.

We now attempt to perform surgeries on the curves that occur below the curve at step length l. Just as in the box case, these curves have either surgery available to them since all possible surgeries will intersect the curve directly adjacent to it (above it in the dot graph) exactly once. Now we are ready perform surgeries on the curves that occur after the curve at step length l. The curve directly adjacent to

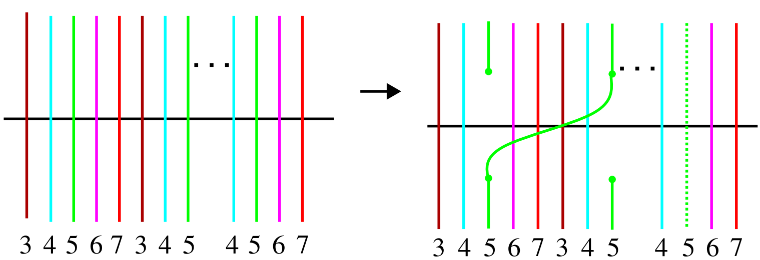


Figure 16. The intersection sequence corresponding to the above hexagon in the dot graph. Odd surgery on curve 5 intersects curve 6 (and 4) exactly once, and thus it is nonseparating. An even surgery would do the same, so any required surgery to delete the last 5 vertices in the dot graph works. The ellipses represent curves in the tail of the hexagon, and the curve 4 is seen in the intersection sequence because we assume the tail length is nonzero.

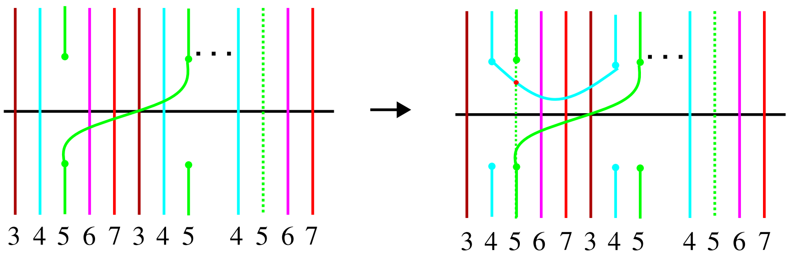


Figure 17. Now surgery on curve 4 is performed. This time an even surgery is demonstrated. As above this curve intersects the old curve 5 exactly one time, so it is nonseparating. An odd surgery would do the same. The same follows for curve 3.

the curve at step length l (above it in the dot graph) may cause issues. We break this into two cases:

Subcase 2.1: Let m denote the total length of the hexagon, and let l denote the step length. If $m > l+1$, then the surgeries follow exactly as in $\mathcal{C}(S)$. This is clear since all the curves obtained from surgery above the curve at step length l intersect an adjacent curve (above or below in the dot graph) exactly once. Notice that for the curve directly above the curve at step length l, α_{k+l+1}, we use the intersection with the curve above it α_{k+l+2} to show it is nonseparating. For curves above α_{k+l+1} in the dot graph, say α_j, we look at the intersection with α_{j-1} to show it is nonseparating. See Figure 18.

Subcase 2.2: Let m denote the total length of the hexagon, and let l denote the step length. If $m = l+1$, then we introduce a new surgery on the curve α_{k+l+1}.

All the curves below the curve at step length l have the surgeries performed on them as before, each is nonseparating. We now need to perform a surgery on the curve α_{k+l+1} and argue that is nonseparating.

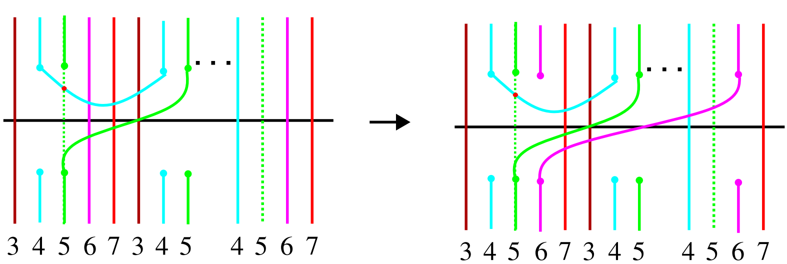


Figure 18. Surgery on curve 6 is performed. This curve intersects the curve 7 exactly one time, so it is nonseparating. An odd surgery would do the same. Notice the surgery would intersect the old curve 5 twice, so it is not possible to argue that it is nonseparating with curve 5.

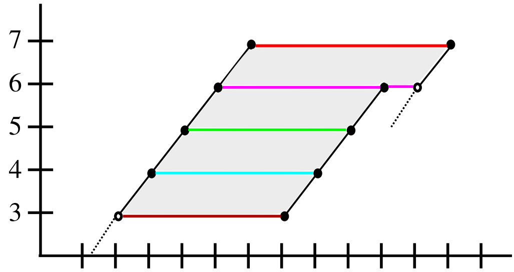


Figure 19. Surgery on the curves 3 through 5 will be just fine, since either type of surgery will intersect the curve directly above it exactly once. But surgery on 7 will intersect 6 twice.

If the curve α_{k+l+1} requires odd surgery, then it is nonseparating because the surgered curve α'_{k+l+1} intersects α_{k+l+1} exactly once as in Proposition 2.5. In the case where α_{k+l+1} requires even surgery and the even surgery required forces the new curve α'_{k+l+1} to be separating, will we consider the part of α_{k+l} (the curve at step length l) that is "inside" of the curve α'_{k+l+1} (We call the "inside" of α'_{k+l+1} the part of the surface disjoint from the curve α'_{k+l}). Since we are assuming our hexagon has a tail, we will also see the curve α_{k+l-1} intersect γ one time inside the region along gamma between the second occurrence of α_{k+l} and the third occurrence of α_{k+1}. If we were able to perform even surgery on these two parts of α_{k+l}, the resulting curve is nonseparating since it intersects α_{k+l-1} exactly once. By the assumption that α'_{k+l+1} is separating, the orientation of α_{k+l} inside of α'_{k+l+1} must be consistent with an even surgery, otherwise α_{k+l} would intersect α_{k+l+1}. See Figure 20.

Let the new nonseparating curve that is obtained by joining the ends of curve α_{k+l} inside of α'_{k+l+1} be denoted by β. We want β to be a replacement curve for curve α_{k+l+1} so it must be disjoint from α'_{k+l} and α_{k+l+2}. Clearly, β is disjoint

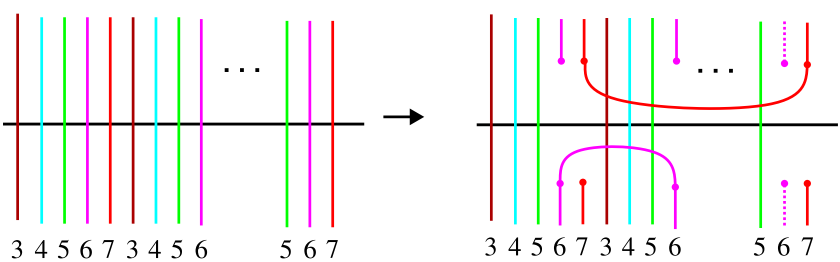


Figure 20. After performing surgery on curve 6, and doing the required even surgery on curve 7 (in the figure it is ++), notice the two parts of curve 6 bounded by the new curve 7′. If we assume that 7′ is separating then the two parts of 6 inside 7′ must be oriented to allow for ++ surgery.

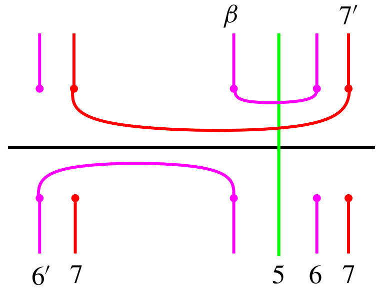


Figure 21. The curve β is disjoint from curve $6'$ since $7'$ is disjoint from $6'$ and β, and β is on the other side of $7'$ than $6'$.

from α'_{k+l} since α'_{k+l+1} is disjoint from α'_{k+l}, β is disjoint from α'_{k+l+1} and β is on the other side of α'_{k+l+1} then α'_{k+l}. This is demonstrated in Figure 21.

Now we need to argue that the curve β is disjoint from the curve α_{k+l+2}. Observe that α'_{k+l+1} must be disjoint from α_{k+l+2}. Indeed, it is obtained by surgery on the curve α_{k+l+1} with an arc of γ, both of which are disjoint from α_{k+l+2}, because α_{k+l+2} does not enter the hexagon. Thus, if α_{k+l+2} intersects β it is contained entirely inside α'_{k+l+1} and therefore disjoint from α'_{k+l}. This is a contradiction. Since then the curve α_{k+l+1} is not required in a geodesic path. We would have a path $\alpha_0, \alpha'_1, \ldots, \alpha'_{k+l}, \alpha_{k+l+2}, \ldots, \alpha_n$ of length $n-1$ contradicting the assumption that the distance between v_0 and v_n is exactly n. So, β is a nonseparating curve disjoint from α_{k+l+2} and α'_{k+l}. Therefore, β is a suitable replacement for α_{k+l+1}.

The case for a hexagon of type 2 follows the exact same argument. This covers all the cases, thus proving Lemma 2.1.

3. Conclusion

We have demonstrated that efficient geodesics exist in the nonseparating curve complex. Moreover, we demonstrated that given any geodesic in the complex of curves we may obtain an efficient geodesic whose vertices, with the possible exception of the endpoints, are all nonseparating curves.

Question 1. Birman, Margalit, and Menasco [2016, Theorem 1.1] prove that, given vertices v_0 and v_n at distance n in $\mathcal{C}(S)$, there are at most n^{6g-6} vertices that may appear as the first curve in an initially efficient geodesic. Can this bound be reduced when we restrict to $\mathcal{N}(S)$?

Question 2. Does replacing a separating curve with a nonseparating curve in a path ever increase the complexity measure?

Question 3. Find an example of an efficient geodesic that contains a separating curve. Are there any restrictions on the number of separating curves in a efficient geodesics?

4. Acknowledgements

We thank William Menasco for suggesting the problem, as well as many conversations about the efficient geodesic algorithm, flushing out some errors in earlier drafts of this paper, and discussing the new required surgery. We are also very thankful to the referees for their careful reading of the manuscript and help at making the exposition clearer.

References

[Birman et al. 2016] J. Birman, D. Margalit, and W. Menasco, "Efficient geodesics and an effective algorithm for distance in the complex of curves", *Math. Ann.* **366**:3-4 (2016), 1253–1279. MR Zbl

[Glenn et al. 2017] P. Glenn, W. W. Menasco, K. Morrell, and M. J. Morse, "MICC: a tool for computing short distances in the curve complex", *J. Symbolic Comput.* **78** (2017), 115–132. MR Zbl

[Hamenstädt 2014] U. Hamenstädt, "Hyperbolicity of the graph of nonseparating multicurves", *Algebr. Geom. Topol.* **14**:3 (2014), 1759–1778. MR Zbl

[Masur and Minsky 2000] H. A. Masur and Y. N. Minsky, "Geometry of the complex of curves, II: Hierarchical structure", *Geom. Funct. Anal.* **10**:4 (2000), 902–974. MR Zbl

[Rasmussen 2020] A. J. Rasmussen, "Uniform hyperbolicity of the graphs of nonseparating curves via bicorn curves", *Proc. Amer. Math. Soc.* **148**:6 (2020), 2345–2357. MR Zbl

[Wright 2023] A. Wright, "Spheres in the curve graph and linear connectivity of the Gromov boundary", preprint, 2023. arXiv 2304.03004

Received October 1, 2023. Revised June 20, 2024.

SETH HOVLAND
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BUFFALO
BUFFALO, NY
UNITED STATES
sethhovl@buffalo.edu

GREG VINAL
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BUFFALO
BUFFALO, NY
UNITED STATES
gregoryv@buffalo.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.251

ON MULTIPLICITY-FREE WEIGHT MODULES OVER QUANTUM AFFINE ALGEBRAS

XINGPENG LIU

Our goal is to construct and study the multiplicity-free weight modules of quantum affine algebras. For this, we introduce the notion of shiftability condition with respect to a symmetrizable generalized Cartan matrix, and investigate its applications on the study of quantum affine algebra structures and the realizations of the infinite-dimensional multiplicity-free weight modules. We also compute the highest ℓ-weights of the infinite-dimensional multiplicity-free weight modules as highest ℓ-weight modules.

1. Introduction

Let $U_q(\mathfrak{g})$ be the quantum affine algebra (without derivation) associated to an affine Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ in which q is not a root of unity. In this note, we are concerned with infinite-dimensional multiplicity-free weight representations, i.e., those with all of their weight subspaces one-dimensional, over $U_q(\mathfrak{g})$. As we shall see, these representations are the basic representations toward the infinite-dimensional modules of quantum affine algebras.

In the classical cases, the multiplicity-free weight representations over finite-dimensional simple Lie algebras, or more generally, the bounded weight representations have been extensively studied in [Benkart et al. 1997; Britten et al. 1994; Grantcharov and Serganova 2006; 2010]. These representations play a crucial role in the classification of simple weight modules of finite-dimensional simple Lie algebras (see [Mathieu 2000]). For the quantum groups of finite type, Futorny, Hartwig, and Wilson [Futorny et al. 2015] gave a classification of all infinite-dimensional irreducible multiplicity-free weight representations of type A_n. Recently, the infinite-dimensional multiplicity-free weight representations of the quantum groups of types A_n, B_n and C_n were constructed in [Chen et al. 2024].

As an important class of multiplicity-free weight modules, the q-oscillator representations over $U_q(\mathfrak{g})$ of types $A_n^{(1)}$, $C_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$ have been obtained in the works of T. Hayashi [1990] and A. Kuniba and M. Okado [2018; 2013; 2015]. Our goal is to construct infinite-dimensional multiplicity-free weight representations

MSC2020: primary 17B10, 17B37; secondary 16T20, 20G42.
Keywords: quantum affine algebra, representation theory, multiplicity-free weight modules.

of $U_q(\mathfrak{g})$ in a general way. For this, associated to each symmetrizable generalized Cartan matrix, we introduce a system of equations in a Laurent polynomial ring $\mathcal{A}$ (essentially, the Cartan part of $U_q(\mathfrak{g})$) by the shift operators. We say that the corresponding generalized Cartan matrix satisfies the *shiftability condition* if the system of equations has solutions (see Section 4A). One result of this note is that an affine Cartan matrix satisfies the shiftability condition if and only if the relevant Dynkin diagram is one of the types mentioned above (see Theorem 4.2). The solutions allow us to define $U_q(\mathfrak{g})$-module structures on $\mathcal{A}$, and to relate the quantum affine algebra structures with the n-fold quantized oscillator algebra. Our method for the construction is parallel with the earlier work concerning U^0-free modules [Chen et al. 2024]. Namely, we can get the multiplicity-free weight modules of $U_q(\mathfrak{g})$ by applying the "weighting" procedure to the above modules on $\mathcal{A}$. In particular, the q-oscillator representations can also be reconstructed.

For the study of weight representations of quantum affine algebras, the concepts of ℓ-weights and ℓ-weight vectors have proven especially useful, allowing one to refine the spectral data properly in weight representations. For example, we have the classification of irreducible finite-dimensional representations (see [Chari and Pressley 1991; 1998]) and infinite-dimensional weight representations of quantum affine algebras [Hernandez 2005; Mukhin and Young 2014] by highest ℓ-weights (their highest ℓ-weights are determined by *Drinfeld polynomials* and *rational functions*, respectively). In this note, we shall compute explicitly the highest ℓ-weight of the q-oscillator representations. For the type $A_n^{(1)}$, the highest ℓ-weights of q-oscillator representations also were discussed in [Boos et al. 2016; 2017; Kwon and Lee 2023].

The paper is organized as follows. In Section 2, we give some necessary notation, and review two presentations of quantum affine algebras. In Section 3, we recall the definition of highest ℓ-weight representations. Then we obtain the classification of highest ℓ-modules with finite weight multiplicities in general. In Section 4, we introduce the notion of shiftability condition, and present the solutions to the corresponding system of equations, which allow us to study the compatible structures of quantum affine algebras with the n-fold quantized oscillator algebra. In Section 5, the infinite-dimensional multiplicity-free weight modules are constructed. In Section 6, we compute the highest ℓ-weight of the q-oscillator representations.

Conventions. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of integers, real numbers and complex numbers respectively. Denote $\mathbb{C}\setminus\{0\}$ by $\mathbb{C}^\times$ and the set of nonnegative integers by $\mathbb{Z}_{\geq 0}$. Finally, δ_{ij} is the Kronecker symbol.

2. Preliminaries and notation

First, let us recall some necessary notation and two presentations of quantum affine algebras based on [Beck and Nakajima 2004; Drinfeld 1987; Kac 1990].

2A. ***Affine Kac–Moody algebras.*** Let $\mathfrak{g} = \mathfrak{g}(X_N^{(r)})$ be an affine Kac–Moody algebra with respect to the generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ of type $X_N^{(r)}$, where $I = \{0, 1, \dots, n\}$ is an indexed set and $X_N^{(r)}$ is a Dynkin diagram from [Kac 1990, Table Aff r], except in the case of $X_N^{(r)} = A_{2n}^{(2)}$ $(n \geq 1)$, where we reverse the numbering of the simple roots.

Let $\{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ (resp. $\{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$) denote the set of simple roots (resp. simple coroots) such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the *root lattice* of $\mathfrak{g}$. Set $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. Assume that $\delta = \sum a_i \alpha_i$ and $c = \sum a_i^\vee \alpha_i^\vee$ are the smallest positive imaginary root and a central element of $\mathfrak{g}$, where a_i and $a_i^\vee$ are the numerical labels of the Dynkin diagrams of $X_N^{(r)}$ and its dual, respectively. Let $\{\omega_i\}_{i \in I}$ denote the *fundamental weights* of $\mathfrak{g}$, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in I$.

Let W be the *affine Weyl group* of $\mathfrak{g}$ (which is a subgroup of the general linear group of $\mathfrak{h}^*$) generated by the *simple reflections* $s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$, $\lambda \in \mathfrak{h}^*$, $i \in I$. Note that $w(\delta) = \delta$ for all $w \in W$. Set $I_0 = I \setminus \{0\}$. Denote by $\mathring{W}$ the subgroup of W generated by the simple reflections s_i for $i \in I_0$. It is a finite group.

Take the nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ invariant under the action of W, which is normalized uniquely by $(\lambda, \delta) = \langle \lambda, c \rangle$ for $\lambda \in \mathfrak{h}^*$. Define D as the diagonal matrix $\operatorname{diag}(d_0, \dots, d_n)$ with $d_i = a_i^{-1} a_i^\vee$. Then $(\alpha_i, \alpha_j) = d_i a_{ij}$ for all $i, j \in I$. Let $\triangle$ be the *root system* of $\mathfrak{g}$, $\triangle^\pm = \triangle \cap (\pm Q_+)$, and let $\triangle^{\mathrm{re}} = \triangle \setminus \mathbb{Z}\delta$ be the set of *real roots*. For each $\alpha \in \triangle^{\mathrm{re}}$ we set $\widetilde{d}_\alpha = \max\bigl(1, \frac{1}{2}(\alpha, \alpha)\bigr)$. In particular, write $\widetilde{d}_i$ simply for $\widetilde{d}_{\alpha_i}$. Then

$$\widetilde{d}_i = \begin{cases} 1 & \text{if } r = 1 \text{ or } X_N^{(r)} = A_{2n}^{(2)}, \\ d_i & \text{otherwise.} \end{cases}$$

Denote by $\mathring{A} = (a_{ij})_{i,j \in I_0}$ the Cartan matrix of finite type, and let $\mathring{\mathfrak{g}}$ be the associated simple finite-dimensional Lie algebra. Then $\{\alpha_i\}_{i \in I_0}$ is a set of simple roots for $\mathring{\mathfrak{g}}$. Let $\mathring{Q} = \bigoplus_{i \in I_0} \mathbb{Z}\alpha_i$ be the root lattice for $\mathring{\mathfrak{g}}$, and let $\widetilde{P}$ be the weight lattice of the euclidean space $\mathbb{R} \otimes_{\mathbb{Z}} \mathring{Q} \subset \mathfrak{h}^*$ defined as $\widetilde{P} = \bigoplus_{i \in I_0} \mathbb{Z}\widetilde{\omega}_i$, where $(\widetilde{\omega}_i, \alpha_j) = \delta_{ij} \widetilde{d}_i$. Then $\mathring{Q}$ can be naturally embedded into $\widetilde{P}$, which provides a W-invariant action on $\mathfrak{h}^*$ by $x(\lambda) = \lambda - (x, \lambda)\delta$ for $x \in \widetilde{P}, \lambda \in \mathfrak{h}^*$.

Define the *extended Weyl group* by $\widetilde{W} = \mathring{W} \ltimes \widetilde{P}$. We also have $\widetilde{W} = W \ltimes \mathscr{T}$, where $\mathscr{T} = \{w \in \widetilde{W} \mid w(\triangle^+) \subset \triangle^+\}$, which is a subgroup of the group of the Dynkin diagram automorphisms. An expression for $w \in \widetilde{W}$ is called *reduced* if $w = \tau s_{i_1} \cdots s_{i_l}$, where $\tau \in \mathscr{T}$ and l is minimal. We call the minimal integer l the *length* of w, and denote it by $l(w)$.

2B. ***Quantum affine algebras.*** The *quantum affine algebra* $U_q(\mathfrak{g})$ in the Drinfeld–Jimbo realization [Drinfeld 1985; Jimbo 1985] is the unital associative algebra over $\mathbb{C}$ generated by X_i^+, X_i^-, $K_i^{\pm 1}$, $i \in I$, with the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \tag{2-1}$$

$$K_i X_j^{\pm} K_i^{-1} = q_i^{\pm a_{ij}} X_j^{\pm}, \tag{2-2}$$

$$X_i^+ X_j^- - X_j^- X_i^+ = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2-3}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-a_{ij}-k} = 0 \quad \text{for } i \neq j, \tag{2-4}$$

where $q \in \mathbb{C}^\times$ is not a root of unity and $q_i = q^{d_i}$. We have used the standard notation:

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q^! = [m]_q [m-1]_q \cdots [1]_q, \quad \begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q^!}{[r]_q^! [m-r]_q^!}.$$

In particular, we denote $[m]_{q_i}$ by $[m]_i$ for simplicity.

Let U^0 be the commutative subalgebra of $U := U_q(\mathfrak{g})$ generated by K_i, K_i^{-1}, $i \in I$. It is clear that each element in U^0 is a linear combination of the monomials $K_\beta := K_0^{b_0} K_1^{b_1} \cdots K_n^{b_n}$ for $\beta = \sum_{i \in I} b_i \alpha_i \in Q$. In particular, K_δ is a central element in U. Let U^+ (resp. U^-) denote the span of monomials in X_i^+ (resp. X_i^-). Recall that U has a canonical triangular decomposition $U \cong U^- \otimes U^0 \otimes U^+$. For later use, we note that U^+ is graded by Q_+ in the usual way: $U^+ = \bigoplus_{\beta \in Q_+} U_\beta^+$.

Let us recall the Hopf algebra structure of U with the coproduct Δ, the antipode S, and the counit ϵ defined as follows:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(X_i^+) = X_i^+ \otimes 1 + K_i \otimes X_i^+,$$
$$\Delta(X_i^-) = X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-,$$
$$S(X_i^+) = -K_i^{-1} X_i^+, \quad S(X_i^-) = -X_i^- K_i, \quad S(K_i) = K_i^{-1},$$
$$\epsilon(X_i^+) = 0 = \epsilon(X_i^-), \quad \epsilon(K_i) = 1.$$

There exists another presentation of U due to Drinfeld [1987]. Just like the realizations of the affine Kac–Moody algebras $\mathfrak{g}$ as (twisted) loop algebras, this presentation of U is generated by the Drinfeld's "loop-like" generators.

Consider the root datum (X_N, σ) with σ a diagram automorphism of X_N of order r. Let $\bar{A} = (\bar{a}_{ij})_{1 \le i,j \le N}$ be the Cartan matrix of the type X_N, and let ω be a fixed primitive r-th root of unity. Note that if $r = 1$ (i.e., σ is an identity) we have $N = n$ and $\bar{A} = \mathring{A}$; if $r > 1$, then X_N is one of the simply laced types: A_N $(N \ge 2)$, D_{n+1} $(n \ge 2)$, or E_6. We use $\bar{i} \in I_0$ to stand for one representative of the σ-orbit of i on $\{1, 2, \dots, N\}$ such that $\bar{i} \le \sigma^s(i)$ for any s. Take the set of simple roots $\{\bar{\alpha}_i\}_{1 \le i \le N}$ and the normalized bilinear form $(\ ,\ )$ (by abuse of notation) such that $(\bar{\alpha}_i, \bar{\alpha}_j) = d_i a_{ij}$ if $r = 1$ and otherwise $(\bar{\alpha}_i, \bar{\alpha}_j) = \bar{a}_{ij}$ for $1 \le i, j \le N$.

The quantum affine algebra U (add the central elements $K_\delta^{\pm 1/2}$) is isomorphic to the algebra generated by $x_{i,k}^{\pm}$ $(1 \le i \le N,\ k \in \mathbb{Z})$, $h_{i,k}$ $(1 \le i \le N,\ k \in \mathbb{Z} \setminus \{0\})$, $k_i^{\pm 1}$ $(1 \le i \le N)$, and the central elements $C^{\pm 1/2}$, subject to the following relations:

$$x^{\pm}_{\sigma(i),k}=\omega^k x^{\pm}_{i,k},\quad h^{\pm}_{\sigma(i),k}=\omega^k h^{\pm}_{i,k},\quad k^{\pm 1}_{\sigma(i)}=k^{\pm 1}_i,$$
$$k_i k_i^{-1}=k_i^{-1}k_i=1,\quad k_i k_j=k_j k_i,\quad k_i h_{j,l}=h_{j,l}k_i,$$
$$k_i x^{\pm}_{j,k}=q_{\bar{i}}^{\pm a_{\bar{i}\bar{j}}}x^{\pm}_{j,k}k_i,$$

$$[h_{i,k},h_{j,l}]=\delta_{k,-l}\frac{1}{k}\biggl(\sum_{s=1}^{r}\biggl[\frac{k(\bar{\alpha}_i,\bar{\alpha}_{\sigma^s(j)})}{d_{\bar{i}}}\biggr]_{\bar{i}}\omega^{ks}\biggr)\frac{C^k-C^{-k}}{q_{\bar{i}}-q_{\bar{i}}^{-1}},\tag{2-5}$$

$$[h_{i,k},x^{\pm}_{j,l}]=\pm\frac{1}{k}\biggl(\sum_{s=1}^{r}\biggl[\frac{k(\bar{\alpha}_i,\bar{\alpha}_{\sigma^s(j)})}{d_{\bar{i}}}\biggr]_{\bar{i}}\omega^{ks}\biggr)C^{\mp|k|/2}x^{\pm}_{j,k+l},$$

$$[x^{+}_{i,k},x^{-}_{j,l}]=\biggl(\sum_{s=1}^{r}\frac{\delta_{\sigma^s(i)j}\,\omega^{sl}}{\tilde{d}_{\bar{i}}}\biggr)\frac{C^{(k-l)/2}\psi^{+}_{i,k+l}-C^{-(k-l)/2}\psi^{-}_{i,k+l}}{q_{\bar{i}}-q_{\bar{i}}^{-1}},$$

where the $\psi^{\pm}_{i,k}$ are the elements determined by the following identity of the formal power series in z:

$$\sum_{k=0}^{\infty}\psi^{\pm}_{i,\pm k}z^{\pm k}=k_i^{\pm 1}\exp\biggl(\pm(q_{\bar{i}}-q_{\bar{i}}^{-1})\sum_{l=1}^{\infty}h_{i,\pm l}z^{\pm l}\biggr),\tag{2-6}$$

together with the *quantum Serre–Drinfeld relations*, whose explicit forms will not be used in this paper. One can refer to [Drinfeld 1987] for more details and to [Beck 1994; Jing 1998] and [Damiani 2000; 2012; 2015][1] for a proof.

Under the isomorphism, we have $X_i^{\pm}=x^{\pm}_{i,0}$, $K_i^{\pm 1}=k_i^{\pm 1}$ for $i\in I_0$, and $K_\delta=C$. Note that $\psi^{+}_{i,-k}=\psi^{-}_{i,k}=0$ for any positive integers k, and $\psi^{\pm}_{i,0}=k_i^{\pm 1}$ from the identity (2-6).

From the relations in the Drinfeld presentation, U is essentially generated by $x^{\pm}_{i,\tilde{d}_i k}$ $(i\in I_0, k\in\mathbb{Z})$, $h_{i,\tilde{d}_i k}$ $(i\in I_0, k\in\mathbb{Z}\setminus\{0\})$, $k_i^{\pm 1}$ $(i\in I_0)$, and the central elements $C^{\pm 1/2}$ (see [Damiani 2012, Proposition 4.25]). Moreover, the quantum affine algebra U has a triangular decomposition [Chari and Pressley 1994; 1998]:

$$U\cong U(\leqslant)\otimes U(0)\otimes U(\geqslant),\tag{2-7}$$

where $U(\geqslant)$ (resp. $U(\leqslant)$) is the subalgebra generated by $x^{+}_{i,\tilde{d}_i k}$ (resp. $x^{-}_{i,\tilde{d}_i k}$), $i\in I_0$, $k\in\mathbb{Z}$, and $U(0)$ is the subalgebra generated by $C^{\pm 1/2}$ and $k_i^{\pm}$, $h_{i,k}$, $i\in I_0$, $k\in\mathbb{Z}\setminus\{0\}$.

3. Highest ℓ-weight representations with finite weight multiplicities

In this section, we recall basic notation of representations over quantum affine algebras: weight modules, ℓ-weights, and highest ℓ-weight modules. Most of the

[1] The author used the notation $\widetilde{H}^{\pm}_{i,l}$, $H_{i,l}$, which is related with the notation $\psi^{\pm}_{i,l}$, $h_{i,l}$ by $\widetilde{H}^{\pm}_{i,l}=C^{l/2}k_i^{\mp 1}\psi^{\pm}_{i,l}$ and $H_{i,l}=C^{l/2}h_{i,l}$.

definitions and results in this section are well-known; one can refer to [Chari and Pressley 1991; Mukhin and Young 2014].

3A. ***Highest ℓ-weight modules.*** We begin with the notion of highest ℓ-weight modules. Thanks to the Hopf algebra structure of U^0 (inherited from U), the set of all *algebra characters* of U^0, i.e., all algebra homomorphisms from U^0 to $\mathbb{C}$, has an abelian group structure. The addition and the inverse are given by

$$(\lambda+\mu)(u)=(\lambda\otimes\mu)\circ\Delta(u),\quad (-\lambda)(u)=\lambda\circ S(u)$$

for any algebra characters λ,μ, and $u\in U^0$. Denote this group simply by $(\mathscr{X},+)$. Any $\beta\in\mathfrak{h}^*$ induces a character in $\mathscr{X}$ by assigning K_i to $q^{(\beta,\alpha_i)}$ for $i\in I$, which is unique up to a constant multiple of δ, so we still denote it by $\beta\in\mathscr{X}$.

For a U-module V and $\lambda\in\mathscr{X}$, define

$$V_\lambda=\{v\in V\mid u.v=\lambda(u)v \text{ for all } u\in U^0\}.$$

By the defining relations (2-2) we have $X_i^{\pm}.V_\lambda\subset V_{\lambda\pm\alpha_i}$. If V_λ is nonzero, then we say λ is a *weight* of V, and V_λ is a *weight space* of *weight* λ. A nonzero vector $v\in V_\lambda$ is called a *weight vector* of weight λ. If the weight space V_λ is finite-dimensional, then $\dim V_\lambda$ is called the *multiplicity* of the weight λ. Call V a *weight module* if $V=\bigoplus_\lambda V_\lambda$. Moreover, a weight module V is said to be *multiplicity-free* if $\dim V_\lambda\le 1$ for all $\lambda\in\mathscr{X}$.

Throughout this note, we assume that the central element C acts trivially on a U-module. So any weight λ of a U-module is *level-zero*, that is, $\lambda(K_\delta)=1$.

Note that the actions of the $\psi_{i,k}^{\pm}$ on a U-module commute with each other by (2-5) and (2-6). For a weight λ of V with finite multiplicity, we may refine the weight space V_λ as

$$V_\lambda=\bigoplus_{\boldsymbol{\gamma}:\mathrm{wt}(\boldsymbol{\gamma})=\lambda}V_{\boldsymbol{\gamma}},$$

$$V_{\boldsymbol{\gamma}}=\{v\in V_\lambda\mid \forall 1\le i\le N, k\ge 0, \exists m\in\mathbb{Z}_{>0}, (\psi_{i,\pm k}^{\pm}-\gamma_{i,\pm k}^{\pm})^m.v=0\},$$

where $\boldsymbol{\gamma}=(\gamma_{i,\pm k}^{\pm})_{1\le i\le N,k\in\mathbb{Z}_{\ge 0}}$ is any N-tuple of sequences of complex numbers satisfying that $\gamma_{i,0}^{+}\gamma_{i,0}^{-}=1$ and $\gamma_{\sigma(i),\pm k}^{\pm}=\omega^{\pm k}\gamma_{i,\pm k}^{\pm}$ for all $1\le i\le N$, and we associate $\boldsymbol{\gamma}$ with a level-zero weight $\mathrm{wt}(\boldsymbol{\gamma})\in\mathscr{X}$ by setting $\mathrm{wt}(\boldsymbol{\gamma})(K_i)=\gamma_{i,0}^{+}$ for all $i\in I_0$. Call such a sequence $\boldsymbol{\gamma}$ an *ℓ-weight* and $V_{\boldsymbol{\gamma}}$ the *ℓ-weight space* of $\boldsymbol{\gamma}$ if $V_{\boldsymbol{\gamma}}$ is not zero.

Given an ℓ-weight $\boldsymbol{\gamma}$, the defining relations in the Drinfeld presentation imply that $\boldsymbol{\gamma}$ is completely determined by the tuple of complex numbers $(\gamma_{i,\pm\tilde{d}_i k}^{\pm})_{i\in I_0,k\in\mathbb{Z}_{\ge 0}}$. Note that the $\gamma_{i,k}^{\pm}$ for $\tilde{d}_i\nmid k$ are zero. Hence we may write $\boldsymbol{\gamma}\equiv(\gamma_{i,\pm\tilde{d}_i k}^{\pm})_{i\in I_0,k\in\mathbb{Z}_{\ge 0}}$ directly without any ambiguity.

Now we can define the highest ℓ-weight modules.

Definition 3.1. We say V is a *highest ℓ-weight modules* of highest ℓ-weight $\boldsymbol{\gamma}$ if $V = U.v$ for some nonzero vector $v \in V$ such that $x^+_{i,k}.v = 0$ for $1 \le i \le N$, $k \in \mathbb{Z}$, and $\psi^{\pm}_{i,\pm k}.v = \gamma^{\pm}_{i,\pm k}v$ for $1 \le i \le N$, $k \in \mathbb{Z}_{\ge 0}$. By (2-7), $\dim V_{\boldsymbol{\gamma}} = 1$, so v is unique up to a scalar; we call it the *highest ℓ-weight vector* of V.

3B. ***The classification theorem: rationality.*** In this subsection we give the classification of simple highest ℓ-weight modules with finite weight multiplicity, which appeared in [Mukhin and Young 2014] for untwisted cases.

We say an ℓ-weight $\boldsymbol{f} = (f^{\pm}_{i,\pm\tilde{d}_i k})_{i\in I_0, k\in\mathbb{Z}_{\ge 0}}$ is *rational* if there is a tuple of complex-valued rational functions $(f_i(z))_{i\in I_0}$ in a formal variable z such that for each $i \in I_0$, $f_i(z)$ is regular at 0 and ∞, $f_i(0)f_i(\infty) = 1$, and

$$\sum_{k=0}^{\infty} f^+_{i,\tilde{d}_i k} z^k = f_i(z) = \sum_{k=0}^{\infty} f^-_{i,-\tilde{d}_i k} z^{-k}$$

in the sense that the left- and right-hand sides are the Laurent expansions of $f_i(z)$ at 0 and ∞, respectively.

Let $\mathcal{R}$ be the set of all rational ℓ-weights. Then $\mathcal{R}$ forms an abelian group with the group operation $(\boldsymbol{f}, \boldsymbol{g}) \mapsto \boldsymbol{f}\boldsymbol{g}$ being given by componentwise multiplication of the corresponding tuples of rational functions. We will not always distinguish between a rational ℓ-weight $\boldsymbol{f}$ and the corresponding tuple $(f_i(z))_{i\in I_0}$ of rational functions.

Recall from [Chari and Pressley 1991; 1998] that simple finite-dimensional modules of U are highest ℓ-weight modules, and their highest ℓ-weights $\boldsymbol{f}$ are parametrized by the tuples of the *Drinfeld polynomials*. More precisely, there exists a tuple of polynomials $(P_i(z))_{i\in I_0}$ with all $P_i(z)$ having constant coefficient 1 such that $\boldsymbol{f}$ satisfies that for $i \in I_0$,

$$f_i(z) = \begin{cases} q_n^{2\deg P_n}(P_n(q_n^{-4}z)/P_n(z)) & \text{if } (X_N^{(r)}, i) = (A_{2n}^{(2)}, n), \\ q_i^{\deg P_i}(P_i(q_i^{-2}z)/P_i(z)) & \text{otherwise.} \end{cases}$$

Therefore, the highest ℓ-weight of any simple finite-dimensional module is rational.

In general, we have the following theorem.

Theorem 3.2. *Let V be an irreducible highest ℓ-weight module. Then all weight spaces of V are finite-dimensional if and only if its highest ℓ-weight $\boldsymbol{f}$ belongs to $\mathcal{R}$.*

Proof. For the nontwisted cases, one can refer to [Mukhin and Young 2014, Theorem 3.7]. The proof of the twisted cases is essentially parallel to that of the untwisted cases thanks to the triangular decomposition (2-7) of the Drinfeld realization. □

4. Shiftability conditions and algebra homomorphisms

In this section, the notion of the shiftability condition with respect to a generalized Cartan matrix will be introduced, and the compatible structures of the quantum affine algebras with the n-fold q-oscillator algebras are given from the q-shiftability condition.

4A. ***Shiftability conditions.*** Let $A=(a_{ij})_{i,j\in I}$ be any symmetrizable generalized Cartan matrix. Let $\mathcal{A}$ be the Laurent polynomial ring over $\mathbb{C}$ in the variables x_i, $i\in I$, i.e., $\mathcal{A}=\mathbb{C}[x_i^{\pm 1}, i\in I]$. For each $i\in I$, consider the algebra automorphism $\zeta_i:\mathcal{A}\to\mathcal{A}$ given by $\zeta_i(x_j)=q_i^{-\delta_{ij}}x_j$ for $j\in I$. For any distinct $i,j\in I$, we say a pair of Laurent polynomials (f,g) in $\mathcal{A}$ is (i,j)-*shiftable* if f, g satisfy

$$fg=\zeta_j^{-1}(f)\zeta_i^{-1}(g).$$

Set $\{x\}_i:=(x-x^{-1})/(q_i-q_i^{-1})$ for any unit x in $\mathcal{A}$, and for simplicity, write $\{x\}=(x-x^{-1})/(q-q^{-1})$. Define the elements $y_i, y_i^{-1}\in\mathcal{A}$ by

$$y_i^{\pm 1}=\prod_{j\in I}x_j^{\pm a_{ji}}.$$

Consider the following system of equations with respect to the variables ϕ_i, $i\in I$, in $\mathcal{A}$:

$$\begin{cases}\zeta_i(\phi_i)-\phi_i=\{y_i\}_i,\\ \phi_i\phi_j=\zeta_j^{-1}(\phi_i)\zeta_i^{-1}(\phi_j),\end{cases}\qquad i,j\in I,\ i\neq j. \tag{4-1}$$

In general, this system of equations does not always have a solution. It depends on the choice of the generalized Cartan matrix A. Therefore, we can say A admits the q-*shiftability condition* when the corresponding system (4-1) has a solution.

By a quick computation, we obtain a family of solutions to (4-1) for A of types A_2 and $A_1^{(1)}$.

Example 4.1. (i) For the type A_2, a pair of Laurent polynomials (ϕ_1,ϕ_2) where $\phi_1=\{qbx_1\}\{bx_1^{-1}x_2\}$ and $\phi_2=\{qbx_1^{-1}x_2\}\{bx_2^{-1}\}$ for each scalar $b\in\mathbb{C}^\times$ is a solution.

(ii) For the type $A_1^{(1)}$, consider the Laurent polynomials $\phi_0=\{qbx_0x_1^{-1}\}\{bx_0^{-1}x_1\}$ and $\phi_1=\{qbx_0^{-1}x_1\}\{bx_0x_1^{-1}\}$ for any scalar $b\in\mathbb{C}^\times$. It is easy to check that (ϕ_0,ϕ_1) is a solution.

In what follows, the q-shiftability condition for the generalized Cartan matrices of affine types will be investigated. Now assume that A is an affine Cartan matrix as in Section 2. Then we have the first main result in this section.

Theorem 4.2. *There exists an $(n+1)$-tuple of Laurent polynomials in $\mathcal{A}$ satisfying the system (4-1) if and only if A is of the type $A_n^{(1)}$ $(n\geq 1)$, $C_n^{(1)}$ $(n\geq 2)$, $A_{2n}^{(2)}$ $(n\geq 1)$ or $D_{n+1}^{(2)}$ $(n\geq 2)$.*

The proof of Theorem 4.2 will be given in the Appendix. Here we list all tuples of Laurent polynomials $(\phi_i)_{i\in I}$ satisfying (4-1) for each affine Cartan matrix A in the theorem above.

For the type $A_n^{(1)}$ $(n \geq 1)$:

$$\bigl(\{qb_A z_0\}\{b_A z_1\}, \{qb_A z_1\}\{b_A z_2\}, \dots, \{qb_A z_n\}\{b_A z_0\}\bigr).$$

For the type $C_n^{(1)}$ $(n \geq 2)$:

$$\bigl(\{q_0 b_C z_1^{-1}\}_0\{b_C z_1\}_0, \{q_1 b_C z_1\}_1\{b_C z_2\}_1, \dots, \{q_{n-1} b_C z_{n-1}\}_{n-1}\{b_C z_n\}_{n-1}, \{q_n b_C z_n\}_n\{b_C z_n^{-1}\}_n\bigr).$$

For the type $A_{2n}^{(2)}$ $(n \geq 1)$:

$$\left(\{\iota q^{-\frac{3}{2}} z_1\}_0\{\iota q^{-\frac{1}{2}} z_1\}_0, \{\iota q^{\frac{1}{2}} z_1\}_1\{\iota q^{-\frac{1}{2}} z_2\}_1, \dots, \{\iota q^{\frac{1}{2}} z_{n-1}\}_{n-1}\{\iota q^{-\frac{1}{2}} z_n\}_{n-1}, \frac{\iota}{q_n - q_n^{-1}}\{\iota q^{\frac{1}{2}} z_n\}_n\right).$$

For the type $D_{n+1}^{(2)}$ $(n \geq 2)$:

$$\left(\frac{\iota}{q_0 - q_0^{-1}}\{\iota q^{-1} z_1\}_0, \{\iota q z_1\}_1\{\iota q^{-1} z_2\}_1, \dots, \{\iota q z_{n-1}\}_{n-1}\{\iota q^{-1} z_n\}_{n-1}, \frac{\iota}{q_n - q_n^{-1}}\{\iota q z_n\}_n\right).$$

Here, $\iota = \sqrt{-1}$. The elements $z_i \in \mathcal{A}$ involved in the above solutions and the relations in our notation are given as follows for each type:

For $A_n^{(1)}$,

$$z_i = x_{i-1}^{-1} x_i, \quad z_0 = (z_1 \cdots z_n)^{-1}, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n z_0^{-1}, \quad b_{A_n^{(1)}} \in \mathbb{C}^\times.$$

For $C_n^{(1)}$,

$$z_i = x_{i-1}^{-1} x_i, \quad y_0 = z_1^{-2}, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n^2, \quad b_{C_n^{(1)}} = q^{-1/4} \text{ or } \iota q^{-1/4}.$$

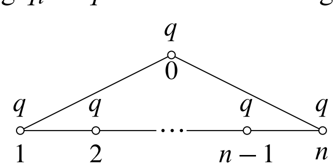
For $A_{2n}^{(2)}$,

$$z_i = x_{i-1}^{-1} x_i, \quad z_n = x_{n-1}^{-1} x_n^2, \quad y_0 = z_1^{-2}, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n, \quad b_{A_{2n}^{(2)}} = \iota q^{-\frac{1}{2}}.$$

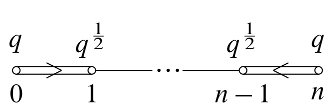
For $D_{n+1}^{(2)}$,

$$z_1 = x_0^{-2} x_1, \quad z_i = x_{i-1}^{-1} x_i, \quad z_n = x_{n-1}^{-1} x_n^2, \quad y_i = z_i z_{i+1}^{-1}, \quad y_n = z_n, \quad y_0 = z_1^{-1}, \quad b_{D_{n+1}^{(2)}} = \iota q^{-1}.$$

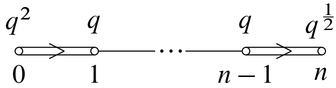
By our convention, the Dynkin diagrams of the above four types and the corresponding $q_i = q^{d_i}$ are the following:



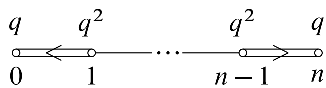
$(A_n^{(1)})$



$(C_n^{(1)})$



$(A_{2n}^{(2)})$



$(D_{n+1}^{(2)})$

Remark 4.3. One can also consider the shiftability condition for a generalized Cartan matrix A in the classical sense. More precisely, consider the polynomial ring $\mathcal{A}^+ = \mathbb{C}[x_i, i \in I]$, and the algebra automorphisms $\zeta_i : \mathcal{A}^+ \to \mathcal{A}^+$ defined by $\zeta_i(x_j) = x_j - \delta_{ij}$ for all $j \in I$. Write $y_i = \sum_{j \in I} a_{ij} x_j$. Then a similar system of equations in $\mathcal{A}^+$ (replace $\{y_i\}_i$ in (4-1) by y_i) can be obtained.

4B. ***Quantized oscillator algebra and algebra homomorphisms.*** One interesting application of the q-shiftability condition is to study the compatible structures of quantum affine algebras of types $X_N^{(r)}$ with the n-fold quantized oscillator algebra.

Fix $\nu \in \mathbb{C}^\times$. The (*symmetric*) *quantized oscillator algebra* $\mathscr{B}_\nu$ is the unital associative algebra over $\mathbb{C}$ generated by four elements $\boldsymbol{a}^+$, $\boldsymbol{a}$, $\boldsymbol{k}^{\pm 1}$ subject to the relations:

$$[\boldsymbol{a}, \boldsymbol{a}^+]_\nu = \boldsymbol{k}, \quad [\boldsymbol{a}, \boldsymbol{a}^+]_{\nu^{-1}} = \boldsymbol{k}^{-1}, \quad \boldsymbol{k}\boldsymbol{k}^{-1} = \boldsymbol{k}^{-1}\boldsymbol{k} = 1,$$
$$\boldsymbol{k}\boldsymbol{a}\boldsymbol{k}^{-1} = \nu^{-1}\boldsymbol{a}, \quad \boldsymbol{k}\boldsymbol{a}^+\boldsymbol{k}^{-1} = \nu\boldsymbol{a}^+,$$

where $[x, y]_\nu := xy - \nu^{-1} yx$. Then $\boldsymbol{a}^+\boldsymbol{a} = \{\boldsymbol{k}\}$, $\boldsymbol{a}\boldsymbol{a}^+ = \{\nu\boldsymbol{k}\}$ and $\{\boldsymbol{k}\}\boldsymbol{a}^+ = \boldsymbol{a}^+\{\nu\boldsymbol{k}\}$, $\boldsymbol{a}\{\boldsymbol{k}\} = \{\nu\boldsymbol{k}\}\boldsymbol{a}$ in $\mathscr{B}_\nu$. Here we define $\{\boldsymbol{x}\} = \{\boldsymbol{x}\}_\nu = (\boldsymbol{x} - \boldsymbol{x}^{-1})/(\nu - \nu^{-1})$ for $\boldsymbol{x} = \boldsymbol{k}$ or $\nu\boldsymbol{k}$.

One can easily check the following results.

Lemma 4.4. (i) *There exists a unique* $\mathbb{C}$*-algebra automorphism* (*an involution*) $\vartheta : \mathscr{B}_\nu \to \mathscr{B}_\nu$ *such that* $\vartheta(\boldsymbol{a}^+) = -\boldsymbol{a}$, $\vartheta(\boldsymbol{a}) = \boldsymbol{a}^+$ *and* $\vartheta(\boldsymbol{k}) = \nu^{-1}\boldsymbol{k}^{-1}$.

(ii) *For any* $b \in \mathbb{C}^\times$ *and* $m \in \mathbb{Z}$, *there exists a family of* $\mathbb{C}$*-algebra automorphisms* $\theta_{b,m} : \mathscr{B}_\nu \to \mathscr{B}_\nu$ *such that* $\theta_{b,m}(\boldsymbol{a}) = b\boldsymbol{k}^m\boldsymbol{a}$, $\theta_{b,m}(\boldsymbol{a}^+) = b^{-1}\boldsymbol{a}^+\boldsymbol{k}^{-m}$ *and* $\theta_{b,m}(\boldsymbol{k}^{\pm 1}) = \boldsymbol{k}^{\pm 1}$.

Consider the algebra $\mathscr{B}_\nu^{\otimes n}$ of the n-fold tensor product of $\mathscr{B}_\nu$. Denote the generators of its i-th component by $\boldsymbol{a}_i^+$, $\boldsymbol{a}_i$ and $\boldsymbol{k}_i^{\pm 1}$, which satisfy the above relations. Let $U_q(X_N^{(r)})$ be the quantum affine algebra U of the type $X_N^{(r)}$ in Theorem 4.2. For convenience, if X is of the type A then we shall deal with $A_{n-1}^{(1)} (n \geq 2)$ instead of $A_n^{(1)} (n \geq 1)$ from now on.

Fix a solution $(\phi_i)_{i \in I}$ in Section 4A. We define the algebra homomorphism $\pi_{X_N^{(r)}} : U_q(X_N^{(r)}) \to \mathscr{B}_\nu^{\otimes n}$ as follows: regard ϕ_i and $\zeta_i(\phi_i)$ as the images of $X_i^- X_i^+$ and $X_i^+ X_i^-$ respectively under $\pi_{X_N^{(r)}}$ by setting $\boldsymbol{k}_i = q^{-1} b_A^{-1} z_i^{-1}$ for the type $A_{n-1}^{(1)}$, and $\boldsymbol{k}_i = \imath\nu^{-1/2} z_i^{-1}$ otherwise (here we consider the solution with $b_C = \imath q^{-1/4}$ for the type $C_n^{(1)}$), where ν is defined as in the following proposition for each type. Then the relations (2-3) for $i=j$ hold under $\pi_{X_N^{(r)}}$ since ϕ_i satisfies $\zeta_i(\phi_i)-\phi_i=\{y_i\}_i$. In this sense, $\mathcal{A}_0:=\mathbb{C}[\boldsymbol{k}_1^{\pm1},\dots,\boldsymbol{k}_n^{\pm1}]$ is a subalgebra of $\mathcal{A}$, and $\pi_{X_N^{(r)}}(K_i)=y_i$ for $i\in I$. On the other hand, we choose $\pi_{X_N^{(r)}}(X_i^{\pm})\in\mathscr{B}_\nu^{\otimes n}$ satisfying

$$\pi_{X_N^{(r)}}(X_i^{\pm})f=\zeta_i^{\pm1}(f)\pi_{X_N^{(r)}}(X_i^{\pm})$$

for any $f\in\mathcal{A}_0$. The above choice yields that the relations (2-1)–(2-4) hold. Then we get the following algebra homomorphisms, which were obtained in [Hayashi 1990; Kuniba et al. 2015].

Proposition 4.5 [Hayashi 1990; Kuniba et al. 2015]. *For a parameter z, there exist algebra homomorphisms from $U_q(X_N^{(r)})$ to $\mathscr{B}_\nu^{\otimes n}[z,z^{-1}]$ defined as follows.*

For the type $A_{n-1}^{(1)}$ and $\nu=q$:

$$\pi_{A_{n-1}^{(1)},z}:U_q(A_{n-1}^{(1)})\to\mathscr{B}_\nu^{\otimes n}[z,z^{-1}],$$

$$X_i^+\mapsto z^{\delta_{i,0}}\boldsymbol{a}_i\boldsymbol{a}_{i+1}^+,\quad X_i^-\mapsto z^{-\delta_{i,0}}\boldsymbol{a}_i^+\boldsymbol{a}_{i+1},\quad K_i\mapsto\boldsymbol{k}_i^{-1}\boldsymbol{k}_{i+1}.$$

For this type, we always read the index i as i modulo n.

For the type $C_n^{(1)}$ and $\nu=q^{\frac12}$:

$$\pi_{C_n^{(1)},z}:U_q(C_n^{(1)})\to\mathscr{B}_\nu^{\otimes n}[z,z^{-1}],$$

$$\begin{aligned}
&X_0^+\mapsto z(\boldsymbol{a}_1^+)^2/[2]_\nu, && X_0^-\mapsto z^{-1}\boldsymbol{a}_1^2/[2]_\nu, && K_0\mapsto-\nu\boldsymbol{k}_1^2,\\
&X_i^+\mapsto\boldsymbol{a}_i\boldsymbol{a}_{i+1}^+, && X_i^-\mapsto\boldsymbol{a}_i^+\boldsymbol{a}_{i+1}, && K_i\mapsto\boldsymbol{k}_i^{-1}\boldsymbol{k}_{i+1},\\
&X_n^+\mapsto\boldsymbol{a}_n^2/[2]_\nu, && X_n^-\mapsto(\boldsymbol{a}_n^+)^2/[2]_\nu, && K_n\mapsto-\nu^{-1}\boldsymbol{k}_n^{-2}.
\end{aligned}$$

For the type $A_{2n}^{(2)}$ and $\nu=q$:

$$\pi_{A_{2n}^{(2)},z}:U_q(A_{2n}^{(2)})\to\mathscr{B}_\nu^{\otimes n}[z,z^{-1}],$$

$$\begin{aligned}
&X_0^+\mapsto z(\boldsymbol{a}_1^+)^2/[2]_\nu, && X_0^-\mapsto z^{-1}\boldsymbol{a}_1^2/[2]_\nu, && K_0\mapsto-\nu\boldsymbol{k}_1^2,\\
&X_i^+\mapsto\boldsymbol{a}_i\boldsymbol{a}_{i+1}^+, && X_i^-\mapsto\boldsymbol{a}_i^+\boldsymbol{a}_{i+1}, && K_i\mapsto\boldsymbol{k}_i^{-1}\boldsymbol{k}_{i+1},\\
&X_n^+\mapsto\imath\tau_\nu\boldsymbol{a}_n, && X_n^-\mapsto\boldsymbol{a}_n^+, && K_n\mapsto\imath\nu^{-\frac12}\boldsymbol{k}_n^{-1},
\end{aligned}$$

For the type $D_{n+1}^{(2)}$ and $\nu=q^2$:

$$\pi_{D_{n+1}^{(2)},z}:U_q(D_{n+1}^{(2)})\to\mathscr{B}_\nu^{\otimes n}[z,z^{-1}],$$

$$\begin{aligned}
&X_0^+\mapsto z\boldsymbol{a}_1^+, && X_0^-\mapsto\imath\tau_\nu z^{-1}\boldsymbol{a}_1, && K_0\mapsto-\imath\nu^{\frac12}\boldsymbol{k}_1,\\
&X_i^+\mapsto\boldsymbol{a}_i\boldsymbol{a}_{i+1}^+, && X_i^-\mapsto\boldsymbol{a}_i^+\boldsymbol{a}_{i+1}, && K_i\mapsto\boldsymbol{k}_i^{-1}\boldsymbol{k}_{i+1},\\
&X_n^+\mapsto\imath\tau_\nu\boldsymbol{a}_n, && X_n^-\mapsto\boldsymbol{a}_n^+, && K_n\mapsto\imath\nu^{-\frac12}\boldsymbol{k}_n^{-1}.
\end{aligned}$$

Here, $\tau_\nu=(\nu+1)/(\nu-1)$. □

5. Multiplicity-free weight modules

In this section, we construct the multiplicity-free weight representations over U from the solutions and the algebra homomorphisms in the previous section. Throughout, we assume that U is the quantum affine algebra of type $X_N^{(r)}$ in Proposition 4.5.

5A. *Module structures on $\mathcal{A}_0$.* In order to construct the multiplicity-free weight representations, we first consider the auxiliary U-module structures on $\mathcal{A}_0 = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_n^{\pm 1}]$.

Let us fix some notation here. Note that α_0 and α_n are long roots in the type $C_n^{(1)}$, while both of them are short roots in $D_{n+1}^{(2)}$. In addition, by our assumption, α_0 is long and α_n is short in $A_{2n}^{(2)}$. We define a pair $\kappa := (\kappa_1, \kappa_2)$ of signs such that κ_1, κ_2 are equal to 0 or 1, depending on the length of the roots α_0 and α_n for each type, that is,

$$(\kappa_1, \kappa_2) = (1, 1) \quad \text{for } C_n^{(1)},$$
$$(\kappa_1, \kappa_2) = (1, 0) \quad \text{for } A_{2n}^{(2)},$$
$$(\kappa_1, \kappa_2) = (0, 0) \quad \text{for } D_{n+1}^{(2)}.$$

Fix a solution $(\phi_i)_{i \in I}$ of (4-1), and recall the units z_i for each type, and the shift operators ζ_i defined in Section 4A. Put $b = b_{X_N^{(r)}}$. Then we have:

Theorem 5.1. *Let z be a parameter valued in $\mathbb{C}^\times$. For an n-tuple $\boldsymbol{f} = (f_i)_{1 \le i \le n}$ such that f_i is equal either to 1 or to $bz_i - b^{-1}z_i^{-1}$ for $1 \le i \le n$, there exists a U-module structure on the algebra $\mathcal{A}_0$ for each type defined in the following*:

For the type $A_{n-1}^{(1)}$,

$$X_i^+.u = z^{\delta_{i,0}} f_i \zeta_i\left(\frac{\{bz_{i+1}\}}{f_{i+1}}\right)\zeta_i(u), \quad X_i^-.u = z^{-\delta_{i,0}} \zeta_i^{-1}\left(\frac{\{bz_i\}}{f_i}\right) f_{i+1}\zeta_i^{-1}(u),$$

and $K_i^{\pm 1}.u = y_i^{\pm 1}u$, for any $u \in \mathcal{A}_0$.

For other types,

$$X_0^+.u = z\frac{\zeta_0(\phi_0)\zeta_0(u)}{\zeta_1^{-1}(f_1)^{\kappa_1}\zeta_0(f_1)}, \qquad X_0^-.u = z^{-1} f_1\zeta_1(f_1)^{\kappa_1}\zeta_0^{-1}(u),$$
$$X_i^+.u = f_i\zeta_i\left(\frac{\{bz_{i+1}\}_i}{f_{i+1}}\right)\zeta_i(u), \quad X_i^-.u = \zeta_i^{-1}\left(\frac{\{bz_i\}_i}{f_i}\right) f_{i+1}\zeta_i^{-1}(u),$$
$$X_n^+.u = f_n\zeta_{n-1}^{-1}(f_n)^{\kappa_2}\zeta_n(u), \qquad X_n^-.u = \frac{\phi_n\zeta_n^{-1}(u)}{\zeta_n^{-1}(f_n)\zeta_{n-1}(f_n)^{\kappa_2}},$$

and $K_i^{\pm 1}.u = y_i^{\pm 1}u$, for any $u \in \mathcal{A}_0$.

Proof. Taking $u = 1$ in the above construction we have the precise expressions of the actions $X_i^{\pm}.1$. In addition, for any $u \in \mathcal{A}_0$ we have $X_i^{\pm}.u = \zeta_i^{\pm 1}(u)X_i^{\pm}.1$. We can check each defining relation directly. For the relations (2-2), we have

$$\begin{aligned}(K_i X_j^{\pm} K_i^{-1}).u &= y_i (X_j^{\pm}.(y_i^{-1} u))\\ &= y_i \zeta_j^{\pm 1}(y_i^{-1} u) X_j^{\pm}.1 = y_i \zeta_j^{\pm 1}(y_i^{-1}) X_j^{\pm}.u = q_i^{\pm a_{ij}} X_j^{\pm}.u.\end{aligned}$$

For the relations (2-3), we consider three cases:

(1) If $i = j$, then we have

$$(X_i^{+} X_i^{-} - X_i^{-} X_i^{+}).u = u\big(\zeta_i (X_i^{-}.1) X_i^{+}.1 - \zeta_i^{-1}(X_i^{+}.1) X_i^{-}.1\big) = u(\zeta_i(\phi_i) - \phi_i).$$

Here, $\phi_i := \zeta_i^{-1}(X_i^{+}.1) X_i^{-}.1,\ i \in I$, is just a solution to the system (4-1), by construction, which implies (2-3) for $i = j$, as desired.

(2) If $|i - j| > 1$, then we have $\zeta_i(f_k) = f_k$ and $\zeta_i(\{bz_k\}_k) = \{bz_k\}_k$ for $k = j, j+1$. Similarly, $\zeta_j(f_k) = f_k$ and $\zeta_j(\{bz_k\}_k) = \{bz_k\}_k$ for $k = i, i+1$. Therefore

$$X_i^{+} X_j^{-}.u = \zeta_i \zeta_j^{-1}(u) \zeta_i (X_j^{-}.1) X_i^{+}.1 = \zeta_i \zeta_j^{-1}(u)(X_j^{-}.1)(X_i^{+}.1) = X_j^{-} X_i^{+}.u.$$

(3) If $|i - j| = 1$, we need to do more detailed calculations for each type. First assume $a_{ij} a_{ji} = 1$. Then we have to show

$$\begin{aligned}&\zeta_i \zeta_j^{-1}\left(\frac{\{bz_j\}_j}{f_j}\right)\zeta_i(f_{j+1}) f_i \zeta_i\left(\frac{\{bz_{i+1}\}_i}{f_{i+1}}\right)\\ &\qquad = \zeta_j^{-1}\left(\frac{\{bz_j\}_j}{f_j}\right) f_{j+1} \zeta_j^{-1}(f_i) \zeta_i \zeta_j^{-1}\left(\frac{\{bz_{i+1}\}_i}{f_{i+1}}\right).\end{aligned}$$

If $j = i + 1$, then $\zeta_i(f_{j+1}) = f_{j+1},\ \zeta_j^{-1}(f_i) = f_i$, and $\zeta_i(f_j) = \zeta_j^{-1}(f_j)$, while for $i = j+1$, we have $\zeta_j^{-1}(f_{i+1}) = f_{i+1},\ \zeta_i(f_j) = f_j$, and $\zeta_i(f_i) = \zeta_j^{-1}(f_i)$. Both cases imply the above equality holds. When $X_N^{(r)} \neq A_{n-1}^{(1)}$, a direct computation yields the following equalities:

$$\begin{aligned}\{bz_1\}_1 \zeta_1 \phi_0 &= \phi_0 \zeta_0^{-1}\{bz_1\}_1, & f_1^{\kappa_1} \zeta_0^{\pm 1} \zeta_1^{\pm 1}(f_1) &= f_1 \zeta_1^{\mp 1}(f_1)^{\kappa_1},\\ \{bz_n\}_{n-1}\phi_n &= \zeta_n^{-1}(\{bz_n\}_{n-1})\zeta_{n-1}^{-1}(\phi_n), & f_n^{\kappa_2} \zeta_{n-1}^{\pm 1} \zeta_n^{\pm 1}(f_n) &= f_n \zeta_{n-1}^{\mp 1}(f_n)^{\kappa_2}.\end{aligned}$$

Then similar arguments for the case that $a_{ij} a_{ji} = 2$ are true. Any tuple $(\phi_i)_i$ satisfying (4-1) and the choice of $(f_i)_i$ also guarantee that these actions hold under the quantum Serre relations (2-4). □

Denote by $S_z(\boldsymbol{f})$ the U-module on $\mathcal{A}_0$ related to z and $\boldsymbol{f}$ given by Theorem 5.1. Recall $\pi_{X_N^{(r)}}(K_i) = y_i$ in the construction in Section 4B. Then

$$\pi_{X_N^{(r)}}(K_\delta) = \prod_{i \in I} y_i^{a_i} = \prod_{i \in I}\prod_{j \in I} x_j^{a_{ji} a_i} = \prod_{j \in I} x_j^{\sum_{i \in I} a_{ji} a_i} = 1. \tag{5-1}$$

In particular, K_δ acts trivially on $S_z(\boldsymbol{f})$. Therefore, $S_z(\boldsymbol{f})$ is finitely U^0-generated instead of U^0-diagonalizable when restricted as a U^0-module.

Now let us explain the "weighting" procedure mentioned in the introduction. That is, to a U-module $S_z(\boldsymbol{f})$, we associate a weight module $\mathcal{W}(S_z(\boldsymbol{f}))$ in the following way. Consider the algebra homomorphism from U^0 to $\mathcal{A}_0$ that assigns K_i to y_i for $i \in I_0$, which induces a natural group homomorphism from the group of characters of $\mathcal{A}_0$ to $\mathscr{X}$. For any character φ of $\mathcal{A}_0$, denote by $\mathfrak{m}_\varphi$ ($:= \ker\varphi$) the corresponding maximal ideal of $\mathcal{A}_0$. Extending $\alpha_j \in \mathscr{X}$ to a character of $\mathcal{A}_0$ by setting $\alpha_j(y_i) = q_j^{a_{ji}}$ (still denoting it by α_j), we have

$$K_i.\mathfrak{m}_\varphi S_z(\boldsymbol{f}) \subset \mathfrak{m}_\varphi S_z(\boldsymbol{f}), \quad X_i^{\pm}.\mathfrak{m}_\varphi S_z(\boldsymbol{f}) \subset \mathfrak{m}_{\varphi\pm\alpha_i} S_z(\boldsymbol{f}).$$

Define

$$\mathcal{W}(S_z(\boldsymbol{f})) = \bigoplus_{\varphi} S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f}),$$

where φ is taken over all characters of $\mathcal{A}_0$.

Corollary 5.2. *For any U-module $S_z(\boldsymbol{f})$, we have that $\mathcal{W}(S_z(\boldsymbol{f}))$ is a weight module, and all its simple subquotients are multiplicity-free.*

Proof. It is clear that $S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f})$ is 1-dimensional and K_i acts diagonally. In particular, K_δ acts by 1. The first assertion follows from the previous statements, and the λ-weight space $\mathcal{W}(S_z(\boldsymbol{f}))_\lambda = \bigoplus_{\overline{\varphi}=\lambda} S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f})$, where $\overline{\varphi}$ means the image of φ in $\mathscr{X}$. Since we have

$$X_i^{\pm}.S_z(\boldsymbol{f})/\mathfrak{m}_\varphi S_z(\boldsymbol{f}) \subset S_z(\boldsymbol{f})/\mathfrak{m}_{\varphi\pm\alpha_i} S_z(\boldsymbol{f}),$$

the second assertion follows. ☐

Remark 5.3. In fact, the U-module $\mathcal{W}(S_z(\boldsymbol{f}))$ is a q-analog of the *coherent family* in the sense of [Nilsson 2016]. This "weighting" procedure was first suggested by O. Mathieu [2000].

Now let us study the possible highest weights of $\mathcal{W}(S_z(\boldsymbol{f}))$ when restricted as a $U_q(\mathring{\mathfrak{g}})$-module. Assume that the weight vector $1 + \mathfrak{m}_\varphi S_z(\boldsymbol{f})$ of $\mathcal{W}(S_z(\boldsymbol{f}))$ is a highest weight vector for some φ. Then we have $X_i^{+}.(1 + \mathfrak{m}_\varphi S_z(\boldsymbol{f})) = 0$ for $i \in I_0$, which implies that

$$(\varphi + \alpha_i)(\zeta_i(\phi_i)) = 0 \quad \text{for } i \in I_0. \tag{5-2}$$

The weight $\lambda = \overline{\varphi}$ is level-zero and is determined uniquely by the values $\lambda(K_i)$, $i \in I_0$. Therefore, all level-zero weights can be seen automatically as weights over $U_q(\mathring{\mathfrak{g}})$. As a result, we can obtain the following result.

Corollary 5.4. *Let $\lambda \in \mathscr{X}$ be a weight of $\mathcal{W}(S_z(\boldsymbol{f}))$ for some $\boldsymbol{f}$. If λ is a highest $U_q(\mathring{\mathfrak{g}})$-weight, then up to twistings by the automorphisms of $U_q(\mathring{\mathfrak{g}})$, we have*

(1) *for the type $A_{n-1}^{(1)}$, the weight λ is of the form $\omega_0 + a\omega_s - (a+1)\omega_{s+1}$ for some $a \in \mathbb{C}$ and $s \in I$ up to a constant multiple of δ;*

(2) *for the type* $C_n^{(1)}$, *the weight* λ *has the form* $\frac{1}{2}\omega_0+\omega_{n-1}-\frac{3}{2}\omega_n$ *or* $\frac{1}{2}(\omega_0-\omega_n)$, *up to a constant multiple of* δ;

(3) *for the type* $A_{2n}^{(2)}$ (*resp.* $D_{n+1}^{(2)}$), *the weight* $\lambda=(\lambda(K_0),\dots,\lambda(K_n))$ *is defined as*

$$(-q,1,\dots,1,\imath q^{-1/2})\quad (\textit{resp. } (-\imath q,1,\dots,1,\imath q^{-1})).$$

Proof. The result can be deduced directly from (5-2). For example, in the type $A_{n-1}^{(1)}$, the equations (5-2) become

$$\begin{cases} m_0m_1\cdots m_{n-1}=1, \\ \{qbm_i\}\{bm_{i+1}\}=0, \quad 1\le i\le n-1, \end{cases} \tag{5-3}$$

where we denote $\varphi(z_i)$ by m_i for $i\in I$. Let $\lambda=\overline{\varphi}$. Then $\lambda(K_i)=m_im_{i+1}^{-1}$. To solve the equations (5-3), we consider two cases: if $\{qbm_1\}$ is not zero, then $\{bm_i\}=0$ for $i\ge 2$; if $\{qbm_1\}$ is zero, then we assume that s is the maximal index such that $\{qbm_s\}$ is zero, and then $\{qbm_i\}=0$ for $i\le s$ and $\{bm_j\}=0$ for $j\ge s+2$. In the first case, the possible solutions are $m_1=\pm b^{n-1}$ and $m_i=\pm b^{-1}$ for $i\ne 1$. Then, up to twistings by sign automorphisms of $U_q(\mathring{\mathfrak{g}})$, the weight λ is given by

$$\lambda(K_0)=b^{-n},\qquad \lambda(K_1)=b^n,\qquad \lambda(K_i)=1\quad\text{for } i\ge 2,$$

which is of the form $(a+1)\omega_0-(a+1)\omega_1$ for some $a\in\mathbb{C}$. In the second case, $m_i=\pm q^{-1}b^{-1}$ for $0\le i\le s$, $m_{s+1}=\pm q^{s+1}b^{n-1}$, and $m_j=\pm b^{-1}$ otherwise. Then, up to signs, the weight λ is the following:

$$\lambda(K_s)=q^{-s-2}b^{-n},\qquad \lambda(K_{s+1})=q^{s+1}b^n,\qquad \lambda(K_i)=1\quad\text{for } i\ne s,s+1,$$

which is exactly of the form in (1). So assertion (1) follows. □

All simple subquotients obtained in Corollary 5.4 can be realized as q-oscillator representations by using the Fock space representations of $\mathscr{B}_\nu$ and the algebra homomorphisms in Proposition 4.5 (see, e.g., [Kuniba 2018]). In the following subsection, we shall recall the q-oscillator representations.

5B. ***Realization of multiplicity-free weight modules.*** Let $F=\bigoplus_{m\in\mathbb{Z}_{\ge 0}}\mathbb{C}|m\rangle$ be the Fock space representation of $\mathscr{B}_\nu$ on which the generators $\boldsymbol{a}^+$ and $\boldsymbol{a}$ act as the creation and annihilation operators, respectively, and the element $\boldsymbol{a}^+\boldsymbol{a}=\{\boldsymbol{k}\}_\nu$ corresponds to the number operator; more precisely, for any $m\in\mathbb{Z}_{\ge 0}$,

$$\boldsymbol{a}^+.|m\rangle=|m+1\rangle,\quad \boldsymbol{a}.|m\rangle=[m]_\nu|m-1\rangle,\quad \boldsymbol{k}^{\pm 1}.|m\rangle=\nu^{\pm m}|m\rangle.$$

In particular, $\boldsymbol{a}.|0\rangle=0$.

Denote this representation by $\rho : \mathscr{B}_\nu \to \mathrm{End}_{\mathbb{C}}(F)$. For any $b \in \mathbb{C}^\times$ and $\varepsilon \in \{0, 1\}$, we denote by $\rho_{\varepsilon,b}$ the composition $\rho \circ \vartheta^\varepsilon \circ \theta_{b,0}$ (see Lemma 4.4). Then F has a new $\mathscr{B}_\nu$-module structure via $\rho_{\varepsilon,b}$.

Definition 5.5. Let z be a parameter valued in $\mathbb{C}^\times$. We define the representation $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ of U on the space $F^{\otimes n}$ via the composition of the algebra homomorphisms $\pi_z := \pi_{X_N^{(r)},z}$ defined in Proposition 4.5 and

$$\mathscr{B}_\nu^{\otimes n}[z, z^{-1}] \xrightarrow{\rho_{\varepsilon_1,b_1}\otimes\cdots\otimes\rho_{\varepsilon_n,b_n}} \mathrm{End}_{\mathbb{C}}(F^{\otimes n}),$$

where $\boldsymbol{\varepsilon} = (\varepsilon_i)_i \in \{0, 1\}^n$, and $\boldsymbol{b} = (b_i)_i \in (\mathbb{C}^\times)^n$.

For any n-tuple $(m_i)_i \in (\mathbb{Z}_{\geq 0})^n$, we use $|\boldsymbol{m}\rangle := |m_1\rangle \otimes \cdots \otimes |m_n\rangle$ for the basis vector of $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$. Let $\boldsymbol{e}_j$ be the j-th standard vector in $\mathbb{Z}^n$ with 1 at the j-th term and 0 otherwise for $1 \leq j \leq n$. Moreover, set $\boldsymbol{0}$ for $(0, \ldots, 0) \in \mathbb{Z}^n$.

For $n \geq 2$, note that U-module actions of $X_i^{\pm 1}$, K_i for $1 \leq i \leq n-1$ on $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$, by Definition 5.5, can be written down uniformly as follows:

$$\begin{aligned}(5\text{-}4)\quad & X_i^+.|\boldsymbol{m}\rangle \\ &\quad = (-1)^{\varepsilon_i} b_i b_{i+1}^{-1} [m_i/m_i^{\varepsilon_i}]_\nu [m_{i+1}^{\varepsilon_{i+1}}]_\nu |\boldsymbol{m} - (-1)^{\varepsilon_i}\boldsymbol{e}_i + (-1)^{\varepsilon_{i+1}}\boldsymbol{e}_{i+1}\rangle,\end{aligned}$$

$$\begin{aligned}(5\text{-}5)\quad & X_i^-.|\boldsymbol{m}\rangle \\ &\quad = (-1)^{\varepsilon_{i+1}} b_i^{-1} b_{i+1} [m_i^{\varepsilon_i}]_\nu [m_{i+1}/m_{i+1}^{\varepsilon_{i+1}}]_\nu |\boldsymbol{m} + (-1)^{\varepsilon_i}\boldsymbol{e}_i - (-1)^{\varepsilon_{i+1}}\boldsymbol{e}_{i+1}\rangle,\end{aligned}$$

$$K_i.|\boldsymbol{m}\rangle = \nu^{-(-1)^{\varepsilon_i}(m_i+\varepsilon_i)+(-1)^{\varepsilon_{i+1}}(m_{i+1}+\varepsilon_{i+1})}|\boldsymbol{m}\rangle, \tag{5-6}$$

for $\boldsymbol{m} \in (\mathbb{Z}_{\geq 0})^n$, where ν is defined in Proposition 4.5 for each type, and $|\boldsymbol{m}\rangle$ for $\boldsymbol{m} \notin (\mathbb{Z}_{\geq 0})^n$ can be read as 0. Here we remark that the $U_q(A_{n-1}^{(1)})$-module actions of $X_0^{\pm}$, K_0 on $|\boldsymbol{m}\rangle$ also have the above forms, where we understand the indices $i, i+1$ as $n, 1 \pmod n$, respectively.

Regard $U_q(A_{n-1})$ as the subalgebra of $U_q(\mathfrak{g})$ $(n \geq 2)$ via forgetting the actions of the Drinfeld–Jimbo generators indexed by 0 and n. One can check that $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ as a $U_q(A_{n-1})$-module is a multiplicity-free weight module. In fact, $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ has the following direct sum decomposition:

$$\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{l=-\infty}^{\infty} \mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}, \quad \mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}}=l} \mathbb{C}|\boldsymbol{m}\rangle.$$

For each $\boldsymbol{m} \in \mathbb{Z}^n$, write $|\boldsymbol{m}|_{\boldsymbol{\varepsilon}} = \sum_i (-1)^{\varepsilon_i} m_i$. Each $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is an irreducible, multiplicity-free weight $U_q(A_{n-1})$-module by formulae (5-4)–(5-6) as q (or ν) is not a root of unity.

Fix $1 \leq i \leq n$. Define the algebra homomorphism $\delta_i : \mathcal{A}_0 \to \mathbb{C}$ by $\boldsymbol{k}_j \mapsto \nu^{\delta_{ij}}$ for $1 \leq j \leq n$. It induces an algebra character $\widetilde{\delta}_i \in \mathscr{X}$ by

$$\widetilde{\delta}_i : U^0 \xrightarrow{\pi|_{U^0}} \mathcal{A}_0 \xrightarrow{\delta_i} \mathbb{C}.$$

Then we have:

Proposition 5.6. *For $\boldsymbol{\varepsilon} \in \{0,1\}^n$ and $\boldsymbol{b} \in (\mathbb{C}^\times)^n$, we have*:

(i) *$\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is a weight module with $\dim(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda \le 1$ for any $\lambda \in \mathscr{X}$.*

(ii) *If $\dim(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda = 1$, then there exists $\boldsymbol{m} \in (\mathbb{Z}_{\ge 0})^n$ such that $(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda = \mathbb{C}|\boldsymbol{m}\rangle$ with*

$$\lambda = \sum_{i=1}^{n} (-1)^{\varepsilon_i} (m_i + \varepsilon_i)\widetilde{\delta}_i. \tag{5-7}$$

Proof. Clearly, $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is a weight module. By (5-6) and the actions of K_0 and K_n, which are defined for $X_N^{(r)} \ne A_{n-1}^{(1)}$ by

$$K_0.|\boldsymbol{m}\rangle = -\iota^{1-\kappa_1} \nu^{(-1)^{\varepsilon_1}(\kappa_1+1)(m_1+1/2)}|\boldsymbol{m}\rangle, \tag{5-8}$$

$$K_n.|\boldsymbol{m}\rangle = \iota^{1+\kappa_2} \nu^{-(-1)^{\varepsilon_n}(\kappa_2+1)(m_n+1/2)}|\boldsymbol{m}\rangle, \tag{5-9}$$

where κ_1 and κ_2 are defined in Section 5A, the relative weight of $|\boldsymbol{m}\rangle$ is given by the right-hand side of the equality (5-7). By the above statement, $\dim(\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}})_\lambda \le 1$ for any $\lambda \in \mathscr{X}$. $\square$

Consider the following decomposition of $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$:

$$\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \mathcal{F}^{z,+}_{\boldsymbol{\varepsilon},\boldsymbol{b}} \oplus \mathcal{F}^{z,-}_{\boldsymbol{\varepsilon},\boldsymbol{b}}, \quad \mathcal{F}^{z,+}_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}} \equiv 0 \ (\mathrm{mod}\ 2)} \mathbb{C}|\boldsymbol{m}\rangle, \quad \mathcal{F}^{z,-}_{\boldsymbol{\varepsilon},\boldsymbol{b}} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}} \equiv 1 \ (\mathrm{mod}\ 2)} \mathbb{C}|\boldsymbol{m}\rangle.$$

For $0 \le s \le n$, let $\boldsymbol{\varepsilon}_{>s} \in \{0,1\}^n$ satisfy

$$\varepsilon_1 = \cdots = \varepsilon_s = 0, \quad \varepsilon_{s+1} = \cdots = \varepsilon_n = 1.$$

For example, $\boldsymbol{\varepsilon}_{>0} = (1,\dots,1)$ and $\boldsymbol{\varepsilon}_{>n} = (0,\dots,0)$.

Then we have:

Proposition 5.7. *For any $\boldsymbol{\varepsilon} \in \{0,1\}^n$ and $\boldsymbol{b} \in (\mathbb{C}^\times)^n$, we have*:

(i) *As a $U_q(A_{n-1}^{(1)})$-module, $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is irreducible for any admissible $l \in \mathbb{Z}$ (defined in (5-10)); it is a highest ℓ-weight module with a highest ℓ-weight vector $v_{l,s}$ if and only if $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>s}$ for some $0 \le s \le n$, where*

$$v_{l,s} = \begin{cases} |l\boldsymbol{e}_s\rangle & \text{if } l \ge 0 \text{ and } 0 < s \le n, \\ |-l\boldsymbol{e}_{s+1}\rangle & \text{if } l < 0 \text{ and } 0 \le s < n. \end{cases} \tag{5-10}$$

(ii) *As $U_q(C_n^{(1)})$-modules, $\mathcal{F}^{z,+}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ and $\mathcal{F}^{z,-}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ are irreducible; they are highest ℓ-weight modules with highest ℓ-weight vector $v^+ = |\boldsymbol{0}\rangle$ and $v^- = |\boldsymbol{e}_n\rangle$ respectively whenever $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$.*

(iii) *As a $U_q(A_{2n}^{(2)})$ or $U_q(D_{n+1}^{(2)})$-module, $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is irreducible; it is a highest ℓ-weight module with a highest ℓ-weight vector $v = |\boldsymbol{0}\rangle$ whenever $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$.*

Proof. Note K_δ acts trivially on $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ by (5-1). The defining relations (2-5) imply that the actions of $h_{i,k}$, $1 \le i \le N$, $k \in \mathbb{Z} \setminus \{0\}$ on $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ commute pairwise. Hence $\mathcal{F}^z_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is an ℓ-weight U-module. It is clear that $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is closed under the action of $U_q(A^{(1)}_{n-1})$. The irreducibility of $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ can be checked by the actions (5-4)–(5-6). Note that for $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>0}$ (resp. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$), $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is finite-dimensional for $l < 0$ (resp. $l \ge 0$). As a $U_q(A_{n-1})$-module (ignore the actions of $X_0^\pm$, K_0), $\mathcal{F}^{z,(l)}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ is a highest weight module if and only if $\varepsilon_1 \le \cdots \le \varepsilon_n$, and the corresponding highest weight vector can be chosen as (5-10), which is also a highest ℓ-weight vector by weight consideration.

For $X_N^{(r)} \ne A^{(1)}_{n-1}$, the actions of X_0^+ and X_n^+ are given by

$$X_0^+.|\boldsymbol{m}\rangle = z\frac{b_1^{-\kappa_1-1}}{[\kappa_1+1]_\nu}\prod_{j=0}^{\kappa_1}[(m_1-j)^{\varepsilon_1}]_\nu|\boldsymbol{m}+(-1)^{\varepsilon_1}(\kappa_1+1)\boldsymbol{e}_1\rangle,$$

and $X_n^+.|\boldsymbol{m}\rangle = x|\boldsymbol{m}-(-1)^{\varepsilon_n}(\kappa_2+1)\boldsymbol{e}_n\rangle$, where $a \in \mathbb{C}^\times$ is defined by

$$x = (-1)^{\varepsilon_n(1-\kappa_2)}b_n^{\kappa_2+1}\frac{\iota^{1-\kappa_2}}{[\kappa_2+1]_\nu}\tau_{\nu,\kappa_2}\prod_{j=0}^{\kappa_2}[(m_n-j)/(m_n-j)^{\varepsilon_n}]_\nu,$$

and $\tau_{\nu,\kappa_2} = (\nu-\kappa_2+1)/(\nu+\kappa_2-1)$. Similarly, we can obtain the actions of X_0^- and X_n^-. Assertions (ii) and (iii) can be deduced directly from the above actions. $\square$

6. Highest $\boldsymbol{\ell}$-weights

In this section, we focus on the irreducible highest ℓ-weight representations constructed in the previous section, and compute their highest ℓ-weights explicitly.

6A. ***Multiplicity-free highest $\boldsymbol{\ell}$-weight modules.*** Fix $0 < s < n$. Let $\mathcal{W}_s := F^{\otimes n}$ be the $U_q(A^{(1)}_{n-1})$-module defined as follows (see also [Kwon and Lee 2023]):

$$\begin{aligned}
X_0^+.|\boldsymbol{m}\rangle &= |\boldsymbol{m}+\boldsymbol{e}_1+\boldsymbol{e}_n\rangle, & X_0^-.|\boldsymbol{m}\rangle &= -[m_1][m_n]|\boldsymbol{m}-\boldsymbol{e}_1-\boldsymbol{e}_n\rangle,\\
X_s^+.|\boldsymbol{m}\rangle &= -[m_s][m_{s+1}]|\boldsymbol{m}-\boldsymbol{e}_s-\boldsymbol{e}_{s+1}\rangle, & X_s^-.|\boldsymbol{m}\rangle &= |\boldsymbol{m}+\boldsymbol{e}_s+\boldsymbol{e}_{s+1}\rangle,\\
X_i^+.|\boldsymbol{m}\rangle &= [m_i]|\boldsymbol{m}-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}\rangle, & X_i^-.|\boldsymbol{m}\rangle &= [m_{i+1}]|\boldsymbol{m}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}\rangle,\\
X_j^+.|\boldsymbol{m}\rangle &= [m_{j+1}]|\boldsymbol{m}+\boldsymbol{e}_j-\boldsymbol{e}_{j+1}\rangle, & X_j^-.|\boldsymbol{m}\rangle &= [m_j]|\boldsymbol{m}-\boldsymbol{e}_j+\boldsymbol{e}_{j+1}\rangle,
\end{aligned}$$

and

$$\begin{aligned}
K_0.|\boldsymbol{m}\rangle &= q^{m_1+m_n+1}|\boldsymbol{m}\rangle, & K_s.|\boldsymbol{m}\rangle &= q^{-m_s-m_{s+1}-1}|\boldsymbol{m}\rangle,\\
K_i.|\boldsymbol{m}\rangle &= q^{m_{i+1}-m_i}|\boldsymbol{m}\rangle, & K_j.|\boldsymbol{m}\rangle &= q^{m_j-m_{j+1}}|\boldsymbol{m}\rangle,
\end{aligned}$$

where $1 \le i < s < j \le n-1$ and $\boldsymbol{m} \in (\mathbb{Z}_{\ge 0})^n$. From the actions (5-4)–(5-6), $\mathcal{W}_s$ is just the twisting of the module $\mathcal{F}^1_{\boldsymbol{\varepsilon},\boldsymbol{b}}$, where $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>s}$ and $\boldsymbol{b} = (1,\ldots,1)$, by the automorphism of $U_q(A_{n-1}^{(1)})$ sending $X_k^{\pm}$ to $-X_k^{\pm}$ for $s \le k \le n$ along with other Drinfeld–Jimbo generators fixed. Let $\mathcal{W}_s^{(l)}$ denote the l-th irreducible component of $\mathcal{W}_s$, i.e., $\mathcal{W}_s^{(l)} = \bigoplus_{|\boldsymbol{m}|_{\boldsymbol{\varepsilon}}=l} \mathbb{C}|\boldsymbol{m}\rangle$.

Let $(X_N^{(r)}, \nu)$ be one of the types in Proposition 4.5 except $(A_{n-1}^{(1)}, q)$. Let $\mathcal{W} = F^{\otimes n}$ be the $U_q(X_N^{(r)})$-module defined as (see [Kuniba and Okado 2015]):

$$\begin{aligned}
X_0^+.|\boldsymbol{m}\rangle &= \frac{1}{[\kappa_1+1]_\nu}|\boldsymbol{m}+(\kappa_1+1)\boldsymbol{e}_1\rangle,\\
X_0^-.|\boldsymbol{m}\rangle &= -(-1)^{|\kappa|}\frac{\iota^{1-\kappa_1}}{[\kappa_1+1]_\nu}\tau_{\nu,\kappa_1}\prod_{j=0}^{\kappa_1}[m_1-j]_\nu|\boldsymbol{m}-(\kappa_1+1)\boldsymbol{e}_1\rangle,\\
K_0.|\boldsymbol{m}\rangle &= (-1)^{|\kappa|}\iota^{1-\kappa_1}\nu^{(\kappa_1+1)(m_1+1/2)}|\boldsymbol{m}\rangle,\\
X_i^+.|\boldsymbol{m}\rangle &= [m_i]_\nu|\boldsymbol{m}-\boldsymbol{e}_i+\boldsymbol{e}_{i+1}\rangle && (1\le i\le n-1),\\
X_i^-.|\boldsymbol{m}\rangle &= [m_{i+1}]_\nu|\boldsymbol{m}+\boldsymbol{e}_i-\boldsymbol{e}_{i+1}\rangle && (1\le i\le n-1),\\
K_i.|\boldsymbol{m}\rangle &= \nu^{-m_i+m_{i+1}}|\boldsymbol{m}\rangle && (1\le i\le n-1),\\
X_n^+.|\boldsymbol{m}\rangle &= \frac{\iota^{1+\kappa_2}}{[\kappa_2+1]_\nu}\tau_{\nu,\kappa_2}\prod_{j=0}^{\kappa_2}[m_n-j]_\nu|\boldsymbol{m}-(\kappa_2+1)\boldsymbol{e}_n\rangle,\\
X_n^-.|\boldsymbol{m}\rangle &= \frac{1}{[\kappa_2+1]_\nu}|\boldsymbol{m}+(\kappa_2+1)\boldsymbol{e}_n\rangle,\\
K_n.|\boldsymbol{m}\rangle &= \iota^{1-\kappa_2}\nu^{-(\kappa_2+1)(m_n+1/2)}|\boldsymbol{m}\rangle,
\end{aligned}$$

where $\boldsymbol{m} \in (\mathbb{Z}_{\ge 0})^n$, $|\kappa| = \kappa_1+\kappa_2$, and $\tau_{\nu,\kappa_i} = (\nu-\kappa_i+1)/(\nu+\kappa_i-1)$. Here the κ_i are defined in Section 5A. This module can be obtained from $\mathcal{F}^1_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ with $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{>n}$ and $\boldsymbol{b} = (1,\dots,1)$ by the automorphism of $U_q(X_N^{(r)})$ defined by

$$\begin{aligned}
X_0^- &\mapsto (-1)^{|\kappa|+1}X_0^-, & K_0 &\mapsto (-1)^{|\kappa|+1}K_0,\\
X_n^+ &\mapsto (-1)^{\kappa_2}X_n^+, & K_n &\mapsto (-1)^{\kappa_2}K_n,
\end{aligned}$$

with other generators fixed. For the type $C_n^{(1)}$, denote the irreducible components $\mathcal{F}^{1,\pm}_{\boldsymbol{\varepsilon},\boldsymbol{b}}$ of the $U_q(C_n^{(1)})$-module $\mathcal{W}$ by $\mathcal{W}^{\pm}$, for convenience.

Lemma 6.1. *Let $L(\boldsymbol{f})$ be an irreducible highest ℓ-weight U-module with $\boldsymbol{f} = (f_i(z))_{i\in I_0} \in \mathcal{R}$. If $\dim L(\boldsymbol{f})_{\mathrm{wt}(\boldsymbol{f})-\alpha_i} = 1$ for some $i \in I_0$ then $f_i(z)$ satisfies*

$$f_i(z) = f_{i,0}^+\frac{1-(a-b)z}{1-az}, \tag{6-1}$$

where $a, b \in \mathbb{C}$ satisfy $f^+_{i,2\tilde{d}_i} = af^+_{i,\tilde{d}_i}$ and $f^+_{i,\tilde{d}_i} = bf^+_{i,0}$.

Proof. Suppose that $v \in L(\boldsymbol{f})$ is a nonzero ℓ-weight vector of $\boldsymbol{f}$. Note that $\{x^-_{i,k}.v, k\in\mathbb{Z}\}$ spans the weight space $L(\boldsymbol{f})_{\mathrm{wt}(\boldsymbol{f})-\alpha_i}$. If $\dim(L(\boldsymbol{f})_{\mathrm{wt}(\boldsymbol{f})-\alpha_i}) = 1$, then there exist $j \in \mathbb{Z}$ such that $x^-_{i,\tilde{d}_i j}.v$ is nonzero, and $a \in \mathbb{C}$ such that

$$x^-_{i,\tilde{d}_i (j+1)}.v = a x^-_{i,\tilde{d}_i j}.v. \tag{6-2}$$

Consider the actions of $x^+_{i,\tilde{d}_i k}$, $k \in \mathbb{Z}$, on (6-2). The defining relations (2-5) imply

$$f^+_{i,\tilde{d}_i (k+j+1)} - f^-_{i,\tilde{d}_i (k+j+1)} = a(f^+_{i,\tilde{d}_i (k+j)} - f^-_{i,\tilde{d}_i (k+j)})$$

for any $k \in \mathbb{Z}$. Because $f^-_{i,k} = f^+_{i,-k} = 0$ for $k > 0$, we have $f^+_{i,\tilde{d}_i (k+1)} = a f^+_{i,\tilde{d}_i k}$ for any $k > 0$. Taking the series $f_i(z) = \sum_{k=0}^{\infty} z^k f^+_{i,\tilde{d}_i k}$, we have

$$\begin{aligned} f_i(z)(1-az) &= \sum_{k=0}^{\infty} z^k f^+_{i,\tilde{d}_i k} - a \sum_{k=0}^{\infty} z^{k+1} f^+_{i,\tilde{d}_i k} \\ [-3pt] &= f^+_{i,0} + \sum_{k=1}^{\infty} z^k (f^+_{i,\tilde{d}_i k} - a f^+_{i,\tilde{d}_i (k-1)}) \\ &= f^+_{i,0} + (f^+_{i,\tilde{d}_i} - a f^+_{i,0}) z. \end{aligned}$$

Hence $f_i(z)$ has the rational form (6-1). □

6B. ***Highest ℓ-weights.*** Let us first study some properties of the Weyl group and the description of the root vectors of quantum affine algebras, which will enable us to compute the highest ℓ-weight explicitly.

Lemma 6.2. *Let $i, j \in I$, and $i \neq j$.*

(1) *If $a_{ij} a_{ji} = 1$, then $s_j s_i \alpha_j = \alpha_i$.*

(2) *If $a_{ij} a_{ji} = 2$, then $s_i s_j s_i \alpha_j = \alpha_j$.*

Proof. Both (1) and (2) are easy to deduce from $s_j s_i \alpha_j = (a_{ij} a_{ji} - 1)\alpha_j - a_{ij}\alpha_i$ and

$$s_i s_j s_i \alpha_j = (a_{ij} a_{ji} - 1)\alpha_j + (2 - a_{ij} a_{ji}) a_{ij} \alpha_i,$$

respectively. □

Recall the braid group operators associated to $\widetilde{W}$ introduced by Lusztig [1990]. For each simple reflection s_i, there is an algebra automorphism $T_i = T_{s_i}$ of U defined by

$$T_i X_i^+ = -X_i^- K_i, \quad T_i X_i^- = -K_i^{-1} X_i^+, \quad T_i K_\beta = K_{s_i \beta},$$

$$T_i X_j^+ = \sum_{k=0}^{-a_{ij}} (-1)^{k-a_{ij}} q_i^{-k} (X_i^+)^{(-a_{ij}-k)} X_j^+ (X_i^+)^{(k)} \quad (i \neq j),$$

$$T_i X_j^- = \sum_{k=0}^{-a_{ij}} (-1)^{k-a_{ij}} q_i^{k} (X_i^-)^{(k)} X_j^- (X_i^-)^{(-a_{ij}-k)} \quad (i \neq j),$$

where $\beta \in Q$ and $(X_i^\pm)^{(k)} = (X_i^\pm)^k / [k]_i^!$. Then $\Phi T_i = T_i^{-1} \Phi$, where Φ is the

$\mathbb{C}$-linear anti-automorphism of U sending $X_i^{\pm}$ to $X_i^{\pm}$, K_i to K_i^{-1} for $i \in I$. For any $\tau \in \mathscr{T}$, define T_τ by $T_\tau(X_i^{\pm}) = X_{\tau(i)}^{\pm}$ and $T_\tau(K_i) = K_{\tau(i)}$.

For later use, we list some well-known properties of braid group operators (see [Lusztig 1993; Beck 1994]). Choose one element $w \in \widetilde{W}$. If $\tau s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression of w, then the automorphism $T_w = T_\tau T_{i_1} T_{i_2} \cdots T_{i_m}$ of U is independent on the choice of the reduced expression of w. In particular, one reduced expression can be transformed to another by a finite sequence of braid relations. If $w(\alpha_i) = \alpha_j$ then $T_w(X_i^+) = X_j^+$. Moreover, if $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression and $l(ws_i) = l(w) + 1$, then we have $T_w X_i^+ \in U^+$.

Remark 6.3. For $i \leq j$, put $\boldsymbol{s}_{(i,j)} = s_i s_{i+1} \cdots s_j$.

(i) In the type $A_{n-1}^{(1)}$, a reduced expression of $\widetilde{\omega}_i$ for $1 \leq i \leq n-1$ can be chosen as

$$\widetilde{\omega}_i = \tau^i \boldsymbol{s}_{(1,n-i)}^{-1} \boldsymbol{s}_{(2,n-i+1)}^{-1} \cdots \boldsymbol{s}_{(i,n-1)}^{-1},$$

where τ is the diagram automorphism of $A_{n-1}^{(1)}$ sending j to $j+1 \pmod n$ for $j \in I$ (see [Jang et al. 2023, Section 3.3]).

(ii) In the type $A_{2n}^{(2)}$, the reduced expression of $\widetilde{\omega}_n$ can be chosen (see [Damiani 2000, Corollary 4.2.4]) as

$$\widetilde{\omega}_n = (s_0 s_1 \cdots s_n)^n.$$

(iii) In the type $C_n^{(1)}$ (resp. $D_{n+1}^{(2)}$), the reduced expressions of $\widetilde{\omega}_{n-1}$ and $\widetilde{\omega}_n$ can be chosen as

$$\widetilde{\omega}_{n-1} = (\boldsymbol{s}_{(0,n)} s_{n-1})^{n-1} \quad \text{and} \quad \widetilde{\omega}_n = \tau s_n \boldsymbol{s}_{(n-1,n)} \boldsymbol{s}_{(n-2,n)} \cdots \boldsymbol{s}_{(1,n)},$$

respectively, where τ is the diagram automorphism of $C_n^{(1)}$ (resp. $D_{n+1}^{(2)}$) sending i to $n-i$ for $i \in I$.

Now, let us define the root vectors in U. We refer the reader to [Beck and Nakajima 2004] for the construction of root vectors X_β^+, $\beta \in \triangle$ (i.e., the E_β defined therein). In particular, the real root vectors $X^+_{k\widetilde{d}_i\delta \pm \alpha_i}$ are described explicitly by

$$X^+_{k\widetilde{d}_i\delta+\alpha_i} = T_{\widetilde{\omega}_i}^{-k} X_i^+ \quad (k \geq 0), \qquad X^+_{k\widetilde{d}_i\delta-\alpha_i} = T_{\widetilde{\omega}_i}^{k} T_i^{-1} X_i^+ \quad (k > 0).$$

Then $X^+_{k\widetilde{d}_i\delta \pm \alpha_i} \in U^+$. The imaginary root vectors are defined by

$$\widetilde{\psi}_{i,k\widetilde{d}_i} = X^+_{k\widetilde{d}_i\delta-\alpha_i} X_i^+ - q_i^{-2} X_i^+ X^+_{k\widetilde{d}_i\delta-\alpha_i} \quad (k > 0), \tag{6-3}$$

and define the elements $X^+_{i,k\widetilde{d}_i\delta}$ by the following formal series in z:

$$\exp\left((q_i - q_i^{-1}) \sum_{k \geq 1} X^+_{i,k\widetilde{d}_i\delta} z^k \right) = 1 + \sum_{k \geq 1} (q_i - q_i^{-1}) \widetilde{\psi}_{i,k\widetilde{d}_i} z^k. \tag{6-4}$$

Under the isomorphism of two presentations of $U_q(\mathfrak{g})$, the generators $\psi^+_{i,k\tilde{d}_i}$ and the imaginary root vectors $\widetilde{\psi}_{i,k\tilde{d}_i}$ are related (see [Beck 1994; Damiani 2012]); more precisely, for $k > 0$ and $i \in I_0$, we have

$$\psi^+_{i,k\tilde{d}_i} = o(i)^k (q_i - q_i^{-1}) C^{-k\tilde{d}_i/2} k_i \widetilde{\psi}_{i,k\tilde{d}_i}, \tag{6-5}$$

where $o : I_0 \to \{\pm 1\}$ is a map such that $o(i) = -o(j)$ whenever

(i) $a_{ij} \leq 0$ implies that $o(i)o(j) = -1$,

(ii) in the twisted cases different from $A^{(2)}_{2n}$, if $a_{ij} = -2$ then $o(i) = 1$.

Note that $o(n) = 1$ in the type $D^{(2)}_{n+1}$ as $a_{n,n-1} = -2$. Thus, we can deduce that the map $o : I_0 \to \{\pm 1\}$ is uniquely determined in the type $D^{(2)}_{n+1}$.

In Lemma 6.1, the scalars a and b can be described by the root vectors according to the above relations, which will become more computable in our case. Let $v \in L(\boldsymbol{f})$ be a nonzero ℓ-weight vector of $\boldsymbol{f}$. Since $C^{1/2}$ acts trivially on $L(\boldsymbol{f})$ and k_i commutes with $\widetilde{\psi}_{i,k\tilde{d}_i}$, this implies that

$$X^+_{2\tilde{d}_i\delta-\alpha_i}.v = o(i) a X^+_{\tilde{d}_i\delta-\alpha_i}.v \quad \text{and} \quad \widetilde{\psi}_{i,\tilde{d}_i}.v = o(i)\frac{b}{q_i - q_i^{-1}} v. \tag{6-6}$$

Lemma 6.4. *For any $i \in I_0$ and $k \in \mathbb{Z}_{>0}$, the root vectors $X^+_{k\tilde{d}_i\delta-\alpha_i}$ in $U_q(X^{(r)}_N)$ have the following relations*:

$$X^+_{(k+1)\tilde{d}_i\delta-\alpha_i} = \begin{cases} \frac{1}{[3]^!_n}[X^+_{k\delta-\alpha_n}, [X^+_{\delta-\alpha_n}, X^+_n]_q] & \text{if } (X^{(r)}_N, i) = (A^{(2)}_{2n}, n), \\ \frac{1}{[2]_i}[X^+_{k\tilde{d}_i\delta-\alpha_i}, [X^+_{\tilde{d}_i\delta-\alpha_i}, X^+_i]_{q_i^2}] & \text{otherwise.} \end{cases}$$

Proof. We may use the following relations [Damiani 2000, Proposition 2.2.4, Corollary 3.2.4] (see [Beck 1994]): for $k \in \mathbb{Z}_{>0}$,

$$\begin{cases} [X^+_{k\delta-\alpha_n}, X^+_{n,\delta}] = [3]^!_n X^+_{(k+1)\delta-\alpha_n} & \text{if } (X^{(r)}_N, i) = (A^{(2)}_{2n}, n), \\ [X^+_{k\tilde{d}_i\delta-\alpha_i}, X^+_{i,\tilde{d}_i\delta}] = [2]_i X^+_{(k+1)\tilde{d}_i\delta-\alpha_i} & \text{otherwise.} \end{cases} \tag{6-7}$$

Note that $X^+_{i,\tilde{d}_i\delta} \in U$ is defined by the formal series (6-4). Because, in (6-3), $\widetilde{\psi}_{i,k\tilde{d}_i} = [X^+_{k\tilde{d}_i\delta-\alpha_i}, X^+_i]_{q_i^2}$, by comparing the coefficients of z in (6-4), we can get

$$X^+_{i,\tilde{d}_i\delta} = \widetilde{\psi}_{i,\tilde{d}_i} = [X^+_{\tilde{d}_i\delta-\alpha_i}, X^+_i]_{q_i^2}, \tag{6-8}$$

which implies the lemma by (6-7) and (6-8). □

Let $\alpha \in Q_+$. We introduce the *height* $\operatorname{ht}\alpha$ of α as $\operatorname{ht}\alpha = \sum_{i\in I} m_i$ if $\alpha = \sum_{i\in I} m_i\alpha_i$. Define a subset $Q_+(\alpha)$ of Q_+ as follows:

$$Q_+(\alpha) = \{\beta \in Q_+ \mid \operatorname{ht}\alpha - \operatorname{ht}\beta = 1, \alpha - \beta \neq \alpha_0\}.$$

Let $U^+(\alpha)$ be the subspace of U^+_α defined as $U^+(\alpha) = \sum_{\beta\in Q_+(\alpha)} U^+_\beta X^+_{\alpha-\beta}$.

Lemma 6.5. (1) *For $i \in I_0$, the root vector $X^+_{\delta-\alpha_i}$ in $U_q(A^{(1)}_{n-1})$ has the following form*:

$$X^+_{\delta-\alpha_i} \equiv (-q^{-1})^{n-2}(X^+_{i+1}\cdots X^+_{n-1})(X^+_{i-1}\cdots X^+_2 X^+_1)X^+_0 \pmod{U^+(\delta-\alpha_i)}.$$

(2) *In $U_q(C^{(1)}_n)$*,

$$X^+_{\delta-\alpha_{n-1}} \equiv q^{-n}(X^+_n X^+_{n-2}\cdots X^+_1)(X^+_{n-1}X^+_{n-2}\cdots X^+_1)X^+_0 \pmod{U^+(\delta-\alpha_{n-1})},$$

$$X^+_{\delta-\alpha_n} \equiv \left(\frac{q^{-1}}{[2]_1}\right)^{n-1}(X^+_{n-1})^2(X^+_{n-2})^2\cdots(X^+_1)^2X^+_0 \pmod{U^+(\delta-\alpha_n)}.$$

(3) *In $U_q(A^{(2)}_{2n})$*,

$$X^+_{\delta-\alpha_n} \equiv q^{-2n}(X^+_{n-1}X^+_{n-2}\cdots X^+_1)(X^+_n X^+_{n-1}\cdots X^+_1)X^+_0 \pmod{U^+(\delta-\alpha_n)}.$$

(4) *In $U_q(D^{(2)}_{n+1})$*,

$$X^+_{\delta-\alpha_n} \equiv q^{-2n+2}X^+_{n-1}X^+_{n-2}\cdots X^+_1 X^+_0 \pmod{U^+(\delta-\alpha_n)}.$$

Proof. Thanks to the reduced expressions of $\widetilde{\omega}_i$ in Remark 6.3, the lemma can be deduced directly by the definition. One can refer to [Jang et al. 2023, Lemma 4.7] for assertion (1). To see the remaining assertions we define the operators $\mathcal{D}^{(1)}_i$ and $\mathcal{D}^{(2)}_i$ of U for $i \in I_0$ by $\mathcal{D}^{(1)}_i(X) = [X, X^+_i]_{q_i}$ and $\mathcal{D}^{(2)}_i(X) = [[X, X^+_i], X^+_i]_q$ for any $X \in U$, respectively. In the type $C^{(1)}_n$, we note that $T_\tau \mathcal{D}^{(s)}_i = \mathcal{D}^{(s)}_{n-i} T_\tau$ and $T_j \mathcal{D}^{(s)}_i = \mathcal{D}^{(s)}_i T_j$ for $|i-j| > 1$ and $s = 1, 2$. Write $T_{\boldsymbol{s}_{(i,j)}} = T_{(i,j)}$ for simplicity. Then

$$T_{(0,n)}T_{n-1}\mathcal{D}^{(s)}_n = \mathcal{D}^{(s)}_n T_{(0,n)}T_{n-1},$$

since $\boldsymbol{s}_{(0,n)}s_{n-1}\alpha_n = \alpha_n$, and for any $0 < i < n-1$, we have $(\boldsymbol{s}_{(0,n)}s_{n-1})^{i-1}\alpha_1 = \alpha_i$ and $\boldsymbol{s}_{(i+1,n-1)}\boldsymbol{s}_{(i,n-2)}\alpha_{n-2} = \alpha_i$ by using Lemma 6.2, so we get

$$\begin{aligned}(T_{(0,n)}T_{n-1})^i T_{(0,n-2)}X^+_{n-1} &= (T_{(0,n)}T_{n-1})^{i-1}T_{(0,n)}T_{(0,n-3)}X^+_{n-2}\\ &= (T_{(0,n)}T_{n-1})^{i-1}T_{(0,n-2)}T_{(0,n-3)}T_{n-1}X^+_{n-2}\\ &= -(T_{(0,n)}T_{n-1})^{i-1}[T_{(0,n-2)}X^+_{n-1}, X^+_1]_{q_{n-1}}\\ &= -\mathcal{D}^{(1)}_i(T_{(0,n)}T_{n-1})^{i-1}T_{(0,n-2)}X^+_{n-1},\end{aligned}$$

and

$$\begin{aligned}T_{(i+1,n)}T_{(i,n-1)}X^+_n &= T_{(i+1,n-1)}T_{(i,n-2)}T_nT_{n-1}X^+_n\\ &= T_{(i+1,n-1)}T_{(i,n-2)}T^{-1}_{n-1}X^+_n\\ &= \frac{1}{[2]_{n-1}}[[T_{(i+1,n-1)}X^+_n, X^+_i], X^+_i]_q\\ &= \frac{1}{[2]_1}\mathcal{D}^{(2)}_i T_{(i+1,n-1)}X^+_n.\end{aligned}$$

Finally, the definition of the root vectors and Remark 6.3 imply that

$$\begin{aligned}
X^+_{\delta-\alpha_{n-1}} &= (T_{(0,n)}T_{n-1})^{n-2}T_{(0,n)}X^+_{n-1}\\
&= -(T_{(0,n)}T_{n-1})^{n-2}T_{(0,n-2)}\mathcal{D}^{(1)}_n X^+_{n-1}\\
&= -\mathcal{D}^{(1)}_n(T_{(0,n)}T_{n-1})^{n-2}T_{(0,n-2)}X^+_{n-1}\\
&= \mathcal{D}^{(1)}_n\mathcal{D}^{(1)}_{n-2}(T_{(0,n)}T_{n-1})^{n-3}T_{(0,n-2)}X^+_{n-1}\\
&\ \ \vdots\\
&= (-1)^{n-1}\mathcal{D}^{(1)}_n\mathcal{D}^{(1)}_{n-2}\cdots\mathcal{D}^{(1)}_1 T_{(0,n-2)}X^+_{n-1}\\
&= \mathcal{D}^{(1)}_n\mathcal{D}^{(1)}_{n-2}\cdots\mathcal{D}^{(1)}_1\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_{1,q}X^+_0,
\end{aligned}$$

where $\mathcal{D}^{(1)}_{i,q}(X) = [X, X^+_i]_q$ for $X \in U$, and

$$\begin{aligned}
X^+_{\delta-\alpha_n} &= T_\tau T_n T_{(n-1,n)}\cdots T_{(2,n)}T_{(1,n-1)}X^+_n\\
&= \frac{1}{[2]_1}T_\tau T_n T_{(n-1,n)}\cdots T_{(3,n)}\mathcal{D}^{(2)}_1 T_{(2,n-1)}X^+_n\\
&= \frac{1}{[2]_1}\mathcal{D}^{(2)}_{n-1}T_\tau T_n T_{(n-1,n)}\cdots T_{(3,n)}T_{(2,n-1)}X^+_n\\
&\ \ \vdots\\
&= \Big(\frac{1}{[2]_1}\Big)^{n-1}\mathcal{D}^{(2)}_{n-1}\mathcal{D}^{(2)}_{n-2}\cdots\mathcal{D}^{(2)}_1 X^+_0,
\end{aligned}$$

which implies assertion (2). Similarly, we can prove that

$$X^+_{\delta-\alpha_n} = -\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_1\mathcal{D}^{(1)}_{n,q_1}\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_2\mathcal{D}^{(1)}_{1,q_0}(X^+_0) \quad \text{in } U_q(A^{(2)}_{2n}), \tag{6-9}$$

$$X^+_{\delta-\alpha_n} = (-1)^{n-1}\mathcal{D}^{(1)}_{n-1}\cdots\mathcal{D}^{(1)}_1(X^+_0) \quad \text{in } U_q(D^{(2)}_{n+1}), \tag{6-10}$$

which imply (3) and (4). □

Remark 6.6. In order to simplify computations in the following theorem for the type $A^{(2)}_{2n}$ $(n \geq 2)$, we actually only need two terms of $X^+_{\delta-\alpha_n}$,

$$q^{-2n}(X^+_{n-1}X^+_{n-2}\cdots X^+_1)(X^+_n X^+_{n-1}\cdots X^+_1)X^+_0 - q^{-2n+1}(X^+_{n-1}\cdots X^+_1)^2 X^+_0 X^+_n, \tag{6-11}$$

which can be deduced directly from formula (6-9).

Now we compute the highest ℓ-weights of the q-oscillator representations defined in Section 6A.

Theorem 6.7. (1) *Fix $0<s<n$ and $l\in\mathbb{Z}$. The $U_q(A_{n-1}^{(1)})$-module $\mathcal{W}_s^{(l)}$ has highest ℓ-weight $\boldsymbol{f}=(f_i(z))_{i\in I_0}$ given by*

$$f_i(z)=\frac{c_{i,l}+u}{1+c_{i,l}u}\quad \textit{with}\ \ c_{i,l}=\begin{cases} q^{\delta_{i,s-1}l-\delta_{i,s}(l+1)} & \textit{if } l\geq 0,\\ q^{\delta_{i,s}(l-1)-\delta_{i,s+1}l} & \textit{if } l<0\end{cases}$$

for $1\leq i\leq n-1$, where $u=o(s)(-q^{-1})^n z$.

(2) *The highest ℓ-weight of the $U_q(C_n^{(1)})$-module $\mathcal{W}^+$ (resp. $\mathcal{W}^-$) is given by*

$$\left(1,\dots,1,\frac{q^{-1/2}+u}{1+q^{-1/2}u}\right)\quad \left(\textit{resp.}\ \left(1,\dots,1,\frac{q^{1/2}+u}{1+q^{1/2}u},\frac{q^{-3/2}+u}{1+q^{-3/2}u}\right)\right),$$

where $u=o(n)q^{-n-1}z$.

(3) *The highest ℓ-weight of the $U_q(A_{2n}^{(2)})$-module (resp. $U_q(D_{n+1}^{(2)})$-module) $\mathcal{W}$ is given by*

$$\left(1,\dots,1,\frac{\imath q_n^{-1}+u}{1+\imath q_n^{-1}u}\right),$$

where $u=o(n)\imath\tau_q q^{-2n-1}z$ (resp. $u=q^{-2n}z$).

Proof. The proof of the first assertion can be found in [Kwon and Lee 2023, Theorem 4.10]. For (2), we have verified in Proposition 5.7 that $v^+=|\mathbf{0}\rangle$ and $v^-=|\boldsymbol{e}_n\rangle$ are highest ℓ-weight vectors of $U_q(C_n^{(1)})$-modules $\mathcal{W}^+$ and $\mathcal{W}^-$ respectively. Therefore, it follows from Lemma 6.1 and formulae (6-6) that we only need to compute the actions of $X^+_{2\tilde{d}_i\delta-\alpha_i}$ and $\widetilde{\psi}_{i,\tilde{d}_i}$ on $v^\pm$.

Note that $X_j^+.v^\pm=0$ for all $j\in I_0$. By using Lemma 6.5 we have

$$X^+_{\delta-\alpha_n}.v^+=\frac{q^{-n+1}}{[2]_1}|2\boldsymbol{e}_n\rangle,\quad X^+_{\delta-\alpha_n}.v^-=\frac{q^{-n+1}}{[2]_1}|3\boldsymbol{e}_n\rangle,$$

and then

$$\widetilde{\psi}_{n,1}.v^+=\frac{q^{-n-1}}{[2]_1}v^+,\quad \widetilde{\psi}_{n,1}.v^-=\frac{q^{-n-1}}{[2]_1}[3]_1v^-.$$

Therefore, we have

$$\begin{aligned}X^+_{2\delta-\alpha_n}.v^+&=\frac{1}{[2]}[X^+_{\delta-\alpha_n},\widetilde{\psi}_{n,1}].v^+\\ &=\frac{1}{[2]}\left(\frac{q^{-n-1}}{[2]_1}X^+_{\delta-\alpha_n}.v^+-\frac{q^{-n+1}}{[2]_1}\widetilde{\psi}_{n,1}.|2\boldsymbol{e}_n\rangle\right)\\ &=\frac{q^{-n+1}}{[2][2]_1}\left(q^{-2}+1-q^{-2}\frac{[4]_1[3]_1}{[2]_1}\right)X^+_{\delta-\alpha_n}.v^+\\ &=-q^{-n-3/2}X^+_{\delta-\alpha_n}.v^+,\end{aligned}$$

and

$$X^+_{2\delta-\alpha_n}.v^- = \frac{1}{[2]}[X^+_{\delta-\alpha_n}, \widetilde{\psi}_{n,1}].v^- = \frac{q^{-n+1}}{[2][2]_1}(q^{-2}[3]_1 X^+_{\delta-\alpha_n}.v^- - \widetilde{\psi}_{n,1}.|3\boldsymbol{e}_n\rangle)$$
$$= \frac{q^{-n+1}}{[2][2]_1}\left(q^{-2}[3]_1 + [3]_1 - q^{-2}\frac{[5]_1[4]_1}{[2]_1}\right)X^+_{\delta-\alpha_n}.v^-$$
$$= -q^{-n-5/2}X^+_{\delta-\alpha_n}.v^-.$$

On the other hand,

$$X^+_{\delta-\alpha_{n-1}}.v^- = -q^{-n}|\boldsymbol{e}_{n-1}\rangle, \quad \widetilde{\psi}_{n-1,1}.v^- = q^{-n-1}v^-,$$

and then

$$X^+_{2\delta-\alpha_{n-1}}.v^- = \frac{1}{[2]_{n-1}}[X^+_{\delta-\alpha_{n-1}}, \widetilde{\psi}_{n-1,1}].v^-$$
$$= \frac{q^{-n}}{[2]_{n-1}}(q^{-1}X^+_{\delta-\alpha_{n-1}}.v^- + \widetilde{\psi}_{n-1,1}.|\boldsymbol{e}_{n-1}\rangle)$$
$$= \frac{q^{-n}}{[2]_{n-1}}(q^{-1}+1)X^+_{\delta-\alpha_{n-1}}.v^-$$
$$= q^{-n-1/2}X^+_{\delta-\alpha_{n-1}}.v^-.$$

In other cases, we can check that $X^+_{\delta-\alpha_i}.v = 0$, so $\widetilde{\psi}_{i,1}.v = 0$ and $X^+_{2\delta-\alpha_i}.v = 0$. Thus, we get (2) as desired.

To get (3), let $v := |0\rangle$. We first focus on the type $D^{(2)}_{n+1}$. By Lemma 6.5(4),

$$X^+_{\delta-\alpha_n}.v = q^{-2n+2}|2\boldsymbol{e}_n\rangle, \quad \widetilde{\psi}_{n,1}.v = -q^{-2}X^+_n X^+_{\delta-\alpha_n}.v = -\imath\tau_\nu q^{-2n}v,$$

and then

$$X^+_{2\delta-\alpha_n}.v = \frac{1}{[2]}[X^+_{\delta-\alpha_n}, \widetilde{\psi}_{n,1}].v = \frac{1}{[2]}(-\imath\tau_\nu q^{-2n}X^+_{\delta-\alpha_n}.v - q^{-2n+2}\widetilde{\psi}_{n,1}.|\boldsymbol{e}_n\rangle),$$
$$= \frac{q^{-2n+2}}{[2]}(-\imath\tau_\nu q^{-2} - \imath\tau_\nu + \imath\tau_\nu[2]_\nu q^{-2})X^+_{\delta-\alpha_n}.v$$
$$= -\imath q^{-2n-1}X^+_{\delta-\alpha_n}.v.$$

For $i \neq n$, we have $X^+_{\widetilde{d}_i\delta-\alpha_i}.v = 0$, and then $\widetilde{\psi}_{i,\widetilde{d}_i}.v = 0$ and $X^+_{2\widetilde{d}_i\delta-\alpha_i}.v = 0$. This proves assertion (3) for the type $D^{(2)}_{n+1}$. For the type $A^{(2)}_{2n}$, Lemma 6.5(3) yields

$$X^+_{\delta-\alpha_n}.v = q^{-2n}\imath\tau_q|\boldsymbol{e}_n\rangle, \quad \widetilde{\psi}_{n,1}.v = -q^{-1}X^+_n X^+_{\delta-\alpha_n}.v = \tau_q^2 q^{-2n-1}v.$$

Note that all terms in the expression of $X^+_{\delta-\alpha_n}$ vanish on the vector $|\boldsymbol{e}_n\rangle$ except for the two terms in (6-11). We can compute the following action by using (6-11):

$$X^+_{\delta-\alpha_n}.|\boldsymbol{e}_n\rangle = \imath\tau_q q^{-2n-1}|2\boldsymbol{e}_n\rangle.$$

Therefore,

$$\begin{aligned}X^+_{2\delta-\alpha_n}.v &= \frac{1}{[3]^!_n}[X^+_{\delta-\alpha_n}, \widetilde{\psi}_{n,1}].v \\ &= \frac{1}{[3]^!_n}(\tau_q^2 q^{-2n-1} X^+_{\delta-\alpha_n}.v - \imath\tau_q q^{-2n}\widetilde{\psi}_{n,1}.|\boldsymbol{e}_n\rangle), \\ &= \frac{q^{-2n}\tau_q}{[3]^!_n}(q^{-1}+1-q^{-2}[2])X^+_{\delta-\alpha_n}.v \\ &= q^{-2n-3/2}\tau_q X^+_{\delta-\alpha_n}.v.\end{aligned}$$

Then $a = o(n)\tau_q q^{-2n-3/2}$ and $b = o(n)\tau_q q^{-2n}(q^{-1/2}+q^{-3/2})$. Assertion (3) for the type $A^{(2)}_{2n}$ follows from Lemma 6.1 and Corollary 5.4(3). □

Appendix: Proof of Theorem 4.2

Proof of Theorem 4.2. For generalized Cartan matrices of finite types, the corresponding system of equations (4-1) has been solved in [Chen et al. 2024]. The method followed in this appendix is parallel with the one there.

Given an affine Cartan matrix A. Fix one $i \in I$ and denote by $\mathcal{A}_{x_i}$ the subalgebra of $\mathcal{A}$ generated by all $x_j^{\pm 1}$ for $j \neq i$. Then there is a natural isomorphism $\mathcal{A} \cong \mathcal{A}_{x_i}[x_i^{\pm 1}]$. Suppose that $(\phi_i)_{i\in I}$ is any solution to the system (4-1).

We have the following two crucial lemmas.

Lemma A.1. *Any $\phi \in \mathcal{A}$ satisfying $\zeta_i(\phi) - \phi = \{y_i\}_i$ has the form*

$$\phi = \beta_i^+ y_i + \phi_0 + \beta_i^- y_i^{-1},$$

where $\phi_0 \in \mathcal{A}_{x_i}$, $\beta_i^+ = -q_i(q_i - q_i^{-1})^{-2}$ and $\beta_i^- = -q_i^{-1}(q_i - q_i^{-1})^{-2}$.

Proof. Let $\phi = \sum_k \phi_k x_i^k$ with $\phi_k \in \mathcal{A}_{x_i}$. Then $\zeta_i(\phi) - \phi = \{y_i\}_i$ implies that

$$\sum_k (q_i^{-k}-1)\phi_k x_i^k = \frac{1}{q_i - q_i^{-1}} x_i^2 \prod_{j\neq i} x_j^{a_{ji}} - \frac{1}{q_i - q_i^{-1}} x_i^{-2} \prod_{j\neq i} x_j^{-a_{ji}}.$$

Hence ϕ_k is zero unless $k = 0, \pm 2$ and

$$\phi_2 = \beta_i^+ \prod_{j\neq i} x_j^{a_{ji}}, \quad \phi_{-2} = \beta_i^- \prod_{j\neq i} x_j^{-a_{ji}}.$$

So the lemma is proved. □

Therefore, we may always assume that ϕ_i in the system (4-1) satisfies $\phi_i = \beta_i^+ y_i + \phi_{i,0} + \beta_i^- y_i^{-1}$, where $\phi_{i,0} \in \mathcal{A}_{x_i}$.

Note that any pair (ϕ_i, ϕ_j) is (i, j)-shiftable. This condition can further restrict the choices of $\phi_{i,0}$ and $\phi_{j,0}$ when the nodes i and j are not connected in the Dynkin diagram of A, namely, $a_{ij} = 0$. More precisely, we have:

Lemma A.2. *If $a_{ij}=0$, then both $\phi_{i,0}$ and $\phi_{j,0}$ lie in $\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}$.*

Proof. Since $a_{ij}=0$, we have $a_{ji}=0$, and $y_i\in\mathcal{A}_{x_j}$ and $y_j\in\mathcal{A}_{x_i}$. If $\phi_{i,0}=0$, there is nothing to do. Assume that $\phi_{i,0}$ is not zero. We rewrite $\phi_{i,0}$ uniquely as the Laurent polynomial in x_j, i.e., a unique form in $\mathcal{A}_{x_j}[x_j^{\pm 1}]$. Take the nonzero term in this form of $\phi_{i,0}$ such that x_j has the highest (resp. lowest) power, denoted by $\phi_{i,\max}$ (resp. $\phi_{i,\min}$). Then the shiftability of (ϕ_i,ϕ_j) implies that

$$q_j^m\beta_j^+\phi_{i,\max}y_j=\beta_j^+\phi_{i,\max}y_j,\quad q_j^l\beta_j^-\phi_{i,\min}y_j^{-1}=\beta_j^-\phi_{i,\min}y_j^{-1},$$

where $m=\deg_{x_j}\phi_{i,\max}$ and $l=\deg_{x_j}\phi_{i,\max}$. Hence $m=0=l$. Then we can conclude that $\phi_{i,0}\in\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}$ as desired. Similarly, we have $\phi_{j,0}\in\mathcal{A}_{x_i}\cap\mathcal{A}_{x_j}$. $\square$

Let us first focus on the rank-two cases. Fix $i\neq j$ in I and $J=\{i,j\}$. Due to Lemma A.2 we may assume that the nodes i and j are connected. Without loss of generality, we set $\lambda=a_{ij}$, $\mu=a_{ji}$ and $|\lambda|\geq|\mu|$. Then

$$A_J=\begin{pmatrix}2&\lambda\\ \mu&2\end{pmatrix}\quad\text{where }\ 1\leq\lambda\mu\leq 4.$$

Then $y_i=x_i^2x_j^{\mu}$ and $y_j=x_i^{\lambda}x_j^2$ in this case. Assume that φ_i and φ_j have the forms as in Lemma A.1, i.e.,

$$\varphi_l=\beta_l^+y_l+\phi_{l,0}+\beta_l^-y_l^{-1},$$

where $\varphi_{l,0}\in\mathcal{A}_{x_l}$, $l\in J$, and let φ_i and φ_j satisfy the equality

$$\varphi_i\varphi_j=\zeta_j^{-1}(\varphi_i)\zeta_i^{-1}(\varphi_j).\tag{$*$}$$

Record the (x_i,x_j)-degrees of a monomial u in $\mathcal{A}$ by a *degree vector*

$$\begin{pmatrix}\deg_{x_i}u\\ \deg_{x_j}u\end{pmatrix}.$$

Then a Laurent polynomial f corresponds to a matrix with each column vector representing for the (x_i,x_j)-degrees of certain term of f. Moreover, if $\varphi_{i,0}$ is not zero (resp. $\varphi_{j,0}$ is not zero), then we use one vector with a parameter s (resp. t),

$$\begin{pmatrix}0\\ s\end{pmatrix}\quad\left(\text{resp. }\begin{pmatrix}t\\ 0\end{pmatrix}\right),$$

to stand for the possible (x_i,x_j)-degrees of $\varphi_{i,0}$ (resp. $\varphi_{j,0}$). For example, by Lemma A.2, if $a_{ij}=0$, then s and t always equal 0. Therefore, we obtain the following matrix with possible (x_i,x_j)-degrees of terms of $\varphi_i\varphi_j$:

$$\begin{pmatrix}q_i^{\lambda}q_j^{\mu}&q_i^tq_j^{\mu}&1&q_i^{\lambda}q_j^s&q_i^tq_j^s&q_i^{-\lambda}q_j^s&1&q_i^tq_j^{-\mu}&q_i^{-\lambda}q_j^{-\mu}\\ 2+\lambda&2+t&2-\lambda&\lambda&t&-\lambda&\lambda-2&t-2&-\lambda-2\\ \mu+2&\mu&\mu-2&s+2&s&s-2&2-\mu&-\mu&-2-\mu\end{pmatrix},$$

where the first row contains the corresponding *shifted coefficients* in $\zeta_i^{-1}(\varphi_j)\zeta_j^{-1}(\varphi_i)$. The terms with shifted coefficient 1 can be canceled on the left- and right-hand sides of the equality $(*)$, so we may omit such terms. Therefore, we have the matrix

$$\begin{pmatrix} q_j^{2\mu} & q_i^t q_j^\mu & q_i^\lambda q_j^s & q_i^t q_j^s & q_i^{-\lambda} q_j^s & q_i^t q_j^{-\mu} & q_j^{-2\mu} \\ 2+\lambda & 2+t & \lambda & t & -\lambda & t-2 & -\lambda-2 \\ \mu+2 & \mu & s+2 & s & s-2 & -\mu & -2-\mu \end{pmatrix}. \tag{M1}$$

One useful statement is that if a shifted coefficient is not 1, then the corresponding degree vector has to be equal to another one in the matrix (M1) by the equality $(*)$. Therefore, we can determine all possible (x_i, x_j)-degrees of $\varphi_{i,0}$ and $\varphi_{j,0}$ as follows:

types	(λ,μ)	possible values of s and t
A_2	$(-1,-1)$	$s,t \in \{1,-1\}$
$B_2\ (= C_2)$	$(-2,-1)$	$\varphi_{i,0}=0,\ t=\pm 2$ or $s=\pm 1,\ t=0$
G_2	$(-3,-1)$	none
$A_1^{(1)}$	$(-2,-2)$	$s=0=t$
$A_2^{(2)}$	$(-4,-1)$	$\varphi_{i,0}=0,\ t=0$

Note that there is no (x_i, x_j)-degree vector for the type G_2 satisfying the above statement, so neither for the types $G_2^{(1)}$ and $D_4^{(3)}$. We have obtained all solutions for the type $A_1^{(1)}$ in Example 4.1. For type $A_2^{(2)}$, we substitute the reduced forms of $\varphi_{i,0}$ and $\varphi_{j,0}$, i.e., $\varphi_{i,0}=0$, $\varphi_{j,0}\in\mathbb{C}^\times$, into (4-1), and then get

$$\varphi_i = \frac{\imath}{q_i - q_i^{-1}}\{\imath q^{\frac{1}{2}} x_i^2 x_j^{-1}\}_i, \quad \varphi_j = \{\imath q^{-\frac{1}{2}} x_i^2 x_j^{-1}\}_j \{\imath q^{\frac{3}{2}} x_i^{-2} x_j\}_j.$$

By our assumption in Section 2, we have $i=1,\ j=0$ and $\phi_0=\varphi_j,\ \phi_1=\varphi_i$ for the type $A_2^{(2)}$.

Let us turn to the higher-rank cases. The next result tells us how to "glue" the rank-two cases together.

Lemma A.3. *Let $j\in I$ be a node which connects to the other two distinct nodes i and l in the Dynkin diagram. Assume that $\phi_{j,0}\neq 0$ and the pair of integers (m,t) is the (x_i, x_l)-degree of any nonzero (monomial) term of $\phi_{j,0}$. Then we have $mt\le 0$.*

Proof. Otherwise, assume that $mt>0$ and the corresponding nonzero term of $\phi_{j,0}$ is $\phi_{j,0}^{(1)}$. Without loss of generality, we may let $m>0$ and $t>0$. Consider the term $\beta_i^- y_i^{-1}\phi_{j,0}^{(1)}$ of $\phi_i\phi_j$ which has the factor $x_i^{m-2}x_j^{-a_{ji}}x_l^{t-a_{li}}$. So we have that the shifted coefficient $q_i^m q_j^{-a_{ji}} = q_i^{m-a_{ij}}$ in $\zeta_j^{-1}\phi_i\zeta_i^{-1}\phi_j$ is not 1. However, there is no other term in $\phi_i\phi_j$ whose (x_i,x_j,x_l)-degree vector equals $(m-2,-a_{ji},t-a_{li})$. This is a contradiction. Hence $mt\le 0$. $\square$

Lemma A.3 implies that there is no solution to the system (4-1) for A whose Dynkin diagram contains D_4 or F_4 as a subdiagram.

Up to this point, we have ruled out all affine Cartan matrices except those of types $A_n^{(1)}$ $(n \geq 1)$, $C_n^{(1)}$ $(n \geq 2)$, $A_{2n}^{(2)}$ $(n \geq 1)$ or $D_{n+1}^{(2)}$ $(n \geq 2)$. Now we can substitute the reduced forms of the $\phi_{i,0}$ into the system (4-1) to determine the coefficients of the possible terms. Then we obtain all solutions as listed below Theorem 4.2. Therefore, Theorem 4.2 is proved, as desired.

Acknowledgements

This paper was partially supported by the China Postdoctoral Science Foundation under grant number 2024M751285 and the NSF of China (11931009, 12161141001, 12171132 and 11771410). The author would like to thank Professor Yun Gao and Professor Hongjia Chen for discussions and encouragement. The author would also like to thank the referee for helpful suggestions which improved the exposition of this paper.

References

[Beck 1994] J. Beck, "Braid group action and quantum affine algebras", *Comm. Math. Phys.* **165**:3 (1994), 555–568. MR Zbl

[Beck and Nakajima 2004] J. Beck and H. Nakajima, "Crystal bases and two-sided cells of quantum affine algebras", *Duke Math. J.* **123**:2 (2004), 335–402. MR Zbl

[Benkart et al. 1997] G. Benkart, D. Britten, and F. Lemire, "Modules with bounded weight multiplicities for simple Lie algebras", *Math. Z.* **225**:2 (1997), 333–353. MR Zbl

[Boos et al. 2016] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, "Oscillator versus prefundamental representations", *J. Math. Phys.* **57**:11 (2016), art. id. 111702. MR Zbl

[Boos et al. 2017] H. Boos, F. Göhmann, A. Klümper, K. S. Nirov, and A. V. Razumov, "Oscillator versus prefundamental representations, II: Arbitrary higher ranks", *Journal of Mathematical Physics* **58**:9 (2017), art. id. 093504. MR Zbl

[Britten et al. 1994] D. J. Britten, J. Hooper, and F. W. Lemire, "Simple C_n modules with multiplicities 1 and applications", *Canad. J. Phys.* **72**:7-8 (1994), 326–335. MR Zbl

[Chari and Pressley 1991] V. Chari and A. Pressley, "Quantum affine algebras", *Comm. Math. Phys.* **142**:2 (1991), 261–283. MR Zbl

[Chari and Pressley 1994] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge Univ. Press, 1994. MR Zbl

[Chari and Pressley 1998] V. Chari and A. Pressley, "Twisted quantum affine algebras", *Comm. Math. Phys.* **196**:2 (1998), 461–476. MR Zbl

[Chen et al. 2024] H. Chen, Y. Gao, X. Liu, and L. Wang, "U^0-free quantum group representations", *J. Algebra* **642** (2024), 330–366. MR Zbl

[Damiani 2000] I. Damiani, "The R-matrix for (twisted) affine quantum algebras", pp. 89–144 in *Representations and quantizations* (Shanghai, 1998), edited by J. Wang and Z. Lin, China High. Educ. Press, Beijing, 2000. MR Zbl

[Damiani 2012] I. Damiani, "Drinfeld realization of affine quantum algebras: the relations", *Publ. Res. Inst. Math. Sci.* **48**:3 (2012), 661–733. MR Zbl

[Damiani 2015] I. Damiani, "From the Drinfeld realization to the Drinfeld–Jimbo presentation of affine quantum algebras: injectivity", *Publ. Res. Inst. Math. Sci.* **51**:1 (2015), 131–171. MR Zbl

[Drinfeld 1985] V. G. Drinfeld, "Hopf algebras and the quantum Yang–Baxter equation", *Dokl. Akad. Nauk SSSR* **283**:5 (1985), 1060–1064. In Russian. MR Zbl

[Drinfeld 1987] V. G. Drinfeld, "A new realization of Yangians and of quantum affine algebras", *Dokl. Akad. Nauk SSSR* **296**:1 (1987), 13–17. In Russian. MR Zbl

[Futorny et al. 2015] V. Futorny, J. Hartwig, and E. Wilson, "Irreducible completely pointed modules of quantum groups of type A", *J. Algebra* **432** (2015), 252–279. MR Zbl

[Grantcharov and Serganova 2006] D. Grantcharov and V. Serganova, "Category of $\mathfrak{sp}(2n)$-modules with bounded weight multiplicities", *Mosc. Math. J.* **6**:1 (2006), 119–134, 222. MR Zbl

[Grantcharov and Serganova 2010] D. Grantcharov and V. Serganova, "Cuspidal representations of $\mathfrak{sl}(n+1)$", *Adv. Math.* **224**:4 (2010), 1517–1547. MR Zbl

[Hayashi 1990] T. Hayashi, "q-analogues of Clifford and Weyl algebras—spinor and oscillator representations of quantum enveloping algebras", *Comm. Math. Phys.* **127**:1 (1990), 129–144. MR Zbl

[Hernandez 2005] D. Hernandez, "Representations of quantum affinizations and fusion product", *Transform. Groups* **10**:2 (2005), 163–200. MR Zbl

[Jang et al. 2023] I.-S. Jang, J.-H. Kwon, and E. Park, "Unipotent quantum coordinate ring and prefundamental representations for types $A_n^{(1)}$ and $D_n^{(1)}$", *Int. Math. Res. Not.* **2023**:2 (2023), 1119–1172. MR Zbl

[Jimbo 1985] M. Jimbo, "A q-difference analogue of $U(\mathfrak{g})$ and the Yang–Baxter equation", *Lett. Math. Phys.* **10**:1 (1985), 63–69. MR Zbl

[Jing 1998] N. Jing, "On Drinfeld realization of quantum affine algebras", pp. 195–206 in *The Monster and Lie algebras* (Columbus, OH, 1996), edited by J. Ferrar and K. Harada, Ohio State Univ. Math. Res. Inst. Publ. **7**, de Gruyter, Berlin, 1998. MR Zbl

[Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge Univ. Press, 1990. MR Zbl

[Kuniba 2018] A. Kuniba, "Tetrahedron equation and quantum R matrices for q-oscillator representations mixing particles and holes", *Symm. Integr. Geom. Methods Appl.* **14** (2018), art. id. 067. MR Zbl

[Kuniba and Okado 2013] A. Kuniba and M. Okado, "Tetrahedron equation and quantum R matrices for infinite-dimensional modules of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$", *J. Phys. A* **46**:48 (2013), art. id. 485203. MR Zbl

[Kuniba and Okado 2015] A. Kuniba and M. Okado, "Tetrahedron equation and quantum R matrices for q-oscillator representations of $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ and $U_q(D_{n+1}^{(2)})$", *Comm. Math. Phys.* **334**:3 (2015), 1219–1244. MR Zbl

[Kuniba et al. 2015] A. Kuniba, M. Okado, and S. Sergeev, "Tetrahedron equation and quantum R matrices for modular double of $U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$ and $U_q(C_n^{(1)})$", *Lett. Math. Phys.* **105**:3 (2015), 447–461. MR Zbl

[Kwon and Lee 2023] J.-H. Kwon and S.-M. Lee, "Affinization of q-oscillator representations of $U_q(\mathfrak{gl}_n)$", *Lett. Math. Phys.* **113**:3 (2023), art. id. 58. MR Zbl

[Lusztig 1990] G. Lusztig, "Quantum groups at roots of 1", *Geom. Dedicata* **35**:1-3 (1990), 89–113. MR Zbl

[Lusztig 1993] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics **110**, Birkhäuser, Boston, 1993. MR Zbl

[Mathieu 2000] O. Mathieu, "Classification of irreducible weight modules", *Ann. Inst. Fourier* (*Grenoble*) **50**:2 (2000), 537–592. MR Zbl

[Mukhin and Young 2014] E. Mukhin and C. A. S. Young, "Affinization of category $\mathcal{O}$ for quantum groups", *Trans. Amer. Math. Soc.* **366**:9 (2014), 4815–4847. MR Zbl

[Nilsson 2016] J. Nilsson, "$\mathcal{U}(\mathfrak{h})$-free modules and coherent families", *J. Pure Appl. Algebra* **220**:4 (2016), 1475–1488. MR Zbl

Received February 27, 2023. Revised December 15, 2023.

XINGPENG LIU
SHENZHEN INTERNATIONAL CENTER FOR MATHEMATICS
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
SHENZHEN
CHINA

xpliu127@ustc.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.283

DIFFERENTIAL GEOMETRIC APPROACH TO THE DEFORMATION OF A PAIR OF COMPLEX MANIFOLDS AND HIGGS BUNDLES

TAKASHI ONO

Let X be a complex manifold and (E, θ) be a Higgs bundle over X. We study the deformation of the triple (X, E, θ). We introduce the differential graded Lie algebra (DGLA) which governs the deformation. We construct the Kuranishi family of it and prove it contains all the information of small deformations of (X, E, θ).

1. Introduction

Let X be a complex manifold and $(E, \bar{\partial}_E)$ be a holomorphic vector bundle on it. Let $\bar{\partial}_{\mathrm{End}(E)}$ be the natural holomorphic structure on $\mathrm{End}(E)$ induced by E. Let $A^{1,0}(\mathrm{End}(E))$ be the smooth sections of $\mathrm{End}(E) \otimes \Omega^{1,0}$. A Higgs field θ on $(E, \bar{\partial}_E)$ is an additional structure on E such that $\theta \in A^{1,0}(\mathrm{End}(E))$, $\bar{\partial}_{\mathrm{End}(E)}\theta = 0$ and the integrability condition $\theta \wedge \theta = 0$ is satisfied. The Higgs field was introduced in [Hitchin 1987] for the Riemann surfaces case and generalized to the higher dimensional case in [Simpson 1988]. We call a triple (X, E, θ) a *holomorphic-Higgs triple*.

We study the deformation of holomorphic-Higgs triples. Our goal is to derive the differential graded Lie algebra (DGLA) which governs the deformation of a given holomorphic-Higgs triple and construct the Kuranishi family of it. For that sake, we apply the Kodaira–Spencer theory [1958a; 1958b; 1960]. The advantage of studying the deformation in the style of Kodaira–Spencer theory is that we can construct the DGLA differential geometrically. Hence we can use the theory of Kuranishi [1965] to construct the Kuranishi space.

There is a lot of interesting work in studying the deformation of pairs of a complex manifold and a holomorphic bundle over it. Such pairs were studied algebraically in [Huybrechts and Thomas 2010; Li 2008; Martinengo 2009; Sernesi 2006], analytically in [Huang 1995; Siu and Trautmann 1981], and in the style of Kodaira–Spencer theory [Chan and Suen 2016].

MSC2020: primary 32G08; secondary 32G05, 58A14.
Keywords: deformation of complex structure, Kuranishi space.

The deformation of holomorphic-Higgs triples was studied algebraically in [Martinengo 2012]. In her work, the DGLA was also obtained. The difference between her work and our work is that she obtained the DGLA purely algebraically while we obtained it differential geometrically.

We first prove the tuple $(L, d_L, [\,\cdot\,,\cdot\,]_L)$ is a DGLA. We show that this is the DGLA which governs the deformation afterwards. We prepare some notation. Let (X, E, θ) be a holomorphic-Higgs triple. Let K be a hermitian metric on E and ∂_K be a (1,0)-part of the Chern connection associated to $\bar\partial_E$ and K, and for $\phi \in A^{0,i}(TX)$, we define $\{\partial_K, \phi\lrcorner\} := \partial_K(\phi\lrcorner) + (-1)^i \phi\lrcorner\partial_K$. Let $\partial_K^{\mathrm{End}(E)} : A^0(\mathrm{End}(E)) \to A^{1,0}(\mathrm{End}(E))$ be the differential operator induced by ∂_K. Let $[\,\cdot\,,\cdot\,]$ be the standard Lie bracket on $A^*(\mathrm{End}(E)) = \bigoplus_i A^i(\mathrm{End}(E))$ and $[\,\cdot\,,\cdot\,]_{\mathrm{SN}}$ be the standard Schouten–Nijenhuys bracket on $A^{0,*}(TX) = \bigoplus_i A^{0,i}(TX)$.

Theorem 1.1 (Theorem 3.1). *Let* $L^i = \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}(E)) \oplus A^{0,i}(TX)$ *and* $L := \bigoplus_i L^i$. *Let* $(A, \phi) \in L^i$ *and* $(B, \psi) \in L^j$. *We set*

$$[(A,\phi),(B,\psi)]_L := \Big((-1)^i\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}A - [A,B], [\phi,\psi]_{\mathrm{SN}}\Big)$$

We define $B_K \in A^{0,1}(\mathrm{Hom}(TX, \mathrm{End}(E))$ *and* $\mathbb{C}$*-linear map* $C_K : A^{0,p}(TX) \to A^{1,p}(\mathrm{End}(E))$ *such that they act on* $v \in A^{0,p}(TX)$ *as*

$$B_K(v) := (-1)^p v\lrcorner F_{d_K}, \quad C_K(v) := \{\partial_K^{\mathrm{End}(E)}, v\lrcorner\}\theta.$$

We define $d_L : L \to L$ *as,*

$$d_L := \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} + \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix}.$$

Then $(L, d_L, [\,\cdot\,,\cdot\,]_L)$ *is a DGLA.*

This DGLA is the DGLA which governs the deformation of the holomorphic-Higgs triple. Actually, we have the following.

Theorem 1.2 (see Theorem 3.6 for precise statement). *Let* $(A, \phi) \in L^1$. *Then* (A, ϕ) *defines a holomorphic-Higgs triple if and only if* (A, ϕ) *satisfies the Maurer–Cartan equation*

$$d_L(A,\phi) - \tfrac{1}{2}[(A,\phi),(A,\phi)]_L = 0.$$

Since the governing DGLA is constructed differential geometrically, We can apply the technique of [Kodaira and Spencer 1958a; 1958b; 1960; Kuranishi 1965] to construct the universal family (= *Kuranishi family*) for a triple (X, E, θ).

Let Δ_L be the Laplacian induced by d_L. Since Δ_L is an elliptic operator, $\mathbb{H}^i := \ker(\Delta_L : L^i \to L^i)$ is finite dimensional. Let $\{\eta_1, \ldots, \eta_n\}$ be a basis of $\mathbb{H}^1$. Let d_L^* be the formal adjoint of d_L w.r.t. the L^2 metric, $H : L^i \to \mathbb{H}^i$ be the projection and

G be the Green's operator associated to Δ_L. The next result is based on [Kuranishi 1965].

Proposition 1.1 (Propositions 4.1 and 4.2). *Let $t = (t_1, \ldots, t_n) \in \mathbb{C}^n$ and $\epsilon_1(t) := \sum_i t_i \eta_i$. For all $|t| \ll 1$ we have a $\epsilon(t)$ such that $\epsilon(t)$ satisfies the equation*

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

Moreover, $\epsilon(t)$ is holomorphic respect to the variable t and $\epsilon(t)$ satisfies the Maurer–Cartan equation

$$d_L \epsilon(t) - \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L = 0$$

if and only if $H[\epsilon(t), \epsilon(t)]_L = 0$.

Let $\Delta \subset \mathbb{C}^n$ be a ball small enough so that $\epsilon(t)$ is holomorphic on Δ. We define $\mathcal{S} \subset \Delta$ as

$$\mathcal{S} := \{t \in \Delta \mid H[\epsilon(t), \epsilon(t)]_L = 0\}.$$

$\mathcal{S}$ might not be smooth, however, it is a complex analytic space. Let $X_{\epsilon(t)}$, $E_{\epsilon(t)}$, $\theta_{\epsilon(t)}$ be the complex manifold, the holomorphic bundle, and the Higgs field which $\epsilon(t)$ defines, respectively. By combining the above results we have a family of holomorphic-Higgs triple $\{(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})\}_{t \in \mathcal{S}}$. We call this family the *Kuranishi family* of (X, E, θ) and $\mathcal{S}$ the *Kuranishi space*.

We recall some properties of the Kuranishi family and space for a compact complex manifold X. Kuranishi [1965] constructed the Kuranishi family and space for arbitrary compact complex manifold and proved the semiuniversality: any other deformation of X is obtained by the pullback of the Kuranishi family. Hence the Kuranishi family contains all the information of small deformations of X. We prove the abbreviated version of the semiuniversality of Kuranishi space. We show that $\{(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})\}_{t \in \mathcal{S}}$ has all the information of small deformations of (X, E, θ).

Let $|\cdot|_k$ be the k-th Sobolev norm on L^1. We assume $k \gg 1$.

Theorem 1.3 (Theorem 4.2). *Let (X, E, θ) be a holomorphic-Higgs triple. Let $\mathcal{S}$ be a Kuranishi family for (X, E, θ). Let $\eta \in L^1$ be a Maurer–Cartan element. If $|\eta|_k$ is small enough, then there is a $t \in \mathcal{S}$ such that $(X_\eta, E_\eta, \theta_\eta)$ is isomorphic to $(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})$ (see Section 4 for the meaning of isomorphic).*

Some applications of Theorem 1.3. Higgs bundles play a core role in the nonabelian Hodge correspondence. Let X be a compact Kähler manifold. The nonabelian Hodge correspondence states there is a one-to-one correspondence in the following objects on X: semisimple representations of the fundamental group of X, flat bundles with a harmonic metric (a.k.a. harmonic bundle), and polystable Higgs bundles with vanishing Chern classes. Here, a harmonic metric is a metric of a flat bundle such that it induces a harmonic map from X to a certain homogenous space.

This correspondence for Riemann surfaces was proved by Hitchin [1987], and the higher dimensional case by Simpson [1988; 1992].

In [Ono 2023], we study the structure of the Kuranishi space for a holomorphic-Higgs triple (X, E, θ) such that X is a compact Kähler manifold and (E, θ) is a polystable Higgs bundle with vanishing Chern classes. We show that the Kuranishi space of such a holomorphic-Higgs triple is isomorphic to the product of the Kuranishi space of X and the Kuranishi space of the Higgs bundle. This predicts that once we construct the moduli space which parametrizes a pair of a Kähler manifold and a polystable Higgs bundle with 0 Chern classes, it locally splits into the moduli space of the Kähler manifold and the moduli space of the Higgs bundle. This phenomenon is interesting since we cannot expect such decomposition globally.

Plan of the paper. In Section 2, we define and study the deformation of holomorphic-Higgs triples. We prove the Newlander–Nirenberg-type theorem in this context (Proposition 2.6). In Section 3, we construct the DGLA which governs the deformation of the holomorphic-Higgs triple. In Section 4, we apply the work of Kuranishi and construct the Kuranishi space for a given holomorphic-Higgs triple and prove its local completeness.

2. Deformation of holomorphic-Higgs triple

For a smooth manifold X, we define $A^p(X)$ to be a space of smooth p-forms on X, and for a smooth vector bundle $E \to X$, we define $A^p(E)$ to be a space of smooth p-forms which take values in E.

Definition 2.1. Let X be a compact complex manifold. Let $\bar{\partial}_{\mathrm{End}(E)}$ be the complex structure on $\mathrm{End}(E)$ induced by E. A Higgs bundle (E, θ) over X is a pair such that

- E is a holomorphic bundle over X,
- θ is a Higgs field such that $\theta \in A^{1,0}(\mathrm{End}(E))$, $\bar{\partial}_{\mathrm{End}(E)}\theta = 0$, and $\theta \wedge \theta = 0$.

We call a triple (X, E, θ) a holomorphic-Higgs triple.

We fix a metric K on E and assume X to be compact throughout this paper.

Definition 2.2. Let (X, E, θ) be a holomorphic-Higgs triple. A family of deformations of holomorphic-Higgs triples $(\mathcal{X}, \mathcal{E}, \Theta)$ over a small ball Δ centered at the origin of $\mathbb{C}^d$ consists of a complex manifold $\mathcal{X}$, a proper submersive holomorphic map

$$\pi : \mathcal{X} \to \Delta$$

and a Higgs bundle $(\mathcal{E}, \Theta)$ over $\mathcal{X}$ such that $\pi^{-1}(0) = X$, $\mathcal{E}|_{\pi^{-1}(0)} = E$, $\Theta|_{\pi^{-1}(0)} = \theta$.

By Ehresmann's theorem and as in [Kodaira 1986, Chapter 7, Lemma 7.1], if we choose Δ small enough, we have maps $F : X \times \Delta \to \mathcal{X}$ and $P : E \times \Delta \to \mathcal{E}$

such that the diagram below commutes, F is a diffeomorphism and P is a smooth bundle isomorphism:

$$\begin{tikzcd} E \times \Delta \arrow[r, "P"] \arrow[d] & \mathcal{E} \arrow[d] \\ X \times \Delta \arrow[r, "F"] \arrow[d] & \mathcal{X} \arrow[dl] \\ \Delta & \end{tikzcd}$$

We can induce a complex structure on $X \times \{t\}$ and a Higgs bundle structure on $E \times \{t\}$ using $F|_{X\times\{t\}} : X \times \{t\} \to \mathcal{X}_t = \pi^{-1}(t)$ and $P|_{E\times\{t\}} : E \times \{t\} \to \mathcal{E}|_{\pi^{-1}(t)}$. We denote this family of holomorphic-Higgs triple $\{(X_t, E_t, \theta_t)\}_{t\in\Delta}$.

Since $\{X_t\}_{t\in\Delta}$ is a deformation of the complex manifold X, we have a family of Maurer–Cartan element $\{\phi_t\}_{t\in\Delta}$ such that each ϕ_t determines the complex structure of X_t.

Let $A^{1,0}(X_t) := \{\alpha \in A^1(X) \mid \alpha \text{ is a (1,0)-form of } X_t\}$, $\pi_X^{1,0} : A^1(X) \to A^{1,0}(X)$, and $\pi_X^{0,1} : A^1(X) \to A^{0,1}(X)$ be the natural projection.

Lemma 2.1. *$\alpha \in A^{1,0}(X_t)$ if and only if $\pi_X^{0,1}(\alpha) = \phi_t \lrcorner \pi_X^{1,0}(\alpha)$.*

Proof. It is enough to prove it locally. Let $x \in X$ and U_x be an open neighborhood of x. Let $(\xi_1, \dots, \xi_n)$, $(z_1, \dots, z_n)$ be local coordinates on U_x and $(\xi_1, \dots, \xi_n)$ be a complex coordinate for X_t and $(z_1, \dots, z_n)$ be a complex coordinate for X.

Let $\alpha = \sum_i f_i d\xi_i$. We have

$$\pi_X^{0,1}(\alpha) = \sum_{i,j} f_i \frac{\partial \xi_i}{\partial \bar{z}_j} d\bar{z}_j \quad \text{and} \quad \pi_X^{1,0}(\alpha) = \sum_{i,j} f_i \frac{\partial \xi_i}{\partial z_j} dz_j.$$

Recall that $\phi_t = \sum_{i,j} \phi_{t,j}^i \frac{\partial}{\partial z_i} \otimes d\bar{z}_j$, $(\phi_{t,j}^i) = \bigl(\frac{\partial \xi_i}{\partial z_k}\bigr)^{-1} \bigl(\frac{\partial \xi_k}{\partial \bar{z}_j}\bigr)$. See [Kodaira 1986] for more details.

Hence

$$\begin{aligned} \phi_t \lrcorner \pi_X^{1,0}(\alpha) &= \Bigl(\sum_{j,k} \phi_{t,k}^j \frac{\partial}{\partial z_j} \otimes d\bar{z}_k\Bigr) \lrcorner \Bigl(\sum_{i,j} f_i \frac{\partial \xi_i}{\partial z_j} dz_j\Bigr) \\ &= \sum_{i,j,k} f_i \frac{\partial \xi_i}{\partial z_j} \phi_{t,k}^j d\bar{z}_k \\ &= \sum_{i,k} f_i \frac{\partial \xi_i}{\partial \bar{z}_k} d\bar{z}_k = \pi_X^{0,1}(\alpha). \end{aligned}$$

To prove the converse, we only have to prove that if $\omega \in A^{0,1}(X_t)$ and $\pi_X^{0,1}(\omega) = \phi_t \lrcorner \pi_X^{1,0}(\omega)$ stands then $\omega = 0$. Let $\omega = \sum_i h_i d\bar{\xi}_i$ and assume $\pi_X^{0,1}(\omega) = \phi_t \lrcorner \pi_X^{1,0}(\omega)$.

We have $\pi_X^{0,1}(\omega) = \sum_{i,j} h_i \frac{\partial \bar{\xi}_i}{\partial \bar{z}_j} d\bar{z}_j$ and $\phi_t \lrcorner \pi_X^{1,0}(\omega) = \sum_{i,j,k} h_i \frac{\partial \bar{\xi}_i}{\partial z_j} \phi_{t,k}^j d\bar{z}_k$. Since $\pi_X^{0,1}(\omega) = \phi_t \lrcorner \pi_X^{1,0}(\omega)$, we have

$$0 = \pi_X^{0,1}(\omega) - \phi_t \lrcorner \pi_X^{1,0}(\omega) = \sum_k \left\{ \sum_i h_i \left(\frac{\partial \bar{\xi}_i}{\partial \bar{z}_k} - \sum_j \frac{\partial \bar{\xi}_i}{\partial z_j} \phi_{t,k}^j \right) \right\} d\bar{z}_k.$$

Hence

$$\left(\frac{\partial \bar{\xi}_i}{\partial \bar{z}_k} - \sum_j \frac{\partial \bar{\xi}_i}{\partial z_j} \phi_{t,k}^j \right) (h_1, \ldots, h_n)^T = 0.$$

Since ϕ_t defines a near complex structure with respect to the original one

$$\det \left(\frac{\partial \bar{\xi}_i}{\partial \bar{z}_k} - \sum_j \frac{\partial \bar{\xi}_i}{\partial z_j} \phi_{t,k}^j \right) \neq 0.$$

Hence $(h_1, \ldots, h_n) = 0$. This implies $\omega = 0$. □

Lemma 2.2. *Let $\alpha \in A^{1,0}(X_t)$. α is a holomorphic 1-form on X_t if and only if $(\bar{\partial} + l_{\psi_t}) \pi_X^{1,0}(\alpha) = 0$. Here $l_{\psi_t} = \partial(\psi_t \lrcorner) - \psi_t \lrcorner \partial$.*

Proof. As in Lemma 2.1, we only have to prove it locally. We use the notation in the proof of Lemma 2.1.

Let $\alpha = \sum_i f_i d\xi_i$ and let $\alpha^{1,0} = \pi_X^{1,0}(\alpha)$. We first calculate $(\bar{\partial} + l_{\psi_t})(\alpha^{1,0})$. Since $\alpha^{1,0} = \sum_{i,j} f_i \frac{\partial \xi_i}{\partial z_j} dz_j$, we have

$$\bar{\partial}\alpha^{1,0} = \bar{\partial} \left(\sum_{i,j} f_i \frac{\partial \xi_i}{\partial z_j} dz_j \right) = \sum_{i,j,k} \left\{ \frac{\partial f_i}{\partial \bar{z}_k} \frac{\partial \xi_i}{\partial z_j} + f_i \frac{\partial^2 \xi_i}{\partial \bar{z}_k \partial z_j} \right\} d\bar{z}_k \wedge dz_j,$$

$$\begin{aligned} l_{\phi_t}(\alpha^{1,0}) &= l_{\phi_t} \left(\sum_{i,j} f_i \frac{\partial \xi_i}{\partial z_j} dz_j \right) \\ &= \partial \left(\phi_t \lrcorner \sum_{i,j} f_i \frac{\partial \xi_i}{\partial z_j} dz_j \right) - \phi_t \lrcorner \left\{ \sum_{i,j,k} \left(\frac{\partial f_i}{\partial z_k} \frac{\partial \xi_i}{\partial z_j} + f_i \frac{\partial^2 \xi_i}{\partial z_k \partial z_j} \right) dz_k \wedge dz_j \right\} \\ &= \partial \left(\sum_{i,j,k} f_i \frac{\partial \xi_i}{\partial z_j} \phi_{t,k}^j d\bar{z}_k \right) - \phi_t \lrcorner \left\{ \sum_{i,j,k} \left(\frac{\partial f_i}{\partial z_k} \frac{\partial \xi_i}{\partial z_j} \right) dz_k \wedge dz_j \right\}, \end{aligned}$$

and

$$\begin{aligned} \partial \left(\sum_{i,j,k} f_i \frac{\partial \xi_i}{\partial z_j} \phi_{t,k}^j d\bar{z}_k \right) &= \partial \left(\sum_{i,k} f_i \frac{\partial \xi_i}{\partial \bar{z}_k} d\bar{z}_k \right) \\ &= \sum_{i,j,k} \frac{\partial f_i}{\partial z_j} \frac{\partial \xi_i}{\partial \bar{z}_k} dz_j \wedge d\bar{z}_k + \sum_{i,j,k} f_i \frac{\partial^2 \xi_i}{\partial \bar{z}_k \partial z_j} dz_j \wedge d\bar{z}_k \\ &= \sum_{i,j,k} \frac{\partial f_i}{\partial z_k} \frac{\partial \xi_i}{\partial z_j} \sum_l \phi_{t,l}^k d\bar{z}_l \wedge dz_j - \sum_{i,j,k} \frac{\partial f_i}{\partial z_k} \frac{\partial \xi_i}{\partial z_j} \sum_l \phi_{t,l}^j d\bar{z}_l \wedge dz_k \\ &= \sum_{i,j,k,l} \frac{\partial f_i}{\partial z_k} \frac{\partial \xi_i}{\partial z_j} \phi_{t,l}^k d\bar{z}_l \wedge dz_j - \sum_{i,k,l} \frac{\partial f_i}{\partial z_k} \frac{\partial \xi_i}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_k. \end{aligned}$$

Hence

$$
\begin{aligned}
(1)\quad (\bar\partial + l_{\phi_t})(\alpha^{1,0}) &= \sum_{i,j,k} \frac{\partial f_i}{\partial \bar z_k}\frac{\partial \xi_i}{\partial z_j} d\bar z_k \wedge dz_j + \sum_{i,j,k} f_i \frac{\partial^2 \xi_i}{\partial \bar z_k \partial z_j} d\bar z_k \wedge dz_j \\
&\quad + \sum_{i,j,k} \frac{\partial f_i}{\partial z_j}\frac{\partial \xi_i}{\partial \bar z_k} dz_j \wedge d\bar z_k + \sum_{i,j,k} f_i \frac{\partial^2 \xi_i}{\partial \bar z_k \partial z_j} dz_j \wedge d\bar z_k \\
&\quad - \sum_{i,j,k,l} \frac{\partial f_i}{\partial z_k}\frac{\partial \xi_i}{\partial z_j}\phi^k_{t,l} d\bar z_l \wedge dz_j + \sum_{i,k,l} \frac{\partial f_i}{\partial z_k}\frac{\partial \xi_i}{\partial \bar z_l} d\bar z_l \wedge dz_k \\
&= \sum_{i,j,l} \frac{\partial f_i}{\partial \bar z_l}\frac{\partial \xi_i}{\partial z_j} d\bar z_l \wedge dz_j - \sum_{i,j,k,l} \frac{\partial f_i}{\partial z_k}\frac{\partial \xi_i}{\partial z_j}\phi^k_{t,l} d\bar z_l \wedge dz_j \\
&= \sum_{j,l}\sum_i \frac{\partial \xi_i}{\partial z_j}\left(\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t.l}\right) d\bar z_l \wedge dz_j.
\end{aligned}
$$

If we assume α to be a holomorphic 1-form on X_t, this implies that $\{f_i\}_i$ are holomorphic functions on X_t. Hence we have

$$
\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t,l} = 0.
$$

Hence by (1), when α is a holomorphic 1-form on X_t, $(\bar\partial + l_{\phi_t})(\alpha^{1,0}) = 0$.

Conversely, if we assume $(\bar\partial + l_{\phi_t})(\alpha^{1,0}) = 0$, by (1) we have

$$
\frac{\partial \xi_i}{\partial z_j}\left(\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t,l}\right) = 0.
$$

Since ϕ_t defines a near complex structure to X, we have $\det\left(\frac{\partial \xi_i}{\partial z_j}\right) \neq 0$. Hence $\frac{\partial f_i}{\partial \bar z_l} - \sum_k \frac{\partial f_i}{\partial z_k}\phi^k_{t,l} = 0$. This shows that $\{f_i\}$ are holomorphic function on X_t and α is a holomorphic 1-form on X_t. □

By Lemma 2.1, θ_t can be decomposed as $\theta_t = \omega_t + \phi_t \lrcorner \omega_t$, where $\omega_t = \pi_X^{1,0}(\theta_t)$. We define an operator $\bar D_t : A^0(E) \to A^1(E)$ as

$$
\bar D_t(s) = \bar D_t(s^k e_k) := (\partial + l_{\phi_t}) s^k \otimes e_k + \omega_t \wedge s, \quad s \in A^0(E).
$$

Here, $\{e_k\}$ is a local holomorphic frame of E_t and we used the Einstein summation rule.

Proposition 2.1. *$\bar D_t$ is a well defined operator, that is, $\bar D_t$ is independent of the holomorphic frame of E_t. Also $\bar D_t$ satisfies the Leibniz rule:*

$$
\bar D_t(\alpha \wedge s) = (\bar\partial + l_{\phi_t})\alpha \otimes s + (-1)^p \alpha \wedge \bar D_t(s)
$$

for every $\alpha \in A^p(X)$ and $s \in A^0(E)$.

Proof. To prove well-definedness, we need to show that $\overline{D}_t$ is independent of the choice of a local holomorphic frame $\{e_k\}$ of E_t. Take another local holomorphic frame $\{f_j\}$ of E_t. Let h^k_j be a holomorphic function of X_t such that $f_j = h^k_j e_j$. Then for local section $s \in A(E)$, $s = \tilde{s}^j f_j = s^k e_k$, we have $\tilde{s}_j h^k_j = s_k$, thus we have

$$\begin{aligned}\overline{D}_t(\tilde{s}^j f_j) &= (\bar{\partial} + l_{\phi_t})\tilde{s}^j \otimes f_j + \omega_t \wedge (\tilde{s}^j f_j)\\ &= (\bar{\partial} + l_{\phi_t})\tilde{s}^j \otimes h^k_j e_k + \omega_t \wedge s\\ &= (\bar{\partial} + l_{\phi_t})(\tilde{s}^j h^k_j) \otimes e_k + \omega_t \wedge (s^k e_k)\\ &= (\bar{\partial} + l_{\phi_t})(s^k) \otimes e_k + \omega_t (s^k e_k)\\ &= \overline{D}_t(s^k e_k).\end{aligned}$$

Hence $\overline{D}_t$ is well defined.

The Leibniz rule for $\overline{D}_t$ follows from the fact that $\alpha \in A^p(X)$, $\beta \in A^q(X)$, $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ stands and $\bar{\partial} + l_{\phi_t}$ satisfies the Leibniz rule:

$$(\bar{\partial} + l_{\phi_t})(\alpha \wedge \beta) = (\bar{\partial} + l_{\phi_t})(\alpha) \wedge \beta + (-1)^p \alpha \wedge (\bar{\partial} + l_{\phi_t})(\beta). \qquad \square$$

Proposition 2.2. $\overline{D}_t^2 = 0$.

Proof. We calculate $\overline{D}_t^2$ locally and show $\overline{D}_t^2 = 0$. Since $\overline{D}_t$ satisfies the Leibniz rule, we only have to prove $(\bar{\partial} + l_{\phi_t})^2 = 0$ and $\overline{D}_t^2(s) = 0$ for $s \in A^0(E)$.

First we prove $(\bar{\partial} + l_{\phi_t})^2 = 0$. According to [Martinengo 2012], we have

$$(\bar{\partial} + l_{\phi_t})^2 = l_{\bar{\partial}_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t]}. \tag{2}$$

Since ϕ_t is a Maurer–Cartan element, we have $\bar{\partial}_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t] = 0$. Hence $(\bar{\partial} + l_{\phi_t})^2 = 0$.

Next we prove $\overline{D}_t^2(s) = 0$ for $s \in A^0(E)$. Let $\{e_k\}$ be a holomorphic frame for E_t. Assume that s and ω_t has a trivialization as $s = s^k e_k$ and $\omega_t = g_i dz_i$, $g_i = (a^s_{i,t})$ respect to the frame $\{e_k\}$. Here $s^k, a^s_{i,t} \in A^0(X)$ and $g_i \in A^0(\mathrm{End}(E))$. Since $\omega_t = \pi_X^{1,0}(\theta_t)$ and θ_t is a Higgs field we have $\omega_t \wedge \omega_t = 0$. Applying Lemma 2.2 and the fact that $\overline{D}_t$ satisfies the Leibniz rule, we have

$$\begin{aligned}\overline{D}_t^2(s) &= \overline{D}_t^2(s^k \otimes e_k)\\ &= \overline{D}_t((\bar{\partial}+l_{\phi_t})s^k \otimes e_k + \omega_t \wedge s)\\ &= (\bar{\partial}+l_{\phi_t})^2 s^k \otimes e_k + \omega_t \wedge (\bar{\partial}+l_{\phi_t})s^k \otimes e_k + (\bar{\partial}+l_{\phi_t})(a^s_{i,k} s^k dz_i) \otimes e_s \omega_t \wedge \omega_t \wedge s\\ &= \omega_t \wedge (\bar{\partial}+l_{\phi_t})s^k \otimes e_k + (\bar{\partial}+l_{\phi_t})(a^s_{i,k} dz_i) \wedge s^k \otimes e_s\\ &\qquad - \omega_t \wedge (\bar{\partial}+l_{\phi_t})s^k \otimes e_k + \omega_t \wedge \omega_t \wedge s\\ &= 0.\end{aligned}$$

Since $s \in A(E)$ is an arbitrary smooth section, this proves the claim. $\square$

Proposition 2.3. *We define $A_t := \overline{D}_t - \bar\partial_E - \{\partial_K, \phi_t\} - \theta$. Then $A_t \in A^1(\mathrm{End}(E))$. Here ∂_K is a (1,0)-part of the Chern connection which is uniquely determined by $\bar\partial_E$ and the hermitian metric K. $\{\partial_K, \phi_t \lrcorner\}$ is the operator such that $\{\partial_K, \phi_t \lrcorner\} = \partial_K(\phi_t \lrcorner) - \phi_t \lrcorner \partial_K$.*

Proof. Let $f \in A^0(X)$ and $s \in A^0(E)$. Using the Leibniz rule and the fact that the contraction is only taken in the (1,0)-part, we have

$$\begin{aligned}
A_t(fs) &= (\bar\partial + l_{\phi_t}) f \otimes s + f\overline{D}_t(s) - \bar\partial f \otimes s - f\bar\partial_E s + \phi_t \lrcorner \partial_K(fs) - \theta \wedge (fs) \\
&= (\bar\partial - \phi_t \lrcorner \partial) f \otimes s + f\overline{D}_t(s) - \bar\partial f \otimes s - f\bar\partial_E s + \phi_t \lrcorner (\partial f \otimes s + f\partial_K s) - f\theta \wedge s \\
&= f(\overline{D}_t - \bar\partial_E - \{\partial_K, \phi_t\} - \theta)s \\
&= fA_t(s).
\end{aligned}$$

This shows that $A_t \in A^1(\mathrm{End}(E))$. □

We summarize the results so far.

Proposition 2.4. *Let (X, E, θ) be a holomorphic-Higgs triple. Let $(\mathcal{X}, \mathcal{E}, \Theta)$ be a deformation family of (X, E, θ) over Δ and $\{(X_t, E_t, \theta_t)\}_{t\in\Delta}$ be the family obtained from $(\mathcal{X}, \mathcal{E}, \Theta)$. Combining ϕ_t and θ_t, we can construct a well-defined differential operator $\overline{D}_t$ such that $(\overline{D}_t)^2 = 0$. Let $A_t := \overline{D}_t - \bar\partial_E - \{\partial_K, \phi_t \lrcorner\} - \theta$. Then $A_t \in A^1(\mathrm{End}(E))$.*

We want the converse of the above proposition. Suppose we have a given smooth family $A_t \in A^{0,1}(\mathrm{End}E)$, $B_t \in A^{1,0}(\mathrm{End}E)$ and $\phi_t \in A^{0,1}(TX)$ parametrized by $t \in \Delta$.

We define the operator $\overline{D}_t : A^0(E) \to A^1(E)$ as

$$\overline{D}_t := \bar\partial_E + \{\partial_K, \phi_t \lrcorner\} + A_t + \theta + B_t.$$

We extend $\overline{D}_t$ to $A^p(E)$ in an obvious way so that it satisfies the Leibniz rule:

$$\overline{D}_t(\alpha \otimes s) = (\bar\partial + l_{\phi_t})\alpha \otimes s + (-1)^p \alpha \wedge \overline{D}_t(s).$$

We want to show that if $\overline{D}_t^2 = 0$, (A_t, B_t, ϕ_t) defines a holomorphic-Higgs triple (X_t, E_t, θ_t). First of all, we have:

Proposition 2.5. *If $\overline{D}_t^2 = 0$, ϕ_t defines a holomorphic structure on X. We denote this complex manifold by X_t.*

Proof. Let $f \in A^0(X)$ and $s \in A^0(E)$. Since $\overline{D}_t^2 = 0$, it satisfies the Leibniz rule:

$$0 = \overline{D}_t^2(f \otimes s) = (\bar\partial + l_{\phi_t})^2 f \otimes s.$$

Since f and s are arbitrary function and section, we have $(\bar\partial + l_{\phi_t})^2 = 0$. By (2), we have

$$0 = (\bar\partial + l_{\phi_t})^2 = l_{\bar\partial_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t]}.$$

Hence $\bar{\partial}_{TX}\phi_t - \frac{1}{2}[\phi_t, \phi_t] = 0$. Hence ϕ_t defines a integrable complex structure on X. □

Next, we show that E admits a holomorphic structure over X_t and we can induce a Higgs field on it. Let us define $\overline{D}'_t : A^0(E) \to A^{0,1}(E)$ as $\overline{D}'_t := \bar{\partial}_E + \{\partial_K, \phi_t \lrcorner\} + A_t$. We remark that $\overline{D}_t = \overline{D}'_t + \theta + B_t$. The next claim was proved in [Moroianu 2007].

Lemma 2.3. $\ker(\overline{D}'_t)$ *generates* $A^0(E)$ *locally.*

Proof. See the proof of [Chan and Suen 2016, Lemma 3.11.]. □

The above lemma tells us that for every $x \in X$ we have an open neighborhood U of x and a frame $\{e_k\}$ on U such that $\{e_k\} \subset \ker(\overline{D}'_t)$. Let $\{e_k\}$ be a local frame of E such that $\{e_k\} \subset \ker(\overline{D}'_t)$. Let $\bar{\partial}_t$ be the Dolbeault operator of X_t. We can then define $\bar{\partial}_{E_t}$ by

$$\bar{\partial}_{E_t}(s^k e_k) := \bar{\partial}_t s^k \otimes e_k.$$

Let $\{f_j\} \subset \ker(\overline{D}'_t)$ be an another local frame of E, then there exist (h^k_j) such that $f_j = h^k_j e_k$. Applying $\overline{D}'_t$, we have

$$\overline{D}'_t(f_j) = \overline{D}'_t(h^k_j e_j) = (\bar{\partial} - \phi_t \lrcorner \partial) h^k_j \otimes e_k.$$

Since e_k is a local frame, we have $(\bar{\partial} - \phi_t \lrcorner \partial) h^k_j = 0$, which is equivalent to $\bar{\partial}_t h^k_j = 0$. We can now check $\bar{\partial}_{E_t}$ is well defined. Let $s \in A^0(E)$ and assume s has local trivialization as $s = \tilde{s}_j f_j = s_k e_k$. Applying $\bar{\partial}_{E_t}$ we have

$$\bar{\partial}_{E_t}(s^k e_k) = \bar{\partial}_t s^k \otimes e_k = \bar{\partial}_t(\tilde{s}_j h^k_j) \otimes e_k = \bar{\partial}_t \tilde{s}_j \otimes h^k_j e_k = \bar{\partial}_t \tilde{s}_j \otimes f_j = \bar{\partial}_{E_t}(\tilde{s}_j f_j).$$

This proves the well-definedness. By definition, $\bar{\partial}_E$ satisfies the Leibniz rule:

$$\bar{\partial}_{E_t}(\alpha \otimes s) = \bar{\partial}_t \alpha \otimes s + (-1)^p \alpha \wedge \bar{\partial}_{E_t} s$$

and $\bar{\partial}^2_{E_t} = 0$ since ϕ_t defines an integral almost complex structure on X. Hence, by the linearized version of the Newlander–Nirenberg theorem, $E_t = (E, \bar{\partial}_{E_t})$ is a holomorphic bundle over X_t.

We want to show next that $\theta_t = \theta + B_t + \phi_t \lrcorner(\theta + B_t)$ is a Higgs field for E_t under the above assertion. By Lemma 2.1, θ_t is a (1,0)-form of X_t which takes value in $\mathrm{End}(E)$.

Let $e_k \subset \ker(\overline{D}'_t)$ be a local frame of E and assume $\theta + B_t$ is written as $\theta + B_t = \sum_i g_i dz_i$ ($g_i \in A^0(\mathrm{End}(E))$) respect to this frame. By Lemma 2.2, to show θ_t is a Higgs field on E_t, it is enough to show $(\bar{\partial} + l_{\phi_t}) g_i dz_i = 0$ and $(\theta + B_t) \wedge (\theta + B_t) = 0$.

Since $\overline{D}_t$ satisfies the Leibniz rule

$$\begin{aligned} 0 = \overline{D}^2_t(e_k) &= \overline{D}_t(\overline{D}_t(e_k)) = \overline{D}_t((\theta + B_t)(e_k)) = \overline{D}_t(g_i dz_i(e_k)) \\ &= (\bar{\partial} + l_{\phi_t})(g_i dz_i) e_k - g_i dz_i \wedge \overline{D}_t(e_k) \\ &= (\bar{\partial} + l_{\phi_t})(g_i dz_i) e_k - (\theta + B_t) \wedge (\theta + B_t)(e_k). \end{aligned}$$

Hence θ_t is a Higgs field for E_t and (X_t, E_t, θ_t) is a holomorphic-Higgs triple. In summary, we have proved the following,

Proposition 2.6. *Suppose we have a given smooth family $A_t \in A^{0,1}(\mathrm{End}(E))$, $B_t \in A^{1,0}(\mathrm{End}(E))$, $\phi_t \in A^{0,1}(TX)$ parametrized by t. If the induced differential operator $\overline{D}_t : A^p(E) \to A^{p+1}(E)$ satisfies $\overline{D}_t^2 = 0$ and the Leibniz rule*

$$\overline{D}_t(\alpha \wedge s) = (\bar{\partial} + l_{\phi_t})\alpha \otimes s + (-1)^p \alpha \wedge \overline{D}_t(s),$$

then E admits a holomorphic structure over the complex manifold X_t, which we denote by E_t, and a Higgs field θ_t such that (X_t, E_t, θ_t) is a holomorphic-Higgs triple.

3. DGLA and the Maurer–Cartan equation

Let us recall the definition of DGLA.

Definition 3.1. A differential graded Lie algebra (DGLA) $(V, [\cdot\,,\cdot], d)$ is the date of $\mathbb{Z}$-graded vector space $L = \bigoplus_{i\in\mathbb{Z}} L^i$ with a bilinear bracket $[\cdot\,,\cdot] : V \times V \to V$ and a linear map d such that:

1. $[a, b] + (-1)^{ij}[b, a] = 0$ for $a \in V^i$, $b \in V^j$.
2. The graded Jacobi identity holds:
$$[a, [b, c]] = [[a, b], c] + (-1)^{ij}[b, [a, c]], \quad a \in V^i,\ b \in V^j,\ c \in V^k.$$
3. $d(V^i) \subset V^{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^i [a, db]$ for $a \in V^i$, $b \in V^j$. The map d is called the differential of V.

We recall the definition of the Maurer–Cartan equation of a DGLA.

Definition 3.2. The Maurer–Cartan equation of a DGLA V is

$$da - \tfrac{1}{2}[a, a] = 0, \quad a \in V^1.$$

The solutions of the Maurer–Cartan equation are called the Maurer–Cartan elements of the DGLA L.

We derive the Maurer–Cartan equation and DGLA which governs the deformation of the holomorphic-Higgs triple. The next proposition is important to construct the DGLA. Before we state it, we introduce some notation. Let $\partial_K^{\mathrm{End}(E)} : A^0(\mathrm{End}(E)) \to A^{1,0}(\mathrm{End}(E))$ be the differential operator induced by ∂_K. Let F_{d_K} be the curvature of the Chern connection. Let the bracket $[\cdot\,,\cdot]$ be the canonical Lie bracket defined on $A^*(\mathrm{End}(E))$ and $[\cdot\,,\cdot]_{\mathrm{SH}}$ be the standard Schouten–Nijenhuys bracket defined on $A^{0,*}(TX)$.

Proposition 3.1. *Suppose we have a $A \in A^{0,1}(\mathrm{End}(E))$, $B \in A^{1,0}(\mathrm{End}(E))$ and $\phi \in A^{0,1}(TX)$. Let $\overline{D}$ be the differential operator defined as*

$$\overline{D} := \bar{\partial}_E + \{\partial_K, \phi\lrcorner\} + \theta + A + B.$$

$\overline{D}^2 = 0$ *holds if and only if the following two equations hold*:

$$\bar\partial_{\mathrm{End}(E)}(A+B) - \phi\lrcorner F_{d_K} + [\theta, A+B] + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\theta + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(A+B) + \tfrac{1}{2}[A+B, A+B] = 0,$$

$$\bar\partial_{TX}\phi - \tfrac{1}{2}[\phi, \phi] = 0.$$

From now on we denote $[\,\cdot\,,\cdot\,]_{\mathrm{SH}}$ as $[\,\cdot\,,\cdot\,]$ if there is no confusion. The proof of the above proposition will be given at the end of the section.

Let us define some notation. Let L^i be $L^i := \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}(E)) \oplus A^{0,i}(TX))$. Let for $\phi \in A^{0,i}(TX)$, $\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\} := \partial_K^{\mathrm{End}(E)}(\phi\lrcorner) + (-1)^i \phi\lrcorner\partial_K^{\mathrm{End}(E)}$. Define the bracket $[\,\cdot\,,\cdot\,]_L : L^i \times L^j \to L^{i+j}$ by

$$[(A,\phi), (B,\psi)]_L := \big((-1)^i\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}A - (-1)^{(i+1)j}\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}B - [A,B], [\phi,\psi]\big).$$

We define $B_K \in A^{0,1}(\mathrm{Hom}(TX, \mathrm{End}(E))$ and the $\mathbb{C}$-linear map $C_K : A^{0,p}(TX) \to A^{1,p}(\mathrm{End}(E))$ such that they act on $v \in A^{0,p}(TX)$ as

$$B_K(v) := (-1)^p v\lrcorner F_{d_K}, \quad C_K(v) := \{\partial_K^{\mathrm{End}(E)}, v\lrcorner\}\theta.$$

We define the linear operator $d_L : L \to L$ as

$$d_L := \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} + \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix}$$

Theorem 3.1. $\big(L = \bigoplus_i L_i, [\,\cdot\,,\cdot\,]_L, d_L\big)$ *is a DGLA.*

We separate the proof of the theorem into the two propositions below. Before going to the proof, we introduce some formulas which are useful for the proof.

Lemma 3.1 [Martinengo 2012, Lemma 3.1]. *Let $i_\xi(\omega) = \xi\lrcorner\omega$ for all $\omega \in A^*(X)$. For every $\xi, \eta \in A^{0,*}(TX)$,*

$$i_{[\xi,\eta]} = [i_\xi, [\partial, i_\eta]], \quad [i_\xi, i_\eta] = 0. \tag{3}$$

We slightly modify Lemma 3.1 so that we can use it in our proof.

Lemma 3.2. *Let X be a complex manifold and E be a holomorphic bundle over X. Let K be a hermitian metric on E and ∂_K be a* (1,0)*-part of the Chern connection. By considering the degree of the differential form of* (3), *for any $\omega \in A^*(E)$ and any $\phi \in A^{0,j}(TX)$ and $\psi \in A^{0,k}(TX)$, we have*

$$[\phi,\psi]\lrcorner\omega = \phi\lrcorner\partial_K(\psi\lrcorner\omega) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner\omega)) - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner\omega) - (-1)^{jk+k}\psi\lrcorner\phi\lrcorner\partial_K\omega.$$

We obtain the corollaries below by applying Lemma 3.2.

Corollary 3.2. *Let $A \in A^i(\mathrm{End}(E))$, $\phi \in A^{0,j}(TX)$ and $\psi \in A^{0,k}(TX)$. Then we have*

$$\begin{aligned}&\{\partial_K^{\mathrm{End}(E)}, [\phi, \psi]\lrcorner\}A \\ &\qquad = \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}A - (-1)^{jk}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}A.\end{aligned}$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

Applying Lemma 3.2 to the left-hand side of the equation we have

$$\begin{aligned}&\{\partial_K, [\phi, \psi]\lrcorner\}A \\ &\quad = \partial_K([\phi, \psi]\lrcorner A) + (-1)^{j+k}[\phi, \psi]\lrcorner\partial_K A \\ &\quad = \partial_K\big\{\phi\lrcorner\partial_K(\psi\lrcorner A) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner A)) \\ &\qquad\qquad - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner A) - (-1)^{jk+k}\psi\lrcorner\phi\lrcorner\partial_K A\big\} \\ &\qquad\qquad\qquad + (-1)^{j+k}\{\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner\partial_K A)) \\ &\qquad\qquad\qquad\qquad - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\} \\ &\quad = \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A)) - (-1)^{jk}\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A)) - (-1)^{jk+k}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A) \\ &\qquad\qquad + (-1)^{j+k}\big\{\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) - (-1)^{jk+k}\partial_K(\psi\lrcorner(\phi\lrcorner\partial_K A)) \\ &\qquad\qquad\qquad - (-1)^{jk}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\big\} \\ &\quad = \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A)) - (-1)^{jk}\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A)) - (-1)^{jk+k}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A) \\ &\qquad\qquad + (-1)^{j+k}\big\{\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) - (-1)^{jk+j}\partial_K(\psi\lrcorner(\phi\lrcorner\partial_K A)) \\ &\qquad\qquad\qquad - (-1)^{jk+j+k}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\big\}.\end{aligned}$$

We apply (3) for the computation of the right-hand side of the equation.

$$\begin{aligned}&\{\partial_K, \phi\lrcorner\}\{\partial_K, \psi\lrcorner\}A - (-1)^{jk}\{\partial_K, \psi\lrcorner\}\{\partial_K, \phi\lrcorner\}A \\ &\ = \{\partial_K, \phi\lrcorner\}(\partial_K\psi\lrcorner A + (-1)^k\psi\lrcorner\partial_K)A - (-1)^{jk}\{\partial_K, \psi\lrcorner\}(\partial_K\phi\lrcorner A + (-1)^j\phi\lrcorner\partial_K A) \\ &\ = \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A)) + (-1)^k\partial_K(\phi\lrcorner\psi\lrcorner\partial_K A) + (-1)^{j+k}\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) \\ &\qquad - (-1)^{jk}\{\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A)) + (-1)^j\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A) + (-1)^{j+k}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A)\} \\ &\ = \partial_K(\phi\lrcorner\partial_K(\psi\lrcorner A)) - (-1)^{jk+k}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A) + (-1)^{j+k}\phi\lrcorner\partial_K(\psi\lrcorner\partial_K A) \\ &\qquad - (-1)^{jk}\partial_K(\psi\lrcorner\partial_K(\phi\lrcorner A)) - (-1)^{jk+j}\partial_K(\psi\lrcorner\phi\lrcorner\partial_K A) \\ &\qquad\qquad - (-1)^{jk+j+k}\psi\lrcorner\partial_K(\phi\lrcorner\partial_K A).\end{aligned}$$

Hence we have equality holds. □

Corollary 3.3. *Let F_{d_K} be the curvature of the Chern connection. Let $\phi \in A^{0,i}(TX)$ and $\psi \in A^{0,j}(TX)$. Then we have*

$$[\phi, \psi]\lrcorner F_{d_K} = (-1)^i \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\psi\lrcorner F_{d_K} - (-1)^{ij+j}\{\partial_K^{\mathrm{End}(E)}, \psi\lrcorner\}\phi\lrcorner F_{d_K}.$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

Recall that F_{d_K} is a $(1,1)$-form which takes values in $\mathrm{End}(E)$.

Applying Lemma 3.2 to the left-hand side of the equation and by the Bianchi identity we have

$$\begin{aligned}[\phi, \psi]\lrcorner F_{d_K} &= \phi\lrcorner\partial_K(\psi\lrcorner F_{d_K}) \\ &\qquad - (-1)^{ij+j}\partial_K(\psi\lrcorner(\phi\lrcorner F_{d_K})) - (-1)^{ij}\psi\lrcorner\partial_K(\phi\lrcorner F_{d_K}) \\ &\qquad\qquad - (-1)^{ij+j}\psi\lrcorner\phi\lrcorner\partial_K F_{d_K} \\ &= \phi\lrcorner\partial_K(\psi\lrcorner F_{d_K}) - (-1)^{ij}\psi\lrcorner\partial_K(\phi\lrcorner F_{d_K}).\end{aligned}$$

By direct computation for the right-hand side of the equation, we have

$$\begin{aligned}&(-1)^i\{\partial_K, \phi\lrcorner\}\psi\lrcorner F_{d_K} - (-1)^{ij+j}\{\partial_K, \psi\lrcorner\}\phi\lrcorner F_{d_K} \\ &\quad = (-1)^i\partial_K(\phi\lrcorner\psi\lrcorner F_{d_K}) + \phi\lrcorner\partial_K\psi\lrcorner F_{d_K} \\ &\qquad\qquad - (-1)^{ij+j}\partial_K(\psi\lrcorner\phi\lrcorner F_{d_K}) - (-1)^{ij}\psi\lrcorner\partial_K\phi\lrcorner F_{d_K} \\ &\quad = \phi\lrcorner\partial_K\psi\lrcorner F_{d_K} - (-1)^{ij}\psi\lrcorner\partial_K\phi\lrcorner F_{d_K}.\end{aligned}$$

Hence we have the desired equality. □

By direct computation, we obtain some corollaries.

Corollary 3.4. *Let $A \in A^i(\mathrm{End}(E))$, $B \in A^j(\mathrm{End}(E))$ and $\phi \in A^{0,k}(TX)$. Then*

$$\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}[A, B] = [\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}A, B] + (-1)^{ik}[A, \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}B]. \tag{4}$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

By using local trivialization we have

$$\begin{aligned}&\{\partial_K, \phi\lrcorner\}[A, B] \\ &= \partial_K(\phi\lrcorner[A, B]) + (-1)^k\phi\lrcorner\partial_K[A, B] \\ &= \partial(\phi\lrcorner[A, B]) + [K^{-1}\partial K, \phi\lrcorner[A, B]] + (-1)^k\phi\lrcorner(\partial[A, B]) + [K^{-1}\partial K, [A, B]] \\ &= [\partial(\phi\lrcorner A), B] + (-1)^{i+k-1}[\phi\lrcorner A, \partial B] + (-1)^{i+ik}[\partial A, \phi\lrcorner B] + (-1)^{ik}[A, \partial(\phi\lrcorner B)] \\ &\qquad + (-1)^k[\phi\lrcorner\partial A, B] + (-1)^{ik+i+1}[\partial A, \phi\lrcorner B] + (-1)^{k+i}[\phi\lrcorner A, \partial B] \\ &\qquad\qquad + (-1)^{k+ki}[A, \phi\lrcorner\partial B] + (-1)^k[\phi\lrcorner K^{-1}\partial K, [A, B]] \\ &= [\partial(\phi\lrcorner A), B] + (-1)^{ik}[A, \partial(\phi\lrcorner B)] + (-1)^k[\phi\lrcorner\partial A, B] + (-1)^{ik+k}[A, \phi\lrcorner\partial B] \\ &\qquad + (-1)^k[[\phi\lrcorner K^{-1}\partial K, A], B] + (-1)^{ki+k}[A, [\phi\lrcorner K^{-1}\partial K, B]] \\ &= [\{\partial_K, \phi\lrcorner\}A, B] + (-1)^{ik}[A, \{\partial_K, \phi\lrcorner\}B].\end{aligned}$$

Hence we have the desired equality. □

Corollary 3.5. *Let $A \in A^i(\mathrm{End}(E))$ and $\phi \in A^{0,j}(TX)$. Then*

$$\begin{aligned}&\bar\partial_{\mathrm{End}(E)}\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}A\\ &\qquad = (-1)^j\{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\bar\partial_{\mathrm{End}(E)}A - \{\partial_K^{\mathrm{End}(E)}, \bar\partial_{TX}\phi\lrcorner\}A - [\phi\lrcorner F_{d_K}, A].\end{aligned}$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

We prove the above equality by using local trivialization:

$$\begin{aligned}
&\bar\partial_{\mathrm{End}(E)}\{\partial_K, \phi\lrcorner\}A\\
&\quad= \bar\partial_{\mathrm{End}(E)}\{\partial_K(\phi\lrcorner) + (-1)^j\phi\lrcorner\partial_K A\}\\
&\quad= \bar\partial_{\mathrm{End}(E)}\{\partial(\phi\lrcorner A) + (-1)^j[\phi\lrcorner K^{-1}\partial K, A] + (-1)^j\phi\lrcorner\partial A\}\\
&\quad= -\partial(\bar\partial_{TX}\phi\lrcorner A) + (-1)^j\partial(\phi\lrcorner\bar\partial_{\mathrm{End}(E)}A) + (-1)^j[\bar\partial_{EndE}(\phi\lrcorner K^{-1}\partial K), A]\\
&\qquad\qquad + [\phi\lrcorner K^{-1}\partial K, \bar\partial_{\mathrm{End}(E)}A] + (-1)^j\bar\partial_{TX}\phi\lrcorner\partial A - \phi\lrcorner\bar\partial_{\mathrm{End}(E)}\partial A\\
&\quad= (-1)^j\{\partial_K, \phi\lrcorner\}\bar\partial_{\mathrm{End}(E)}A - [\phi\lrcorner F_{d_K}, A] - \partial(\bar\partial_{TX})\phi\lrcorner A\\
&\qquad\qquad + (-1)^j[(\bar\partial_{TX}\phi)\lrcorner K^{-1}\partial K, A] + (-1)^j\bar\partial_{TX}\phi\lrcorner\partial A\\
&\quad= (-1)^j\{\partial_K, \phi\lrcorner\}\bar\partial_{\mathrm{End}(E)}A - \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}A - [\phi\lrcorner F_{d_K}, A].
\end{aligned}$$

Hence we have the desired equality. □

Proposition 3.2. *The bracket $[\,\cdot\,,\,\cdot\,]_L : L \times L \to L$ satisfies the following*:

1. *For every $(A, \phi) \in L^i$, $(B, \psi) \in L^j\,(i, j \in \mathbb{Z})$,*

$$[(A, \phi), (B, \psi)]_L + (-1)^{pq}[(B, \psi), (A, \phi)]_L = 0.$$

2. *The graded Jacobi identity holds*: *for every $(A, \phi) \in L^i$, $(B, \psi) \in L^j$, $(C, \tau) \in L^k$, and i, j, k,*

$$\begin{aligned}&[(A, \phi), [(B, \psi), (C, \tau)]_L]_L\\ &\qquad = [[(A, \phi), (B, \psi)]_L, (C, \tau)]_L + (-1)^{ij}[(B, \psi), [(A, \phi), (C, \tau)]_L]_L\end{aligned}$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

1. is obvious from the definition. We prove 2.

We first calculate each component. First we have

$$\begin{aligned}
(5)\quad &[(A, \phi), [(B, \psi), (C, \tau)]_L]_L\\
&\quad= \big[(A, \phi), ((-1)^j\{\partial_K, \psi\lrcorner\}C - (-1)^{(j+1)k}\{\partial_K, \tau\lrcorner\}B - [B, C], [\psi, \tau])\big]_L\\
&\quad= \begin{pmatrix}\alpha\\ [\phi, [\psi, \tau]]\end{pmatrix}
\end{aligned}$$

where

$$\alpha = (-1)^i\{\partial_K, \phi\lrcorner\}\{(-1)^j\{\partial_K, \psi\lrcorner\}C - (-1)^{(j+1)k}\{\partial_K, \tau\lrcorner\}B - [B, C]\} \\ - (-1)^{(i+1)(j+k)}\{\partial_K, [\psi, \tau]\lrcorner\}A \\ - [A, (-1)^j\{\partial_K, \psi\lrcorner\}C - (-1)^{(j+1)k}\{\partial_K, \tau\lrcorner\}B - [B, C]]$$

Next we have

$$\big[[(A, \phi), (B, \psi)]_L, (C, \tau)\big]_L = \begin{pmatrix} \beta \\ [[\phi, \psi], \tau] \end{pmatrix} \tag{6}$$

where

$$\beta = (-1)^{i+j}\{\partial, [\phi, \psi]\lrcorner\}C \\ - (-1)^{(i+j+1)k}\{\partial_K, \tau\lrcorner\}\{(-1)^i\{\partial_K, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K, \psi\lrcorner\}A - [A, B]\} \\ - [(-1)^i\{\partial_K, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K, \psi\lrcorner\}A - [A, B], C].$$

We also have

$$(-1)^{ij}\big[(B, \psi), [(A, \phi), (C, \tau)]_L\big]_L = \begin{pmatrix} \gamma \\ [\psi, [\phi, \tau]] \end{pmatrix} \tag{7}$$

where

$$\gamma = (-1)^{ij}((-1)^i\{\partial_K, \psi\lrcorner\}\{(-1)^i\{\partial_K, \phi\lrcorner\}C - (-1)^{(i+1)k}\{\partial_K, \phi\lrcorner\}A - [A, C]\} \\ - (-1)^{(j+1)(i+k)}\{\partial_K, [\phi, \tau]\lrcorner\}B \\ - [B, (-1)^i\{\partial_K, \psi\lrcorner\}(-1)^i\{\partial_K, \phi\lrcorner\}C - (-1)^{(i+1)k}\{\partial_K, \phi\lrcorner\}A - [A, C]].$$

Hence by (5), (6), and (7) we only have to prove the equations

$$\{\partial_K, [\phi, \psi]\lrcorner\}A = \{\partial_K, \phi\lrcorner\}\{\partial_K, \psi\lrcorner\}A - (-1)^{jk}\{\partial_K, \psi\lrcorner\}\{\partial_K, \phi\lrcorner\}A,$$
$$\{\partial_K, \phi\lrcorner\}[A, B] = [\{\partial_K, \phi\lrcorner\}A, B] + (-1)^{ik}[A, \{\partial_K, \phi\lrcorner\}B],$$
$$[A, [B, C]] = [[A, B], C] + (-1)^{ij}[B, [A, C]],$$
$$[\phi, [\psi, \tau]] = [[\phi, \psi], \tau] + (-1)^{ij}[\psi, [\phi, \tau]].$$

The above equations follow from Corollaries 3.2 and 3.4 and the fact that the Schouten–Nijenhuis bracket satisfies the Jacobi identity. Hence we proved that $[\,\cdot\,,\cdot\,]_L$ satisfies the Jacobi identity. □

Proposition 3.3. *d_L is a differential with respect to the bracket $[\,\cdot\,,\cdot\,]_L$, that is,*

1. $d_L(L^i) \subset L^{i+1}$,
2. $d_L \circ d_L = 0$,
3. *for every $(A, \phi) \in L^i$, $(B, \psi) \in L^j$ and i, j,*

$$d_L[(A, \phi), (B, \psi)]_L = [d_L(A, \phi), (B, \psi)]_L + (-1)^i[(A, \phi), d_L(B, \psi)]_L$$

Proof. We denote $\partial_K^{\mathrm{End}(E)}$ as ∂_K in this proof.

1. is obvious from the definition of d_L.

We prove 2. for $d_L \circ d_L : L^1 \to L^3$. Let $(A, \phi) \in L^1$. We calculate $d_L \circ d_L(A, \phi)$:

$$d_L(A,\phi) = \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{T(X)} \end{pmatrix} + \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}$$

$$\cdot d_L\left(\begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{T(X)}\phi \end{pmatrix} + \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{T(X)} \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{T(X)}\phi \end{pmatrix} \tag{8}$$

$$+ \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{TX}\phi \end{pmatrix} + \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix} \tag{9}$$

$$+ \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix}. \tag{10}$$

Let us show (8) $=$ (9) $=$ (10) $= 0$:

$$\begin{aligned}
(8) &= \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{TX}\phi \end{pmatrix} \\
&= \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} \circ \bar\partial_{\mathrm{End}(E)} A + \bar\partial_{\mathrm{End}(E)}(\phi\lrcorner F_{d_K}) + B_K(\bar\partial_{T(X)}\phi) \\ \bar\partial_{TX} \circ \bar\partial_{TX}\phi \end{pmatrix} \\
&= \begin{pmatrix} \bar\partial_{TX}\phi \lrcorner F_{d_K} + \phi\lrcorner \bar\partial_{\mathrm{End}(E)} F_{d_k} - \bar\partial_{TX}\phi\lrcorner F_{d_K} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \phi \lrcorner \bar\partial_{\mathrm{End}(E)} F_{d_k} \\ 0 \end{pmatrix} = 0.
\end{aligned}$$

The last equation comes from the Bianchi identity. Next, we show (9) $= 0$:

$$\begin{aligned}
(9) &= \begin{pmatrix} \theta & C_K \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} A - \phi \lrcorner F_{d_K} \\ \bar\partial_{TX}\phi \end{pmatrix} + \begin{pmatrix} \bar\partial_{\mathrm{End}(E)} & B_K \\ 0 & \bar\partial_{TX} \end{pmatrix} \begin{pmatrix} [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} [\theta, \bar\partial_{\mathrm{End}(E)} A] - [\theta, \phi\lrcorner F_{d_K}] + \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}\theta \\ 0 \end{pmatrix} \\
&\qquad + \begin{pmatrix} \bar\partial_{\mathrm{End}(E)}[\theta, A] + \bar\partial_{\mathrm{End}(E)}(\{\partial_K, \phi\lrcorner\}\theta) \\ 0 \end{pmatrix}.
\end{aligned}$$

Since θ is a Higgs field, $\bar\partial_{\mathrm{End}(E)}[\theta, A] = -[\theta, \bar\partial_{\mathrm{End}(E)} A]$. Hence we have

$$(9) = \begin{pmatrix} -[\theta, \phi\lrcorner F_{d_K}] + \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}\theta + \bar\partial_{\mathrm{End}(E)}(\{\partial_K, \phi\lrcorner\}\theta) \\ 0 \end{pmatrix}. \tag{11}$$

By direct computation using the local realization we have

$$\begin{aligned}\{\partial_K, \bar\partial_{TX}\phi\lrcorner\}\theta &= \partial_K(\bar\partial_{TX}\phi\lrcorner\theta) + \bar\partial_{TX}\phi\lrcorner(\partial_K\theta)\\ &= \partial(\bar\partial_{TX}\phi\lrcorner\theta) + [K^{-1}\partial K, \bar\partial_{TX}\phi\lrcorner\theta] + \bar\partial_{TX}\phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta]).\end{aligned}$$

and

$$\begin{aligned}\bar\partial_{\mathrm{End}(E)}(\{\partial_K, \phi\lrcorner\}\theta) &= \bar\partial_{\mathrm{End}(E)}\{\partial(\phi\lrcorner\theta) + [K^{-1}\partial K, \phi\lrcorner\theta] - \phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta])\}\\ &= -\partial\bar\partial(\phi\lrcorner\theta) + [F_{d_K}, \phi\lrcorner\theta] - [K^{-1}\partial K, \bar\partial_{\mathrm{End}(E)}(\phi\lrcorner\theta)]\\ &\qquad - \bar\partial_{TX}\phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta]) - \phi\lrcorner[F_{d_K}, \theta]\\ &= -\partial(\bar\partial_{TX}\phi\lrcorner\theta) - [\phi\lrcorner F_{d_K}, \theta] - [K^{-1}\partial K, \bar\partial_{TX}\phi\lrcorner\theta]\\ &\qquad - \bar\partial_{TX}\phi\lrcorner(\partial\theta + [K^{-1}\partial K, \theta]).\end{aligned}$$

Hence by (11) and the above two displays, we obtain that (9) $= 0$.

Next, we show (10) $= 0$:

$$\text{(12)}\qquad (10) = \begin{pmatrix}\theta & C_K\\ 0 & 0\end{pmatrix}\begin{pmatrix}[\theta, A]+\{\partial_K, \phi\lrcorner\}\theta\\ 0\end{pmatrix} = \begin{pmatrix}[\theta, \{\partial_K, \phi\lrcorner\}\theta]\\ 0\end{pmatrix}.$$

By direct computation using the local realization we have

$$\begin{aligned}&[\theta, \{\partial_K, \phi\lrcorner\}\theta]\\ &= \theta\wedge\{\partial_K, \phi\lrcorner\}\theta - \{\partial_K, \phi\lrcorner\}\theta\wedge\theta\\ &= \theta\wedge\{\partial(\phi\lrcorner\theta) + [K^{-1}\partial K, \phi\lrcorner\theta] - \phi\lrcorner\partial\theta - \phi\lrcorner[K^{-1}\partial K, \theta]\}\\ &\qquad - \{\partial(\phi\lrcorner\theta) + [K^{-1}\partial K, \phi\lrcorner]\theta - \phi\lrcorner\partial\theta - \phi\lrcorner[K^{-1}\partial K, \theta]\}\wedge\theta\\ &= \theta\wedge\{\partial(\phi\lrcorner\theta) - \phi\lrcorner\partial\theta - [\phi\lrcorner K^{-1}\partial K, \theta]\} - \{\partial(\phi\lrcorner\theta) - \phi\lrcorner\partial\theta - [\phi\lrcorner K^{-1}\partial K, \theta]\}\wedge\theta\\ &= \theta\wedge\partial(\phi\lrcorner\theta) - \theta\wedge\phi\lrcorner\partial\theta - \partial(\phi\lrcorner\theta)\wedge\theta + (\phi\lrcorner\partial\theta)\wedge\theta.\end{aligned}$$

Since $\theta\wedge\theta = 0$, we have

$$\begin{aligned}0 &= \partial(\phi\lrcorner(\theta\wedge\theta)) - \phi(\partial(\theta\wedge\theta))\\ &= \partial(\phi\lrcorner\theta)\wedge\theta - (\phi\lrcorner\theta)\wedge\partial\theta + \partial\theta\wedge\phi\lrcorner\theta - \theta\wedge\partial(\phi\lrcorner\theta)\\ &\qquad - (\phi\lrcorner\partial\theta)\wedge\theta - \partial\theta\wedge(\phi\lrcorner\theta) + (\phi\lrcorner\theta)\wedge\partial\theta + \theta\wedge(\phi\lrcorner\partial\theta)\\ &= \partial(\phi\lrcorner\theta)\wedge\theta - \theta\wedge\partial(\phi\lrcorner\theta) - (\phi\lrcorner\partial\theta)\wedge\theta + \theta\wedge\phi\lrcorner\partial\theta.\end{aligned}$$

Hence by (12) and the above two displays, we obtain that (10) $= 0$. This completes the proof of 2.

Next we prove 3.

Let $(A, \phi) \in L^i$ and $(B, \psi) \in L^j$. We first calculate each component of 3.

First we have

$$d_L[(A, \phi), (B, \psi)]_L = \begin{pmatrix}\alpha\\ \bar\partial_{TX}[\phi, \psi]\end{pmatrix}$$

where

$$\alpha = (\bar\partial_{\mathrm{End}(E)} + \theta)\big((-1)^i\{\{\partial_K, \phi\lrcorner\}B - (-1)^{(i+1)j}\{\partial_K, \psi\lrcorner\}A - [A, B]\}\big) \\ + (-1)^{i+j}[\phi, \psi]\lrcorner F_{d_K} + \{\partial_K, [\phi, \psi]\lrcorner\}\theta.$$

Next we have

$$[d_L(A, \phi), (B, \psi)]_L = \begin{pmatrix} \beta \\ [\bar\partial_{TX}\phi, \psi]) \end{pmatrix},$$

where

$$\beta = (-1)^{i+1}\{\partial_K, \bar\partial_{TX}\phi\lrcorner\}B \\ - (-1)^{(i+2)j}\{\partial_K, \psi\lrcorner\}(\bar\partial_{\mathrm{End}(E)}A + (-1)^i\phi\lrcorner F_{d_K} + [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta) \\ - [\bar\partial_{\mathrm{End}(E)}A + (-1)^i\phi\lrcorner F_{d_K} + [\theta, A] + \{\partial_K, \phi\lrcorner\}\theta, B].$$

We also have

$$(-1)^i[(A, \phi), d_L(B, \psi)]_L = \begin{pmatrix} \gamma \\ (-1)^i[\phi, \bar\partial_{TX}\psi] \end{pmatrix},$$

where

$$\gamma = \{\partial_K, \phi\lrcorner\}(\bar\partial_{\mathrm{End}(E)}B + (-1)^j\psi\lrcorner F_{d_K} + [\theta, B] + \{\partial_K, \phi\lrcorner\}\theta) \\ - (-1)^{(i+1)(j+1)+i}\{\partial_K, \bar\partial_{TX}\psi\lrcorner\}A \\ - (-1)^i[A, \bar\partial_{\mathrm{End}(E)}B + (-1)^j\psi\lrcorner F_{d_K} + [\theta, B] + \{\partial_K, \psi\lrcorner\}\theta].$$

Hence by the above equations, we have to prove

$$\begin{aligned} \bar\partial_{\mathrm{End}(E)}\{\partial_K, \phi\lrcorner\}A &= (-1)^j\{\partial_K, \phi\}\bar\partial_{\mathrm{End}(E)}A - \{\partial_K, \bar\partial_{TX}\phi\lrcorner\}A - [\phi\lrcorner F_{d_K}, A], \\ \{\partial_K, \phi\lrcorner\}[\theta, A] &= [\{\partial_K, \phi\lrcorner\}\theta, A] + (-1)^i[\theta, \{\partial_K, \phi\lrcorner\}A], \\ [\phi, \psi]\lrcorner F_{d_K} &= (-1)^i\{\partial_K, \phi\lrcorner\}\psi\lrcorner F_{d_K} - (-1)^{ij+j}\{\partial_K, \psi\lrcorner\}\phi\lrcorner F_{d_K}, \\ [\theta, [A, B]] &= [[\theta, A], B] + (-1)^i[A, [\theta, B]], \\ \bar\partial_{\mathrm{End}(E)}[A, B] &= [\bar\partial_{\mathrm{End}(E)}A, B] + (-1)^i[A, \bar\partial_{\mathrm{End}(E)}B], \\ \bar\partial_{TX}[\phi, \psi] &= [\bar\partial_{TX}\phi, \psi] + (-1)^i[\phi, \bar\partial_{TX}\psi]. \end{aligned}$$

These equations follow from Corollaries 3.2–3.5 and the fact that $\bar\partial_{\mathrm{End}(E)}$ and $\bar\partial_{TX}$ satisfy the Leibniz rule and the canonical bracket satisfies the Jacobi identity. □

Propositions 3.2 and 3.3 show us that $(L, [\cdot, \cdot]_L, d_L)$ is a DGLA. Hence we proved Theorem 3.1. Combining Propositions 2.6 and 3.1 with Theorem 3.1, we have:

Theorem 3.6. *Given a holomorphic-Higgs triple (X, E, θ) and a smooth family of elements $\{A_t, B_t, \phi_t\}_{t\in\Delta} \subset A^{0,1}(\mathrm{End}(E)) \oplus A^{1,0}(\mathrm{End}(E)) \oplus A^{0,1}(TX)$. Then,*

(A_t, B_t, ϕ_t) defines a holomorphic-Higgs triple if and only if (A_t, B_t, ϕ_t) satisfies the Maurer–Cartan equation

$$d_L(A_t + B_t, \phi_t) - \tfrac{1}{2}[(A_t + B_t, \phi_t), (A_t + B_t, \phi_t)] = 0. \tag{13}$$

We now give the proof of Proposition 3.1.

Proof of Proposition 3.1. We calculate $\bar{D}^2$. Using Corollaries 3.2 and 3.5, we have

$$\begin{aligned}
\bar{D}^2 &= (\bar{\partial}_E + \{\partial_K, \phi\lrcorner\} + A + \theta + B)^2 \\
&= \bar{\partial}_E\bar{\partial}_E + \bar{\partial}_E\{\partial_K, \phi\lrcorner\} + \{\partial_K, \phi\lrcorner\}\bar{\partial}_E + \{\partial_K, \phi\lrcorner\}\{\partial_K, \phi\lrcorner\} \\
&\qquad + \{\partial_K, \phi\lrcorner\}(A + \theta + B) + (A + \theta + B)\{\partial_K, \phi\lrcorner\} \\
&\qquad\qquad + \bar{\partial}_E(A + \theta + B) + (A + \theta + B)\bar{\partial}_E + [\theta, A + B] + \tfrac{1}{2}[A + B, A + B] \\
&= -\{\partial_K, \bar{\partial}_{TX}\phi\lrcorner\} - \phi\lrcorner F_{d_K} + \tfrac{1}{2}\{\partial_K, [\phi, \phi]\lrcorner\} + \bar{\partial}_{\mathrm{End}(E)}(\theta + A + B) \\
&\qquad\qquad + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(\theta + A + B) + [\theta, A + B] + \tfrac{1}{2}[A + B, A + B] \\
&= -\big\{\partial_K, \big(\bar{\partial}_{TX}\phi - \tfrac{1}{2}[\phi, \phi]\big)\lrcorner\big\} \\
&\qquad + \bar{\partial}_{\mathrm{End}(E)}(A + B) - \phi\lrcorner F_{d_K} + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\theta + [\theta, A + B] \\
&\qquad\qquad + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(A + B) + \tfrac{1}{2}[A + B, A + B].
\end{aligned}$$

Hence by the above calculation, $\bar{D}^2 = 0$ is equivalent to

$$\begin{aligned}
&\bar{\partial}_{\mathrm{End}(E)}(A + B) - \phi\lrcorner F_{d_K} + [\theta, A + B] \\
&\qquad + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}\theta + \{\partial_K^{\mathrm{End}(E)}, \phi\lrcorner\}(A + B) + \tfrac{1}{2}[A + B, A + B] = 0, \\
&\qquad\qquad \bar{\partial}_{TX}\phi - \tfrac{1}{2}[\phi, \phi] = 0.
\end{aligned}$$

Hence we have the proof. □

4. Kuranishi family

4A. ***Construction of Kuranishi family.*** Kuranishi [1965] constructed a universal family for any complex manifold X over a possible singular analytic space. We want to construct a family of holomorphic-Higgs triples over a certain singular space which becomes a universal family in this context.

Here we recall some differential operators and inequalities we need. These are commonly used in classical Hodge theory. We choose a hermitian metric g on X and a hermitian metric K on E. Using these two metrics, we can define an inner product $(\cdot\,,\cdot)$ on $L = \bigoplus_i L^i$. We remark that L^i and L^j are orthogonal with respect to this inner product. We first define the formal adjoint of d_L with respect to $(\cdot\,,\cdot)$ by

$$(d_L\alpha, \beta) = (\alpha, d_L^*\beta).$$

Then the Laplacian Δ_L is defined by

$$\Delta_L = d_L \circ d_L^* + d_L^* \circ d_L.$$

This is an elliptic self-adjoint operator. Hence by Theorem 4.12 in [Wells 1980, Chapter 4], it has a finite dimensional kernel $\mathbb{H}^i$. We call the elements of $\mathbb{H}^i$ a *harmonic form*. Let $\widetilde{L}^i$ be a completion of L^i with respect to the inner product $(\cdot\,,\cdot)$, and let $H : L^i \to \mathbb{H}^i$ be the harmonic projection. The Green's operator $G : L^i \to L^i$ is defined by

$$I = H + \Delta_L \circ G = H + G \circ \Delta_L,$$

where I is the identity for L^i. H and G can be extended to the bounded operator $H, G : \widetilde{L}^i \to \widetilde{L}^i$. G commutes with d_L and d_L^*.

Now let $\{\eta_1, \dots, \eta_n\} \subset \mathbb{H}^1$ be a basis and $\epsilon_1(t) := \sum_{j=1}^n t_j \eta_j \in \mathbb{H}^1$. Consider

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

We define the Hölder norm $\|\cdot\|_{k,\alpha}$ as in [Morrow and Kodaira 2006]. We have the inequalities

$$\|d_L^* \epsilon\|_{k,\alpha} \le C_1 \|\epsilon\|_{k+1.\alpha},$$
$$\|[\epsilon, \delta]\|_{k,\alpha} \le C_2 \|\epsilon\|_{k+1,\alpha} \|\delta\|_{k+1,\alpha}.$$

Douglis and Nirenberg [1955] proved the a priori estimate

$$\|\epsilon\|_{k,\alpha} \le C_3(\|\Delta_L \epsilon\|_{k-2,\alpha} + \|\epsilon\|_{0,\alpha}).$$

Applying these and following the proof of Proposition 2.3 in [Morrow and Kodaira 2006, Chapter 4] one can deduce an estimate for the Green's operator G:

$$\|G\epsilon\|_{k,\alpha} \le C_4 \|\epsilon\|_{k-2,\alpha},$$

where all C_i's are positive constants which depend only on k and α.

Then by applying the proof of Proposition 2.4 in [Morrow and Kodaira 2006, Chapter 4] or using the implicit function theorem for Banach spaces as in [Kuranishi 1965], we obtain a unique solution $\epsilon(t)$ which satisfies

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L,$$

which is analytic in the variable in t. The solution $\epsilon(t)$ is also a smooth section for L^1. By applying the Laplacian to the above equation, we get

$$\Delta_L \epsilon(t) - \tfrac{1}{2} d_L^* [\epsilon(t), \epsilon(t)]_L = 0.$$

Since $\epsilon(t)$ is holomorphic in t, we have

$$\sum_{i,j} \frac{\partial^2}{\partial t_i \partial \bar{t}_j} \epsilon(t) = 0.$$

Hence we have

$$\left(\Delta_L + \sum_{i,j} \frac{\partial^2}{\partial t_i \partial \bar{t}_j}\right)\epsilon(t) - \tfrac{1}{2} d_L^*[\epsilon(t), \epsilon(t)]_L = 0.$$

Since the operator

$$\Delta_L + \sum_{i,j} \frac{\partial^2}{\partial t_i \partial \bar{t}_j}$$

is elliptic, we can say that $\epsilon(t)$ is smooth by elliptic regularity.

From the discussion so far, we have:

Proposition 4.1. *Let $\{\eta_1, \dots, \eta_n\} \subset \mathbb{H}^1$ be a basis. Let $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ and $\epsilon_1(t) := \sum_i t_i \eta_i$. For all $|t| \ll 1$ we have a $\epsilon(t)$ such that $\epsilon(t)$ satisfies*

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

Moreover, $\epsilon(t)$ is holomorphic with respect to the variable t.

Following Kuranishi [1965] we have:

Proposition 4.2. *If we take $|t|$ small enough, the solution $\epsilon(t)$ that satisfies*

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L$$

solves the Maurer–Cartan equation if and only if $H[\epsilon(t), \epsilon(t)]_L = 0$. Here H is the harmonic projection.

Proof. Suppose the Maurer–Cartan equation holds. Then

$$d_L \epsilon(t) - \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L = 0.$$

Hence we have

$$H[\epsilon(t), \epsilon(t)]_L = 2H d_L \epsilon(t) = 0.$$

Conversely, suppose that $H[\epsilon(t), \epsilon(t)]_L = 0$. We have to show

$$\delta(t) := d_L \epsilon(t) - \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L = 0.$$

Since $\epsilon(t)$ is a solution to

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L$$

and $\epsilon_1(t)$ is d_L-closed, applying d_L we get

$$d_L \epsilon(t) = \tfrac{1}{2} d_L d_L^* G[\epsilon(t), \epsilon(t)]_L.$$

Hence

$$2\delta(t) = d_L d_L^* G[\epsilon(t), \epsilon(t)]_L - [\epsilon(t), \epsilon(t)]_L.$$

By the Hodge decomposition, we can write

$$[\epsilon(t), \epsilon(t)]_L = H[\epsilon(t), \epsilon(t)]_L + \Delta_L G[\epsilon(t), \epsilon(t)]_L = \Delta_L G[\epsilon(t), \epsilon(t)]_L.$$

Therefore

$$\begin{aligned} 2\delta(t) &= d_L d_L^* G[\epsilon(t), \epsilon(t)]_L - \Delta_L G[\epsilon(t), \epsilon(t)]_L \\ &= -d_L^* d_L G[\epsilon(t), \epsilon(t)]_L \\ &= -d_L^* G d_L[\epsilon(t), \epsilon(t)]_L \\ &= -2d_L^* G[d_L\epsilon(t), \epsilon(t)]_L. \end{aligned}$$

Hence we get

$$\begin{aligned} \delta(t) &= -d_L^* G[d_L\epsilon(t), \epsilon(t)]_L \\ &= -d_L^* G\big[\delta(t) + \tfrac{1}{2}[\epsilon(t), \epsilon(t)]_L, \epsilon(t)\big]_L \\ &= -d_L^* G[\delta(t), \epsilon(t)]_L. \end{aligned}$$

We used the Jacobi identity in the last equality. Using the estimate

$$\|[\xi, \eta]\|_{k,\alpha} \le C_{k,\alpha}\|\xi\|_{k+1,\alpha}\|\eta\|_{k+1,\alpha},$$

we get

$$\|\delta(t)\|_{k,\alpha} \le C_{k,\alpha}\|\delta(t)\|_{k,\alpha}\|\epsilon(t)\|_{k,\alpha}.$$

If we take $|t|$ small enough such that $C_{k,\alpha}\|\epsilon(t)\|_{k,\alpha} < 1$, we obtain $\delta(t) = 0$. This stands for all $|t|$ small enough. This finishes the proof. □

In the case when $H[\epsilon(t), \epsilon(t)]_L = 0$ for all t or $\mathbb{H}^2 = 0$, we have:

Corollary 4.1. *Let n be the dimension of $\mathbb{H}^1$. If $H[\epsilon(t), \epsilon(t)]_L = 0$ for all t, we have a family of deformation of holomorphic-Higgs triples over a small ball Δ centered at the origin of $\mathbb{C}^n$.*

Proof. If $H[\epsilon(t), \epsilon(t)] = 0$ for all t, $\epsilon(t) = (A_t + B_t, \phi_t)$ satisfies the Maurer–Cartan equation and so we obtain family of holomorphic-Higgs triple (X_t, E_t, θ_t). Since ϕ_t is holomorphic for variable t, applying the Newlander–Nirenberg theorem, we can define a complex structure on $\mathcal{X} := X \times \Delta$ such that $X_t = \mathcal{X}|_{X\times\{t\}}$. Let $\mathcal{E} := E \times \Delta$. By applying the linearized Newlander–Nirenberg theorem as in [Moroianu 2007], we have a local frame $\{e(x, t)\}$ of $\mathcal{E}$ on $\mathcal{X}$ such that for each t, $\{e(x, t)\} \subset \ker(\overline{D}_t') = \ker(\bar\partial_{E_t})$ and is holomorphic respect to variable t. Let $\sigma : \mathcal{X} \to \mathcal{E}$ be a smooth section and locally trivialized as $\sigma(x, t) = \sum_k s^k(x, t)e_k(x, t)$ where s^k are smooth function on $\mathcal{X}$. We define $\bar\partial_{\mathcal{E}} : A(\mathcal{E}) \to A^{0,1}_{\mathcal{X}}(\mathcal{E})$ as

$$\bar\partial_{\mathcal{E}}(\sigma(x, t)) := \sum_k \bar\partial_{\mathcal{X}} s^k(x, t) \otimes e_k(x, t).$$

Note that $\bar{\partial}_{\mathcal{E}}$ is well defined and $\bar{\partial}_{\mathcal{E}}|_{E\times t} = \bar{\partial}_{E_t}$. It is clear that $\bar{\partial}_{\mathcal{E}}^2 = 0$ so that $\mathcal{E}$ is a holomorphic bundle over $\mathcal{X}$.

Let $\Theta = \theta + B_t + \phi_t \lrcorner (\theta + B_t)$. Since ϕ_t, B_t is holomorphic respect to the variable t and $\theta + B_t + \phi_t \lrcorner (\theta + B_t)$ is a Higgs field for (X_t, E_t), we have $\bar{\partial}_{\mathrm{End}\mathcal{E}}\Theta = 0$, $\Theta \wedge \Theta = 0$. Hence Θ is a Higgs field for $(\mathcal{X}, \mathcal{E})$.

Let $\pi : \mathcal{X} = X \times \Delta \to \Delta$ be a natural projection, this is a holomorphic submersion. Also $\pi^{-1}(0) = X$, $\mathcal{E}|_{\pi^{-1}(0)} = E$ and $\Theta|_{\pi^{-1}(0)} = \theta$ stands. Hence we have a family of deformation of a holomorphic-Higgs triple over Δ. □

In general, the condition $\mathbb{H}^2 = 0$ may not be satisfied. However, we can define a possible singular analytic space

$$\mathcal{S} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)]_L = 0\}.$$

Let $X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)}$ be the complex manifold, the holomorphic bundle, and the Higgs field defined by $\epsilon(t)$. By the above results, we have a family of holomorphic-Higgs triples $\{(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})\}_{t\in\mathcal{S}}$. We call this family the *Kuranishi family* of (X, E, θ) and $\mathcal{S}$ the *Kuranishi space*.

4B. *Local completeness of Kuranishi family.* We give a proof of the local completeness of the Kuranishi family for the deformation of the triple (X, E, θ). Here we follow Kuranishi's method.

Recall that in Section 4A we proved that for a given $\epsilon_1(t) = \sum_i t_i \eta_i \in \mathbb{H}^1 = \ker(\Delta_L : L^1 \to L^1)$ the existence of solutions $\epsilon(t)$ to

$$\epsilon(t) = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon(t), \epsilon(t)]_L$$

and proved that $\epsilon(t)$ satisfies the Maurer–Cartan equation if and only if

$$H[\epsilon(t), \epsilon(t)]_L = 0.$$

Hence we obtain a family of holomorphic-Higgs triples over

$$\mathcal{S} := \{t \in \Delta : H[\epsilon(t), \epsilon(t)]_L = 0\}.$$

Before we state the main theorem of this paper, we introduce the Sobolev norm for L and collect some estimates.

First, let us recall the Sobolev norm on Euclidean space. Let U be an open subset of $\mathbb{R}^n$ and f and g be a complex-valued smooth function on $\overline{U}$. Here, $\overline{U}$ is a closure of U. We set

$$(f, g)_k := \sum_{|\alpha|<k} \int_U D^\alpha f \cdot \overline{D^\alpha g}\, dx,$$

where we use the multi-index notation $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i > 0$, $|\alpha| = \sum_i \alpha_i$ and $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$.

Then we define k-th Sobolev norm $|\cdot|_k$ as

$$|f|_k = |f|_k^U := \sqrt{(f, f)_k}. \tag{14}$$

Let V be a relatively compact open subset of U. By [Morrow and Kodaira 2006, Chapter 4, Lemma 3.1], we have an estimate such that

$$|fg|_k^V \leq c|f|_k^U \cdot |g|_k^U, \quad k \geq n+2, \tag{15}$$

where c is a constant.

By using a partition of unity and the metric of E and X, we can define k-th Sobolev $|\eta|_k$ for any $\eta \in L^i = \bigoplus_{p+q=i} A^{p,q}(\mathrm{End}(E)) \oplus A^{0,i}(TX)$. We list some estimates that we need. Let c_k be a constant. Then the following estimates hold (see [Morrow and Kodaira 2006] for more details):

$$\begin{aligned} &|[\phi, \psi]_k| \leq c_k|\phi|_{k+1}|\psi|_{k+1}, \quad k \geq 2n+2,\ \dim_{\mathbb{C}} X = n, \\ &|H\phi| \leq c_k|\phi|_k, \\ &|d_L^* G\phi|_k \leq c_k|\phi|_{k-1}. \end{aligned} \tag{16}$$

From now on, we choose a k large enough such that the above estimates hold.

Let $\eta := (A+B, \phi) \in L^1$ be a Maurer–Cartan element and assume $|\eta|_k$ is small enough so that η can define a holomorphic-Higgs triple. Let $X_\eta, E_\eta, \theta_\eta$ be the complex manifold, the holomorphic bundle, and the Higgs field which η defines, respectively. We denote this holomorphic-Higgs triple $(X_\eta, E_\eta, \theta_\eta)$. Let $\eta' \in L^1$ be another Maurer–Cartan element and assume that η' also defines a holomorphic-Higgs triple $(X_{\eta'}, E_{\eta'}, \theta_{\eta'})$. We denote as $(X_\eta, E_\eta, \theta_\eta) \cong (X_{\eta'}, E_{\eta'}, \theta_{\eta'})$ when there is a biholomorphic map $F : X_\eta \to X'_\eta$, a holomorphic bundle isomorphism $\Phi : E_\eta \to E'_\eta$ which is compatible with F and $\theta_\eta = \widehat{\Phi}^{-1} \circ F^*(\theta'_\eta) \circ \widehat{\Phi}$ holds. Here $\widehat{\Phi} : E_\eta \to F^*(E_{\eta'})$ is the holomorphic bundle isomorphism induced by Φ. $F^*(E_{\eta'})$ is the pull back of the bundle $E_{\eta'}$ by F.

Now we state the main theorem of this paper.

Theorem 4.2. *Let $\eta := (A+B, \phi) \in L^1$ be a Maurer–Cartan element. If $|\eta|_k$ is small enough, then there exists some $t \in \mathcal{S}$ such that $(X_\eta, E_\eta, \theta_\eta) \cong (X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)})$.*

Proposition 4.3. *Let $\epsilon_1(t) \in \mathbb{H}^1$, $t \in S$. Assume that ϵ solves the equation*

$$\epsilon = \epsilon_1(t) + \tfrac{1}{2} d_L^* G[\epsilon, \epsilon]_L.$$

If $|\epsilon|_k$ is small enough, then the solution is unique.

Proof. Suppose ϵ is another solution. Let $\delta = \epsilon - \epsilon(t)$. Then

$$\begin{aligned} \delta &= \tfrac{1}{2} d_L^* G\big([\epsilon, \epsilon]_L - [\epsilon(t), \epsilon(t)]_L\big) \\ &= \tfrac{1}{2} d_L^* G\big([\delta, \epsilon(t)]_L + [\epsilon(t), \delta]_L + [\delta, \delta]_L\big) \\ &= \tfrac{1}{2} d_L^* G\big(2[\delta, \epsilon(t)]_L + [\delta, \delta]_L\big). \end{aligned}$$

Estimating $|\delta|_k$ gives

$$\begin{aligned}|\delta|_k &\le D_k\big(|\delta|_k|\epsilon(t)|_k + |\delta|_k^2\big)\\ &\le D_k|\delta|_k\big(|\epsilon(t)|_k + |\delta|_k\big).\end{aligned}$$

If $|\epsilon(t)|_k$ is small enough, the above estimate holds if and only if $|\delta|_k = 0$. This proves the proposition. □

Proposition 4.4. *Suppose $\eta \in L^1$ satisfies the Maurer–Cartan equation (13). If $d_L^*\eta = 0$ and $|\eta|_k$ is small enough, then $\eta = \epsilon(t)$ for some $t \in S$.*

Proof. Since η satisfies the Maurer–Cartan equation, we have

$$d_L\eta - \tfrac{1}{2}[\eta,\eta]_L = 0.$$

Since $d_L^*\eta = 0$, we have

$$\begin{aligned}\Delta_L\eta &= d_L^*d_L\eta + d_Ld_L^*\eta\\ &= \tfrac{1}{2}d_L^*[\eta,\eta]_L.\end{aligned}$$

Hence

$$\eta - H\eta = G\Delta_L\eta = \tfrac{1}{2}Gd_L^*[\eta,\eta]_L.$$

Let $\psi := H\eta$. Then $\eta = \psi + \frac{1}{2}Gd_L^*[\eta,\eta]_L$. By the assumption such that $|\eta|_k$ is small, $|\psi|_k$ is small by (16). Hence $\psi = \epsilon_1(t)$ for $|t|$ small enough. Hence by Proposition 4.3, $\eta = \epsilon(t)$ for some $t \in S$. □

In general $d_L^*\eta \neq 0$ so we must try something else. We follow the idea of [Kuranishi 1965]. Let us recall how we solved this problem in the complex manifold setting. The idea is that for a given Maurer–Cartan element $\phi \in A^{0,1}(TX)$, we deform ϕ along a diffeomorphism $f : X \to X$.

Let X_ϕ be a complex manifold such that the complex structure comes from ϕ. Let $f : X \to X$ be a diffeomorphism. We can induce a complex structure on X by f. We denote the corresponding Maurer–Cartan element as $\phi \circ f$. Note that $f : X_{\phi\circ f} \to X_\phi$ is a biholomorphic map.

Kuranishi showed that for every Maurer–Cartan elements ϕ with $|\phi|_k$ small, there is a diffeomorphism f such that $\bar\partial_{TX}(\phi \circ f) = 0$. We recall how we obtain such f.

Let $g = (g_{i\bar j})$ be a fixed hermitian metric on X. Let $\xi = \sum_i \xi_i(z)\frac{\partial}{\partial z_i} \in A^0(TX)$ and $\bar\xi$ be the conjugate. Let $z_0 \in X$. Let $c(t) = c(t, z_0, \xi) = (c_1(t), \dots, c_n(t))$ be the geodesic curve starting from z_0 with initial velocity $\xi + \bar\xi$. Let $f_\xi(z_0) := c(1, z_0, \xi)$. Since X is compact, f_ξ is a diffeomorphism. By using Taylor expansion for f_ξ, we obtain

$$\phi \circ f_\xi = \phi + \bar\partial_{TX}\xi + R(\phi,\xi) \tag{17}$$

where $R(t\phi, t\xi) = t^2 R_1(\phi, \xi, t)$ if t is a real number and both R, R_1 are smooth map on X. In [Kuranishi 1965], it was shown that there is a $\xi \in A^0(TX)$ such that $\bar\partial_{TX}(\phi \circ f_\xi) = 0$ for any ϕ with $|\phi|_k$ small by the implicit function theorem between Banach spaces.

Let $\eta = (A + B, \phi) \in L^1$ be a Maurer–Cartan elements and assume $|\eta|_k$ is small enough so that η can define a holomorphic-Higgs triple $(X_\eta, E_\eta, \theta_\eta)$. By Kuranishi's work we have a $\xi \in A^0(TX)$ such that $\bar\partial_{TX}(\phi \circ f_\xi) = 0$.

Let $P_\xi : E \to E$ be the parallel transport of the Chern connection along f_ξ. Let $\upsilon \in A^0(\mathrm{End}(E))$ and $\exp(\upsilon) := \sum_{n=0}^{\infty} \frac{\upsilon^n}{n!} \in A^0(\mathrm{End}(E))$. $\exp(\upsilon) : E \to E$ is an automorphism and the inverse is given as $\exp(-\upsilon)$. Note that $(\upsilon, \xi) \in L^0$.

Let $\Phi := P_\xi \circ \exp(\upsilon)$. Since P_ξ is an isomorphism and compatible with f_ξ, Φ also is. Hence there is a smooth bundle isomorphism $\widehat{\Phi} : E \to f_\xi^* E_\eta$ which is induced by Φ. Hence we can induce a holomorphic-Higgs triple structure on (X, E, θ) via Φ and f_ξ. This holomorphic-Higgs triple is isomorphic to $(X_\eta, E_\eta, \theta_\eta)$. We denote the corresponding Maurer–Cartan element as $\eta_\gamma := ((A + B) \star \Phi, \phi \circ f_\xi)$. We show the existence of $\gamma := (\upsilon, \xi) \in L^0$ such that $d_L^* \eta_\gamma = 0$.

We first prove the next proposition.

Proposition 4.5. *Let $\eta_\gamma = ((A + B) \star \Phi, \phi \circ f_\xi)$, $\eta = (A + B, \phi)$ and $\gamma = (\upsilon, \xi)$ be as above. Then we have*

$$((A + B) \star \Phi, \phi \circ f_\xi) = (A + B, \phi) + d_L(\upsilon, \xi) + R((A, B, \phi), (\upsilon, \xi)). \tag{18}$$

The error term R is of order t^2 in the sense that

$$R(t(A, B, \phi), t(\upsilon, \xi)) = t^2 R_1((A, B, \phi), (\upsilon, \xi), t).$$

where t is a real number and R_1 is a smooth map.

Proof. Before going to the proof, we prepare some terminologies. Let $A \in A^{0,1}(\mathrm{End}(E))$, $B \in A^{1,0}(\mathrm{End}(E))$, $\upsilon \in A(\mathrm{End}(E))$, $\phi \in A^{0,1}(TX)$ and $\xi \in A^0(TX)$. The map $R((A, B, \phi), (\upsilon, \xi))$ is a smooth map on X such that R depends on A, B, υ, ϕ and ξ and R is of order t^2 in the sense that

$$R(t(A, B, \phi), t(\upsilon, \xi)) = t^2 R_1((A, B, \phi), (\upsilon, \xi), t),$$

where t is a real number and $R_1((A, B, \phi), (\upsilon, \xi), t)$ is a smooth map defined on X. We assume that the same property holds for $R((A, \phi), (\upsilon, \xi))$, $R((B, \phi), (\upsilon, \xi))$.

The map $R'((A, B, \phi), (\upsilon, \xi))$ is a smooth map defined on some open set of X such that R' depends on A, B, υ, ϕ, and ξ and R' is of order t^2 in the sense that

$$R'(t(A, B, \phi), t(\upsilon, \xi)) = t^2 R'_1((A, B, \phi), (\upsilon, \xi), t),$$

where t is a real number and $R'_1((A,B,\phi),(\upsilon,\xi),t)$ is a smooth map defined on some open set of X. We assume that the same property holds for $R'((A,\phi),(\upsilon,\xi))$, $R'((B,\phi),(\upsilon,\xi))$, $R'(\upsilon,\xi)$ and $R'(\phi,\xi)$.

By (17), we only have to prove

$$\begin{aligned}(19)\quad &(A+B)\star\Phi\\ &=A+B+\bar\partial_{\mathrm{End}(E)}\upsilon+\xi\lrcorner F_{d_K}+[\theta,\upsilon]+\{\partial_K^{\mathrm{End}(E)},\xi\lrcorner\}\theta+R((A,B,\phi),(\upsilon,\xi)).\end{aligned}$$

First we prove

$$(20)\qquad A\star\Phi=A+\bar\partial_{\mathrm{End}(E)}\upsilon+\xi\lrcorner F_{d_K}+R((A,\phi),(\upsilon,\xi)).$$

Let U' and U be open sets of X such that $U'\subset U$ and $f_\xi(U')\subset U$. We calculate $(A\star\Phi-A)(z)$ for $z\in U'$. Let $\{e_k\}$ be a holomorphic frame on U for $E_{\eta'}$. Since $E_{\eta'}$'s complex structure is induced by Φ, $\Phi:E_{\eta'}\to E_\eta$ is a holomorphic bundle isomorphism. Hence $\Phi(e_k)$ is a holomorphic section for E_η. Hence we have

$$(21)\qquad \bar\partial_E e_k+\{\partial_K,(\phi\circ f_\xi)\lrcorner\}e_k+(A\star\Phi)e_k=0,$$

$$(22)\qquad \Phi^{-1}\circ(\bar\partial_E+\{\partial_k,\phi\lrcorner\}+A)\circ\Phi(e_k)=0.$$

By (17), (21) is equivalent to

$$(23)\qquad \bar\partial_E e_k+\{\partial_K,\phi\lrcorner\}e_k+\{\partial_K,\bar\partial_{TX}\xi\lrcorner\}e_k+(A\star\Phi)e_k+R'(\phi,\xi)(e_k)=0.$$

Let P'_ξ be the first order of P_ξ. Since $\Phi=P_\xi\circ\exp(\upsilon)$, we have an expansion for $\Phi(e_k)$ such that

$$\Phi(e_k)=P_\xi\circ\exp(\upsilon)(e_k)=e_k+P'_\xi(e_k)+\upsilon(e_k)+R'(\upsilon,\xi)(e_k).$$

Hence (22) is equivalent to

$$\begin{aligned}&\bar\partial_E e_k+\{\partial_K,\phi\lrcorner\}e_k+Ae_k+\bar\partial_E(P'_\xi e_k)-P'_\xi\bar\partial_E e_k\\ &\qquad+\bar\partial_E\upsilon e_k-\upsilon\bar\partial_E(e_k)+R'((A,\phi),(\upsilon,\xi))(e_k)=0.\end{aligned}$$

Since $\bar\partial_E(\upsilon e_k)-\upsilon\bar\partial_E e_k=(\bar\partial_{\mathrm{End}(E)}\upsilon)e_k$, we have

$$\begin{aligned}(24)\quad &\bar\partial_E e_k+\{\partial_K,\phi\lrcorner\}e_k+Ae_k+\bar\partial_E(P'_\xi e_k)-P'_\xi\bar\partial_E e_k\\ &\qquad+(\bar\partial_{\mathrm{End}(E)}\upsilon)e_k+R'((A,\phi),(\upsilon,\xi))(e_k)=0.\end{aligned}$$

Hence by (23), (24),

$$\begin{aligned}(25)\quad &(A\star\Phi)e_k-Ae_k+\{\partial_K,\bar\partial_{TX}\xi\lrcorner\}e_k-\bar\partial_E(P'_\xi e_k)+P'_\xi\bar\partial_E e_k\\ &\qquad-(\bar\partial_{\mathrm{End}(E)}\upsilon)e_k+R'((A,\phi),(\upsilon,\xi))(e_k)=0.\end{aligned}$$

We have to prove $\{\partial_K,\bar\partial_{TX}\xi\lrcorner\}-\bar\partial_E\circ P'_\xi+P'_\xi\circ\bar\partial_E=-\xi\lrcorner F_{d_K}$. We prove this for a holomorphic frame $\{e'_k\}$ for E on U. Since P_ξ is the parallel transport along f_ξ

respect to the Chern connection, we have $P'_\xi(e'_k) = -\xi \lrcorner K^{-1}\partial K(e'_k)$. (See [Spivak 1999] for more details). Hence we have

$$\begin{aligned}\{\partial_K, \bar\partial_{TX}\xi\lrcorner\}e'_k - \bar\partial_E \circ P'_\xi e'_k + P'_\xi \circ \bar\partial_E e'_k &= -\bar\partial_{TX}\xi\lrcorner\partial_K e'_k + \bar\partial_E(\xi\lrcorner K^{-1}\partial K e'_k)\\ &= -\bar\partial_{TX}\xi\lrcorner\partial_K e'_k + \bar\partial_{TX}\xi\lrcorner K^{-1}\partial K e'_k - \xi\lrcorner(\bar\partial_{\mathrm{End}(E)}(K^{-1}\partial K))e'_k\\ &= (-\xi\lrcorner F_{d_K})e'_k.\end{aligned}$$

Hence by (25), we have

$$(A \star \Phi)e_k = Ae_k + (\bar\partial_{\mathrm{End}(E)}\upsilon)e_k + \xi\lrcorner F_{d_K}(e_k) + R'((A,\phi),(\upsilon,\xi))(e_k).$$

Since $\{e_k\}$ is an arbitrary holomorphic frame on $E_{\eta'}$ and $R'((A,\phi),(\upsilon,\xi))(e_k)$ is a local expression of $A\star\Phi - A - \bar\partial_{\mathrm{End}(E)}\upsilon - \xi\lrcorner F_{d_K}$ we proved (20).

Next, we prove that

$$B\star\Phi = B + [\theta,\upsilon] + \{\partial_K^{\mathrm{End}(E)}, \xi\lrcorner\}\theta + R((B,\phi),(\upsilon,\xi)). \tag{26}$$

We recall that $\widehat{\Phi} : E_{\eta'} \to f_\xi^*(E_\eta)$ is a holomorphic bundle isomorphism and $\theta_{\eta'} = \widehat{\Phi}^{-1}\circ f_\xi^*(\theta_\eta)\circ\widehat{\Phi}$. Let $\theta_{\eta'}^{1,0}$ is the $(1,0)$-part of $\theta_{\eta'}$ respect to the original complex structure, then we have $B\star\Phi = \theta_{\eta'}^{1,0} - \theta$. We calculate $B\star\Phi$ locally.

Let (U,z) be a local coordinate and $U' \subset U$. We assume $f_\xi(U') \subset U$ and $\xi = \sum_i \xi_i(z)\big(\frac{\partial}{\partial z_i}\big)$. By the definition of f_ξ, for $z \in U'$, we have

$$f_\xi(z) = \big(z_1 + \xi_1(z) + O(|\xi|^2), \dots, z_n + \xi_n(z) + O(|\xi|^2)\big).$$

Let $\{e_k\}$ be a holomorphic frame on U for E. Let $g_i, B_i \in A^0(\mathrm{End}(E))$. Assume that θ is locally expressed as $\sum_i g_i(z)dz_i$ and B as $\sum_i B_i(z)dz_i$ respect to this frame.

Let the bracket $[\,\cdot\,,\cdot\,]$ be the canonical Lie bracket defined on $A^*(\mathrm{End}(E))$. Since $\theta_\eta = \theta + B + \phi\lrcorner(\theta+B) = (g_i+B_i)dz_i + (g_i+B_i)\phi^i_j d\bar z_j$ and $\widehat{\Phi}$ is induced by Φ, we have

$$\begin{aligned}\theta_{\eta'}(e_k) &= \widehat{\Phi}^{-1}\circ f_\xi^*(\theta_\eta)\circ\widehat{\Phi}(e_k)\\ &= \widehat{\Phi}^{-1}\circ\{(g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j}\}\circ\widehat{\Phi}(e_k)\\ &= ((g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j})(e_k)\\ &\quad + [(g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j}, P'_\xi](e_k)\\ &\quad + [(g_i+B_i)(f_\xi(z))df_{\xi,i}(z) + (g_i+B_i)(f_\xi(z))\phi^i_j(f_\xi(z))d\bar f_{\xi,j}, \upsilon](e_k)\\ &\quad + R'((B,\phi),(\upsilon,\xi))(e_k).\end{aligned}$$

Hence

$$\begin{aligned}
(27)\quad \theta^{1,0}_{\eta'}(e_k) &= \left((g_i+B_i)(f_\xi(z))dz_i+(g_i+B_i)(f_\xi(z))\frac{\partial\xi_i}{\partial z_j}dz_j\right)(e_k)\\
&\quad+\left[((g_i+B_i)(f_\xi(z))dz_i+(g_i+B_i)(f_\xi(z))\frac{\partial\xi_i}{\partial z_j}dz_j,\, P'_\xi\right](e_k)\\
&\quad+\left[((g_i+B_i)(f_\xi(z))dz_i+(g_i+B_i)(f_\xi(z))\frac{\partial\xi_i}{\partial z_j}dz_j,\, \upsilon\right](e_k)\\
&\quad+R'((B,\phi),(\upsilon,\xi))(e_k)\\
&= \left((g_i+B_i)(f_\xi(z))dz_i+g_i(f_\xi(z))\frac{\partial\xi_i}{\partial z_j}dz_j\right)(e_k)\\
&\quad+\big[g_i(f_\xi(z))dz_i,\, P'_\xi\big](e_k)+\big[g_i(f_\xi(z))dz_i,\, \upsilon\big](e_k)\\
&\quad+R'((B,\phi),(\upsilon,\xi))(e_k).
\end{aligned}$$

Since $f_\xi(z)=c(z,\xi,1)$, we have the Taylor expansion at $t=0$ for $g_i(f_\xi(z))$ and $B_i(f_\xi(z))$:

$$g_i(f_\xi(z))=g_i(z)+\xi_j(z)\frac{\partial g_i}{\partial z_j}(z)+O(|\xi|^2),$$

$$B_i(f_\xi(z))=B_i(z)+\xi_j(z)\frac{\partial B_i}{\partial z_j}(z)+O(|\xi|^2).$$

Hence by (27), $\theta^{1,0}_{\eta'}(e_k)$ becomes

$$\begin{aligned}
\theta^{1,0}_{\eta'}(e_k) &= \left(g_i(z)dz_i+\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+B_i(z)dz_i+g_i(z)\frac{\partial\xi_i}{\partial z_j}dz_j\right)(e_k)\\
&\quad+[g_i(z)dz_i,\, P'_\xi](e_k)+[g_i(z)dz_i,\, \upsilon](e_k)+R'((B,\phi),(\upsilon,\xi))(e_k)\\
&=(\theta+B)(e_k)+\left(\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+g_i(z)\frac{\partial\xi_i}{\partial z_j}dz_j\right)(e_k)\\
&\quad+[\theta,\, P'_\xi](e_k)+[\theta,\upsilon](e_k)+R'((B,\phi),(\upsilon,\xi))(e_k).
\end{aligned}$$

Hence

$$\begin{aligned}
(28)\quad B\star\Phi(e_k) &= \theta^{1,0}_{\eta}(e_k)-\theta(e_k)\\
&= B(e_k)+\left(\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+g_i(z)\frac{\partial\xi_i}{\partial z_j}dz_j\right)(e_k)\\
&\quad+[\theta,\, P'_\xi](e_k)+[\theta,\upsilon](e_k)+R'((B,\phi),(\upsilon,\xi))(e_k).
\end{aligned}$$

Hence the only thing we have to prove is

$$\left(\xi_j\frac{\partial g_i}{\partial z_j}(z)dz_i+g_i(z)\frac{\partial\xi_i}{\partial z_j}dz_j\right)(e_k)+[\theta,\, P'_\xi](e_k)=(\{\partial_K^{\mathrm{End}(E)},\xi\lrcorner\}\theta)(e_k).$$

Since $\{e_k\}$ is a local holomorphic frame for E, we have

$$\begin{aligned}
&\left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k) + [\theta, P'_\xi](e_k) \\
&\quad = \left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k) + \theta(-\xi \lrcorner K^{-1}\partial K)(e_k) + \xi \lrcorner K^{-1}\partial K(\theta(e_k)) \\
&\quad = \left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right)(e_k) + [\xi \lrcorner K^{-1}\partial K, \theta](e_k)
\end{aligned}$$

and

$$\begin{aligned}
&\{\partial_K, \xi \lrcorner\}\theta \\
&\quad = \partial_K(\xi \lrcorner \theta) + \xi \lrcorner \partial_K(\theta) \\
&\quad = \partial(\xi_i(z)g_i) + [K^{-1}\partial K, \xi \lrcorner \theta] + \xi_j(z)\frac{\partial g_i}{\partial z_j}dz_i - \xi_i(z)\frac{\partial g_i}{\partial z_j}dz_j + \xi \lrcorner [K^{-1}\partial K, \theta] \\
&\quad = \frac{\partial}{\partial z_j}(\xi_i(z)g_i(z))dz_j + \xi_j(z)\frac{\partial g_i}{\partial z_j}dz_i - \xi_i(z)\frac{\partial g_i}{\partial z_j}dz_j + [\xi \lrcorner K^{-1}\partial K, \theta] \\
&\quad = \left(\xi_j \frac{\partial g_i}{\partial z_j}(z)dz_i + g_i(z)\frac{\partial \xi_i}{\partial z_j}dz_j\right) + [\xi \lrcorner K^{-1}\partial K, \theta].
\end{aligned}$$

Hence we have the desired equality. Hence by (28) we have

$$B \star \Phi(e_k) = B(e_k) + [\theta, \upsilon](e_k) + (\{\partial_K^{\mathrm{End}(E)}, \xi \lrcorner\}\theta)(e_k) + R'((B, \phi), (\upsilon, \xi))(e_k).$$

Since $\{e_k\}$ is an arbitrary local holomorphic frame on E and $R'((B, \phi), (\upsilon, \xi))(e_k)$ is a local expression of $B \star \Phi - B - [\theta, \upsilon] - (\{\partial_K^{\mathrm{End}(E)}, \xi \lrcorner\}\theta)$, we proved (26).

Hence by (20) and (26) we proved (19). This completes the proof. □

Recall that $\mathbb{H}^0 = \ker(\Delta_L) \subset L^0$. Let F^0 be the orthogonal complement of $\mathbb{H}^0$ w.r.t. the inner product $(\cdot, \cdot)$. Note that $\ker(H) = F^0$. H is the harmonic projection. Then, for $\gamma \in F^0$,

$$\eta = G\Delta_L\gamma + H\gamma = G\Delta_L\gamma.$$

Since d_L^* is zero on L^0, $d_L^*(\gamma) = 0$. Hence

$$\Delta_L\gamma = d_L^* d_L\gamma.$$

This yields,

$$\gamma = G d_L^* d_L \gamma. \tag{29}$$

From now on, we think of L^1, F^0 as normed by the k-th Sobolev norm and by the $(k-1)$-th Sobolev norm. Let L^1_{k-1}, L^1_k, F^0_k be the completion of L^1, F^0 with respect to the corresponding norms.

Proposition 4.6. *Let $\eta(\gamma) := \eta + d_L\gamma + R(\eta, \gamma)$. There are neighborhoods of the origin U and V in L^1 and F^0 such that for any $\eta \in U$ there is a $\gamma \in V$ such that*

$$d_L^*(\eta(\gamma)) = 0. \tag{30}$$

Proof. Let $\gamma := (\upsilon, \xi) \in F^0$. By the definition of $\eta(\gamma)$, (30) is equivalent to

$$0 = d_L^*(\eta(\gamma)) = d_L^*\eta + d_L^* d_L\gamma + d_L^* R(\eta, \gamma).$$

By (29),

$$\gamma = G d_L^* d_L \gamma = -G d_L^* \eta - G d_L^* R(\eta, \gamma).$$

Thus (30) is equivalent to

$$\gamma + G d_L^* \eta + G d_L^* R(\eta, \gamma) = 0.$$

Let U_1 and V_1 are neighborhoods of the origin of L_k^1 and F_k^0. By the local property of $R(\eta, \gamma)$ which we observed in Proposition 4.5, we can define C^1 map $h : U_1 \times V_1 \to L_{k-1}^1$ by

$$h(\eta, \gamma) = \gamma + G d_L^* \eta + G d_L^* R(\eta, \gamma).$$

By the order condition on the error term R, the identity map is the derivative of h concerning γ at $(0, 0)$. Hence by the implicit function theorem for Banach spaces, there exists an open neighborhood U_0 of $0 \in L_k^1$ and a continuous map $g : U_0 \to V_1$ such that $g(0) = 0$ and such that $h(\eta, \gamma) = 0$ if and only if $\gamma = g(\eta)$ for all $\eta \in U_0$ (see [Lang 1993] for details).

Let $U := U_0 \cap L^0$ and $V := g(U_0) \cap F^0$. Let $\eta \in U$ and $\gamma := g(\eta)$. By the previous section, we have $h(\eta, \gamma) = 0$. If we take U_0 small enough, $\Delta_L + d_L^* R(\eta, \cdot) + d_L^* \eta$ is a quasilinear elliptic operator. By elliptic regularity, γ is smooth. Hence $\gamma \in V$. Hence this completes the proof. □

We can now give the proof of Theorem 4.2.

Proof of Theorem 4.2. Let $\eta \in L^1$ be a Maurer–Cartan element and $|\eta|_k \ll 1$. By Proposition 4.4, we only have to prove the theorem for $d_{T(E)}^* \eta \neq 0$. By Proposition 4.6, we have a $\gamma = (\upsilon, \xi) \in L^0$ such that

$$d_L^* \eta + d_L^* d_L \gamma + d_L^* R(\eta, \gamma) = 0.$$

Let $\Phi := P_\xi \circ \exp(\upsilon)$. We can induce a structure of holomorphic-Higgs triple on (X, E, θ) that is isomorphic to $(X_\eta, E_\eta, \theta_\eta)$ by Φ and f_ξ. We denote the corresponding Maurer–Cartan element as η_γ. By Proposition 4.5, we have

$$\eta_\gamma = \eta + d_L\gamma + R(\eta, \gamma).$$

We can easily see that $d_L^* \eta_\gamma = 0$. Hence by Proposition 4.4, we have some $t \in \mathcal{S}$ such that $(X_{\epsilon(t)}, E_{\epsilon(t)}, \theta_{\epsilon(t)}) \cong (X_\eta, E_\eta, \theta_\eta)$. This completes the proof. □

Acknowledgements

The author would like to express his gratitude to his supervisor Hisashi Kasuya for his encouragement and patience. The author thanks Masataka Iwai and Takahiro Saito for their kindness and encouragement. The author would like to thank the referee for comments which helped to improve the paper and patience in reading this manuscript.

References

[Chan and Suen 2016] K. Chan and Y.-H. Suen, "A differential-geometric approach to deformations of pairs (X, E)", *Complex Manifolds* **3**:1 (2016), 16–40. MR Zbl

[Douglis and Nirenberg 1955] A. Douglis and L. Nirenberg, "Interior estimates for elliptic systems of partial differential equations", *Comm. Pure Appl. Math.* **8** (1955), 503–538. MR Zbl

[Hitchin 1987] N. J. Hitchin, "The self-duality equations on a Riemann surface", *Proc. London Math. Soc.* (3) **55**:1 (1987), 59–126. MR Zbl

[Huang 1995] L. Huang, "On joint moduli spaces", *Math. Ann.* **302**:1 (1995), 61–79. MR Zbl

[Huybrechts and Thomas 2010] D. Huybrechts and R. P. Thomas, "Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes", *Math. Ann.* **346**:3 (2010), 545–569. MR Zbl

[Kodaira 1986] K. Kodaira, *Complex manifolds and deformation of complex structures*, Grundl. Math. Wissen. **283**, Springer, 1986. MR

[Kodaira and Spencer 1958a] K. Kodaira and D. C. Spencer, "On deformations of complex analytic structures, I", *Ann. of Math.* (2) **67** (1958), 328–401. MR Zbl

[Kodaira and Spencer 1958b] K. Kodaira and D. C. Spencer, "On deformations of complex analytic structures, II", *Ann. of Math.* (2) **67** (1958), 403–466. Zbl

[Kodaira and Spencer 1960] K. Kodaira and D. C. Spencer, "On deformations of complex analytic structures, III: Stability theorems for complex structures", *Ann. of Math.* (2) **71** (1960), 43–76. MR Zbl

[Kuranishi 1965] M. Kuranishi, "New proof for the existence of locally complete families of complex structures", pp. 142–154 in *Proc. Conf. Complex Analysis* (Minneapolis, MN, 1964), edited by A. Aeppli et al., Springer, 1965. MR Zbl

[Lang 1993] S. Lang, *Real and functional analysis*, 3rd ed., Graduate Texts in Mathematics **142**, Springer, 1993. MR Zbl

[Li 2008] S. Li, "On the deformation theory of pair (X, E)", preprint, 2008. arXiv 0809.0344

[Martinengo 2009] E. Martinengo, *Higher brackets and moduli space of vector bundles*, Ph.D. thesis, Sapienza Università di Roma, 2009, available at https://www1.mat.uniroma1.it/PhD/TESI/ARCHIVIO/martinengoelena.pdf.

[Martinengo 2012] E. Martinengo, "Infinitesimal deformations of Hitchin pairs and Hitchin map", *Internat. J. Math.* **23**:7 (2012), art. id. 1250053. MR Zbl

[Moroianu 2007] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts **69**, Cambridge University Press, 2007. MR Zbl

[Morrow and Kodaira 2006] J. Morrow and K. Kodaira, *Complex manifolds*, AMS Chelsea, Providence, RI, 2006. MR Zbl

[Ono 2023] T. Ono, "Structure of the Kuranishi Spaces of pairs of Kähler manifolds and polystable Higgs bundles", preprint, 2023. arXiv 2310.06439

[Sernesi 2006] E. Sernesi, *Deformations of algebraic schemes*, Grundl. Math. Wissen. **334**, Springer, 2006. MR Zbl

[Simpson 1988] C. T. Simpson, "Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization", *J. Amer. Math. Soc.* **1**:4 (1988), 867–918. MR Zbl

[Simpson 1992] C. T. Simpson, "Higgs bundles and local systems", *Inst. Hautes Études Sci. Publ. Math.* 75 (1992), 5–95. MR Zbl

[Siu and Trautmann 1981] Y. T. Siu and G. Trautmann, *Deformations of coherent analytic sheaves with compact supports*, Mem. Amer. Math. Soc. **238**, Amer. Math. Soc., Providence, RI, 1981. MR Zbl

[Spivak 1999] M. Spivak, *A comprehensive introduction to differential geometry*, *II*, 3rd ed., Publish or Perish, Wilmington, DE, 1999.

[Wells 1980] R. O. Wells, Jr., *Differential analysis on complex manifolds*, 2nd ed., Graduate Texts in Mathematics **65**, Springer, 1980. MR Zbl

Received November 1, 2023. Revised June 13, 2024.

TAKASHI ONO
u708091f@ecs.osaka-u.ac.jp
DEPARTMENT OF MATHEMATICS
OSAKA UNIVERSITY
OSAKA
JAPAN

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.317

UNIFORM EXTENSION OF DEFINABLE $C^{m,\omega}$-WHITNEY JETS

ADAM PARUSIŃSKI AND ARMIN RAINER

We show that definable Whitney jets of class $C^{m,\omega}$, where m is a nonnegative integer and ω is a modulus of continuity, are the restrictions of definable $C^{m,\omega}$-functions; "definable" refers to an arbitrary given o-minimal expansion of the real field. This is true in a uniform way: any definable bounded family of Whitney jets of class $C^{m,\omega}$ extends to a definable bounded family of $C^{m,\omega}$-functions. We also discuss a uniform C^m-version and how the extension depends on the modulus of continuity.

1. Introduction

Let an o-minimal expansion of the real field be fixed. Throughout the paper, a set $X \subseteq \mathbb{R}^n$ is called *definable* if it is definable in this fixed o-minimal structure. A map $\varphi : X \to \mathbb{R}^m$ is definable if its graph $\Gamma(\varphi) := \{(x, \varphi(x)) : x \in X\}$ is a definable subset of $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$ (this natural identification is used throughout the paper). We assume familiarity with the basics of o-minimal structures; see [5] and [4].

Due to Kurdyka and Pawłucki [9; 10] and Thamrongthanyalak [14] we have the definable C^m Whitney extension theorem:

Theorem 1.1. *Let $0 \le m \le p$ be integers. Let $E \subseteq \mathbb{R}^n$ be a definable closed set. Any definable Whitney jet of class C^m on E extends to a definable C^m-function on $\mathbb{R}^n$ which is of class C^p outside E.*

We prove a $C^{m,\omega}$-version of this result.

Theorem 1.2. *Let $0 \le m \le p$ be integers. Let ω be a modulus of continuity. Let $E \subseteq \mathbb{R}^n$ be a definable closed set. Any definable Whitney jet of class $C^{m,\omega}$ on E extends to a definable $C^{m,\omega}$-function on $\mathbb{R}^n$ which is of class C^p outside E.*

By a *modulus of continuity* we always mean a positive, continuous, increasing, and concave function $\omega : (0, \infty) \to (0, \infty)$ such that $\omega(t) \to 0$ as $t \to 0$. We say that *ω is a modulus of continuity for a function $f : S \to \mathbb{R}$*, defined on a subset $S \subseteq \mathbb{R}^n$, if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \le C\,\omega(|x - y|) \quad \text{for all } x, y \in S. \tag{1-1}$$

MSC2020: primary 03C64, 14P10, 32B20; secondary 26B35, 26E25, 46E15.
Keywords: o-minimal structures, Whitney extension theorem, $C^{m,\omega}$-extension of Whitney jets, uniform boundedness of the extension.

The class $C^{m,\omega}$ consists of C^m-functions that are globally bounded together with its partial derivatives up to order m and whose partial derivatives of order m satisfy a global ω-Hölder condition of the type (1-1). See Section 3.

We use Theorem 1.2 in [11] to show that a definable function $f : E \to \mathbb{R}$ on a definable closed set $E \subseteq \mathbb{R}^n$ that has a $C^{1,\omega}$-extension to $\mathbb{R}^n$ also has a definable $C^{1,\omega}$-extension. (In [11] we assume that ω is definable, but not in the present paper.) In fact, this application was one of our main motivations for proving Theorem 1.2.

We will show that the definable extension of Whitney jets of class $C^{m,\omega}$ can be done in a bounded way:

Theorem 1.3. *Let $0 \le m \le p$ be integers. Let ω be a modulus of continuity. Let $(E_a)_{a\in A}$ be a definable family of closed subsets of $\mathbb{R}^n$. For any definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ there exists a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ such that f_a is of class C^p outside E_a for all $a \in A$.*

Clearly, boundedness is understood with respect to the natural norms; see Section 3 for precise definitions. Theorem 1.2 follows as a special case from Theorem 1.3. And already the case that $(F_a)_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on a *fixed set* $E = E_a$, for all $a \in A$, is very interesting. However, the method of proof (by induction on dimension) necessitates to consider the general case that the families of Whitney jets are defined on definable families of sets $(E_a)_{a\in A}$.

The construction of the extension in Theorem 1.3 depends on ω only in a weak sense. We may, for instance, let the modulus of continuity ω_a depend as well on $a \in A$ if we impose that there is a constant $C > 0$ such that $C^{-1} \le \omega_a(1) \le C$ for all $a \in A$. This will be discussed in detail in Section 5B in which we present a more general version of Theorem 1.3. As a consequence, we deduce from Theorem 1.3 a uniform version of the C^m-result Theorem 1.1 on compact sets:

Theorem 1.4. *Let $0 \le m \le p$ be integers. Let $(E_a)_{a\in A}$ be a definable family of compact subsets of $\mathbb{R}^n$. For any definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class C^m on $(E_a)_{a\in A}$ there exists a definable bounded family $(f_a)_{a\in A}$ of C^m-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ such that f_a is of class C^p outside E_a for all $a \in A$.*

Theorem 1.4 is proved in Section 5C. We deduce a local version of Theorem 1.3 in Section 5A and apply Theorem 1.3 in Section 5D to get a definable version of a correspondence, due to Shvartsman [13], between Whitney jets of class $C^{m,\omega}$ and certain Lipschitz maps.

The proof of Theorem 1.3 (which builds upon the one of Theorem 1.1 devised in [9; 10; 14] and also Pawłucki [12] and is very different from Whitney's classical method [16]) rests on two main cornerstones:

(1) *Two versions of Gromov's inequality* [7]*; one classical, the other incorporating the modulus of continuity.* These are inequalities for the derivatives of a definable function. Since the constants that appear in them are universal, it is not difficult to get them uniform for definable families of functions. See Corollary 2.18 and Proposition 2.19.

(2) *Uniform Λ_p-stratification of definable families of sets.* Roughly speaking, definable families of sets admit a stratification into a finite number of cells that are defined by functions satisfying certain estimates (for their derivatives up to order p). The constants in these estimates and the number of cells are independent of the parameter of the family. See Theorem 2.16. This is essential for the uniform extension Theorem 1.3. We think that it is also of independent interest.

It is a natural question if there exists even a continuous and/or linear extension operator for definable Whitney jets of class $C^{m,\omega}$ (or C^m) on a definable closed set $E \subseteq \mathbb{R}^n$. This remains an open problem. The theorem of Bartle and Graves [2] (see also [3, Theorem 1.6]) is not applicable since the normed spaces of definable jets and functions (defined in Section 3) are not complete.

Azagra, Le Gruyer, and Mudarra [1] give an explicit formula for the extension of Whitney jets of class $C^{1,1}$ with an optimal control of the norms; for definable input this explicit formula yields a definable $C^{1,1}$-extension. See [11, Sections 4.2–4].

Let us point out that Pawłucki [12] presents a continuous linear extension operator for (not necessarily definable) Whitney jets on definable closed sets which preserves (up to a multiplicative constant) the modulus of continuity. This extension operator is a finite composite of operators that preserve definability on the one hand or are defined by integration with respect to a parameter (more precisely, convolution) on the other hand; in general, the latter leads out of the original o-minimal structure.

While [12] was a important source of inspiration for handling the modulus of continuity, the main difficulty (besides getting everything uniformly bounded) was to replace the convolution operators by definable operations which at the same time allow for preserving the modulus of continuity.

The paper is organized as follows. In Section 2 the main geometric tools are prepared: Gromov's inequality and uniform Λ_p-stratification. We present in Section 3 background on definable bounded families of Whitney jets of class $C^{m,\omega}$, most notably, how they behave under pullback along a definable family of Λ_p-regular maps. The proof of Theorem 1.3 is carried out in Section 4. In the final Section 5 we give the mentioned applications, discuss dependence on the modulus of continuity, and prove Theorem 1.4.

Notation. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ be the set of nonnegative integers. We denote by $d(x, S) := \inf_{y \in S} |x - y|$ the Euclidean distance in $\mathbb{R}^n$ of a point x to a subset S of $\mathbb{R}^n$, with the convention $d(x, \varnothing) := +\infty$. The open Euclidean ball with center

$x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The closure of a set S is denoted by $\overline{S}$ and the frontier of S by $\partial S := \overline{S} \setminus S$. If S is a subset of $\mathbb{R}^k$, we write $S \times 0$ for the set $\{(u, w) \in \mathbb{R}^k \times \mathbb{R}^\ell : u \in \mathbb{R}^k, w = 0\}$. The graph of a map φ is denoted by $\Gamma(\varphi)$. For real-valued nonnegative functions f, g we write $f \lesssim g$ if $f \le Cg$ for some universal constant $C > 0$. In particular, it should be always understood that C is independent of $a \in A$, i.e., the parameter which we consistently use in parameterized families of sets and maps. We write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. We use standard multi-index notation and in this context $(i) \in \mathbb{N}^n$ is the multi-index $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th entry.

2. Uniform Λ_p-stratifications

The existence of uniform Λ_p-stratifications (Theorem 2.16) is based on an inequality of Gromov [7], of which we need two versions, and on uniform L-regular decomposition due to Kurdyka and Parusiński [8].

2A. ***Definable families of sets and maps.*** Let A be a definable subset of $\mathbb{R}^N$. A family $(E_a)_{a\in A}$ of definable sets $E_a \subseteq \mathbb{R}^n$ is called a *definable family* if the associated set

$$E := \bigcup_{a\in A} \{a\} \times E_a \tag{2-1}$$

is a definable subset of $\mathbb{R}^N \times \mathbb{R}^n$. Conversely, any definable subset $E \subseteq \mathbb{R}^N \times \mathbb{R}^n$ defines a definable family $(E_a)_{a\in A}$ by setting $A := \{a \in \mathbb{R}^N : \exists x \in \mathbb{R}^n, (a, x) \in E\}$ and $E_a := \{x \in \mathbb{R}^n : (a, x) \in E\}$, where $a \in A$. If we allow $E_a = \varnothing$, we may just take $A = \mathbb{R}^N$.

A family $(E'_a)_{a\in A}$ of subsets $E'_a \subseteq E_a$ is said to be a *definable subfamily* of $(E_a)_{a\in A}$ if the associated set E' (defined in analogy to (2-1)) is a definable subset of E.

A family $(\varphi_a)_{a\in A}$ of definable maps $\varphi_a : E_a \to \mathbb{R}^m$ is called a *definable family* if the map $\varphi : E \to \mathbb{R}^m$, where E is the associated set (2-1) and

$$\varphi(a, u) := \varphi_a(u), \quad u \in E_a, \tag{2-2}$$

is definable. This is consistent with the first paragraph, since

$$\begin{aligned}\Gamma(\varphi) &= \bigl\{(a, u, \varphi(a, u)) \in \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^m : (a, u) \in E\bigr\}\\ &= \bigcup_{a\in A} \bigl\{(a, u, \varphi_a(u)) \in \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^m : u \in E_a\bigr\}\\ &= \bigcup_{a\in A} \{a\} \times \bigl\{(u, \varphi_a(u)) \in \mathbb{R}^n \times \mathbb{R}^m : u \in E_a\bigr\} = \bigcup_{a\in A} \{a\} \times \Gamma(\varphi_a).\end{aligned}$$

2B. ***Gromov's inequality.*** We need two versions of an inequality due to Gromov [7]. We start with a C^m-version.

Lemma 2.1 [9, Lemma 2]. *Let $m \ge 1$. Let $f : I \to \mathbb{R}$ be a C^{m+1}-function, where $I = [t_0 - r, t_0 + r] \subseteq \mathbb{R}$, $r > 0$, is an interval. Suppose that, for all $j = 2, \dots, m+1$, we have either $f^{(j)} \ge 0$ on I or $f^{(j)} \le 0$ on I. Then*

$$|f^{(m)}(t_0)| \le 2^{\binom{m+2}{2}-2} \frac{\sup_{t\in I} |f(t)|}{r^m}.$$

We combine Lemma 2.1 with the following lemma in order to get a $C^{m,\omega}$-version in Lemma 2.3.

Lemma 2.2. *Let $f : I \to \mathbb{R}$ be a C^2-function, where $I = [t_0 - r, t_0 + r] \subseteq \mathbb{R}$, $r > 0$, is an interval, and let ω be a modulus of continuity for f. Suppose that $f'' \ge 0$ on I or $f'' \le 0$ on I. Then*

$$|f'(t_0)| \le \frac{\omega(r)}{r}.$$

Proof. We may assume that $t_0 = 0$. Suppose that $f'' \ge 0$ on I. Then f is convex and, for $0 < s < r$,

$$\frac{f(s) - f(0)}{s} \le \frac{f(r) - f(0)}{r} \le \frac{\omega(r)}{r}.$$

Letting $s \to 0$, we find that $f'(0) \le \omega(r)/r$. The same reasoning applied to $f(-t)$ shows that also $-f'(0) \le \omega(r)/r$ so that the assertion is proved.

The case $f'' \le 0$ follows from the previous one by considering $-f$. □

Lemma 2.3. *Let $m \ge 1$. Let $f : I \to \mathbb{R}$ be a C^{m+1}-function, where $I = [t_0 - r, t_0 + r] \subseteq \mathbb{R}$, $r > 0$, is an interval, and let ω be a modulus of continuity for f. Suppose that, for all $j = 2, \dots, m+1$, we have either $f^{(j)} \ge 0$ on I or $f^{(j)} \le 0$ on I. Then*

$$|f^{(m)}(t_0)| \le 2^{\binom{m+1}{2}+m-2} \frac{\omega(r)}{r^m}.$$

Proof. If $m = 1$, then the statement is immediate from Lemma 2.2. If $m \ge 2$, then, by Lemma 2.1 applied to f' and in turn Lemma 2.2, we have

$$|f^{(m)}(t_0)| \le \frac{2^{\binom{m+1}{2}-2}}{\left(\frac{r}{2}\right)^{m-1}} \sup_{|t-t_0|\le\frac{r}{2}} |f'(t)| \le \frac{2^{\binom{m+1}{2}-2}}{\left(\frac{r}{2}\right)^{m-1}} \cdot \frac{\omega\left(\frac{r}{2}\right)}{\frac{r}{2}} \le 2^{\binom{m+1}{2}+m-2} \frac{\omega(r)}{r^m}$$

as claimed, since ω is increasing. □

2C. *Uniform bounds for definable families of functions.*

Proposition 2.4. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$ and let $U \subseteq \mathbb{R}^N \times \mathbb{R}^k$ be the associated definable set (see (2-1)). Let $(\varphi_a)_{a\in A}$ be a definable family of functions $\varphi_a : U_a \to \mathbb{R}$ and let $\varphi : U \to \mathbb{R}$ be the associated definable function (see (2-2)). Let $\alpha \in \mathbb{N}^k$ with $|\alpha| = p$. There exists a definable subset $Z \subseteq U$*

such that, for all $a \in A$, Z_a is closed in U_a, $\dim Z_a < k$, *φ_a is C^p on $U_a \setminus Z_a$, and, for each open ball $B = B(u, r)$, $r > 0$, contained in $U_a \setminus Z_a$, we have*

$$|\partial^\alpha \varphi_a(u)| \le C(k, p) \sup_{v \in B} |\varphi_a(v)| \, r^{-|\alpha|}. \tag{2-3}$$

Proof. Consider the definable set

$$\begin{aligned} X &:= \{(b, v) \in U : (a, u) \mapsto \varphi(a, u) \text{ is not } C^{p+1} \text{ at } (b, v) \text{ in } u\} \\ &= \{(b, v) \in U : \varphi_b \text{ is not } C^{p+1} \text{ at } v\} \end{aligned}$$

and note that

$$X_a = \{u \in U_a : \varphi_a \text{ is not } C^{p+1} \text{ at } u\}$$

is closed in U_a and, by o-minimality, $\dim X_a < k$.

The operator ∂^α is a linear combination of directional derivatives d_v^p for a finite collection V of suitably chosen unit directions v in $\mathbb{R}^k$. Let $\varphi_1, \dots, \varphi_s$ be an enumeration of all functions $d_v^j \varphi : U \setminus X \to \mathbb{R}$, $j = 2, \dots, p+1$, $v \in V$ (where d_v^j acts only in the u-variable: $d_v^j \varphi(a, u) = \partial_t^j|_{t=0} \varphi(a, u + tv)$). For $i = 1, \dots, s$, set

$$\begin{aligned} Y_i &:= \{(a, u) \in U \setminus X : \exists \epsilon > 0 \, \forall v \in B(u, \epsilon), \varphi_i(a, v) = 0\}, \\ Z_i &:= \{(a, u) \in U \setminus X : \varphi_i(a, u) = 0\} \setminus Y_i, \\ Z &:= X \cup \bigcup_{i=1}^s Z_i. \end{aligned}$$

Then Z is a definable subset of U and, for all $a \in A$, Z_a is closed in U_a and $\dim Z_a < k$.

Now (2-3) follows easily by applying Lemma 2.1 to $t \mapsto \varphi(a, u + tv)$. □

Corollary 2.5. *Let $(U_a)_{a \in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$ and let $(\varphi_a)_{a \in A}$ be a definable family of C^1-functions $\varphi_a : U_a \to \mathbb{R}$. Suppose that there is a constant $M > 0$ such that*

$$|\partial_j \varphi_a(u)| \le M, \quad a \in A, \ u \in U_a, \ j = 1, \dots, k.$$

Let p be a positive integer. There exists a definable family $(Z_a)_{a \in A}$ of closed definable sets $Z_a \subseteq U_a$ of dimension $\dim Z_a < k$ *such that, for all $a \in A$, φ_a is of class C^p on $U_a \setminus Z_a$ and*

$$|\partial^\gamma \varphi_a(u)| \le C(k, p) \, M \, d(u, Z_a \cup \partial U_a)^{1-|\gamma|}, \quad u \in U_a \setminus Z_a, \ 1 \le |\gamma| \le p.$$

Proof. Apply Proposition 2.4 to $\partial_j \varphi_a$. □

Remark 2.6. We may assume that Z_a is not empty so that $d(u, Z_a \cup \partial U_a)$ is always finite. We will tacitly make this assumption in all subsequent results of this type.

Proposition 2.7. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$ and $(\varphi_a)_{a\in A}$ a definable family of continuous functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a positive integer. Then there exists a definable family $(Z_a)_{a\in A}$ of closed subsets $Z_a \subseteq U_a$ of dimension* $\dim Z_a < k$ *such that, for all $a \in A$, φ_a is C^p on $U_a \setminus Z_a$ and*

$$|\partial^\gamma \varphi_a(x)| \le C(k,p)\,\frac{\omega(d(x, Z_a \cup \partial U_a))}{d(x, Z_a \cup \partial U_a)^{|\gamma|}}, \quad x \in U_a \setminus Z_a,\ 1 \le |\gamma| \le p, \tag{2-4}$$

where ω is a modulus of continuity for φ_a.

Proof. Follow the proof of Proposition 2.4 and use Lemma 2.3. □

Remark 2.8. We want to emphasize that the construction of $(Z_a)_{a\in A}$ is independent of ω.

2D. Λ_p-*regular mappings.* Let $U \subseteq \mathbb{R}^k$ be an open set. Let p be a positive integer. A C^p-mapping $\varphi : U \to \mathbb{R}^n$ is said to be Λ_p-*regular* if there exists a constant $C > 0$ such that

$$|\partial^\gamma \varphi(u)| \le C\, d(u, \partial U)^{1-|\gamma|}, \quad u \in U,\ 1 \le |\gamma| \le p. \tag{2-5}$$

Λ_p-regular maps behave nicely on quasiconvex sets. Let us first recall the definition of quasiconvexity.

Definition 2.9 (quasiconvex sets). A set $E \subseteq \mathbb{R}^n$ is called *quasiconvex* if there is a constant $C > 0$ such any two points $x, y \in E$ can be joined in E by a rectifiable path of length at most $C\,|x - y|$.

Let $\varphi : U \to \mathbb{R}^n$ by Λ_p-regular. If $E \subseteq U$ is a quasiconvex subset, then $\varphi|_E$ is Lipschitz on E and extends continuously to a map $\overline{\varphi}$ on $\overline{E}$.

2E. Λ_p-*cells.* Let p be a positive integer. We define recursively Λ_p-cells in $\mathbb{R}^n$:

A definable subset $S \subseteq \mathbb{R}^n$ is an *open Λ_p-cell in* $\mathbb{R}^n$ if,

- in the case $n = 1$, S is an open interval in $\mathbb{R}$,
- in the case $n > 1$, S is of the form

$$S = (\psi_1, \psi_2, T) := \{(x', x_n) : x' \in T,\ \psi_1(x') < x_n < \psi_2(x')\},$$

where T is an open Λ_p-cell in $\mathbb{R}^{n-1}$ and each ψ_i, $i = 1, 2$, is either a Λ_p-regular definable function $T \to \mathbb{R}$ or identically $-\infty$ or $+\infty$, and $\psi_1 < \psi_2$ on T. (Here $x' = (x_1, \dots, x_{n-1})$.)

Note that S is quasiconvex. If ψ_i is finite, then it is Lipschitz on T and has a continuous extension $\overline{\psi}_i$ to $\overline{T}$.

A definable subset S of $\mathbb{R}^n$ is a *k-dimensional Λ_p-cell in* $\mathbb{R}^n$, where $k = 0, \dots, n-1$, if

$$S = \{(u, w) : u \in T,\ w = \varphi(u)\} = \Gamma(\varphi),$$

where $u = (x_1, \dots, x_k)$, $w = (x_{k+1}, \dots, x_n)$, T is an open Λ_p-cell in $\mathbb{R}^k$, and $\varphi : T \to \mathbb{R}^{n-k}$ is a Λ_p-regular definable map.

Definition 2.10 (Λ_p-cell with constant C). A *Λ_p-cell S in $\mathbb{R}^n$* is an open or a k-dimensional Λ_p-cell in $\mathbb{R}^n$. We say that S is a Λ_p-cell in $\mathbb{R}^n$ *with constant C* if all the Λ_p-regular maps involved in the recursive definition of S satisfy (2-5) with the same constant C.

2F. ***Associated functions.*** We associate with any open Λ_p-cell S in $\mathbb{R}^n$ a sequence of $2n+1$ definable functions $\rho_j : \overline{S} \to [0, \infty]$, for $j = 0, 1, 2, \dots, 2n$. We put $\rho_0 \equiv 1$ and define ρ_j for $j \ge 1$ as follows:

Case $n = 1$: If $S = (a_1, a_2)$ we set

$$\rho_1(x) := \begin{cases} x - a_1 & \text{if } a_1 \in \mathbb{R}, \\ +\infty & \text{if } a_1 = -\infty, \end{cases}$$
$$\rho_2(x) := \begin{cases} a_2 - x & \text{if } a_2 \in \mathbb{R}, \\ +\infty & \text{if } a_2 = +\infty. \end{cases}$$

Case $n > 1$: If $S = (\psi_1, \psi_2, T)$ and σ_j, $j = 1, \dots, 2n-2$, are the functions associated with T, we set $\rho_j(x) = \sigma_j(x')$, for $j = 1, \dots, 2n-2$, and

$$\rho_{2n-1}(x) := \begin{cases} x_n - \overline{\psi}_1(x') & \text{if } \psi_1 \text{ is finite}, \\ +\infty & \text{if } \psi_1 \equiv -\infty, \end{cases}$$
$$\rho_{2n}(x) := \begin{cases} \overline{\psi}_2(x') - x_n & \text{if } \psi_2 \text{ is finite}, \\ +\infty & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

Remark 2.11. We add the function ρ_0 (which is not present in [9; 10; 14]) in order to handle the extension from unbounded Λ_p-cells (see the proof of Theorem 1.3).

Each of the functions ρ_j, that is finite, is Λ_p-regular on S and Lipschitz on $\overline{S}$; see [9, Lemma 4]. There is a positive constant $C > 0$ such that

$$\frac{1}{C} \min_{j \ge 1} \rho_j(x) \le d(x, \partial S) \le \min_{j \ge 1} \rho_j(x), \quad x \in \overline{S}, \tag{2-6}$$

where $d(x, \varnothing) = +\infty$ by convention; see [9, Lemma 3]. Consequently,

$$\frac{1}{C} \min_{j \ge 0} \rho_j(x) \le \min\{1, d(x, \partial S)\} \le \min_{j \ge 0} \rho_j(x), \quad x \in \overline{S}. \tag{2-7}$$

If ρ_j for $j \ge 1$ is finite, then there exists a positive constant $C > 0$ such that

$$\left| \partial^\gamma \left(\frac{1}{\rho_j} \right)(x) \right| \le C\, d(x, \partial S)^{-|\gamma|-1}, \quad x \in S,\ |\gamma| \le p; \tag{2-8}$$

see [9, Lemma 5] and (2-10). It follows that for all finite ρ_j, $j \geq 0$, we have

$$\left|\partial^\gamma\left(\frac{1}{\rho_j}\right)(x)\right| \leq C \min\{1, d(x, \partial S)\}^{-|\gamma|-1}, \quad x \in S,\ |\gamma| \leq p. \tag{2-9}$$

Remark 2.12. The constants C in (2-6)–(2-9) only depend on the constants of the Λ_p-regular maps involved in the definition of S.

For later reference, we recall that for a nonvanishing C^p-function r we have, for $1 \leq |\gamma| \leq p$,

$$\partial^\gamma\left(\frac{1}{r}\right) = \sum_{j=1}^{|\gamma|}\left(\sum_{\substack{\delta_1+\dots+\delta_j=\gamma\\ \delta_1\neq 0,\dots,\delta_j\neq 0}} a^\gamma_{\delta_1,\dots,\delta_j}\partial^{\delta_1}r\cdots\partial^{\delta_j}r\right)r^{-j-1}, \tag{2-10}$$

where $a^\gamma_{\delta_1,\dots,\delta_j}$ are integers that only depend on γ and $\delta_1, \dots, \delta_j$.

2G. *Λ_p-stratification of definable sets.* Recall that a definable C^p-*stratification* of a definable set $E \subseteq \mathbb{R}^n$ is a finite decomposition $\mathscr{S}$ of E into definable C^p-submanifolds of $\mathbb{R}^n$, called strata, such that, for each stratum $S \in \mathscr{S}$, the frontier $(\partial S) \cap E$ in E is the union of some strata of dimension $< \dim S$. A stratification is called *compatible* with a collection of finitely many definable subsets of E if each subset is a union of strata.

A definable C^p-stratification $\mathscr{S}$ of E is called a Λ_p-*stratification* if each stratum $S \in \mathscr{S}$ is a Λ_p-cell in $\mathbb{R}^n$ in some linear coordinate system.

Theorem 2.13 [9, Proposition 4; 10, Theorem 3]. *Let $E \subseteq \mathbb{R}^n$ be a definable set and let $E_1, \dots, E_\ell$ be definable subsets of E. Then there exists a Λ_p-stratification $\mathscr{S}$ of E that is compatible with $E_1, \dots, E_\ell$.*

2H. *Uniform Λ_p-stratifications of definable families of sets.* We prove a uniform version of Theorem 2.13. Let us first recall a result on uniform L-regular decompositions.

Theorem 2.14 [8, Proposition 1.4]. *Let $E^i \subseteq \mathbb{R}^N \times \mathbb{R}^n$, where $i \in I$, be a finite collection of definable sets. Then there exist finitely many disjoint definable sets $B^j \subseteq \mathbb{R}^N \times \mathbb{R}^n$, where $j \in J$, and linear orthogonal mappings $\varphi^j : \mathbb{R}^n \to \mathbb{R}^n$, where $j \in J$, such that*:

(1) *For every $a \in \mathbb{R}^N$, each $\varphi^j(B^j_a)$ is a standard L-regular cell in $\mathbb{R}^n$ with constant $C = C(n)$.*

(2) *For every $a \in \mathbb{R}^N$, the family of B^j_a, where $j \in J$, is a stratification of $\mathbb{R}^n$.*

(3) *For any $i \in I$, there exists $J_i \subseteq J$ such that $E^i_a = \bigcup_{j \in J_i} B^j_a$ for every $a \in \mathbb{R}^N$.*

Here a *standard L-regular cell in $\mathbb{R}^n$ with constant $C = C(n)$* (which is terminology used in [8]) is by definition nothing else than a Λ_1-cell with constant $C = C(n)$.

Definition 2.15 (uniform Λ_p-stratification). Let $(E_a)_{a\in A}$ be a definable family of sets $E_a \subseteq \mathbb{R}^n$ and let $E \subseteq \mathbb{R}^N \times \mathbb{R}^n$ be the associated definable set (see (2-1)). Let p be a positive integer.

A finite collection $\mathscr{S} = \{S^j\}_{j\in J}$ of disjoint definable sets $S^j \subseteq \mathbb{R}^N \times \mathbb{R}^n$ is called a *uniform Λ_p-stratification of $(E_a)_{a\in A}$* if

(1) there exist linear orthogonal maps $\varphi^j : \mathbb{R}^n \to \mathbb{R}^n$, $j \in J$, such that, for each $a \in A$ and each $j \in J$, $\varphi^j(S^j_a)$ is a Λ_p-cell in $\mathbb{R}^n$ with constant C independent of $a \in A$,

(2) for each $a \in A$, the family S^j_a, $j \in J$, is a stratification of E_a.

For all $a \in A$, let $\mathscr{S}_a := \{S^j_a\}_{j\in J}$. Abusing notation, we will also say that $(\mathscr{S}_a)_{a\in A}$ *is a uniform Λ_p-stratification of $(E_a)_{a\in A}$*.

Let I be a finite index set and, for each $i \in I$, let $(E^i_a)_{a\in A}$ be a definable subfamily of $(E_a)_{a\in A}$. A uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(E_a)_{a\in A}$ is said to be *compatible with $(E^i_a)_{a\in A}$, $i \in I$*, if additionally

(3) for each $i \in I$, there exists a subset $J_i \subseteq J$ such that $E^i_a = \bigcup_{j\in J_i} S^j_a$ for each $a \in A$.

By Theorem 2.14, there always exist uniform Λ_1-stratifications. We shall see that there exist uniform Λ_p-stratifications for all $p \ge 1$.

Theorem 2.16. *Let $(E_a)_{a\in A}$ be a definable family of sets $E_a \subseteq \mathbb{R}^n$ and let $(E^i_a)_{a\in A}$, $i \in I$, be a finite collection of definable subfamilies of $(E_a)_{a\in A}$. Let p be a positive integer. Then there exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(E_a)_{a\in A}$ compatible with $(E^i_a)_{a\in A}$, $i \in I$.*

Proof. Let $k = \max_{a\in A} \dim E_a$. We proceed by induction on k. If $k = 0$, then all E_a are finite and the number of elements of E_a is bounded by a constant independent of a. In that case, the assertion is trivially true.

Suppose that $k > 0$. We claim that there exist a finite collection of disjoint definable sets $T^j \subseteq \mathbb{R}^N \times \mathbb{R}^n$, $j \in J$, and linear orthogonal maps $\varphi^j : \mathbb{R}^n \to \mathbb{R}^n$, $j \in J$, such that, for each $a \in A$ and each $j \in J$,

- T^j_a is either empty or open in E_a and compatible with E^i_a, $i \in I$,
- if $T^j_a \ne \varnothing$ then $\varphi^j(T^j_a)$ is a k-dimensional Λ_p-cell in $\mathbb{R}^n$ with constant C independent of $a \in A$, and
- $\dim E_a \setminus \bigcup_{j\in J} T^j_a < k$.

We allow $T^j_a = \varnothing$ to account for the case $\dim E_a < k$.

Then we can use the induction hypothesis for the definable family $(E'_a)_{a\in A}$, where $E'_a := E_a \setminus \bigcup_{j\in J} T^j_a$, and the definable subfamilies $(E^i_a \cap E'_a)_{a\in A}$, $i \in I$, and $((\partial T^j_a) \cap E'_a)_{a\in A}$, $j \in J$. The statement follows.

Let us prove the claim. Theorem 2.14 implies that the claim holds for $p = 1$: let T^j, $j \in J$, be the corresponding sets with all the properties as listed in the claim. Now we apply Corollary 2.5 and induction on the dimension. In fact, for each fixed $j \in J$, $(T^j_a)_{a\in A}$ is a definable family of k-dimensional Λ_1-cells T^j_a in $\mathbb{R}^n$ that are open in E_a. After the change of coordinates φ^j, we may assume that T^j_a is a Λ_1-cell with constant C independent of $a \in A$. By Corollary 2.5, there is a definable family $(Z^j_a)_{a\in A}$ of closed definable sets $Z^j_a \subseteq T^j_a$, $\dim Z^j_a < k$, such that the functions defining the cell T^j_a are Λ_p-regular with uniform constants independent of $a \in A$ in the complement of Z^j_a. Thus there exists a definable family $(S^j_a)_{a\in A}$ of subsets $S^j_a \subseteq T^j_a$ such that, for all $a \in A$, S^j_a is a finite disjoint union of k-dimensional definable Λ_p-cells $S^{j,\ell}_a$ that are open in E_a with constant C independent of $a \in A$ and $\dim T^j_a \setminus S^j_a < k$. Thus the number of connected components $S^{j,\ell}_a$ of S^j_a is uniformly bounded by a constant independent of $a \in A$. This implies the claim. □

2I. *Consequences.* We may use Theorem 2.16 in order to refine Proposition 2.4, Corollary 2.5, and Proposition 2.7.

Corollary 2.17. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a nonnegative integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and*

$$|\partial^\gamma \varphi_a(u)| \le C(k, p)\, \frac{\sup\{|\varphi_a(v)| : v \in S_a,\ |v-u| < d(u, \partial S_a)\}}{d(u, \partial S_a)^{|\gamma|}}, \quad u \in S_a,\ |\gamma| \le p.$$

Proof. This follows from Theorem 2.16 and Proposition 2.4. □

Corollary 2.18. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of C^1-functions $\varphi_a : U_a \to \mathbb{R}$. Suppose that there is a constant $M > 0$ such that*

$$|\partial_j \varphi_a(u)| \le M, \quad a \in A,\ u \in U_a,\ j = 1, \dots, k.$$

Let p be a positive integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and

$$|\partial^\gamma \varphi_a(u)| \le C(k, p)\, M\, d(u, \partial S_a)^{1-|\gamma|}, \quad u \in S_a,\ 1 \le |\gamma| \le p.$$

Proof. This follows from Theorem 2.16 and Corollary 2.5. □

Proposition 2.19. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of continuous functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a* *positive integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and*

$$|\partial^\gamma \varphi_a(u)| \le C(k,p)\,\frac{\omega(d(u,\partial S_a))}{d(u,\partial S_a)^{|\gamma|}}, \quad u \in S_a,\ 1 \le |\gamma| \le p,$$

where ω is a modulus of continuity for φ_a.

Proof. This follows from Theorem 2.16 and Proposition 2.7. □

We will need another uniform fact:

Proposition 2.20. *Let $(U_a)_{a\in A}$ be a definable family of open sets $U_a \subseteq \mathbb{R}^k$. Let $(\varphi_a)_{a\in A}$ be a definable family of functions $\varphi_a : U_a \to \mathbb{R}$. Let p be a positive integer. There exists a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(U_a)_{a\in A}$ such that, for all $a \in A$ and each open stratum $S_a \in \mathscr{S}_a$, φ_a is C^p on S_a and, for all $j = 1, \dots, k$, either*

$$|\partial_j \varphi_a| \ge 1 \quad \text{on } S_a \qquad \text{or} \qquad |\partial_j \varphi_a| < 1 \quad \text{on } S_a. \tag{2-11}$$

Proof. Let $U \subseteq \mathbb{R}^N \times \mathbb{R}^n$ and $\varphi : U \to \mathbb{R}$ be the associated definable set and function (see (2-1) and (2-2)). Let $X \subseteq U$ be the set defined in the proof of Proposition 2.4.

For $j = 1, \dots, k$, let $\partial_j \varphi(a,u) := \partial_j \varphi_a(u)$ and set

$$\begin{aligned} Y_j &:= \{(a,u) \in U \setminus X : \exists \epsilon > 0\, \forall v \in B(u,\epsilon),\ \partial_j\varphi(a,v) = 1\},\\ Z_j &:= \{(a,u) \in U \setminus X : \partial_j\varphi(a,u) = 1\} \setminus Y_j,\\ Z &:= X \cup \bigcup_{j=1}^k Z_j. \end{aligned}$$

Then Z is a definable subset of U and, for all $a \in A$, Z_a is closed in U_a and $\dim Z_a < k$. Now the statement follows from Theorem 2.16. □

3. Bounded definable families of Whitney jets

Recall that a modulus of continuity ω is by definition a positive, continuous, increasing, and concave function $\omega : (0,\infty) \to (0,\infty)$ such that $\omega(t) \to 0$ as $t \to 0$.

3A. *$C^{m,\omega}$-functions.* Let ω be a modulus of continuity. Let $U \subseteq \mathbb{R}^n$ be an open set. Let $C^{0,\omega}(U)$ be the set of all continuous bounded functions $f : U \to \mathbb{R}$ such that

$$|f|_{C^{0,\omega}(U)} := \inf\big\{C > 0 : |f(x) - f(y)| \le C\,\omega(|x-y|) \text{ for all } x, y \in U\big\} < \infty.$$

For a nonnegative integer m, the set $C^{m,\omega}(U)$ consists of all C^m-functions such that $\partial^\alpha f$ is globally bounded for all $|\alpha| \le m$ and $\partial^\alpha f \in C^{0,\omega}(U)$ for all $|\alpha| = m$.

Then $C^{m,\omega}(U)$ is a Banach space with the norm

$$\|f\|_{C^{m,\omega}(U)} := \sup_{x\in U}\sup_{|\alpha|\le m}|\partial^\alpha f(x)| + \sup_{|\alpha|=m}|\partial^\alpha f|_{C^{0,\omega}(U)}.$$

We say that $f \in C^{m,\omega}(U)$ is *m-flat* outside an open set $V \subseteq U$ if all $\partial^\alpha f$, $|\alpha| \le m$, vanish on $U \setminus V$.

Assume that the open set $U \subseteq \mathbb{R}^n$ is definable. We denote by $C^{m,\omega}_{\text{def}}(U)$ the subspace of $C^{m,\omega}(U)$ consisting of the definable functions in the latter space. Note that the normed space $C^{m,\omega}_{\text{def}}(U)$ is not complete.

Definition 3.1 (bounded families of $C^{m,\omega}$-functions). Let $m \in \mathbb{N}$ and ω a modulus of continuity. A family $(f_a)_{a\in A}$ of $C^{m,\omega}$-functions $f_a : U_a \to \mathbb{R}$, where $U_a \subseteq \mathbb{R}^n$ is open, is said to be a *bounded family of $C^{m,\omega}$-functions* if

$$\sup_{a\in A}\|f_a\|_{C^{m,\omega}(U_a)} < \infty.$$

We say that $(f_a)_{a\in A}$ is a *definable bounded family of $C^{m,\omega}$-functions* if it is a bounded family of $C^{m,\omega}$-functions and, additionally, the families $(U_a)_{a\in A}$ and $(f_a)_{a\in A}$ are definable. Moreover, $(f_a)_{a\in A}$ is called *m-flat outside* $(V_a)_{a\in A}$ if, for each $a \in A$, $V_a \subseteq U_a$ is open and f_a is m-flat on $U_a \setminus V_a$. We will say that $(f_a)_{a\in A}$ is C^p *outside* $(E_a)_{a\in A}$ if, for each $a \in A$, $E_a \subseteq U_a$ is closed and f_a is C^p on $U_a \setminus E_a$.

3B. ***Whitney jets of class $C^{m,\omega}$.*** Let E be a locally closed subset of $\mathbb{R}^n$. An *m-jet on E* is a collection $F = (F^\alpha)_{|\alpha|\le m}$ of continuous functions $F^\alpha : E \to \mathbb{R}$. An m-jet $F = (F^\alpha)_{|\alpha|\le m}$ on E is said to be *flat* on a subset $E' \subseteq E$ if all functions F^α, $|\alpha| \le m$, vanish on E'.

For $a \in E$ we denote by $T^m_a F$ the Taylor polynomial

$$T^m_a F(x) = \sum_{|\alpha|\le m}\frac{1}{\alpha!}F^\alpha(a)(x-a)^\alpha, \quad x \in \mathbb{R}^n,$$

and define the m-jet

$$R^m_a F := F - J^m_E(T^m_a F),$$

where $J^m_E(f) := (\partial^\alpha f|_E)_{|\alpha|\le m}$ for $f \in C^m(\mathbb{R}^n)$.

A $C^{m,\omega}$ (or C^m) function $f : \mathbb{R}^n \to \mathbb{R}$ is an *extension to $\mathbb{R}^n$ of F* if $J^m_E(f) = F$. A necessary and sufficient condition for an m-jet to have a $C^{m,\omega}$-extension to $\mathbb{R}^n$ is to be a Whitney jet of class $C^{m,\omega}$ [6; 16]: by definition, an m-jet $F = (F^\alpha)_{|\alpha|\le m}$ on E is a *Whitney jet of class $C^{m,\omega}$ on E* if there exists $C > 0$ such that

$$\sup_{x\in E}\sup_{|\alpha|\le m}|F^\alpha(x)| \le C \tag{3-1}$$

and, for all $x, y \in E$ and $|\alpha| \le m$,

$$|(R^m_x F)^\alpha(y)| \le C\,\omega(|x-y|)\,|x-y|^{m-|\alpha|}. \tag{3-2}$$

Remark 3.2. A condition equivalent to (3-2) is

$$|T_x^m F(z) - T_y^m F(z)| \le C' \omega(|x-y|) (|z-x|^m + |z-y|^m)$$

for all $x, y \in E$ and $z \in \mathbb{R}^n$; see [15, Proposition IV.1.5]. Moreover, (3-2) holds if and only if

$$|F^0(x) - T_y^m F(x)| \le C \omega(|x-y|) |x-y|^m \quad \text{for all } x, y \in E,$$

and, if $m \ge 1$,

$$\partial_i F := (F^{\alpha+(i)})_{|\alpha| \le m-1} \text{ is a Whitney jet of class } C^{m-1,\omega} \quad \text{for all } i = 1, \dots, n.$$

If E is quasiconvex with constant C' (see Definition 2.9), then (3-2) follows from

$$|F^\alpha(x) - F^\alpha(y)| \le C'' \omega(|x-y|), \quad x, y \in E, \ |\alpha| = m;$$

see [15, IV (2.5.1)]; then the constant C in (3-2) depends only on n, m, C', and C''.

It is not hard to see that the set of all Whitney jets of class $C^{m,\omega}$ on E with the natural addition and the multiplication $FG := J_E^m(T^m F \cdot T^m G)$ is an $\mathbb{R}$-algebra.

Let F be an m-jet on $E \subseteq \mathbb{R}^n$. Let $G_1, \dots, G_n$ be m-jets on $A \subseteq \mathbb{R}^k$ such that $(G_1^0, \dots, G_n^0)(A) \subseteq E$. The *composite* $F \circ G = F \circ (G_1, \dots, G_n)$ of m-jets F and G on A is defined by

$$(F \circ G)(x) := J_A^m(T_{G^0(x)}^m F \circ T_x^m G)(x).$$

Note that

$$T_y^m (F \circ G)(x) = \pi_m \big(T_{G^0(y)}^m F(T_y^m G(x)) \big),$$

where π_m is the natural truncation operator (which truncates monomials of order $> m$). One can show (using Remark 3.2) that, for $m \ge 1$, the composite $F \circ G$ is a Whitney jet of class $C^{m,\omega}$ if F and G are Whitney jets of class $C^{m,\omega}$. We will not use this fact, but the pullback of Whitney jets of class $C^{m,\omega}$ along a Λ_m-regular map will be crucial; see Proposition 3.5.

Definition 3.3 (bounded families of Whitney jets of class $C^{m,\omega}$). A family $(F_a)_{a \in A}$ of Whitney jets F_a of class $C^{m,\omega}$ on $E_a \subseteq \mathbb{R}^n$ is said to be a *bounded family of Whitney jets of class* $C^{m,\omega}$ if the constant $C > 0$ in (3-1) and (3-2) can be chosen independent of $a \in A$, that is,

$$\sup_{a \in A} \sup_{x \in E_a} \sup_{|\gamma| \le m} |F_a^\gamma(x)| < \infty \tag{3-3}$$

and

$$\sup_{a \in A} \sup_{x \ne y \in E_a} \sup_{|\gamma| \le m} \frac{|(R_x^m F_a)^\gamma(y)|}{\omega(|x-y|) |x-y|^{m-|\gamma|}} < \infty. \tag{3-4}$$

We say that $(F_a)_{a\in A}$ is a *definable bounded family of Whitney jets of class* $C^{m,\omega}$ if it is a bounded family of Whitney jets of class $C^{m,\omega}$ and, additionally, the families $(E_a)_{a\in A}$ and $(F_a^\gamma)_{a\in A}$, $|\gamma|\le m$, are definable. We say that it is *flat* on a subfamily $(E'_a)_{a\in A}$ of $(E_a)_{a\in A}$ if F_a is flat on E'_a for all $a\in A$.

A (definable bounded) family $(f_a)_{a\in A}$ of $C^{m,\omega}$-functions $f_a:\mathbb{R}^n\to\mathbb{R}$ is called a (*definable bounded*) *family of* $C^{m,\omega}$*-extensions to* $\mathbb{R}^n$ *of* $(F_a)_{a\in A}$ if f_a is a $C^{m,\omega}$-extension of F_a to $\mathbb{R}^n$, for each $a\in A$.

3C. ***Separation.*** Let X, Y, Z be subsets of $\mathbb{R}^n$. Recall that X and Y are said to be *Z-separated* if there exists $C>0$ such that

$$d(x,Y)\ge C\,d(x,Z),\quad x\in X, \tag{3-5}$$

or equivalently, if there is $C'>0$ such that

$$d(x,X)+d(x,Y)\ge C'\,d(x,Z),\quad x\in\mathbb{R}^n.$$

If X and Y are $X\cap Y$-separated, then we will simply say that X and Y are *separated*.

Definition 3.4 (uniformly separated families of sets). Let $(X_a)_{a\in A}$, $(Y_a)_{a\in A}$, and $(Z_a)_{a\in A}$ be definable families of subsets of $\mathbb{R}^n$. Then $(X_a)_{a\in A}$ and $(Y_a)_{a\in A}$ are said to be *uniformly* $(Z_a)_{a\in A}$*-separated* if, for all $a\in A$, X_a and Y_a are Z_a-separated with a constant $C>0$ (in (3-5)) independent of $a\in A$. We will say that $(X_a)_{a\in A}$ and $(Y_a)_{a\in A}$ are *uniformly separated* if they are uniformly $(X_a\cap Y_a)_{a\in A}$-separated.

3D. ***Pullback along a definable family of Λ_p-regular maps.*** Let $\varphi:U\to\mathbb{R}^\ell$ be a Λ_{m+1}-regular map, where $U\subseteq\mathbb{R}^k$ is open and quasiconvex. Let $\overline{\varphi}:\overline{U}\to\mathbb{R}^\ell$ be the continuous extension of φ; see Section 2D. Consider

$$\varphi_+:U\times\mathbb{R}^\ell\to U\times\mathbb{R}^\ell,\quad (u,w)\mapsto(u,w+\varphi(u)),$$

and

$$\overline{\varphi}_+:\overline{U}\times\mathbb{R}^\ell\to\overline{U}\times\mathbb{R}^\ell,\quad (u,w)\mapsto(u,w+\overline{\varphi}(u)).$$

Let M be a closed subset of $U\times\mathbb{R}^\ell$ and F an m-jet on M. The *pullback of F along* φ_+ is the m-jet

$$\varphi_+^*F:=F\circ J_N^m(\varphi_+)$$

on $N:=\varphi_+^{-1}(M)=\big\{(u,w)\in U\times\mathbb{R}^\ell:(u,w+\varphi(u))\in M\big\}$.

We shall need the following result on the pullback of a definable bounded family of Whitney jets of class $C^{m,\omega}$ along a definable family $(\varphi_a)_{a\in A}$ of Λ_{m+1}-regular maps. For each $a\in A$, let $\varphi_{a,+}$ and $\overline{\varphi}_{a,+}$ be defined in analogy to φ_+ and $\overline{\varphi}_+$.

Proposition 3.5 [12, Proposition 4.3]. *Let $(U_a)_{a\in A}$ be a definable family of open quasiconvex sets $U_a\subseteq\mathbb{R}^k$ with constant* (*in Definition 2.9*) *independent of $a\in A$. Let $(\varphi_a)_{a\in A}$ be a definable family of Λ_{m+1}-regular maps $\varphi_a:U_a\to\mathbb{R}^\ell$ with constant* (*in* (2-5)) *independent of* $a \in A$. *Let* $(M_a)_{a\in A}$ *be a definable family of closed quasiconvex subsets* M_a *of* $U_a \times \mathbb{R}^\ell$ *such that* $(\overline{M}_a)_{a\in A}$ *and* $(\partial U_a \times \mathbb{R}^\ell)_{a\in A}$ *are uniformly separated.*

If $(F_a)_{a\in A}$ *is a definable bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(\overline{M}_a)_{a\in A}$ *which is flat on* $(\partial M_a)_{a\in A}$, *then* $(G_a)_{a\in A}$ *is a definable bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(N_a)_{a\in A}$, *where* $G_a := \varphi^*_{a,+}F_a$ *and* $N_a := \varphi^{-1}_{a,+}(M_a)$. *If, moreover, for each* $a \in A$, $t_a : U_a \to (0,\infty)$ *is a function satisfying* $t_a(u) \le d(u, \partial U_a)$ *for every* $u \in U_a$ *and*

$$|F_a^\kappa(u,w)| \lesssim \omega(t_a(u))t_a(u)^{m-|\kappa|}, \quad (u,w) \in M_a,\ |\kappa| \le m,$$

then, for each $a \in A$,

$$|G_a^\kappa(u,w)| \lesssim \omega(t_a(u))t_a(u)^{m-|\kappa|}, \quad (u,w) \in N_a,\ |\kappa| \le m.$$

Proof. Follows from the proof of [12, Proposition 4.3]. □

We will be interested in the case that $M_a = \Gamma(\varphi_a)$, $a \in A$. Then $(G_a)_{a\in A}$ extends to a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{N}_a = \overline{U}_a \times 0)_{a\in A}$ which is flat on $(\partial N_a = \partial U_a \times 0)_{a\in A}$. This follows from the following lemma and Hestenes' lemma (e.g., [14, Theorem 1.10]); see [12, Remark 4.2].

Lemma 3.6. *Let* $(E_a)_{a\in A}$ *be a family of locally closed, quasiconvex sets* $E_a \subseteq \mathbb{R}^n$ *with constant* (*in Definition 2.9*) *independent of* $a \in A$. *Suppose that* $(F_a)_{a\in A}$ *is a bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(E_a)_{a\in A}$ *such that, for all* $a \in A$ *and* $|\alpha| \le m$, F_a^α *has a continuous extension* $\overline{F}_a^\alpha$ *to* $\overline{E}_a$. *Then* $(\overline{F}_a)_{a\in A}$ *is a bounded family of Whitney jets of class* $C^{m,\omega}$ *on* $(\overline{E}_a)_{a\in A}$.

Proof. Let $x, y \in \overline{E}_a$. There exist sequences $(x_k), (y_k) \subseteq E_a$ such that $x_k \to x$ and $y_k \to y$. By assumption, there exist a constant $C_1 > 0$, independent of $a \in A$ and of k, and a rectifiable path σ_k joining x_k and y_k in E_a such that for the length of σ_k we have

$$\ell(\sigma_k) \le C_1\,|x_k - y_k|.$$

Let $F = (F^\alpha)_{|\alpha|\le m}$ be a Whitney jet of class $C^{m,\omega}$ on $(E_a)_{a\in A}$. By [15, IV (2.5.1)], for all $|\alpha| \le m$,

$$|(R^m_{x_k}F_a)^\alpha(y_k)| \le n^{\frac{m-|\alpha|}{2}} C_1^{m-|\alpha|}|x_k - y_k|^{m-|\alpha|} \sup_{\xi\in\sigma_k}\sup_{|\beta|=m} |F_a^\beta(\xi) - F_a^\beta(x_k)|.$$

We may assume that σ_k is parameterized by $t \in [0,1]$ with $\sigma_k(0) = x_k$ and $\sigma_k(1) = y_k$. By (3-4), for $t \in [0,1]$,

$$\begin{aligned}\sup_{|\beta|=m} |F^\beta(\sigma_k(t)) - F^\beta(x_k)| &\le C_2\,\omega(|\sigma_k(t)-x_k|) \le C_2\,\omega(\ell(\sigma_k|_{[0,t]}))\\ &\le C_2\,\omega(\ell(\sigma_k)) \le C_2\,\omega(C_1|x_k-y_k|) \le C_3\,\omega(|x_k-y_k|),\end{aligned}$$

for constants $C_i > 0$ independent of $a \in A$. Thus

$$|(R^m_{x_k} F_a)^\alpha (y_k)| \le n^{\frac{m-|\alpha|}{2}} C_1^{m-|\alpha|} |x_k - y_k|^{m-|\alpha|} C_3\, \omega(|x_k - y_k|)$$

and letting $k \to \infty$ shows that (3-4) is satisfied for $(\bar{F}_a)_{a\in A}$. It is clear that (3-3) is satisfied. □

3E. *Cutoff.* We finish this section with a technical cutoff result which will be used in the proof of Theorem 1.3.

Proposition 3.7 [12, Proposition 3.9]. *Let $(U_a)_{a\in A}$ be a definable family of open quasiconvex sets $U_a \subseteq \mathbb{R}^k$ with constant (in Definition 2.9) independent of $a \in A$. Let $(h_a)_{a\in A}$ be a definable family of C^m-functions $h_a : U_a \times \mathbb{R}^\ell \to \mathbb{R}$ and $(\rho_a)_{a\in A}$ a definable family of C^{m+1}-functions $\rho_a : U_a \to \mathbb{R}$. Let $(t_a)_{a\in A}$ be a definable family of positive Lipschitz functions $t_a : U_a \to (0,\infty)$ with Lipschitz constants independent of $a \in A$ such that $t_a(u) \le d(u, \partial U_a)$ for all $a \in A$ and $u \in U_a$. For $\epsilon > 0$, consider the definable family $(\Delta^\epsilon_a)_{a\in A}$, where*

$$\Delta^\epsilon_a := \{(u,w) \in U_a \times \mathbb{R}^\ell : |w| < \epsilon\, t_a(u)\}.$$

Assume that, for all $a \in A$,

$$\left|\partial^\alpha \left(\frac{1}{\rho_a}\right)(u)\right| \lesssim t_a(u)^{-|\alpha|-1}, \quad u \in U_a,\ |\alpha| \le m+1. \tag{3-6}$$

Let $\xi : \mathbb{R} \to \mathbb{R}$ be a definable C^m-function with compact support, fix $1 \le i \le \ell$, and set, for all $a \in A$,

$$f_a(u,w) := \xi\left(\frac{w_i}{\rho_a(u)}\right) h_a(u,w), \quad (u,w) \in U_a \times \mathbb{R}^\ell.$$

If $(h_a)_{a\in A}$ is a definable bounded family of $C^{m,\omega}$-functions on $(\Delta^\epsilon_a)_{a\in A}$ such that, for each $a \in A$,

$$|\partial^\gamma h_a(u,w)| \lesssim \omega(t_a(u)) t_a(u)^{m-|\gamma|}, \quad (u,w) \in \Delta^\epsilon_a,\ |\gamma| \le m,$$

then $(f_a)_{a\in A}$ is a definable bounded family of $C^{m,\omega}$-functions on $(\Delta^\epsilon_a)_{a\in A}$ such that, for each $a \in A$,

$$|\partial^\gamma f_a(u,w)| \lesssim \omega(t_a(u)) t_a(u)^{m-|\gamma|}, \quad (u,w) \in \Delta^\epsilon_a,\ |\gamma| \le m.$$

Proof. It suffices to repeat the proof of Proposition 3.9 in [12] (as well as Lemma 3.5 and Proposition 3.6 which are used in the proof). □

Remark 3.8. Proposition 3.5 and Proposition 3.7 remain true if we remove everywhere the attribute "definable".

4. Bounded definable extension of Whitney jets

This section is devoted to the proof of Theorem 1.3. Let us recall the setup: Let $0 \le m \le p$ be integers and ω a modulus of continuity. Let $(E_a)_{a\in A}$ be a definable family of closed subsets of $\mathbb{R}^n$. Let $(F_a)_{a\in A}$ be a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$. We will show that there exists a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(E_a)_{a\in A}$.

For each $a \in A$, let $\operatorname{supp} F_a$ denote the closure of $\bigcup_{|\kappa|\le m}\{x \in E_a : F_a^\kappa(x) \ne 0\}$ and let $(E'_a)_{a\in A}$ be a definable subfamily of $(E_a)_{a\in A}$ consisting of closed subsets E'_a of E_a such that $\operatorname{supp} F_a \subseteq E'_a$.

Let $A' := \{a \in A : \operatorname{supp} F_a = \varnothing\}$. The family $(F_a)_{a\in A'}$ can be extended by $(0)_{a\in A'}$ to $\mathbb{R}^n$. So we may assume that, for all $a \in A$, $\operatorname{supp} F_a \ne \varnothing$ and thus $E'_a \ne \varnothing$.

We proceed by induction on $k := \max_{a\in A} \dim E'_a$ and show:

($\mathbf{I}_k$) *Let $(E_a)_{a\in A}$ be a definable family of closed subsets E_a of $\mathbb{R}^n$ and $(F_a)_{a\in A}$ a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$. Let $(E'_a)_{a\in A}$ be a definable subfamily of $(E_a)_{a\in A}$ of closed subsets E'_a of E_a such that* $\operatorname{supp} F_a \subseteq E'_a$ *and* $\dim E'_a \le k$*, for all $a \in A$. Then there exists a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(E'_a)_{a\in A}$.*

Let us fix an integer $p \ge m+1$. (We need that $p \ge m+1$ in the proof. The case $p = m$ in Theorem 1.3 is evidently a trivial consequence.)

Overview of the proof. Before we actually start the proof of ($\mathbf{I}_k$), let us give a brief general overview. Besides the induction on the dimension k, we use, for fixed k, an induction on the number of the k-dimensional strata of E'_a. This is possible since this number is uniformly bounded independently of $a \in A$, thanks to Theorem 2.16. In this way, we can reduce the proof to the case that E'_a has dimension k and is the closure of a single stratum S_a that is the graph of a Λ_p-regular map φ_a. We can assume that the Whitney jet F_a is flat on ∂S_a; see Proposition 4.2. This case is then checked in three gradually more general steps:

(1) In the first step, we assume that $\varphi_a \equiv 0$ and $E'_a = E_a$.

(2) In the second step, we still suppose that $\varphi_a \equiv 0$ but allow that E'_a is a proper subset of E_a.

(3) The general case in the final third step is reduced to the previous steps by means of Proposition 3.5.

Induction basis ($\mathbf{I}_0$). If $k = 0$, then each E'_a is a finite set (but E_a might be infinite) and there is a constant which bounds the number $|E'_a|$ of elements of E'_a independently of $a \in A$, by uniform finiteness; see [5, 4.4].

Let us make induction on $s := \max_{a\in A} |E'_a|$. The base case $s = 1$ is treated in the following lemma.

Lemma 4.1. *Let $(E_a)_{a\in A}$ be a definable family of closed subsets E_a of $\mathbb{R}^n$ and $(E'_a)_{a\in A}$ a definable subfamily of $(E_a)_{a\in A}$ such that, for each $a \in A$, $E'_a = \{x_a\}$. Let $(F_a)_{a\in A}$ be a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ such that* supp $F_a \subseteq \{x_a\}$, *for all $a \in A$. Then there exists a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(E'_a)_{a\in A}$.*

Proof. Note that, for each $a \in A$, x_a is an isolated point of E_a, by continuity of F_a.

Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a definable C^p-function that equals 1 in a neighborhood of 0 and has support contained in the unit ball. For each $a \in A$, set

$$d_a := \begin{cases} \min\{1, d(x_a, E_a \setminus \{x_a\})\} & \text{if } E_a \setminus \{x_a\} \neq \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Define, for each $a \in A$,

$$f_a(x) := \chi\left(\frac{x - x_a}{d_a}\right) \cdot T^m_{x_a} F_a(x), \quad x \in \mathbb{R}^n.$$

Then $(f_a)_{a\in A}$ is a definable family of C^p-functions $f_a : \mathbb{R}^n \to \mathbb{R}$ such that each f_a has support contained in the ball $B_a := B(x_a, d_a)$ with radius d_a around x_a and extends the jet F_a. We will prove that the family $(f_a)_{a\in A}$ is bounded in $C^{m,\omega}(\mathbb{R}^n)$.

Let $\gamma \in \mathbb{N}^n$, where $|\gamma| \le m$. Then

$$\partial^\gamma f_a(x) = \sum_{\alpha+\beta=\gamma} \binom{\gamma}{\alpha} d_a^{-|\alpha|}\, \partial^\alpha \chi\left(\frac{x - x_a}{d_a}\right) \partial^\beta T^m_{x_a} F_a(x).$$

By (3-4) (for $y \in E_a \setminus \{x_a\}$ with $d_a = |x_a - y|$ if $d_a < 1$) and (3-3) (if $d_a = 1$),

$$|F_a^\beta(x_a)| \le C\,\omega(d_a) d_a^{m-|\beta|}, \quad |\beta| \le m,$$

for a constant $C > 0$ independent of $a \in A$. For the rest of the proof, C will denote a constant independent of $a \in A$; its actual value may change. Thus, for $x \in B_a$,

$$|\partial^\beta T^m_{x_a} F_a(x)| = \left| \sum_{|\kappa|\le m-|\beta|} \frac{1}{\kappa!} F_a^{\kappa+\beta}(x_a)(x - x_a)^\kappa \right| \le C\,\omega(d_a) d_a^{m-|\beta|}, \quad |\beta| \le m.$$

It follows that, for all $x \in \mathbb{R}^n$,

$$|\partial^\gamma f_a(x)| \le C\,\omega(d_a) d_a^{m-|\gamma|} \le C\,\omega(1), \quad |\gamma| \le m. \tag{4-1}$$

Now assume that $|\gamma| = m$. To see that $|\partial^\gamma f_a|_{C^{0,\omega}(\mathbb{R}^n)}$ is bounded by a constant independent of $a \in A$, it suffices to estimate, for $\alpha + \beta = \gamma$,

$$D(x, y) := \left| d_a^{-|\alpha|}\, \partial^\alpha \chi\left(\frac{x - x_a}{d_a}\right) \partial^\beta T^m_{x_a} F_a(x) - d_a^{-|\alpha|}\, \partial^\alpha \chi\left(\frac{y - x_a}{d_a}\right) \partial^\beta T^m_{x_a} F_a(y) \right|.$$

Let us first assume that $x, y \in \bar{B}_a$. Then

$$d_a^{-|\alpha|} \left| \partial^\alpha \chi\left(\frac{x - x_a}{d_a}\right) - \partial^\alpha \chi\left(\frac{y - x_a}{d_a}\right) \right| |\partial^\beta T^m_{x_a} F_a(x)| \\ \le C\, \frac{\omega(d_a)}{d_a} |x - y| \le 2C\, \omega(|x - y|),$$

since ω is concave and $|x - y| \le 2d_a$. On the other hand,

$$d_a^{-|\alpha|} \left| \partial^\alpha \chi\left(\frac{y - x_a}{d_a}\right) \right| |\partial^\beta T^m_{x_a} F_a(x) - \partial^\beta T^m_{x_a} F_a(y)| \\ \le C\, d_a^{-|\alpha|} \sum_{|\kappa| \le m - |\beta|} \frac{1}{\kappa!} |F_a^{\kappa+\beta}(x_a)| |(x - x_a)^\kappa - (y - x_a)^\kappa| \\ \le C'\, \frac{\omega(d_a)}{d_a}\, |x - y| \le 2C'\, \omega(|x - y|).$$

So $D(x, y) \le C\, \omega(|x - y|)$ for a constant $C > 0$ independent of $a \in A$.

If x and y lie outside of B_a, then $D(x, y) = 0$. If $x \in B_a$ and $y \notin \bar{B}_a$ and z is the point, where the line segment $[x, y]$ meets ∂B_a, then

$$D(x, y) = D(x, z) \le C\, \omega(|x - z|) \le C\, \omega(|x - y|).$$

This ends the proof. □

Assume that $s > 1$. For each $a \in A$, choose a numbering of the elements of $E'_a = \{x_{a,1}, \dots, x_{a,s_a}\}$, where $s_a \le s$. By the induction hypothesis, $(F_a|_{E_a \setminus \{x_{a,2},\dots,x_{a,s_a}\}})_{a\in A}$ admits a definable bounded family $(f_a^1)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(\{x_{a,1}\})_{a\in A}$. Then $(F_a - J^m_{E_a}(f_a^1))_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ which is flat on $(E_a \setminus \{x_{a,2}, \dots, x_{a,s_a}\})_{a\in A}$ and has a definable bounded family $(f_a^2)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(\{x_{a,2}, \dots, x_{a,s_a}\})_{a\in A}$, again by the induction hypothesis. Thus, $(f_a^1 + f_a^2)_{a\in A}$ is the desired definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$.

This ends the induction on s and the base case $(\mathbf{I}_0)$ of the induction on k.

Setup for the induction step. Let $k > 0$ and suppose that $(\mathbf{I}_{k-1})$ holds. We will prove $(\mathbf{I}_k)$. This will be accomplished by showing Proposition 4.2 below, but first we make a few preparatory reductions.

By Theorem 2.16, there is a uniform Λ_p-stratification $(\mathscr{S}_a)_{a\in A}$ of $(E_a)_{a\in A}$ compatible with $(E'_a)_{a\in A}$ such that, for each $a \in A$ and each $|\kappa| \le m$, F_a^κ is of class C^p on the strata in $\mathscr{S}_a$.

By $(\mathbf{I}_{k-1})$, we may assume that $\dim E'_a = k$ for all $a \in A$ and there is a definable bounded $C^{m,\omega}$-extension $(f_a^0)_{a\in A}$ to $\mathbb{R}^n$ of the restriction of $(F_a)_{a\in A}$ to $(E_a \setminus P_a)_{a\in A}$, where

$$P_a = \bigcup \{S_a \in \mathscr{S}_a : S_a \subseteq E'_a,\ \dim S_a = k\}.$$

Replacing F_a by $F_a - J^m_{E_a}(f^0_a)$, for each $a \in A$, we may assume that F_a is flat on all strata $S_a \in \mathscr{S}_a$, $S_a \subseteq E'_a$, with $\dim S_a < k$ and also on $E_a \setminus E'_a$.

Let us now see that we may furthermore reduce to the case that, for each $a \in A$, E'_a is the closure of just one k-dimensional stratum S_a and that F_a is flat on its frontier. Indeed, the number s_a of k-dimensional strata of E'_a is uniformly bounded by a constant not depending on $a \in A$. We may use induction on $s := \max_{a\in A} s_a$ of which the above statement is the base case that we take for granted for the moment. The induction step works just as for finite sets E'_a: for each $a \in A$, let $S_{a,1}, \dots, S_{a,s_a}$ be a numbering of the k-dimensional strata of E'_a. By the induction hypothesis, $(F_a|_{E_a\setminus R_a})_{a\in A}$, where $R_a := \bigcup_{i\ge 2} S_{a,i}$, admits a definable bounded family $(f^1_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(E'_a \setminus R_a)_{a\in A}$. Then $(F_a - J^m_{E_a}(f^1_a))_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a)_{a\in A}$ which is flat on $(E_a \setminus R_a)_{a\in A}$ and has a definable bounded family $(f^2_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(R_a)_{a\in A}$, again by the induction hypothesis. Thus, $(f^1_a + f^2_a)_{a\in A}$ is the desired definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$.

In the case that $k = n$, S_a is open in $\mathbb{R}^n$ and extending F_a by 0 outside S_a, for all $a \in A$, yields a definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on $(\mathbb{R}^n)_{a\in A}$ so that $(F^0_a)_{a\in A}$ is the desired family of $C^{m,\omega}$-extensions. This follows from Hestenes' lemma (e.g., [14, Theorem 1.10]); indeed, if $x \in S_a$ and $y \notin \overline{S}_a$ and z is the point, where the line segment $[x, y]$ meets ∂S_a, then, by (3-4) and Remark 3.2, for any $u \in \mathbb{R}^n$,

$$\begin{aligned}|T^m_x F_a(u) - T^m_y F_a(u)| = |T^m_x F_a(u)| &= |T^m_x F_a(u) - T^m_z F_a(u)|\\ &\le C\,\omega(|x-z|)(|u-x|^m + |u-z|^m)\\ &\le 2C\,\omega(|x-y|)(|u-x|^m + |u-y|^m),\end{aligned}$$

for a constant $C > 0$ independent of $a \in A$, since $|u - z| \le \max\{|u - x|, |u - y|\}$.

Consequently, we may assume that $\ell := n - k > 0$.

We reduced the proof to showing the following. (We may assume that S_a is a Λ_p-cell in a fixed orthogonal system of coordinates of $\mathbb{R}^n$, which is independent of $a \in A$, thanks to Theorem 2.16.)

Proposition 4.2. *Let $(E_a)_{a\in A}$ be a definable family of closed sets E_a in $\mathbb{R}^n$. Let $(E'_a)_{a\in A}$ be a definable subfamily of $(E_a)_{a\in A}$ of closed subsets E'_a of E_a with* $\dim E'_a = k$ *such that $E'_a = \overline{S}_a$, where*

$$S_a = \{(u, \varphi_a(u)) \in \mathbb{R}^k \times \mathbb{R}^\ell : u \in T_a\} = \Gamma(\varphi_a),$$

and $(\varphi_a)_{a\in A}$ is a definable family of Λ_p-regular maps $\varphi_a : T_a \to \mathbb{R}^\ell$, T_a an open Λ_p-cell in $\mathbb{R}^k$, and all constants in the definition of T_a and φ_a are independent of $a \in A$. Then any definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on

$(E_a)_{a\in A}$ such that, for all $a \in A$, supp $F_a \subseteq E'_a$, *F_a is flat on ∂S_a, and F_a^κ, $|\kappa| \le m$, is C^p on S_a, admits a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is C^p outside $(E'_a)_{a\in A}$.*

The proposition is proved in three gradually more general steps:

Step 1: $\varphi_a \equiv 0$ and $E'_a = E_a$ for all $a \in A$.

Step 2: $\varphi_a \equiv 0$ for all $a \in A$.

Step 3: The general case.

Step 1. For all $a \in A$, $E_a = E'_a = \overline{T}_a \times 0$, where $T_a \subseteq \mathbb{R}^k$ is an open Λ_p-cell with constant C independent of $a \in A$. We will prove Proposition 4.2 in this special case with the additional property that $(f_a)_{a\in A}$ is m-flat outside $(\Delta(T_a \times 0))_{a\in A}$, where

$$\Delta(T_a \times 0) := \big\{(u, w) \in T_a \times \mathbb{R}^\ell : |w| < \min\{1, d(u, \partial T_a)\}\big\}. \tag{4-2}$$

For each $a \in A$, we write

$$F_a = (F_a^{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{N}^k\times\mathbb{N}^\ell,|\alpha|+|\beta|\le m}.$$

Fix $\beta \in \mathbb{N}^\ell$ with $|\beta| \le m$. Let $F_{a,\beta}$ be the m-jet which results from F_a by setting all $F_a^{(\alpha,\beta')}$ equal to 0 whenever $\beta' \ne \beta$. Then $(F_{a,\beta})_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{T}_a \times 0)_{a\in A}$. Indeed, for each $a \in A$, the definable Whitney jet F_a of class $C^{m,\omega}$ on $\overline{T}_a \times 0$ can be identified with a collection $\widetilde{F}_{a,\beta}$, $|\beta| \le m$, where $\widetilde{F}_{a,\beta}$ is a definable $C^{m-|\beta|,\omega}$-function on $\overline{T}_a$ such that $\partial^\alpha \widetilde{F}_{a,\beta}(u) = F_a^{(\alpha,\beta)}(u, 0)$ for all $u \in \overline{T}_a$ and $\alpha \in \mathbb{N}^k$, $|\alpha| \le m - |\beta|$; see [10, Remark 5] and [6, pp. 87–88]. It suffices to prove that, for each β, $(F_{a,\beta})_{a\in A}$ admits a definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is m-flat outside $(\Delta(T_a \times 0))_{a\in A}$ and C^p outside $(\overline{T}_a \times 0)_{a\in A}$. Thus, we may suppose that, for each $a \in A$, $F_a^{(\alpha,\beta')} = 0$ whenever $\beta' \ne \beta$. By assumption, $F_a^{(\alpha,\beta)}$ is C^p on $T_a \times 0$.

By Theorem 2.16, Corollary 2.17, and Proposition 2.19, there is a uniform Λ_p-stratification $(\mathscr{D}_a)_{a\in A}$ of $(\overline{T}_a)_{a\in A}$ such that, for all $a \in A$, each open k-dimensional $D_a \in \mathscr{D}_a$, and all α, β, $F_a^{(\alpha,\beta)}$ is C^p on $D_a \times 0$, and, for all $u \in D_a$ and $\gamma \in \mathbb{N}^k$ with $|\gamma| \le p$, we have

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u, 0)| \le L \,\frac{\sup\{|F_a^{(\alpha,\beta)}(v, 0)| : v \in D_a,\ |v - u| < d(u, \partial D_a)\}}{d(u, \partial D_a)^{|\gamma|}}, \tag{4-3}$$

and, if $1 \le |\gamma| \le p$,

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u, 0)| \le L \,\frac{\omega(d(u, \partial D_a))}{d(u, \partial D_a)^{|\gamma|}}, \tag{4-4}$$

where $L > 0$ is a constant independent of $a \in A$.

For each $a \in A$, let $Z_a := \bigcup\{D_a \in \mathscr{D}_a : \dim D_a < k\}$. Setting

$$G_a(x) := \begin{cases} F_a(x) & \text{if } x \in Z_a \times 0, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Delta(T_a \times 0), \end{cases}$$

defines a definable bounded family $(G_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}$ on $\big((Z_a \times 0) \cup (\mathbb{R}^n \setminus \Delta(T_a \times 0))\big)_{a\in A}$. This follows from Hestenes' lemma (e.g., [14, Theorem 1.10]) and the following reasoning. Clearly, $(G_a)_{a\in A}$ satisfies (3-3). To see (3-4) it suffices to consider the case that $x \in Z_a \times 0$ and $y \in \mathbb{R}^n \setminus \Delta(T_a \times 0)$ and to show that

$$|F_a^\kappa(x)| \le C\,\omega(|x-y|)\,|x-y|^{m-|\kappa|}, \quad |\kappa| \le m, \tag{4-5}$$

for a constant $C > 0$ independent of $a \in A$. We have

$$|F_a^\kappa(x)| \le C\,\omega(d(u, \partial T_a))d(u, \partial T_a)^{m-|\kappa|}, \quad |\kappa| \le m,$$

by (3-4), since $(F_a)_{a\in A}$ is flat on $(\partial T_a \times 0)_{a\in A}$, and

$$|F_a^\kappa(x)| \le C = \frac{C}{\omega(1)} \cdot \omega(1) 1^{m-|\kappa|}, \quad |\kappa| \le m, \tag{4-6}$$

by (3-3). Then (4-5) follows, since we have $|x-y| \ge c\,\min\{1, d(u, \partial T_a)\}$ for a universal constant $c > 0$, by (4-2).

By $(\mathbf{I}_{k-1})$, there exists a definable bounded family $(g_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(G_a)_{a\in A}$ that is C^p outside $(Z_a \times 0)_{a\in A}$. So, instead of $(F_a)_{a\in A}$, it is enough to consider $(F_a - J^m_{E_a}(g_a))_{a\in A}$.

If D_a and D'_a are distinct open k-dimensional strata in $\mathscr{D}_a$, then $\Delta(D_a \times 0) \subseteq \Delta(T_a \times 0)$ and $\overline{\Delta(D_a \times 0)} \cap \overline{\Delta(D'_a \times 0)} \subseteq Z_a \times 0$. Thus it suffices to find, separately for each $(D_a)_{a\in A}$, a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $\big((F_a - J^m_{E_a}(g_a))|_{\overline{D}_a\times 0}\big)_{a\in A}$ that is m-flat outside $(\Delta(D_a \times 0))_{a\in A}$ and C^p outside $(\overline{D}_a \times 0)_{a\in A}$.

For each $a \in A$, set

$$h_a(u, w) := \frac{1}{\beta!} F_a^{(0,\beta)}(u, 0) w^\beta - g_a(u, w),$$

and define $f_a : \mathbb{R}^n \to \mathbb{R}$ by

$$f_a(u, w) := \begin{cases} r_a(u, w) h_a(u, w) & \text{if } u \in D_a, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$r_a(u, w) := \prod_{i=1}^{\ell} \prod_{j=0}^{2k} \xi\Big(C\sqrt{\ell}\,\frac{w_i}{\rho_{a,j}(u)}\Big) \tag{4-7}$$

with $\xi : \mathbb{R} \to \mathbb{R}$ a semialgebraic C^p-function which is 1 near 0 and vanishes outside $(-1, 1)$, $\rho_{a,0}, \rho_{a,1}, \dots, \rho_{a,2k}$ the functions associated with the open Λ_p-cell D_a (see Section 2F), and C is the constant from (2-7) which may be taken independent from $a \in A$, since it is determined by the constants in the definition of the Λ_p-cells D_a, $a \in A$; see Remark 2.12. Note that the m-jet of h_a at $(u, 0)$ coincides with $(F_a - J^m_{E_a}(g_a))(u, 0)$ for all $u \in D_a$.

By construction, $(f_a)_{a\in A}$ is a definable family. We will see that it is a bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $\big((F_a - J^m_{E_a}(g_a))|_{\bar{D}_a\times 0}\big)_{a\in A}$. It is m-flat outside $(\Delta(D_a \times 0))_{a\in A}$, thanks to the properties of r_a, and it is C^p outside $(\bar{D}_a \times 0)_{a\in A}$. Indeed, if $(u, w) \in (D_a \times \mathbb{R}^\ell) \setminus \Delta(D_a \times 0)$, then, by (2-7),

$$\sqrt{\ell}\max_{1\le i\le \ell} |w_i| \ge |w| \ge \min\{1, d(u, \partial D_a\} \ge \frac{1}{C}\min_{0\le j\le 2k} \rho_{a,j}(u)$$

so that r_a is identically zero on $(D_a \times \mathbb{R}^\ell) \setminus \Delta(D_a \times 0)$. It remains to check that the family $(f_a)_{a\in A}$ is contained and bounded in $C^{m,\omega}(\mathbb{R}^n)$. To this end, we need two lemmas.

Lemma 4.3. *For each $a \in A$, h_a is of class $C^{m,\omega}$ on $\Delta(D_a \times 0)$ and the $C^{m,\omega}$-norm of h_a on $\Delta(D_a \times 0)$ is bounded by a constant independent of $a \in A$.*

Proof. By construction, each h_a is of class C^m. Since $(g_a)_{a\in A}$ is a bounded family of $C^{m,\omega}$-functions on $\mathbb{R}^n$, it suffices to consider $(u, w) \mapsto F_a^{(0,\beta)}(u, 0)w^\beta$. We have to check that there is a constant $C > 0$ such that, for all $a \in A$, all $\kappa = (\sigma, \tau) \in \mathbb{N}^k \times \mathbb{N}^\ell$, $|\kappa| \le m$, and all $(u, w) \in \Delta(D_a \times 0)$,

$$\big|\partial^\kappa(F_a^{(0,\beta)}(u, 0)w^\beta)\big| \le C, \tag{4-8}$$

and, if $|\kappa| = m$, for all $x_i = (u_i, w_i) \in \Delta(D_a \times 0)$, $i = 1, 2$,

$$\big|\partial^\kappa(F_a^{(0,\beta)}(u_1, 0)w_1^\beta) - \partial^\kappa(F_a^{(0,\beta)}(u_2, 0)w_2^\beta)\big| \le C\,\omega(|x_1 - x_2|). \tag{4-9}$$

Fix $\kappa = (\sigma, \tau)$ with $|\kappa| \le m$. We may assume that $\tau \le \beta$. Let us decompose σ as $\sigma = \alpha + \gamma$, where $\alpha, \gamma \in \mathbb{N}^k$, $|\alpha| \le m - |\beta|$, and α is maximal with this property. Thus, if $|\gamma| > 0$ then $|\alpha| + |\beta| = m$. To see (4-8), observe that, by (4-3), (3-3), and $|w| < \min\{1, d(u, \partial D_a)\}$,

$$\begin{aligned}|\partial^\gamma F_a^{(\alpha,\beta)}(u, 0)w^{\beta-\tau}| &\le L\,\frac{\sup\big\{|F_a^{(\alpha,\beta)}(v, 0)| : v \in D_a,\ |v - u| < d(u, \partial D_a)\big\}}{d(u, \partial D_a)^{|\gamma|}}|w|^{|\beta-\tau|}\\ &\le CL\,|w|^{|\beta-\tau|-|\gamma|} \le CL,\end{aligned}$$

where $C > 0$ is the supremum in (3-3); indeed, if $\gamma \ne 0$ then $|\alpha| + |\beta| = m$ and thus $|\beta - \tau| \ge |\gamma|$.

Let us prove (4-9). Now $|\kappa| = m$ and $|\alpha| + |\beta| = m$, whence $|\beta - \tau| = |\gamma|$. Then it is enough to show

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_1, 0) w_1^{\beta-\tau} - \partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0) w_2^{\beta-\tau}| \le C\,\omega(|x_1 - x_2|). \tag{4-10}$$

If $\gamma = 0$, this follows from (3-4). So let us assume that $|\gamma| \ge 1$.

Set $t_a(u) := \frac{1}{2} d(u, \partial D_a)$. Then

$$|t_a(u_1) - t_a(u_2)| \le \tfrac{1}{2}|u_1 - u_2|. \tag{4-11}$$

Note that, for $i = 1, 2$,

$$|w_i| < d(u_i, \partial D_a) = 2t_a(u_i). \tag{4-12}$$

We consider two cases.

Case 1. Suppose that $t_a(u_i) \le |x_1 - x_2|$ for $i = 1, 2$. Then, by (4-4) and (4-12),

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_i, 0) w_i^{\beta-\tau}| \le L\,\omega(2t_a(u_i)) \le 2L\,\omega(|x_1 - x_2|),$$

since ω is concave and increasing.

Case 2. Assume (without loss of generality) that $t_a(u_1) > |x_1 - x_2|$. Then $|u_1 - u_2| \le |x_1 - x_2| < t_a(u_1) = \frac{1}{2} d(u_1, \partial D_a)$ so that the line segment $[x_1, x_2]$ is contained in $D_a \times \mathbb{R}^\ell$. Furthermore, if $u \in [u_1, u_2]$ then, by (4-11),

$$|t_a(u_1) - t_a(u)| \le \tfrac{1}{2}|u_1 - u| \le \tfrac{1}{2}|x_1 - x_2| < \tfrac{1}{2} t_a(u_1),$$

whence

$$\tfrac{1}{2} t_a(u_1) < t_a(u) < \tfrac{3}{2} t_a(u_1), \qquad u \in [u_1, u_2].$$

The left-hand side of (4-10) is bounded by

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_1, 0) - \partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)|\,|w_1|^{|\beta-\tau|} + |\partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)|\,|w_1^{\beta-\tau} - w_2^{\beta-\tau}|.$$

By (4-4) and (4-12),

$$\begin{aligned}
&|\partial^\gamma F_a^{(\alpha,\beta)}(u_1, 0) - \partial^\gamma F_a^{(\alpha,\beta)}(u_2, 0)|\,|w_1|^{|\beta-\tau|} \\
&\qquad\lesssim \sup_{u \in [u_1,u_2]} \sum_{j=1}^k |\partial^{\gamma+(j)} F_a^{(\alpha,\beta)}(u, 0)|\,|u_1 - u_2|\, t_a(u_1)^{|\gamma|} \\
&\qquad\lesssim \sup_{u \in [u_1,u_2]} \frac{\omega(2t_a(u))}{t_a(u)^{|\gamma|+1}} |u_1 - u_2|\, t_a(u_1)^{|\gamma|} \\
&\qquad\lesssim \frac{\omega(t_a(u_1))}{t_a(u_1)} |x_1 - x_2| \\
&\qquad\le \omega(|x_1 - x_2|),
\end{aligned}$$

since ω is concave. Again, by (4-4) and (4-12),

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u_2,0)|\,|w_1^{\beta-\tau}-w_2^{\beta-\tau}| \\ \lesssim \frac{\omega(2t_a(u_2))}{t_a(u_2)^{|\gamma|}}|w_1-w_2|\,t_a(u_1)^{|\gamma|-1} \lesssim \frac{\omega(t_a(u_1))}{t_a(u_1)}|x_1-x_2| \le \omega(|x_1-x_2|).$$

The proof is complete. □

The proof shows that (4-9) actually holds on the larger set $\{(u,w)\in D_a\times\mathbb{R}^\ell : |w|<d(u,\partial D_a)\}$.

Lemma 4.4. *For each $a\in A$,*

$$|\partial^\kappa h_a(u,w)| \le C\,\omega(d(u,\partial D_a))d(u,\partial D_a)^{m-|\kappa|}, \tag{4-13}$$

for all $(u,w)\in\Delta(D_a\times 0)$, all $\kappa\in\mathbb{N}^n$, $|\kappa|\le m$, and a constant $C>0$ independent of $a\in A$.

Proof. Fix $x=(u,w)\in\Delta(D_a\times 0)$. If $d(u,\partial D_a)<d(u,\partial T_a)$, then let $u'\in\partial D_a$ be such that $|u-u'|=d(u,\partial D_a)$ and set $x'=(u',0)$. The open line segment (x,x') is contained in $\Delta(D_a\times 0)$. Since $u'\in T_a$, where $F_a^{(0,\beta)}$ is of class C^p, and h_a is of class $C^{m,\omega}$ on $\Delta(D_a\times 0)$ with $C^{m,\omega}$-norm bounded by a constant independent of $a\in A$, by Lemma 4.3, we may conclude the assertion from Taylor's theorem.

So we assume that $d(u,\partial D_a)=d(u,\partial T_a)$. Let $u'\in\partial T_a$ such that $|u-u'|=d(u,\partial T_a)$. Let us first assume that $\kappa=(\sigma,\tau)\in\mathbb{N}^k\times\mathbb{N}^\ell$ with $|\kappa|=m$. By construction, $\partial^\kappa g_a(u',0)=0$ so that

$$|\partial^\kappa g_a(u,w)| = |\partial^\kappa g_a(u,w)-\partial^\kappa g_a(u',0)| \lesssim \omega(|u-u'|)=\omega(d(u,\partial D_a)),$$

where we used that $|w|<d(u,\partial D_a)=|u-u'|$. Hence it suffices to consider $\partial^\kappa(F_a^{(0,\beta)}(u,0)w^\beta)$ or equivalently $\partial^\gamma F_a^{(\alpha,\beta)}(u,0)w^{\beta-\tau}$, where $\alpha,\gamma\in\mathbb{N}^k$ are such that $\alpha+\gamma=\sigma$, $|\alpha|+|\beta|=m$, and $\tau\le\beta$. Thus $|\beta-\tau|=|\gamma|$. If $|\gamma|\ge 1$, (4-4) implies

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u,0)w^{\beta-\tau}| \le L\,\frac{\omega(d(u,\partial D_a))}{d(u,\partial D_a)^{|\gamma|}}|w|^{|\gamma|} \le L\,\omega(d(u,\partial D_a)),$$

and, if $\gamma=0$, (3-4) gives

$$|\partial^\gamma F_a^{(\alpha,\beta)}(u,0)w^{\beta-\tau}| = |F_a^{(\alpha,\beta)}(u,0)| \lesssim \omega(d(u,\partial T_a))=\omega(d(u,\partial D_a)),$$

since $(F_a)_{a\in A}$ is flat on $(\partial T_a\times 0)_{a\in A}$.

To prove the statement for $|\kappa|<m$, we proceed by induction on $m-|\kappa|$. Suppose that the assertion is already shown for every $\lambda\in\mathbb{N}^n$ with $|\kappa|<|\lambda|\le m$. Since the open line segment (x,x') connecting $x=(u,w)$ and $x'=(u',0)$ is contained in

$\Delta(D_a \times 0)$, we have, by induction hypothesis, where $x'' = (u'', w'')$,

$$
\begin{aligned}
|\partial^\kappa h_a(u,w)| &\le \sup_{x'' \in (x,x')} \sum_{j=1}^{n} |\partial^{\kappa+(j)} h_a(u'',w'')|\,|x-x'| \\
&\lesssim \sup_{x'' \in (x,x')} \omega(d(u'',\partial D_a)) d(u'',\partial D_a)^{m-|\kappa|-1} d(u,\partial D_a) \\
&\lesssim \omega(d(u,\partial D_a)) d(u,\partial D_a)^{m-|\kappa|},
\end{aligned}
$$

since $d(u'',\partial D_a) \le d(u,\partial D_a)$. □

It follows from Lemmas 4.3 and 4.4 that, for each $a \in A$,

$$|\partial^\kappa h_a(u,w)| \le C\,\omega\big(\min\{1, d(u,\partial D_a)\}\big) \min\{1, d(u,\partial D_a)\}^{m-|\kappa|}, \tag{4-14}$$

for all $(u,w) \in \Delta(D_a \times 0)$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. Indeed, by Lemma 4.3,

$$|\partial^\kappa h_a(u,w)| \le C = \frac{C}{\omega(1)} \cdot \omega(1) 1^{m-|\kappa|}, \quad |\kappa| \le m,$$

for all $(u,w) \in \Delta(D_a \times 0)$, which, together with (4-13), gives (4-14).

Now Proposition 3.7 (see also Remark 3.8) implies that the family $(f_a)_{a \in A}$ is bounded in $C^{m,\omega}(\mathbb{R}^n)$. Indeed, Lemmas 4.3 and 4.4 and (4-14) guarantee that the assumptions of Proposition 3.7 are satisfied, where $t_a(u) = \min\{1, d(u,\partial D_a)\}$. Condition (3-6) holds thanks to (2-9) and Remark 2.12. We also get

$$|\partial^\kappa f_a(u,w)| \le C\,\omega(d(u,\partial D_a)) d(u,\partial D_a)^{m-|\kappa|}, \tag{4-15}$$

for all $(u,w) \in \Delta(D_a \times 0)$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$.

Step 2. For all $a \in A$, $E'_a = \overline{S}_a = \overline{T}_a \times 0$, but possibly E'_a is a proper subset of E_a for some $a \in A$. Consider the definable family $(r_a)_{a \in A}$ of functions $r_a : T_a \to (0,\infty)$ given by

$$r_a(u) := \begin{cases} \inf\{|w| : (u,w) \in E_a \setminus S_a\} & \text{if } \{w : (u,w) \in E_a \setminus S_a\} \ne \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Since F_a is flat on $E_a \setminus S_a$ we have (by (3-3) and (3-4))

$$|F_a^\kappa(u,0)| \le C\,\omega(r_a(u)) r_a(u)^{m-|\kappa|} \tag{4-16}$$

for all $u \in T_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. (In the case that $\{w : (u,w) \in E_a \setminus S_a\} = \varnothing$, it follows from (3-3) and we have to replace C by $C/\omega(1)$.)

By Theorem 2.16 and Proposition 2.20, there is a uniform Λ_p-stratification of $(\overline{T}_a)_{a \in A}$ such that

$$\overline{T}_a = Q_{a,1} \cup \cdots \cup Q_{a,s} \cup Z_a,$$

where, for each $a \in A$ and each $i = 1, \dots, s$, Z_a is closed with $\dim Z_a < k$, each $Q_{a,i}$ is an open k-dimensional Λ_p-cell with constant independent of $a \in A$, r_a is C^p on $Q_{a,i}$, and either

(i) $|\partial_j r_a| \le 1$, for each $j = 1, \dots, k$, on $Q_{a,i}$, in which case we may assume that $|\partial^\alpha r_a(u)| d(u, \partial Q_{a,i})^{|\alpha|-1}$, $1 \le |\alpha| \le p$, is bounded on $Q_{a,i}$ by a constant independent of $a \in A$, by Corollary 2.18, or

(ii) $|\partial_j r_a(u)| > 1$ for some j on $Q_{a,i}$.

By $(\mathbf{I}_{k-1})$, we may assume that $(F_a)_{a\in A}$ is flat on $(Z_a \times 0)_{a \in A}$ and hence on $(\partial Q_{a,i} \times 0)_{a\in A}$ for each $i = 1, \dots, s$.

Now it is enough to show that, for every $i = 1, \dots, s$, $(F_a|_{E_a \cap (\overline{Q}_{a,i} \times \mathbb{R}^\ell)})_{a\in A}$ admits a definable bounded family $(f_{a,i})_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ that is m-flat outside $(\Delta(Q_{a,i} \times 0))_{a\in A}$ and C^p outside $(\overline{Q}_{a,i} \times 0)_{a\in A}$. To this end, we fix i and drop it from the notation.

Step 1 gives a definable bounded family $(g_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a|_{\overline{Q}_a \times 0})_{a\in A}$ that is m-flat outside $(\Delta(Q_a \times 0)_{a\in A}$ and C^p outside $(\overline{Q}_a \times 0)_{a\in A}$. By Taylor's formula and (4-16), for each $a \in A$,

$$|\partial^\kappa g_a(u, w)| \le C\, \omega(r_a(u))\, r_a(u)^{m-|\kappa|} \tag{4-17}$$

for all $(u, w) \in Q_a \times \mathbb{R}^\ell$, $|w| < C' r_a(u)$, and all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, where $C, C' > 0$ are independent of $a \in A$. Similarly, we have

$$|\partial^\kappa g_a(u, w)| \le C\, \omega(d(u, \partial Q_a))\, d(u, \partial Q_a)^{m-|\kappa|} \tag{4-18}$$

for all $(u, w) \in Q_a \times \mathbb{R}^\ell$, $|w| < C' d(u, \partial Q_a)$, and all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, where $C, C' > 0$ are independent of $a \in A$.

In Case (ii), one can easily see (see [10, p. 94]) that, for each $a \in A$, $r_a(u) \ge d(u, \partial Q_a)$ for $u \in Q_a$, so that $E_a \setminus S_a \subseteq \mathbb{R}^n \setminus \Delta(Q_a \times 0)$. That means that g_a is a $C^{m,\omega}$-extension to $\mathbb{R}^n$ of $F_a|_{E_a \cap (\overline{Q}_a \times 0)}$, and we are done.

In Case (i), a modification is necessary: we define, for each $a \in A$,

$$f_a(u, w) := \begin{cases} \displaystyle\prod_{i=1}^{\ell} \xi\left(\sqrt{\ell}\frac{w_i}{r_a(u)}\right) \cdot g_a(u, w) & \text{if } u \in Q_a, \\ 0 & \text{otherwise,} \end{cases}$$

where $\xi : \mathbb{R} \to \mathbb{R}$ is a semialgebraic C^p-function that is 1 near 0 and vanishes outside $(-1, 1)$. Note that $(f_a)_{a\in A}$ is a definable family of functions $f_a : \mathbb{R}^n \to \mathbb{R}$. Moreover, we set

$$\Delta'(Q_a \times 0) := \{(u, w) \in Q_a \times \mathbb{R}^\ell : |w| < t_a(u)\}$$

with

$$t_a(u) := \min\{r_a(u), d(u, \partial Q_a)\},$$

and claim that, for each $a \in A$, f_a is of class $C^{m,\omega}$ on $\Delta'(Q_a \times 0)$ with $C^{m,\omega}$-norm bounded by a constant independent of $a \in A$, and

$$|\partial^\kappa f_a(u,w)| \le C\,\omega(t_a(u))\,t_a(u)^{m-|\kappa|} \tag{4-19}$$

for $(u,w) \in \Delta'(Q_a \times 0)$ and all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, where $C > 0$ is independent of $a \in A$.

To see this, let us first assume that $r_a(u) < d(u, \partial Q_a)$ so that $t_a(u) = r_a(u)$. Since we are in Case (i), we find that, thanks to (2-10),

$$\left|\partial^\alpha\left(\frac{1}{r_a}\right)(u)\right| \le C\,r_a(u)^{-|\alpha|-1}, \quad u \in Q_a,\ |\alpha| \le p,$$

for a constant $C > 0$ independent of $a \in A$. Thus, the claim follows from (4-17) and Proposition 3.7.

If $r_a(u) \ge d(u, \partial Q_a)$ (that is, $t_a(u) = d(u, \partial Q_a)$), then similarly

$$\left|\partial^\alpha\left(\frac{1}{r_a}\right)(u)\right| \le C\,d(u, \partial Q_a)^{-|\alpha|-1}, \quad u \in Q_a,\ |\alpha| \le p,$$

Then we infer the claim from (4-18) and Proposition 3.7.

We conclude that $(f_a)_{a\in A}$ is the required family of definable bounded $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a|_{E_a \cap (\overline{Q}_a \times \mathbb{R}^\ell)})_{a\in A}$ that is m-flat outside $(\Delta(Q_a \times 0))_{a\in A}$ and C^p outside $(\overline{Q}_a \times 0)_{a\in A}$. This ends Step 2.

Step 3. The general case of Proposition 4.2: for all $a \in A$, $S_a = \Gamma(\varphi_a)$, $E'_a = \overline{S}_a \subseteq E_a$, where $\varphi_a : T_a \to \mathbb{R}^\ell$ is not necessarily identically 0. Consider the definable family $(s_a)_{a\in A}$ of functions $s_a : S_a \to (0, \infty)$ given by

$$s_a(x) := \min\{d(x, E_a \setminus S_a), d(x, \partial S_a)\}, \quad x \in S_a.$$

For each $a \in A$, let $\overline{\varphi}_a : \overline{T}_a \to \mathbb{R}^\ell$ be the continuous extension of φ_a; see Section 2D. Furthermore, we consider the maps

$$\varphi_{a,\pm} : T_a \times \mathbb{R}^\ell \to T_a \times \mathbb{R}^\ell, \quad (u,w) \mapsto (u, w \pm \varphi_a(u))$$

and

$$\overline{\varphi}_{a,\pm} : \overline{T}_a \times \mathbb{R}^\ell \to \overline{T}_a \times \mathbb{R}^\ell, \quad (u,w) \mapsto (u, w \pm \overline{\varphi}_a(u)).$$

Note that $\overline{\varphi}_{a,+}$ is a bi-Lipschitz homeomorphism with inverse $\overline{\varphi}_{a,-}$ and Lipschitz constants independent of $a \in A$.

Since F_a is flat on $E_a \setminus S_a$ and on ∂S_a, we have (by (3-4))

$$|F_a^\kappa(x)| \le C\,\omega(s_a(x)) s_a(x)^{m-|\kappa|} \tag{4-20}$$

for all $x \in S_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. Setting

$$t_a(u) := s_a(u, \varphi_a(u)), \quad u \in T_a,$$

we have

$$\left|F_a^\kappa(u,\varphi_a(u))\right| \le C\,\omega(t_a(u))t_a(u)^{m-|\kappa|}, \quad u \in T_a,\ |\kappa| \le m. \tag{4-21}$$

The uniformity of the constants in the definition of T_a and φ_a implies that $(\bar{S}_a)_{a\in A}$ and $(\partial T_a \times \mathbb{R}^\ell)_{a\in A}$ are uniformly separated. Observe that (by the definition of s_a) $t_a(u) \le C'\,d(u,\partial T_a)$ for $C' > 0$ independent of $a \in A$, since $\overline{\varphi}_{a,+}$ is a bi-Lipschitz homeomorphism with Lipschitz constants independent of $a \in A$.

Thus Proposition 3.5 (and Lemma 3.6) implies that $(G_a)_{a\in A}$, where $G_a := \varphi_{a,+}^*(F_a|_{S_a})$, is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(T_a \times 0)_{a\in A}$ and extends to a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\overline{T}_a \times 0)_{a\in A}$ which is flat on $(\partial T_a \times 0)_{a\in A}$ and such that

$$|G_a^\kappa(u,0)| \le C\,\omega(t_a(u))t_a(u)^{m-|\kappa|} \tag{4-22}$$

for all $u \in T_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. For each $a \in A$, set $\widetilde{E}_a := \overline{\varphi}_{a,-}(E_a \cap (\overline{T}_a \times \mathbb{R}^\ell))$. Since $\overline{\varphi}_{a,+}$ is a bi-Lipschitz homeomorphism with constants independent of $a \in A$, we may conclude

$$|G_a^\kappa(u,0)| \le C\,\omega\big(d((u,0),\widetilde{E}_a \setminus (T_a \times 0))\big)d((u,0),\widetilde{E}_a \setminus (T_a \times 0))^{m-|\kappa|} \tag{4-23}$$

for all $u \in T_a$, all $\kappa \in \mathbb{N}^n$, $|\kappa| \le m$, and a constant $C > 0$ independent of $a \in A$. Thus $(\widetilde{G}_a)_{a\in A}$, where

$$\widetilde{G}_a(u,w) := \begin{cases} G_a(u,0) & \text{if } (u,w) \in \overline{T}_a \times 0, \\ 0 & \text{if } (u,w) \in \widetilde{E}_a \setminus (\overline{T}_a \times 0), \end{cases}$$

is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(\widetilde{E}_a)_{a\in E}$ that is flat on $(\widetilde{E}_a \setminus (T_a \times 0))_{a\in A}$.

By Step 2, there exists a definable bounded family $(\tilde{g}_a)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(\widetilde{G}_a)_{a\in A}$ that is m-flat on $(\widetilde{E}_a \setminus (T_a \times 0))_{a\in A}$ as well as outside $(T_a \times \mathbb{R}^\ell)_{a\in A}$ and C^p outside $(\overline{T}_a \times 0)_{a\in A}$.

For each $a \in A$, define $f_a : \mathbb{R}^n \to \mathbb{R}$ by

$$f_a(u,w) := \begin{cases} (\tilde{g}_a \circ \varphi_{a,-})(u,w) & \text{if } (u,w) \in T_a \times \mathbb{R}^\ell, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f_a)_{a\in A}$ is a definable bounded family of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(\bar{S}_a)_{a\in A}$, which follows again from Proposition 3.5 (with $M_a = T_a \times \mathbb{R}^\ell$ and $U_a = T_a$).

This completes the proof of Proposition 4.2, hence of $(\mathbf{I}_k)$, and thus the proof of Theorem 1.3.

5. Further applications

We present a local version of Theorem 1.3, we discuss the dependence of the bounded extension on the modulus of continuity which leads to the proof of Theorem 1.4, and finally we obtain a definable version of a correspondence between Whitney jets of class $C^{m,\omega}$ and certain Lipschitz maps, which was first observed by Shvartsman [13].

5A. *Definable $C^{m,\omega}_{\mathrm{loc}}$-extensions.* Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^{m,\omega}_{\mathrm{loc}}(U)$ the space of functions $f : U \to \mathbb{R}$ such that $f|_V \in C^{m,\omega}(V)$, for all relatively compact open subsets $V \subseteq U$.

Let $E \subseteq \mathbb{R}^n$ be a closed set. An m-jet F on E is called a (*definable*) *Whitney jet of class $C^{m,\omega}_{\mathrm{loc}}$ on E* if $F|_K$ is a (definable) Whitney jet of class $C^{m,\omega}$ on K, for all (definable) compact subsets $K \subseteq E$. A $C^{m,\omega}_{\mathrm{loc}}$-function $f : \mathbb{R}^n \to \mathbb{R}$ is a *$C^{m,\omega}_{\mathrm{loc}}$-extension to $\mathbb{R}^n$ of F* if $J^m_E(f) = F$.

Let $(E_a)_{a\in A}$ be a family of closed sets $E_a \subseteq \mathbb{R}^n$. A family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}_{\mathrm{loc}}$ on E_a is called a (*definable*) *bounded family of Whitney jets of class $C^{m,\omega}_{\mathrm{loc}}$* if $(F_a|_{K_a})_{a\in A}$ is a (definable) bounded family of Whitney jets of class $C^{m,\omega}$ for each (definable) subfamily $(K_a)_{a\in A}$ of $(E_a)_{a\in A}$ consisting of (definable) compact sets $K_a \subseteq E_a$.

A family $(f_a)_{a\in A}$ of $C^{m,\omega}_{\mathrm{loc}}$-functions $f_a : \mathbb{R}^n \to \mathbb{R}$ is called a (*definable*) *bounded family of $C^{m,\omega}_{\mathrm{loc}}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$* if f_a is a $C^{m,\omega}_{\mathrm{loc}}$-extension to $\mathbb{R}^n$ of F_a, for each $a \in A$, and, for each (definable) relatively compact subset $V \subseteq \mathbb{R}^n$, $(f_a|_V)_{a\in A}$ is a (definable) bounded family of $C^{m,\omega}$-functions.

Corollary 5.1. *Let $0 \le m \le p$ be integers. Let ω be a modulus of continuity. Let $(E_a)_{a\in A}$ be a definable family of closed subsets E_a of $\mathbb{R}^n$. For any definable bounded family $(F_a)_{a\in A}$ of Whitney jets of class $C^{m,\omega}_{\mathrm{loc}}$ on $(E_a)_{a\in A}$ there exists a definable bounded family $(f_a)_{a\in A}$ of $C^{m,\omega}_{\mathrm{loc}}$-extensions to $\mathbb{R}^n$ of $(F_a)_{a\in A}$ that is C^p outside $(E_a)_{a\in A}$.*

Proof. For integers $k \ge 1$, consider the definable sets $U_k := \{x \in \mathbb{R}^n : k-2 < |x| < k\}$; note that U_1 is the unit ball, U_2 is a punctured ball, and U_k, for $k \ge 3$, are annuli centered at the origin. The sets U_k, for $k \ge 1$, form an open cover of $\mathbb{R}^n$ with the property that $U_k \cap U_\ell \ne \varnothing$ if and only if $|k-\ell| \le 1$. Fix an integer $p \ge m+1$. There exists a partition of unity $\{\varphi_k\}_{k\ge1}$ of class C^p subordinated to the cover $\{U_k\}_{k\ge1}$, where each φ_k is definable: $\varphi_k \in C^p(\mathbb{R}^n)$, $\varphi_k \ge 0$, $\operatorname{supp}\varphi_k \subseteq U_k$, for all $k \ge 1$, the family $\{\operatorname{supp}\varphi_k\}_{k\ge1}$ is locally finite, and $\sum_{k\ge1}\varphi_k = 1$. For instance, let $h : \mathbb{R} \to \mathbb{R}$ be a nonnegative definable function of class C^p such that $\operatorname{supp} h = \left[-\frac{3}{4}, \frac{3}{4}\right]$ and set $\psi_1(x) := h(|x|^2)$ and $\psi_k(x) := h(|x| - (k-1))$, for $k \ge 2$. Then $\psi := \sum_{k\ge1}\psi_k$ is of class C^p and everywhere positive (locally it is a finite sum). Thus $\varphi_k := \psi_k/\psi$ is as required; it is definable, since in a neighborhood of $\operatorname{supp}\varphi_k = \operatorname{supp}\psi_k$ the denominator ψ is represented by a finite sum of definable functions.

Let $(F_a)_{a\in A}$ be a definable bounded family of Whitney jets of class $C^{m,\omega}_{\mathrm{loc}}$ on $(E_a)_{a\in A}$. For each $k \ge 1$, $(F_a|_{\overline{U}_k})_{a\in A}$ is a definable bounded family of Whitney jets of class $C^{m,\omega}$ on $(E_a \cap \overline{U}_k)_{a\in A}$. By Theorem 1.3, there exists a definable bounded family $(f_a^k)_{a\in A}$ of $C^{m,\omega}$-extensions to $\mathbb{R}^n$ of $(F_a|_{\overline{U}_k})_{a\in A}$ such that f_a^k is of class C^p outside $E_a \cap \overline{U}_k$ for all $a \in A$; if $E_a \cap \overline{U}_k = \varnothing$ we set $f_a^k := 0$. Let $f_a := \sum_{k=1}^{\infty} \varphi_k f_a^k$, for $a \in A$. The function f_a is of class $C^{m,\omega}_{\mathrm{loc}}$ on $\mathbb{R}^n$ and C^p outside E_a, since the defining sum is finite on every compact set and $p \ge m+1$. Let $x \in E_a$. There exist a neighborhood U of x and $k \ge 1$ such that $U \subseteq U_k \cup U_{k+1} \cup U_{k+2}$ and $U \cap \bigcup_{\ell \notin \{k,k+1,k+2\}} U_\ell = \varnothing$. So, for each $\kappa \in \mathbb{N}^n$ with $|\kappa| \le m$,

$$\begin{aligned}
\partial^\kappa f_a(x) &= \sum_{i=0}^{2} \sum_{\sigma \le \kappa} \binom{\kappa}{\sigma} \partial^\sigma \varphi_{k+i}(x) \partial^{\kappa-\sigma} f_a^{k+i}(x) \\
&= \sum_{i=0}^{2} \sum_{\sigma \le \kappa} \binom{\kappa}{\sigma} \partial^\sigma \varphi_{k+i}(x) F_a^{\kappa-\sigma}(x) \\
&= \sum_{\sigma \le \kappa} \binom{\kappa}{\sigma} \partial^\sigma \Big(\sum_{i=0}^{2} \varphi_{k+i}(x) \Big) F_a^{\kappa-\sigma}(x) \\
&= \sum_{\sigma \le \kappa} \binom{\kappa}{\sigma} \partial^\sigma (1) F_a^{\kappa-\sigma}(x) \\
&= F_a^\kappa(x).
\end{aligned}$$

Thus, f_a is a $C^{m,\omega}_{\mathrm{loc}}$-extension to $\mathbb{R}^n$ of F_a.

Fix a definable relatively compact subset $V \subseteq \mathbb{R}^n$. There exists $K \in \mathbb{N}$ such that $\overline{V} \cap U_k = \varnothing$ for all $k > K$. In particular, $f_a(x) := \sum_{k=1}^{K} \varphi_k(x) f_a^k(x)$, for all $x \in V$ and $a \in A$. Hence, $(f_a|_V)_{a\in A}$ is a definable bounded family of $C^{m,\omega}$-functions. $\square$

Remark 5.2. We do not say that f_a is definable as a global function $f_a : \mathbb{R}^n \to \mathbb{R}$, because the gluing argument (based on the partition of unity) involves an infinite sum.

5B. ***Dependence on the modulus of continuity.*** The main result, Theorem 1.3, only depends in a weak sense on the modulus of continuity ω, namely, the uniform constant C occasionally must be multiplied by $\omega(1)$ or by $\omega(1)^{-1}$; see (4-1), (4-6), (4-14), and (4-16).

Thus, we can allow in Theorem 1.3 that, for each $a \in A$, F_a is a Whitney jet of class C^{m,ω_a} on E_a, where ω_a is a modulus of continuity and there is a constant $C > 0$ independent of $a \in A$ such that

$$C^{-1} \le \omega_a(1) \le C, \quad a \in A. \tag{5-1}$$

Then the statement is the following:

Theorem 5.3. *Let $0 \le m \le p$ be integers. Let $(\omega_a)_{a\in A}$ be a family of moduli of continuity satisfying (5-1). Let $(E_a)_{a\in A}$ be a definable family of closed subsets E_a of $\mathbb{R}^n$. For any definable family $(F_a)_{a\in A}$ of Whitney jets F_a of class C^{m,ω_a} on E_a such that*

$$\sup_{a\in A}\sup_{x\in E_a}\sup_{|\gamma|\le m}|F_a^\gamma(x)|<\infty, \tag{5-2}$$

and

$$\sup_{a\in A}\sup_{x\neq y\in E_a}\sup_{|\gamma|\le m}\frac{|(R_x^m F_a)^\gamma(y)|}{\omega_a(|x-y|)\,|x-y|^{m-|\gamma|}}<\infty, \tag{5-3}$$

there exists a definable family $(f_a)_{a\in A}$ of C^{m,ω_a}-extensions f_a to $\mathbb{R}^n$ of F_a such that f_a is of class C^p outside E_a, for all $a \in A$, and

$$\sup_{a\in A}\|f_a\|_{C^{m,\omega_a}(\mathbb{R}^n)}<\infty. \tag{5-4}$$

5C. ***Proof of Theorem 1.4.*** Let $(F_a)_{a\in A}$ be a definable family of Whitney jets of class C^m on $(E_a)_{a\in A}$, where $E_a \subseteq \mathbb{R}^n$ is compact. We say that the family $(F_a)_{a\in A}$ is *bounded* if

$$\sup_{a\in A}\sup_{x\in E_a}\sup_{|\gamma|\le m}|F_a^\gamma(x)|<\infty \tag{5-5}$$

and

$$\sup_{a\in A}\sup_{x\neq y\in E_a}\sup_{|\gamma|\le m}\frac{|(R_x^m F_a)^\gamma(y)|}{|x-y|^{m-|\gamma|}}<\infty. \tag{5-6}$$

Proof of Theorem 1.4. We modify slightly an argument used in Proposition IV.1.5 of [15]. For each $a \in A$, consider

$$\sigma_a(t) := \sup_{\substack{x\neq y\in E_a\\ |x-y|\le t}}\sup_{|\gamma|\le m}\frac{|(R_x^m F_a)^\gamma(y)|}{|x-y|^{m-|\gamma|}}, \qquad t>0, \qquad \sigma_a(0):=0.$$

Then $\sigma_a : [0,\infty) \to [0,\infty)$ is an increasing function that is continuous at 0 and

$$\sigma_a(t)=\sigma_a(\operatorname{diam} E_a), \qquad t\ge \operatorname{diam} E_a.$$

Thus also $\tau_a : [0,\infty) \to [0,\infty)$, defined by

$$\tau_a(t) := \begin{cases}\sigma_a(t) & \text{if } t<1,\\ \max\{1,\sigma_a(t)\} & \text{if } t\ge 1,\end{cases}$$

is increasing and continuous at 0 with

$$\tau_a(t)\le \max\{1,\sigma_a(\operatorname{diam} E_a)\}, \qquad t\ge 0. \tag{5-7}$$

Let ω_a be the least concave majorant of τ_a which is finite, thanks to (5-7). Then ω_a is a modulus of continuity and

$$\sup_{a\in A}\sup_{x\neq y\in E_a}\sup_{|\gamma|\le m}\frac{|(R_x^m F_a)^\gamma(y)|}{\omega_a(|x-y|)\,|x-y|^{m-|\gamma|}}\le 1.$$

Moreover, $\omega_a(1)\ge 1$ and, by (5-7),

$$\omega_a(t)\le \max\{1,\sigma_a(\operatorname{diam} E_a)\}\le C,\qquad t\ge 0,$$

for a constant $C>0$ independent of $a\in A$, thanks to (5-6). In particular, (5-1) is satisfied.

Thus Theorem 5.3 implies that there is a definable family $(f_a)_{a\in A}$ such that, for each $a\in A$, f_a is a C^{m,ω_a}-extension to $\mathbb{R}^n$ of F_a, C^p outside E_a, and

$$\sup_{a\in A}\|f_a\|_{C^{m,\omega_a}(\mathbb{R}^n)}<\infty.$$

In particular, $(f_a)_{a\in A}$ is a bounded family of C^m-functions. □

5D. *Definable Whitney jets as Lipschitz maps.* We end with a few observations on a definable version of a correspondence, due to Shvartsman [13], between Whitney jets of class $C^{m,\omega}$ and certain Lipschitz maps. Here the notation follows closely the one of [13].

Let ω be a modulus of continuity and m a positive integer. For $\alpha\in\mathbb{N}^n$ with $|\alpha|<m$ let ψ_α be the inverse of the (strictly increasing) function $s\mapsto s^{m-|\alpha|}\omega(s)$ and put $\varphi_\alpha:=\omega\circ\psi_\alpha$. For $|\alpha|=m$, set $\varphi_\alpha(t):=t$.

Let $\mathcal{P}_m$ denote the space of real polynomials of degree at most m in n variables. For $T_i=(P_i,x_i)\in\mathcal{P}_m\times\mathbb{R}^n$, $i=1,2$, define

$$\delta_\omega(T_1,T_2):=\max\Bigl\{\omega(|x_1-x_2|),\max_{\substack{|\alpha|\le m\\ i=1,2}}\varphi_\alpha\bigl(|\partial^\alpha(P_1-P_2)(x_i)|\bigr)\Bigr\}.$$

Then we get a metric d_ω on $\mathcal{P}_m\times\mathbb{R}^n$ by setting

$$d_\omega(T,T'):=\inf\sum_{j=0}^{k-1}\delta_\omega(T_j,T_{j+1}),$$

where the infimum is taken over all finite sequences $T=T_1,T_2,\dots,T_k=T'$ in $\mathcal{P}_m\times\mathbb{R}^n$. It turns out (see [13, Theorem 2.1]) that

$$d_\omega((P,x),(P',x'))\le\delta_\omega((P,x),(P',x'))\le d_\omega((e^nP,x),(e^nP',x')).$$

Let $\mathcal{T}_{m,n}$ be the metric space $(\mathcal{P}_m\times\mathbb{R}^n,d_\omega)$. For a nonempty subset $X\subseteq\mathbb{R}^n$, we denote by X_ω the metric space $(X,(x,y)\mapsto\omega(|x-y|))$. Let $\mathbf{Lip}(X_\omega,\mathcal{T}_{m,n})$ be the space of Lipschitz maps $T : x \mapsto (P_x, z_x)$ such that $\max_{|\alpha|\le m} \sup_{x\in X} |\partial^\alpha P_x(x)| < \infty$, equipped with the norm

$$\|T\|^*_{\mathrm{LO}(X)} := \max_{|\alpha|\le m} \sup_{x\in X} |\partial^\alpha P_x(x)| \\ + \inf\bigl\{\lambda > 0 : d_\omega(\lambda^{-1}T(x), \lambda^{-1}T(y)) \le \omega(|x-y|) \text{ for all } x, y \in X\bigr\},$$

where $\lambda^{-1}T(x) := (\lambda^{-1}P_x, z_x)$. Let $T^m_x f$ be the Taylor polynomial of order m at x of a C^m-function f.

Now let us recall a result of [13].

Proposition 5.4 [13, Propositions 1.9 and 2.8]. *Let $X \subseteq \mathbb{R}^n$ be a closed set. Given a family of polynomials $\{P_x \in \mathcal{P}_m : x \in X\}$, there exists $f \in C^{m,\omega}(\mathbb{R}^n)$ such that $T^m_x f = P_x$ for all $x \in X$ if and only if the map $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$. We have*

$$\inf\{\|f\|_{C^{m,\omega}(\mathbb{R}^n)} : T^m_x f = P_x \textit{ for all } x \in X\} \approx \|T\|^*_{\mathrm{LO}(X)}$$

in the sense that either side is bounded by a constant $C(m,n)$ times the other side. If, moreover, $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$, then T has an extension $\widetilde{T} : x \mapsto (\widetilde{P}_x, x)$ in $\mathbf{Lip}(\mathbb{R}^n_\omega, \mathcal{T}_{m,n})$ satisfying

$$\|\widetilde{T}\|^*_{\mathrm{LO}(\mathbb{R}^n)} \le C(m,n)\, \|T\|^*_{\mathrm{LO}(X)}.$$

These results are based on the classical extension theorem for Whitney jets of class $C^{m,\omega}$. As a consequence of Theorem 1.2, we may conclude the following definable version, where, provided that X is definable, $\mathbf{Lip}_{\mathrm{def}}(X_\omega, \mathcal{T}_{m,n})$ is the subspace of definable maps $T : x \mapsto (P_x, z_x)$ in $\mathbf{Lip}(X_\omega, \mathcal{T}_{m,n})$, which means that z_x and the coefficients of P_x are definable maps in x. Recall that $C^{m,\omega}_{\mathrm{def}}(\mathbb{R}^n)$ is the subspace of $C^{m,\omega}(\mathbb{R}^n)$ consisting of all definable functions in $C^{m,\omega}(\mathbb{R}^n)$.

Proposition 5.5. *Let $X \subseteq \mathbb{R}^n$ be a definable closed set. Given a definable family of polynomials $\{P_x \in \mathcal{P}_m : x \in X\}$, there exists $f \in C^{m,\omega}_{\mathrm{def}}(\mathbb{R}^n)$ such that $T^m_x f = P_x$ for all $x \in X$ if and only if the map $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}_{\mathrm{def}}(X_\omega, \mathcal{T}_{m,n})$. If, moreover, $T : x \mapsto (P_x, x)$ belongs to $\mathbf{Lip}_{\mathrm{def}}(X_\omega, \mathcal{T}_{m,n})$, then T has an extension $\widetilde{T} : x \mapsto (\widetilde{P}_x, x)$ in $\mathbf{Lip}_{\mathrm{def}}(\mathbb{R}^n_\omega, \mathcal{T}_{m,n})$.*

Concerning the existence of uniform bounds for the norms, remarks similar to the ones in [11, Section 4.4] apply. But Theorem 1.3 implies the following supplement.

Proposition 5.6. *Suppose that in the setting of Proposition 5.5, the family of polynomials depends definably on additional parameters $a \in A$, i.e., a definable family of polynomials $\{P^a_x \in \mathcal{P}_m : x \in X,\ a \in A\}$ is given. Then there exists a bounded family $(f^a)_{a\in A}$ of definable $C^{m,\omega}$-functions $f^a : \mathbb{R}^n \to \mathbb{R}$ such that*

$$T^m_x f^a = P^a_x \quad \textit{for all } x \in X \textit{ and } a \in A$$

if and only if $(T^a : x \mapsto (P^a_x, x))_{a\in A}$ *forms a bounded subset of* $\mathbf{Lip}_{\mathrm{def}}(X_\omega, \mathcal{T}_{m,n})$. *If, moreover,* $(T^a : x \mapsto (P^a_x, x))_{a\in A}$ *forms a bounded subset of* $\mathbf{Lip}_{\mathrm{def}}(X_\omega, \mathcal{T}_{m,n})$, *then there is a family* $(\widetilde{T}^a : x \mapsto (\widetilde{P}^a_x, x))_{a\in A}$ *of extensions* $\widetilde{T}^a$ *of* T^a *which forms a bounded subset of* $\mathbf{Lip}_{\mathrm{def}}(\mathbb{R}^n_\omega, \mathcal{T}_{m,n})$.

Acknowledgements

This research was funded in whole or in part by the Austrian Science Fund (FWF) DOI 10.55776/P32905. For open access purposes, the author has applied a CC BY public copyright license to any author-accepted manuscript version arising from this submission. A large part of this work has been done at Mathematisches Forschungsinstitut Oberwolfach (*Oberwolfach Research Fellows* (*OWRF*) *ID* 2244*p*, *October* 31 – *November* 19, 2022). We are grateful for the support and the excellent working conditions.

References

[1] D. Azagra, E. Le Gruyer, and C. Mudarra, "Explicit formulas for $C^{1,1}$ and $C^{1,\omega}_{\mathrm{conv}}$ extensions of 1-jets in Hilbert and superreflexive spaces", *J. Funct. Anal.* **274**:10 (2018), 3003–3032. MR Zbl

[2] R. G. Bartle and L. M. Graves, "Mappings between function spaces", *Trans. Amer. Math. Soc.* **72** (1952), 400–413. MR Zbl

[3] A. Brudnyi and Y. Brudnyi, *Methods of geometric analysis in extension and trace problems, I*, Monogr. Math. **102**, Birkhäuser, Basel, 2012. MR Zbl

[4] L. van den Dries, *Tame topology and o-minimal structures*, Lond. Math. Soc. Lect. Note Ser. **248**, Cambridge Univ. Press, 1998. MR Zbl

[5] L. van den Dries and C. Miller, "Geometric categories and o-minimal structures", *Duke Math. J.* **84**:2 (1996), 497–540. MR Zbl

[6] G. Glaeser, "Étude de quelques algèbres tayloriennes", *J. Anal. Math.* **6** (1958), 1–124. MR Zbl

[7] M. Gromov, "Entropy, homology and semialgebraic geometry", pp. 225–240 in *Séminaire Bourbaki*, 1985/1986, Astérisque **145-146**, Soc. Math. France, Paris, 1987. MR Zbl

[8] K. Kurdyka and A. Parusiński, "Quasi-convex decomposition in o-minimal structures: application to the gradient conjecture", pp. 137–177 in *Singularity theory and its applications* (Sapporo, Japan, 2003), Adv. Stud. Pure Math. **43**, Math. Soc. Japan, Tokyo, 2006. MR Zbl

[9] K. Kurdyka and W. Pawłucki, "Subanalytic version of Whitney's extension theorem", *Studia Math.* **124**:3 (1997), 269–280. MR Zbl

[10] K. Kurdyka and W. Pawłucki, "O-minimal version of Whitney's extension theorem", *Studia Math.* **224**:1 (2014), 81–96. MR Zbl

[11] A. Parusiński and A. Rainer, "Definable Lipschitz selections for affine-set valued maps", 2023. To appear in *Israel J. Math.* arXiv 2306.09155

[12] W. Pawłucki, "A linear extension operator for Whitney fields on closed o-minimal sets", *Ann. Inst. Fourier* (*Grenoble*) **58**:2 (2008), 383–404. MR Zbl

[13] P. Shvartsman, "The Whitney extension problem and Lipschitz selections of set-valued mappings in jet-spaces", *Trans. Amer. Math. Soc.* **360**:10 (2008), 5529–5550. MR Zbl

[14] A. Thamrongthanyalak, "Whitney's extension theorem in o-minimal structures", *Ann. Polon. Math.* **119**:1 (2017), 49–67. MR Zbl

[15] J.-C. Tougeron, *Idéaux de fonctions différentiables*, Ergebnisse der Math. (2) **71**, Springer, 1972. MR Zbl

[16] H. Whitney, "Analytic extensions of differentiable functions defined in closed sets", *Trans. Amer. Math. Soc.* **36**:1 (1934), 63–89. MR Zbl

Received November 2, 2023. Revised June 25, 2024.

ADAM PARUSIŃSKI
LJAD
UNIVERSITÉ CÔTE D'AZUR
NICE
FRANCE

adam.parusinski@unice.fr

ARMIN RAINER
FACULTY OF MATHEMATICS
UNIVERSITY OF VIENNA
VIENNA
AUSTRIA

armin.rainer@univie.ac.at

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.355

NONCOMMUTATIVE TENSOR TRIANGULAR GEOMETRY: CLASSIFICATION VIA NOETHERIAN SPECTRA

JAMES ROWE

Given a monoidal triangulated category T with Noetherian spectrum, we show that there is an order-preserving bijection between the collection of all Thomason subsets of the noncommutative spectrum of T and the collection of all thick two-sided semiprime ideals of T and that it is universal among all such spaces classifying the ideals in question.

1. Introduction

One application of tensor triangular geometry (tt-geometry) is the classification of various types of subcategories via topological spaces. Following specific classification results in fields such as algebraic geometry (for example, [Thomason 1997]) and topology [Hopkins and Smith 1998], Balmer [2005] proved generic classification theorems within the framework of tt-geometry. Since then, the theory of classification within tt-geometry has expanded to include techniques such as the use of residue functors [Balmer and Favi 2011] and categorical actions [Stevenson 2013], further increasing the range of subcategories for which classification is possible.

This article is concerned with the theory of *monoidal* triangulated categories, or *mt-categories*. Within these categories, the tensor product functor $\otimes$ is no longer required to be symmetric. The general theory of these categories has been of significant interest in recent times, with foundational results established by Nakano, Vashaw, and Yakimov [Nakano et al. 2022a]. These general results include a classification theorem applicable to a wide collection of weak support data [Nakano et al. 2022a, 6.2.1], as well as characterisations of those categories in which the tensor product functor interacts well with support data [Nakano et al. 2022b, 3.1.1].

The classification theorems in both the symmetric and monoidal settings use the connections between a mt-category T and its collection of prime ideals $\mathrm{Spc}(\mathsf{T})$ considered as a topological space. Many of the classifications of subcategories rely on controlling the properties of this space or the prime ideals from which it is

MSC2020: 18G80, 18M05.

Keywords: noncommutative tensor triangular geometry, monoidal triangulated categories, Noetherian spectra.

formed. For example, in the recent work of Mallick and Ray [2023], radical thick tensor ideals of mt-categories are classified using a point-free approach, under the assumption that every prime ideal is completely prime: given a prime ideal $\mathcal{P}$, if there are objects a, b such that $a \otimes b \in \mathcal{P}$, then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

At the level of topological spaces, classification theorems often assume that the spaces used to classify subcategories are *Noetherian* topological spaces. Such spaces are commonly encountered in algebraic geometry: given a commutative Noetherian ring R, the spectrum $\mathrm{Spec}(R)$ is a Noetherian topological space. This is the case in results from both the monoidal setting [Nakano et al. 2022a, 6.2.1], and the symmetric setting (via classifying support data [Balmer 2005, 5.2]), as well as in the formulation of visible points [Balmer and Favi 2011, 7.14].

We follow the original work of Balmer [2005] in the symmetric case and prove analogous statements in the monoidal setting for those mt-categories with Noetherian spectrum, Within this framework we obtain the following classification:

Theorem (Theorem 4.6). *Let* T *be a monoidal triangulated category such that the Balmer spectrum* $\mathrm{Spc}(\mathsf{T})$ *is a Noetherian topological space. Then there is an order-preserving bijection between the collection of all Thomason subsets of the spectrum* $\mathrm{Spc}(\mathsf{T})$ *and the collection of all thick two-sided semiprime ideals of* T.

We also show that if we are given a classification via some other well-behaved support datum (X, σ) then the universal map becomes a homeomorphism.

Theorem (Theorem 5.7). *Suppose* (X, σ) *is a support datum which classifies all thick two-sided semiprime tensor ideals of* T *and the space* X *is Noetherian and* T_0. *Then the universal map* $f_\sigma : X \to \mathrm{Spc}(\mathsf{T})$ *is a homeomorphism.*

In proving these results, we also show that under the Noetherian assumption, the noncommutative spectrum $\mathrm{Spc}(\mathsf{T})$ is a spectral space (Theorem 3.7). These results are all obtained in the *small* setting; we only use thick subcategories and do not require the use of larger structure such as localising subcategories.

2. Preliminaries on mt-categories

Let T be a triangulated category. Many examples in the literature, such as derived categories of commutative rings and the stable homotopy category of spectra, can be equipped with a tensor product functor that interacts well with the triangulated structure of the category. In the case where the tensor product functor is symmetric, these categories are referred to as *tensor triangulated categories* (or tt-categories), while in the general setting where the tensor product need not be symmetric such categories are referred to as *monoidal triangulated categories* (or mt-categories). We will review several of the key definitions and main results from [Nakano et al. 2022a] that will be used throughout this paper.

Definition 2.1. An *essentially small monoidal triangulated category* is a triple of the form $(\mathsf{T}, \otimes, \mathbf{1})$, where T is an essentially small triangulated category and $(\otimes, \mathbf{1})$ is a monoidal structure on T such that $\otimes$ is an exact functor in each variable.

In other words, it is a tensor triangulated category, but the tensor product operation need not be symmetric.

Investigations into both tt-categories and mt-categories often include some specification of the size of the categories to be studied. We may consider a category to be essentially small, and focus only on types of *thick subcategories*, as in [Balmer 2005]. Alternatively, we may consider categories that have a greater level of structure, namely those categories that are rigidly-compactly generated in which the collection of compact objects are also carry the structure of a tt-category or mt-category. This approach via *big* tt-categories is used in work such as [Balmer and Favi 2011] and [Stevenson 2013], and can be used to investigate a variety of different types of subcategory. This is also the approach used in the classification of [Nakano et al. 2022a] in the monoidal setting. We have chosen to use only the essentially small approach, obtaining our results without appealing to this greater level of structure. Although we only obtain a full classification for those cases in which the spectrum is Noetherian, this aligns with the original result of Balmer [2005] that thick subcategories should be controlled by the smaller data of the original category without any higher structure.

With our notion of an mt-category in place, we can now fix the types of subcategory we would like to classify. Aligning with the monoidal classification of [Nakano et al. 2022a] previously discussed, we are aiming to classify certain *semiprime ideals* via the collection of *prime ideals*. Just as when moving from the study of commutative rings to noncommutative rings, the definition of a prime ideal must change as we move from tt-geometry to mt-geometry. We make this precise below:

Definition 2.2 [Nakano et al. 2022a, 1.2]. Let T be an essentially small mt-category.

(1) A *thick two-sided ideal* of T is a thick subcategory closed under left and right tensoring with arbitrary objects of T.

(2) A *prime ideal* of T is a proper thick ideal $\mathcal{P}$ such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ implies $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$ for all thick two-sided ideals $\mathcal{I}$ and $\mathcal{J}$ or T.

(3) A *semiprime* ideal of T is an intersection of prime ideals of T.

(4) A *completely prime* ideal of T is a proper thick ideal $\mathcal{P}$ such that $A \otimes B \in \mathcal{P}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$ for all objects $A, B \in \mathsf{T}$.

In other words, when we talk about prime and semiprime ideals in this setting of mt-geometry, the notion aligns with the usual definitions from noncommutative algebra. While Balmer [2005] classified radical thick tensor ideals (necessarily two-sided) in the symmetric case, we will classify all two-sided thick semiprime ideals. In both cases, the classification is with respect to the collection of prime ideals, considered as a topological space.

Definition 2.3 [Nakano et al. 2022a]. The *noncommutative Balmer spectrum* of an essentially small mt-category T is the set of prime ideals of T. We denote the noncommutative spectrum by $\mathrm{Spc}(\mathsf{T})$. The spectrum can be equipped with a topology generated by the collection of closed subsets

$$V(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{P} \cap \mathcal{S} = \varnothing\}$$

for all subsets $\mathcal{S}$ of T.

In the symmetric case the Balmer spectrum is indeed a spectral space (see Definition 3.1) in the sense of Hochster [1969] (see for instance [Balmer 2005] for a direct proof and [Buan et al. 2007, 6.7] for a proof via lattices). The connection to algebraic geometry can be concretely realised via classifications such as the celebrated result of Thomason [1997], which in the framework of tt-geometry results in a homeomorphism

$$X \xrightarrow{\sim} \mathrm{Spc}(\mathrm{D}^{\mathsf{perf}}(X)),$$

where X is a topologically Noetherian scheme and $\mathrm{D}^{\mathsf{perf}}(X)$ is the derived category of perfect complexes over X, equipped with the usual tensor product $\otimes = \otimes^{L}_{\mathcal{O}_X}$ [Balmer 2005, 5.4, 5.6].

The other key element of the classifications in both the symmetric and monoidal settings is the notion of a *support datum*. There is a natural support datum associated to the noncommutative Balmer spectrum:

Definition 2.4 [Nakano et al. 2022a]. For an essentially small mt-category T, the *small noncommutative support* of an object t is given by

$$\mathrm{supp}(t) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid t \notin \mathcal{P}\}.$$

Note that this is just the restriction to objects of the map V of Definition 2.3.

This noncommutative support carries many desirable properties, interacting with distinguished triangles and sums in the same way as the commutative version of the support. The key distinguishing difference from the symmetric case occurs when taking the intersection of supports, which is investigated further in [Nakano et al. 2022b]. We formally set out the properties below:

Lemma 2.5 [Nakano et al. 2022a, 4.1.2]. *The small noncommutative support satisfies the following properties*:

(1) $\mathrm{supp}(0) = \varnothing$ *and* $\mathrm{supp}(\mathbf{1}) = \mathrm{Spc}(\mathsf{T})$.

(2) $\mathrm{supp}(t \oplus s) = \mathrm{supp}(t) \cup \mathrm{supp}(s)$ *for all* $t, s \in \mathsf{T}$.

(3) $\mathrm{supp}(\Sigma t) = \mathrm{supp}(t)$.

(4) *If* $t \to s \to r \to \Sigma t$ *is a distinguished triangle, then* $\operatorname{supp}(t) \subseteq \operatorname{supp}(s) \cup \operatorname{supp}(r)$.

(5) $\bigcup_{r\in \mathsf{T}} \operatorname{supp}(t \otimes r \otimes s) = \operatorname{supp}(t) \cap \operatorname{supp}(s)$ *for all* $t, s \in \mathsf{T}$.

(6) *For all* $t \in \mathsf{T}$ *the subset* $\operatorname{supp}(t)$ *is closed.*

When working with examples, it is not always the case that one can immediately compute a classification directly via the Balmer spectrum. One of the key features of the results in both the symmetric setting [Balmer 2005, 5.2] and the monoidal setting [Nakano et al. 2022a, 6.2.1] is that if a classification can be obtained with a Noetherian topological space equipped with a function that behaves *like* the Balmer support (with potentially some additional hypotheses), then this space must have been homeomorphic to the Balmer spectrum all along. To that end, the general notion of a *support datum* is introduced:

Definition 2.6 [Nakano et al. 2022a, 4.1.1]. A support datum on a mt-category T is a pair (X, σ), where X is a topological space and σ is an assignment $\sigma : \mathsf{T} \to \mathcal{X}$, where $\mathcal{X}$ is the collection of all closed subsets of X, such that σ satisfies the following additional properties:

(1) $\sigma(0) = \varnothing$ and $\sigma(\mathbf{1}) = X$.

(2) $\sigma(a \oplus b) = \sigma(a) \cup \sigma(b)$ for all $a, b \in \mathsf{T}$.

(3) $\sigma(\Sigma a) = \sigma(a)$ for all $a \in \mathsf{T}$.

(4) If $a \to b \to c \to \Sigma a$ is a distinguished triangle in T, then $\sigma(a) \subseteq \sigma(b) \cup \sigma(c)$.

(5) $\bigcup_{c\in \mathsf{T}} \sigma(a \otimes c \otimes b) = \sigma(a) \cap \sigma(b)$.

Note that in [Nakano et al. 2022a] it is not required that support data take values in *closed subsets*.

The property on the intersection of pairs of supports can be extended to finite intersections via a simple inductive argument:

Lemma 2.7. *Given a support datum* (X, σ) *and a finite collection of objects* $r_1, r_2, \ldots, r_n$, *we have*

$$\bigcap_{i=1}^{n} \sigma(r_i) = \bigcup_{c_1, c_2, \ldots, c_{n-1} \in \mathsf{T}} \sigma(r_1 \otimes c_1 \otimes r_2 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes r_n).$$

Proof. We proceed by induction, where the base case $\sigma(r_1) \cap \sigma(r_2)$ is satisfied by Definition 2.6. Suppose the result holds for the $n-1$ case. Then

$$\bigcap_{i=1}^{n} \sigma(r_i) = \left(\bigcap_{i=1}^{n-1} \sigma(r_i) \right) \cap \sigma(r_n)$$

$$
\begin{aligned}
&= \Bigg(\bigcup_{c_1,c_2,\ldots,c_{n-2}\in\mathsf{T}} \sigma(r_1\otimes c_1\otimes\cdots\otimes c_{n-2}\otimes r_{n-1}) \Bigg) \cap \sigma(r_n) \\
&= \bigcup_{c_1,c_2,\ldots,c_{n-2}\in\mathsf{T}} \Big(\sigma(r_1\otimes c_1\otimes\cdots\otimes c_{n-2}\otimes r_{n-1}) \cap \sigma(r_n) \Big) \\
&= \bigcup_{c_1,c_2,\ldots,c_{n-2}\in\mathsf{T}} \; \bigcup_{c_{n-1}\in\mathsf{T}} \sigma(r_1\otimes c_1\otimes r_2\otimes c_2\otimes\cdots\otimes c_{n-1}\otimes r_n) \\
&= \bigcup_{c_1,c_2,\ldots,c_{n-1}\in\mathsf{T}} \sigma(r_1\otimes c_1\otimes r_2\otimes c_2\otimes\cdots\otimes c_{n-1}\otimes r_n). \qquad \square
\end{aligned}
$$

The homeomorphism alluded to between the spectrum and other spaces that classify ideals is realised by the fact that amongst all support data, the natural support associated to the Balmer spectrum is universal.

Theorem 2.8 [Nakano et al. 2022a, 4.2.2]. *Let (X, σ) be a support datum on T such that $\sigma(t)$ is closed for every object $t \in \mathsf{T}$. Then there is a unique continuous map $f_\sigma : X \to \operatorname{Spc}(\mathsf{T})$ satisfying $\sigma(t) = f_\sigma^{-1}(\operatorname{supp}(t))$ for all $t \in \mathsf{T}$. In other words, $(\operatorname{Spc}(\mathsf{T}), \operatorname{supp})$ is the final support datum among all such support data. The map f_σ is given by*

$$f_\sigma(x) = \{t \in \mathsf{T} \mid x \notin \sigma(t)\}.$$

The definitions and results presented so far contain a mixture of properties defined idealwise (such as prime ideals) and properties that are defined objectwise (such as the properties of support data). Just as in the case of noncommutative rings, prime and semiprime ideals can also be characterised objectwise, allowing convenient translation between the two concepts.

Theorem 2.9 [Nakano et al. 2022a, 1.2.1]. *Let T be an essentially small mt-category. Then the following hold*:

(1) *A proper thick ideal $\mathcal{P}$ of T is prime if and only if, given objects $A, B \in \mathsf{T}$, we have $A \otimes C \otimes B \in \mathcal{P}$ for all $C \in \mathsf{T}$ implies $A \in \mathcal{P}$ or $B \in \mathcal{P}$.*

(2) *A proper thick ideal $\mathcal{P}$ of T is semiprime if and only if, given $A \in \mathsf{T}$, we have $A \otimes C \otimes A \in \mathcal{P}$ for all $C \in \mathsf{T}$ implies $A \in \mathcal{P}$.*

(3) *The noncommutative Balmer spectrum $\operatorname{Spc}(\mathsf{T})$ is always nonempty.*

All of the results reviewed so far hold for all mt-categories, irrespective of the properties of their noncommutative Balmer spectrum. Our objective for the remainder of the paper is to demonstrate that for those mt-categories T with Noetherian spectrum $\operatorname{Spc}(\mathsf{T})$, the noncommutative Balmer spectrum is a spectral space classifying all two-sided thick semiprime ideals, and that any Noetherian and T_0 space classifying these ideals via support data is homeomorphic to the Balmer spectrum.

3. Noetherian noncommutative spectra are spectral

Given a commutative ring R, the topological properties of the spectrum $\mathrm{Spec}(R)$ have been the subject of considerable study, particularly in the work of Hochster [1967; 1969], in which the topological spaces that share various properties with ring spectra are classified. To this end, the definition of a spectral space is introduced:

Definition 3.1 [Hochster 1969]. Let X be a topological space and let $K(X)$ denote the set of all quasicompact open subsets of X. The topological space X is *spectral* if it satisfies all of the following conditions:

(1) X is quasicompact and T_0. By T_0 we mean that given any two points $x, y \in X$ there is an open subset of X containing one of these points, but not the other.

(2) $K(X)$ is a basis of open subsets for X.

(3) $K(X)$ is closed under finite intersections.

(4) X is a sober space. That is, every irreducible closed subset of X has a necessarily unique generic point.

Given a commutative ring R, the usual spectrum $\mathrm{Spec}(R)$ is spectral. Moreover, Hochster [1967] proved in his original thesis that given any spectral space X, there exists a commutative ring R such that X is homeomorphic to $\mathrm{Spec}(R)$. The topological properties possessed by spectral spaces are essential to various elements of foundational classification results [Balmer 2005, 5.1, 5.2; Nakano et al. 2022a, 6.2.1].

Although when using general classifying support data we still need to assume the presence of a Noetherian spectral space, in the specific case of the noncommutative Balmer spectrum, we will show that the Noetherian assumption is sufficient. Specifically, we will show that if an mt-category T has Noetherian spectrum $\mathrm{Spc}(\mathsf{T})$, then the spectrum $\mathrm{Spc}(\mathsf{T})$ is a spectral space.

We begin by translating some of the topological properties of the Balmer spectrum investigated in [Balmer 2005] into the monoidal setting.

Let T be an essentially small monoidal triangulated category with spectrum $\mathrm{Spc}(\mathsf{T})$. We aim to show that $\mathrm{Spc}(\mathsf{T})$ is a spectral topological space. As seen in Definition 2.3, a basis of closed sets is given by $\{V(\mathcal{S}) \mid \mathcal{S} \subseteq \mathsf{T}\}$, where

$$V(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{P} \cap \mathcal{S} = \varnothing\}.$$

The corresponding basis of open sets is $\{U(\mathcal{S}) \mid \mathcal{S} \subseteq \mathsf{T}\}$, where

$$U(\mathcal{S}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{P} \cap \mathcal{S} \neq \varnothing\}.$$

Given an object $s \in \mathsf{T}$, we will simplify the above notation and write $V(s)$ for the basic closed set $V(\{s\})$ and write $U(s)$ for the basic open set $U(\{s\})$.

Immediately we have $V(\mathcal{S}) = \bigcap_{s\in\mathcal{S}} V(s)$ and $U(\mathcal{S}) = \bigcup_{s\in\mathcal{S}} U(s)$. Therefore the sets of the form $V(s) = \operatorname{supp}(s)$ form a basis of closed sets for the topology by Lemma 2.5, while the sets of the form $U(s)$ form a basis of open sets for the topology.

For a collection of objects $\mathcal{E}$ define $\operatorname{supp}(\mathcal{E}) = \bigcup_{s\in\mathcal{E}} \operatorname{supp}(s)$.

Lemma 3.2. *Let $Y \subseteq \operatorname{Spc}(\mathsf{T})$. Then the closure of Y is given by*

$$\overline{Y} = \bigcap_{Y\subseteq \operatorname{supp}(t)} \operatorname{supp}(t).$$

Proof. This follows immediately from the fact that the sets of the form $\operatorname{supp}(s)$ are a basis of closed sets for the topology on $\operatorname{Spc}(\mathsf{T})$. □

Proposition 3.3. *For any point $\mathcal{P} \in \operatorname{Spc}(\mathsf{T})$, the closure of $\mathcal{P}$ is given by*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

In particular, if $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$ then $\mathcal{P}_1 = \mathcal{P}_2$. That is, the space $\operatorname{Spc}(\mathsf{T})$ is T_0.

Proof. The proof is identical to [Balmer 2005, 2.9]. Fix a prime ideal $\mathcal{P}$. Consider the set $\mathcal{S}_0 = \mathsf{T}\setminus\mathcal{P}$ and the associated basic closed subset $V(\mathcal{S}_0) = \{\mathcal{Q} \mid \mathcal{Q}\cap\mathcal{S}_0 = \varnothing\}$. Clearly $\mathcal{P} \in V(\mathcal{S}_0)$. If there is a subset $\mathcal{S} \subseteq \mathsf{T}$ such that $\mathcal{P} \in V(\mathcal{S})$, then $\mathcal{S} \subseteq \mathcal{S}_0$ and so $V(\mathcal{S}_0) \subseteq V(\mathcal{S})$. Therefore $V(\mathcal{S}_0)$ is the smallest closed subset containing $\mathcal{P}$ and is the closure of $\mathcal{P}$. We have

$$\overline{\{\mathcal{P}\}} = V(\mathcal{S}_0) = \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{Q} \subseteq \mathcal{P}\}.$$

The fact that $\operatorname{Spc}(\mathsf{T})$ is T_0 follows immediately. □

We will make use of the following theorem from [Nakano et al. 2022a], which is the nonsymmetric version of [Balmer 2005, 2.2].

Theorem 3.4 [Nakano et al. 2022a, 3.2.3]. *Suppose that $\mathcal{M}$ is a multiplicative subset of T and suppose $\mathcal{I}$ is a proper thick two-sided tensor ideal of T such that $\mathcal{I}\cap\mathcal{M} = \varnothing$. The set*

$$X(\mathcal{M},\mathcal{I}) = \{\mathcal{J} \text{ a thick two-sided tensor ideal of } \mathsf{T} \mid \mathcal{I} \subseteq \mathcal{J} \text{ and } \mathcal{J}\cap\mathcal{M} = \varnothing\}$$

has a maximal element, and moreover this maximal element is prime.

Proposition 3.5. *Nonempty irreducible subsets of $\operatorname{Spc}(\mathsf{T})$ have unique generic points. That is, the noncommutative Balmer spectrum is always a sober space. Indeed for a nonempty closed subset $Z \subseteq \operatorname{Spc}(\mathsf{T})$ the following are equivalent:*

(1) *Z is irreducible.*

(2) *For all $t, s \in \mathsf{T}$, if $U(t\oplus s)\cap Z = \varnothing$, then $U(t)\cap Z = \varnothing$ or $U(s)\cap Z = \varnothing$.*

(3) *The collection $\mathcal{P} = \{t \in \mathsf{T} \mid U(t)\cap Z \neq \varnothing\}$ is a thick prime $\otimes$-ideal.*

Moreover, when these conditions hold, $Z = \overline{\{\mathcal{P}\}}$.

Proof. The proof is very similar to [Balmer 2005, 2.18], although some extra care is needed when proving certain ideals are prime. We have already seen that $\mathrm{Spc}(\mathsf{T})$ is T_0 and so uniqueness of generic points is immediate.

(1) $\Longrightarrow$ (2): Z irreducible means that for any open subsets $U_1, U_2 \in \mathrm{Spc}(\mathsf{T})$, if $Z \cap U_1 \cap U_2 = \varnothing$ then $Z \cap U_1 = \varnothing$ or $Z \cap U_2 = \varnothing$. This gives (2), since $U(t \oplus s) = U(t) \cap U(s)$.

(2) $\Longrightarrow$ (3): This will be slightly more involved than the proof in [Balmer 2005]. Condition (2) gives $t, s \in \mathcal{P}$ implies $t \oplus s \in \mathcal{P}$. Using this, we see that if $t, s \in \mathcal{P}$ and $t \to s \to r \to \Sigma t$ is a distinguished triangle, then $r \in \mathrm{thick}^{\otimes}(t \oplus s)$, hence $U(t \oplus s) \subseteq U(r)$ and since $U(t \oplus s) \cap Z \neq \varnothing$, we get $U(r) \cap Z \neq \varnothing$, and so $r \in \mathcal{P}$.

The fact that $\mathcal{P}$ is closed under summands is immediate as $U(t \oplus s) \cap Z \neq \varnothing$ implies $(U(t) \cap U(s)) \cap Z \neq \varnothing$ and therefore $U(t) \cap Z \neq \varnothing$ and $U(s) \cap Z \neq \varnothing$. Therefore, both t and s are objects in $\mathcal{P}$.

It remains to show that $\mathcal{P}$ is a two-sided ideal, and that it is prime. Fix $t \in \mathcal{P}$ and $x \in \mathsf{T}$. We have

$$\begin{aligned}
\varnothing &\neq Z \cap U(t) \\
&\subseteq Z \cap (U(t) \cup U(x)) \\
&= Z \cap \Big(\bigcap_{s \in \mathsf{T}} U(t \otimes s \otimes x) \Big) \quad \text{(by Lemma 2.5)} \\
&\subseteq Z \cap U(t \otimes x).
\end{aligned}$$

Therefore, $t \otimes x \in \mathcal{P}$. An almost identical argument shows that $x \otimes t \in \mathcal{P}$ and so $\mathcal{P}$ is indeed a two-sided ideal. Now we deal with primeness. Let $\mathcal{I}, \mathcal{J}$ be thick $\otimes$-ideals such that $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$. In particular, for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$ we have $U(i \otimes j) \cap Z \neq \varnothing$. Now,

$$\bigcup_{i \in \mathcal{I}, j \in \mathcal{J}} \mathrm{supp}(i \otimes j) = \bigcup_{i \in \mathcal{I}} \mathrm{supp}(i) \cap \bigcup_{j \in \mathcal{J}} \mathrm{supp}(j);$$

see for example [Nakano et al. 2022a, 4.4.2]. Therefore,

$$\bigcap_{i \in \mathcal{I}, j \in \mathcal{J}} U(i \otimes j) = \bigcap_{i \in \mathcal{I}} U(i) \cup \bigcap_{j \in \mathcal{J}} U(j).$$

As we assumed $\mathcal{I} \otimes \mathcal{J} \subseteq \mathcal{P}$ we must have $\mathcal{P} \in U(i \otimes j)$ for all i and j. Therefore $\mathcal{P} \in U(i)$ for all i or $\mathcal{P} \in U(j)$ for all j, which is equivalent to asking that $\mathcal{I} \subseteq \mathcal{P}$ or $\mathcal{J} \subseteq \mathcal{P}$, and so we conclude that $\mathcal{P}$ is indeed prime.

(3) $\Longrightarrow$ (1): We prove that $Z = \overline{\{\mathcal{P}\}}$, which proves (1) and the final statement of the proposition. Let $\mathcal{Q} \in Z$. For $a \in \mathcal{Q}$, we have $\mathcal{Q} \in U(a) \cap Z \neq \varnothing$, and hence $a \in \mathcal{P}$. We have proved $\mathcal{Q} \subseteq \mathcal{P}$, that is, $\mathcal{Q} \in \overline{\{\mathcal{P}\}}$ by Proposition 3.3 for any $\mathcal{Q} \in Z$, and

so $Z \subseteq \overline{\{\mathcal{P}\}}$. Conversely, it suffices to prove $\mathcal{P} \in Z$. To see this, let $s \in \mathsf{T}$ be an object such that $Z \subseteq \operatorname{supp}(s)$. Such objects exist by Lemma 3.2. Then $U(s) \cap Z = \varnothing$, which means $s \notin \mathcal{P}$ or, equivalently, $\mathcal{P} \in \operatorname{supp}(s)$. Therefore, by Lemma 3.2,

$$\mathcal{P} \in \bigcap_{Z \subseteq \operatorname{supp}(s)} \operatorname{supp}(s) = \bar{Z} = Z. \qquad \square$$

So far, all of the topological properties proved hold for the spectrum $\operatorname{Spc}(\mathsf{T})$ of any mt-category T. We will now introduce the Noetherian condition.

Definition 3.6. A topological space X is Noetherian if any of the following equivalent conditions hold:

(1) X satisfies the descending chain condition for closed subsets. That is, for any sequence
$$Y_1 \supseteq Y_2 \supseteq \cdots$$
of closed subsets Y_i of X, there exists an integer m such that for all integers $n \geq m$ we have $Y_m = Y_n$.

(2) Every subspace of X is quasicompact.

(3) Every open subset of X is quasicompact.

By assuming that $\operatorname{Spc}(\mathsf{T})$ is Noetherian, it immediately follows that $\operatorname{Spc}(\mathsf{T})$ is quasicompact, as are all of the basic open subsets $U(\mathcal{S})$, including those of the form $U(t)$ for all objects $t \in \mathsf{T}$.

Theorem 3.7. *If* $\operatorname{Spc}(\mathsf{T})$ *is a Noetherian topological space*, *then it is a spectral space.*

Proof. We verify the conditions required to be spectral.

(1) As $\operatorname{Spc}(\mathsf{T})$ is Noetherian, it is quasicompact. The space is T_0 by Proposition 3.3.

(2) Under the Noetherian assumption, every open subset is quasicompact and so it is immediate that the collection $K(\operatorname{Spc}(\mathsf{T}))$ of all quasicompact open subsets is a basis for $\operatorname{Spc}(\mathsf{T})$.

(3) Given quasicompact basic open sets of the form $U(t)$ and $U(s)$ we have that $U(s) \cap U(t) = U(s \oplus t)$. For a quasicompact basic open set of the form $U(\mathcal{S}) = \bigcup_{s \in \mathcal{S}} U(s)$, by quasicompactness there exists a finite subset $\mathcal{S}' \subseteq \mathcal{S}$ such that $U(\mathcal{S}) = U(\mathcal{S}') = \bigcup_{s \in \mathcal{S}'} U(s)$. Given another such quasicompact basic open set $U(\mathcal{T})$, with finite refinement $\mathcal{T}'$, we obtain

$$U(\mathcal{S}) \cap U(\mathcal{T}) = \bigcup_{s \in \mathcal{S}'} U(s) \cap \bigcup_{t \in \mathcal{T}'} U(t) = \bigcup_{s \in \mathcal{S}'} \bigcup_{t \in \mathcal{T}'} (U(s) \cap U(t)) = \bigcup_{s \in \mathcal{S}'} \bigcup_{t \in \mathcal{T}'} U(s \oplus t).$$

As $\mathcal{S}'$ is a finite set, and $\mathcal{T}'$ is a finite refinement, both unions are finite. As $U(\mathcal{S})$ and $U(\mathcal{T})$ are quasicompact, it follows that the intersection is quasicompact.

(4) By Proposition 3.5, the spectrum $\mathrm{Spc}(\mathsf{T})$ is always a sober space, irrespective of Noetherianity and therefore every nonempty irreducible subset of $\mathrm{Spc}(\mathsf{T})$ has a unique generic point.

Since all of the conditions are satisfied, the spectrum $\mathrm{Spc}(\mathsf{T})$ is a spectral space. □

There are other conditions on the mt-category T which can lead to $\mathrm{Spc}(\mathsf{T})$ being spectral. For example, [Buan et al. 2007, 6.7] proves that the spectrum is spectral under the assumption that the tensor product is symmetric, or that the mt-category T has a generator.

4. Classifying thick two-sided ideals

The objective of this section is to classify all *semiprime* thick tensor ideals of a mt-category T in terms of Thomason subsets of the spectrum $\mathrm{Spc}(\mathsf{T})$, under the assumption that the Balmer spectrum is a Noetherian topological space. If the mt-category T is rigid, then the classification actually covers *all* thick two-sided tensor ideals, providing a monoidal Noetherian analogue to Balmer's original classification [2005, 4.10] in the symmetric case.

As with the previous section, we will first obtain results on tensor ideals in general mt-categories, without assuming that the Balmer spectrum is Noetherian.

Lemma 4.1. *Given a collection of objects $\mathcal{E} \subseteq \mathsf{T}$ there is an equality*

$$\mathrm{supp}(\mathcal{E}) = \{\mathcal{P} \in \mathrm{Spc}(\mathsf{T}) \mid \mathcal{E} \not\subseteq \mathcal{P}\}.$$

Proof. The proof is identical to [Balmer 2005, 4.6]. We have $\mathcal{P} \in \mathrm{supp}(\mathcal{E})$ if and only if there exists an object $a \in \mathcal{E}$ such that $\mathcal{P} \in \mathrm{supp}(a)$ which means $a \notin \mathcal{P}$, by definition of the support. □

Definition 4.2. Let $\mathcal{J}$ be a thick tensor ideal of T. We denote by $\sqrt{\mathcal{J}}$ the semiprime ideal

$$\sqrt{\mathcal{J}} = \bigcap_{\mathcal{J} \subseteq \mathcal{P} \in \mathrm{Spc}(\mathsf{T})} \mathcal{P}.$$

Definition 4.3. Let $Y \subseteq \mathrm{Spc}(\mathsf{T})$ be a subset. Define the full subcategory T_Y by

$$\mathsf{T}_Y = \{t \in \mathsf{T} \mid \mathrm{supp}(t) \subseteq Y\}.$$

Lemma 4.4. (1) *The subcategory T_Y is a thick two-sided tensor ideal.*

(2) *There is an equality*

$$\mathsf{T}_Y = \bigcap_{\mathcal{P} \notin Y} \mathcal{P} \quad \textit{where } \mathcal{P} \in \mathrm{Spc}(\mathsf{T}).$$

Proof. (1) The statement is similar to [Nakano et al. 2022a, 6.1.1]. The fact that T_Y is a thick subcategory follows immediately from the usual properties of support.

Now let $s \in \mathsf{T}_Y$ and $t \in \mathsf{T}$. Then

$$\operatorname{supp}(s \otimes t) = \operatorname{supp}(s \otimes \mathbf{1} \otimes t) \subseteq \bigcup_{c \in \mathsf{T}} \operatorname{supp}(s \otimes c \otimes t) = \operatorname{supp}(s) \cap \operatorname{supp}(t) \subseteq Y.$$

Therefore, $\operatorname{supp}(s \otimes t) \subseteq Y$ and $s \otimes t \in \mathsf{T}_Y$. That is, T_Y is a right ideal. A similar argument shows that T_Y is a left ideal.

(2) The proof is identical to [Balmer 2005, 4.8]. For an object $t \in \mathsf{T}$, we have $t \in \mathsf{T}_Y$ if and only if $\operatorname{supp}(t) \subseteq Y$. Therefore, for all $\mathcal{P} \in \operatorname{Spc}(\mathsf{T}) \setminus Y$, $t \in \mathsf{T}_Y$ if and only if $\mathcal{P} \notin \operatorname{supp}(t)$ and $t \notin \mathcal{P}$. Hence, $t \in \bigcap_{\mathcal{P} \notin Y} \mathcal{P}$ and the conclusion holds. □

Note that if (X, σ) is a support datum on T then the above lemma can be adjusted to show that the full subcategory $\{t \in \mathsf{T} \mid \sigma(t) \subseteq Y\}$ is a thick two-sided ideal.

Proposition 4.5. *Let $\mathcal{J}$ be a thick tensor ideal of T. Then*

$$\mathsf{T}_{\operatorname{supp}(\mathcal{J})} = \sqrt{\mathcal{J}}.$$

Proof. The proof is identical to [Balmer 2005, 4.9]. By Lemma 4.4, we have

$$\mathsf{T}_{\operatorname{supp}(\mathcal{J})} = \bigcap_{\mathcal{P} \notin \operatorname{supp}(\mathcal{J})} \mathcal{P}.$$

Applying Lemma 4.1, gives $\operatorname{supp}(\mathcal{J}) = \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{J} \nsubseteq \mathcal{Q}\}$ and so

$$\mathsf{T}_{\operatorname{supp}(\mathcal{J})} = \bigcap_{\mathcal{P} \notin \{\mathcal{Q} \in \operatorname{Spc}(\mathsf{T}) \mid \mathcal{J} \nsubseteq \mathcal{Q}\}} \mathcal{P}.$$

The result then immediately follows from the definition of $\sqrt{\mathcal{J}}$. □

We denote by $\mathcal{T}$ the collection of all Thomason subsets of $\operatorname{Spc}(\mathsf{T})$. Recall that a subset $Y \subseteq \operatorname{Spc}(\mathsf{T})$ is Thomason if $Y = \bigcup Y_i$ such that each Y_i is closed and the open complement $\operatorname{Spc}(\mathsf{T}) \setminus Y_i$ is quasicompact. We denote by $\mathcal{S}$ the collection of all semiprime ideals of T.

With the general results in position, we now consider the case in which the Balmer spectrum is Noetherian and obtain the classification result.

Theorem 4.6. *Let T be a monoidal triangulated category such that the Balmer spectrum $\operatorname{Spc}(\mathsf{T})$ is a Noetherian topological space. Let $\mathcal{T}$ denote the collection of all Thomason subsets of $\operatorname{Spc}(\mathsf{T})$ and let $\mathcal{S}$ denote the collection of all semiprime ideals of T. Then there is an order-preserving bijection $\mathcal{T} \xrightarrow{\sim} \mathcal{S}$ given by*

$$Y \to \mathsf{T}_Y$$

whose inverse is

$$\mathcal{J} \to \operatorname{supp}(\mathcal{J}).$$

Proof. The first map is well defined because T_Y is semiprime by Lemma 4.4. The second map is well defined as the complement of an object's support is quasicompact under the Noetherian assumption. Both maps are clearly inclusion preserving. It remains to show the maps are mutually inverse.

Given a semiprime ideal $\mathcal{J}$ the composite $\mathsf{T}_{\mathrm{supp}(\mathcal{J})}$ is equal to $\sqrt{\mathcal{J}}$ by Proposition 4.5. By assumption, $\mathcal{J}$ is semiprime and so $\mathcal{J} = \sqrt{\mathcal{J}}$, and so the composite

$$\mathcal{J} \to \mathrm{supp}(\mathcal{J}) \to \mathsf{T}_{\mathrm{supp}(\mathcal{J})}$$

is the identity. Now let Y be a Thomason subset of $\mathrm{Spc}(\mathsf{T})$. It remains to show that the composition

$$Y \to \mathsf{T}_Y \to \mathrm{supp}(\mathsf{T}_Y)$$

is the identity. For an object $t \in \mathsf{T}$ we have by definition $t \in \mathsf{T}_Y$ if and only if $\mathrm{supp}(t) \subseteq Y$. Therefore,

$$\mathrm{supp}(\mathsf{T}_Y) = \bigcup_{t \in \mathsf{T}_Y} \mathrm{supp}(t) \subseteq Y.$$

Now we need to show that $Y \subseteq \mathrm{supp}(\mathsf{T}_Y)$. That is, for each prime ideal $\mathcal{P} \in Y$ we must find a compact object x such that $\mathcal{P} \in \mathrm{supp}(x)$ and $\mathrm{supp}(x) \subseteq Y$. As Y is a Thomason subset of $\mathrm{Spc}(\mathsf{T})$, there exist closed subsets Y_i such that $Y = \bigcup Y_i$ and the complement of each Y_i is a quasicompact open subset U_i. Fix a prime $\mathcal{P} \in Y$. Then there exists an index i such that

$$\mathcal{P} \in Y_i = \mathrm{Spc}(\mathsf{T}) \setminus U_i.$$

By assumption, $\mathrm{Spc}(\mathsf{T})$ is Noetherian, so there exists a finite collection of objects $\{r_1, \dots, r_n\}$ such that $U_i = \bigcup_{j=1}^n U(r_j)$. Therefore

$$Y_i = \mathrm{Spc}(\mathsf{T}) \setminus U_i = \mathrm{Spc}(\mathsf{T}) \setminus \left(\bigcup_{j=1}^n U(r_j) \right) = \bigcap_{j=1}^n (\mathrm{Spc}(\mathsf{T}) \setminus U(r_j)) = \bigcap_{j=1}^n \mathrm{supp}(r_j).$$

By Lemma 2.7, there exists compact objects $c_1, \dots, c_{n-1}$ such that for

$$x = r_1 \otimes c_1 \otimes \cdots \otimes c_{n-1} \otimes r_n,$$

we have $\mathcal{P} \in \mathrm{supp}(x)$. Moreover,

$$\mathrm{supp}(x) \subseteq \bigcap_{j=1}^n \mathrm{supp}(r_j) = Y_i \subseteq Y$$

and so $x \in \mathsf{T}_Y$, thus completing the proof. □

Proposition 4.7 [Nakano et al. 2022b, 4.1.1]. *Suppose* T *is rigid, so that every object is either left or right dualisable. Then every thick two-sided tensor ideal is semiprime.*

Corollary 4.8. *Let* T *be a rigid mt-category with Noetherian spectrum* $\mathrm{Spc}(\mathsf{T})$. *Then the order-preserving bijection of Theorem 4.6 classifies all thick two-sided tensor ideals of* T.

5. Classifying support data and the universal map

We can now investigate the universality of the spectrum with respect to classifying support data. This section provides the monoidal analogue of [Balmer 2005, 5.2], and recovers the general classification result of [Nakano et al. 2022a, 6.2.1]. Note that although the statements of the results are for mt-categories, the majority of the proofs are identical to the arguments in the symmetric setting of Balmer, with only the proofs of Proposition 5.6 and Theorem 5.7 requiring alterations for the monoidal setting.

Definition 5.1. A subset $Y \subseteq X$ of a topological space X is *specialisation closed* if it is the union of closed sets, or equivalently if $y \in Y$ implies $\overline{\{y\}} \subseteq Y$. Given a topological space X we denote by $\mathcal{X}_{\mathrm{sp}}$ the collection of all specialisation closed subsets of X.

Recall that we denote the collection of all thick semiprime ideals of T by $\mathcal{S}$.

Definition 5.2. Let (X, σ) be a support datum on T. We say that (X, σ) is a *classifying support datum* if the following two conditions hold:

(1) The space X is Noetherian and spectral.

(2) We have a bijection $\Theta : \mathcal{X}_{\mathrm{sp}} \to \mathcal{S}$ defined by

$$\Theta(Y) = \{t \in \mathsf{T} \mid \sigma(t) \subseteq Y\}$$

with inverse

$$\Theta^{-1}(\mathcal{J}) = \sigma(\mathcal{J}) = \bigcup_{j \in \mathcal{J}} \sigma(j).$$

Lemma 5.3. *Suppose* (X, σ) *is a classifying support datum on* T. *Then every closed subset* $Z \subseteq X$ *is of the form* $Z = \sigma(t)$ *for some object* $t \in \mathsf{T}$.

Proof. This is the first claim of [Balmer 2005, 5.2]. The proof is identical and included for completeness. Because X is Noetherian, every closed subset has a finite number of irreducible components. Since $\sigma(t_1) \cup \sigma(t_2) \cup \cdots \cup \sigma(t_n) = \sigma(t_1 \oplus t_2 \oplus \cdots \oplus t_n)$ for any finite collection of objects in T, it therefore suffices to prove the lemma for closed sets of the form $Z = \overline{\{x\}}$ for some $x \in X$.

As (X, σ) is classifying, we have

$$\overline{\{x\}} = Z = \Theta^{-1}\Theta(Z) = \bigcup_{t \in \Theta(Z)} \sigma(t).$$

Therefore there exists $t \in \mathsf{T}$ such that $x \in \sigma(t) \subseteq Z$. Hence

$$\overline{\{x\}} \subseteq \sigma(t) \subseteq Z = \overline{\{x\}},$$

proving the lemma. □

Corollary 5.4. *Suppose* $\mathrm{Spc}(\mathsf{T})$ *is a Noetherian topological space. Then every open subset of* $\mathrm{Spc}(\mathsf{T})$ *is of the form* $U(t)$ *for some object* $t \in \mathsf{T}$.

Proof. For $\mathrm{Spc}(\mathsf{T})$ Noetherian, Theorem 4.6 tells us $(\mathrm{Spc}(\mathsf{T}), \mathrm{supp})$ is a classifying support datum on T. Thus, given an open subset $U \subseteq \mathrm{Spc}(\mathsf{T})$, Lemma 5.3 tells us that $\mathrm{Spc}(\mathsf{T}) \setminus U = \mathrm{supp}(t)$ for some object $t \in \mathsf{T}$. Then $U = \mathrm{Spc}(\mathsf{T}) \setminus \mathrm{supp}(t) = U(t)$. Under the Noetherian assumption, $U(t)$ is quasicompact. □

Proposition 5.5. *If* (X, σ) *is a classifying support datum on* T, *then the universal map* $f_\sigma : X \to \mathrm{Spc}(\mathsf{T})$ *is injective.*

Proof. This is the same as the proof of injectivity in [Balmer 2005, 5.2] and is included for completeness. For $x \in X$ define $Y(x) = \{y \in X \mid x \notin \overline{\{y\}}\}$. Clearly $Y(x)$ is specialisation closed. Fix an object $t \in \mathsf{T}$. We will show that $\sigma(t) \subseteq Y(x)$ if and only if $x \notin \sigma(t)$. Since $x \notin Y(x)$, if $\sigma(t) \subseteq Y(x)$ then $x \notin \sigma(t)$. Conversely, as $\sigma(t)$ is specialisation closed, if $x \notin \sigma(t)$ we have $x \notin \overline{\{y\}}$ for all $y \in \sigma(t)$ and so by definition $\sigma(t) \subseteq Y(x)$. Therefore

$$\Theta(Y(x)) = \{t \in \mathsf{T} \mid \sigma(t) \subseteq Y(x)\} = \{t \in \mathsf{T} \mid x \notin \sigma(t)\} = f_\sigma(x).$$

As (X, σ) is classifying, if $f_\sigma(x_1) = f_\sigma(x_2)$ then $Y(x_1) = Y(x_2)$ and $\overline{\{x_1\}} = \overline{\{x_2\}}$. The space X must be T_0 as (X, σ) is classifying, so $\overline{\{x_1\}} = \overline{\{x_2\}}$ implies $x_1 = x_2$ and the map f_σ is injective. □

Proposition 5.6. *If* (X, σ) *is a classifying support datum on* T, *then the universal map* $f_\sigma : X \to \mathrm{Spc}(\mathsf{T})$ *is surjective.*

Proof. Fix a prime $\mathcal{P} \in \mathrm{Spc}(\mathsf{T})$. As (X, σ) is classifying, there exists a specialisation closed subset $Y \subseteq X$ such that $\mathcal{P} = \Theta(Y)$. As $\mathcal{P}$ is proper, the set $X \setminus Y$ is nonempty. Let $x, y \in X \setminus Y$. By Lemma 5.3, there exist objects $s, t \in \mathsf{T}$ such that $\overline{\{x\}} = \sigma(s)$ and $\overline{\{y\}} = \sigma(t)$. Let $\mathcal{I}$ and $\mathcal{J}$ denote the thick two-sided ideals generated by s and t, respectively. By [Nakano et al. 2022a, 4.3.2],

$$\overline{\{x\}} = \sigma(s) = \sigma(\mathcal{I}) \quad \text{and} \quad \overline{\{y\}} = \sigma(t) = \sigma(\mathcal{J}),$$

and so neither $\mathcal{I}$ nor $\mathcal{J}$ are contained in $\mathcal{P}$. As $\mathcal{P}$ is prime, $\mathcal{I} \otimes \mathcal{J} \not\subseteq \mathcal{P}$ and so $\sigma(\mathcal{I} \otimes \mathcal{J}) \not\subseteq Y$. Therefore, there exists a point $z \in X \setminus Y$ such that

$$z \in \sigma(\mathcal{I} \otimes \mathcal{J}) = \overline{\{x\}} \cap \overline{\{y\}}$$

and hence

$$\overline{\{z\}} \subseteq \overline{\{x\}} \cap \overline{\{y\}}.$$

As in [Balmer 2005, 5.2], as the space X is Noetherian, the nonempty family of sets

$$\{\overline{\{x\}} \mid x \in X \setminus Y\}$$

admits a minimal element which must be the lower bound for inclusion. That is, there exists $x \in X \setminus Y$ such that, for all $y \in X \setminus Y$, we have $x \in \overline{\{y\}}$. Hence,

$$X \setminus Y \subseteq \{y \in X \mid x \in \overline{\{y\}}\}.$$

The reverse inclusion holds because $x \notin Y$, and Y is specialisation closed. Thus,

$$X \setminus Y = \{y \in X \mid x \in \overline{\{y\}}\}$$

and so

$$Y = \{y \in X \mid x \notin \overline{\{y\}}\} = Y(x).$$

Hence,

$$\mathcal{P} = \Theta(Y) = \Theta(Y(x)) = f_\sigma(x),$$

where the final equality is demonstrated in the proof of Proposition 5.5. We conclude that f_σ is surjective. $\square$

Theorem 5.7. *Let (X, σ) be a classifying support datum on T. Then the universal map $f_\sigma : X \to \operatorname{Spc}(\mathsf{T})$ is a homeomorphism.*

Proof. By Proposition 5.5 the universal continuous map f_σ is injective, and by Proposition 5.6 the map is surjective. Therefore, f_σ is bijective. By Theorem 2.8 we have $\sigma(t) = f_\sigma^{-1}(\operatorname{supp}(t))$ for all $t \in \mathsf{T}$. Consequently, $f_\sigma(\sigma(t)) = \operatorname{supp}(t)$ and so f_σ is a closed map as by Lemma 5.3 every closed subset of X is of the form $\sigma(t)$ for some $t \in \mathsf{T}$. We conclude that f_σ is a homeomorphism. $\square$

The requirement that the topological space component of a classifying support datum be Noetherian is satisfied by many examples of interest in both symmetric and monoidal cases. These include the cases of the derived category of perfect complexes over a topologically Noetherian scheme [Thomason 1997] and the stable module category of a finite group scheme [Friedlander and Pevtsova 2007] in the symmetric case and the stable module categories of various quantum groups and Hopf algebras in the monoidal case [Nakano et al. 2022a]. It should be noted that proving that the Noetherian condition is satisfied is often an extensive process (as is the case in all of the stable module category examples mentioned). Moreover, not every mt-category possesses a Noetherian spectrum, including examples in the symmetric case of tt-categories. For example, while the stable homotopy category of finite spectra admits a classification theorem, it has a non-Noetherian spectrum (see [Hopkins and Smith 1998] for the original classification and [Balmer 2010, Section 9] for the tt-geometric context).

However, if the Noetherian condition on the given topological space is satisfied, Theorem 5.7 will guarantee that the space is homeomorphic to the Balmer spectrum.

Acknowledgements

Many thanks to Greg Stevenson and Kent Vashaw for their insightful comments and suggestions. Many thanks also to the referee for their suggested improvements. The author was supported by Engineering and Physical Sciences Research Council (EPSRC) grants EP/N509668/1 and EP/R513222/1.

References

[Balmer 2005] P. Balmer, "The spectrum of prime ideals in tensor triangulated categories", *J. Reine Angew. Math.* **588** (2005), 149–168. MR Zbl

[Balmer 2010] P. Balmer, "Spectra, spectra, spectra—tensor triangular spectra versus Zariski spectra of endomorphism rings", *Algebr. Geom. Topol.* **10**:3 (2010), 1521–1563. MR Zbl

[Balmer and Favi 2011] P. Balmer and G. Favi, "Generalized tensor idempotents and the telescope conjecture", *Proc. Lond. Math. Soc.* (3) **102**:6 (2011), 1161–1185. MR Zbl

[Buan et al. 2007] A. B. Buan, H. Krause, and Ø. Solberg, "Support varieties: an ideal approach", *Homology Homotopy Appl.* **9**:1 (2007), 45–74. MR Zbl

[Friedlander and Pevtsova 2007] E. M. Friedlander and J. Pevtsova, "Π-supports for modules for finite group schemes", *Duke Math. J.* **139**:2 (2007), 317–368. MR Zbl

[Hochster 1967] M. Hochster, *Prime ideal structure in commutative rings*, Ph.D. thesis, Princeton University, 1967, available at https://www.proquest.com/docview/302250870. MR Zbl

[Hochster 1969] M. Hochster, "Prime ideal structure in commutative rings", *Trans. Amer. Math. Soc.* **142** (1969), 43–60. MR Zbl

[Hopkins and Smith 1998] M. J. Hopkins and J. H. Smith, "Nilpotence and stable homotopy theory, II", *Ann. of Math.* (2) **148**:1 (1998), 1–49. MR Zbl

[Mallick and Ray 2023] V. M. Mallick and S. Ray, "Noncommutative tensor triangulated categories and coherent frames", *C. R. Math. Acad. Sci. Paris* **361** (2023), 1415–1427. MR Zbl

[Nakano et al. 2022a] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "Noncommutative tensor triangular geometry", *Amer. J. Math.* **144**:6 (2022), 1681–1724. MR Zbl

[Nakano et al. 2022b] D. K. Nakano, K. B. Vashaw, and M. T. Yakimov, "Noncommutative tensor triangular geometry and the tensor product property for support maps", *Int. Math. Res. Not.* **2022**:22 (2022), 17766–17796. MR Zbl

[Stevenson 2013] G. Stevenson, "Support theory via actions of tensor triangulated categories", *J. Reine Angew. Math.* **681** (2013), 219–254. MR Zbl

[Thomason 1997] R. W. Thomason, "The classification of triangulated subcategories", *Compositio Math.* **105**:1 (1997), 1–27. MR Zbl

Received September 18, 2023. Revised June 6, 2024.

JAMES ROWE
SCHOOL OF MATHEMATICS & STATISTICS
UNIVERSITY OF GLASGOW
UNITED KINGDOM
james.rowe@glasgow.ac.uk

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.373

REGULARITY OF MANIFOLDS WITH INTEGRAL SCALAR CURVATURE BOUND AND ENTROPY LOWER BOUND

SHANGZHI ZOU

We generalize the work of Lee, Naber and Neumayer on regularity of manifolds with lower-bounded scalar curvature and almost Euclidean entropy. We show the same result in the case of integral bounded scalar curvature.

In addition, we also obtain a compactness theorem and an a prior L^p scalar curvature bound estimate for $p < 1$.

1. Introduction

Local regularity is important in the study of manifolds under curvature restrictions. Cheeger [1970] demonstrated that the injectivity radius is uniformly bounded below by a positive constant for compact manifolds with bounded sectional curvature, noncollapsing volume, and bounded diameter. Gromov [1981] further showed a $C^{1,\alpha}$ harmonic radius estimate on such manifolds. These results play a crucial role in the proof of compactness and finiteness theorems (see also [Greene and Wu 1988; Kasue 1989; Peters 1987]). Anderson [1990] extended the $C^{1,\alpha}$ regularity to manifolds with bounded Ricci curvature and bounded injectivity. In the case that the manifold admits only a lower bound on Ricci curvature, Anderson and Cheeger [1992] have given a C^α harmonic radius estimate under an additional assumption on the injectivity radius. Moreover, Cheeger and Colding [Colding 1997; Cheeger and Colding 1997] further proved that for manifolds with almost nonnegative Ricci curvature, the unit geodesic ball is Gromov–Hausdorff close to the Euclidean ball if and only if they are close in volume. This is also true for manifolds with integral Ricci curvature lower bound, which is proved by Tian and Zhang [2016].

Regularity of manifolds with bounded scalar curvature would be much more difficult. Recently, Lee, Naber, and Neumayer [Lee et al. 2023] showed that the unit ball of a complete manifold is close to the Euclidean ball in the d_p-distance whenever the scalar curvature, as well as the Perelman ν-functional, is almost nonnegative. This gives a weaker regularity than the usual Gromov–Hausdorff closeness. This paper aims to generalize their result to the integral scalar curvature bound case.

MSC2020: 53C23.
Keywords: regularity theorem, bounded integral scalar curvature.

Recall the Perelman $\mathcal{W}$-functional (see [Perelman 2002] or [Chow et al. 2010]). For a complete manifold (M, g), $\tau > 0$, and $f \in C^\infty(M)$,

$$\mathcal{W}(g, f, \tau) = \int_M \big(\tau(R + |\nabla f|^2) + f - n\big)(4\pi\tau)^{-n/2} e^{-f}\, d\mathrm{vol}_g.$$

Letting $u = (4\pi\tau)^{-n/4} e^{-f/2}$, the Perelman $\mathcal{W}$-function can be reformulated as

$$\mathcal{W}(g, u, \tau) = \int_M \big(\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2\big)\, d\mathrm{vol}_g - \tfrac{1}{2} n \log(4\pi\tau). \tag{1.1}$$

The Perelman entropy $\mu(g, \tau)$ is given by

$$\mu(g, \tau) = \inf\left\{\mathcal{W}(g, u, \tau) \;\middle|\; \int_M u^2\, d\mathrm{vol}_g = 1\right\}.$$

Finally, the Perelman ν-functional is given by

$$\nu(g, \tau) = \inf\{\mu(g, \tau') : \tau' \in (0, \tau)\},$$

which satisfies the rescaling invariance $\nu(\alpha g, \alpha\tau) = \nu(g, \tau)$ for all $\alpha > 0$. For any complete Riemannian manifold (M, g) with bounded geometry, the Perelman ν-functional is nonpositive. Furthermore, a rigidity result asserts that for such a manifold, if there exists $\tau > 0$ such that $\nu(g, \tau) = 0$, then (M, g) must be Euclidean space; see [Perelman 2002; Chau et al. 2011; Lee et al. 2023]. When the complete manifold admits only nonnegative scalar curvature, Cheng Liang [Cheng 2022] showed that the manifold must be isometric to Euclidean space whenever the Euclidean isoperimetric inequality holds.

To investigate the stability of the above rigidity result of manifolds with almost nonnegative scalar curvature, Lee, Naber and Neumayer consider the Gromov–Hausdorff convergence under the d_p-distance.

Definition 1.2 (d_p-distance). Given a Riemannian manifold (M^n, g) and a real number $p \in (n, \infty]$, we define the d_p-distance between any $x, y \in M$ by

$$d_p(x, y) = \sup\left\{|f(x) - f(y)| : \int_M |\nabla f|^p\, d\mathrm{vol}_g \le 1,\ f \in W^{1,p}_{\mathrm{loc}}(M) \cap C^0_{\mathrm{loc}}(M)\right\}.$$

Note that this distance makes sense for any space equipped with a $W^{1,p}$ structure, ensuring the integrability and differentiability of functions. Let $\mathcal{B}_{p,g}(x, r)$ denote the ball centered at x of radius r with respect to d_p, i.e.,

$$\mathcal{B}_{p,g}(x, r) = \{y \in M : d_p(x, y) < r\}.$$

Then the rescaled metric $\tilde{g} = r^{-2} g$ satisfies $\mathcal{B}_{p,\tilde{g}}(x, \rho) = \mathcal{B}_{p,g}(x, \rho r^{1-n/p})$ for any $\rho > 0$.

For a complete Riemannian manifold (M, g), let $B_r(x)$ be the geodesic ball centered at $x \in M$ of radius r, let $R_- = \max\{-R, 0\}$, and let

$$\|R_-\|_{g,q,r} := \sup_{x\in M}\left\{ r^{2-n/q}\left(\int_{B_r(x)} |R_-|^q\right)^{1/q}\right\}.$$

We defined the upper bound of capacity as follows.

Definition 1.3 (capacity). Let (M^n, g) be a Riemannian manifold. For fixed $r > 0$ and $N \in \mathbb{N}^+$, if for any $x \in M$, there exists $\{x_i\}_{i=1}^N \subset B_{2r}(x)$ such that $\{B_r(x_i)\}_{i=1}^N$ forms a covering of $B_{2r}(x)$, then we denote the upper bound of capacity by

$$\mathrm{Cap}_{(M,g)}(r) \le N.$$

The main result of this paper is:

Theorem 1.4 (regularity). *Let (M^n, g) be a complete n-manifold with bounded curvature and fix $\varepsilon, r, N > 0$, $p > n$ and $q > n/2$. There exists $\delta = \delta(n, \varepsilon, N, p, q)$ such that if*

$$\nu(g, 2r^2) \ge -\delta, \quad \|R_-\|_{g,q,r} \le \delta, \quad \mathrm{Cap}_{(M,g)}(r) \le N,$$

then for all $x \in M$ and $0^n \in \mathbb{R}^n$, we have

$$d_{GH}\big((\mathcal{B}_{p,g}(x, r^{1-n/p}), d_{p,g}), (\mathcal{B}_{p,g_{\mathrm{euc}}}(0^n, r^{1-n/p}), d_{p,g_{\mathrm{euc}}})\big) \le \varepsilon r^{1-n/p},$$

and for any $0 < s \le r^{1-n/p}$,

$$1-\varepsilon \le \frac{\mathrm{vol}_g(\mathcal{B}_{p,g}(x, s))}{\mathrm{vol}_{g_{\mathrm{euc}}}(\mathcal{B}_{p,g_{\mathrm{euc}}}(0^n, s))} \le 1+\varepsilon.$$

Remark 1.5. Note that all the statements exhibit scaling invariance, allowing us to assume $r = 1$ in our proofs.

As in the proof of the regularity theorem for manifolds with a pointwise lower bound on scalar curvature [Lee et al. 2023, Theorem 1.7], the main step in our argument is to establish an integral estimate for the Ricci curvature along the Ricci flow; see Lemma 4.1. Once the estimate holds, the proof of Theorem 1.4 is identical with that in [Lee et al. 2023, Sections 5–7] and thus omitted.

Furthermore, we can immediately obtain results analogous to those presented in [Lee et al. 2023].

Theorem 1.6 (compactness). *Fix $\varepsilon, r, N > 0$, $p > n$ and $q > n/2$. There exists $\delta = \delta(n, \varepsilon, N, p, q)$ such that if a sequence of complete pointed Riemannian manifolds $\{(M_i, g_i, x_i)\}$ with bounded curvature satisfies*

$$\nu(g_i, 2r^2) \ge -\delta, \quad \|(R_i)_-\|_{g_i,q,r} \le \delta, \quad \mathrm{Cap}_{(M_i,g_i)}(r) \le N,$$

then there is a subsequence of $\{(M_i, g_i, x_i)\}$ that converges in the pointed d_p sense to (X, g, x), where X is a pointed rectifiable Riemannian space.

See [Lee et al. 2023] for definitions of pointed d_p convergence and pointed rectifiable Riemannian spaces.

For closed manifolds, we establish a prior $L^p(p<1)$ bounds for scalar curvature:

Theorem 1.7. *Fix $n \geq 2$, $\varepsilon > 0$, $p \in (0,1)$ and $q > \max(n/2, 2p)$. Let (M^n, g) be a closed Riemannian n-manifold. There exists $\delta = \delta(n, \varepsilon, p, q) > 0$ such that if*

$$\operatorname{vol}(M)^{2/n} \cdot \left(\fint_M |R_-|^q \right)^{1/q} \leq \delta \quad \textit{and} \quad \nu(g, 2\operatorname{vol}(M)^{2/n}) \geq -\delta,$$

then

$$\operatorname{vol}(M)^{2/n} \cdot \left(\fint_M |R|^p \right)^{1/p} \leq \varepsilon.$$

2. Preliminaries

In this paper, unless specified differently, (M^n, g) will always denote a complete Riemannian manifold of dimension n with bounded curvature.

A Ricci flow $(M^n, g(t))_{t\in[0,T]}$ is a family of smooth metrics $g(t)$ on a smooth manifold M^n satisfying the evolution equation

$$\partial_t g(t) = -2\operatorname{Ric}_{g(t)}.$$

Along the Ricci flow, the scalar curvature and the volume form evolve by

$$\partial_t R = \Delta_{g(t)} R + 2|\operatorname{Ric}_{g(t)}|^2, \quad \partial_t \, d\mathrm{vol}_{g(t)} = -R_{g(t)} \, d\mathrm{vol}_{g(t)}. \tag{2.1}$$

Consider the heat operator $\partial_t - \Delta_{g(t)}$ coupled to the Ricci flow. Correspondingly, the operator $-\partial_t - \Delta_{g(t)} + R_{g(t)}$ is called the conjugate heat operator. In particular, for $u, v \in C_0^2(M \times [0,T])$, we have

$$\int_M v(\partial_t - \Delta_{g(t)}) u \, d\mathrm{vol}_{g(t)} - \int_M u(-\partial_t - \Delta_{g(t)} + R_{g(t)}) v \, d\mathrm{vol}_{g(t)} = \frac{d}{dt} \int_M uv \, d\mathrm{vol}_{g(t)}.$$

Let $K(\cdot\,,\cdot\,; y, s)$ denote the *heat kernel* based at (y, s), i.e.,

$$(\partial_t - \Delta_{x,g(t)}) K(x,t;y,s) = 0, \qquad \lim_{t\to s^+} K(\cdot\,, t; y, s) = \delta_y.$$

The heat kernel exists and is positive; see [Guenther 2002]. For fixed (x, t), the function $K(x, t; \cdot\,, \cdot)$ is also the *conjugate heat kernel*, i.e.,

$$(-\partial_s - \Delta_{y,g(s)} + R_{g(s)}(y)) K(x,t;y,s) = 0, \qquad \lim_{s\to t^-} K(x,t;\cdot\,,s) = \delta_x.$$

For any $0 \leq s < t < T$ we have

$$\int_M K(x,t;\cdot\,,s) \, d\mathrm{vol}_{g(s)} = 1. \tag{2.2}$$

Set $\tau(t) = T - t$, and let $u = u(x,t)$ be a solution of the conjugate heat equation along the flow. Chau, Tam and Yu [Chau et al. 2011, Theorem 7.1] show that if $u(\cdot, T) \in C_0^\infty(M)$, then the Perelman $\mathcal{W}$-functional, defined in (1.1), is monotone along the Ricci flow:

$$\mathcal{W}(g(s), u(s), \tau(s)) \le \mathcal{W}(g(t), u(t), \tau(t)) \quad \text{for all } 0 \le s \le t \le T.$$

By taking a compactly supported minimizing sequence for $\mu(g(t), \tau(t))$, we see that $\mu(\tau)$ is also monotone:

$$\mu(g(s), \tau(s)) \le \mu(g(t), \tau(t)) \quad \text{for all } 0 \le s \le t \le T. \tag{2.3}$$

In particular, for Ricci flow $(M, g(t))_{t\in(0,2)}$ and $s \in (0, 1]$, we have $\nu(g(s), 1) \ge \nu(g(0), 2)$.

We restate some basic results of Ricci flow. For the proofs, see [Lee et al. 2023, Section 3].

Proposition 2.4. *Fix $n \ge 2$ and $\lambda > 0$. There exists $\delta = \delta(n, \lambda) > 0$ such that if (M, g) satisfies $\nu(g, 2) \ge -\delta$, then the Ricci flow $(M, g(t))$ with $g(0) = g$ exists for $t \in (0, 1]$ and has the scale-invariant estimate*

$$\sup_{x\in M} |\mathrm{Rm}_{g(t)}(x)| \le \frac{\lambda}{t} \quad \textit{for all } t \in (0, 1]. \tag{2.5}$$

Moreover, for any $x_0 \in M$ and $t \in (0, 1]$, there is a diffeomorphism

$$\phi : B_{g(t)}(x_0, 16t^{\frac{1}{2}}) \to \Omega \subset \mathbb{R}^n$$

such that $\phi(x_0) = 0$ and

$$\tfrac{1}{2}\phi^* g_{\mathrm{euc}}(\phi(x)) \le g(t)(x) \le \tfrac{3}{2}\phi^* g_{\mathrm{euc}}(\phi(x)) \quad \textit{for all } x \in B_{g(t)}(x_0, 16t^{\frac{1}{2}}).$$

In particular, there exists $C = C(n)$ such that

$$C^{-1} r^n \le \mathrm{vol}_{g(t)}(B_{g(t)}(x, r)) \le C r^n \quad \textit{for all } r \in (0, 16t^{\frac{1}{2}}).$$

Combine (2.5) with Shi's estimate [1989], there exists $C = C(k)$ such that $|\nabla^k \mathrm{Rm}_{g(t)}|$ are uniformly bounded by $C/t^{1+k/2}$ for all k. Thus, all Ricci flows are assumed to have bounded curvature throughout the entire paper.

Let $\hat{R}_-(t) = \sup_{x\in M} R_-(x, t)$. It is important to note that the L^1 norm of $K(\cdot, t; y, s)$ has an upper bound:

$$\int_M K(\cdot, t; y, s)\, d\mathrm{vol}_{g(s)} \le \exp\biggl(\int_s^t \hat{R}_-\biggr). \tag{2.6}$$

In the compact case, this result can be derived from the following computation:

$$\begin{aligned}\frac{d}{dt}\int_M K(\cdot, t; y, s)\, d\mathrm{vol}_{g(t)} &= \int_M \triangle K - RK\, d\mathrm{vol}_{g(t)} \\ &\le \hat{R}_-(t) \int_M K(\cdot, t; y, s)\, d\mathrm{vol}_{g(t)}.\end{aligned}$$

If M is noncompact, consider an exhaustion of M by smooth domains with compact closure: $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset M$. Let $K_{\Omega_i}(x,t;y,s)$ be the corresponding Dirichlet heat kernel on Ω_i. Since Ω_i is compact, we have

$$\int_{\Omega_i} K_{\Omega_i}(\,\cdot\,,t;y,s)\,d\mathrm{vol}_{g(s)} \le e^{\int_s^t \hat{R}_-}.$$

By maximum principle, K_{Ω_i} is an increasing sequence, and K is the limit of K_{Ω_i} as i tend to infinity. By the monotone convergence theorem, we can ascertain that the L^1 norm of $K(\,\cdot\,,t;y,s)$ also satisfies (2.6).

3. Heat kernel estimates for Ricci flow

In this section, we establish the heat kernel's upper and lower bounds. The Gaussian upper bounds for the heat kernel are primarily derived from the heat kernel estimates by Bamler, Cabezas-Rivas, and Wilking in [Bamler et al. 2019, Proposition 3.1]. Incorporating this result with the log Sobolev inequality (see also [Cao and Zhang 2011; Zhang 2011, Theorem 4.2.1]), we achieve a more precise estimation of the heat kernel, specifically about integral scalar curvature, as detailed in the following lemma.

Lemma 3.1. *Fix $n \ge 2$. There exist $\delta = \delta(n) > 0$ and $C = C(n) > 0$ such that if $(M^n, g(t))_{t\in[0,1]}$ satisfies $\nu(g(0),2) \ge -\delta$, then*

$$K(x,t;y,s) \le \frac{C}{(t-s)^{n/2}} \exp\biggl(-\frac{d^2_{g(s)}(x,y)}{C(t-s)} + \int_s^t \hat{R}_-\biggr) \quad \textit{for all } 0 \le s < t \le 1.$$

Proof. Up to scaling, we only need to show that if $\nu(g(0),2) \ge -\delta$, then

$$K(x,1;y,0) \le C\exp\biggl(-C^{-1}d^2_{g(0)}(x,y) + \int_0^1 \hat{R}_-\biggr). \tag{3.2}$$

Let $p(s) = 1/(1-s)$ for $s \in [0,1)$, and let $u = u(x,t)$ be a positive solution of the heat equation with Dirichlet boundary condition

$$(\partial_t - \Delta)u = 0 \quad \text{in } \Omega_i \times (0,1), \qquad u = 0 \quad \text{on } \partial\Omega_i \times [0,1].$$

Let $\tau = s(1-s)$ and $v(x,s) = u^{p(s)/2}/\|u^{p(s)/2}\|_2$. By (2.3), we compute

$$\begin{aligned}
\frac{d}{ds}\log\|u\|_{p(s)} &= \frac{p'}{p^2}\int_M (v^2\log v^2) - \frac{p-1}{p^2}\int_M (Rv^2 + 4|\nabla v|^2) - \frac{1}{p^2}\int_M Rv^2 \\
&\le -s(1-s)\int_M (Rv^2 + 4|\nabla v|^2) + \int_M v^2\log v^2 + \hat{R}_-(s) \\
&= -\bigl(\mathcal{W}(g(s),v(s),\tau) + \tfrac{1}{2}n\log(4\pi\tau) + n\bigr) + \hat{R}_-(s) \\
&\le \delta - \tfrac{1}{2}n\log\tau + \hat{R}_-(s).
\end{aligned}$$

Integrating from $s=0$ to $s=1$, we find

$$\|u(\,\cdot\,,1)\|_\infty \le \exp\left(\delta+n+\int_0^1 \hat{R}_-\right)\|u(\,\cdot\,,0)\|_1.$$

Since

$$\sup_{x,y} K_{\Omega_i}(x,1;y,0)=\sup_{u\neq 0}\frac{\|u(\,\cdot\,,1)\|_\infty}{\|u(\,\cdot\,,0)\|_1},$$

letting i tend to infinity, it follows that

$$K(x,1;y,0)\le \exp\left(\delta+n+\int_0^1 \hat{R}_-\right). \tag{3.3}$$

This implies the desired bound (3.2) if $d_{g(0)}(x,y)$ is controlled. Therefore, our task reduces to estimating $K(x,1;\cdot,0)$ when $d_{g(0)}(x,y)$ is large. The proof closely parallels that of [Bamler et al. 2019, Proposition 3.1]. The main deviation lies in the application of formula (2.6) to transform [Bamler et al. 2019, (3.11)] into the form

$$\begin{aligned}\mathcal{I}[B_k] &\le \int_{M/B_{g(0)}(x,r_k)} K(x,1;\cdot,t_k)K(\cdot,t_k;y,t_{k+1})\,d\mathrm{vol}_{g(t_k)}\\ &\le a_k\exp\left(\int_{t_{k+1}}^{t_k}\hat{R}_-\right).\end{aligned} \tag{3.4}$$

Thus, further details of the proof are omitted here. □

In preparation for the lower bound estimate, we need the following.

Proposition 3.5. *Let $(M,g(t))_{t\in(0,T]}$ be a Ricci flow. Then the following properties hold.*

(1) (*interpolation inequality* [Zhang 2011, Theorom 6.5.1]) *Let u be a positive solution to the heat equation $(\partial_t-\Delta)u=0$. Then, for $x,y\in M$ and $0<t\le T$, letting $U=\sup_{M\times[0,T]}u$, we have*

$$u(y,t)\le u(x,t)^{\frac12}U^{\frac12}\exp\left(\frac{d^2_{g(t)}(x,y)}{t}\right).$$

(2) (*Perelman's differential Harnack inequality* [Perelman 2002, Corollary 9.4]) *Let $\gamma(s)$ be any smooth curve and suppose $w(y,s)=(4\pi(T-s))^{-n/2}e^{-h(y,s)}$ satisfies the conjugate heat equation $(\partial_s+\Delta-R)w=0$. Then*

$$-\frac{d}{ds}h(\gamma(s),s)\le\frac12(R(\gamma(s),s)+|\dot\gamma(s)|^2)-\frac{1}{2(T-s)}h(\gamma(s),s).$$

Then, we establish the lower bound estimate on the heat kernel by using the same argument as in the proof of [Zhang 2012, Theorem 1.1].

Lemma 3.6. *Fix $n \geq 2$ and $\lambda > 0$. There exist $\delta = \delta(n, \lambda) > 0$ and $C = C(n) > 0$ such that if $(M, g(t))_{t\in(0,1]}$ satisfies $\nu(g(0), 2) \geq -\delta$, then for any $0 < s < t \leq 1$,*

$$K(x, t; y, s) \geq \frac{C}{(t-s)^{n/2}}\left(\frac{s}{t}\right)^{2\lambda} \exp\left(\frac{-4d^2_{g(t)}(x, y)}{t-s}\right).$$

Proof. By Proposition 2.4, for any $\lambda > 0$, we may choose δ small enough that the Ricci flow enjoys the scale invariant curvature bounds $|\mathrm{Rm}| \leq \lambda/t$. Combining this with Lemma 3.1,

$$K(x, t; y, s) \leq \frac{C}{(t-s)^{n/2}}\left(\frac{t}{s}\right)^{\lambda} \quad \text{for all } 0 < s < t \leq 1. \tag{3.7}$$

Let $K(x, t; y, l) = (4\pi(t-l))^{-n/2}e^{-h(y,l)}$ for $l \in [s, t]$, and let $\gamma(l)$ be the fixed point x. By Proposition 3.5,

$$-\frac{d}{dl}h(x, l) \leq \frac{1}{2}R(x, l) - \frac{1}{2(t-l)}h(x, l).$$

Integrating from s to t, we have $h(x, s) \leq \frac{1}{2}\lambda \log(t/s)$. Consequently,

$$K(x, t; x, s) \geq (4\pi(t-s))^{-n/2}\left(\frac{s}{t}\right)^{\lambda/2}. \tag{3.8}$$

Note that the function $K(y', t'; x, s)$ for $(y', t') \in M \times [(t+s)/2, t]$ is a positive solution to the heat equation. Then Proposition 3.5 implies

$$\begin{aligned} &K(y, t; y, s) \\ &\quad \leq K(x, t; y, s)^{\frac{1}{2}} \cdot \left(\sup_{(x',t')\in M\times[(t+s)/2,t]} K(x', t'; y, s)\right)^{\frac{1}{2}} \exp\left(\frac{2d^2_{g(t)}(x, y)}{t-s}\right). \end{aligned}$$

Combining this with (3.7) and (3.8), we get the lower bound of $K(x, t; y, s)$. ☐

For $0 < t \leq 1$ and $x_0 \in M$, let $\varphi : M \times \{t\} \to [0, 1]$ be a cutoff function such that $\varphi(y) \equiv 1$ for $y \in B_{g(t)}(x_0, 8t^{\frac{1}{2}})$ and $\operatorname{supp}\varphi \subset B_{g(t)}(x_0, 16t^{\frac{1}{2}})$. Let $\varphi : M \times [0, t] \to \mathbb{R}$ be the solution of the conjugate heat equation $(\partial_s + \Delta - R)\varphi = 0$ with terminal data $\varphi(y)$.

Applying Lemma 3.6, we can derive the following estimates. For the proof, see [Lee et al. 2023, Proposition 4.4].

Proposition 3.9. *Fix $n \geq 2$ and $\lambda > 0$. There exist $\delta = \delta(n, \lambda) > 0$ and $C = C(n) > 0$ such that if $(M, g(t))_{t\in(0,1]}$ satisfies $\nu(g(0), 2) \geq -\delta$, then for all $(y, s) \in B_{g(t)}(x_0, 4t^{\frac{1}{2}}) \times (0, t)$,*

$$\varphi(y, s) \geq C\left(\frac{s}{t}\right)^{2\lambda}. \tag{3.10}$$

In addition, if the manifold mentioned above also has almost nonnegative integral scalar curvature and an upper capacity bound, we can refine the heat kernel estimates in Lemma 3.1 to obtain lower bounds on scalar curvature along the Ricci flow. Moreover, the lower scalar curvature bounds ensure that the volume of a given set does not expend too much along the Ricci flow.

Lemma 3.11. *Fix r, $N > 0$ $n > 2$ and $q > n/2$. There exist $\delta = \delta(n, N, q)$ and $C = C(n, N, q)$ such that if $(M^n, g(t))_{t\in(0,1/r^2]}$ satisfies*

$$\nu(g(0), 2r^2) \geq -\delta, \quad \|R_-\|_{g(0),q,r} \leq \delta, \quad \mathrm{Cap}_{(M,g(0))}(r) \leq N,$$

then for $0 < t \leq 1/r^2$,

$$R(x, t) \geq -C\delta r^{-(2q-n)/q} t^{-n/(2q)} \quad \textit{and} \quad d\mathrm{vol}_{g(t)} \leq e^{C\delta}\, d\mathrm{vol}_{g(s)}.$$

Proof of Lemma 3.11. Up to rescaling, we may assume $r = 1$. We will use the upper bound of capacity to prove that for each $k \in \mathbb{N}^+$ there exists a finite subset $\{x_i\}_{i=1}^{N^k} \subset B_{g(0)}(x, k+1)$ such that

$$B_{g(0)}(x, k+1) \subseteq \bigcup_{i=1}^{N^k} B_{g(0)}(x_i, 1). \tag{3.12}$$

We argue by induction. For $k = 1$ and $y \in M$, by the definition of the capacity, there exists $\{y_j\}_{j=1}^N$ such that $B_{g(0)}(y, 2) \subset \bigcup_{j=1}^N B_g(y_j, 1)$. For $k > 1$, if there exists $\{x_i\}_{i=1}^{N^{k-1}}$ such that $B_{g(0)}(x, k)$ can be covered by $\{B_{g(0)}(x_i, 1)\}_{i=1}^{N^{k-1}}$, then by the triangle inequality we have $B_{g(0)}(x, k+1) \subset \bigcup_{i=1}^{N^{k-1}} B_{g(0)}(x_i, 2)$. Since for each $B_{g(0)}(x_i, 2)$ there is a finite cover $\{B_{g(0)}(y_{i,j}, 1)\}_{j=1}^N$, $\{B_{g(0)}(y_{i,j}, 1)\}_{i,j}$ form a cover for $B_{g(0)}(x, k+1)$. Thus (3.12) follows.

Let w be the solution to the heat equation with the initial data $w(y, 0) = R_-(y, 0)$. By maximum principle, $-R(x, t) \leq w(x, t)$ pointwise. Let $S(t) = \int_0^t \hat{R}_-$ and $A_x(k, k+1) = B_{g(0)}(x, k+1) \setminus B_{g(0)}(x, k)$ for $k \in \mathbb{N}^+$. By Hölder's inequality, (3.12), (2.2) and Lemma 3.1,

$$\begin{aligned}
w(x, t) &= \int_M K(x, t; y, 0) R_-(y, 0)\, d\mathrm{vol}_{g(0)}(y) \\
&\leq \sum_{k=0}^{\infty} \|R_-\|_{L^q(B_g(x,k+1))} \left(\int_{A_x(k,k+1)} K^{q/(q-1)} \right)^{(q-1)/q} \\
&\leq \sum_{k=0}^{\infty} \sup_{A_x(k,k+1)} K^{1/q} \left(\int_M K \right)^{(q-1)/q} \left(\sum_{i=1}^{N^k} \|R_-\|^q_{L^q(B_g(x_i^k,1))} \right)^{1/q} \\
&\leq \sum_{k=0}^{\infty} C_0^{1/q} \delta t^{-n/(2q)} \exp\!\left(-\frac{k^2}{C_0 q t} + \frac{S(t)}{q} \right) N^{k/q} \leq C\delta t^{-n/(2q)} e^{S(t)/q},
\end{aligned}$$

where $C_0 = C_0(n)$ and $C = C(n, N, q)$. In particular, we obtain

$$\hat{R}_-(t) \le \max_{x\in M}(w(x,t)) \le C\delta t^{-n/(2q)} e^{S(t)/q}. \tag{3.13}$$

Thus $\frac{d}{dt} q \cdot e^{-S(t)/q} = -e^{-S(t)/q}\hat{R}_-(t) \ge -C\delta t^{-n/(2q)}$. Integrating from 0 to t,

$$S(t) \le -q\log\left(1 - \frac{2C\delta}{2q-n}\right) \quad \text{for all } 0 < t \le 1.$$

If $\delta \le (2q-n)/(4C)$, then $S(t) \le q$. Substituting this in (3.13), $R_-(\cdot\,, t) \le \hat{R}_-(t) \le 2C\delta t^{-n/(2q)}$. By the evolved equation of volume form (2.1), we have $\partial_\tau \, d\mathrm{vol}_{g(\tau)} \le 2C\delta\tau^{-n/(2q)} \, d\mathrm{vol}_{g(\tau)}$. Integrating τ from s to t completes the proof. $\square$

4. Integral estimate for Ricci curvature under Ricci flow

In this section, we prove an integral estimate for the Ricci curvature, which is scale invariant. The proof of Lemma 4.1 is analogous to [Lee et al. 2023, Theorem 4.1]. In our case, we replace the use of pointed lower bound of initial scalar curvature by the lower bound of $w(x,t)$ along the Ricci flow, as shown in (3.13). For the sake of completeness, we include the proof.

Lemma 4.1 (integral Ricci estimate). *Fix $n > 2$, $\varepsilon, r, N > 0$, $q > n/2$ and $\theta \in \left[0, \frac{1}{2}\right)$. If $(M^n, g(t))_{t\in(0,1/r^2]}$ satisfies*

$$\nu(g(0), 2r^2) \ge -\delta, \quad \|R_-\|_{g(0),q,r} \le \delta, \quad \mathrm{Cap}_{(M,g(0))}(r) \le N,$$

then for any $(x,s) \in M \times (0, 1/r^2]$,

$$\int_0^s \left(\frac{\tau}{s}\right)^{-\theta} ⨍_{B_{g(s)}(x,4s^{1/2})} |\mathrm{Ric}_{g(\tau)}| \, d\mathrm{vol}_{g(\tau)} \, d\tau \le \varepsilon^2.$$

As the evolution equation of the scalar curvature contains the term of $|\mathrm{Ric}|^2$, we can combine the scalar curvature estimate and the heat kernel estimate along the Ricci flow to estimate $|\mathrm{Ric}|^2\varphi$. Combining this estimate with Hölder's inequality, we prove Lemma 4.1:

Proof of Lemma 4.1. Up to rescaling the flow, we may assume that $t = 1$. By Proposition 2.4 and the volume comparison in Lemma 3.11, there exists a constant $C_0 = C_0(n, q, N)$ such that for any $(x,s) \in M \times [0, 1]$,

$$\mathrm{vol}_{g(s)}(B_{g(1)}(x,4)) \ge e^{-C_0\delta}\,\mathrm{vol}_{g(1)}(B_{g(1)}(x,4)) \ge C_0. \tag{4.2}$$

For fixed $0 < \lambda \le \frac{1}{4} - \frac{1}{2}\theta$, let $\theta_0 = \theta + \lambda$, and then we have $1 - 2\theta_0 \ge \frac{1}{2} - \theta > 0$. Choose δ small enough that Proposition 3.9 holds for this choice of λ. By Hölder's inequality and (4.2) we get

$$\begin{aligned}(4.3)\quad &\int_0^1 s^{-\theta} \fint_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}| \, d\mathrm{vol}_{g(s)} \, ds \\ &\qquad \le C_0^{-1}(1-2\theta_0)^{-\frac{1}{2}} \left(\int_0^1 s^{2\lambda} \int_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}|^2 \, d\mathrm{vol}_{g(s)} \, ds \right)^{\frac{1}{2}}.\end{aligned}$$

Let φ be the cutoff function that evolves by the conjugate heat equation as in Proposition 3.9. Then there exists $C_1 = C_1(n)$ such that

$$\begin{aligned}\int_0^1 s^{2\lambda} &\int_{B_{g(1)}(x,4)} |\mathrm{Ric}_{g(s)}|^2 \, d\mathrm{vol}_{g(s)} \, ds \\ &\le C_1 \int_0^1 \int_M |\mathrm{Ric}_{g(s)}|^2 \varphi \\ &= \frac{1}{2} C_1 \int_0^1 \int_M (\partial_s - \triangle) R_{g(s)} \varphi \\ &= \frac{1}{2} C_1 \left(\int_M R_{g(1)} \varphi(\,\cdot\,,1) \, d\mathrm{vol}_{g(1)} - \int_M R_{g(0)} \varphi(\,\cdot\,,0) \, d\mathrm{vol}_{g(0)} \right).\end{aligned}$$

By Proposition 2.4, we find that $\int_M R_{g(1)} \varphi(\,\cdot\,,1) \, d\mathrm{vol}_{g(1)} \le \lambda \, \mathrm{vol}_{g(1)}(B_{g(1)}(x,16)) \le C_1 \lambda$. By (3.13), there exists a constant $C_2 = C_2(n,q,N)$ such that

$$\begin{aligned}-\int_M &R_{g(0)} \varphi(\,\cdot\,,0) \, d\mathrm{vol}_{g(0)} \\ &= \int_M R_-(x,0) \int_M K(y,1;x,0) \varphi(y,1) \, d\mathrm{vol}_{g(1)}(y) \, d\mathrm{vol}_{g(0)}(x) \\ &= \int_M \varphi(y,1) \int_M K(y,1;x,0) R_-(x,0) \, d\mathrm{vol}_{g(0)}(x) \, d\mathrm{vol}_{g(1)}(y) \le C_2 \delta.\end{aligned}$$

By choosing λ and δ appropriately small, we conclude the proof. □

5. L^p bound for the scalar curvature

For closed manifold, we derive an a prior L^p $(p<1)$ bound of scalar curvature, Theorem 1.7, which we restate below for convenience. The proof of this theorem closely parallels that of [Lee et al. 2023, Theorem 4.7]. The main difference is that, since the curvature here is only bounded below in an integral sense, we need to estimate the L^p norm of $R+w$ along the Ricci flow, where w evolves by the heat equation with initial data R_-.

Theorem 1.7. *Fix $n \ge 2$, $\varepsilon > 0$, $p \in (0,1)$ and $q > \max(n/2, 2p)$. Let (M^n, g) be a closed Riemannian n-manifold. There exists $\delta = \delta(n, \varepsilon, p, q) > 0$ such that if*

$$\mathrm{vol}(M)^{2/n} \cdot \left(\fint_M |R_-|^q \right)^{1/q} \le \delta \quad \textit{and} \quad \nu(g, 2\,\mathrm{vol}(M)^{2/n}) \ge -\delta,$$

then

$$\mathrm{vol}(M)^{2/n}\cdot\left(\fint_M |R|^p\right)^{1/p}\le\varepsilon.$$

Proof. Up to rescaling, we may assume that $\mathrm{vol}(M)=1$. Choosing $\delta\le\varepsilon/2$, by Hölder's inequality we have $\left(\int_M|R_-|^p\right)^{1/p}\le\left(\int_M|R_-|^q\right)^{1/q}\le\varepsilon/2$, and it is suffices to show that $\int_M|R_+|^p\,d\mathrm{vol}_g\le(\varepsilon/2)^p$, where $R_+=\max(R,0)$. By Proposition 2.4 and Lemma 3.1, for any fixed $\lambda>0$, we may choose δ small enough that the Ricci flow $(M,g(t))$ with $g(0)=g$ exists for $t\in(0,1]$ and there exists a constant $C_0=C_0(n)$ such that

$$|\mathrm{Rm}|\le\lambda/t,\qquad\sup_{x,y\in M}\{K(x,t;y,0)\}\le C_0t^{-n/2}\exp\left(\int_0^t\hat{R}_-\right)\qquad\text{for all }0<t\le 1.$$

Let $w(x,t)$ be the solution of the heat equation with initial data $w(x,0)=R_-(x,0)$ and $f(x,t)=R(x,t)+w(x,t)$, the evolve equation of scalar curvature in (2.1) implies that $(\partial_t-\triangle)f=2|\mathrm{Ric}|^2\ge 0$. By the maximum principle, we have $f(x,t)\ge R_+(x,t)$. Thus, we only need to show $\int_M f^p\,d\mathrm{vol}_{g(0)}\le(\varepsilon/2)^p$.

For any $p\in(0,1)$, we see that f^p is a supersolution of the heat equation:

$$(\partial_t-\triangle)f^p=pf^{p-1}(\partial_t-\triangle)f-p(p-1)f^{p-1}|\nabla f|^2\ge 2pf^{p-1}|\mathrm{Ric}|^2\ge 0.$$

Combining this with Young's inequality, we have

$$\begin{aligned}(5.1)\quad &\int_M f^p\,d\mathrm{vol}_{g(0)}\\ &=\int_M f^p\,d\mathrm{vol}_{g(1)}-\int_0^1\int_M\big((\partial_t-\triangle)f^p-Rf^p\big)\,d\mathrm{vol}_{g(t)}\,dt\\ &\le\int_M f^p\,d\mathrm{vol}_{g(1)}+\int_0^1\int_M Rf^p\,d\mathrm{vol}_{g(t)}\,dt\\ &\le\int_M f^p\,d\mathrm{vol}_{g(1)}+\int_0^1\int_M(R^{p+1}+Rw^p)\,d\mathrm{vol}_{g(t)}\,dt\\ &\le\int_M f^p\,d\mathrm{vol}_{g(1)}+\frac{p}{q}\int_0^1\int_M w^q\,d\mathrm{vol}_{g(t)}\,dt\\ &\qquad+\int_0^1\int_M\left(R^{p+1}+\frac{q-p}{q}R^{q/(q-p)}\right)d\mathrm{vol}_{g(t)}\,dt.\end{aligned}$$

To bound the right-hand side of (5.1), let $S(t)=\int_0^t\hat{R}_-$. Then Lemma 3.1 implies

$$\begin{aligned}w(x,t)&=\int_M K(x,t;y,0)R_-(y,0)\,d\mathrm{vol}_{g(0)}(y)\\ &\le\left(\int_M|R_-(\cdot,0)|^q\,d\mathrm{vol}_{g(0)}\right)^{1/q}\left(\int_M K(x,t;\cdot,0)^{q/(q-1)}\,d\mathrm{vol}_{g(0)}\right)^{(q-1)/q}\\ &\le\delta\max_{y\in M}K(x,t;y,0)^{1/q}\le C_0^{1/q}\delta t^{-n/(2q)}e^{S(t)/q}.\end{aligned}$$

Similar to the argument in Lemma 3.11, there exists $C_1 = C_1(n, q)$ such that for $x \in M$ and $0 < t \le 1$, we have

$$R_-(x,t) \le w(x,t) \le C_1 \delta t^{-n/(2q)}, \quad d\mathrm{vol}_{g(t)} \le e^{C_1\delta}\, d\mathrm{vol}_{g(0)}. \tag{5.2}$$

By (2.6), we find $C_2 = C_2(n, q)$ such that for any $0 \le s < t \le 1$ we have

$$\int_M K(\cdot\,,t;y,s)\, d\mathrm{vol}_{g(t)} \le \exp\biggl(\int_s^t R_-\biggr) \le \exp\biggl(\int_s^t C_1\delta\tau^{-n/(2q)}\biggr)\, d\tau \le C_2.$$

Then we bound each term on the right-hand side of (5.1) separately. For the first term, by (5.2) we see that

$$\int_M f^p\, d\mathrm{vol}_{g(1)} = \int_M (R+w)^p\, d\mathrm{vol}_{g(1)} \le \int_M (\lambda + C_1\delta)^p\, d\mathrm{vol}_{g(1)} \le (\lambda + C_1\delta)^p e^{C_1\delta}.$$

For the second term, by Hölder's inequality, we have

$$\begin{aligned}
&\int_M w^q(\cdot\,,t)\, d\mathrm{vol}_{g(t)}\\
&= \int_M \biggl(\int_M R_-(y,0) K(x,t;y,0)\, d\mathrm{vol}_{g(0)}(y)\biggr)^q d\mathrm{vol}_{g(t)}(x)\\
&\le \int_M \biggl(\int_M R_-^q(y,0) K(x,t;y,0)\, d\mathrm{vol}_{g(0)}(y)\biggr)\\
&\qquad\qquad\qquad \cdot \biggl(\int_M K(x,t;y,0)\, d\mathrm{vol}_{g(0)}(y)\biggr) d\mathrm{vol}_{g(t)}(x)\\
&= \int_M R_-^q(y,0) \int_M K(x,t;y,0)\, d\mathrm{vol}_{g(t)}(x)\, d\mathrm{vol}_{g(0)}(y)\\
&\le C_2 \int_M R_-^q(y,0)\, d\mathrm{vol}_{g(0)}(y) \le C_2\delta^q.
\end{aligned}$$

For the third term, let $\varphi : M \times (0,1) \to \mathbb{R}$ be the solution to the conjugate heat equation with terminal data $\varphi(x,1) = 1$ on $M \times \{1\}$. Using the same proof as [Lee et al. 2023, Proposition 4.4], there exists a constant $C_3 = C_3(n)$ such that for all $y \in M$ and $t \in (0,1]$ we have $\varphi(y,t) \ge C_3 t^{2\lambda}$. Moreover, by using the same argument as in the proof of Lemma 4.1, (5.2) implies that there exists a constant $C_4 = C_4(n,q)$ such that

$$\int_0^1 t^{2\lambda} \int_M |R|^2\, d\mathrm{vol}_{g(t)}\, dt \le C_3^{-1} \int_0^1 \int_M |R|^2 \varphi(y,t)\, d\mathrm{vol}_{g(t)}\, dt \le C_4(\lambda+\delta). \tag{5.3}$$

For any $0 < \alpha < 2$ and $0 < \lambda < (2-\alpha)/(4\alpha)$, let $\theta = 2\lambda\alpha/(2-\alpha)$. Then, by (5.3),

$$\begin{aligned}
\int_0^1 \int_M |R|^\alpha\, d\mathrm{vol}_{g(t)}\, dt &\le \biggl(\int_0^1 \int_M t^{-\theta}\, d\mathrm{vol}_s\, ds\biggr)^{\lambda\alpha/\theta} \biggl(\int_0^1 s^{2\lambda} \int_M |R|^2\, d\mathrm{vol}_s\, ds\biggr)^{\alpha/2}\\
&\le 2e^{C_1\delta} C_4^{\alpha/2} (\lambda+\delta)^{\alpha/2}.
\end{aligned}$$

In particular, there exists a constant $C_5 = C_5(n, p, q)$ such that

$$\int_0^1 \int_M R^{p+1} + \frac{q-p}{q} R^{q/(q-p)} \operatorname{vol}_{g(t)}\, dt \le C_5(\lambda+\delta)^{(p+1)/2} + C_5(\lambda+\delta)^{q/(2q-2p)}.$$

Finally, by choosing λ and δ sufficiently small, we conclude the proof. □

Acknowledgements

The author would like to thank Zhenlei Zhang for his guidance and encouragement throughout the entire research process.

References

[Anderson 1990] M. T. Anderson, "Convergence and rigidity of manifolds under Ricci curvature bounds", *Invent. Math.* **102**:2 (1990), 429–445. MR Zbl

[Anderson and Cheeger 1992] M. T. Anderson and J. Cheeger, "C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below", *J. Differential Geom.* **35**:2 (1992), 265–281. MR Zbl

[Bamler et al. 2019] R. H. Bamler, E. Cabezas-Rivas, and B. Wilking, "The Ricci flow under almost non-negative curvature conditions", *Invent. Math.* **217**:1 (2019), 95–126. MR Zbl

[Cao and Zhang 2011] X. Cao and Q. S. Zhang, "The conjugate heat equation and ancient solutions of the Ricci flow", *Adv. Math.* **228**:5 (2011), 2891–2919. MR Zbl

[Chau et al. 2011] A. Chau, L.-F. Tam, and C. Yu, "Pseudolocality for the Ricci flow and applications", *Canad. J. Math.* **63**:1 (2011), 55–85. MR Zbl

[Cheeger 1970] J. Cheeger, "Finiteness theorems for Riemannian manifolds", *Amer. J. Math.* **92** (1970), 61–74. MR Zbl

[Cheeger and Colding 1997] J. Cheeger and T. H. Colding, "On the structure of spaces with Ricci curvature bounded below, I", *J. Differential Geom.* **46**:3 (1997), 406–480. MR Zbl

[Cheng 2022] L. Cheng, "Pseudolocality theorems of Ricci flows on incomplete manifolds", 2022. arXiv 2210.15397

[Chow et al. 2010] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, III. Geometric-analytic aspects*, Mathematical Surveys and Monographs **163**, Amer. Math. Soc., Providence, RI, 2010. MR Zbl

[Colding 1997] T. H. Colding, "Ricci curvature and volume convergence", *Ann. of Math.* (2) **145**:3 (1997), 477–501. MR Zbl

[Greene and Wu 1988] R. E. Greene and H. Wu, "Lipschitz convergence of Riemannian manifolds", *Pacific J. Math.* **131**:1 (1988), 119–141. MR Zbl

[Gromov 1981] M. Gromov, *Structures métriques pour les variétés riemanniennes*, edited by J. Lafontaine and P. Pansu, Textes Mathématiques **1**, CEDIC, Paris, 1981. MR Zbl

[Guenther 2002] C. M. Guenther, "The fundamental solution on manifolds with time-dependent metrics", *J. Geom. Anal.* **12**:3 (2002), 425–436. MR Zbl

[Kasue 1989] A. Kasue, "A convergence theorem for Riemannian manifolds and some applications", *Nagoya Math. J.* **114** (1989), 21–51. MR Zbl

[Lee et al. 2023] M.-C. Lee, A. Naber, and R. Neumayer, "d_p-convergence and ϵ-regularity theorems for entropy and scalar curvature lower bounds", *Geom. Topol.* **27**:1 (2023), 227–350. MR Zbl

[Perelman 2002] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", 2002. arXiv math/0211159

[Peters 1987] S. Peters, "Convergence of Riemannian manifolds", *Compositio Math.* **62**:1 (1987), 3–16. MR Zbl

[Shi 1989] W.-X. Shi, "Ricci deformation of the metric on complete noncompact Riemannian manifolds", *J. Differential Geom.* **30**:2 (1989), 303–394. MR Zbl

[Tian and Zhang 2016] G. Tian and Z. Zhang, "Regularity of Kähler–Ricci flows on Fano manifolds", *Acta Math.* **216**:1 (2016), 127–176. MR Zbl

[Zhang 2011] Q. S. Zhang, *Sobolev inequalities*, *heat kernels under Ricci flow*, *and the Poincaré conjecture*, CRC Press, Boca Raton, FL, 2011. MR Zbl

[Zhang 2012] Q. S. Zhang, "Bounds on volume growth of geodesic balls under Ricci flow", *Math. Res. Lett.* **19**:1 (2012), 245–253. MR Zbl

Received January 16, 2024. Revised July 23, 2024.

SHANGZHI ZOU
SCHOOL OF MATHEMATICAL SCIENCES
CAPITAL NORMAL UNIVERSITY
BEIJING, CHINA
2210501005@cnu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 330, No. 2, 2024
https://doi.org/10.2140/pjm.2024.330.389

CORRECTION TO THE ARTICLE A HECKE ALGEBRA ISOMORPHISM OVER CLOSE LOCAL FIELDS

RADHIKA GANAPATHY

Volume **319**:2 (2022), 307–332

The proof of Lemma 2.5 of the author's article "A Hecke algebra isomorphism over close local fields" (*Pacific J. Math.* 319:2 (2022), 307–332) is incorrect. We use a slight variant of the original approach to correct the proof. This leads to some modifications to some parts of Section 3 of the original article, and these are given in Section 2 of this note. With these modifications, Theorem 4.1 of the original article holds.

We retain the notation in [Ganapathy 2022, Section 2]. Let T be a torus over F. Then T is determined by the Γ_F-module $X_*(T)$. Let $\mathscr{T}^{\mathrm{ft}}$ be the Néron–Raynaud model of T and $\mathscr{T}$ its identity component. Let $m \geq 1$ be such that T splits over an at most m-ramified Galois extension of F. Then the action of Γ_F on $X_*(T)$ factors through Γ_F/I_F^m. For any field F' that is at least m-close to F, we obtain a torus T' over F' via the action of $\Gamma_{F'} \to \Gamma_{F'}/I_{F'}^m \xrightarrow[\cong]{\mathrm{Del}_m^{-1}} \Gamma_F/I_F^m$ on $X_*(T)$. This torus splits over an at most m-ramified extension of F'. Let $\mathscr{T}'^{\mathrm{ft}}$ be the Néron–Raynaud model of T' and $\mathscr{T}'$ its identity component.

Theorem 0.1 [Chai and Yu 2001, Section 9]. *Let $m \geq 1$ and let h be as in* [Chai and Yu 2001, Section 8]. *Assume $e \geq m+3h$. Then for any nonarchimedean local field F' that is e-close to F, the group schemes $\mathscr{T}^{\mathrm{ft}} \times_{\mathfrak{O}_F} \mathfrak{O}_F/\mathfrak{p}_F^m$ and $\mathscr{T}'^{\mathrm{ft}} \times_{\mathfrak{O}_{F'}} \mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m$ are isomorphic. In particular,*

$$\mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_F/\mathfrak{p}_F^m) \cong \mathscr{T}'^{\mathrm{ft}}(\mathfrak{O}_{F'}/\mathfrak{p}_{F'}^m)$$

as groups. This isomorphism continues to hold when we replace $\mathscr{T}^{\mathrm{ft}}$ by $\mathscr{T}$.

In [Ganapathy 2022, Section 2C], we had constructed a group-theoretic section of the Kottwitz homomorphism $\kappa_{T,F} : T(F) \to X_*(T)_{I_F}^{\sigma}$ and had used Theorem 0.1 for the neutral component $\mathscr{T}$ to give a proof of Lemma 2.5 in the same article. If T splits over an unramified extension of F or is an induced torus over F, the results in [Ganapathy 2022, Section 2] go through. However, the Kottwitz homomorphism for

MSC2020: primary 11F70; secondary 22E50.
Keywords: close local fields, Kazhdan, Hecke algebra.

a general torus need not admit a group-theoretic section, as the following example illustrates.

Example 0.2. Let $\breve{F}$ be the completion of the maximal unramified subextension of $\mathbb{Q}_2$. Let $L = \breve{F}(\sqrt{-1})$. Then L is a wildly ramified quadratic extension of $\breve{F}$. Let $T = \mathrm{Nm}^1_{L/\breve{F}}\mathbb{G}_m$ denote the norm-1 torus. Let γ be the nontrivial element of $\mathrm{Gal}(L/\breve{F})$. Then $X_*(T)_{I_F} \cong \mathbb{Z}/2\mathbb{Z}$. Note that $\kappa_{T,\breve{F}}$ has a group-theoretic section if and only if $-1 \in T(\breve{F})$ does not lie in $T(\breve{F})_1$. Note that

$$T(\breve{F})_1 = \{y \in L^\times \mid x\gamma(x)^{-1} = y \text{ for some } x \in L^\times\}.$$

Since $-1 = (\sqrt{-1})\gamma(\sqrt{-1})^{-1}$, -1 indeed lies in $T(\breve{F})_1$. We conclude that $\kappa_{T,\breve{F}}$ does not admit a group-theoretic section.

The error in [Ganapathy 2022, Section 2] is that Lemma 2.3 is false in general (the $n_{\tilde{\lambda}}$ defined in the line above Lemma 2.3 may not be well-defined). Consequently, Lemma 2.4 cannot be salvaged to yield a well-defined set of representatives for the torsion elements of $X_*(T)_{I_F}$ that forms a group and is σ-stable.

1. Proof of [Ganapathy 2022, Lemma 2.5]

Let T be a torus over F and let $\widetilde{F}$ be the splitting extension of $T_{\breve{F}}$ in the completion of F_s. Fix a uniformizer $\varpi_{\widetilde{F}}$ of $\widetilde{F}$. Consider the Kottwitz homomorphism $\kappa_{T,\breve{F}} : T(\breve{F}) \to X^*(T)_{I_F}$. Let $X_*(T)_{I_F}/\mathrm{tor}$ denote the quotient of $X_*(T)_{I_F}$ by its torsion subgroup. Note that $X_*(T)_{I_F}/\mathrm{tor}$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(X^*(T)^{I_F}, \mathbb{Z})$. This leads to the valuation homomorphism $\omega_{T,\breve{F}} : T(\breve{F}) \to \mathrm{Hom}_{\mathbb{Z}}(X^*(T)^{I_F}, \mathbb{Z})$. Note that $\mathrm{Ker}(\omega_{T,\breve{F}}) = T(\breve{F})_b = \mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}})$ is the maximal bounded subgroup of $T(\breve{F})$ and it contains $T(\breve{F})_1$. We will construct a group-theoretic section of the valuation homomorphism. We will then use Theorem 0.1 for $\mathscr{T}^{\mathrm{ft}}$ to prove [Ganapathy 2022, Lemma 2.5] over $\breve{F}$. We will show that this isomorphism over $\breve{F}$ is σ-equivariant to obtain the required isomorphism over F (see Lemmas 1.2 and 1.3).

1A. ***A group-theoretic section of the valuation homomorphism and its consequences.*** Let $\breve{\lambda}_1, \dots, \breve{\lambda}_n \in X_*(T)_{I_F}$ be such that their images $\breve{\lambda}^t_1, \dots, \breve{\lambda}^t_n$ form a basis of $X_*(T)_{I_F}/\mathrm{tor}$. Fix $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n \in X_*(T)$ such that $\mathrm{pr}(\tilde{\lambda}_i) = \breve{\lambda}_i$, where $\mathrm{pr} : X_*(T) \to X_*(T)_{I_F}$ is the natural surjection. Define $n_{\tilde{\lambda}_i} = \tilde{\lambda}_i(\varpi_{\widetilde{F}})$. Define $n_{\breve{\lambda}^t_i} = n_{\breve{\lambda}_i} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, n_{\tilde{\lambda}_i}$. For $\breve{\lambda}^t \in X_*(T)_{I_F}/\mathrm{tor}$, write $\breve{\lambda}^t = \sum_i c_i \breve{\lambda}^t_i$ and define $n_{\breve{\lambda}^t} = \prod_i n^{c_i}_{\breve{\lambda}^t_i}$. Note that $n_0 = 1$ by construction.

Lemma 1.1. *The set* $\mathscr{S} := \{n_{\breve{\lambda}^t} \mid \breve{\lambda}^t \in X_*(T)_{I_F}/\mathrm{tor}\}$ *is a subgroup of* $T(\breve{F})$. *The map* $\nabla_{T,\breve{F}} : X_*(T)_{I_F}/\mathrm{tor} \to \mathscr{S}$, $\breve{\lambda}^t \mapsto n_{\breve{\lambda}^t}$, *is a group isomorphism.*

Proof. It is clear that $\mathscr{S}$ is a subgroup of $T(\breve{F})$. It is also clear that $\nabla_{T,\breve{F}}$ is a surjective group homomorphism. We just need to see that it is injective. Suppose $n_{\breve{\lambda}^t} = 1$.

We need to show that $\breve{\lambda}^t = 0$. Write $\breve{\lambda}^t = \sum_i c_i \breve{\lambda}^t_i$. The natural pairing between $X_*(T)$ and $X^*(T)$ induces a perfect pairing $\langle\cdot,\cdot\rangle : X_*(T)_{I_F}/\mathrm{tor} \times X^*(T)^{I_F} \to \mathbb{Z}$. Let $\breve{\chi}_1, \ldots, \breve{\chi}_n \in X^*(T)^{I_F}$ be such that $\langle \tilde{\lambda}_j, \breve{\chi}_k\rangle = \delta_{j,k}$, $1 \le j,k \le n$. Now $n_{\breve{\lambda}^t} = \prod_i \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, n_{\tilde{\lambda}_i}^{c_i} = 1$. This implies that $1 = \breve{\chi}_j(n_{\breve{\lambda}^t}) = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \breve{\chi}_j(n_{\tilde{\lambda}_j})^{c_j} = (\mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \varpi_{\widetilde{F}}^{c_j})$. This forces $c_j = 0$. Since j was arbitrary, this shows that $\breve{\lambda}^t = 0$. □

Lemma 1.2. *Let T be a torus over F. Let $\mathscr{T}^{\mathrm{ft}}$ be as above and for $m \ge 1$, let $\breve{T}_m = \mathrm{Ker}(\mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}}) \to \mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}}/\mathfrak{p}^m_{\breve{F}}))$. Let $e \ge m + 4h$. If $\breve{F}$ and $\breve{F}'$ are e-close, we have an isomorphism*

$$\breve{\mathscr{T}}_m : T(\breve{F})/\breve{T}_m \to T'(\breve{F}')/\breve{T}'_m.$$

Proof. Since $\mathscr{T}^{\mathrm{ft}}(\mathfrak{O}_{\breve{F}}) = T(\breve{F})_b$, we have by Theorem 0.1 (which also holds over $\breve{F}$; see [Chai and Yu 2001]) an isomorphism

$$T(\breve{F})_b/\breve{T}_m \to T'(\breve{F}')_b/\breve{T}'_m. \tag{1-1}$$

Since T splits over an at most m-ramified extension of F, the action of Γ_F on $X_*(T)$ factors through Γ_F/I_F^m. Since the action of Γ_F/I_F^m on $X_*(T)$ is Del_m-equivariant, we have $X_*(T)_{I_F} \cong X_*(T)_{I_{F'}}$ and $X_*(T)_{I_F}/\mathrm{tor} \cong X_*(T)_{I_{F'}}/\mathrm{tor}$ via Del_m. We identify these groups via these isomorphisms. Let $\varpi_{\widetilde{F}'}$ be a uniformizer of $\widetilde{F}'$ such that $\varpi_{\widetilde{F}} \bmod \mathfrak{p}^{rm}_{\widetilde{F}} \mapsto \varpi_{\widetilde{F}'} \bmod \mathfrak{p}'^{rm}_{\widetilde{F}}$ where $r = [\widetilde{F} : \breve{F}]$. For $1 \le i \le n$, define $n'_{\tilde{\lambda}_i} = \tilde{\lambda}_i(\varpi_{\widetilde{F}'})$, $n'_{\breve{\lambda}^t_i} = \mathrm{Nm}_{\widetilde{F}'/\breve{F}'}\, n_{\tilde{\lambda}_i}$. Form the subgroup $\mathscr{S}' \subset T(\breve{F}')$ as before. Since $\nabla_{T,\breve{F}}$, $\nabla_{T',\breve{F}'}$ are group isomorphisms, we get

$$T(\breve{F})/\breve{T}_m \cong X_*(T)_{I_F}/\mathrm{tor} \times T(\breve{F})_b/\breve{T}_m,$$

and similarly over $\breve{F}'$. These observations, combined with (1-1), finish the proof of the lemma. □

Lemma 1.3. *The isomorphism $\breve{\mathscr{T}}_m : T(\breve{F})/\breve{T}_m \to T'(\breve{F}')/\breve{T}'_m$ of Lemma 1.2 is σ-equivariant. It induces a group isomorphism $\mathscr{T}_m : T(F)/T_m \to T'(F')/T'_m$.*

Proof. We know that the isomorphism in (1-1) is σ-equivariant. We need to see that for $\breve{\lambda}^t \in \mathscr{S}$, $\sigma(n_{\breve{\lambda}}) \bmod \breve{T}_m \mapsto \sigma'(n'_{\breve{\lambda}}) \bmod \breve{T}'_m$. It suffices to see this for $\breve{\lambda}^t_i$, $1 \le i \le n$. Fix i and let $\breve{\lambda}^t = \breve{\lambda}^t_i$. Write

$$\sigma(\breve{\lambda}^t) = \sum_j c_j \breve{\lambda}^t_j. \tag{1-2}$$

Let $\tilde{\sigma}$ be any lift of σ to Γ_F/I_F^m and we denote its action on $X_*(T)$ as $\tilde{\sigma}$. We know

$$\sigma(n_{\breve{\lambda}^t}) = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\sigma}(n_{\tilde{\lambda}}) = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\sigma}(\tilde{\lambda})(\tilde{\sigma}(\varpi_{\widetilde{F}}))$$

and

$$n_{\sigma(\breve{\lambda}^t)} = \prod_j \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, n_{\tilde{\lambda}_j}^{c_j} = \prod_j \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, \tilde{\lambda}_j(\varpi_{\widetilde{F}})^{c_j}.$$

Equation (1-2) implies that $\tilde{\sigma}(\tilde{\lambda}) - \sum_j c_j \tilde{\lambda}_j \in X_*(T)(I_F)$, so

$$\tilde{\sigma}(\tilde{\lambda}) - \sum_j c_j \tilde{\lambda}_j = \sum_k d_k(\gamma_k(\tilde{\mu}_k) - \tilde{\mu}_k),$$

for suitable $\gamma_k \in I_F/I_F^m$ and $\tilde{\mu}_k \in X_*(T)$. Now,

$$\tilde{\sigma}(\tilde{\lambda})(\tilde{\sigma}(\varpi_{\widetilde{F}})) = \prod_j \tilde{\lambda}_j(\tilde{\sigma}(\varpi_{\widetilde{F}})^{c_j}) \cdot \prod_k (\gamma_k(\mu_k) - \mu_k)(\tilde{\sigma}(\varpi_{\widetilde{F}})^{d_k})$$

Define

$$u_{\tilde{\lambda},\tilde{\sigma}} = \prod_j \tilde{\lambda}_j((\tilde{\sigma}(\varpi_{\widetilde{F}})\varpi_{\widetilde{F}}^{-1})^{c_j}) \prod_k \mu_k\big(\gamma_k^{-1}(\tilde{\sigma}(\varpi_{\widetilde{F}}))(\tilde{\sigma}(\varpi_{\widetilde{F}})^{-1})^{d_k}\big)$$

and define $u_{\breve{\lambda},\sigma} = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\, u_{\tilde{\lambda},\tilde{\sigma}}$. Then we have $\sigma(n_{\breve{\lambda}^t}) = u_{\breve{\lambda},\sigma} \cdot n_{\sigma(\breve{\lambda}^t)}$.

By construction of $\breve{\mathscr{T}}_m$, we have $n_{\sigma(\breve{\lambda}^t)} \bmod \breve{T}_m \mapsto n'_{\sigma(\breve{\lambda}^t)} \bmod \breve{T}'_m$. Further $u_{\tilde{\lambda},\tilde{\sigma}} \in T(\widetilde{F})_1$. Recall that $r = [\widetilde{F} : \breve{F}]$. With $\varpi_{\widetilde{F}}$ and $\varpi_{\widetilde{F}'}$ as above, the map $X_*(T) \to T(\widetilde{F})$, $\tilde{\lambda} \mapsto \tilde{\lambda}(\varpi_{\widetilde{F}})$, is a group-theoretic section of the Kottwitz homomorphism over $\widetilde{F}$, and using the Chai–Yu isomorphism $T(\tilde{F})_1/\tilde{T}_{rm} \cong T'(\widetilde{F}')_1/\tilde{T}'_{rm}$ we obtain that

$$\tilde{\mathscr{T}}_{rm} : T(\widetilde{F})/\tilde{T}_{rm} \cong T(\widetilde{F}')/\tilde{T}'_{rm}$$

as groups. Since under the isomorphism $\mathfrak{O}_{\widetilde{F}}/\mathfrak{p}_{\widetilde{F}}^{rm} \cong \mathfrak{O}_{\widetilde{F}'}/\mathfrak{p}_{\widetilde{F}'}^{rm}$, we have

$$\tilde{\sigma}(\varpi_{\widetilde{F}})\varpi_{\widetilde{F}}^{-1} \bmod \mathfrak{p}_{\widetilde{F}}^{rm} \mapsto \tilde{\sigma}'(\varpi'_{\widetilde{F}})\varpi'^{-1}_{\widetilde{F}} \bmod \mathfrak{p}_{\widetilde{F}'}^{rm},$$
$$\gamma_k^{-1}(\tilde{\sigma}(\varpi_{\widetilde{F}}))(\tilde{\sigma}(\varpi_{\widetilde{F}}))^{-1} \bmod \mathfrak{p}_{\widetilde{F}}^{rm} \mapsto \gamma_k'^{-1}(\tilde{\sigma}'(\varpi'_{\widetilde{F}}))(\tilde{\sigma}'(\varpi'_{\widetilde{F}}))^{-1} \bmod \mathfrak{p}_{\widetilde{F}'}^{rm}$$

we have that $u_{\tilde{\lambda},\tilde{\sigma}} \bmod \tilde{T}_{rm} \mapsto u_{\tilde{\lambda}',\tilde{\sigma}'} \bmod \tilde{T}'_{rm}$ via $\tilde{\mathscr{T}}_{rm}$. By the functoriality of the Chai–Yu isomorphism [2001, Section 9.2], we have the commutative diagram

$$\begin{array}{ccc} T(\widetilde{F})_1/\tilde{T}_{rm} & \xrightarrow{\mathrm{Nm}} & T(\breve{F})_1/\breve{T}_m \\ \cong\downarrow & & \downarrow\cong \\ T'(\widetilde{F}')_1/\tilde{T}'_{rm} & \xrightarrow{\mathrm{Nm}} & T'(\breve{F}')_1/\breve{T}'_m \end{array}$$

It follows that $u_{\breve{\lambda},\sigma} \bmod \breve{T}_m \mapsto u'_{\breve{\lambda},\sigma'} \bmod \breve{T}'_m$. We have proved that $\sigma(n_{\breve{\lambda}^t}) \bmod \breve{T}_m \mapsto \sigma'(n'_{\breve{\lambda}^t}) \bmod \breve{T}'_m$ for all $\breve{\lambda}^t = \breve{\lambda}_i^t$, $1 \le i \le n$. Hence this same claim holds for all $\breve{\lambda}^t \in X_*(T)_{I_F}/\mathrm{tor}$. This implies that $\breve{\mathscr{T}}_m$ is σ-equivariant. The claim that $\breve{\mathscr{T}}_m$ restricts to an isomorphism $\mathscr{T}_m : T(F)/T_m \to T'(F')/T'_m$ follows from the fact that $H^1(\sigma, \breve{T}_m) = 1$ (see [Serre 1979, Chapter XII, §3, Lemma 3]). □

1B. ***Some remarks.*** Assume $e \ge m + 4h$. We have σ-equivariant isomorphisms $\breve{\mathscr{T}}_m$ and $\breve{\mathscr{T}}_{m+h}$ constructed above (we also have $\tilde{\mathscr{T}}_{rm}$ and $\tilde{\mathscr{T}}_{r(m+h)}$). Let $t \in T(\breve{F})_b$ with

$\kappa_{T,\breve{F}}(t) = \breve{\mu}$. Write $t = \mathrm{Nm}_{\widetilde{F}/\breve{F}}\ \tilde{t}$, with $\tilde{t} \in T(\widetilde{F})$. By functoriality of the Chai–Yu isomorphism (for $T_{\breve{F}} \hookrightarrow R_{\breve{F}} = \mathrm{Res}_{\widetilde{F}/\breve{F}} T_{\widetilde{F}}$), we have the commutative diagram

$$\begin{tikzcd} T(\breve{F})_b/\breve{T}_{m+h} \arrow[r, "i"] \arrow[d, "\breve{\mathscr{T}}_{m+h}"'] & T(\widetilde{F})_b/\tilde{T}_{r(m+h)} \arrow[d, "\tilde{\mathscr{T}}_{r(m+h)}"] \\ T'(\breve{F}')_b/\breve{T}'_{m+h} \arrow[r, "i'"] & T'(\widetilde{F}')_b/\tilde{T}'_{r(m+h)} \end{tikzcd}$$

As explained in [Aubert and Varma 2024, Theorem 2.5.3], it follows from the arguments in [Chai and Yu 2001, Section 8] that $T(\breve{F})_b \cap \tilde{T}_{r(m+h)} \subset \breve{T}_m$. Let $\tilde{t}' \in T(\widetilde{F}')$ be such that $\tilde{\mathscr{T}}_{r(m+h)}(\tilde{t} \bmod \tilde{T}_{r(m+h)}) = \tilde{t}' \bmod \tilde{T}'_{r(m+h)}$. Using the Galois equivariance of $\tilde{\mathscr{T}}_{r(m+h)}$ and the commutativity of the above diagram, we have $\breve{\mathscr{T}}_{m+h}(t \bmod T(\breve{F})_b \cap \tilde{T}_{r(m+h)}) = t' \bmod T'(\breve{F}')_b \cap \tilde{T}'_{r(m+h)}$ where $t' = \mathrm{Nm}_{\widetilde{F}'/\breve{F}'}\ \tilde{t}'$. Hence $\breve{\mathscr{T}}_m(t \bmod \breve{T}_m) = \breve{\mathscr{T}}_{m+h}(t \bmod \breve{T}_m) = t' \bmod \breve{T}'_m$. By Diagram (7.3.1) in [Kottwitz 1997], $\kappa_{T',\breve{F}'}(t') = \breve{\mu}$.

Now, let $t \in T(\breve{F})$. Write $t = t_1 n_{\breve{\mu}^t}$ for suitable $t_1 \in T(\breve{F})_b$ and $\breve{\mu}^t \in X_*(T)_{I_F}/\mathrm{tor}$. Then $\kappa_{T,\breve{F}}(t) = \kappa_{T,\breve{F}}(t_1) + \breve{\mu}$. Also $t \bmod \breve{T}_m \mapsto (t_1' \bmod \breve{T}'_m)(n'_{\breve{\mu}^t} \bmod \breve{T}'_m)$ for a suitable $t_1' \in T'(\breve{F}')_b$. Then $\kappa_{T',\breve{F}'}(t_1' n'_{\breve{\mu}^t}) = \kappa_{T',\breve{F}'}(t_1') + \breve{\mu}$. By the preceding paragraph, we see that $\kappa_{T,\breve{F}}(t_1) = \kappa_{T',\breve{F}'}(t_1')$. Hence $\breve{\mathscr{T}}_m$ is compatible with the Kottwitz homomorphism $\kappa_{T,\breve{F}}$. Also $\mathscr{T}_m$ is compatible with $\kappa_{T,F}$.

2. Modifications to [Ganapathy 2022, Section 3]

2A. ***Modifications to*** [Ganapathy 2022, Section 3A]**.** The correction given in Section 1 leads to some corrections in [Ganapathy 2022, Section 3]. One important modification is that we need to replace the set of representatives $\{n_{\breve{\lambda}} \mid \breve{\lambda} \in X_*(T)_{I_F}\}$ and $\{n_{\breve{\lambda}_{\mathrm{ad}}} \mid \breve{\lambda} \in X_*(T_{\mathrm{ad}})_{I_F}\}$ used in the proofs in [Ganapathy 2022, Section 3A] with the set of representatives given in Lemma 2.1. Let M, M^*, A, S, T, B and σ be as in [Ganapathy 2022, Section 3]. So M^* is an inner form of a quasisplit connected, reductive group M with $M_{\mathrm{ad}} \cong \mathrm{Res}_{L/F}\,\mathrm{PGL}_n$ for a finite separable extension L/F. Let $\widetilde{F} \supset L\breve{F}$ be the splitting extension of $T_{\breve{F}}$. Let $e = [L : L \cap \breve{F}]$ and $f = [L \cap \breve{F} : F]$. Fix a uniformizer $\varpi_{\widetilde{F}}$ of $\widetilde{F}$.

Lemma 2.1. *Let* $\omega_{T,\breve{F}} : T(\breve{F}) \to X_*(T)_{I_F}/\mathrm{tor}$ *and* $\omega_{T_{\mathrm{ad}},\breve{F}} = \kappa_{T_{\mathrm{ad}},\breve{F}} : T_{\mathrm{ad}}(\breve{F}) \to X_*(T_{\mathrm{ad}})_{I_F}$ *be the valuation homomorphisms on* T *and* T_{ad}, *respectively. There exist group-theoretic sections* $\nabla_{T,\breve{F}} : X_*(T)_{I_F}/\mathrm{tor} \to T(\breve{F})$ *and* $\nabla_{T_{\mathrm{ad}},\breve{F}} : X_*(T_{\mathrm{ad}})_{I_F} \to T_{\mathrm{ad}}(\breve{F})$ *of* $\omega_{T,\breve{F}}$ *and* $\omega_{T_{\mathrm{ad}},\breve{F}}$, *respectively, such that* $\nabla_{T,\breve{F}}$ *and* $\nabla_{T_{\mathrm{ad}},\breve{F}}$ *agree on the subset* $X_*(T_{\mathrm{sc}})_{I_F}$.

Proof. Let us begin by noting that $X_*(T_{\mathrm{ad}})$ has a $\mathbb{Z}$-basis permuted by Γ_F and $X_*(T_{\mathrm{ad}})_{I_F}$ is torsion-free and admits a $\mathbb{Z}$-basis permuted by σ. Note that $M_{\mathrm{ad},\widetilde{F}} = \prod_{1 \le i \le e, 1 \le j \le f} M^{(i,j)}_{\mathrm{ad},\widetilde{F}}$ where each $M^{(i,j)}_{\mathrm{ad},\widetilde{F}} \cong \mathrm{PGL}_n/\widetilde{F}$. Following the notation of

[Bourbaki 2002], for $1 \le i \le e$, $1 \le j \le f$, let

$$\tilde{\lambda}_{\mathrm{ad},n-1}^{(i,j)} = \epsilon_1^{(i,j)} - \frac{1}{n}(\epsilon_1^{(i,j)} + \epsilon_2^{(i,j)} + \cdots + \epsilon_n^{(i,j)}),$$

and, for $1 \le k \le n-2$,

$$\tilde{\lambda}_{\mathrm{ad},k}^{(i,j)} = \epsilon_k^{(i,j)} - \epsilon_{k+1}^{(i,j)}.$$

The set

$$\{\tilde{\lambda}_{\mathrm{ad},k}^{(i,j)} \mid 1 \le k \le n-1,\ 1 \le i \le e,\ 1 \le j \le f\}$$

yields a $\mathbb{Z}$-basis of $X_*(T_{\mathrm{ad}})$. Let $\mathrm{pr} : X_*(T_{\mathrm{ad}}) \to X_*(T_{\mathrm{ad}})_{I_F}$ be the natural projection. For $1 \le k \le n-1$ and $1 \le j \le f$, let $\breve{\lambda}_{\mathrm{ad},k}^{(j)} = \mathrm{pr}(\tilde{\lambda}_{\mathrm{ad},k}^{(1,j)})$. Then the set

$$\{\breve{\lambda}_{\mathrm{ad},k}^{(j)} \mid 1 \le k \le n-1,\ 1 \le j \le f\}$$

yields a $\mathbb{Z}$-basis of $X_*(T_{\mathrm{ad}})_{I_F}$. Let

$$n_{\breve{\lambda}_{\mathrm{ad},k}^{(j)}} = \mathrm{Nm}_{\tilde{F}/\breve{F}}\, \tilde{\lambda}_{\mathrm{ad},k}^{(1,j)}(\varpi_{\tilde{F}}), \quad 1 \le k \le n-2, \quad \text{and} \quad n_{\breve{\lambda}_{\mathrm{ad},n-1}^{(j)}} = \mathrm{Nm}_{\tilde{F}/\breve{F}}\, \tilde{\lambda}_{\mathrm{ad},n-1}^{(1,j)}(\varpi_{\tilde{F}}).$$

The elements $n_{\breve{\lambda}_{\mathrm{ad},k}^{(j)}}$, $1 \le k \le n-1$, $1 \le j \le f$, are used to obtain a set of representatives

$$\{n_{\breve{\lambda}_{\mathrm{ad}}} \mid \breve{\lambda}_{\mathrm{ad}} \in X_*(T_{\mathrm{ad}})_{I_F}\}$$

that form a group; see Lemma 1.1. Let $\nabla_{T_{\mathrm{ad}},\breve{F}} : X_*(T_{\mathrm{ad}})_{I_F} \to T_{\mathrm{ad}}(\breve{F})$, $\breve{\lambda}_{\mathrm{ad}} \mapsto n_{\breve{\lambda}_{\mathrm{ad}}}$, denote this group-theoretic section of $\omega_{T_{\mathrm{ad}},\breve{F}}$.

Next note that $X_*(T_{\mathrm{sc}})_{I_F} \subset X_*(T)_{I_F}/\mathrm{tor}$. Hence the elements $\breve{\lambda}_{\mathrm{ad},k}^{(j)}$, $1 \le k \le n-2$, $1 \le j \le f$, lie in $X_*(T)_{I_F}/\mathrm{tor}$. Also, $j(X_*(T)_{I_F}/\mathrm{tor})$ is of finite index in $X_*(T_{\mathrm{ad}})_{I_F}$, so there exists a nonnegative integer r, which we may choose as small as possible, such that for each $1 \le j \le f$, $r \cdot \breve{\lambda}_{\mathrm{ad},n-1}^{(j)} = j(\breve{\lambda}_{n-1}^{(j)})$ for a $\breve{\lambda}_{n-1}^{(j)} \in X_*(T)_{I_F}/\mathrm{tor}$. For the same r, there exists $\tilde{\lambda}_{n-1}^{(1,j)} \in X_*(T)$ such that $j(\tilde{\lambda}_{n-1}^{(1,j)}) = r \cdot \tilde{\lambda}_{\mathrm{ad},n-1}^{(1,j)}$ and $\mathrm{pr}(\tilde{\lambda}_{n-1}^{(1,j)}) = \breve{\lambda}_{n-1}^{(j)}$. For $1 \le k \le n-2$,

$$\tilde{\lambda}_{\mathrm{ad},k}^{(1,1)} \in X_*(T), \quad \mathrm{pr}(\tilde{\lambda}_{\mathrm{ad},k}^{(1,1)}) = \breve{\lambda}_{\mathrm{ad},k}^{(1)} \quad \text{and} \quad j(\breve{\lambda}_{\mathrm{ad},k}^{(1)}) = \breve{\lambda}_{\mathrm{ad},k}^{(1)}.$$

Set

$$n_{\breve{\lambda}_{\mathrm{ad},k}^{(j)}} = \mathrm{Nm}_{\tilde{F}/\breve{F}}\, \tilde{\lambda}_{\mathrm{ad},k}^{(1,j)}(\varpi_{\tilde{F}}), \quad 1 \le k \le n-2, \quad \text{and} \quad n_{\breve{\lambda}_{n-1}^{(j)}} = \mathrm{Nm}_{\tilde{F}/\breve{F}}\, \tilde{\lambda}_{n-1}^{(1,j)}(\varpi_{\tilde{F}}).$$

Now, the set $\{\breve{\lambda}_{\mathrm{ad},k}^{(j)} \mid 1 \le k \le n-2,\ 1 \le j \le f\} \cup \{\breve{\lambda}_{n-1}^{(j)} \mid 1 \le j \le f\}$ is $\mathbb{Z}$-linearly independent. Further, it may be extended to a basis of $X_*(T)_{I_F}/\mathrm{tor}$. For the remaining basis elements of $X_*(T)_{I_F}/\mathrm{tor}$, we choose representatives as in Section 1A. This then yields a set of representatives $\{n_{\breve{\lambda}} \mid \breve{\lambda} \in X_*(T)_{I_F}/\mathrm{tor}\}$ that forms a group. Let $\nabla_{T,\breve{F}} : X_*(T)_{I_F}/\mathrm{tor} \to T(\breve{F})$, $\breve{\lambda} \to n_{\breve{\lambda}}$ denote this group-theoretic section of $\omega_{T,\breve{F}}$. By construction, we have $\nabla_{T,\breve{F}}$ and $\nabla_{T_{\mathrm{ad}},\breve{F}}$ agree on $X_*(T_{\mathrm{sc}})_{I_F}$. This finishes the proof of the lemma. $\square$

Lemmas 3.1 and 3.2 in [Ganapathy 2022] are not affected.

Let $\Omega_{\breve{M}}$ and $\Omega_{\breve{M},\mathrm{ad}}$ be as in [Ganapathy 2022, Section 3A]. We fix a σ-stable alcove $\breve{a}$ in $\mathscr{A}(S, \breve{F})$ and identify $\Omega_{\breve{M}}$ with $\Omega_{\breve{a}}$ and $\Omega_{\breve{M}_{\mathrm{ad}}}$ with $\Omega_{\breve{a},\mathrm{ad}}$. Let $\breve{\nu}_{\mathrm{ad}} = t_{\breve{\eta}_{\mathrm{ad}}}\breve{z}$ be as in [Ganapathy 2022, Section 3A]. With notation as in Lemma 2.1, $\breve{\eta}_{\mathrm{ad}} = \breve{\lambda}^{(1)}_{\mathrm{ad},n-1}$. Let $\breve{z} = \breve{z}^{(1)} = s_1^{(1)} \cdots s_{n-1}^{(1)}$. Let $n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}} \in T_{\mathrm{ad}}(\breve{F})$ be as in Lemma 2.1. We fix a system of pinnings $\{x_{\breve{a}} \mid \breve{a} \in \breve{\Phi}(M, S)\}$ that is σ-stable as in [Ganapathy 2022, Section 3A]. Let $n_{\breve{z}^{(1)}} = n_{s_1^{(1)}} \cdots n_{s_{n-1}^{(1)}}$. Let $\sigma^* = \mathrm{Ad}(n_{\breve{\nu}_{\mathrm{ad}}}) \circ \sigma$ where $n_{\breve{\nu}_{\mathrm{ad}}} = n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}} n_{\breve{z}^{(1)}}$, and let $M^* = M^{\sigma^*}_{\breve{F}}$. Let $\Omega_M = \Omega^{\sigma}_{\breve{M}}$ and $\Omega_{M^*} = \Omega^{\sigma^*}_{\breve{M}}$. Similarly define $\Omega_{M,\mathrm{ad}}$ and $\Omega_{M^*,\mathrm{ad}}$. By [Ganapathy 2022, Lemma 3.2] we have $\Omega_M = \Omega_{M^*}$ and $\Omega_{M,\mathrm{ad}} = \Omega_{M^*,\mathrm{ad}} \cong \mathbb{Z}/n\mathbb{Z}$. The group $j(\Omega_M) \subset \Omega_{M,\mathrm{ad}}$ is cyclic. Assume $[\Omega_{M,\mathrm{ad}} : j(\Omega_M)] = r$ and that $j(\Omega_M) \neq 0$. Let $\breve{\tau}_0 \in \Omega_M \subset \Omega^{\sigma}_{\breve{M}}$ be such that $j(\breve{\tau}_0)$ is a generator of $j(\Omega_M)$. Then $j(\breve{\tau}_0) = \breve{\nu}^r_{\mathrm{ad}}\sigma(\breve{\nu}_{\mathrm{ad}})^r \cdots \sigma^{k-1}(\breve{\nu}_{\mathrm{ad}})^r$. Write $\breve{\tau}_0 = t_{\breve{\lambda}_0}\breve{y}_0$, where $\breve{\lambda}_0 \in X_*(T)^{\sigma}_{I_F}$ and $\breve{y}_0 \in W(M, S)$. Note that $\breve{y}_0 = (\breve{z}^{(1)})^r\sigma(\breve{z}^{(1)})^r \cdots \sigma^{f-1}(\breve{z}^{(1)})^r$. We may and do assume that $\breve{\lambda}_0 \in (X_*(T)_{I_F}/\mathrm{tor})^{\sigma}$. Let $n_{\breve{\lambda}_0} \in T(\breve{F})$ be as in Lemma 2.1. Note that $n_{\breve{\lambda}_0}$ may not be fixed by σ. Let $n_{\breve{y}_0} := n^r_{\breve{z}^{(1)}}\sigma(n^r_{\breve{z}^{(1)}}) \cdots \sigma^{f-1}(n^r_{\breve{z}^{(1)}})$.

Lemma 2.2. *Let $\breve{\tau}_0$ be as in the preceding paragraph. There exists $v \in T(\breve{F})_1$ such that $n_{\breve{\tau}_0} = vn_{\breve{\lambda}_0}n_{\breve{y}_0} \in M^*(F)$ and $\kappa_{M^*,F}(n_{\breve{\tau}_0}) = \breve{\tau}_0$.*

Proof. Recall that we have fixed representatives $\{n_{\breve{\lambda}} \mid \breve{\lambda} \in X_*(T)_{I_F}/\mathrm{tor}\}$ that forms a group. Note that $\sigma(\breve{\lambda}_0) = \breve{\lambda}_0$ and $\sigma(\breve{y}_0) = \breve{y}_0$. Let us compute $\sigma^*(n_{\breve{\lambda}_0}n_{\breve{y}_0})$. Using the definition of $n_{\breve{y}_0}$, we have $\sigma(n_{\breve{y}_0}) = n_{\breve{y}_0}$. Using [Ganapathy 2022, Lemma 3.1(b)], we have

$$\sigma^*(n_{\breve{\lambda}_0}n_{\breve{y}_0}) = \sigma^*(n_{\breve{\lambda}_0})n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}-\breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})}n_{\breve{y}_0}.$$

Now,

$$u = \sigma^*(n_{\breve{\lambda}_0})n^{-1}_{\sigma^*(\breve{\lambda}_0)} \in T(\breve{F})_1$$

since its image under $\kappa_{T,\breve{F}}$ is 0. Since $H^1(\sigma^*, T(\breve{F})_1) = 1$, there exists $v \in T(\breve{F})_1$ such that $\sigma^*(v)v^{-1} = u^{-1}$. Now $\sigma^*(vn_{\breve{\lambda}_0}) = vu^{-1}\sigma^*(n_{\breve{\lambda}_0}) = vn_{\sigma^*(\breve{\lambda}_0)}$. Then

$$\sigma^*(vn_{\breve{\lambda}_0}n_{\breve{y}_0}) = vn_{\sigma^*(\breve{\lambda}_0)}n_{\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}-\breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})}n_{\breve{y}_0} = vn_{\sigma^*(\breve{\lambda}_0)+\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}-\breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})}n_{\breve{y}_0} = vn_{\breve{\lambda}_0}n_{\breve{y}_0}.$$

The second equality follows from Lemma 2.1 and that $\breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - \breve{y}_0(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1}) \in X_*(T_{\mathrm{sc}})_{I_F} \subset X_*(T)_{I_F}/\mathrm{tor}$. To get the third equality, note that from the proof of [Ganapathy 2022, Lemma 3.2], $\sigma^*(\breve{\lambda}_0) - \sigma(\breve{\lambda}_0) = \breve{\lambda}^{(1)}_{\mathrm{ad},n-1} - (\mathrm{Ad}(z^{(1)})(\breve{y}_0))(\breve{\lambda}^{(1)}_{\mathrm{ad},n-1})$ but $\sigma(\breve{\lambda}_0) = \breve{\lambda}_0$ and $\mathrm{Ad}(z^{(1)})(\breve{y}_0) = \breve{y}_0$. This finishes the proof of the lemma. □

Now, given $\breve{\tau} = t_{\breve{\lambda}}\breve{w} \in \Omega_M$ with $t_{\breve{\lambda}} \in X_*(T)_{I_F}$ and $\breve{w} \in W(M, S)$, we have $j(\breve{\tau}) = sj(\breve{\tau}_0)$ for a unique nonnegative integer s with $0 \le s < n/r$. Let $\breve{\mu} = \breve{\tau} - s\breve{\tau}_0$. Write $\breve{\mu} = t_{\breve{\mu}_0} \cdot \breve{w}_0 \in \Omega_M$. Then $j(\breve{\tau}) = sj(\breve{\tau}_0)$ implies that $\breve{w} = \breve{y}_0^s$, so $\breve{w}_0 = 1$ and the element $\breve{\mu}$ is just given by the translation $t_{\breve{\mu}_0} \in X_*(T)_{I_F}$. We identify $\breve{\mu}$ and $\breve{\mu}_0$. Since σ fixes $\breve{\tau}$ and $\breve{\tau}_0$, we have $\sigma(\breve{\mu}) = \breve{\mu}$. We claim that $\sigma^*(\breve{\mu}) = \breve{\mu}$. To see this, note that since $j(\breve{\mu})=0$, we have that $j(\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})-\breve{\mu})=0$, but since $\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})-\breve{\mu}\in X_*(T_{\mathrm{sc}})_{I_F}$, and since j acts as identity on $X_*(T_{\mathrm{sc}})_{I_F}$, it follows that $\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})-\breve{\mu}=0$. This then implies that $\sigma^*(\breve{\mu})=\mathrm{Ad}(\breve{z}^{(1)})(\breve{\mu})=\breve{\mu}$. So $\breve{\mu}\in X_*(T)_{I_F}^{\sigma^*}$. Set $n_{\breve{\tau}}=n_{\breve{\mu}}n_{\breve{\tau}_0}^s$ with $n_{\breve{\mu}}\in T^*(F)$ satisfies $\kappa_{T^*,F}(n_{\breve{\mu}})=\breve{\mu}$.

Proposition 2.3. *Let $\breve{\tau}\in\Omega_{M^*}=\Omega_M$. Then $\sigma^*(n_{\breve{\tau}})=n_{\breve{\tau}}$. In particular, $n_{\breve{\tau}}\in M^*(F)$ and $\widetilde{p}:\Omega_{M^*}\to M^*(F)$, $\breve{\tau}\mapsto n_{\breve{\tau}}$, is a (set-theoretic) section of $\kappa_{M^*,F}$.*

Proof. It suffices to prove that $\sigma^*(n_{\breve{\tau}_0})=n_{\breve{\tau}_0}$, but this is Lemma 2.2. □

2B. ***Modifications to* [Ganapathy 2022, Section 3B].** Via Del_m, we have isomorphisms $X_*(T)\cong X_*(T')$ and $X_*(T_{\mathrm{ad}})\cong X_*(T'_{\mathrm{ad}})$ that are Γ_F/I_F^m-equivariant, and $\Omega_{\breve{M}}\cong\Omega_{\breve{M}'}$ and $\Omega_{\breve{M}_{\mathrm{ad}}}\cong\Omega_{\breve{M}'_{\mathrm{ad}}}$. We identify these groups via these isomorphisms. We construct $\nabla_{T',\breve{F}'}:X_*(T)_{I_F}/\mathrm{tor}\to T'(\breve{F}')$, $\breve{\lambda}\mapsto n'_{\breve{\lambda}}$, and $\nabla_{T'_{\mathrm{ad}},\breve{F}'}:X_*(T_{\mathrm{ad}})_{I_F}\to T'_{\mathrm{ad}}(\breve{F}')$, $\breve{\lambda}\mapsto n_{\breve{\lambda}}$, exactly as in Lemma 2.1, but with $\varpi_{\widetilde{F}}$ replaced with $\varpi_{\widetilde{F}'}$ where $\varpi_{\widetilde{F}} \bmod \mathfrak{p}_{\widetilde{F}}^{rm}\mapsto\varpi_{\widetilde{F}'} \bmod \mathfrak{p}_{\widetilde{F}}'^{rm}$ as in Lemma 1.2. Let $\breve{\tau}_0$ be as in Lemma 2.2. Let $n'_{\breve{\lambda}_0}, n'_{\sigma^*(\breve{\lambda}_0)}\in T'(\breve{F}')$ be such that under $\breve{\mathscr{T}}_m$, $n_{\breve{\lambda}_0} \bmod \breve{T}_m\mapsto n'_{\breve{\lambda}_0} \bmod \breve{T}'_m$, and similarly for $n'_{\sigma^*(\breve{\lambda}_0)}$. Then, since $\breve{\mathscr{T}}_m$ is σ^*-equivariant, we have $u \bmod \breve{T}_m\mapsto u' \bmod \breve{T}'_m$, where $u'=\sigma'^*(n'_{\breve{\lambda}_0})n'^{-1}_{\sigma'^*(\breve{\lambda}_0)}$. By the proof of the fact that $H^1(\sigma^*,T(\breve{F})_1)=1$ [Serre 1979, Chapter XII, §3, Lemma 3], it follows that we may choose $v'\in T(\breve{F})_1$ such that $\sigma^*(v')v'^{-1}=u'^{-1}$ and such that $v \bmod \breve{T}_m\mapsto v' \bmod \breve{T}'_m$. Let $n'_{\breve{y}_0}=n'^{r}_{\breve{z}^{(1)}}\sigma'(n'_{\breve{z}^{(1)}})^r\cdots\sigma^{f-1}(n'_{\breve{z}^{(1)}})^r$. Set $n'_{\breve{\tau}_0}=v'n'_{\breve{\lambda}_0}n'_{\breve{y}_0}$. Given $\breve{\tau}\in\Omega_M$, we may write $\breve{\tau}=\breve{\mu}+s\breve{\tau}_0$ for a unique $0\le s<n/r$ as in the paragraph preceding Proposition 2.3. Set $n'_{\breve{\tau}}=n'_{\breve{\mu}}n'^{s}_{\breve{\tau}_0}$ where $n'_{\breve{\mu}}\in T'^*(F')$ with $\mathscr{T}_m^*(n_{\breve{\mu}} \bmod T_m^*)\mapsto n'_{\breve{\mu}} \bmod T'^*_m$. Note that $\kappa_{T'^*,F'}(n'_{\breve{\mu}})=\breve{\mu}$ by Section 1B. By Proposition 2.3, $n'_{\breve{\tau}}\in M'^*(F')$.

Proposition 2.4 [Ganapathy 2022, Proposition 3.4]. *Let $m\ge 1$ and let $e\ge m+4h$. If the fields F and F' are e-close, then we have an isomorphism $M^*(F)/M_m^*\cong M'^*(F')/M'^*_m$.*

Proof. The proof given in [Ganapathy 2022, Proposition 3.4] works with straightforward modifications.

Consider the set theoretic section $\widetilde{p}:\Omega_{M^*}\to M^*(F)$ in Proposition 2.3 and let p be its composition with the natural projection $M^*(F)\to M^*(F)/M_m^*$. Similarly, we get $\tilde{p}':\Omega_{M^*}\xrightarrow{\widetilde{p}} M'^*(F')$ and p'.

It suffices to prove that the sections p and p' satisfy (a) and (b) of [Ganapathy 2022, Proposition 3.4].

To see (a), it suffices to prove that

$$\begin{array}{ccc} M^*(F)_1/M_m^* & \xrightarrow{\cong} & M'^*(F')_1/M'^*_m \\ \big\downarrow{\scriptstyle \mathrm{Inn}(n_{\breve{\tau}})} & & \big\downarrow{\scriptstyle \mathrm{Inn}(n'_{\breve{\tau}})} \\ M^*(F)_1/M_m^* & \xrightarrow{\cong} & M'^*(F')_1/M'^*_m \end{array}$$

is commutative for $\breve{\tau} \in \Omega_{M^*}$. Let $\breve{P}$ be the Iwahori subgroup of $M(\breve{F})$ $(= M^*(\breve{F}))$ attached to the σ-stable alcove $\breve{a}$ and let $\breve{P}'$ be the corresponding Iwahori subgroup of $M'(\breve{F}')$. Then by [Ganapathy 2019, Theorem 4.5], we have that $\breve{P}/\breve{P}_m \cong \breve{P}'/\breve{P}'_m$. Since $\breve{\nu}_{\mathrm{ad}} \in \Omega_{\breve{a},\mathrm{ad}}$, the alcove $\breve{a}$ is also σ^*-stable. By Propositions 4.10 and 6.2 in [Ganapathy 2019], the isomorphism $\breve{P}/\breve{P}_m \cong \breve{P}'/\breve{P}'_m$ is σ- and σ^*-equivariant. This implies that $\breve{P} \cap M^*(F) = M^*(F)_1$, $\breve{P}_m \cap M^*(F) = M^*_m$ and similarly that $\breve{P}' \cap M'^*(F') = M'^*(F')_1$, $\breve{P}'_m \cap M'^*(F') = M'^*_m$. Since $\breve{\tau} \in \Omega_{M^*} = \Omega_{\breve{a}}^{\sigma^*} \subset \Omega_{\breve{a}}$, we see that $n_{\breve{\tau}}$ normalizes $\breve{P}$ and $\breve{P}_m$. To finish the proof of (a), it suffices to observe that the following diagram is commutative:

$$\begin{tikzcd} \breve{P}/\breve{P}_m \arrow[r, "\cong"] \arrow[d, "\mathrm{Inn}(n_{\breve{\tau}})"] & \breve{P}'/\breve{P}'_m \arrow[d, "\mathrm{Inn}(n'_{\breve{\tau}})"] \\ \breve{P}/\breve{P}_m \arrow[r, "\cong"] & \breve{P}'/\breve{P}'_m \end{tikzcd}$$

This follows by arguing as in the proof of [Ganapathy 2019, Proposition 6.2].

Let us prove (b). The element $n_{\breve{y}_0}^{n/r}$ equals $\breve{a}^\vee(-1) \in M^*(F)_1$ for a suitable $\breve{a} \in \breve{\Phi}(M, S)$.

Let $\breve{\tau}_1, \breve{\tau}_2 \in \Omega_{M^*}$. As in the proof of Proposition 2.3, write $\breve{\tau}_i = \breve{\mu}_i + s_i\breve{\tau}_0$, and $\breve{\tau}_1 + \breve{\tau}_2 = \breve{\mu} + s\breve{\tau}_0$. Note that $s \bmod (n/r) \equiv s_1 + s_2 \bmod (n/r)$.

Recall that $n_{\breve{\mu}}, n_{\breve{\mu}_1}, n_{\breve{\mu}_2} \in T^*(F)$ and $n_{\breve{\mu}} \bmod T^*_m \mapsto n'_{\breve{\mu}} \bmod T'^*_m$ and for $i = 1, 2$, $n_{\breve{\mu}_i} \bmod T^*_m \mapsto n'_{\breve{\mu}_i} \bmod T'^*_m$. Write $n^s_{\breve{\tau}_0} = t_s n^s_{\breve{y}_0}$ where $t_s \in T(\breve{F})$. Similarly write $n'^s_{\breve{\tau}_0} = t'_s n'^s_{\breve{y}_0}$. Then it is straightforward to see that $t_s \bmod \breve{T}_m \mapsto t'_s \bmod \breve{T}'_m$ via $\breve{\mathscr{T}}_m$. The same claim holds for t_{s_i}, $i = 1, 2$. Also, $\breve{a}^\vee(-1) \bmod \breve{T}_m \to \breve{a}'^\vee(-1) \bmod \breve{T}_m$. Finally, we note that $n_{\breve{\tau}_1+\breve{\tau}_2} n^{-1}_{\breve{\tau}_1} n^{-1}_{\breve{\tau}_2} \in M^*(F)_1 \cap T(\breve{F})$ and by [Ganapathy 2019, Proof of Proposition 6.2 and Corollary 6.3], we see that on the subgroup $M^*(F)_1 \cap T(\breve{F})$ the isomorphism of [Ganapathy 2019, Corollary 6.3] restricts to $\mathscr{T}^*_m$. Hence the sections p, p' satisfy (b). □

Acknowledgements

I thank Sandeep Varma for asking me some questions about [Ganapathy 2022] that allowed me to identify the error there. I thank Siyan Daniel Li-Huerta, Dipendra Prasad, and Sandeep Varma for their feedback on previous drafts of this note. I was supported through the DST-SERB grant SPF/2022/000142.

References

[Aubert and Varma 2024] A.-M. Aubert and S. Varma, "On congruent isomorphisms of tori", preprint, 2024. arXiv 2401.08306

[Bourbaki 2002] N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer, Berlin, 2002. MR Zbl

[Chai and Yu 2001] C.-L. Chai and J.-K. Yu, "Congruences of Néron models for tori and the Artin conductor", *Ann. of Math.* (2) **154**:2 (2001), 347–382. MR Zbl

[Ganapathy 2019] R. Ganapathy, "Congruences of parahoric group schemes", *Algebra Number Theory* **13**:6 (2019), 1475–1499. MR Zbl

[Ganapathy 2022] R. Ganapathy, "A Hecke algebra isomorphism over close local fields", *Pacific J. Math.* **319**:2 (2022), 307–332. MR Zbl

[Kottwitz 1997] R. E. Kottwitz, "Isocrystals with additional structure, II", *Compositio Math.* **109**:3 (1997), 255–339. MR Zbl

[Serre 1979] J.-P. Serre, *Local fields*, Graduate Texts in Mathematics **67**, Springer, Berlin, 1979. MR Zbl

Received April 4, 2024.

RADHIKA GANAPATHY
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF SCIENCE
BENGALURU
INDIA
radhikag@iisc.ac.in

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 330 No. 2 June 2024