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ON RELATIVE COMMUTANTS OF SUBALGEBRAS IN GROUP AND TRACIAL CROSSED PRODUCT VON NEUMANN ALGEBRAS

TATTWAMASI AMRUTAM AND JACOPO BASSI

Let Γ be a discrete group acting on a compact Hausdorff space X. Given $x \in X$ and $\mu \in \mathrm{Prob}(X)$, we introduce the notion of contraction of μ towards x with respect to unitary elements of a group von Neumann algebra not necessarily coming from group elements. Using this notion, we study relative commutants of subalgebras in tracial crossed product von Neumann algebras. The results are applied to negatively curved groups and $\mathrm{SL}(d, \mathbb{Z})$ for $d \geq 2$.

1. Introduction

Operator algebras associated with discrete groups, or more generally discrete group actions, reveal essential properties of the underlying group. Probably the first evidence of this connection is that amenability has neat characterizations at the operator-algebraic level: injectivity of the group von Neumann algebra and nuclearity of the reduced group C^*-algebra. Nowadays, it is known that many other group properties have analog descriptions in terms of group C^*-algebras, for example, a-T-menability and property T [20]. Also, free groups' full and reduced C^*-algebras can detect their order. It is a significant open problem raised by A. Connes whether nonisomorphic ICC property T groups have nonisomorphic von Neumann algebras (see [24] for examples of ICC property (T) groups with nonisomorphic von Neumann algebras).

Dynamical systems represent a powerful tool for the study of rigidity properties of groups. As an example, rigidity results for certain discrete subgroups of $\mathrm{SL}(2, \mathbb{R})$ can be obtained by looking at the C^*-crossed products associated with certain actions (see [9; 10; 29]). Among the possible dynamical systems, an important role is played by boundary actions; for example, the topological amenability of the left action of a discrete group Γ on its Stone-Čech boundary $\partial_\beta\Gamma$ is equivalent to exactness and topological amenability of the left-right action on the same space

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(which is usually referred to as biexactness or property $\mathcal{S}$) implies the Akemann–Ostrand (AO) property [4], (i.e., temperedness of the representation of $\Gamma \times \Gamma$ on the Calkin algebra of $l^2(\Gamma)$), which ensures solidity (hence primeness) of the group von Neumann algebra. A significant open problem in the theory is deciding whether these three properties coincide (see [11]).

The solidity of the group von Neumann algebra is a very rigid property, which, for example, captures to some extent the dimension of the ambient group in the case of a lattice in a simple Lie group: it is automatic for discrete subgroups of simple rank-1 Lie groups, and it is automatically denied by the existence of infinite subgroups with nonamenable centralizers. Weakenings of the AO property have been considered in the literature and lead to the definition of properly proximal groups [18], for which some weaker rigidity properties hold as well. More recently, the notion of biexact von Neumann algebra was introduced in [26], where examples of von Neumann algebras that are solid but not biexact were given. However, it is still not known if there are nonbiexact groups that give rise to solid von Neumann algebras.

One of the most significant breakthroughs of recent years is the recognition of the central role of proximality arguments in the study of rigidity properties of discrete groups through the lens of dynamical systems, which lead, for example, to the identification of the Furstenberg boundary of a discrete group with the equivariant Hamana-injective envelope of the complex numbers (see, for example, [12; 33; 38]).

Given a probability measure, $\mu \in \text{Prob}(X)$, proximality of μ is nothing but the contraction of this measure with a specific sequence of group elements. In addition, if these group elements leave every finite subset of the group Γ, then it can be shown that the corresponding unitary elements converge to zero weakly. We generalize this notion to the context of general unitary elements inside the group von Neumann algebra, which does not necessarily come from group elements.

Definition 2.3. *Let Γ be a discrete countable group acting on a compact Hausdorff space X endowed with a probability measure μ. Let (u_n) be a sequence of unitaries in $L\Gamma$. We say that u_n is μ-contracting (towards a point $x \in X$) if for every $\epsilon > 0$, every $F \subset \Gamma$ finite and for every $f \in C(X)$ there is N such that for every $n \geq N$ we have $\|u_n|_A\|_2 > 1 - \epsilon$, where $A = \{\gamma \in \Gamma : |\gamma^{-1}\eta\mu(f) - f(x)| < \epsilon \text{ for all } \eta \in F\}$.*

If u_n's come from the group elements, then the notion of μ-contraction agrees with that of the Γ-contraction. Motivated by the notion of solid von Neumann algebra, we employ proximality arguments to study the position of relative commutants of subalgebras in group (and more generally crossed product) von Neumann algebras.

Theorem 1.1. *Let Γ be a discrete countable group acting on a compact Hausdorff space X. Let $\{u_n\} \subset L(\Gamma)$ be a μ-contracting sequence for the action on X for some probability measure μ on X. Let $(\mathcal{N}, \tilde{\tau})$ be a tracial von Neumann algebra, and*

$\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$ *be a trace-preserving action. Then* $\{u_n : n \in \mathbb{N}\}' \cap (\mathcal{N} \rtimes \Gamma) \subset \mathcal{N} \rtimes \Gamma_x$, *where* x *is determined by the fact that* (u_n) *is* μ*-contracting towards* x, *and that* $\Gamma_x = \{s \in \Gamma : sx = x\}$.

Organization of the paper. We prove some preliminary technicalities in Section 2. In Section 3, using a suitable notion of convergence of a measure to a point under a sequence of unitaries, inspired by [18], we show that the commutant of certain subalgebras of tracial crossed product algebras is contained in the von Neumann algebra associated to the stabilizer of the limiting point. In Section 4, we consider the case of “negatively curved groups” and show that in this case, the position of the relative commutants of subgroup algebras is reminiscent of an averaging property, in the spirit of the Powers’ averaging property (see, for example, [3; 30; 39]). In Section 5, we consider the case of infinite subgroups of $\mathrm{SL}(d, \mathbb{Z})$, in which case we prove that the position of the relative commutant of a subgroup algebra depends on the dynamics of the subgroup in a particular partial flag. Per the results appearing in [14], we also see that a weak form of solidity holds for $\mathrm{SL}(3, \mathbb{Z})$.

The authors believe that the techniques developed in this manuscript should have a deep connection with the approach appearing in [15] and [13]. This connection will be investigated in a future work.

2. Preliminaries and technicalities

Let Γ be a discrete group. By a Γ-space X, (also denoted as $\Gamma \curvearrowright X$ sometimes) on a compact Hausdorff space X, we mean a group homomorphism $\pi : \Gamma \to \mathrm{Homeo}(X)$. We often abuse the notation by ignoring π and write sx instead of $\pi(s)\,x$ for $s \in \Gamma$ and $x \in X$.

Definition 2.1. Let Γ be a discrete countable group acting on a second countable compact Hausdorff space X endowed with a probability measure μ. A sequence $(\gamma_n) \subset \Gamma$ is said to be a μ*-pointwise contracting sequence* if there is $x \in X$ such that $\gamma_n y \to x$ for μ-almost every $y \in X$. In this case we say that (γ_n) is μ-pointwise-contracting towards x.

If μ is pointwise contracted by the sequence $\{\gamma_n\}$, then it is also pointwise contracted by $\{\gamma_n \gamma\}$ for any group element $\gamma \in \Gamma$. Moreover, the point where it converges is also unchanged. In other words, the contracting sequence is invariant with respect to the right multiplication by the group elements. We make this precise below.

Lemma 2.2. *Let* Γ *be a discrete countable group acting on a second countable compact space* X *and* μ *a* Γ*-quasiinvariant probability measure on* X. *Suppose that there is a* μ*-pointwise-contracting sequence* $(\gamma_n) \subset \Gamma$ *towards a point* $x \in X$. *Then for every* $\gamma \in \Gamma$ *we have that* $\lim_n \gamma_n \gamma \mu = \delta_x$ *exists and is independent of* γ.

Proof. There is $x \in X$ such that for μ-almost every $y \in X$ we have $\lim_n \gamma_n y \to x$. Let $E \subset X$ be the subset of μ-measure 0 such that $\lim_n \gamma_n y = x$ for every $y \notin E$. Since μ is quasiinvariant we have that for every $\gamma \in \Gamma$ there is a subset E' of measure zero, namely $\gamma^{-1}E$, such that $\lim_n \gamma_n \gamma y = x$ for every $y \notin E'$. Hence, for every $f \in C(X)$ and every $\gamma \in \Gamma$ we have $f(\gamma_n \gamma y) \to f(x)$ μ-almost everywhere. It follows from Lebesgue dominated convergence Theorem that $\lim_n \gamma_n \gamma \mu = \lim_n \gamma_n \mu$ in the weak*-topology. □

Group von Neumann algebra. We briefly recall the construction of the group von Neumann algebra. Let $\ell^2(\Gamma)$ be the space of square summable $\mathbb{C}$-valued functions on Γ. There is a natural action $\Gamma \curvearrowright \ell^2(\Gamma)$ by left translation:

$$\lambda_g \, \xi(h) := \xi(g^{-1}h), \quad \xi \in \ell^2(\Gamma), \qquad g, h \in \Gamma.$$

The group von Neumann algebra $L(\Gamma)$ is generated (as a von Neumann algebra inside $\mathbb{B}(\ell^2(\Gamma))$, by the left regular representation λ of Γ. The group von Neumann algebra $L(\Gamma)$ comes equipped with a canonical trace $\tau_0 : L(\Gamma) \to \mathbb{C}$ defined by

$$\tau_0(\lambda_g) = \begin{cases} 0 & \text{if } g \neq e, \\ 1 & \text{if } g = e. \end{cases}$$

It is worth noting that a natural embedding of $L(\Gamma)$ into $\ell^2(\Gamma)$ exists via the map $x \mapsto x\delta_e$. So, any element $x \in L(\Gamma)$ can be expressed as $x = \sum_{g \in \Gamma} x_g \lambda(g)$, where $\lambda(g) \in L(\Gamma)$ correspond to the canonical unitaries of $L(\Gamma)$ and $x_g = \tau_0(x\lambda(g)^*)$ are the Fourier coefficients of x. The above sum converges in ℓ^2-norm ($\|\cdot\|_2$) and not with respect to the strong operator or weak operator topology, as mentioned in [5, Remark 1.3.7]. This expansion is commonly referred to as the Fourier expansion of x.

We will make use of the following notion, which generalizes Definition 2.1 to sequences of unitaries in a group von Neumann algebra, which do not come from group elements.

Definition 2.3. Let Γ be a discrete countable group acting on a compact Hausdorff space X endowed with a probability measure μ. Let (u_n) be a sequence of unitaries in $L\Gamma$. We say that u_n is *μ-contracting* (towards a point $x \in X$) if for every $\epsilon > 0$, every $F \subset \Gamma$ finite and for every $f \in C(X)$ there is N such that for every $n \geq N$ we have $\|u_n|_A\|_2 > 1 - \epsilon$, where $A = \{\gamma \in \Gamma : |\gamma^{-1}\eta\mu(f) - f(x)| < \epsilon \text{ for all } \eta \in F\}$.

Lemma 2.4. *Let Γ be a discrete countable group acting on a second countable compact space X and μ a Γ-quasiinvariant probability measure on X. Let $\Lambda \subset \Gamma$ be a subgroup with the property that there is a point $x \in X$ such that every diverging sequence λ_n in Λ is μ-pointwise-contracting towards x. Then every sequence of unitaries in $L\Lambda$ which goes to zero weakly is μ-contracting towards x.*

Proof. Let $\epsilon > 0$. Since every diverging sequence in Λ is μ-contracting towards x, it follows from Lemma 2.2 that for every finite set $G \subset \Gamma$ and every $f \in C(X)$ there is a finite set $F \subset \Lambda$ satisfying $|\lambda^{-1}\gamma\mu(f) - f(x)| < \epsilon$ for every $\lambda \in \Lambda\backslash F$, $\gamma \in G$. Now, since $u_n \to 0$ weakly, there is $N \in \mathbb{N}$ such that $\|u_n|_F\|_2 < \epsilon$ for every $n > N$. The result follows. □

A group Γ is called a convergence group if it admits an action $\Gamma \curvearrowright X$ such that for every distinct sequence of elements $\{g_n\} \subset \Gamma$, we can find two elements $a, b \in X$ and a subsequence g_{n_k} such that $g_{n_k}|_{X\setminus\{b\}} \to a$ uniformly on every compact subset of $X \setminus \{b\}$ (see [40]). In this case, a is called the attracting point, and b is the repelling point. An element s in a convergence group Γ is called parabolic if s has exactly one fixed point on X. Moreover, an element s in a convergence group Γ is called loxodromic if it has exactly two fixed points on X, denoted by x_s^+ and x_s^-. Moreover, x_s^+ is the attractive point for s, and x_s^-, the repelling point (see [40, Lemma 2D]).

Example 2.5. Let Γ be a convergence group, and $s \in \Gamma$ a parabolic element. Let us call it x_s^+. Let $\Lambda = \langle s \rangle$. Using [40, Lemma 2F], we see that $\{s^n\}_{n\in\mathbb{Z}}$ is a convergence sequence with the attractive and repelling point of $\{s^n\}_{n\in\mathbb{Z}}$ the same as that of $\{x_s^+\}$. Let us assume that Γ is nonelementary, i.e., the set of limit points LX (the collection of all attracting points on X) has more than two points. Using [40, Theorem 2S], we see that LX is an infinite perfect set. Let μ be a Γ-quasiinvariant probability measure on X such that $\mu(x_s^+) \neq 0$. It follows from Definition 2.1 that every diverging sequence λ_n in Λ is μ-pointwise-contracting towards x_s^+. Let $\mathcal{M} \leq L(\Lambda)$ be a diffuse von Neumann subalgebra. Let $u_n \in \mathcal{M}$ be a sequence of unitaries in $\mathcal{M}$ which converges to 0 weakly. It follows from Lemma 2.4 that u_n is μ-contracting towards x_s^+. Similarly, if $s \in \Gamma$ is loxodromic, then there is a quasiinvariant probability measure μ on X and points x_s^+, x_s^- such that the sequence (s^n) is μ-pointwise-contracting towards x_s^+ and (s^{-n}) is μ-pointwise-contracting towards x_s^{-1}.

Now, suppose that X is a Γ-space, and $\mu \in \text{Prob}(X)$, the Poisson transformation $P_\mu : C(X) \to \mathbb{B}(\ell^2(\Gamma))$ is defined by $P_\mu(f)(\delta_t) = \mu(t^{-1}f)\,\delta_t$ for $t \in \Gamma$. It is well known that P_μ is a Γ-equivariant unital positive map. Whenever μ can be contracted using the unitaries, P_μ satisfies a kind of singularity phenomenon.

Lemma 2.6. *Let Γ be a discrete countable group acting on a compact Hausdorff space X. Let $\mu \in \text{Prob}(X)$. Let $\{u_n\}_n$ be a sequence of unitaries in $L(\Gamma)$ be such that $u_n \to 0$ weakly and is μ-contracting towards x. Then, $u_n P_\mu(f)\, u_n^* \xrightarrow{SOT} f(x)$ for every $f \in C(X)$.*

Proof. Let $\{u_n\}_n \in L(\Gamma)$ be a sequence of unitaries satisfying the above assumptions. Let us write $u_n = \sum_{t\in\Gamma} u_n(t)\,\lambda(t)$, where $u_n(t) = \tau_0(u_n\lambda(t)^*)$ for each $t \in \Gamma$.

Moreover, the convergence of the above series is in the $\|\cdot\|_2$-norm induced by the canonical trace τ_0. Moreover,

$$u_n^* = \sum_{t\in\Gamma} \overline{u_n(t)}\, \lambda(t^{-1}).$$

Let $f \in C(X)$ and $\xi \in l^2(\Gamma)$ be given. Choose $\epsilon > 0$. Let $M = \max\{\sup_{t\in\Gamma}|\xi(t)|, 1\}$. Choose ϵ' such that $2\epsilon' M\|f\|_\infty < \frac{\epsilon}{2}$. Let F be a finite subset of Γ such that

$$\sum_{t\notin F} |\xi(t)|^2 < (\epsilon')^2.$$

Choose also $\epsilon'' > 0$ such that

$$(\epsilon'' 2\|f\|_\infty + \epsilon'')\, M|F| < \frac{\epsilon}{2}.$$

Let $A = \{s \in \Gamma : s^{-1}t\mu(f) - f(x) < \epsilon \text{ for all } t \in F\}$. Since u_n is μ-contracting towards $x \in X$, we can find a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have

$$\|u_n|_A\|_2 > 1 - \epsilon''.$$

We now see that

$$\begin{aligned}
&\Big\|\sum_{t\in\Gamma}(u_n P_\mu(f)\, u_n^*\delta_t - f(x))\,\xi(t)\,\delta_t\Big\|_2 \\
&\overset{\text{(i)}}{\leq} \Big\|\sum_{t\in F}(u_n P_\mu(f)\, u_n^*\delta_t - f(x))\,\xi(t)\,\delta_t\Big\|_2 + \Big\|\sum_{t\notin F}(u_n P_\mu(f)\, u_n^*\delta_t - f(x))\,\xi(t)\,\delta_t\Big\|_2.
\end{aligned}$$

Let us observe that

$$\Big\|\sum_{t\notin F}(u_n P_\mu(f)\, u_n^*\delta_t - f(x))\,\xi(t)\,\delta_t\Big\|_2 \overset{\text{(ii)}}{\leq} 2\|f\|_\infty \sqrt{\sum_{t\notin F}|\xi(t)|^2}.$$

On the other hand,

$$\begin{aligned}
&\Big\|\sum_{t\in F}(u_n P_\mu(f)\, u_n^* - f(x))\,\xi(t)\,\delta_t\Big\|_2 \\
&\leq \sum_{t\in F}\|(u_n P_\mu(f)\, u_n^* - f(x))\,\xi(t)\,\delta_t\|_2 \\
&= \sum_{t\in F}\|(P_\mu(f)\, u_n^* - f(x)\, u_n^*)\,\xi(t)\,\delta_t\|_2 \\
&\leq \sum_{t\in F}\Bigg(\Big\|\sum_{s\notin A}\overline{u_n(s)}(P_\mu(f) - f(x))\lambda(s^{-1})\,\xi(t)\,\delta_t\Big\|_2 \\
&\qquad\qquad + \Big\|\sum_{s\in A}\overline{u_n(s)}(P_\mu(f) - f(x))\lambda(s^{-1})\,\xi(t)\,\delta_t\Big\|_2\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t\in F}\Bigg(\Big\|\sum_{s\notin A}\overline{u_n(s)}(P_\mu(f)-f(x))\,\xi(t)\,\delta_{s^{-1}t}\Big\|_2 \\
&\qquad\qquad + \Big\|\sum_{s\in A}\overline{u_n(s)}(P_\mu(f)-f(x))\,\xi(t)\,\delta_{s^{-1}t}\Big\|_2\Bigg) \\
&= \sum_{t\in F}\Bigg(\Big\|\sum_{s\notin A}\overline{u_n(s)}(s^{-1}t\mu(f)-f(x))\,\xi(t)\,\delta_{s^{-1}t}\Big\|_2 \\
&\qquad\qquad + \Big\|\sum_{s\in A}\overline{u_n(s)}(s^{-1}t\mu(f)-f(x))\,\xi(t)\,\delta_{s^{-1}t}\Big\|_2\Bigg) \\
&\overset{\text{(iii)}}{\le} \sum_{t\in F}\Bigg(2\|f\|_\infty\sqrt{\sum_{tu^{-1}\notin A}|\overline{u_n(tu^{-1})}|^2|\xi(t)|^2} \\
&\qquad\qquad + \sqrt{\sum_{tu^{-1}\in A}|\overline{u_n(tu^{-1})}|^2|(u\mu(f)-f(x))|^2|\xi(t)|^2}\Bigg).
\end{aligned}
$$

Since for all $t\in F$, $|(u\mu(f)-f(x))|<\epsilon$ for every u with $\sum_{s\notin A}|u_n(s)|^2<\epsilon''$ and $tu^{-1}\in A$, the inequality (iii) becomes less than or equal to

$$
\begin{aligned}
&\sum_{t\in F}\Bigg(2\|f\|_\infty M\epsilon''+\epsilon\sqrt{\sum_{tu^{-1}\in A}|\overline{u_n(tu^{-1})}|^2\,|\xi(t)|^2}\Bigg) \\
&\qquad \le \sum_{t\in F}\Bigg(2\|f\|_\infty M\epsilon''+M\epsilon\sqrt{\sum_{tu^{-1}\in A}|\overline{u_n(tu^{-1})}|^2}\Bigg) \overset{\text{(iv)}}{\le} |F|M(2\|f\|_\infty\epsilon''+\epsilon'').
\end{aligned}
$$

Hence for every $n\ge n_0$, combining the inequalities (i), (ii) and (iv), we obtain that

$$
\|u_nP_\mu(f)\,u_n^*\,\xi-f(x)\,\xi\|_2\le 2\|f\|_\infty\sqrt{\sum_{t\notin F}|\xi(t)|^2}+|F|M(2\|f\|_\infty\epsilon''+\epsilon'')<\epsilon.
$$

The claim follows. □

3. Crossed product of tracial von Neumann algebras

We apply the results of Section 2 in order to study the position of the relative commutants of certain subalgebras of crossed-product von Neumann algebras.

Theorem 3.1. *Let Γ be a discrete countable group acting on a compact Hausdorff space X. Let μ be a probability measure on X and let (u_n) be a μ-contracting sequence of unitaries in $L\Gamma$. Let $(\mathcal{N},\tilde\tau)$ be a tracial von Neumann algebra, and $\Gamma\curvearrowright(\mathcal{N},\tilde\tau)$ be a trace-preserving action. Then $\{u_n\}'\cap(\mathcal{N}\rtimes\Gamma)\subset\mathcal{N}\rtimes\Gamma_x$, where x is determined by the fact that (u_n) is μ-contracting towards x.*

We shall view $\mathcal{N}\rtimes\Gamma\subset\mathbb{B}(L^2(\mathcal{N},\tilde\tau)\bar\otimes\ell^2\Gamma)$. Moreover, Let X be a Γ-space. Let $P_\mu : C(X)\to\ell^\infty(\Gamma)$ be the Poisson transformation. This gives us a unital completely positive (ucp) Γ-equivariant map from $C(X)$ to $\ell^\infty(\Gamma)$. We can view $\ell^\infty(\Gamma)$ as multiplication operators on $\mathbb{B}(\ell^2(\Gamma))$. For $f\in\ell^\infty(\Gamma)$, the map $M(f):\ell^2(\Gamma)\to\ell^2(\Gamma)$ defined by $M(f)(\delta_t)=f(t)\,\delta_t$ is linear and bounded. Therefore, we obtain a ucp map $M \circ P_\mu : C(X) \to 1 \otimes \mathbb{B}(\ell^2(\Gamma)$. We see that every element of $M \circ P_\mu(C(X))$ commutes with $\mathcal{N}$. We will ignore M for the most part and write $P_\mu(f)$ for the ease of notation. We denote by τ, the canonical trace $\tilde{\tau} \circ \mathbb{E}$. Note that here $\mathbb{E} : \mathcal{N} \rtimes \Gamma \to \mathcal{N}$ is the canonical conditional expectation. Moreover, we consider the $\|\cdot\|_2$-norm induced by τ. We also denote by $\mathbb{E}_x$, the canonical conditional expectation from $\mathcal{N} \rtimes \Gamma$ to $\mathcal{N} \rtimes \Gamma_x$.

A crucial ingredient in these arguments is that the state obtained in the limiting stage satisfies some tracial property. While this is automatic in the case of amenable tracial von Neumann algebras, we cannot expect it to hold for us. Nonetheless, before we head on to the proof, we show the existence of an "almost-hypertrace", the last technical bit.

Proposition 3.2. *Suppose Γ is a discrete countable group acting on a compact Hausdorff space X. Let μ be a probability measure on X and $(u_n) \subset L\Gamma$ a μ-contracting sequence for the action on X. Let $(\mathcal{N}, \tilde{\tau})$ be a tracial von Neumann algebra, and $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$ be a trace-preserving action. Then, there exists a state $\psi \in S\big(\mathbb{B}(L^2(\mathcal{N}, \tilde{\tau})\overline{\otimes}\ell^2\Gamma)\big)$ such that $\psi|_{\mathcal{N}\rtimes\Gamma} = \tau$, $\psi|_{P_\mu(C(X))} = \delta_x$, and $\psi(aP_\mu(f)\, b) = \psi(P_\mu(f)\, ab)$ for all $a \in \{u_n\}'$ and for all $b \in \mathbb{B}(L^2(\mathcal{N}, \tilde{\tau})\overline{\otimes}\ell^2\Gamma)$. In particular, $P_\mu(C(X))$ falls in the multiplicative domain of ψ.*

Proof. Let τ denote the canonical trace $\tilde{\tau} \circ \mathbb{E}$. Let $P_\mu : C(X) \to \mathbb{B}(l^2\Gamma)$ be the Poisson map, which is ucp Γ-equivariant. Let $u_n \in L(\Gamma)$ be a μ-contracting sequence towards x. We identify an operator T on $\mathbb{B}(\ell^2(\Gamma))$ with $\mathrm{id} \otimes T$, which is an operator on $\mathbb{B}(H\overline{\otimes}\ell^2(\Gamma))$. In this way, we view $\mathbb{B}(\ell^2(\Gamma))$ as a Γ-invariant subalgebra of $\mathbb{B}(H\overline{\otimes}\ell^2(\Gamma))$. Moreover, if $T_n \in \mathbb{B}(\ell^2\Gamma)$ is a uniformly bounded sequence such that $T_n \xrightarrow[\mathbb{B}(\ell^2\Gamma)]{\text{SOT}} T$, then, $\mathrm{id} \otimes T_n \xrightarrow[\mathbb{B}(H\overline{\otimes}\ell^2\Gamma)]{\text{SOT}} \mathrm{id} \otimes T$. Therefore, using Lemma 2.6, we see that for every $f \in C(X)$,

$$u_n(P_\mu(f))\, u_n^* \xrightarrow[\text{SOT}]{n\to\infty} f(x) \cdot 1.$$

In particular, since $(u_n(P_\mu(f))\, u_n^*)$ is uniformly bounded, this implies that

$$u_n(P_\mu(f))(P_\mu(f))^* u_n^* \xrightarrow[\text{SOT}]{n\to\infty} f(x)\overline{f(x)} \cdot 1.$$

Consider now the (separable) C^*-algebra A generated by $P_\mu(C(X))$. Note that the state τ is of the form $\hat{1}_{\mathcal{N}} \otimes \delta_e$, and hence, is defined on $\mathbb{B}(L^2(\mathcal{N}, \tilde{\tau})\overline{\otimes}l^2\Gamma)$. Consider, after passing to a subnet if necessary, a weak* limit

$$\psi(\cdot) := \lim_n \tau \circ ad(u_n)(\cdot) = \lim_n \langle (\cdot)\hat{1}_{\mathcal{N}} \otimes \hat{u}_n, \hat{1}_{\mathcal{N}} \otimes \hat{u}_n\rangle \in S\big(\mathbb{B}(L^2(\mathcal{N}, \tilde{\tau})\overline{\otimes}l^2\Gamma)\big).$$

We see that $P_\mu(f)$ is in the multiplicative domain of ψ for every $f \in C(X)$.

Claim. *We have $\psi\big((aP_\mu(f) - P_\mu(f)\, a)(aP_\mu(f) - P_\mu(f)\, a)^*\big) = 0$ for all a which commute with $\{u_n : n \in \mathbb{N}\}$.*

Let us observe that

$$\begin{aligned}\psi\big((aP_\mu(f)-P_\mu(f)\,a)(aP_\mu(f)-P_\mu(f)\,a)^*\big)\\ =\psi(aP_\mu(f)P_\mu(f)^*a^*)-\psi(P_\mu(f)\,aP_\mu(f)^*a^*)\\ -\psi(aP_\mu(f)\,a^*P_\mu(f)^*)+\psi(P_\mu(f)\,aa^*P_\mu(f)^*).\end{aligned}$$

Now, since ϕ is normal and a commutes with $\{u_n : n\in\mathbb{N}\}$, we have

$$\begin{aligned}\psi(aP_\mu(f)P_\mu(f)^*a^*)&=\lim_n\phi(u_n\,aP_\mu(f)P_\mu(f)^*a^*u_n^*)\\ &=\lim_n\phi(au_nP_\mu(f)P_\mu(f)^*u_n^*a^*)=|f(x)|^2\phi(aa^*).\end{aligned}$$

On the other hand, since $P_\mu(f)$ and $P_\mu(f)^*$ are in the multiplicative domain, we have $\psi(P_\mu(f)\,aa^*P_\mu(f)^*)=f(x)\overline{f(x)}\,\psi(aa^*)$. As a commutes with $\{u_n : n\in\mathbb{N}\}$, we see that $\psi(aa^*)=\lim_n\phi(u_n\,aa^*u_n^*)=\phi(aa^*)$. Therefore,

$$\psi(P_\mu(f)\,aa^*P_\mu(f)^*)=f(x)\overline{f(x)}\,\psi(aa^*)=|f(x)|^2\phi(aa^*).$$

Arguing similarly, we see that

$$\begin{aligned}\psi(P_\mu(f)\,aP_\mu(f)^*a^*)&=f(x)\,\psi(aP_\mu(f)^*a^*)\\ &=f(x)\lim_n\phi(u_n\,aP_\mu(f)^*a^*u_n^*)\\ &=f(x)\lim_n\phi(au_nP_\mu(f)^*u_n^*a^*)\\ &=f(x)\overline{f(x)}\,\phi(aa^*)=|f(x)|^2\phi(aa^*).\end{aligned}$$

It also follows similarly that

$$\psi(aP_\mu(f)\,a^*P_\mu(f)^*)=|f(x)|^2\phi(aa^*).$$

Consequently, we see that $\psi\big((aP_\mu(f)-P_\mu(f)\,a)(aP_\mu(f)-P_\mu(f)\,a)^*\big)=0$ for every $a\in\{u_n\}'$. Also, for every $b\in\mathbb{B}(L^2(\mathcal{N},\tilde\tau)\overline{\otimes}l^2\Gamma)$ and $a\in L\Lambda'$, we see that

$$\begin{aligned}|\psi(P_\mu(f)\,ab-aP_\mu(f)\,b)|^2\\ \le\psi\big((P_\mu(f)\,a-aP_\mu(f))(P_\mu(f)\,a-aP_\mu(f))^*\big)\,\psi(b^*b)=0.\end{aligned}$$

Therefore, it follows that $\psi(P_\mu(f)\,ab)=\psi(aP_\mu(f)\,b)$ for all $a\in\{u_n\}'$ and for $b\in\mathbb{B}(L^2(\mathcal{N},\tilde\tau)\overline{\otimes}l^2\Gamma)$. □

Our idea of the proof is motivated by [17, Theorem 1.4].

Proof of Theorem 3.1. From Proposition 3.2, a state $\psi\in S\big(\mathbb{B}(L^2(\mathcal{N},\tilde\tau)\overline{\otimes}\ell^2\Gamma)\big)$ exists such that $\psi|_{\mathcal{N}\rtimes\Gamma}=\tau$, $\psi|_{P_\mu(C(X))}=\delta_x$, and $\psi(aP_\mu(f)\,b)=\psi(P_\mu(f)\,ab)$ for all $a\in\{u_n\}'$ and for all $b\in\mathbb{B}(L^2(\mathcal{N},\tilde\tau)\overline{\otimes}\ell^2\Gamma)$. Let $\mathcal{M}$ denote the von Neumann algebra $\{u_n\}'\cap(\mathcal{N}\rtimes\Gamma)$. Let $u\in\mathcal{M}$ be a unitary element. We shall show that

$\|\mathbb{E}_x(u)\|_2 = 1$ from whence it will follow that $u \in \mathcal{N} \rtimes \Gamma_x$. Let $\epsilon > 0$. Let $u_0 = \sum_{i=1}^n a_i \lambda(s_i) \in \mathcal{N} \rtimes \Gamma$ be such that

$$\|u^* - u_0\|_2 < \epsilon. \tag{1}$$

Let us write $F = \{s_1, s_2, \dots, s_n\}$. Then, we can rewrite

$$u_0 = \sum_{s \in F \cap \Gamma_x} a_s \lambda(s) + \sum_{s \in F \cap \Gamma_x^c} a_s \lambda(s).$$

In particular, we see that $sx \neq x$ for all $s \in F \cap \Gamma_x^c$. Therefore, we can find $f \in C(X)$ with $0 \leq f \leq 1$ such that $f(x) = 1$ and $f(sx) = 0$ for all $s \in F \cap \Gamma_x^c$. Below, we write f instead of $P_\mu(f)$ for ease of notation. Let us now observe that

$$\begin{aligned} |\psi(f(uu_0 - 1))| &\leq \sqrt{\psi\big((uu_0 - 1)^*(uu_0 - 1)\big)}\sqrt{\psi(ff^*)} \\ &= \|uu_0 - 1\|_2 \qquad\qquad (\psi|_{\mathcal{N} \rtimes \Gamma} = \tau) \\ &\leq \|u^* - u_0\|_2 < \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} |\psi(ufu_0)| = |\psi(fuu_0)| &= |\psi(f(uu_0 - 1)) + \psi(f)| \\ &\geq \Big|\psi(f) - |\psi(f(uu_0 - 1))|\Big| \geq 1 - \epsilon. \end{aligned}$$

To reiterate,

$$|\psi(ufu_0)| \geq 1 - \epsilon. \tag{2}$$

On the other hand,

$$\begin{aligned} |\psi(ufu_0)| &\leq \left|\psi\left(uf\Big(\sum_{s \in F \cap \Gamma_x} a_s \lambda(s)\Big)\right)\right| + \left|\psi\left(uf\Big(\sum_{s \in F \cap \Gamma_x^c} a_s \lambda(s)\Big)\right)\right| \\ &\leq |\psi(uf\mathbb{E}_x(u_0))| + \sum_{s \in F \cap \Gamma_x^c} |\psi(ufa_s \lambda(s))|. \end{aligned}$$

Since $f \in C(X)$, $a_s \in \mathcal{N}$ and every element of $C(X)$ commutes with $\mathcal{N}$ (see the paragraph below Theorem 3.1), we see that

$$\begin{aligned} \sum_{s \in F \cap \Gamma_x^c} |\psi(ufa_s \lambda(s))| &= \sum_{s \in F \cap \Gamma_x^c} |\psi(ua_s f \lambda(s))| \\ &= \sum_{s \in F \cap \Gamma_x^c} |\psi(ua_s \lambda(s)\, s^{-1} f)| = \sum_{s \in F \cap \Gamma_x^c} |\psi(ua_s \lambda(s)) f(sx)| = 0. \end{aligned}$$

It follows therefore that $|\psi((ufu_0)| \leq |\psi(uf\mathbb{E}_x(u_0))|$. Combining this along with (1), (2) and the Cauchy–Schwarz inequality, we see that

$$1 - \epsilon \leq |\psi(ufu_0)| \leq |\psi(uf\mathbb{E}_x(u_0))| \leq \sqrt{\psi(uff^*u^*)}\|\mathbb{E}_x(u_0)\|_2 \leq \|\mathbb{E}_x(u)\|_2 + \epsilon.$$

As a result, it follows that $\|\mathbb{E}_x(u)\|_2 \geq 1 - 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $u \in \mathcal{N} \rtimes \Gamma_x$. □

We obtain the following as an immediate result.

Corollary 3.3. *Let Γ be a discrete countable group acting on a compact Hausdorff space X. Let (u_n) be a μ-contracting sequence for the action on X for some probability measure μ on X. Then $\{u_n\}' \cap L\Gamma \subset L(\Gamma_x)$, where x is determined by the fact that u_n is μ-contracting towards x.*

Example 3.4. Let Γ be a convergence group. Then, in view of Example 2.5, given a parabolic element $s \in \Gamma$, for every tracial crossed product $\mathcal{N} \rtimes \Gamma$ and every diffuse subalgebra $\mathcal{M}$ of $L(\langle s \rangle)$, the relative commutant of $\mathcal{M}$ in $\mathcal{N} \rtimes \Gamma$ is injective. If $s \in \Gamma$ is loxodromic, then $L(\langle s \rangle)' \cap (\mathcal{N} \rtimes \Gamma)$ is injective.

4. Relative commutants of subgroups of negatively curved groups

In this section, we examine the relative commutants of subgroups inside groups that satisfy "north pole south pole"-dynamics. We begin with the following singularity phenomenon, which has been exploited in the past to prove rigidity results (see, for example, [1; 7; 32; 33; 34]).

Lemma 4.1. *Let X be a continuous Γ-space. Let $\tau \in S(C(X) \rtimes_r \Gamma)$ such that $\tau|_{C(X)} = a\delta_x + (1-a)\,\delta_y$ for some $x \neq y \in X$. Then, $\tau(\lambda(s)) = 0$ for all $s \in \Gamma$ with $s\{x, y\} \cap \{x, y\} = \varnothing$.*

Proof. Let $s \in \Gamma$ be such that $s\{x, y\} \cap \{x, y\} = \varnothing$. Using Uryhson's lemma, we can find a nonnegative continuous function $f \in C(X)$ with $0 < f < 1$ such that $f|_{\{x,y\}} = 1$ and $f|_{\{sx,sy\}} = 0$. Using the Cauchy–Schwarz inequality, we obtain

$$|\tau(f\lambda(s))|^2 = |\tau(\sqrt{f}\sqrt{f}\lambda(s))|^2 \leq \tau(f)\,\tau\big(\lambda(s^{-1})f\lambda(s)\big) = \tau(f)\,\tau(s^{-1}.f).$$

Since $\tau|_{C(X)} = a\delta_x + (1-a)\,\delta_y$, we obtain that

$$\tau(s^{-1}f) = as^{-1}.f(x) + (1-a)\,s^{-1}.f(y) = af(sx) + (1-a)f(sy) = 0.$$

This shows that $\tau(f\lambda(s)) = 0$. On the other hand, applying the Cauchy–Schwarz inequality again, we have

$$\begin{aligned}|\tau((1-f)\,\lambda(s))|^2 &= |\tau(\sqrt{1-f}\sqrt{1-f}\lambda(s))|^2 \\ &\leq \tau(1-f)\,\tau\big(\lambda(s^{-1})(1-f)\,\lambda(s)\big) = \tau(1-f)\,\tau(s^{-1}.(1-f)).\end{aligned}$$

Let us now see that

$$\tau(1-f) = a(1-f(x)) + (1-a)(1-f(y)) = a(0) + (1-a)(0) = 0.$$

Therefore, we obtain that $\tau((1-f)\,\lambda(s)) = 0$. Now, combining the above two identities, we see that

$$\tau(\lambda(s)) = \tau(f\lambda(s)) + \tau((1-f)\,\lambda(s)) = 0. \qquad \square$$

An action $\Gamma \curvearrowright X$ is said to have "north pole south pole"-dynamics, if for every infinite order element $g \in \Gamma$, there are unique fixed point x_g^+ and x_g^- on the Γ-space X such that $g^n x \xrightarrow{n\to\infty} x_g^+$ for all $x \neq x_g^-$. We denote by $E(g) = \mathrm{Stab}_\Gamma(\{x_g^+, x_g^-\})$, the set wise stabilizer of $\{x_g^+, x_g^-\}$. We denote by $\mathbb{E}_{E(g)}$ the canonical conditional expectation from $C_r^*(\Gamma)$ onto $C_r^*(E(g))$. This also extends to a normal trace-preserving conditional expectation from $L(\Gamma)$ onto $L(E(g))$.

Proposition 4.2. *Let Γ be a discrete group admitting a minimal action $\Gamma \curvearrowright X$ with the north pole south pole dynamics. Let $s \in \Gamma$ be an infinite order element with the property that $t\{x_s^+, x_s^-\} \cap \{x_s^+, x_s^-\} = \varnothing$ for all $t \notin E(s)$. Then, given $a \in C_r^*(\Gamma)$ and $\epsilon > 0$, we can find $\{s_1, s_2, \ldots, s_m\} \subset \langle s \rangle$ such that*

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j)(a - \mathbb{E}_{E(s)}(a))\, \lambda(s_j)^* \right\| < \epsilon.$$

Before we head on to the proof, let us briefly ponder our strategy, similar to that of [30]. Let $\Lambda = \langle s \rangle$. We shall first show that for every bounded linear functional φ on $S(C_r^*(\Gamma))$, we can find a bounded linear functional $\psi \in \overline{\{s.\omega : s \in \Lambda\}}^{\text{weak}^*}$ such that $\psi = \psi \circ \mathbb{E}_\Lambda$. Here, $\mathbb{E}_\Lambda : C_r^*(\Gamma) \to C_r^*(\langle s \rangle)$ is the canonical conditional expectation. The claim would then follow by a usual Hahn–Banach separation argument.

Proof. Let Γ be a discrete group admitting a minimal action $\Gamma \curvearrowright X$ with the north pole south pole dynamics. Since $\Gamma \curvearrowright X$ is minimal, we can view $C(X)$ as multiplication operators on $\mathbb{B}(\ell^2(\Gamma))$. Given a bounded linear functional φ on $C_r^*(\Gamma)$, extend it to a bounded linear functional η on $C(X) \rtimes_r \Gamma$. We can write $\eta = c_1 \omega_1 - c_2 \omega_2 + i c_3 \omega_3 - i c_4 \omega_4$, where $\omega_i \in S(C(X) \rtimes_r \Gamma)$ and $c_i \in \mathbb{C}$ for each $i = 1, 2, 3, 4$. Let $\nu_i = \omega_i|_{C(X)}$ for each $i = 1, 2, 3, 4$. Since s is an infinite order element, there are unique fixed points x_s^+ and x_s^- on the Γ-space X such that $s^n x \xrightarrow{n\to\infty} x_s^+$ for all $x \neq x_s^-$. Using the dominated convergence theorem, it follows that $s^n \nu_i \xrightarrow{\text{weak}^*} a_i\, \delta_{x_s^+} + (1 - a_i)\, \delta_{x_s^-}$, where $a_i = \nu_i(X \setminus x_s^-)$. By passing to a subnet (four times) if required, we can assume that $s^n \omega_i \to \omega_i' \in S(C(X) \rtimes_r \Gamma)$ for each $i = 1, 2, 3, 4$. Observe that $\omega_i'|_{C(X)} = a_i\, \delta_{x_s^+} + (1 - a_i)\, \delta_{x_s^-}$. Now let $\eta' = c_1 \omega_1' - c_2 \omega_2' + i c_3 \omega_3' - i c_4 \omega_4'$. Let $\psi = \eta'|_{C_r^*(\Gamma)}$. We claim that $\psi = \psi \circ \mathbb{E}_{E(s)}$. Note that $t\{x_s^+, x_s^-\} \cap \{x_s^+, x_s^-\} = \varnothing$ for all $t \notin E(s)$. It now follows from Lemma 4.1 that $\omega_i'(\lambda(t)) = 0$ for all $t \notin E(s)$ and for all $i = 1, 2, 3, 4$. From this we see that $\omega_i'|_{C_r^*(\Gamma)} = \omega_i' \circ \mathbb{E}_{E(s)}$ for each $i = 1, 2, 3, 4$. Consequently, it follows that $\psi = \psi \circ \mathbb{E}_{E(s)}$. The claim now follows by a usual Hahn–Banach separation argument (see, for example, [21, Theorem 3.4]). □

It was shown in [2] that the averaging scheme at the level of the group C^*-algebra lifts to the same averaging scheme at the level of the crossed product. We merely reiterate the steps to prove that the averaging established in Proposition 4.2 lifts to the crossed product of tracial von Neumann algebras. Given a tracial von Neumann algebra $(\mathcal{N}, \tilde{\tau})$ and a trace preserving action $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$, we let $\tau = \tilde{\tau} \circ \mathbb{E}$ which is a faithful normal trace on $\mathcal{N} \rtimes \Gamma$. We do all the approximations in the $\|\cdot\|_2$-norm induced by τ. We denote by $\mathbb{E}$, the canonical conditional expectation from $\mathcal{N} \rtimes \Gamma$ onto $\mathcal{N}$. Moreover, we shall use $\tilde{\mathbb{E}}_{E(s)}$ to denote the canonical conditional expectation from $\mathcal{N} \rtimes \Gamma$ onto $\mathcal{N} \rtimes E(s)$.

Theorem 4.3. *Let Γ be a discrete group admitting a minimal action $\Gamma \curvearrowright X$ with the north pole south pole dynamics. Let $s \in \Gamma$ be an infinite order element with the property that $t\{x_s^+, x_s^-\} \cap \{x_s^+, x_s^-\} = \varnothing$ for all $t \notin E(s)$. Let $(\mathcal{N}, \tilde{\tau})$ be a tracial von Neumann algebra and $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$ be a trace-preserving action. Let $\mathcal{M} = \mathcal{N} \rtimes \Gamma$. Then, given $x \in \mathcal{M}$ and $\epsilon > 0$, we can find $\{s_1, s_2, \ldots, s_m\} \subset \langle s \rangle$ such that*

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j)(x - \tilde{\mathbb{E}}_{E(s)}(x))\, \lambda(s_j)^* \right\|_2 < \epsilon.$$

Proof. Let $x \in \mathcal{M}$ and $\epsilon > 0$ be given. We can find a finite set $F \subset \Gamma$ and finitely many elements $\{a_t : t \in F\} \subset \mathcal{N}$ such that

$$\left\| x - \sum_{t \in F} a_t\, \lambda(t) \right\|_2 < \frac{\epsilon}{3}.$$

Since $\mathbb{E} \circ \tilde{\mathbb{E}}_{E(s)} = \mathbb{E}$, it follows that

$$\left\| \tilde{\mathbb{E}}_{E(s)}(x) - \sum_{t \in E(s) \cap F} a_t\, \lambda(t) \right\|_2 < \frac{\epsilon}{3}.$$

Let $M = \sup_{t \in F} \|a_t\|$. Using Proposition 4.2 for $a = \sum_{t \in E(s)^c \cap F} \lambda(t)$, we can find $\{s_1, s_2, \ldots, s_m\} \subset \langle s \rangle$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j) \left(\sum_{t \in E(s)^c \cap F} \lambda(t) \right) \lambda(s_j)^* \right\| < \frac{\epsilon}{3|F|M}.$$

It follows from [30, Lemma 4.1] that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j)\, \lambda(t)\, \lambda(s_j)^* \right\|_2 \leq \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j)\, \lambda(t)\, \lambda(s_j)^* \right\| < \frac{\epsilon}{3|F|M}$$

for all $t \in E(s)^c \cap F$. Therefore, using [2, Lemma 2.1], we obtain that

$$\begin{aligned} \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j) \left(\sum_{t \in E(s)^c \cap F} a_t\, \lambda(t) \right) \lambda(s_j)^* \right\| &\leq \sum_{t \in E(s)^c \cap F} \|a_t\| \left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j)\, \lambda(t)\, \lambda(s_j)^* \right\| \\ &\leq \sum_{t \in E(s)^c \cap F} \|a_t\| \frac{\epsilon}{3|F|M} < \frac{\epsilon}{3}. \end{aligned}$$

Putting all these together, along with an application of triangle inequality, we see that

$$\begin{aligned}
&\left\| \frac{1}{m}\sum_{j=1}^{m} \lambda(s_j)(x - \tilde{\mathbb{E}}_{E(s)}(x))\,\lambda(s_j)^* \right\|_2 \\
&\leq \left\| \frac{1}{m}\sum_{j=1}^{m} \lambda(s_j)\Big(x - \sum_{t\in F} a_t\,\lambda(t)\Big)\,\lambda(s_j)^* \right\|_2 \\
&\qquad + \left\| \frac{1}{m}\sum_{j=1}^{m} \lambda(s_j)\Big(\sum_{t\in F\cap E(s)} a_t\,\lambda(t) - \tilde{\mathbb{E}}_{E(s)}(x)\Big)\,\lambda(s_j)^* \right\|_2 \\
&\qquad + \left\| \frac{1}{m}\sum_{j=1}^{m} \lambda(s_j)\Big(\sum_{t\in E(s)^c\cap F} a_t\,\lambda(t)\Big)\,\lambda(s_j)^* \right\|_2 \\
&\leq \left\| x - \sum_{t\in F} a_t\,\lambda(t) \right\|_2 + \left\| \sum_{t\in F\cap E(s)} a_t\,\lambda(t) - \tilde{\mathbb{E}}_{E(s)}(x) \right\|_2 \\
&\qquad + \left\| \frac{1}{m}\sum_{j=1}^{m} \lambda(s_j)\Big(\sum_{t\in E(s)^c\cap F} a_t\,\lambda(t)\Big)\,\lambda(s_j)^* \right\| \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

The claim follows. □

Consequently, we can determine the position of the relative commutants of specific subgroups of a nonelementary acylindrically hyperbolic group with at least one infinite order element. We briefly recall the definitions and refer the readers to [37] for more details.

Acylindrically hyperbolic groups. An action $\Gamma \curvearrowright (X, d)$ on a metrizable space is considered acylindrical if for every $\epsilon > 0$, there exist $\delta, N > 0$ such that for any $x, y \in X$ with $d(x, y) \geq \delta$, the number of elements $g \in \Gamma$ satisfying $d(x, gx) \leq \epsilon$ and $d(y, gy) \leq \epsilon$ is at most N. A group Γ is called acylindrically hyperbolic if it admits a nonelementary acylindrical action on a hyperbolic space.

Every nonelementary hyperbolic group is acylindrically hyperbolic. Further examples of acylindrically hyperbolic groups include non(virtually) cyclic groups hyperbolic relative to proper subgroups, $\mathrm{Out}(F_n)$ for $n > 1$, many mapping class groups, and non(virtually cyclic) groups acting properly on proper CAT(0)-spaces and containing rank one elements, among others (for more details, refer to [37, Section 8]).

For a group Γ acting on a hyperbolic space S, recall that an infinite order element $g \in \Gamma$ is called loxodromic if it has precisely two fixed points x_g^+, x_g^- on the Gromov boundary ∂S and $g^n x \to x_g^+$ for every $x \in \partial S$ except x_g^-. It turns out that a group being acylindrically hyperbolic is equivalent to the notion of "weak proper discontinuity" introduced by Bestvina and Fujiwara [16]. Let Γ be a group acting on a hyperbolic space S. An element $g \in \Gamma$ is said to have the weak proper discontinuity property (in this case, we say that g is a WPD element) if for every $\epsilon > 0$ and every $x \in S$, there exist $M \in \mathbb{N}$ such that the number of elements $h \in \Gamma$ satisfying $d(x, hx) < \epsilon$ and $d(g^M x, hg^M x) < \epsilon$ is finite.

Osin [37, Theorem 1.2] later established that a group Γ being acylindrically hyperbolic is equivalent to the existence of a loxodromic element $g \in \Gamma$ that satisfies the weak proper discontinuity condition. Moreover, there is a unique maximal virtually cyclic subgroup $E(g) \leq \Gamma$ containing g. Explicitly, $E(g) = \mathrm{Stab}_\Gamma(\{x_g^+, x_g^-\})$ is the set wise stabilizer of $\{x_g^+, x_g^-\}$ (see, for example, [25, Lemma 6.5]).

Corollary 4.4. *Let Γ be a group admitting an action $\Gamma \curvearrowright X$ with north pole south pole dynamics. Let $\Lambda \leq \Gamma$ be a subgroup with one infinite order element $s \in \Lambda$. Assume that $\{tx_s^+, tx_s^-\} \cap \{x_s^+, x_s^-\} = \varnothing$ for all $t \notin \mathrm{Stab}(\{x_s^+, x_s^-\})$. Let $(\mathcal{N}, \tilde{\tau})$ be a tracial von Neumann algebra and $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$ be a trace-preserving action. Let $\mathcal{M} = \mathcal{N} \rtimes \Gamma$. Then, $L(\Lambda)' \cap \mathcal{M} \subset \mathcal{N} \rtimes E(s)$.*

Proof. Since s is an infinite order WPD loxodromic element, it satisfies the north pole south pole dynamics on the Gromov boundary. Let $x \in L(\Lambda)' \cap \mathcal{M}$. Let $s \in \Lambda$ be an infinite order loxodromic element. Let $\epsilon > 0$. Using Theorem 4.3, it follows that we can find $s_1, s_2, \ldots, s_m \subset \langle s \rangle$ such that

$$\left\| \frac{1}{m} \sum_{j=1}^{m} \lambda(s_j)(x - \tilde{\mathbb{E}}_{E(s)}(x))\, \lambda(s_j)^* \right\|_2 < \epsilon. \tag{3}$$

Note that here $\tilde{\mathbb{E}}_{E(s)} : \mathcal{N} \rtimes \Gamma \to \mathcal{N} \rtimes E(s)$ is the canonical conditional expectation. Let us write $\tilde{\mathbb{E}}_{E(s)}(x) = \sum_{t \in E(s)} a_t\, \lambda(t)$, where the convergence is in the $\|\cdot\|_2$-norm. For any $s_j \in \langle s \rangle$, writing it as s^{m_j} for some $m_j \in \mathbb{Z}$, we see that

$$\lambda(s_j)\, \tilde{\mathbb{E}}_{E(s)}(x)\, \lambda(s_j)^* = \sum_{t \in E(s)} \alpha_{s_j}(a_t)\, \lambda(s_j t s_j^{-1}) = \sum_{t \in E(s)} \alpha_{s_j}(a_t)\, \lambda(s^{m_j} t s^{-m_j}).$$

Since $t \in E(s) = \{x_s^+, x_s^-\}$, we see that $s^{m_j} t s^{-m_j} \{x_s^+, x_s^-\} = \{x_s^+, x_s^-\}$. As such, we can now see that $\lambda(s_j)\, \tilde{\mathbb{E}}_{E(s)}(x)\, \lambda(s_j)^* \in \mathcal{N} \rtimes E(s)$ for each $j = 1, 2, \ldots, m$. Writing

$$\sum_{j=1}^{m} \lambda(s_j)\, \tilde{\mathbb{E}}_{E(s)}(x)\, \lambda(s_j)^* = y_{E(s)},$$

since $x \in L(\Lambda)' \cap \mathcal{M}$, it follows from (3) that

$$\|x - y_{E(s)}\|_2 < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it is evident that $x \in \mathcal{N} \rtimes E(s)$. Therefore, we see that $L(\Lambda)' \cap \mathcal{M} \subset \mathcal{N} \rtimes E(s)$. □

In addition, if we assume that $\mathcal{N}$ is amenable, then it can be concluded that the relative commutant $L(\Lambda)' \cap \mathcal{M}$ is amenable.

We now give examples that fit into the above setup. Before doing so, we briefly recall the notion of hyperbolic elements and refer the reader to [27] for more details. Let Γ be a group acting by isometries on a hyperbolic space X. An element $s \in \Gamma$ is called hyperbolic if it fixes exactly two points on the boundary of X, denoted by $\partial\Gamma$.

Example 4.5. Let Γ be a torsion-free hyperbolic group and $\Lambda \leq \Gamma$ be an infinite subgroup. Then, Λ contains an element of infinite order and is loxodromic. Let's call it s. Denote by x_s^+ and x_s^-, the corresponding fixed points on the Gromov boundary $\partial\Gamma$. It is well-known that $E(s) = \mathrm{Stab}_\Gamma(\{x_s^+, x_s^-\})$ (see, for example, [25, Lemma 6.5]). Since $s \in \mathrm{Fix}(x_s^+)$ is a hyperbolic element, and x_s^- is the unique fixed point of x in $\partial\Gamma \setminus x_s^+$, it follows from the proof of [27, Theorem 8.30] that $\mathrm{Fix}(x_s^+) = \mathrm{Fix}(x_s^-)$. We claim that $\{tx_s^+, tx_s^-\} \cap \{x_s^+, x_s^-\} = \varnothing$ for all $t \notin E(s)$. Let $t \notin E(s)$. Since $\mathrm{Fix}(x_s^+) = \mathrm{Fix}(x_s^-)$, it follows that $tx_s^+ \neq x_s^+$ and $tx_s^- \neq x_s^-$. If $tx_s^+ = x_s^-$, then $t^{-1}stx_s^+ = x_s^+$. Therefore, $t^{-1}st \in \mathrm{Fix}(x_s^-)$. Therefore, we see that $t^{-1}stx_s^- = x_s^-$. This further implies that $s(tx_s^-) = (tx_s^-)$. Since s is a loxodromic element, either $tx_s^- = x_s^-$ or $tx_s^- = x_s^+$. If $tx_s^- = x_s^-$, it would follow that $tx_s^+ = x_s^- = tx_s^-$ from whence we would obtain that $x_s^+ = x_s^-$ which would contradict the fact that s is an infinite loxodromic element. Therefore, $tx_s^- = x_s^+$. This shows that $t \in E(s)$ which contradicts our earlier choice of $t \notin E(s)$. If $tx_s^- = x_s^+$, the argument follows analogously by replacing $t^{-1}st$ with tst^{-1}. As such, we can now apply Corollary 4.4 to conclude the relative commutant $L(\Lambda)' \cap \mathcal{N} \rtimes \Gamma$ is contained inside $\mathcal{N} \rtimes E(s)$ for any trace-preserving action $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$. Under the further assumption of amenability of $\mathcal{N}$, it follows that $L(\Lambda)' \cap \mathcal{N} \rtimes \Gamma$ is amenable since $E(s)$ is an amenable subgroup of Γ.

There are many acylindrically hyperbolic groups for which we can find an element $t \notin E(s)$ such that $t\{x_s^+, x_s^-\} \cap \{x_s^+, x_s^-\} \neq \varnothing$. Nevertheless, we can still determine the position of the relative commutant of any diffuse von Neumann subalgebra of $L(\langle s \rangle)$ in these situations. Recall that for an action $\Gamma \curvearrowright X$ with north pole south pole dynamics, an element $s \in \Gamma$ is called parabolic if there exists a unique fixed point $x_s^+ \in X$ such that both $\{s^n x\}_{n\in\mathbb{N}}$ and $\{s^{-n} x\}_{n\in\mathbb{N}}$ converge to x_s^+ as $n \to \infty$ for every $x \in X$.

Corollary 4.6. *Let Γ be a discrete group admitting an action $\Gamma \curvearrowright X$ with the north pole south pole dynamics. Let $s \in \Gamma$ be an infinite order parabolic element. Suppose there exists a quasiinvariant probability measure $\mu \in \mathrm{Prob}(X)$ such that $\mu(x_s^+) \neq 0$. Let $(\mathcal{N}, \tilde{\tau})$ be a tracial von Neumann algebra and $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$ be a trace-preserving action. Let $\mathcal{M} = \mathcal{N} \rtimes \Gamma$. Then, $\mathcal{M}_1' \cap \mathcal{M} \leq \mathcal{N} \rtimes \Gamma_{x_s^+}$ for any diffuse subalgebra $\mathcal{M}_1 \leq L(\langle s \rangle)$.*

Proof. Let $s \in \Gamma$ be an infinite order parabolic element. By assumption, there exists one fixed point x_s^+ on X. Moreover, $s^n y \xrightarrow{n\to\infty} x_s^+$ for all $y \in X$. Let $P_\mu : C(X) \to \mathbb{B}(\ell^2(\Gamma))$ be the associated Poisson-transformation. It follows from the Definition 2.1 that (s^n) is μ contracting towards x_s^+. In particular, every diverging sequence $(\lambda_n) \subset \Lambda = \langle s \rangle$ is μ contracting towards x_s^+. Let $\mathcal{M}_1 \leq L(\langle s \rangle)$ be a diffuse subalgebra. Let $(u_n) \subset \mathcal{U}(\mathcal{M}_1)$ be a sequence of unitaries which converge to 0 weakly. We can now appeal to Lemma 2.4 to conclude that u_n is μ-contracting towards x_s^+. The claim now follows from Theorem 3.1. □

Recall that a CAT(0)-cube complex is a simply connected cell complex whose cells are Euclidean cubes $[0, 1]^d$ of various dimensions. We refer the readers to [22], [19], and [23] for more details on these. One can assign many compact Hausdorff boundaries to a CAT(0)-cube complex (see, for example, [36, Section 1.3]). For our purposes, given a CAT(0) metric space, we consider the action on the visual boundary ∂X (see [19, Chapter 8]) equipped with the cone-topology. If X is Gromov hyperbolic, then ∂X is the classical Gromov boundary of X.

Let Γ be a countable discrete group acting on a proper CAT(0) cube complex X (not necessarily hyperbolic) by isometries. We say that the action is elementary if the limit set LX (the set of accumulation points in ∂X of an orbit of the action) consists of at most two points or if Γ fixes a point on ∂X.

Example 4.7 (CAT(0)-cube complexes). Let Γ be a countable discrete group acting on a proper CAT(0) cube complex X (not necessarily hyperbolic) by isometries in a nonelementary way. Let $s \in \Gamma$ be rank-one isometry. It is well known that any rank-one isometry $g \in \mathrm{Isom}(X)$ has the north pole south pole dynamics (see, for example, [31, Lemma 4.4]). Using [31, Theorem 1.1], we see that the limit set $LX \subset \partial X$ is perfect. It follows from [18, Lemma 2.1] that there is a nonatomic measure $\mu \in \mathrm{Prob}(\partial X)$. Since s is a rank-one isometry, there are two fixed points x_s^+ and x_s^- on the visual boundary ∂X. Moreover, $s^n y \xrightarrow{n\to\infty} x_s^+$ for all $y \neq x_s^- \in \partial X$. Since μ is nonatomic, we see that $\mu(x_s^-) = 0$. Now, let $(\mathcal{N}, \tilde{\tau})$ be a tracial von Neumann algebra, and $\Gamma \curvearrowright (\mathcal{N}, \tilde{\tau})$, a trace-preserving action. Setting $\mathcal{M} = \mathcal{N} \rtimes \Gamma$, it follows from Theorem 3.1 that $L(\langle s \rangle)' \cap \mathcal{M} \leq \mathcal{N} \rtimes \Gamma_{x_s^+}$. If we further assume that $\mathcal{N}$ is amenable, then in this case, since $\Gamma_{x_s^+}$ is amenable (see the argument in the last paragraph of [35, Lemma 5.6]), we conclude that $L(\langle s \rangle)' \cap \mathcal{M}$ is amenable.

It is not difficult to find actions $\Gamma \curvearrowright X$ on CAT(0)-cube complexes X which are nonelementary. For example, if $|\partial X| > 2$ and $\Gamma \curvearrowright X$ cocompactly by isometries, then the action is necessarily nonelementary (see, for example, [8]).

5. The case of SL(d, $\mathbb{Z}$)

This section applies our results to the von Neumann algebras associated with infinite subgroups of $\mathrm{SL}(d, \mathbb{Z})$, $d \geq 2$. We show that for each such subgroup Λ, the relative commutant of $L\Lambda$ is always contained in the von Neumann algebra of the intersection of some parabolic subgroup with $\mathrm{SL}(d,\mathbb{Z})$. In the case $d=3$ or if the subgroup is Zariski dense in $\mathrm{SL}(d,\mathbb{R})$, such parabolic subgroups are always Borel groups.

Proposition 5.1. *Let $d\in\mathbb{N}$ and Γ be an infinite subgroup of $\mathrm{SL}(d,\mathbb{Z})$. Then there is a parabolic subgroup P of $\mathrm{SL}(d,\mathbb{R})$ such that $L\Gamma'\cap L\,\mathrm{SL}(d,\mathbb{Z})\subset L(\mathrm{SL}(d,\mathbb{Z})\cap P)$.*

Proof. Let $G=\mathrm{SL}(d,\mathbb{R})$. We want to show that given a diverging sequence (γ_n) in $\mathrm{SL}(d,\mathbb{Z})$ there are a parabolic subgroup P of G, an $\mathrm{SL}(d,\mathbb{Z})$-quasiinvariant probability measure on G/P and a point $y\in G/P$ such that, up to taking a subsequence, for μ-almost every point x in G/P we have $\lim_n \gamma_n\, x = y$.

Let then (γ_n) be such a sequence and write $\gamma_n=k_n\,a_n\,k'_n$ (KAK decomposition in $\mathrm{SL}(d,\mathbb{R})$), in such a way that the diagonal entries $(\lambda_i^{(n)})$ of a_n are taken in decreasing order: $\lambda_i(n)\geq\lambda_{i+1}^{(n)}$ for every n, for every $i=1,\dots,d$. Up to taking a subsequence we can suppose that $\lambda_i^{(n)}/\lambda_{i+1}^{(n)}$ converges to a point in $(0,\infty]$ for every $i=1,\dots,d-1$, and $k_n\to k$, $k'_n\to k'$ in K. We consider the partition of $\{1,\dots,d\}$ into $I_1,\dots,I_l$ subsets (for some $l\in\mathbb{N}$) defined by the condition that i and $i+j$ belong to the same set I_m if and only if $\lambda_i^{(n)}/\lambda_{i+j}^{(n)}$ converges to a finite number. Then we consider the parabolic subgroup P associated with this partition, i.e., the one given by matrices in $\mathrm{SL}(d,\mathbb{R})$ of the form

$$\begin{pmatrix} \mathrm{GL}_{|I_1|}(\mathbb{R}) & * & * & \dots & * \\ 0 & \mathrm{GL}_{|I_2|}(\mathbb{R}) & * & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \mathrm{GL}_{|I_l|}(\mathbb{R}) \end{pmatrix}.$$

Let now $A=\{g\in\mathrm{SL}(d,\mathbb{R})\mid \det(g_i)\neq 0 \text{ for all } i=1,\dots,d-1\}$, where g_i is the i-th principal minor of g. A is a dense open subset of G. It follows from Gaussian elimination that every element of A can be written as a product of an element of the group T of strictly lower triangular matrices (i.e., the ones having only 1's on the diagonal) and an element from the Borel subgroup B of upper triangular matrices (see the proof of [41, Lemma 5.1.4]). The map $T\to A/P$ is continuous and surjective; it restricts to a continuous surjective map $T\backslash\{T\cap P\}\to A/P\backslash\{eP\}$. Let then $x\in T\backslash\{T\cap P\}$ and write it as $x=(X_{i,j})_{i,j=1}^{l}$, where $X_{i,j}$ is a matrix of size $|I_i|\times|I_j|$; in the same way we write $a_n=(\Lambda_{i,j})_{i,j=1}^{l}$. Define the sequence in P given by $h_n=(H_{i,j})_{i,j=1}^{l}$, where $H_{i,j}=\delta_{i,j}(\Lambda_{i,i}X_{i,i})^{-1}$. Then $a_n\,xh_n\to e$ and so $a_n\,xP\to eP$. Let now $yP\in A/P$ and $C\subset A/P$ be a compact neighborhood of yP. Let U be an open subset around eP with an empty intersection with C. For every $xP\in C$ there is n_{xP} such that $a_{n_{x}P}xP\in U$. Hence the open sets $a_{n_{xP}}^{-1}U$ cover C. It follows that the sequence $k^{-1}k_n\,a_n\,k'_n(k')^{-1}xP$ converges to eP. Hence, the result follows by choosing any $\mathrm{SL}(d,\mathbb{Z})$-quasiinvariant probability measure on G/P which gives zero mass to $G/P\backslash A/P$. □

Proposition 5.2. *Let $n \geq 2$ and let $\Gamma = \mathrm{SL}(n,\mathbb{Z})$. Let also μ be the K-invariant probability measure on the complete n-dimensional flag variety. Then, every subgroup of Γ, which is Zariski dense in $\mathrm{SL}(n,\mathbb{R})$, contains a μ-contracting sequence.*

Proof. If Λ is a Zariski dense subgroup of Γ, then we can apply the procedure in [28, Theorem 3.6] (since the action of $\mathrm{SL}(n,\mathbb{Z})$ on the complete flag variety is transitive) to deduce that Λ has the contraction property (as defined in [28, Definition 3.1]. The result follows from [28, Lemma 3.9] and the proof of Proposition 5.1. □

Corollary 5.3. *Let $\Lambda \subset \mathrm{SL}(d,\mathbb{Z})$ be a Zariski-dense subgroup of $\mathrm{SL}(d,\mathbb{R})$. Then $L\Lambda' \cap L\,\mathrm{SL}(d,\mathbb{Z}) \subset L\Gamma_x$ for some $x \in \mathrm{SL}(d,\mathbb{R})/B_d$ (which is injective), where B_d is the Borel subgroup of upper-triangular matrices in $\mathrm{SL}(d,\mathbb{R})$. In particular, this applies to infinite commensurated subgroups of $\mathrm{SL}(d,\mathbb{Z})$.*

Proof. The proof follows from Theorem 3.1 and [6, Lemma 7.5]. □

A stronger result holds if we assume $d = 3$ in the above proposition.

Proposition 5.4. *Let Γ be an infinite subgroup of $\mathrm{SL}(3,\mathbb{Z})$. Then we find that $L\Gamma' \cap L\,\mathrm{SL}(3,\mathbb{Z}) \subset LB \cap \mathrm{SL}(3,\mathbb{Z})$ for some Borel subgroup $B \subset \mathrm{SL}(3,\mathbb{R})$. In particular, the relative commutant of every infinite subgroup of $\mathrm{SL}(3,\mathbb{Z})$ is injective.*

Proof. It follows from the discussion in [38, Example 7] that every infinite subgroup of $\mathrm{SL}(3,\mathbb{Z})$ contains an element whose singular values are pairwise distinct. The result follows arguing as in the proof of Proposition 5.1. □

Example 5.5 [18, Corollary 6.4]. Let Λ be an infinite subgroup of P, where P is the parabolic subgroup associated to the partition $\{\{1,2\},\{3\}\}$, such that for every element $g \in \Lambda$, 1 is a singular value of g. Then the relative commutant of every diffuse subalgebra of $L\Lambda$ inside $L\,\mathrm{SL}(3,\mathbb{Z})$ is injective. Indeed, by the proof of Proposition 5.1, every divergent sequence in Λ is μ-contracting towards eP. It follows from Lemma 2.4 and Theorem 3.1 that the relative commutant of every diffuse von Neumann subalgebra of $L\Lambda$ is contained in LP, and hence it coincides with its relative commutant inside LP. But LP is solid, and the result follows. Note that if Λ is contained in $\mathbb{Z}^2$ (after identifying P with $\mathrm{SL}(2,\mathbb{Z}) \rtimes \mathbb{Z}^2$), the result follows from another application of Theorem 3.1.

We can give a more general example in the case when $d \geq 2$.

Example 5.6. If $d \geq 2$ and Λ is an infinite subgroup of $\mathrm{SL}(d,\mathbb{Z})$ with the property that for every $g \in \Lambda$, g has only two singular values which are not 1, then the relative commutant of every diffuse subalgebra of $L\Lambda$ is contained in the von Neumann algebra of the parabolic subgroup associated to the partition $\{\{1,\ldots,d-1\},\{d\}\}$. This is for example the case for certain embeddings of $\mathrm{SL}(2,\mathbb{Z})$ inside $\mathrm{SL}(d,\mathbb{Z})$ for $d \geq 2$.

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DIFFERENTIAL CALCULUS FOR GENERALIZED GEOMETRY AND GEOMETRIC LAX FLOWS

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It is of interest to extend classical geometric notions to generalized geometry. Various approaches have been proposed in the recent literature. Employing a class of generalized connections, we describe certain differential complices $(\widetilde{\Omega}^*_{\mathbb{T}}(M), \widetilde{\mathbb{d}}^{\mathbb{T}})$ constructed from $\bigwedge^* \mathbb{T}M$ and study some of their basic properties, where $\mathbb{T}M = TM \oplus T^*M$ is the generalized tangent bundle on M. To illustrate how various constructions fit together from this point of view, we describe within the proposed framework the analogues to the Levi-Civita connection when $\mathbb{T}M$ is endowed with a generalized metric and a structure of exact Courant algebroid, the Chern–Weil homomorphism, a Weitzenböck identity, the Ricci flow as a Lax flow and Ricci soliton, the Hermitian–Einstein equation and the degree of a holomorphic vector bundle.

1. Introduction

In generalized geometry à la Hitchin [24], over a smooth manifold M of real dimension n, the bundle $\mathbb{T}M := TM \oplus T^*M$ is considered the analogue of the classical tangent bundle TM. It fits into the natural exact sequence

$$0 \to T^*M \hookrightarrow \mathbb{T}M \xrightarrow{\pi} TM \to 0$$

and is endowed with the natural pairing

$$\langle x, y\rangle = \langle X+\xi, Y+\eta\rangle := \tfrac{1}{2}(\iota_X\eta + \iota_Y\xi),$$

where $x, y \in C^\infty(\mathbb{T}M)$ and $X, Y \in C^\infty(TM)$, $\xi, \eta \in C^\infty(T^*M)$ are their respective components. The dual of $\mathbb{T}M$ can be identified with itself under the pairing $2\langle\ ,\ \rangle$. Well-known geometric structures on $\mathbb{T}M$ such as generalized complex, Riemannian, Hermitian, and Kähler structures and generalized connections are natural extensions of the corresponding classical notions on TM. There are by now many references in the literature, including the pioneering works by Gualtieri [19; 20; 21; 22] on the subjects. We show that the analogy can be pushed further, with $\mathbb{T}M$ consistently

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taking the role of the tangent bundle, leading to coherent extensions of well-known geometric notions.

One of the motivations of this work is to understand an analogue of the Hermitian–Einstein equation proposed by Hitchin [25, Remark in §3.3] to describe a stability condition on generalized holomorphic bundles. One of the obstacles is that some of the most natural choices for a curvature operator, such as the *naïve* curvature operator (2-10), are in general *not* tensorial, and hence are not directly suitable for such an analogue or defining the corresponding notions of degree, or slope stability.

In the literature, there have been various attempts at extending the notion of curvature tensor and related constructions to generalized geometry. In Streets [35], the generalized Ricci flow (7-13) is put in Lax form, where a generalized Ricci tensor is constructed from the Ricci tensor of $\nabla^{-\phi}$, the metric connection with totally skew torsion $-\phi \in \Omega^3(M)$. Ševera and Valach [37; 38] extended similar constructions to general Courant algebroids. In Garcia-Fernandez [10] and Garcia-Fernandez and Streets [12], many notions related to those discussed in this article were discussed in somewhat different contexts. For instance, the notion of metric compatible generalized connection and their eigendecomposition with respect to the generalized metric $\mathbb{G}$ can be found in [10] (see also Definition 2.4). Based on the generalized torsion in [20] and the notion of *divergence*, Garcia-Fernandez and Streets [10; 12] discussed a notion of generalized Levi-Civita connections associated to a generalized metric $\mathbb{G}$ on a Courant algebroid E. Different from the constructions proposed here (Theorem 2.8), the generalized Levi-Civita connections described in [10; 12] are not uniquely determined by $\mathbb{G}$ and the structure of exact Courant algebroid on $\mathbb{T}M$, but form an affine space modeled on a certain space of 3-tensors defined from the eigenbundles of $\mathbb{G}$ (see the discussion surrounding Proposition 3.15 in [12]). Moreover, Garcia-Fernandez and Streets [10; 12] constructed generalized curvature operators for the Courant algebroid E involving only mixed eigensubbundles of $\mathbb{G}$, and an algebraic Bianchi identity was shown involving these components. The resulting generalized Ricci tensors thus only have components that involve different eigensubbundles of $\mathbb{G}$, which provide a description of the generalized Ricci flow in Lax form, with the Ricci curvature as the Lax operator, as in [10, (5.3)] and [12, Remark 4.8]. Using spinors in generalized geometry, Goto [14; 15] and Wang [39] considered the notion of scalar curvature for generalized Kähler manifolds and related constructions. Besides in [12], the discussion of Ricci soliton in generalized geometry has appeared for example in Apostolov, Streets, and Ustinovskiy [3] and Lee [31]. In Garcia-Fernandez, Jordan, and Streets [13] and Garcia-Fernandez and Molina [11], the Hermitian–Einstein equations are considered respectively in the context of pluriclosed flow and the Hull–Strominger system.

As will be described in more detail below, the framework proposed in this article leads to curvature tensors for a generalized connection on *any* vector bundle V, as a section of $\bigwedge^2 \mathbb{T}M \otimes \mathrm{End}(V)$. Such a curvature tensor provides a natural generalization of the Hermitian–Einstein equation (Definition 6.8) to generalized geometry. It also produces a *generalized Ricci tensor* as a section of the bundle $\bigotimes^2 \mathbb{T}M$, which, analogously to the classical case, is symmetric (Section 4B). Using the generalized Ricci curvature as the Lax operator, the Lax equation recovers again the generalized Ricci flow (7-13). It is interesting to note that the equation in Lax form naturally picks out the mixed components of the generalized Ricci tensor (see Theorem 7.14). Indeed, one of the main advantages of the framework we propose is that the extension to generalized geometry of many classical notions follows closely the classical constructions. Hence, for the benefit of brevity, we often omit computations that are in parallel to the classical situations, such as those in the standard textbooks, e.g., do Carmo [6], Griffiths and Harris [18], Lee [30] and Petersen [33]. A large portion of the article consists of examples illustrating the various extensions. Since this is the first of a series of articles exploring the consequences of the proposed framework, we leave discussion of further consequences to future works.

For simplicity, we will restrict our considerations to compact connected oriented smooth manifolds without boundary, while, aside from cohomology computations, most descriptions are of a local nature. The construction starts with a generalized connection $\mathbb{V}^{\mathbb{T}}$ ([20] or (2-1)) on $\mathbb{T}M$. Suppose that $\mathbb{V}^{\mathbb{T}}$ preserves the pairing $\langle\ ,\ \rangle$ and is *TM-torsion-free* (Definition 2.1). It then induces a differential complex, constructed with $\mathbb{T}M$ in place of TM, as a quotient of $\Omega^*_{\mathbb{T}}(M) := C^\infty\big(\bigwedge^* \mathbb{T}M\big)$.

Theorem 1.1 (Section 2A). *Consider the derivation* $\mathbb{d}^{\mathbb{T}} : \Omega^*_{\mathbb{T}}(M) \to \Omega^{*+1}_{\mathbb{T}}(M)$ *defined by*

$$(\mathbb{d}^{\mathbb{T}}\theta)(x_0, x_1, \dots, x_k) := \sum_i (-1)^i (\mathbb{V}^{\mathbb{T}}_{x_i}\theta)(x_0, x_1, \dots, \hat{x}_i, \dots, x_k),$$

where $\theta \in \Omega^k_{\mathbb{T}}(M)$ *and* $x_j \in C^\infty(\mathbb{T}M)$. *Then* $\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}$ *is tensorial if and only if* $\mathbb{V}^{\mathbb{T}}$ *is TM-torsion-free, in which case, the quotient* $\widetilde{\Omega}^*_{\mathbb{T}}(M)$ *of* $\Omega^*_{\mathbb{T}}(M)$ *by the image of* $\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}$ *is a differential complex with the induced derivation* $\widetilde{\mathbb{d}}^{\mathbb{T}}$, *whose cohomology is denoted by* $\widetilde{H}^*_{\mathbb{T}}(M)$.

The differential calculus thus established on $\mathbb{T}M$ leads to a natural definition (3-2) of the tensorial $\mathbb{V}^{\mathbb{T}}$*-curvature* $\mathcal{F}^{\mathbb{T}}(\mathbb{V}) \in \Omega^2_{\mathbb{T}}(\mathrm{End}(V))$ for any generalized connection $\mathbb{V}$ on any vector bundle V. A side effect of this is that the resulting curvature tensor $\mathcal{F}^{\mathbb{T}}$ now depends on the TM-torsion-free generalized connection $\mathbb{V}^{\mathbb{T}}$ on $\mathbb{T}M$. Nonetheless, the differential Bianchi identity holds in the quotient $\widetilde{\Omega}^*_{\mathbb{T}}(\mathrm{End}(V))$ (Lemma 3.2). Passing to a further quotient $\overline{\Omega}^*_{\mathbb{T}}(\mathrm{End}(V))$ (3-11), the Chern–Weil homomorphism naturally extends. The results in Section 3A can be summarized as the following theorem.

Theorem 1.2. *Any invariant polynomial of the $\mathbb{V}^{\mathbb{T}}$-curvature $\mathcal{F}^{\mathbb{T}}(\mathbb{V})$ defines a class in $\overline{H}^*_{\mathbb{T}}(M)$, the reduced $\mathbb{V}^{\mathbb{T}}$-de Rham cohomology, which coincides with the image under $\overline{\pi}^*$ of the corresponding classical characteristic class in $H^*(M)$.*

When $\mathbb{T}M$ is endowed with a generalized Riemannian structure $\mathbb{G}$ ([19; 21] or (2-24)), and a structure of exact Courant algebroid defined by a closed 3-form $\gamma \in \Omega^3(M)$, there exists an analogue to the classical Levi-Civita connection. On the exact Courant algebroid $(\mathbb{T}M, \langle\ ,\ \rangle, \pi, *_\gamma)$, where $*_\gamma$ is the Dorfman bracket (2-36), for lack of better terminology and risking conflicts with [10; 12], the (*generalized*) *Levi-Civita connection* $\mathbb{V}^\phi$ for $\mathbb{G}$ is the *unique* $\mathbb{G}$-adapted connection on $\mathbb{T}M$ that is metric compatible with $*_\gamma$ (Theorem 2.8)

Theorem 1.3 (Theorem 2.8, Section 2C). *Let $\mathbb{G}$ be a generalized metric on $\mathbb{T}M$. The* ***(generalized) Levi-Civita connection*** *$\mathbb{V}^\phi$ is the unique TM-torsion-free $\mathbb{G}$-metric connection on $\mathbb{T}M$ that is metric compatible with the Dorfman bracket $*_\gamma$. The natural map $\widetilde{\pi}^* : H^*(M) \to \widetilde{H}^*_{\phi,\mathbb{G}}(M)$ is injective. Moreover, $\widetilde{H}^{2n}_{\phi,\mathbb{G}}(M) \cong \mathbb{R}$.*

The notion *ϕ-curvature* refers to the generalized curvature defined with $\mathbb{V}^\phi$ for a generalized connection $\mathbb{V}$ on a vector bundle V. The analogue to the Riemannian curvature in this context is the ϕ-curvature for $\mathbb{V}^\phi$ itself, denoted by $\mathcal{R}^\phi$ (Definition 4.1). The *ϕ-Ricci curvature $\mathcal{R}ic^\phi$* (Definition 4.6) and the corresponding scalar curvature (Section 4D) are defined via the usual contractions of $\mathcal{R}^\phi$. In particular, in close analogy with the classical case, $\mathcal{R}ic^\phi$ is an endomorphism of $\mathbb{T}M$, and the corresponding Ricci tensor is symmetric (Section 4B). To illustrate the natural parallel with the classical situation, we show an analogue to the Weitzenböck identity (Theorem 4.7), i.e., the Bochner and Hodge Laplacians differ by the ϕ-Ricci curvature of $\mathbb{G}$.

We next turn to generalized complex geometry. On a generalized complex manifold $(M, \gamma; \mathbb{J})$ [19; 21], $\mathbb{J}$ is integrable with respect to $*_\gamma$. When a generalized connection $\mathbb{V}^{\mathbb{T}}$ is *$\mathbb{J}$-compatible with $*_\gamma$* (Definition 5.1), $\mathbb{d}^{\mathbb{T}}$ decomposes (Lemma 5.2) according to the types with respect to $\mathbb{J}$. Together with a generalized metric $\mathbb{G}$ commuting with $\mathbb{J}$, the resulting generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$ corresponds classically to an almost bi-Hermitian structure $(M, \gamma; g, I_\pm; b)$, where $b \in \Omega^2(M)$ and g is Hermitian with respect to both almost complex structures $I_\pm$. Letting $\phi = \gamma + db$, the $\mathbb{J}$-compatibility of $\mathbb{V}^\phi$ with $*_\gamma$ (Definition 5.1) is equivalent to a generalized Kähler condition given in [20].

Theorem 1.4 (Theorem 5.7). *On a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, let $\phi = \gamma + db$. Then $(M, \gamma; \mathbb{G}, \mathbb{J})$ is a generalized Kähler manifold if and only if $\mathbb{V}^\phi$ is $\mathbb{J}$-compatible with $*_\gamma$.*

In terms of $I_\pm$, the $\mathbb{J}$-compatibility is equivalent to $\nabla^{\pm\phi} I_\pm = 0$. On a generalized Kähler manifold, $I_\pm$ are integrable. Working with $\mathbb{V}^\phi$, we recover a well-known result obtained via holomorphic reduction in [22], namely, the $I_\pm$-(anti)holomorphic tangent bundles on a generalized Kähler manifold carry natural $I_\mp$-holomorphic structures (Proposition 5.11).

For a $\mathbb{J}$-holomorphic Hermitian vector bundle $(V, \bar\partial_{\mathbb{J}}, h)$ [19; 21], the notion of Chern connection extends naturally (6-5). Over a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, there is a natural contraction $\Lambda_{\mathbb{J}_-}$ on $\Omega^2_{\mathbb{T}}(M)$ (Definition 6.7), which leads to the notion of degree (Definition 6.10) for a $\mathbb{J}$-holomorphic Hermitian vector bundle. The degree is independent of the choice of Hermitian metric on V if the generalized Hermitian manifold is $\mathbb{V}^{\mathbb{T}}$-$\mathbb{J}$-*Gauduchon* (Definition 6.13). For such manifolds, the notions of slope and slope stability naturally extend (Definition 6.14). Given a γ-$\mathbb{J}$-connection $\mathbb{V}^{\mathbb{T}}$ (Definition 5.1) on $\mathbb{T}M$, in analogy with the classical case (Lübke and Teleman [32]), we propose the $\mathbb{V}^{\mathbb{T}}$-$\mathbb{J}$-*Hermitian–Einstein equation* (Definition 6.8) for the Hermitian metric h on V. Similarly to the classical situation, one should expect a version of Kobayashi–Hitchin correspondence to hold in this case (see Hu, Moraru, and Seyyedali [28]). On a generalized Kähler manifold, we show (Proposition 6.9) that these notions relate to their classical counterparts, in particular, the $\mathbb{J}$-Hermitian–Einstein equation is equivalent to an equation proposed by Hitchin [25, Remark in §3.3].

Theorem 1.5 (Section 6C). *Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a $\mathbb{J}$-Gauduchon generalized Kähler manifold and $\omega_\pm \in \Omega^2(M)$ be the Kähler forms for $I_\pm$ respectively. Let $(V, \partial_{\mathbb{J}})$ be a $\mathbb{J}$-holomorphic vector bundle. Then the $\mathbb{J}$-Hermitian–Einstein equation is equivalent to*

$$\frac{\sqrt{-1}}{2}\big(F^C_+(V)\wedge\omega_+^{m-1}+(-1)^\varepsilon F^C_-(V)\wedge\omega_-^{m-1}\big)=c(m-1)!\,\mathrm{Id}_V\,d\mathrm{vol}_g,$$

where $F^C_\pm$ are the classical Chern curvatures with respect to $I_\pm$, $\varepsilon = 0$ if $I_\pm$ induce the same orientation on TM, and $\varepsilon = 1$ otherwise.

Geometric flows such as the mean curvature flow (Brakke [5]) and the Ricci flow (Hamilton [23]) are very important in understanding smooth manifolds and structures associated to them. In generalized geometry, it is natural to consider flows involving structures on $\mathbb{T}M$ such as the generalized metrics or generalized complex structures, e.g., in [10; 12; 36]. In this context, we generally assume that the flow preserves the structure of Courant algebroid on $\mathbb{T}M$ defined by the Dorfman bracket $*_\gamma$.

We describe a general construction of Lax flows of generalized metrics or generalized complex structures in the proposed framework. A Lax flow can be defined from any $\theta \in \Omega^2_{\mathbb{T}}(M)$ (Lemma 7.6) via the induced map $\theta : \mathbb{T}M \to \mathbb{T}M$. In particular, the Lax flow defined by the ϕ-curvature of a Hermitian line bundle generates the action of generalized symmetries on $\mathbb{T}M$ (Theorem 7.10). Even though the Bianchi identities do not hold for $\mathcal{R}^\phi$ in general (Lemma 4.3), it turns out that the ϕ-Ricci tensor $\mathcal{R}c^{\phi}$ is symmetric (Section 4B). The corresponding Lax flow is the *Ricci Lax flow* (7-11), which exactly recovers the *generalized Ricci flow* in the mathematics and physics literature; see, for instance, [12]. Conformal deformations of the Riemannian metric g can be represented as a Lax flow, where the Lax operators are $\mathbb{G}_t$-conformal $\mathbb{T}M$-forms (Definition 7.7). The *Ricci soliton equation* (Streets [34]) can be obtained as a combination of the geometric Lax flows described so far (Definition 7.16), involving the generalized Ricci curvature, generalized curvature of line bundles and conformal $\mathbb{T}M$-forms. We also see that the classical Kähler–Ricci flow can be recast as a geometric Lax flow (Section 7D).

We expect that many classical constructions should admit natural extensions to $\mathbb{T}M$ via the differential calculus developed here. Spinors, which are behind the notion of $\mathbb{J}_-$-contraction in Definition 6.7, relate the geometry on $\mathbb{T}M$ back to $\Omega^*(M)$, and, in particular, lead to the canonical line of a generalized (almost) complex structure [19; 21] as well as the notion of scalar curvature in generalized Kähler geometry [14; 15; 39]. Functionals involving curvatures, such as the Yang–Mills functional, can be extended (Section 3C) and lead to natural questions on extremal/critical (generalized) connections/metrics with respect to them. Explicit examples such as compact Lie groups ([15]; Hu [26]) could provide further insights into understanding these extensions. It should be worth exploring the interaction of the Riemannian, the complex and the Poisson geometric methods in generalized Hermitian geometry. Equations in Lax form admit geometric interpretations (Griffiths [17]), and it would be interesting to understand if this provides new perspective for the related geometric flows. We plan to come back to these topics in future works.

We briefly summarize the structure of the paper. In Section 2, we set up the differential calculus on $\mathbb{T}M$ and compute in Section 2E the group $\widetilde{H}^*_{\gamma,\mathbb{G}}(G)$ for a compact Lie group G, with the bi-invariant metric and the Cartan 3-form γ. The generalized curvature tensors are introduced in Section 3. The rest of the article applies the constructions in various contexts. The analogue to the Riemann curvature is discussed in Section 4, together with the associated Ricci and scalar curvatures, as well as the generalized Bismut connections [20]. In Section 5, we apply the differential calculus to generalized complex and Hermitian manifolds. The degree, stability and Hermitian–Einstein equation for a generalized holomorphic bundle over a generalized Hermitian manifold are discussed in Section 6. In Section 7, we discuss the notion of geometric Lax flows.

2. Differential calculus on $\mathbb{T}M$

Let $V \to M$ be a vector bundle. Recall that a generalized connection on V is a derivation:

$$\nabla : C^\infty(V) \to C^\infty(\mathbb{T}M \otimes V) \quad \text{such that} \quad \nabla(fv) = df \otimes v + f\nabla v, \tag{2-1}$$

where $\mathbb{T}M = TM \oplus T^*M$, $f \in C^\infty(M)$ and $v \in C^\infty(V)$. It is the *lift* of a classical connection ∇_0 on V if

$$\nabla_x v = \nabla_{0,\pi(x)} v \tag{2-2}$$

for all $x \in C^\infty(\mathbb{T}M)$ and $v \in C^\infty(V)$. The generalized connections naturally extend to tensor bundles in the standard fashion.

2A. $\mathbb{T}M$-forms. Under the pairing $2\langle\ ,\ \rangle$, sections of $\bigwedge^* \mathbb{T}M$ can be seen as $\mathbb{T}M$-*forms* and the space of such forms will be suggestively denoted by

$$\Omega^*_{\mathbb{T}}(M) := C^\infty\big(\textstyle\bigwedge^* \mathbb{T}M\big).$$

Let $\nabla^{\mathbb{T}}$ be a generalized connection on $\mathbb{T}M$ preserving $\langle\ ,\ \rangle$. The skew-symmetrization of the covariant derivative by $\nabla^{\mathbb{T}}$ induces the $\nabla^{\mathbb{T}}$-*derivation* $\mathbb{d}^{\mathbb{T}}$. Namely, for $\theta \in \Omega^k_{\mathbb{T}}(M)$,

$$(\mathbb{d}^{\mathbb{T}}\theta)(x_0, x_1, \ldots, x_k) := \sum_i (-1)^i (\nabla^{\mathbb{T}}_{x_i}\theta)(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_k), \tag{2-3}$$

where $x_i \in C^\infty(\mathbb{T}M)$. For $f \in C^\infty(M)$, $\mathbb{d}^{\mathbb{T}}$ coincides with the usual differential:

$$\mathbb{d}^{\mathbb{T}} f := df \in \Omega^1_{\mathbb{T}}(M). \tag{2-4}$$

Moreover, $\mathbb{d}^{\mathbb{T}}$ is a graded derivation on $\Omega^*_{\mathbb{T}}(M)$, that is, for $\theta_1 \in \Omega^k(M)$ and $\theta_2 \in \Omega^*(M)$,

$$\mathbb{d}^{\mathbb{T}}(\theta_1 \wedge \theta_2) = (\mathbb{d}^{\mathbb{T}}\theta_1) \wedge \theta_2 + (-1)^k \theta_1 \wedge (\mathbb{d}^{\mathbb{T}}\theta_2). \tag{2-5}$$

Let the $\nabla^{\mathbb{T}}$-*diamond bracket* $\diamond_{\mathbb{T}}$ be the skew-symmetrization of $\nabla^{\mathbb{T}}$:

$$x \diamond_{\mathbb{T}} y := \nabla^{\mathbb{T}}_x y - \nabla^{\mathbb{T}}_y x. \tag{2-6}$$

Recall that the notion of *generalized torsion* for $\nabla^{\mathbb{T}}$ is introduced in [20], in the context of generalized Kähler conditions (see also Remark 5.8). Here, a different notion of TM-*torsion* is more convenient.

Definition 2.1. Let $\nabla^{\mathbb{T}}$ be a generalized connection on $\mathbb{T}M$. Its TM-*torsion* is

$$\tau_T(x, y) := \pi(x \diamond_{\mathbb{T}} y) - [\pi(x), \pi(y)], \tag{2-7}$$

where $x, y \in C^\infty(\mathbb{T}M)$. Then $\nabla^{\mathbb{T}}$ is TM-*torsion-free* if its TM-torsion vanishes.

Standard computations yield

$$\begin{aligned}(\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}\theta)&(x_0, x_1, \ldots, x_{k+1}) \\ &= \sum_{i<j} (-1)^{i+j} [\tau_T(x_i, x_j)]\theta(x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1}) \\ &\quad - \sum_{i<j<\ell} (-1)^{i+j+\ell} \theta([x_i \diamond_{\mathbb{T}} x_j \diamond_{\mathbb{T}} x_\ell], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, \hat{x}_\ell, \ldots, x_{k+1}),\end{aligned} \tag{2-8}$$

where $[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z]$ is the *Jacobiator* of the diamond bracket $\diamond_{\mathbb{T}}$ (2-6):

$$[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] := (x \diamond_{\mathbb{T}} y) \diamond_{\mathbb{T}} z + \text{c.p.} \tag{2-9}$$

Let $\mathcal{R}^{\mathbb{T}}$ be the *naïve* curvature operator for $\nabla^{\mathbb{T}}$, which may not be tensorial:

$$\mathcal{R}^{\mathbb{T}}_{x,y} z := \nabla^{\mathbb{T}}_x \nabla^{\mathbb{T}}_y z - \nabla^{\mathbb{T}}_y \nabla^{\mathbb{T}}_x z - \nabla^{\mathbb{T}}_{\nabla^{\mathbb{T}}_x y} z + \nabla^{\mathbb{T}}_{\nabla^{\mathbb{T}}_y x} z, \tag{2-10}$$

where $x, y, z \in C^\infty(\mathbb{T}M)$. Then a straightforward rearrangement gives

$$[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] = -\mathcal{R}^{\mathbb{T}}_{x,y} z - \text{c.p.} \tag{2-11}$$

It follows that $\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}$ is tensorial when $\nabla^{\mathbb{T}}$ is TM-torsion-free, since in this case $\mathcal{R}^{\mathbb{T}}$ is tensorial by (3-4). Furthermore, $\nabla^{\mathbb{T}}$ being TM-torsion-free also implies the Jacobiator for $\diamond_{\mathbb{T}}$ has values in T^*M:

$$\pi[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] = \big[[\pi(x), \pi(y)], \pi(z)\big] + \text{c.p.} = 0. \tag{2-12}$$

Any torsion-free affine connection ∇^T on TM lifts to a TM-torsion-free generalized connection on $\mathbb{T}M$:

$$\nabla^{\mathbb{T}}_x y := \nabla^T_X y = \nabla^T_X Y + \nabla^T_X \eta$$

for $x, y \in C^\infty(\mathbb{T}M)$ with $x = X + \xi$ and $y = Y + \eta$. The affine space $\mathscr{D}(\mathbb{T}M)$ of generalized connections on $\mathbb{T}M$ is modeled on the space of bundle homomorphisms

$$\mathscr{D}(\mathbb{T}M) \cong \{A : \mathbb{T}M \otimes \mathbb{T}M \to \mathbb{T}M\},$$

in which the subspace $\mathscr{D}_\tau(\mathbb{T}M)$ of TM-torsion-free ones is modeled on the subspace of the right-hand side consisting of the ones whose skew-symmetric part lies in $\Omega^2_{\mathbb{T}}(T^*M)$:

$$\mathscr{D}(\mathbb{T}M) \supseteq \mathscr{D}_\tau(\mathbb{T}M) \cong \text{Sym}^2_{\mathbb{T}}(\mathbb{T}M) \oplus \Omega^2_{\mathbb{T}}(T^*M), \tag{2-13}$$

where $\text{Sym}^2_{\mathbb{T}}(\mathbb{T}M)$ is the space of symmetric 2-tensors

$$\text{Sym}^2_{\mathbb{T}}(\mathbb{T}M) := \{A : \mathbb{T}M \odot \mathbb{T}M \to \mathbb{T}M\}.$$

The *contraction* by $x \in C^\infty(\mathbb{T}M)$ is a graded derivation on $\Omega^*_{\mathbb{T}}(M)$ defined by

$$\iota_x y := 2\langle x, y\rangle, \tag{2-14}$$

where $y \in \Omega^1_{\mathbb{T}}(M)$. The *Lie derivative* along $x \in C^\infty(\mathbb{T}M)$ is given by

$$\mathcal{L}^{\mathbb{T}}_x \theta := \iota_x \mathbb{d}^{\mathbb{T}} \theta + \mathbb{d}^{\mathbb{T}} \iota_x \theta, \tag{2-15}$$

where $\theta \in \Omega^*_{\mathbb{T}}(M)$. In particular, for $f \in C^\infty(M)$ and $X = \pi(x) \in C^\infty(TM)$,

$$\mathcal{L}^{\mathbb{T}}_x f = Xf.$$

Suppose $\nabla^{\mathbb{T}}$ is TM-torsion-free. Then the familiar relations among the operators $\mathbb{d}^{\mathbb{T}}$, ι_x and $\mathcal{L}^{\mathbb{T}}_x$ almost hold, up to possible terms involving the Jacobiator, similarly to (2-8).

Proposition 2.2. *Let* $x, y, z, w \in C^\infty(\mathbb{T}M)$, $\theta \in \Omega^k_{\mathbb{T}}(M)$, $\alpha \in \Omega^1(M)$ *and* $X = \pi(x)$. *Suppose that* $\nabla^{\mathbb{T}}$ *is* TM*-torsion-free. Then*:

(1) $d\alpha = 0 \implies \mathbb{d}^{\mathbb{T}}\alpha = 0$.

(2) $[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] \in C^\infty(T^*M)$.

(3) $X\langle y, z\rangle = \langle x \diamond_{\mathbb{T}} y, z\rangle + \langle y, \mathcal{L}^{\mathbb{T}}_x z\rangle$.

(4) $\mathcal{L}^{\mathbb{T}}_x \iota_y \theta - \iota_y \mathcal{L}^{\mathbb{T}}_x \theta = \iota_{x \diamond_{\mathbb{T}} y}\theta$.

(5) $\langle \mathcal{L}^{\mathbb{T}}_x y - x \diamond_{\mathbb{T}} y, z\rangle = \langle \nabla^{\mathbb{T}}_y x, z\rangle + \langle \nabla^{\mathbb{T}}_z x, y\rangle$.

(6) $\langle [\mathcal{L}^{\mathbb{T}}_x, \mathcal{L}^{\mathbb{T}}_y] z, w\rangle = \langle \mathcal{L}^{\mathbb{T}}_{x \diamond_{\mathbb{T}} y} z, w\rangle + \iota_{[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} w]} z$.

(7) *For* $x_1, \dots, x_{k+1} \in C^\infty(\mathbb{T}M)$,

$$([\mathbb{d}^{\mathbb{T}}, \mathcal{L}^{\mathbb{T}}_x]\theta)(x_1, \dots, x_{k+1}) = \sum_{i<j} (-1)^{i+j+1} \theta\big([x \diamond_{\mathbb{T}} x_i \diamond_{\mathbb{T}} x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}\big).$$

Proof. The verification follows from standard computations and is left for the reader. □

If $\nabla^{\mathbb{T}}$ is TM-torsion-free, by item (1) of Proposition 2.2, the cohomology of $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\mathbb{T}})$ is well defined for degrees $k < 2$. Furthermore, the Jacobiator (2-9) defines a degree 2 map $\mathscr{J}^{\mathbb{T}}$ on $\Omega^*_{\mathbb{T}}(M)$, which commutes with $\mathbb{d}^{\mathbb{T}}$,

$$\begin{aligned}(\mathscr{J}^{\mathbb{T}}\theta)(x_0, \dots, x_{k+1}) &:= (\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}\theta)(x_0, \dots, x_{k+1}) \\ &= \sum_{i<j<\ell} (-1)^{i+j+\ell+1} \theta\big([x_i \diamond_{\mathbb{T}} x_j \diamond_{\mathbb{T}} x_\ell], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \hat{x}_\ell, \dots, x_{k+1}\big),\end{aligned} \tag{2-16}$$

where $\theta \in \Omega^k_{\mathbb{T}}(M)$ and $x_j \in C^\infty(\mathbb{T}M)$. It follows that $\mathbb{d}^{\mathbb{T}}$ induces a differential $\widetilde{\mathbb{d}}^{\mathbb{T}}$ on the quotient space of the $\nabla^{\mathbb{T}}$*-reduced* $\mathbb{T}M$*-forms*:

$$0 \to \operatorname{img} \mathscr{J}^{\mathbb{T}} \to \Omega^*_{\mathbb{T}}(M) \xrightarrow{\mathscr{Q}^{\mathbb{T}}} \widetilde{\Omega}^*_{\mathbb{T}}(M) \to 0, \tag{2-17}$$

where $\mathscr{Q}^{\mathbb{T}}$ denotes the quotient map. In particular,

$$\widetilde{\Omega}^k_{\mathbb{T}}(M) = \Omega^k_{\mathbb{T}}(M) \quad \text{for } k \leqslant 2. \tag{2-18}$$

Definition 2.3. Let $\nabla^{\mathbb{T}}$ be a TM-torsion-free generalized connection on $\mathbb{T}M$. The complex $(\widetilde{\Omega}^*_{\mathbb{T}}(M), \widetilde{\mathbb{d}}^{\mathbb{T}})$ is the $\nabla^{\mathbb{T}}$-*de Rham complex* and its k-th cohomology is the *k-th* $\mathbb{V}^{\mathbb{T}}$*-de Rham cohomology of* M:

$$\widetilde{H}^k_{\mathbb{T}}(M) := \frac{\ker\big(\widetilde{\mathbb{d}}^{\mathbb{T}} : \widetilde{\Omega}^k_{\mathbb{T}}(M) \to \widetilde{\Omega}^{k+1}_{\mathbb{T}}(M)\big)}{\operatorname{img}\big(\widetilde{\mathbb{d}}^{\mathbb{T}} : \widetilde{\Omega}^{k-1}_{\mathbb{T}}(M) \to \widetilde{\Omega}^k_{\mathbb{T}}(M)\big)}. \tag{2-19}$$

Because $\mathscr{J}^{\mathbb{T}}$ is of degree 2, when $k < 2$, $\widetilde{H}^k_{\mathbb{T}}(M)$ computes the corresponding cohomology groups of $\Omega^*_{\mathbb{T}}(M)$. By (2-4), it is evident that $\widetilde{H}^0_{\mathbb{T}}(M) = \mathbb{R} = H^0(M)$, which consists of the constant functions. Item (1) in Proposition 2.2 then gives a natural inclusion

$$H^1(M) \subseteq \widetilde{H}^1_{\mathbb{T}}(M), \tag{2-20}$$

in which the equality may not hold in general (see Proposition 2.12).

In general, the map π induces a natural injection $\pi^* : \Omega^k(M) \hookrightarrow \Omega^k_{\mathbb{T}}(M)$ for all k:

$$(\pi^*\alpha)(x_1, \dots, x_k) := \alpha(\pi(x_1), \dots, \pi(x_k)), \tag{2-21}$$

where $\alpha \in \Omega^k(M)$ and $x_j \in C^\infty(\mathbb{T}M)$. Alternatively, π^* is induced from the inclusion $T^*M \hookrightarrow \mathbb{T}M$. When $\mathbb{V}^{\mathbb{T}}$ is TM-torsion-free, π^* commutes with the derivations

$$\pi^*(d\alpha) = \mathbb{d}^{\mathbb{T}}(\pi^*\alpha). \tag{2-22}$$

Thus π^* defines a morphism of cochain complices after passing to the quotient

$$\widetilde{\pi}^* : \Omega^k(M) \to \widetilde{\Omega}^k_{\mathbb{T}}(M),$$

which induces the corresponding maps on the cohomology groups:

$$\widetilde{\pi}^* : H^k(M) \to \widetilde{H}^k_{\mathbb{T}}(M). \tag{2-23}$$

2B. $\mathbb{G}$***-adapted connections.*** To restrict $\mathbb{V}^{\mathbb{T}}$ further, consider a generalized metric $\mathbb{G}$ [19; 20]. In the *standard splitting* of $\mathbb{T}M$, which is defined by the inclusion of TM as the first factor of $\mathbb{T}M = TM \oplus T^*M$, $\mathbb{G}$ corresponds to a pair (g, b) of Riemannian metric g on M and 2-form $b \in \Omega^2(M)$:

$$\mathbb{G}(x, y) := \tfrac{1}{2}[g(X, Y) + g^{-1}(\xi - \iota_X b, \eta - \iota_Y b)]. \tag{2-24}$$

Let the $\mathbb{G}$-*splitting* of $\mathbb{T}M$ be

$$s_0 : TM \to \mathbb{T}M \quad \text{given by} \quad X \mapsto X + \iota_X b, \tag{2-25}$$

under which $\mathbb{G}$ can be written as

$$\mathbb{G}(x, y) = \tfrac{1}{2}[g(X, Y) + g^{-1}(x - s_0(X), y - s_0(Y))].$$

Definition 2.4. A generalized connection $\nabla^{\mathbb{T}}$ on $\mathbb{T}M$ is a $\mathbb{G}$-*metric connection* [10; 12] if it preserves both $\mathbb{G}$ and $\langle\ ,\ \rangle$, i.e.,

$$X\langle y,z\rangle = \langle \nabla^{\mathbb{T}}_x y, z\rangle + \langle y, \nabla^{\mathbb{T}}_x z\rangle, \quad X\mathbb{G}(y,z) = \mathbb{G}(\nabla^{\mathbb{T}}_x y, z) + \mathbb{G}(y, \nabla^{\mathbb{T}}_x z), \tag{2-26}$$

where $x, y, z \in C^\infty(\mathbb{T}M)$ and $X = \pi(x)$.

A $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$ preserves the ±1-eigenbundles $C_\pm$ of $\mathbb{G}$:

$$C_\pm := \{s_\pm(X) := (s_0 \pm g)(X) = X + (b \pm g)X : X \in TM\}. \tag{2-27}$$

Hence, $\nabla^{\mathbb{T}}$ admits a $\mathbb{G}$-*eigendecomposition* [10; 12; 20] into four metric connections $\nabla^{\bullet}_{\star}$ on TM:

$$\nabla^{\bullet}_{\star,X} Y := \pi(\nabla^{\mathbb{T}}_{s_\star(X)} s_\bullet(Y)), \tag{2-28}$$

where $\star$ and $\bullet$ respectively stand for $+$ or $-$. Furthermore, the corresponding $\mathbb{d}^{\mathbb{T}}$ admits the induced $\mathbb{G}$-eigendecomposition.

Lemma 2.5. *Let $\nabla^{\mathbb{T}}$ be a $\mathbb{G}$-metric connection. Then the operator $\mathbb{d}^{\mathbb{T}}$ decomposes into components as follows*:

$$\mathbb{d}^{\mathbb{T}} = \mathbb{d}^{\mathbb{T}}_+ + \mathbb{d}^{\mathbb{T}}_- : \Omega^{p,q}_{\mathbb{G}}(M) \to \Omega^{p+1,q}_{\mathbb{G}}(M) \oplus \Omega^{p,q+1}_{\mathbb{G}}(M), \tag{2-29}$$

where

$$\Omega^{p,q}_{\mathbb{G}}(M) := C^\infty\big(\textstyle\bigwedge^p C_+ \otimes \bigwedge^q C_-\big) \cong \Omega^p(M) \otimes \Omega^q(M). \tag{2-30}$$

Proof. Consider $\theta \in \Omega^{p,0}_{\mathbb{G}}(M)$. For $k > 1$, since $\nabla^{\mathbb{T}}$ preserves $C_\pm$, it is straightforward to verify that for $x^j_\pm = s_\pm(X_j)$, where $X_j \in C^\infty(TM)$,

$$(\mathbb{d}^{\mathbb{T}}\theta)(x^0_-, x^1_-, \dots, x^{k-1}_-, x^k_+, \dots, x^q_+) = 0.$$

Thus $\mathbb{d}^{\mathbb{T}}\theta$ cannot contain any components in $\Omega^{p-k+1,k}_{\mathbb{G}}(M)$ for $k > 1$. The general situation follows from the analogue for $\theta' \in \Omega^{0,q}_{\mathbb{G}}(M)$ and noticing that $\mathbb{d}^{\mathbb{T}}$ is a derivation (2-5). □

In its $\mathbb{G}$-eigendecomposition, a $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$ is TM-torsion-free if and only if $\nabla^+_+ = \nabla^-_- = \nabla$ is the Levi-Civita connection for g and for all $X, Y \in C^\infty(TM)$,

$$\nabla^-_{+,X} Y - \nabla^+_{-,Y} X - [X, Y] = 0.$$

It follows that a 3-form $\phi \in \Omega^3(M)$, which may not be closed, can be defined by

$$\phi(X,Y,Z) := 2g((\nabla^+_- - \nabla)_X Y, Z) = 2g((\nabla^-_+ - \nabla)_Y X, Z). \tag{2-31}$$

The mixed components in the $\mathbb{G}$-eigendecomposition can be expressed in terms of ϕ, e.g.,

$$\nabla^+_{-,X} Y = \nabla^{+\phi}_X Y := \nabla_X Y + \tfrac{1}{2} g^{-1}\iota_Y \iota_X \phi, \tag{2-32}$$

which is a metric connection on TM with totally skew torsion ϕ. The computations are summarized in the following theorem/definition.

Theorem 2.6 ($\mathbb{G}$-adapted connections). *The TM-torsion-free $\mathbb{G}$-metric connections on $\mathbb{T}M$ are classified by 3-forms $\phi \in \Omega^3(M)$, which are denoted by $\mathbb{V}^{\phi,\mathbb{G}}$, and are referred to as the $\mathbb{G}$-**adapted connections**. The notation $\mathbb{V}^{\phi,\mathbb{G}}$ is often abbreviated as $\mathbb{V}^{\phi}$ if $\mathbb{G}$ is understood. The $\mathbb{G}$-eigendecomposition of $\mathbb{V}^{\phi}$ is*

$$(\nabla^+_+, \nabla^+_-, \nabla^-_+, \nabla^-_-) = (\nabla, \nabla^{+\phi}, \nabla^{-\phi}, \nabla). \tag{2-33}$$

In the $\mathbb{G}$-splitting (2-25) *of $\mathbb{T}M$, the $\mathbb{G}$-adapted connection takes the form*

$$\begin{aligned}\mathbb{V}^{\phi}_x y = s_0\big[\nabla_X Y + \tfrac{1}{4}g^{-1}(\iota_{g^{-1}\eta}\iota_X\phi - \iota_Y\iota_{g^{-1}\xi}\phi)\big]&\\ + \nabla_X\eta + \tfrac{1}{4}(\iota_Y\iota_X\phi - \iota_{g^{-1}\eta}\iota_{g^{-1}\xi}\phi)&,\end{aligned} \tag{2-34}$$

where $x = s_0(X) + \xi$ and $y = s_0(Y) + \eta \in C^\infty(\mathbb{T}M)$.

Proof. Under the pairing $2\langle\ ,\ \rangle$, the space $\mathscr{D}(\mathbb{T}M)$ of generalized connections on $\mathbb{T}M$ is modeled on

$$\mathscr{D}(\mathbb{T}M) \cong \{A : \mathbb{T}M \to \mathbb{T}M \otimes \mathbb{T}M\}.$$

The subspace $\mathscr{D}_{\mathbb{G}}(\mathbb{T}M)$ of $\mathbb{G}$-metric connections is then modeled on

$$\mathscr{D}_{\mathbb{G}}(\mathbb{T}M) \cong \big\{A : \mathbb{T}M \to \big(\textstyle\bigwedge^2 C_+\big) \oplus \big(\textstyle\bigwedge^2 C_-\big) \cong \operatorname{End}(TM, g)^{\oplus 2}\big\}. \tag{2-35}$$

From (2-13) and (2-35), it follows that space of TM-torsion-free $\mathbb{G}$-metric connections on $\mathbb{T}M$ is modeled on the intersection. The identity (2-31) can be seen also as

$$\phi(X, Y, Z) = 2\langle A_{x_-}y_+, z_+\rangle = -2\langle A_{y_+}x_-, z_-\rangle,$$

which shows that the intersection is isomorphic to $\Omega^3(M)$. The rest follows from straightforward computations. □

Let $\gamma \in \Omega^3(M)$ be a closed 3-form and let $*_\gamma$ be the corresponding *Dorfman bracket* on $C^\infty(\mathbb{T}M)$, where for $X, Y \in C^\infty(TM)$ and $\xi, \eta \in C^\infty(T^*M)$,

$$(X + \xi) *_\gamma (Y + \eta) := [X, Y] + \mathcal{L}_X\eta - d\iota_Y\xi + \iota_Y\iota_X\gamma. \tag{2-36}$$

Definition 2.7. A $\mathbb{G}$-metric connection $\mathbb{V}^{\mathbb{T}}$ is *metric compatible with* $*_\gamma$ if the diamond bracket $\diamond_{\mathbb{T}}$ coincides with $*_\gamma$ on mixed $\mathbb{G}$-eigensections, i.e.,

$$x_+ \diamond_{\mathbb{T}} y_- = x_+ *_\gamma y_-, \tag{2-37}$$

where $x_+ \in C^\infty(C_+)$ and $y_- \in C^\infty(C_-)$.

Using the explicit description (2-34), it is straightforward to verify that

$$x \diamond_\phi y = s_0([X, Y]) + \nabla_X \eta - \nabla_Y \xi + \tfrac{1}{2}(\iota_Y \iota_X \phi - \iota_{g^{-1}\eta} \iota_{g^{-1}\xi} \phi).$$

Hence, ∇^ϕ is metric compatible with $*_{\phi - db}$. The space of $\mathbb{G}$-metric connections that are metric compatible with $*_\gamma$ but not necessarily TM-torsion-free is modeled on the following subspace of (2-35):

$$\mathscr{D}_{\mathbb{G},\gamma}(\mathbb{T}M) \cong \{A : C_+ \to \textstyle\bigwedge^2 C_+\} \oplus \{A : C_- \to \textstyle\bigwedge^2 C_-\} \cong \Omega^1(\mathrm{End}(TM, g))^{\oplus 2}.$$

Theorem 2.8. *For a closed $\gamma \in \Omega^3(M)$, let $\phi = \gamma + db$. Then ∇^ϕ (Theorem 2.6) is the unique TM-torsion-free $\mathbb{G}$-metric connection on $\mathbb{T}M$ that is metric compatible with $*_\gamma$. It is called the* ***(generalized) Levi-Civita connection*** *for $\mathbb{G}$ on the Courant algebroid $(\mathbb{T}M, \langle\ ,\ \rangle, \pi, *_\gamma)$.* □

Remark 2.9. In Theorem 2.8, the exact Courant algebroid $(\mathbb{T}M, \langle\ ,\ \rangle, \pi, *_\gamma)$ can be seen as defining *two levels* of differential structures — represented by the Lie bracket $\langle\ ,\ \rangle$ on $C^\infty(TM)$ and the Dorfman bracket $*_\gamma$ on $C^\infty(\mathbb{T}M)$. The notion of TM-torsion-free specifies the compatibility with the differential structure on TM, while metric compatibility with $*_\gamma$ concerns the differential structure on $\mathbb{T}M$. The uniqueness stemming from these two compatibility conditions is in complete analogue with the classical case, and hence the choice of the notion *generalized Levi-Civita connection*. Moreover, the classical Levi-Civita connection are two of the components in its $\mathbb{G}$-eigendecomposition, as in Theorem 2.6. In [10; 12], the notion *generalized Levi-Civita connection* has been used to describe generalized metric connections satisfying another set of natural conditions, which are not uniquely determined, and come in an affine family modeled on a certain subspace of $C^\infty(\bigwedge^3 \mathbb{T}M)$.

2C. ***∇^ϕ-de Rham cohomology.*** The $\mathbb{G}$-eigendecomposition of $\mathbb{T}M$ induces two left inverses of the natural injection π^* in (2-21). There are two obvious projections $p_\pm : \Omega^k_{\mathbb{T}}(M) \to C^\infty(\bigwedge^k C_\pm)$ for each k. Let $\theta \in \Omega^k_{\mathbb{T}}(M)$, $X_j \in C^\infty(TM)$ and $x^j_\pm = s_\pm(X_j)$ for all $j = 1, \ldots, k$. Then

$$(p_\pm \theta)(x^1_\pm, \ldots, x^k_\pm) := \theta(x^1_\pm, \ldots, x^k_\pm).$$

The projection π induces the natural isomorphisms $\pi_* : C^\infty(\bigwedge^* C_\pm) \cong \Omega^*(M)$:

$$(\pi_* \theta_\pm)(X_1, \ldots, X_k) := \theta_\pm(x^1_\pm, \ldots, x^k_\pm),$$

where $\theta_\pm \in C^\infty(\bigwedge^* C_\pm)$. Then $\pi^\pm_* = \pi_* \circ p_\pm : \Omega^k_{\mathbb{T}}(M) \to \Omega^k(M)$ is given by

$$(\pi^\pm_* \theta)(X_1, \ldots, X_k) := \theta(x^1_\pm, \ldots, x^k_\pm).$$

It is now straightforward to see that

$$\pi_*^{\pm} \circ \pi^* = \mathrm{Id} : \Omega^*(M) \to \Omega^*(M).$$

Let $\mathbb{d}^\phi$ denote the $\mathbb{V}^\phi$-derivation (2-3) induced by $\mathbb{V}^\phi$. The decomposition (2-29) can be explicitly described by the classical de Rham differential and covariant derivatives. For instance, for $\alpha \in \Omega^k(M)$, let $\alpha_+ \in \Omega_{\mathbb{G}}^{p,0}(M)$ such that $\alpha = \pi_*^+(\alpha_+)$. Then $\mathbb{d}_+^\phi \alpha_+$ is essentially the de Rham differential, that is,

$$\text{(2-38)} \quad \begin{aligned}(\mathbb{d}^\phi \alpha_+)(x_+^0, x_+^1, \dots, x_+^k) &= \sum_i (-1)^i (\mathbb{V}^\phi_{x_+^i} \alpha_+)(x_+^0, x_+^1, \dots, \hat{x}_+^i, \dots, x_+^k) \\ &= \sum_i (-1)^i (\nabla_{X_i} \alpha)_+(x_+^0, x_+^1, \dots, \hat{x}_+^i, \dots, x_+^k) \\ &= (d\alpha)_+(x_+^0, x_+^1, \dots, x_+^k),\end{aligned}$$

while the component $\mathbb{d}_-^\phi \alpha_+ \in \Omega_{\mathbb{G}}^{p,1}(M)$ is essentially given by $\nabla^{+\phi}$, that is,

$$\text{(2-39)} \qquad (\mathbb{d}_-^\phi \alpha_+)(x_-^0, x_+^1, \dots, x_+^q) = (\nabla_{X_0}^{+\phi} \alpha)_+(x_+^1, \dots, x_+^q),$$

where $x_\pm^j = s_\pm(X_j)$ for $X_j \in C^\infty(TM)$.

Lemma 2.10. *The Jacobiator for $\diamond_\phi$ is given by*

$$\text{(2-40)} \qquad [x_\pm \diamond_\phi y_\pm \diamond_\phi z_\mp] = \pm 2g(R_{X,Y}^{\mp\phi} Z),$$

where $x_\pm = s_\pm(X)$ for $X \in C^\infty(TM)$ and so on, and $R^{\pm\phi}$ are respectively the classical curvature for $\nabla^{\pm\phi}$. All other components in the $\mathbb{G}$-eigendecomposition vanish.

Proof. This follows by lengthy but standard computations from the definitions. □

Proposition 2.11. *The natural map $\widetilde{\pi}^* : H^*(M) \to \widetilde{H}_{\phi,\mathbb{G}}^*(M)$ in (2-23) is injective.*

Proof. For $\alpha, \beta \in \Omega^*(M)$, suppose that $\pi^*\alpha = \pi^*(d\beta)$. Then the injectivity of π^* implies that $\alpha = d\beta$. The statement then follows from $\operatorname{img} \mathscr{J}^\phi \cap \operatorname{img} \pi^* = \{0\}$. In fact, suppose $\mathscr{J}^\phi \theta = \pi^*\alpha \in \operatorname{img} \mathscr{J}^\phi \cap \operatorname{img} \pi^*$. Then

$$\alpha = \pi_*^+(\pi^*\alpha) = \pi_*^+(\mathscr{J}^\phi \theta) = 0.$$

The last equality is due to $[x_+ \diamond_\phi y_+ \diamond_\phi z_+] = 0$, which follows from Lemma 2.10. □

Since $\widetilde{\Omega}_{\mathbb{T}}^k(M) \cong \Omega_{\mathbb{T}}^k(M)$ for $k \leqslant 2$ and the derivations coincide for $k < 2$, the groups $\widetilde{H}_{\phi,\mathbb{G}}^k(M)$ for $k < 2$ can be determined. When $k = 0$, it is easy to see that $\widetilde{H}_{\phi,\mathbb{G}}^0(M) \cong \mathbb{R}$ consists of the constant functions.

Proposition 2.12. *Let $P_\phi^1(M)$ be the space of ∇-parallel 1-forms on M which also annihilates ϕ, i.e.,*

$$\xi \in P_\phi^1(M) \quad \Longleftrightarrow \quad \nabla\xi = 0 \quad \textit{and} \quad \iota_{g^{-1}\xi}\phi = 0.$$

Then

$$\widetilde{H}^1_{\phi,\mathbb{G}}(M) \cong H^1(M) \oplus P^1_\phi(M). \tag{2-41}$$

Proof. Let $\theta \in \Omega^1_{\mathbb{T}}(M)$. For $X, Y \in C^\infty(TM)$, let $x_\pm = s_\pm(X)$ and so on, and define

$$\alpha(X) := \theta(x_+) \quad \text{and} \quad \beta(X) := \theta(x_-). \tag{2-42}$$

This then gives the identification

$$Q : \Omega^1_{\mathbb{T}}(M) \cong \Omega^1(M) \oplus \Omega^1(M) \quad \text{given by} \quad \theta \mapsto \tfrac{1}{2}(\alpha+\beta, \alpha-\beta). \tag{2-43}$$

Suppose that $\mathbb{d}^\phi\theta = 0$. Then (2-38) implies that

$$d\alpha = d\beta = 0.$$

Hence

$$0 = (\mathbb{d}^\phi_-\theta)(x_+, y_-) = -[\nabla^{+\phi}_Y(\alpha-\beta)](X),$$

i.e., $\nabla^{+\phi}(\alpha-\beta) = 0$. Set $\xi = \alpha - \beta$. Then $d\xi = 0$ implies that

$$0 = (\nabla^{+\phi}_X\xi)(Y) - (\nabla^{+\phi}_Y\xi)(X) = -\phi(X, Y, g^{-1}\xi)$$

for all $X, Y \in C^\infty(TM)$, from which $\xi \in P^1_\phi(M)$ follows. Thus, on $\Omega^1_{\mathbb{T}}(M)$,

$$Q(\ker \mathbb{d}^\phi) = \ker d \oplus P^1_\phi(M).$$

Then (2-41) follows from $Q(df) = (df, 0)$ for $f \in C^\infty(M)$. □

Corollary 2.13. $\widetilde{H}^1_{\phi,\mathbb{G}}(M) \cong H^1(M)$ *if one of the following holds*:

(1) ϕ *is nondegenerate*, i.e., *the following map is injective*:

$$\iota_\bullet\phi : C^\infty(TM) \to \Omega^2(M) \quad \textit{given by} \quad X \mapsto \iota_X\phi.$$

(2) M *admits no nontrivial* ∇*-parallel vector fields.* □

Example 2.14. Let $M = \mathbb{R}^n/\mathbb{Z}^n$, with the induced flat metric. Suppose that $\phi = 0$, and thus the ∇-parallel forms on M are the constant forms. Then

$$\widetilde{H}^1_{0,\mathbb{G}}(M) \cong H^1(M) \oplus T_0M \cong \mathbb{R}^{2n},$$

where T_0M is the tangent space at $0 \in M$. On the other hand, for $n = 3$, let $\phi = \mathrm{vol}_g$, the volume form of the flat metric on M. Then it is nondegenerate. By Corollary 2.13,

$$\widetilde{H}^1_{\mathrm{vol}_g,\mathbb{G}}(M) = H^1(M) \cong \mathbb{R}^3.$$

Let $n = \dim_{\mathbb{R}} M$. The group $H^{2n}_{\mathbb{T}}(M)$ is always well defined for a TM-torsion-free $\nabla^{\mathbb{T}}$:

$$H^{2n}_{\mathbb{T}}(M) := \frac{\Omega^{2n}_{\mathbb{T}}(M)}{\mathrm{img}\big(\mathbb{d}^{\mathbb{T}} : \Omega^{2n-1}_{\mathbb{T}}(M) \to \Omega^{2n}_{\mathbb{T}}(M)\big)}.$$

When $\mathbb{\nabla}^{\mathbb{T}} = \mathbb{\nabla}^{\phi,\mathbb{G}}$ and $d\phi = 0$, the groups $\widetilde{H}^{2n}_{\phi,\mathbb{G}}(M)$ and $H^{2n}_{\phi,\mathbb{G}}(M)$ can be determined.

Proposition 2.15. *Let $\phi \in \Omega^3(M)$ and $d\phi = 0$. Then $\widetilde{H}^{2n}_{\phi,\mathbb{G}}(M) \cong H^{2n}_{\phi,\mathbb{G}}(M) \cong \mathbb{R}$.*

Proof. The derivation $\mathbb{d}^\phi$ on $\Omega^{2n-1}_{\mathbb{T}}(M)$ splits in the decomposition (2-30) as

$$\mathbb{d}^\phi = \mathbb{d}^\phi_+ \oplus \mathbb{d}^\phi_- : \Omega^{n-1,n}_{\mathbb{T}}(M) \oplus \Omega^{n,n-1}_{\mathbb{T}}(M) \to \Omega^{n,n}_{\mathbb{T}}(M) = \Omega^{2n}_{\mathbb{T}}(M).$$

For instance, $\mathbb{d}^\phi_+ : \Omega^{n-1,n}_{\mathbb{T}}(M) \to \Omega^{n,n}_{\mathbb{T}}(M)$ is given by the lifting of $d : \Omega^{n-1}(M) \to \Omega^n(M)$ as in (2-38). Since img $\mathscr{J}^\phi \cap \Omega^{2n}(M) = \{0\}$ by Proposition 4.5, it follows that $\widetilde{\Omega}^{2n}_{\mathbb{T}}(M) = \Omega^{2n}_{\mathbb{T}}(M)$, which implies the statement. □

Let M and M' be two smooth manifolds. It is straightforward to see that if a generalized diffeomorphism $\widetilde{\lambda} = (\lambda, B) : (M, \phi, \mathbb{G}) \to (M', \phi', \mathbb{G}')$ relates the corresponding data on both manifolds, i.e.,

$$\mathbb{G} = \widetilde{\lambda}^* \mathbb{G}' \quad \text{and} \quad \phi = \widetilde{\lambda}^* \phi' = \lambda^* \phi' + dB,$$

it induces isomorphisms throughout the constructions. In particular, it induces the natural isomorphism of cohomology groups $\widetilde{\lambda}^* : \widetilde{H}^*_{\phi',\mathbb{G}'}(M') \xrightarrow{\cong} \widetilde{H}^*_{\phi,\mathbb{G}}(M)$.

2D. *Laplacians.* Analogously to the classical case, a generalized connection $\mathbb{\nabla}$ on V defines the corresponding *Bochner Laplacian* on $C^\infty(V)$. Let $\{X_i\}$ be a local orthonormal frame on TM for the Riemannian metric g. Then

$$\{e^i_\pm := s_\pm(X_i)\}$$

is a local $\mathbb{G}$-orthonormal frame on $\mathbb{T}M$.

Definition 2.16. Let $\mathbb{\nabla}$ be a generalized connection on V. The *Bochner Laplacian* (*for $\mathbb{\nabla}$ with respect to $\mathbb{\nabla}^\phi$*) is defined by

$$\Delta_{\mathbb{\nabla}} v := -\sum_i [(\mathbb{\nabla}_{e^i_+} \mathbb{\nabla}_{e^i_+} - \mathbb{\nabla}_{\mathbb{\nabla}^\phi_{e^i_+} e^i_+})v + (\mathbb{\nabla}_{e^i_-} \mathbb{\nabla}_{e^i_-} - \mathbb{\nabla}_{\mathbb{\nabla}^\phi_{e^i_-} e^i_-})v] \tag{2-44}$$

for $v \in C^\infty(V)$.

The operator defined by (2-44) is independent of the choice of $\{X_i\}$. Because the $\mathbb{\nabla}^\phi_{e^i_\pm} e^i_\pm$ involve only the Levi-Civita connection for g, it is evident that

$$\Delta_{\mathbb{\nabla}} v = (\Delta_+ + \Delta_-)v,$$

where $\Delta_\pm$ are the classical Bochner Laplacians on V for $\nabla_\pm$ respectively. Thus $\Delta_{\mathbb{\nabla}}$ is a second-order elliptic operator, which in general depends on $\mathbb{G}$, but not on ϕ. When $\mathbb{\nabla}$ is a lift of a classical connection, $\Delta_{\mathbb{\nabla}}$ reduces to (a constant multiple of) the corresponding classical Bochner Laplacian.

The operator $\mathbb{d}^\phi$ can alternatively be written as

$$\mathbb{d}^\phi \theta = \sum_j (e^j_+ \wedge \mathbb{\nabla}^\phi_{e^j_+} \theta - e^j_- \wedge \mathbb{\nabla}^\phi_{e^j_-} \theta), \tag{2-45}$$

where $\theta \in \Omega^*_{\mathbb{T}}(M)$. Thus the principal symbol of $\mathbb{d}^\phi$ is

$$\sigma(\mathbb{d}^\phi) = 2\sqrt{-1}\xi \wedge : \bigwedge^k \mathbb{T}M \to \bigwedge^{k+1} \mathbb{T}M,$$

where $\xi \in T^*M$. It follows that whenever $\mathbb{d}^\phi$ squares to 0 it defines an elliptic complex. Analogously to the classical case, for $\theta \in \Omega^*_{\mathbb{T}}(M)$, define

$$\mathbb{d}^{\phi *}\theta := -\frac{1}{2} \sum_j (\iota_{e^j_+} \mathbb{\nabla}^\phi_{e^j_+} \theta + \iota_{e^j_-} \mathbb{\nabla}^\phi_{e^j_-} \theta), \tag{2-46}$$

where the $\frac{1}{2}$ is due to the convention (2-14). The principal symbol of $\mathbb{d}^{\phi *}$ is then

$$\sigma(\mathbb{d}^{\phi *}) = -\sqrt{-1}\iota_{g^{-1}\xi + bg^{-1}\xi} : \bigwedge^k \mathbb{T}M \to \bigwedge^{k-1} \mathbb{T}M,$$

where $\xi \in T^*M$. The operators $\mathbb{d}^\phi$ and $\mathbb{d}^{\phi *}$ are formal adjoints with respect to the pairing $(\ ,\ )_{\mathbb{G}}$ on $\bigwedge^* \mathbb{T}M$ induced by $\mathbb{G}$, for which local orthonormal bases are given by $\{\bigwedge_{i \in I} e^i_+ \wedge \bigwedge_{j \in J} e^j_- : I, J \subseteq \{1, 2, \dots, n\}\}$. Indeed, direct computation shows that for $\theta, \mu \in \Omega^*_{\mathbb{T}}(M)$,

$$(\mathbb{d}^\phi \theta, \mu)_{\mathbb{G}} - (\theta, \mathbb{d}^{\phi *}\mu)_{\mathbb{G}} = \frac{1}{2} \sum_j [X_j (\theta, \iota_{s_0(X_j)}\mu)_{\mathbb{G}} - (\theta, \iota_{s_0(\nabla_{X_j} X_j)}\mu)_{\mathbb{G}}],$$

which is the divergence (with respect to g) of the vector field W defined by

$$g(W, Z) = (\theta, \iota_{s_0(Z)}\mu)_{\mathbb{G}}$$

for all $Z \in C^\infty(TM)$.

Definition 2.17. The $\mathbb{\nabla}^\phi$*-Hodge Laplacian* is the operator on $\Omega^*_{\mathbb{T}}(M)$ given by

$$\Delta^\phi := \mathbb{d}^\phi \mathbb{d}^{\phi *} + \mathbb{d}^{\phi *} \mathbb{d}^\phi. \tag{2-47}$$

Proposition 2.18. *The $\mathbb{\nabla}^\phi$-Hodge Laplacian Δ^ϕ is a second-order elliptic operator.*

Proof. It is clear from the discussion above that the principal symbol of Δ^ϕ is

$$\sigma(\Delta^\phi) = 2\|\xi\|^2_g : \bigwedge^k \mathbb{T}M \to \bigwedge^k \mathbb{T}M,$$

where $\xi \in T^*M$, from which the statement follows. □

Theorem 2.19. *Let M be a closed manifold. Then the following holds*:

$$\widetilde{H}^1_{\phi,\mathbb{G}}(M) \cong \ker \mathbb{d}^\phi|_{\Omega^1_{\mathbb{T}}(M)} \cap \ker \mathbb{d}^{\phi *}|_{\Omega^1_{\mathbb{T}}(M)} \subseteq \ker \Delta^\phi|_{\Omega^1_{\mathbb{T}}(M)}. \tag{2-48}$$

Proof. Let $\theta \in \Omega^1_{\mathbb{T}}(M)$ and consider $\alpha, \beta \in \Omega^1(M)$ as in (2-42). Then it can be shown that

$$\mathbb{d}^{\phi *}\theta = d^*(\alpha + \beta).$$

Combining the identity above with the proof of Proposition 2.12, the first isomorphism follows from the classical Hodge theory. The last inclusion is obvious. □

Even though $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^\phi)$ is generally not a chain complex, Theorem 2.19 nonetheless hints at the analogue of its "cohomology groups".

Definition 2.20. Let $\mathbb{G}$ be a generalized metric on M and $\phi \in \Omega^3(M)$. The $\mathbb{V}^\phi$-*pseudocohomology groups* $\check{H}^*_{\phi,\mathbb{G}}(M)$ of M consist of the common kernels of $\mathbb{d}^\phi$ and $\mathbb{d}^{\phi*}$:

$$\check{H}^k_{\phi,\mathbb{G}}(M) := \ker \mathbb{d}^\phi|_{\Omega^k_{\mathbb{T}}(M)} \cap \ker \mathbb{d}^{\phi*}|_{\Omega^k_{\mathbb{T}}(M)}.$$

The $\mathbb{V}^\phi$-*Laplacian kernels* $\widehat{H}^k_{\phi,\mathbb{G}}(M)$ are the subspaces

$$\widehat{H}^k_{\phi,\mathbb{G}}(M) := \ker \Delta^\phi|_{\Omega^k_{\mathbb{T}}(M)}.$$

Corollary 2.21. *For a closed manifold M, both $\widehat{H}^*_{\phi,\mathbb{G}}(M)$ and $\check{H}^*_{\phi,\mathbb{G}}(M)$ are finite-dimensional.*

Proof. This follows from $\check{H}^k_{\phi,\mathbb{G}}(M) \subseteq \widehat{H}^k_{\phi,\mathbb{G}}(M)$ and the ellipticity of Δ^ϕ. □

Remark 2.22. The inclusions $\check{H}^k_{\phi,\mathbb{G}}(M) \subseteq \widehat{H}^k_{\phi,\mathbb{G}}(M)$ may not be equalities. For $k = 1$, direct computation using the $\mathbb{G}$-eigendecomposition shows that

$$\theta = \widehat{H}^1_{\phi,\mathbb{G}}(M)$$

if and only if

$$\Delta(\alpha+\beta) + \Delta_{+\phi}(\alpha-\beta) - \frac{1}{2}\sum_{j,k}(d\beta)(X_j, X_k)\iota_{X_k}\iota_{X_j}\phi = 0,$$

$$\Delta(\alpha+\beta) - \Delta_{-\phi}(\alpha-\beta) + \frac{1}{2}\sum_{j,k}(d\alpha)(X_j, X_k)\iota_{X_k}\iota_{X_j}\phi = 0,$$

where α, β are as in (2-42) and $\{X_j\}$ is a local g-orthonormal frame of TM. In the case that $d\alpha = d\beta = 0$, it can be shown that the right-hand side is exactly equivalent to $\theta \in \check{H}^1_{\phi,\mathbb{G}}(M)$. Namely, under the identification (2-43),

$$\check{H}^1_{\phi,\mathbb{G}}(M) = \widehat{H}^1_{\phi,\mathbb{G}}(M) \cap (\ker d \oplus \ker d).$$

2E. ***Compact Lie groups.*** Manifolds admitting flat metric connections with nontrivial completely skew torsions are known by Cartan and Schouten [8; 7], which are essentially compact Lie groups and S^7 (see also Agricola and Friedrich [1]).

Suppose that G is a real semisimple Lie group, endowed with the bi-invariant Killing metric g and the corresponding bi-invariant Cartan 3-form $\gamma \in \Omega^3(G)$. Suppose that G is simply connected. In this case, as will become clear below, the computations are very much parallel to those for the classical de Rham cohomology of the doubled group $G \times G$.

The metric connections $\nabla^{\pm\gamma}$, with torsions $\pm\gamma$ respectively, are flat. Let $\mathbb{G}$ be given by g and $b=0$ and denote the corresponding Levi-Civita connection as $\mathbb{\nabla}^{\gamma}$. By Lemma 2.10, for all $x, y, z \in C^\infty(\mathbb{T}G)$,

$$[x \diamond_\gamma y \diamond_\gamma z] = 0,$$

which implies that $\mathscr{J}^\gamma = \mathbb{d}^\gamma \circ \mathbb{d}^\gamma = 0$. Hence $(\Omega^*_{\mathbb{T}}(G), \mathbb{d}^\gamma)$ is the $\mathbb{\nabla}^\gamma$-de Rham complex, whose cohomology is the $\mathbb{\nabla}^\gamma$-de Rham cohomology $\widetilde{H}^*_{\gamma,g}(G)$.

Let X^l_u denote the left-invariant vector field on G such that $X^l_u(e) = u \in \mathfrak{g} := T_eG$. The corresponding right-invariant vector fields are denoted by X^r_u. The Lie algebra structure on $\mathfrak{g}$ is identified with the Lie algebra of left-invariant vector fields:

$$[X^l_u, X^l_v] = X^l_{[u,v]}.$$

For $u, v \in \mathfrak{g}$, set $\theta^r_u := g(X^r_u)$ and $\theta^l_v := g(X^l_v)$. Then

$$x^+_u = X^r_u + \theta^r_u \in C^\infty(C_+) \quad \text{and} \quad x^-_v = X^l_v - \theta^l_v \in C^\infty(C_-).$$

It is straightforward to see that

$$\mathbb{\nabla}^\gamma_{x^+_u} x^+_v = -\tfrac{1}{2} x^+_{[u,v]}, \quad \mathbb{\nabla}^\gamma_{x^-_u} x^-_v = \tfrac{1}{2} x^-_{[u,v]}, \quad \text{and} \quad \mathbb{\nabla}^\gamma_{x^+_u} x^-_v = \mathbb{\nabla}^\gamma_{x^-_u} x^+_v = 0.$$

Let $\mathbb{u} = (u, u')$, $\mathbb{v} = (v, v') \in \mathfrak{g} \oplus \mathfrak{g}$ and

$$x_{\mathbb{u}} = -x^+_u + x^-_{u'},\ x_{\mathbb{v}} = -x^+_v + x^-_{v'} \in C^\infty(\mathbb{T}G).$$

Then direct computation leads to

$$\mathbb{\nabla}^\gamma_{x_{\mathbb{u}}} x_{\mathbb{v}} = \tfrac{1}{2} x_{[\mathbb{u},\mathbb{v}]} \implies x_{\mathbb{u}} \diamond_\gamma x_{\mathbb{v}} = x_{\mathbb{u}} *_\gamma x_{\mathbb{v}} = x_{[\mathbb{u},\mathbb{v}]}, \tag{2-49}$$

where the Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}$ is the direct sum of those on each factor. The last equality in (2-49) can be seen also from the *Courant trivialization* of Alekseev, Bursztyn, and Meinrenken [2]. The γ-curvature can then be computed as

$$\mathcal{R}^\gamma_{x_{\mathbb{u}}, x_{\mathbb{v}}} x_{\mathbb{w}} = -\tfrac{1}{4}\big[[x_{\mathbb{u}}, x_{\mathbb{v}}], x_{\mathbb{w}}\big] = -\tfrac{1}{4} x_{[[\mathbb{u},\mathbb{v}],\mathbb{w}]},$$

which gives the γ-Ricci tensor

$$\mathcal{R}c^\gamma(x_{\mathbb{u}}, x_{\mathbb{v}}) = \tfrac{1}{4}\mathbb{G}(x_{\mathbb{u}}, x_{\mathbb{v}}).$$

In particular, $(G, \gamma; \mathbb{G})$ may be seen as an example of a *γ-Einstein manifold*, where the γ-Ricci curvature is proportional to the generalized metric.

To compute $\widetilde{H}^*_{\gamma,g}(G)$, recall $\mathbb{T}G$ is the dual of itself via $2\langle\ ,\ \rangle$, which leads to

$$(\mathbb{d}^\gamma x_{\mathbb{u}})(x_{\mathbb{v}}, x_{\mathbb{w}}) = -x_{\mathbb{u}}(x_{[\mathbb{v},\mathbb{w}]}).$$

Let $\theta \in \Omega^*_{\mathbb{T}}(G)$ be decomposable as the product of k sections of the form $x_{\mathbb{v}}$. Then

$$(\mathbb{d}^\gamma \theta)(x_{\mathbb{u}_0}, \dots, x_{\mathbb{u}_k}) = \sum_{i<j} (-1)^{i+j} \theta(x_{[\mathbb{u}_i,\mathbb{u}_j]}, x_{\mathbb{u}_0}, \dots, \hat{x}_{\mathbb{u}_i}, \dots, \hat{x}_{\mathbb{u}_j}, \dots, x_{\mathbb{u}_k}).$$

Let $f_{\mathbb{u}} \in \mathfrak{g}^* \oplus \mathfrak{g}^*$ be defined by

$$f_{\mathbb{u}}(\mathbb{v}) := x_{\mathbb{u}}(x_{\mathbb{v}}).$$

It induces an inclusion of the Chevalley–Eilenberg complex of $\mathfrak{g} \oplus \mathfrak{g}$ for the trivial module:

$$\big(\bigwedge^*(\mathfrak{g}^* \oplus \mathfrak{g}^*), \delta\big) \hookrightarrow (\Omega^*_{\mathbb{T}}(G), \mathbb{d}^{\gamma}) \quad \text{given by} \quad f_{\mathbb{u}} \mapsto x_{\mathbb{u}}.$$

Similarly to the classical case, this induces an isomorphism on the cohomology:

$$\widetilde{H}^*_{\gamma,g}(G) \cong H^*(\mathfrak{g} \oplus \mathfrak{g}) \cong H^*(G \times G).$$

The isomorphism above in fact is an isomorphism of rings, where on $\widetilde{H}^*_{\gamma,g}(G)$ the product is induced by the wedge product in $\Omega^*_{\mathbb{T}}(M)$.

3. Curvature tensors

Consider a generalized connection $\mathbb{\nabla}$ on V. Let $\mathbb{\nabla}^{\mathbb{T}}$ be any generalized connection on $\mathbb{T}M$. The $\mathbb{\nabla}^{\mathbb{T}}$-derivation $\mathbb{d}^{\mathbb{T}}$ (2-3) extends to $\Omega^k_{\mathbb{T}}(V) := C^\infty\big(\bigwedge^k \mathbb{T}M \otimes V\big)$:

$$\mathbb{d}^{\mathbb{T}}_{\mathbb{\nabla}}(\theta \otimes v) := (\mathbb{d}^{\mathbb{T}}\theta) \otimes v + (-1)^k \theta \wedge \mathbb{\nabla} v, \tag{3-1}$$

where $v \in C^\infty(V)$. The $\mathbb{\nabla}^{\mathbb{T}}$-*curvature operator* $\mathcal{F}^{\mathbb{T}}(\mathbb{\nabla})$ of $\mathbb{\nabla}$ is then given by

$$\mathcal{F}^{\mathbb{T}}(\mathbb{\nabla}) := \mathbb{d}^{\mathbb{T}}_{\mathbb{\nabla}} \circ \mathbb{\nabla}, \tag{3-2}$$

which generally is not tensorial in v if $\mathbb{\nabla}^{\mathbb{T}}$ is not TM-torsion-free. When $\mathbb{\nabla}$ is understood, it is often dropped from the notation $\mathcal{F}^{\mathbb{T}}(\mathbb{\nabla})$.

In terms of covariant derivatives, the $\mathbb{\nabla}^{\mathbb{T}}$-curvature operator $\mathcal{F}^{\mathbb{T}}$ is given by

$$\mathcal{F}^{\mathbb{T}}_{x,y}(\mathbb{\nabla})v := (\mathbb{\nabla}_x \mathbb{\nabla}_y - \mathbb{\nabla}_y \mathbb{\nabla}_x - \mathbb{\nabla}_{x \diamond_{\mathbb{T}} y})v, \tag{3-3}$$

where $x, y \in C^\infty(\mathbb{T}M)$ and $v \in C^\infty(V)$. It is tensorial if and only if $\mathbb{\nabla}^{\mathbb{T}}$ is TM-torsion-free, in which case, for any $f \in C^\infty(M)$,

$$\mathcal{F}^{\mathbb{T}}_{x,y}(fv) - f\mathcal{F}^{\mathbb{T}}_{x,y}v = ([\pi(x), \pi(y)] - \pi(x \diamond_{\mathbb{T}} y))(f)v = 0. \tag{3-4}$$

The resulting tensor $\mathcal{F}^{\mathbb{T}} \in \Omega^2_{\mathbb{T}}(\mathrm{End}(V))$ is the $\mathbb{\nabla}^{\mathbb{T}}$-*curvature* of $\mathbb{\nabla}$. Similar to (3-1), let $\widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{\nabla}}$ be the extension of $\widetilde{\mathbb{d}}^{\mathbb{T}}$ to $\widetilde{\Omega}^*_{\mathbb{T}}(V) := \widetilde{\Omega}^*_{\mathbb{T}}(M) \otimes C^\infty(V)$. By (2-18), $\mathcal{F}^{\mathbb{T}}$ can be seen as an element in $\widetilde{\Omega}^2_{\mathbb{T}}(\mathrm{End}(V))$ and (3-2) can also be rewritten as

$$\mathcal{F}^{\mathbb{T}} = \widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{\nabla}} \circ \mathbb{\nabla}. \tag{3-5}$$

Example 3.1. Let (V, h) be a Hermitian vector bundle on M. A generalized connection $\mathbb{\nabla}$ on V *preserves* h, or is $(h\text{-})$*unitary* if for $v_j \in C^\infty(V)$ and $x \in \mathbb{T}M$ with $X = \pi(x)$,

$$Xh(v_1, v_2) = h(\mathbb{\nabla}_x v_1, v_2) + h(v_1, \mathbb{\nabla}_x.v_2).$$

Suppose now that V is a Hermitian line bundle and s is a local section of V such that $h(s, s) = 1$. Since $\mathbb{V}$ is unitary, it is determined by a local section $u \in C^\infty(\mathbb{T}M)$, such that for $x \in \mathbb{T}M$,

$$\sqrt{-1}\mathbb{V}_x s = 2\langle x, u\rangle s.$$

In analogy with the classical computation, the $\mathbb{V}^{\mathbb{T}}$-curvature for the line bundle V is then

$$\sqrt{-1}\mathcal{F}^{\mathbb{T}}_{x,y} = 2(\langle y, \mathbb{V}^{\mathbb{T}}_x u\rangle - \langle x, \mathbb{V}^{\mathbb{T}}_y u\rangle) = (\mathbb{d}^{\mathbb{T}} u)(x, y). \tag{3-6}$$

3A. *Chern–Weil homomorphism.* Let $\mathbb{V}^{\mathbb{T}}$ be a TM-torsion-free generalized connection on $\mathbb{T}M$. Since $\mathbb{d}^{\mathbb{T}}$ generally does not square to 0 (2-8), the Bianchi identity generally does not hold for $\mathcal{F}^{\mathbb{T}}$. In terms of covariant derivatives, $\mathbb{d}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}}$ can be expanded into

$$\begin{aligned}(\mathbb{d}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}})_{x,y,z} &= \mathbb{V}_x \mathcal{F}^{\mathbb{T}}_{y,z} - \mathcal{F}^{\mathbb{T}}_{\mathbb{V}^{\mathbb{T}}_x y,z} - \mathcal{F}^{\mathbb{T}}_{y,\mathbb{V}^{\mathbb{T}}_x z} - \mathcal{F}^{\mathbb{T}}_{y,z}\mathbb{V}_x + \text{c.p. in } x, y, z \\ &= -\mathbb{V}_{\mathcal{R}^{\mathbb{T}}_{x,y} z} - \text{c.p. in } x, y, z,\end{aligned} \tag{3-7}$$

where $x, y, z \in C^\infty(\mathbb{T}M)$. By (2-11) and (2-12), this gives

$$(\mathbb{d}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}})_{x,y,z} = \mathbb{V}_{[x\diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z]} = \psi_{[x\diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z]}, \tag{3-8}$$

which leads to the Bianchi identity over $\widetilde{\Omega}^*_{\mathbb{T}}(M)$.

Lemma 3.2. *Let $\mathcal{F}^{\mathbb{T}} \in \Omega^2_{\mathbb{T}}(M) \otimes \operatorname{End}(V)$ be the $\mathbb{V}^{\mathbb{T}}$-curvature of a generalized connection $\mathbb{V}$ on V. Then*

$$\widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}} = 0. \tag{3-9}$$

Proof. It follows from (3-8) that

$$\mathbb{d}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}} \in \operatorname{img} \mathscr{J}^{\mathbb{T}} \otimes \operatorname{End}(V).$$

Thus $\widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}} = \mathscr{Q}^{\mathbb{T}}(\mathbb{d}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}}) = 0$. □

The space $\mathscr{D}(V)$ of generalized connections on V is an affine space modeled on $\Omega^1_{\mathbb{T}}(\operatorname{End}(V))$, which coincides with $\widetilde{\Omega}^1_{\mathbb{T}}(\operatorname{End}(V))$. For $A \in \widetilde{\Omega}^1_{\mathbb{T}}(\operatorname{End}(V))$, a standard computation gives

$$\mathcal{F}^{\mathbb{T}}(\mathbb{V} + A) - \mathcal{F}^{\mathbb{T}}(\mathbb{V}) = \mathbb{d}^{\mathbb{T}}_{\mathbb{V}} A + A \wedge A.$$

It follows that

$$\operatorname{tr}_V(\mathcal{F}^{\mathbb{T}}(\mathbb{V} + A)) - \operatorname{tr}_V(\mathcal{F}^{\mathbb{T}}(\mathbb{V})) = \operatorname{tr}_V(\widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}} A) = \widetilde{\mathbb{d}}^{\mathbb{T}} \operatorname{tr}_V(A). \tag{3-10}$$

As in the classical case, (3-9) implies that

$$\widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}} \operatorname{tr}_V \mathcal{F}^{\mathbb{T}} = \operatorname{tr}_V \widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}} \mathcal{F}^{\mathbb{T}} = 0.$$

The gauge group $\mathrm{Aut}(V)$ acts on $\mathscr{D}(V)$ by pushforward. Namely, for $\lambda \in \mathrm{Aut}(V)$, $x \in \mathbb{T}M$ and $v \in C^\infty(V)$,

$$(\lambda\nabla)_x v := \lambda^{-1}[\nabla_x(\lambda v)].$$

It induces the action on the curvature by conjugation

$$\mathcal{F}^{\mathbb{T}}(\lambda\nabla) = \lambda^{-1}\mathcal{F}^{\mathbb{T}}(\nabla)\lambda.$$

Let $\mathscr{I}^{\mathbb{T}} \subset \Omega^*_{\mathbb{T}}(M)$ be the ideal generated by $\mathrm{img}\,\mathscr{J}^{\mathbb{T}}$,

$$\mathscr{I}^{\mathbb{T}} := \mathrm{img}\,\mathscr{J}^{\mathbb{T}} \wedge \Omega^*_{\mathbb{T}}(M),$$

and define $\overline{\Omega}^*_{\mathbb{T}}(M)$ as the quotient

$$0 \to \mathscr{I}^{\mathbb{T}} \to \Omega^*_{\mathbb{T}}(M) \xrightarrow{\overline{\mathscr{R}}^{\mathbb{T}}} \overline{\Omega}^*_{\mathbb{T}}(M) \to 0. \tag{3-11}$$

Then $\mathbb{d}^{\mathbb{T}}$ induces a differential on $\overline{\Omega}^*_{\mathbb{T}}(M)$,

$$\overline{\mathbb{d}}^{\mathbb{T}} : \overline{\Omega}^*_{\mathbb{T}}(M) \to \overline{\Omega}^*_{\mathbb{T}}(M),$$

whose cohomology is the *reduced* $\nabla^{\mathbb{T}}$*-de Rham cohomology*. The exact sequence

$$0 \to \frac{\mathscr{I}^{\mathbb{T}}}{\mathrm{img}\,\mathscr{J}^{\mathbb{T}}} \to \widetilde{\Omega}^*_{\mathbb{T}}(M) \xrightarrow{\mathscr{P}^{\mathbb{T}}} \overline{\Omega}^*_{\mathbb{T}}(M) \to 0$$

induces the map on the cohomologies

$$\mathscr{P}^{\mathbb{T}}_* : \widetilde{H}^*_{\mathbb{T}}(M) \to \overline{H}^*_{\mathbb{T}}(M).$$

The Chern–Weil homomorphism extends to define characteristic classes for V in $\overline{H}^*_{\mathbb{T}}(M)$.

Definition 3.3. For a Hermitian vector bundle (V, h) over M, its *k-th* $\nabla^{\mathbb{T}}$*-Chern class* is

$$c^{\mathbb{T}}_k(V) := [\mathrm{tr}_V(\sqrt{-1}\mathcal{F}^{\mathbb{T}}(\nabla))^k] \in \overline{H}^{2k}_{\mathbb{T}}(M), \tag{3-12}$$

where ∇ is a generalized connection on V. For real vector bundles, their $\nabla^{\mathbb{T}}$*-Euler* and $\nabla^{\mathbb{T}}$*-Pontrjagin* classes can similarly be defined, as elements of $\overline{H}^*_{\mathbb{T}}(M)$ of appropriate degrees.

By (3-10), the $\nabla^{\mathbb{T}}$-Chern classes do not depend on the choice of ∇ on V. Let ∇ be the lift of a classical connection ∇_0 on V, and let F_0 be the classical curvature of ∇_0. Then

$$\mathcal{F}^{\mathbb{T}}_{x,y} = F_{0;\pi(x),\pi(y)}$$

for $x, y \in C^\infty(\mathbb{T}M)$. This relates $c^{\mathbb{T}}_*(V)$ to the classical Chern classes $c_*(V)$.

Proposition 3.4. *Let (V, h) be a Hermitian vector bundle. Then*

$$c_k^{\mathbb{T}}(V) = \overline{\pi}^* c_k(V) := \mathscr{P}_*^{\mathbb{T}}(\widetilde{\pi}^* c_k(V))$$

for all k. In particular, $c_k^{\mathbb{T}}(V) = 0$ for all k such that $2k > n$. Similarly, the $\mathbb{V}^{\mathbb{T}}$-Euler and Pontrjagin classes are the images of the respective classical classes under $\overline{\pi}^*$. □

3B. *ϕ-curvatures.* A generalized Riemannian metric $\mathbb{G}$ induces an eigendecomposition of a generalized connection $\mathbb{V}$ on V. For $X \in C^\infty(TM)$ and $v \in C^\infty(V)$,

$$\nabla_{\pm,X} v := \mathbb{V}_{s_\pm(X)} v. \tag{3-13}$$

The connections $\nabla_\pm$ depend on b, while their difference does not (see [20]):

$$\psi_X := \tfrac{1}{2}(\nabla_{+,X} - \nabla_{-,X}) = \tfrac{1}{2}(\mathbb{V}_{s_+(X)} - \mathbb{V}_{s_-(X)}) = \mathbb{V}_{g(X)}. \tag{3-14}$$

The average of $\nabla_\pm$ gives the $\mathbb{G}$-*neutral connection* of $\mathbb{V}$,

$$\nabla_{0,X} := \mathbb{V}_{s_0(X)}, \tag{3-15}$$

which leads to

$$\nabla_\pm = \nabla_0 \pm \psi. \tag{3-16}$$

When $\psi = 0$, the generalized connection $\mathbb{V}$ is the lift of a classical connection ∇_0 on V, in which case $\nabla_\pm = \nabla_0$ are independent of b as well. The dependence on b of the $\mathbb{G}$-eigendecomposition of $\mathbb{V}$ can be described in terms of ψ.

Proposition 3.5. *Let $\mathbb{G}$ and $\mathbb{G}'$ be two generalized metrics corresponding to (g, b) and (g, b') respectively. Let $a = b' - b \in \Omega^2(M)$, and define j_a by*

$$j_a := g^{-1}a : TM \to TM \quad \text{given by} \quad X \mapsto g^{-1}(\iota_X a).$$

Let $\nabla_\pm$ and $\nabla'_\pm$ be the respective $\mathbb{G}$-eigendecomposition of $\mathbb{V}$ and $\mathbb{V}'$ on V. Then

$$\nabla'_\pm = \nabla_\pm + \psi_{j_a}.$$ □

Definition 3.6. Let $\mathbb{V}$ be a generalized connection on a vector bundle V over M. Let $\phi \in \Omega^3(M)$ and $\mathbb{G}$ a generalized metric on $\mathbb{T}M$. The ($\mathbb{G}$-*adapted*) ϕ-*curvature* $\mathcal{F}^\phi(\mathbb{V})$ of $\mathbb{V}$ is its $\mathbb{V}^\phi$-curvature (3-2), and is denoted by $\mathcal{F}^\phi$ if $\mathbb{V}$ is understood.

Given the pair of classical connections (∇_+, ∇_-), besides the curvature $F_\pm$ of each of them, there is also a *mixed curvature* $F_{+,-}$ [12; 39]:

$$F_{+,-;X,Y} v := (\nabla_{+,X}\nabla_{-,Y} - \nabla_{-,\nabla_X^{-\phi}Y})v - (\nabla_{-,Y}\nabla_{+,X} - \nabla_{+,\nabla_Y^{+\phi}X})v, \tag{3-17}$$

where $X, Y \in C^\infty(TM)$ and $v \in C^\infty(V)$. It can also be expressed using the tensor ψ:

$$F_{+,-;X,Y} = F_{+,X,Y} - 2(\nabla_{+,X}\psi)_Y - (\iota_{g^{-1}\psi}\phi)(X, Y). \tag{3-18}$$

Let F_0 be the classical curvature for ∇_0 (3-15). It gives another decomposition for the mixed curvature (3-17):

$$F_{+,-;X,Y} = F_{0;X,Y} + (\iota_{g^{-1}\psi}\phi)(X,Y) - [\psi_X,\psi_Y] - [(\nabla_{0,X}\psi)_Y + (\nabla_{0,Y}\psi)_X]. \tag{3-19}$$

Theorem 3.7. *The ϕ-curvature $\mathcal{F}^\phi(\mathbb{V})$ admits $\mathbb{G}$-eigendecomposition in terms of the (mixed) curvatures of the pair (∇_+,∇_-) of classical connections as follows:*

$$\mathcal{F}^\phi_{x_\pm,y_\pm} v = F_{\pm,X,Y} v \quad \textit{and} \quad \mathcal{F}^\phi_{x_+,y_-} v = F_{+,-;X,Y} v, \tag{3-20}$$

where for $X, Y \in TM$, $x_\pm = s_\pm(X)$, etc.

Proof. Straightforward from the definition, via the $\mathbb{G}$-eigendecomposition. □

Example 3.8. Continue with Example 3.1 for $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^\phi$. In this case, the local section $u \in C^\infty(\mathbb{T}M)$ decomposes into

$$u = \tfrac{1}{2}[(g^{-1}\nu_+ + bg^{-1}\nu_+ + \nu_+) - (g^{-1}\nu_- + bg^{-1}\nu_- - \nu_-)],$$

where $\sqrt{-1}\nu_\pm \in \Omega^1(M)$ are the local 1-forms defining the connections $\nabla_\pm$ respectively. It follows that

$$\sqrt{-1}F_\pm = d\nu_\pm,$$

and the mixed component in $\mathcal{F}^\phi$ is given by

$$\sqrt{-1}F_{+,-;X,Y} = [\nabla^{-\phi}_X \nu_-](Y) - [\nabla^{+\phi}_Y \nu_+](X).$$

Note that $F_{+,-}$ is neither symmetric nor skew-symmetric in X and Y, and decomposes into symmetric and skew-symmetric parts as

$$\sqrt{-1}F_{+,-} = -\mathcal{L}_{g^{-1}\psi} g + (\sqrt{-1}F_0 - \iota_{g^{-1}\psi}\phi),$$

where F_0 is the curvature of the $\mathbb{G}$-neutral connection ∇_0 and $\psi = \frac{1}{2}(\nu_+ - \nu_-)$, in the decomposition (3-16) of $\mathbb{V}$.

Corollary 3.9. *In the $\mathbb{G}$-splitting of $\mathbb{T}M$, the ϕ-curvature is*

$$\begin{aligned}\mathcal{F}^\phi_{x,y} = F_{0;X,Y} &+ (\nabla_{0,X}\psi)_{g^{-1}\eta} - (\nabla_{0,Y}\psi)_{g^{-1}\xi} + [\psi_{g^{-1}\xi},\psi_{g^{-1}\eta}] \\ &+ \tfrac{1}{2}\big[(\iota_{g^{-1}\psi}\phi)(X,Y) - (\iota_{g^{-1}\psi}\phi)(g^{-1}\xi, g^{-1}\eta)\big],\end{aligned} \tag{3-21}$$

where $x = s_0(X) + \xi$ and $y = s_0(Y) + \eta$. □

Remark 3.10. In (3-3), only the term $\mathbb{V}_{x\diamond_\phi y}$ depends on $\mathcal{F}^\phi$ on ϕ and $\mathbb{G}$. Let $\mathbb{V}^{\phi'}$ be the $\mathbb{G}'$-adapted ϕ'-connection. Then

$$\mathbb{V}_{x\diamond_\phi y} - \mathbb{V}_{x\diamond_{\phi'} y} = \psi_{g^{-1}(x\diamond_\phi y - x\diamond_{\phi'} y)}.$$

In the classical expansions, the dependence of $\mathcal{F}^\phi$ on ϕ is completely contained in the last term of (3-18), or the second line in (3-21); while the dependence on g is contained in the last term of the second line in (3-21). The dependence of $\mathcal{F}^{\phi}$ on b is more complicated. Nonetheless, it can be derived from (3-21) by relatively lengthy computations, noting that ∇_0 as well as ξ and η in the expression all depend on b.

3C. ***Yang–Mills functional.*** As one further example, it is straightforward to extend the Yang–Mills functional to this context. The generalized metric $\mathbb{G}$ induces natural inner product on $\bigwedge^* \mathbb{T}M$. For a Hermitian bundle (V, h), it induces a natural norm on $\bigwedge^* \mathbb{T}M \otimes \mathrm{End}(V)$, denoted by $\|\bullet\|_h$. The $\nabla^{\mathbb{T}}$-*Yang–Mills functional* on $\mathscr{D}(V)$ is given by

$$\mathcal{YM}_{\mathbb{T}}(\mathbb{V}) := \int_M \|\mathcal{F}^{\mathbb{T}}(\mathbb{V})\|_h^2 \, d\mathrm{vol}_g. \tag{3-22}$$

It is evidently invariant under the gauge action on $\mathscr{D}(V)$. When restricted to the subspace of the lifts of classical connections on V, $\mathcal{YM}_{\mathbb{T}}(\mathbb{V})$ reduces to (a constant multiple of) the classical Yang–Mills functional. It can also be regarded as a functional of the pair $(\mathbb{V}, \nabla^{\mathbb{T}})$ of generalized connections on V and $\mathbb{T}M$ respectively.

When $\nabla^{\mathbb{T}} = \nabla^{\phi}$, it can be represented as

$$\mathcal{YM}_{\phi}(\mathbb{V}) = \mathrm{YM}(\nabla_+) + \mathrm{YM}(\nabla_-) + 2\int_M \|F_{+,-}\|_h^2 \, d\mathrm{vol}_g, \tag{3-23}$$

where $\mathrm{YM}(\bullet)$ denotes the classical Yang–Mills functional. The 2-form b affects only the $\mathbb{G}$-eigendecomposition of $\mathbb{V}$. The right-hand side can be seen as a functional for a pair of classical connections (∇_+, ∇_-), where the last term encodes the dependence on $\phi \in \Omega^3(M)$ (Remark 3.10), as well as the interaction within the pair.

4. Curvatures on $\mathbb{T}M$

For a $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$ on $\mathbb{T}M$, its ϕ-curvature is denoted by $\mathcal{R}^{\mathbb{T},\phi}$ and the associated *curvature tensor* is given by

$$\mathcal{R}^{\mathbb{T},\phi}(x, y, z, w) := \mathbb{G}(\mathcal{R}^{\mathbb{T},\phi}_{x,y} z, w), \tag{4-1}$$

where $x, y, z, w \in C^{\infty}(\mathbb{T}M)$. Similar to the classical situation, it is skew in the first two and the last two entries respectively:

$$\mathcal{R}^{\mathbb{T},\phi}(x, y, z, w) = -\mathcal{R}^{\mathbb{T},\phi}(y, x, z, w) = \mathcal{R}^{\mathbb{T},\phi}(y, x, w, z).$$

Definition 4.1. The *ϕ-Riemannian curvature* $\mathcal{R}^{\phi}$ for (M, g) is the ϕ-curvature for ∇^{ϕ}, and the corresponding curvature tensor is the *ϕ-Riemann tensor*, which is also denoted by $\mathcal{R}^{\phi}$.

4A. ***Bianchi identities.*** Since ∇^{ϕ} preserves $C_{\pm}$, $\mathcal{R}^{\phi}(x, y, z, w)$ vanishes when the last two entries are sections of different $\mathbb{G}$-eigenbundles. The nonvanishing components in the $\mathbb{G}$-eigendecomposition of $\mathcal{R}^\phi$ are given by the classical Riemann tensor R of g as well as the curvature tensors $R^{\pm\phi}$ for $\nabla^{\pm\phi}$.

Proposition 4.2. *Let $X, Y, Z, W \in TM$ and $x_\pm = s_\pm(X) \in C_\pm$, etc. Then*:

(1) $\mathcal{R}^\phi(x_\pm, y_\pm, z_\pm, w_\pm) = R(X, Y, Z, W)$.

(2) $\mathcal{R}^\phi(x_\mp, y_\pm, z_\pm, w_\pm) = R^{\pm\phi}(X, Y, Z, W) \mp \frac{1}{2}(\nabla^{\pm\phi}_X \phi)(Y, Z, W)$.

(3) $\mathcal{R}^\phi(x_\mp, y_\mp, z_\pm, w_\pm) = R^{\pm\phi}(X, Y, Z, W)$.

All other components of $\mathcal{R}^\phi$ vanish.

Proof. Standard computations from the definitions, which is left for the reader. □

By (1) above, the algebraic Bianchi identity for $\mathcal{R}^\phi$ holds when all entries involved are from the same $\mathbb{G}$-eigenbundle (see also [12] Proposition 3.24). The analogues to the algebraic and differential Bianchi identities follow from previous discussion.

Lemma 4.3. *In their respective $\mathbb{G}$-eigendecompositions*:

(1) *For $\mathcal{R}^\phi_{x,y} z + c.p.$*,

$$\mathcal{R}^\phi_{x_\mp, y_\mp} z_\pm + \mathcal{R}^\phi_{y_\mp, z_\pm} x_\mp + \mathcal{R}^\phi_{z_\pm, x_\mp} y_\mp = \pm 2g(R^{\pm\phi}_{X,Y} Z). \tag{4-2}$$

(2) *For $\mathbb{d}^\phi_{\mathbb{V}} \mathcal{F}^\phi$*,

$$(\mathbb{d}^{\phi,\mathbb{G}}_{\mathbb{V}} \mathcal{F}^\phi)_{x_\mp, y_\mp, z_\pm} = \mp 2\mathbb{V}_{g(R^{\pm\phi}_{X,Y} Z)} = \mp 2\psi_{R^{\pm\phi}_{X,Y} Z}. \tag{4-3}$$

All other components vanish.

Proof. The identity (4-2) follows from (2-11) and (2-40). If $d\phi = 0$, (4-2) can also be obtained from the explicit expressions in Proposition 4.2 and the identity below (see [4]):

$$R^{+\phi}(X, Y, Z, W) = R^{-\phi}(Z, W, X, Y) + \tfrac{1}{2}(d\phi)(X, Y, Z, W). \tag{4-4}$$

Then (4-3) follows from (3-8) and (4-2). □

In particular, the differential Bianchi identity holds for the ϕ-curvature when $\psi = 0$, i.e., $\mathbb{V}$ is the lifting of a classical connection on V. Another special case is when $\nabla^{\pm\phi}$ are flat [1; 8; 7], i.e., $R^{\pm\phi} = 0$, and thus both Bianchi identities hold. In this second special case, $\mathcal{R}^\phi$ enjoys all the symmetries of a classical Riemann curvature.

Theorem 4.4. *Suppose that the connections $\nabla^{\pm\phi}$ are flat on TM. Then*:

(1) $\mathcal{R}^\phi(x, y, z, w) = -\mathcal{R}^\phi(y, x, z, w) = \mathcal{R}^\phi(y, x, w, z)$.

(2) $\mathcal{R}^\phi_{x,y} z + \mathcal{R}^\phi_{y,z} x + \mathcal{R}^\phi_{z,x} y = 0$.

(3) $\mathbb{d}^\phi_{\mathbb{V}} \mathcal{R}^\phi = 0$.

(4) $\mathcal{R}^\phi(x, y, z, w) = \mathcal{R}^\phi(z, w, x, y)$.

In this case, $\mathcal{R}^\phi$ defines a symmetric pairing on $\bigwedge^2 \mathbb{T}M$,

$$\mathcal{R}^\phi : \textstyle\bigwedge^2 \mathbb{T}M \otimes \bigwedge^2 \mathbb{T}M \to \mathbb{R} \quad \textit{given by} \quad \mathcal{R}^\phi(x \wedge y, w \wedge z) := \mathcal{R}^\phi(x, y, z, w),$$

which defines the corresponding operator on $\bigwedge^2 \mathbb{T}M$ via $\mathbb{G}$.

Proof. Items (1)–(3) follow from previous discussion and the flatness assumption, while (4) follows from (1)–(3) as in the classical situation. The last statement is a consequence of (1) and (4). □

The following consequence of Lemma 4.3 was used in the proof of Proposition 2.15.

Proposition 4.5. *If $d\phi = 0$, then $\mathbb{d}^\phi$ is a differential at $\Omega_{\mathbb{T}}^{2n-1}(M)$.*

Proof. Let $\{X_j\}$ be a local g-orthonormal frame of TM and $\{e_\pm^j\}$ the induced $\mathbb{G}$-orthonormal frame of $\mathbb{T}M$. Let $\theta \in \Omega_{\mathbb{T}}^{2n-2}(M)$. Then, by (2-11), (2-16) and (4-2),

$$\begin{aligned}
&(\mathbb{d}^\phi \circ \mathbb{d}^\phi \theta)(e_+^1, \dots, e_+^n, e_-^1, \dots, e_-^n) \\
&\quad = -\sum_{i<j,k} (-1)^{i+j+k+n} \theta(-g(R^{-\phi}_{X_i,X_j} X_k), \dots, \hat{e}_+^i, \dots, \hat{e}_+^j, \dots, \hat{e}_-^k, \dots) \\
&\qquad\qquad - \sum_{i,j<k} (-1)^{i+j+k} \theta(g(R^{+\phi}_{X_j,X_k} X_i), \dots, \hat{e}_+^i, \dots, \hat{e}_-^j, \dots, \hat{e}_-^k, \dots) \\
&\quad = \sum_{i,j,k} (-1)^{j+k+n} \big[-R^{-\phi}(X_i, X_j, X_k, X_i) \\
&\qquad\qquad + R^{+\phi}(X_i, X_k, X_j, X_i)\big] \theta(\dots, \hat{e}_+^j, \dots, \hat{e}_-^k, \dots) \\
&\quad = 0.
\end{aligned}$$

Since $d\phi = 0$, the last step above follows from (4-4). □

4B. *Ricci curvature.* The trace of the ϕ-curvature on $\mathbb{T}M$ defines the corresponding ϕ-Ricci curvature.

Definition 4.6. For a $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$, the *ϕ-Ricci curvature $\mathcal{R}ic^{\mathbb{T},\phi}$* : $\mathbb{T}M \to \mathbb{T}M$ for $(M, \mathbb{G})$ is the trace of the ϕ-curvature $\mathcal{R}^{\mathbb{T},\phi}$. For $x, y \in C^\infty(\mathbb{T}M)$, in the local orthonormal frame $\{e_+^i, e_-^j\}$ of $\mathbb{T}M$ induced from a local g-orthonormal frame $\{X_i\}$,

$$\mathcal{R}ic^{\mathbb{T},\phi}(x) := \sum_i [\mathcal{R}^{\mathbb{T},\phi}_{x,e_+^i} e_+^i + \mathcal{R}^{\mathbb{T},\phi}_{x,e_-^i} e_-^i]. \tag{4-5}$$

The *ϕ-Ricci tensor $\mathcal{R}c^{\mathbb{T},\phi}$* $\in C^\infty(\mathbb{T}M \otimes \mathbb{T}M)$ is

$$\mathcal{R}c^{\mathbb{T},\phi}(x, y) := \mathbb{G}(\mathcal{R}ic^{\mathbb{T},\phi}(x), y). \tag{4-6}$$

For the connection ∇^ϕ, these are denoted by $\mathcal{R}ic^\phi$ and $\mathcal{R}c^\phi$ respectively.

Specialized to the $\mathbb{G}$-adapted connection $\mathbb{V}^\phi$, the $\mathbb{G}$-eigendecomposition of $\mathcal{R}c^\phi$ can be determined from that of $\mathcal{R}^\phi$ as follows:

$$\mathcal{R}c^\phi(x_\pm, y_\pm) = \mathrm{Rc}(X, Y) \quad \text{and} \quad \mathcal{R}c^\phi(x_\pm, y_\mp) = \mathrm{Rc}^{\mp\phi}(X, Y), \tag{4-7}$$

where Rc is the Ricci tensor for ∇, $\mathrm{Rc}^{\pm\phi}$ are the Ricci tensors for $\nabla^{\pm\phi}$ respectively:

$$\mathrm{Rc}^{\pm\phi} = \mathrm{Rc} \mp \tfrac{1}{2} d^*\phi - \tfrac{1}{4}\phi^2, \tag{4-8}$$

where

$$\phi^2(X, Y) := \sum_{i,j} \phi(X, X_i, X_j)\phi(Y, X_i, X_j).$$

It follows that $\mathcal{R}c^\phi$ is symmetric; in other words,

$$\langle \mathcal{R}ic^\phi(x), \mathbb{G}y\rangle = \langle \mathbb{G}\,\mathcal{R}ic^\phi(x), y\rangle = \langle \mathbb{G}\,\mathcal{R}ic^\phi(y), x\rangle = \langle \mathbb{G}x, \mathcal{R}ic^\phi(y)\rangle. \tag{4-9}$$

Constructions of generalized Ricci curvature or tensor in the literature, such as in [10; 12; 37; 38], contain only the mixed components, and are generally in the context of generalized Ricci flows. Indeed, as will become clear in Section 7, only the mixed components contribute to the generalized Ricci flow.

Similar to the classical case, the ϕ-Ricci curvature described here appears in a Weitzenböck identity relating two natural Laplacians on $\bigwedge^* \mathbb{T}M$ described in Section 2D.

Theorem 4.7. *On $\Omega^1_{\mathbb{T}}(M)$, the following Weitzenböck identity holds*:

$$\Delta^\phi = \Delta_{\mathbb{V}^\phi} + \mathbb{G}\,\mathcal{R}ic^\phi\,\mathbb{G}, \tag{4-10}$$

where Δ^ϕ is the $\mathbb{V}^\phi$-Hodge Laplacian (2-47) *while $\Delta_{\mathbb{V}^\phi}$ is the Bochner Laplacian* (2-44).

Proof. It can be shown following standard computations that for $\theta \in \Omega^*_{\mathbb{T}}(M)$,

$$\Delta^\phi\,\theta = \Delta_{\mathbb{V}^\phi}\,\theta - \frac{1}{2}\sum_{\alpha,\beta} \mathbb{G}(e_\alpha) \wedge \iota_{e_\beta}(\mathcal{R}^\phi_{e_\alpha, e_\beta}\theta),$$

where e_α, e_β run through $\{e^i_+, e^j_-\}$. Set $\theta = \mathbb{G}(x)$ for $x \in \mathbb{T}M$. Then

$$\begin{aligned}\sum_{\alpha,\beta}\big\langle \mathbb{G}(e_\alpha) \wedge \iota_{e_\beta}(\mathcal{R}^\phi_{e_\alpha,e_\beta}\mathbb{G}(x)), y\big\rangle &= 2\sum_{\alpha,\beta}\langle \mathbb{G}(\mathcal{R}^\phi_{e_\alpha,e_\beta}x), e_\beta\rangle \mathbb{G}(e_\alpha, y)\\ &= 2\sum_\beta \mathcal{R}^\phi(y, e_\beta, x, e_\beta)\\ &= -2\,\mathcal{R}c^\phi(y, x) = -2\langle \mathbb{G}\,\mathcal{R}ic^\phi(x), y\rangle,\end{aligned}$$

from which (4-10) follows. □

The symmetry of $\mathcal{R}c^\phi$ implies that $\mathbb{d}^{\phi *}$ is a differential at $\Omega^1_{\mathbb{T}}(M)$. For $x_i \in C^\infty(\mathbb{T}M)$, $i = 1, \dots, k$, set $y_i = \mathbb{G}(x_i)$. Then a straightforward computation gives

$$(4\text{-}11)\quad \begin{aligned}&(\mathbb{d}^{\phi *} \circ \mathbb{d}^{\phi *})[y_1 \wedge \cdots \wedge y_k]\\ &= \sum_{i<j} (-1)^{i+j} [\mathcal{R}c^\phi(x_j, x_i) - \mathcal{R}c^\phi(x_i, x_j)] y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_k\\ &\quad - \sum_{i<j<\ell} (-1)^{i+j+\ell} \mathbb{G}(\mathcal{R}^\phi_{x_i,x_j} x_\ell + \text{c.p.}) y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge \hat{y}_\ell \wedge \cdots \wedge y_k.\end{aligned}$$

Due to the symmetry of $\mathcal{R}c^\phi$, the terms in the second line vanish. Furthermore, when $k = 2$, the terms in the last line vanish as well.

Proposition 4.8. *The operator $\mathbb{d}^{\phi *}$ is a differential at $\Omega^1_{\mathbb{T}}(M)$, i.e.,*

$$(4\text{-}12)\qquad \theta \in \Omega^2_{\mathbb{T}}(M) \implies (\mathbb{d}^{\phi *} \circ \mathbb{d}^{\phi *})\theta = 0.$$

If the connections $\nabla^{\pm\phi}$ are flat on TM, then $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^\phi)$ and $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\phi *})$ are both chain complices.

Proof. For the last statement, that $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^\phi)$ is a chain complex follows from (2-8), (2-11) and the algebraic Bianchi identity, which is item (2) in Theorem 4.4. The statement for $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\phi *})$ follows from (4-11) and the algebraic Bianchi identity. □

4C. *Bismut connection.* The *generalized Bismut connection* $\mathbb{\nabla}^{\phi,\mathcal{B}}$ introduced by Gualtieri [20] is a $\mathbb{G}$-metric connection and is the lift of a classical connection $\nabla^{\phi,\mathcal{B}}$ on $\mathbb{T}M$:

$$(4\text{-}13)\qquad \mathbb{\nabla}^{\phi,\mathcal{B}}_X s_\pm(Y) = s_\pm(\nabla^{\pm\phi}_X Y).$$

Note $\mathbb{\nabla}^{\phi,\mathcal{B}}$ is compatible with the (almost) Dorfman bracket $*_{\phi-db}$ (Definition 2.7).

Since $\mathbb{\nabla}^{\phi,\mathcal{B}}$ is a lift of a classical connection, by (4-3), the *ϕ-Bismut curvature* $\mathcal{R}^{\phi,\mathcal{B}}$ of $\mathbb{\nabla}^{\phi,\mathcal{B}}$ satisfies the differential Bianchi identity:

$$\mathbb{d}^\phi_{\mathbb{\nabla}} \mathcal{R}^{\phi,\mathcal{B}} = 0.$$

More explicitly, $\mathcal{R}^{\phi,\mathcal{B}}$ is determined by the classical curvature of $\nabla^{\phi,\mathcal{B}}$, which in turn is given by $R^{\pm\phi}$:

$$(4\text{-}14)\qquad \mathcal{R}^{\phi,\mathcal{B}}(x, y, s_\pm(Z), s_\pm(W)) = R^{\pm\phi}(X, Y, Z, W).$$

The $\mathbb{G}$-eigendecomposition of the corresponding *ϕ-Bismut Ricci tensor* $\mathcal{R}c^{\phi,\mathcal{B}}$ is thus

$$(4\text{-}15)\quad \mathcal{R}c^{\phi,\mathcal{B}}(x, s_+(Y)) = \mathrm{Rc}^{+\phi}(X, Y) \quad\text{and}\quad \mathcal{R}c^{\phi,\mathcal{B}}(x, s_-(Y)) = \mathrm{Rc}^{-\phi}(X, Y).$$

4D. ***Scalar curvatures.*** The traces of the ϕ-Ricci curvatures (4-5) give the corresponding *ϕ-scalar curvatures*, which depend on the $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$. For instance, the *ϕ-Riemann scalar curvature $\mathcal{S}^{\phi}$* is the trace of $\mathcal{R}ic^{\phi}$:

$$\mathcal{S}^{\phi} = \sum_j [\mathcal{R}c^{\phi}(e_+^j, e_+^j) + \mathcal{R}c^{\phi}(e_-^j, e_-^j)] = 2S, \tag{4-16}$$

where S is the classical scalar curvature of g. On the other hand, the *ϕ-Bismut scalar curvature $\mathcal{S}^{\phi,\mathcal{B}}$* is the trace of $\mathcal{R}ic^{\phi,\mathcal{B}}$:

$$\mathcal{S}^{\phi,\mathcal{B}} = \sum_j [\mathcal{R}c^{\phi,\mathcal{B}}(e_+^j, e_+^j) + \mathcal{R}c^{\phi,\mathcal{B}}(e_-^j, e_-^j)] = 2S - 3\|\phi\|_g^2,$$

where $\|\phi\|_g$ is the norm of ϕ with respect to g,

$$\|\phi\|_g^2 = \sum_{i<j<k} \phi(X_i, X_j, X_k)^2.$$

5. Generalized complex manifolds

Let $\mathbb{J}$ be a generalized almost complex structure on M [19; 21; 24]. It induces a polarization of $\mathbb{T}_{\mathbb{C}}M := \mathbb{T}M \otimes_{\mathbb{R}} \mathbb{C}$ as the direct sum of its $\pm\sqrt{-1}$-eigenbundles:

$$\mathbb{T}_{\mathbb{C}}M = \mathbb{T}_{\mathbb{J}}^{1,0}M \oplus \mathbb{T}_{\mathbb{J}}^{0,1}M. \tag{5-1}$$

Here $\mathbb{T}_{\mathbb{J}}^{1,0}M$ denotes the $\sqrt{-1}$-eigenbundle of $\mathbb{J}$, and $\mathbb{T}_{\mathbb{J}}^{0,1}M$ its complex conjugate. They are maximally isotropic and are dual to each other under the pairing $2\langle\ ,\ \rangle$ on $\mathbb{T}_{\mathbb{C}}M$. For instance, the space of (0, 1)-*forms with respect to* $\mathbb{J}$ is identified with the sections of $\mathbb{T}_{\mathbb{J}}^{1,0}M$:

$$\Omega_{\mathbb{J}}^{0,1}(M) := C^{\infty}(\mathbb{T}_{\mathbb{J}}^{1,0}M).$$

Similar to the classical case, the $(0,k)$-forms with respect to $\mathbb{J}$ are sections of $\bigwedge^k \mathbb{T}_{\mathbb{J}}^{1,0}M$:

$$\Omega_{\mathbb{J}}^{0,k}(M) := C^{\infty}\big(\textstyle\bigwedge^k \mathbb{T}_{\mathbb{J}}^{1,0}M\big).$$

In general, the type decomposition of $\Omega_{\mathbb{T}}^{*}(M)$ with respect to $\mathbb{J}$ is given by

$$\Omega_{\mathbb{T}}^{*}(M) = \bigoplus_{p,q} \Omega_{\mathbb{J}}^{p,q}(M) := \bigoplus_{p,q} C^{\infty}\big(\textstyle\bigwedge^p \mathbb{T}_{\mathbb{J}}^{0,1}M \otimes \bigwedge^q \mathbb{T}_{\mathbb{J}}^{1,0}M\big). \tag{5-2}$$

For notational convenience, sometimes $\mathbb{T}_{\mathbb{J}}^{1,0}M$ is denoted by L, while $\mathbb{T}_{\mathbb{J}}^{0,1}M$ is denoted by $\bar{L}$.

Definition 5.1. Let $(M, \mathbb{J})$ be a generalized almost complex manifold and let $\gamma \in \Omega^3(M)$ be a closed 3-form. Then a generalized connection $\nabla^{\mathbb{T}}$ on $\mathbb{T}M$ is

$\mathbb{J}$-*compatible with* $*_\gamma$ if $\diamond_{\mathbb{T}}$ coincides with $*_\gamma$ on the sections from the same eigenbundle of $\mathbb{J}$, i.e.,

$$x \diamond_{\mathbb{T}} y = x *_\gamma y \quad \text{and} \quad \bar{x} \diamond_{\mathbb{T}} \bar{y} = \bar{x} *_\gamma \bar{y}, \tag{5-3}$$

where $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M)$. Such generalized connection $\nabla^{\mathbb{T}}$ is a γ-$\mathbb{J}$-*connection* if it furthermore is TM-torsion-free.

It is straightforward to see that (5-3) is equivalent to the following, where $x, y \in C^\infty(\mathbb{T}M)$:

$$(\mathbb{J}x) \diamond_{\mathbb{T}} y + x \diamond_{\mathbb{T}} (\mathbb{J}y) = (\mathbb{J}x) *_\gamma y + x *_\gamma (\mathbb{J}y).$$

Thus, if nonempty, the space $\mathscr{D}_{\mathbb{J},\gamma}(\mathbb{T}M)$ of generalized connections that are $\mathbb{J}$-compatible with $*_\gamma$ is modeled on

$$\mathscr{D}_{\mathbb{J},\gamma}(\mathbb{T}M) \cong \big\{A : \mathbb{T}M \otimes \mathbb{T}M \to \mathbb{T}M \ \text{ such that } \ A_{\mathbb{J}x} y - A_y(\mathbb{J}x) = A_{\mathbb{J}y} x - A_x(\mathbb{J}y)\big\}.$$

It's then evident by (2-13) that the subspace $\mathscr{D}_{\mathbb{J},\gamma,\tau}(\mathbb{T}M)$ of the γ-$\mathbb{J}$-connections, if nonempty, is modeled on

$$\mathscr{D}_{\mathbb{J},\gamma,\tau}(\mathbb{T}M) \cong \mathrm{Sym}^2_{\mathbb{T}}(\mathbb{T}M) \oplus \Omega^{1,1}_{\mathbb{T}}(T^*M), \tag{5-4}$$

where $\Omega^{1,1}_{\mathbb{T}}(T^*M)$ consists of T^*M-valued forms that are compatible with $\mathbb{J}$, i.e.,

$$\theta \in \Omega^{1,1}_{\mathbb{T}}(T^*M) \iff \theta(\mathbb{J}x, \mathbb{J}y) = \theta(x, y).$$

Let $\gamma \in \Omega^3(M)$ be a closed 3-form. Recall that $\mathbb{J}$ is *integrable* (*with respect to* γ) if $\mathbb{T}^{1,0}_{\mathbb{J}} M$ is involutive under the Dorfman bracket $*_\gamma$, i.e.,

$$x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M) \implies x *_\gamma y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M).$$

In this case, $(M, \gamma; \mathbb{J})$ is a *generalized complex manifold* [19; 21; 24].

Lemma 5.2. *Let* $(M, \gamma; \mathbb{J})$ *be a generalized complex manifold with a generalized connection* $\nabla^{\mathbb{T}}$ *on* $\mathbb{T}M$ *that is* $\mathbb{J}$*-compatible with* $*_\gamma$*. Then the operator* $\mathbb{d}^{\mathbb{T}}$ *decomposes into components as follows*:

$$\mathbb{d}^{\mathbb{T}} = \partial^{\mathbb{T}}_{\mathbb{J}} + \bar{\partial}^{\mathbb{T}}_{\mathbb{J}} : \Omega^{p,q}_{\mathbb{J}}(M) \to \Omega^{p+1,q}_{\mathbb{J}}(M) \oplus \Omega^{p,q+1}_{\mathbb{J}}(M). \tag{5-5}$$

Proof. The proof is similar to that of Lemma 2.5, employing (5-3) and the integrability of $\mathbb{J}$. The details are left for the reader. □

The integrability of $\mathbb{J}$ implies that both of its eigenbundles are complex Lie algebroids, with their Lie brackets given by the restriction of $*_\gamma$. The corresponding Lie algebroid de Rham differential for $\bar{L} = \mathbb{T}^{0,1}_{\mathbb{J}} M$ will be denoted by $d_{\bar{L}}$:

$$d_{\bar{L}} : \Omega^{0,k}_{\mathbb{J}}(M) \to \Omega^{0,k+1}_{\mathbb{J}}(M). \tag{5-6}$$

If $\mathbb{\nabla}^{\mathbb{T}}$ is $\mathbb{J}$-compatible with $*_\gamma$, then $d_{\bar{L}}$ coincides with the restriction of $\mathbb{d}^{\mathbb{T}}$ on $\Omega^{0,*}_{\mathbb{J}}(M)$. More precisely, for $\theta \in \Omega^{0,q}_{\mathbb{J}}(M)$, direct computation shows that

$$(d_{\bar{L}}\theta)(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_q) = \sum_j (-1)^j (\mathbb{\nabla}^{\mathbb{T}}_{\bar{x}_j}\theta)(\bar{x}_0, \dots, \hat{\bar{x}}_j, \dots, \bar{x}_q), \tag{5-7}$$

where $x_j \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$ for all j, which gives

$$\bar{\partial}^{\mathbb{T}}_{\mathbb{J}}\theta = d_{\bar{L}}\theta \in \Omega^{0,q+1}_{\mathbb{J}}(M). \tag{5-8}$$

The other component $\partial^{\mathbb{T}}_{\mathbb{J}}\theta \in \Omega^{1,q}_{\mathbb{J}}(M)$ is given by

$$(\partial^{\mathbb{T}}_{\mathbb{J}}\theta)(x_0, \bar{x}_1, \dots, \bar{x}_q) = (\mathbb{\nabla}^{\mathbb{T}}_{x_0}\theta)(\bar{x}_1, \dots, \bar{x}_q) + \sum_j \theta(\bar{x}_1, \dots, \mathbb{\nabla}^{\mathbb{T}}_{\bar{x}_j}x_0, \dots, \bar{x}_q), \tag{5-9}$$

where $x_j \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$. Via complex conjugation, the analogous versions of the identities (5-7), (5-8) and (5-9) are valid for $\bar{\theta} \in \Omega^{q,0}_{\mathbb{J}}(M)$ with $L = \mathbb{T}^{1,0}_{\mathbb{J}}M$ in place of $\bar{L}$.

Example 5.3. Suppose that $\mathbb{\nabla}^{\mathbb{T}}$ is $\mathbb{J}$-compatible with $*_\gamma$ and consider $f \in C^\infty(M)$. It follows from Proposition 2.2 and Lemma 5.2 that

$$\partial^{\mathbb{T}}_{\mathbb{J}}\partial^{\mathbb{T}}_{\mathbb{J}} f = 0,\ \bar{\partial}^{\mathbb{T}}_{\mathbb{J}}\bar{\partial}^{\mathbb{T}}_{\mathbb{J}} f = 0 \quad \text{and} \quad \partial^{\mathbb{T}}_{\mathbb{J}}\bar{\partial}^{\mathbb{T}}_{\mathbb{J}} f + \bar{\partial}^{\mathbb{T}}_{\mathbb{J}}\partial^{\mathbb{T}}_{\mathbb{J}} f = 0. \tag{5-10}$$

The first two identities in the above can also be seen as the consequences of (5-8) and the corresponding version for d_L.

5A. *Generalized (almost) Hermitian manifolds.* Let $(M; \mathbb{G}, \mathbb{J})$ be a *generalized almost Hermitian manifold* [19; 22], i.e., $\mathbb{J}$ and $\mathbb{J}_- := \mathbb{G}\mathbb{J}$ are commuting generalized almost complex structures. The eigenbundles of $\mathbb{J}$ (and $\mathbb{J}_-$) decompose into the common eigenbundles of $\mathbb{G}$ and $\mathbb{J}$. Let

$$\ell_\pm := \mathbb{T}^{1,0}_{\mathbb{J}}M \cap (C_\pm \otimes \mathbb{C}). \tag{5-11}$$

Then, for instance,

$$\mathbb{T}^{1,0}_{\mathbb{J}}M = \ell_+ \oplus \ell_- \quad \text{and} \quad \mathbb{T}^{1,0}_{\mathbb{J}_-}M = \ell_+ \oplus \bar{\ell}_-.$$

The restriction of $\mathbb{J}$ to $C_\pm$ induces two almost complex structures $I_\pm$ on TM:

$$s_\pm(I_\pm X) := \mathbb{J}_+[s_\pm(X)]. \tag{5-12}$$

It follows that $(M; \mathbb{G}, \mathbb{J})$ is equivalent to a pair of almost Hermitian structures $(g, I_\pm)$ together with $b \in \Omega^2(M)$ [19; 22].

Lemma 5.4. *For a generalized almost Hermitian manifold $(M; \mathbb{G}, \mathbb{J})$, the space of $\mathbb{G}$-metric γ-$\mathbb{J}$-connections, if nonempty, is modeled on the following subspace of $\Omega^3(M)$:*

$$\big\{\phi \in \Omega^3(M) : \phi(I_- X, Y, Z) + \phi(X, I_+ Y, Z) = 0 \textit{ for all } X, Y, Z \in C^\infty(TM)\big\}.$$

Proof. This follows from (5-4) and Theorem 2.6. □

Let $\phi = \gamma + db$. Then by Theorem 2.8, $\mathbb{V}^{\phi}$ is metric compatible with $*_\gamma$. Hence, the $\mathbb{J}$-compatibility of $\mathbb{V}^{\phi}$ with $*_\gamma$ is equivalent to $\diamond_\phi$ and $*_\gamma$ coincide on sections of the same common eigenbundle of $\mathbb{J}$ and $\mathbb{G}$, e.g., for $x_\pm, y_\pm \in C^\infty(\ell_\pm)$,

$$x_\pm *_\gamma y_\pm = x_\pm \diamond_\phi y_\pm. \tag{5-13}$$

Lemma 5.5. *On a generalized almost Hermitian manifold* $(M; \mathbb{G}, \mathbb{J})$, *let* $\gamma \in \Omega^3(M)$ *be a closed* 3*-form and* $\phi = \gamma + db$. *Then* $\mathbb{V}^{\phi}$ *is* $\mathbb{J}$*-compatible with* $*_\gamma$ *if and only if* $\nabla^{\pm\phi} I_\pm = 0$.

Proof. For any $X, Y, Z \in C^\infty(TM)$, let $x_\pm = s_\pm(X) \in C^\infty(C_\pm)$ and so on. Then

$$\langle x_\pm *_\gamma y_\pm - x_\pm \diamond_\phi y_\pm, z\rangle = \pm g(\nabla_Z^{\pm\phi} X, Y).$$

Thus (5-13) is equivalent to

$$g(\nabla_Z^{\pm\phi} X_\pm, Y_\pm) = 0$$

for all $Z \in C^\infty(TM)$ and $X_\pm, Y_\pm \in C^\infty(T_{\pm;1,0}M)$ respectively. Hence $\nabla^{\pm\phi}$ preserves $T_{\pm;1,0}M$ respectively, from which the statement follows. □

Remark 5.6. The condition $\nabla^{\pm\phi} I_\pm = 0$ can be rewritten as

$$(\nabla_X I_\pm)Y = \pm\tfrac{1}{2}(I_\pm g^{-1}\iota_Y\iota_X\phi - g^{-1}\iota_{I_\pm Y}\iota_X\phi) \tag{5-14}$$

for $X, Y \in C^\infty(TM)$. Hence, $I_\pm$ are of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ in the classification of almost Hermitian structures by Gray and Hervella [16]. The Nijenhuis tensors of $I_\pm$,

$$N_{I_\pm}(X, Y) := [X, Y] + I_\pm[I_\pm X, Y] + I_\pm[X, I_\pm Y] - [I_\pm X, I_\pm Y],$$

can be expressed in terms of ϕ:

$$g(N_{I_\pm}(X, Y), Z) = \phi(I_\pm X, I_\pm Y, Z) + \phi(I_\pm X, Y, I_\pm Z) + \phi(X, I_\pm Y, I_\pm Z) - \phi(X, Y, Z).$$

This implies that the integrability of $I_\pm$ is equivalent to ϕ being of type $(1, 2)+(2, 1)$ with respect to $I_\pm$ respectively. Furthermore, by Friedrich and Ivanov [9], if almost Hermitian connections for $(g, I_\pm)$ that admit completely skew torsions exist, they must be unique.

In $(M; \mathbb{G}, \mathbb{J})$, when $\mathbb{J}$ is integrable with respect to $\gamma \in \Omega^3(M)$, the structure $(M, \gamma; \mathbb{G}, \mathbb{J})$ defines a *generalized Hermitian manifold*. In this case, the $\mathbb{J}$-compatibility of $\mathbb{V}^{\phi}$ with $*_\gamma$ is equivalent to the integrability of $\mathbb{J}_-$, i.e., it provides a *generalized Kähler condition*.

Theorem 5.7. *On a generalized Hermitian manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$, *let* $\phi = \gamma + db$. *Then* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *is a generalized Kähler manifold if and only if* ∇^ϕ *is* $\mathbb{J}$*-compatible with* $*_\gamma$.

Proof. Starting with the $\mathbb{J}$-compatibility, then (5-13) and the integrability of $\mathbb{J}$ imply that $\ell_\pm$ and $\bar{\ell}_\pm$ are involutive with respect to $*_\gamma$, which further implies that $I_\pm$ are integrable. Let $x_+ = s_+(X_+) \in C^\infty(\ell_+)$ and $y_- = s_-(Y_-) \in C^\infty(\ell_-)$ for $X_+, Y_- \in C^\infty(T_{\mathbb{C}}M)$. Then, by (2-33) and Lemma 5.5,

$$x_+ *_\gamma \bar{y}_- = [\nabla^{-\phi}_{X_+}\overline{Y}_- + (b-g)\nabla^{-\phi}_{X_+}\overline{Y}_-] - [\nabla^{+\phi}_{\overline{Y}_-}X_+ + (b+g)\nabla^{+\phi}_{\overline{Y}_-}X_+] \in \ell_+ \oplus \bar{\ell}_-.$$

Thus $\mathbb{J}_-$ is integrable. The opposite direction is left to the reader. □

Remark 5.8. The generalized Kähler condition in Theorem 5.7 relates to the condition given in [20] as follows. The condition $\nabla^{\pm\phi} I_\pm = 0$ on a generalized almost Hermitian manifold is equivalent to $\nabla^{\phi,\mathcal{B}}\mathbb{J} = 0$, which implies the equivalence of the integrability of $I_\pm$ to the type condition for the *generalized torsion* as defined in [20] — the integrability of $\mathbb{J}$ then follows. In Theorem 5.7, the integrability of $I_\pm$ follows from the integrability of $\mathbb{J}$ and (5-13), obtaining the type of ϕ with respect to $I_\pm$ as a consequence.

Corollary 5.9 [20, Theorem 6.1]. *On a generalized Hermitian manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$, *let* $\phi = \gamma + db$. *Then* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *is a generalized Kähler manifold if and only if* $\nabla^{\phi,\mathcal{B}}\mathbb{J} = 0$.

Proof. As stated in Remark 5.8, $\nabla^{\phi,\mathcal{B}}\mathbb{J} = 0$ is equivalent to $\nabla^{\pm\phi} I_\pm = 0$, which by Lemma 5.5 is equivalent to ∇^ϕ being $\mathbb{J}$-compatible. □

Let $\nabla^{\mathbb{T}}$ be any $\mathbb{G}$-metric connection on a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$. Its $\mathbb{J}$-*Ricci form* $\rho^{\mathbb{T}}_{\mathbb{J}} := \rho_{\mathbb{J}}(\nabla^{\mathbb{T}}) \in \Omega^2_{\mathbb{T}}(M)$ is defined as

$$\rho^{\mathbb{T}}_{\mathbb{J}}(x, y) := \sum_i [\mathcal{R}^{\mathbb{T}}(x, y, \mathbb{J}e^i_+, e^i_+) + \mathcal{R}^{\mathbb{T}}(x, y, \mathbb{J}e^i_-, e^i_-)].$$

The $\mathbb{J}$-*scalar curvature* for $\nabla^{\mathbb{T}}$ is

$$\mathcal{S}^{\mathbb{T}}_{\mathbb{J}} := \sum_i [\rho^{\mathbb{T}}_{\mathbb{J}}(\mathbb{J}e^i_+, e^i_+) + \rho^{\mathbb{T}}_{\mathbb{J}}(\mathbb{J}e^i_-, e^i_-)].$$

5B. ***Generalized Kähler manifolds.*** Recall that the structure $(M, \gamma; \mathbb{G}, \mathbb{J})$ defines a *generalized Kähler manifold* if both $\mathbb{J}$ and $\mathbb{J}_- = \mathbb{G}\mathbb{J}$ are integrable generalized almost complex structures with respect to γ [22]. Let $\phi = \gamma + db$, Theorem 5.7 indicates that ∇^ϕ is $\mathbb{J}$-compatible with $*_\gamma$. In particular, for $\theta \in \Omega^{p,0}_{\mathbb{J}}(M)$, (5-8) gives

$$\partial^\phi_{\mathbb{J}}\theta = d_L\theta \quad \text{and} \quad \bar{\partial}^\phi_{\mathbb{J}}\bar{\theta} = d_{\bar{L}}\bar{\theta}. \tag{5-15}$$

Example 5.10 (see Example 5.3). Set $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^{\phi}$ in (5-10). Then the components in the third identity can further be rewritten in terms of the $\mathbb{G}$-eigendecomposition. For instance,

$$(\partial_{\mathbb{J}}^{\phi}\bar{\partial}_{\mathbb{J}}^{\phi} f)(x_{\pm}, \bar{y}_{\pm}) = (\partial_{\pm}\bar{\partial}_{\pm} f)(X_{\pm}, \overline{Y}_{\pm}), \tag{5-16}$$

where $\partial_{\pm}$ and $\bar{\partial}_{\pm}$ are the operators associated to the classical complex structures $I_{\pm}$, while $X_{\pm}, Y_{\pm} \in C^{\infty}(T_{\pm;1,0}M)$ are sections of the $I_{\pm}$-holomorphic tangent bundles respectively, and $x_{\pm} = s_{\pm}(X_{\pm})$ and so on.

Since $d_{\bar{L}}^2 = 0$, as a consequence of (5-15), the algebraic Bianchi identity (4-2) for $\mathcal{R}^{\phi}$ implies that $\nabla^{+\phi}$ (resp. $\nabla^{-\phi}$) induces a natural I_{-}-holomorphic (resp. I_{+}-holomorphic) structure on the eigenbundles of I_{+} (resp. of I_{-}), providing an alternative proof of this well-known result in [20].

Proposition 5.11. *Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Kähler manifold. Let $\phi = \gamma + db$, let $x, y, z, w \in C^{\infty}(\mathbb{T}_{\mathbb{J}}^{1,0}M)$, and $\bar{x}$, etc. are their complex conjugates. Then*

$$\langle \mathcal{R}^{\phi}_{\bar{x},\bar{y}}\bar{z}, w\rangle + c.p. \text{ in } x, y, z = 0 \quad \text{and} \quad \langle \mathcal{R}^{\phi}_{x,y}z, \bar{w}\rangle + c.p. \text{ in } x, y, z = 0, \tag{5-17}$$

which give rise to

$$R^{-\phi}_{\overline{X}_+,\overline{Y}_+}\overline{Z}_- = 0,\ R^{-\phi}_{\overline{X}_+,\overline{Y}_+}W_- = 0,\ R^{+\phi}_{\overline{X}_-,\overline{Y}_-}\overline{Z}_+ = 0, \quad \text{and} \quad R^{+\phi}_{\overline{X}_-,\overline{Y}_-}W_+ = 0,$$

as well as their complex conjugates, where $X_{\pm}, Y_{\pm}, Z_{\pm}, W_{\pm} \in C^{\infty}(T_{\pm;1,0}M)$.

Proof. To see the first identity in (5-17), note that $w \in \Omega_{\mathbb{J}}^{0,1}(M)$. Then

$$\bar{\partial}_{\mathbb{J}}^{\phi} \circ \bar{\partial}_{\mathbb{J}}^{\phi} w = d_{\bar{L}}^2 w = 0.$$

By (2-11) and (2-16), for $x, y, z \in C^{\infty}(\mathbb{T}_{\mathbb{J}}^{1,0}M)$,

$$\begin{aligned} 0 &= (\bar{\partial}_{\mathbb{J}}^{\phi} \circ \bar{\partial}_{\mathbb{J}}^{\phi} w)(\bar{x}, \bar{y}, \bar{z}) = (\mathbb{d}^{\phi} \circ \mathbb{d}^{\phi} w)(\bar{x}, \bar{y}, \bar{z}) \\ &= w([\bar{x} \diamond_{\phi} \bar{y} \diamond_{\phi} \bar{z}]) = -2\langle \mathcal{R}^{\phi}_{\bar{x},\bar{y}}\bar{z}, w\rangle - \text{c.p. in } x, y, z. \end{aligned}$$

The second identity then follows by taking complex conjugation.

To see the classical curvature identities, restrict x, y, z, w in (5-17) further to the $\mathbb{G}$-eigenbundles (5-11). For instance, consider $x, y \in C^{\infty}(\ell_+)$ and $z, w \in C^{\infty}(\ell_-)$. Set $X_+ = \pi(x)$, $Z_- = \pi(z)$ and so on. Applying (4-2) to the first identity in (5-17) gives

$$-\langle 2g(R^{-\phi}_{\overline{X}_+,\overline{Y}_+}\overline{Z}_-), w_-\rangle = g(R^{-\phi}_{\overline{X}_+,\overline{Y}_+}W_-, \overline{Z}_-) = 0.$$

Since (g, I_-) is Hermitian and $\nabla^{-\phi}$ preserves I_-, it gives the identities involving $R^{-\phi}$. The rest of the identities are obtained similarly. □

For the generalized Bismut connection $\mathbb{V}^{\phi,\mathcal{B}}$, with $\phi = \gamma + db$, (4-14) implies that $\rho_{\mathbb{J}}^{\phi,\mathcal{B}}$ is computed by the sum of the respective classical Bismut–Ricci forms $\rho_\pm$ for $(g, I_\pm)$:

$$\rho_{\mathbb{J}}^{\phi,\mathcal{B}}(x, y) = \rho_+(X, Y) + \rho_-(X, Y).$$

The corresponding $\mathbb{J}$-scalar curvature also decomposes:

$$\mathcal{S}_{\mathbb{J}}^{\phi,\mathcal{B}} = \sum_i [\rho_{\mathbb{J}}^{\phi,\mathcal{B}}(\mathbb{J}e_+^i, e_+^i) + \rho_{\mathbb{J}}^{\phi,\mathcal{B}}(\mathbb{J}e_-^i, e_-^i)] = S_+ + 2S_{+-} + S_-,$$

where $S_\pm$ are the respective classical Bismut scalar curvatures for $(g, I_\pm)$ and S_{+-} is the *mixed Bismut scalar curvature*:

$$S_{+-} := \sum_{i,j} R^{+\phi}(I_-X_i, X_i, I_+X_j, X_j) = \sum_{i,j} R^{-\phi}(I_+X_i, X_i, I_-X_j, X_j).$$

6. $\mathbb{J}$-holomorphic vector bundles

Let (V, h) be a Hermitian vector bundle on a generalized complex manifold $(M, \gamma; \mathbb{J})$. Recall that a $\mathbb{J}$*-holomorphic structure* on V is given by a flat $\mathbb{T}_{\mathbb{J}}^{0,1}M$-connection [19; 25]:

$$\bar\partial_{\mathbb{J}} : C^\infty(V) \to \Omega_{\mathbb{J}}^{0,1}(V) := C^\infty(\mathbb{T}_{\mathbb{J}}^{1,0}M \otimes V) \quad \text{such that} \quad \bar\partial_{\mathbb{J}}(fv) = d_{\bar L} f \otimes v + f\,\bar\partial_{\mathbb{J}} v$$

for $f \in C^\infty(M)$ and $v \in C^\infty(V)$ and such that

$$\bar\partial_{\mathbb{J}} \circ \bar\partial_{\mathbb{J}} = 0, \tag{6-1}$$

where the extension to $\Omega_{\mathbb{J}}^{0,k}(V)$ is given by

$$\bar\partial_{\mathbb{J}} : \Omega_{\mathbb{J}}^{0,k}(V) \to \Omega_{\mathbb{J}}^{0,k+1}(V) \quad \text{such that} \quad \bar\partial_{\mathbb{J}}(\theta \otimes v) = d_{\bar L}\theta \otimes v + (-1)^k \theta \wedge \bar\partial_{\mathbb{J}} v \tag{6-2}$$

for $\theta \in \Omega_{\mathbb{J}}^{0,k}(M)$ and $v \in C^\infty(V)$.

If $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler, via the restriction to $C_\pm$, the $\mathbb{J}$-holomorphic structure $\bar\partial_{\mathbb{J}}$ induces on V an $I_\pm$-holomorphic structure, which will be denoted by $\bar\partial_\pm$ respectively:

$$\bar\partial_{\pm,\bar X_\pm} v := \bar\partial_{\mathbb{J},s_\pm(\bar X_\pm)} v, \tag{6-3}$$

where $X_\pm \in T_{\pm;1,0}M$ and $v \in C^\infty(V)$.

6A. ***A connection on*** $\mathbb{T}_{\mathbb{J}}^{1,0}M$**.** On a generalized Kähler manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, natural $\bar L$-connections can be defined on $\mathbb{T}_{\mathbb{J}}^{1,0}M$ using the diamond bracket $\diamond_{\mathbb{T}}$ associated to any $\mathbb{G}$-metric connection $\mathbb{V}^{\mathbb{T}}$:

$$\bar\partial_\diamond^{\mathbb{T}} : C^\infty(\mathbb{T}_{\mathbb{J}}^{0,1}M) \otimes C^\infty(\mathbb{T}_{\mathbb{J}}^{1,0}M) \to C^\infty(\mathbb{T}_{\mathbb{J}}^{1,0}M), \quad \bar\partial_{\diamond,\bar x}^{\mathbb{T}} y := [\bar x \diamond_{\mathbb{T}} y]_{1,0}, \tag{6-4}$$

where $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$ and $[\bullet]_{1,0}$ denotes taking the $(1,0)$-component with respect to $\mathbb{J}$.

It follows from Proposition 5.11 that $\bar{\partial}^{\phi}_{\diamond}$ defined by ∇^{ϕ} via (6-4) induces an $I_{\pm}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ via its restriction to $\bar{\ell}_{\pm} \cong T_{\pm;0,1}M$. Moreover, by (2-11) and (5-13),

$$\begin{aligned}(\bar{\partial}^{\phi}_{\diamond} \circ \bar{\partial}^{\phi}_{\diamond})_{\bar{x},\bar{y}} z &= [\bar{x} \diamond_{\phi} (\bar{y} \diamond_{\phi} z) - \bar{y} \diamond_{\phi} (\bar{x} \diamond_{\phi} z) - (\bar{x} \diamond_{\phi} \bar{y}) \diamond_{\phi} z]_{1,0} \\ &= [\mathcal{R}^{\phi}_{\bar{x},\bar{y}} z + \mathcal{R}^{\phi}_{\bar{y},z} \bar{x} + \mathcal{R}^{\phi}_{z,\bar{x}} \bar{y}]_{1,0}.\end{aligned}$$

Thus, from (4-2), $\bar{\partial}^{\phi}_{\diamond}$ defines a $\mathbb{J}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ if and only if

$$[g(R^{-\phi}_{\overline{X}_+, Z_+} \overline{Y}_-)]_{1,0} = [g(R^{+\phi}_{\overline{Y}_-, Z_-} \overline{X}_+)]_{1,0} = 0$$

for all $X_{\pm}, Y_{\pm}, Z_{\pm} \in C^\infty(T_{\pm;1,0}M)$. Since $\nabla^{\pm\phi}$ preserves $I_{\pm}$, together with (4-4), the above is equivalent to

$$R^{+\phi}(\overline{Y}_-, \overline{W}_+, \overline{X}_+, Z_+) = R^{-\phi}(\overline{X}_+, \overline{W}_-, \overline{Y}_-, Z_-) = 0$$

for all $X_{\pm}, Y_{\pm}, Z_{\pm}, W_{\pm} \in C^\infty(T_{\pm;1,0}M)$. The computations can be summarized as the following result.

Theorem 6.1. *Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Kähler manifold, and $\phi = \gamma + db$. Let $\bar{\partial}^{\phi}_{\diamond}$ be the $\bar{L}$-connection on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ defined by*

$$\bar{\partial}^{\phi}_{\diamond,\bar{x}} y := [\bar{x} \diamond_{\phi} y]_{1,0}$$

for $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$. It is a $\mathbb{J}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ if and only if

$$R^{\pm\phi}_{\overline{X}_{\mp}, \overline{Y}_{\pm}} \bar{Z}_{\pm} = 0$$

for all $X_{\pm}, Y_{\pm}, Z_{\pm} \in C^\infty(T_{\pm;1,0}M)$. In particular, if $\nabla^{\pm\phi}$ are flat on TM, $\bar{\partial}^{\phi}_{\diamond}$ is a $\mathbb{J}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$. □

6B. *Chern curvature.* Analogously to the classical situation, on a $\mathbb{J}$-holomorphic Hermitian bundle (V, h), there exists a unique *generalized Chern connection*:

$$\nabla^{h,C} := \bar{\partial}_{\mathbb{J}} + \partial_{\mathbb{J}}, \tag{6-5}$$

where $\partial_{\mathbb{J}} : C^\infty(V) \to \Omega^{1,0}_{\mathbb{J}}(V) := C^\infty(\mathbb{T}^{0,1}_{\mathbb{J}}M \otimes V)$ is defined by

$$d_{\bar{L}} h(v_1, v_2) = h(\bar{\partial}_{\mathbb{J}} v_1, v_2) + h(v_1, \partial_{\mathbb{J}} v_2)$$

for all $v_j \in C^\infty(V)$.

Definition 6.2. On a generalized complex manifold $(M, \gamma; \mathbb{J})$, let $\nabla^{\mathbb{T}}$ be a TM-torsion-free generalized connection on $\mathbb{T}M$. For a $\mathbb{J}$-holomorphic Hermitian bundle

$(V, h, \bar{\partial}_{\mathbb{J}})$, its $\nabla^{\mathbb{T}}$-*Chern curvature* $\mathcal{F}^{\mathbb{T},C}$ is the $\nabla^{\mathbb{T}}$-curvature of its generalized Chern connection (6-5).

When $\nabla^{\mathbb{T}}$ is $\mathbb{J}$-compatible with $*_\gamma$, (5-8) implies that the extension (3-1) of $\nabla^{\mathbb{T}}$ to $\Omega^*_{\mathbb{T}}(V)$ by $\nabla^{h,C}$ is compatible with the extension (6-2) of $\bar{\partial}_{\mathbb{J}}$ to $\Omega^{0,*}_{\mathbb{J}}(V)$.

Proposition 6.3. *Let* $\nabla^{\mathbb{T}}$ *be a* γ-$\mathbb{J}$-*connection on a generalized complex manifold* $(M, \gamma; \mathbb{J})$. *The corresponding* $\nabla^{\mathbb{T}}$-*Chern curvature* $\mathcal{F}^{\mathbb{T},C}$ *is of type* $(1, 1)$ *with respect to* $\mathbb{J}$, *i.e.*,

$$\mathcal{F}^{\mathbb{T},C} \in \Omega^{1,1}_{\mathbb{J}}(\mathrm{End}(V)) := C^\infty\big(\mathbb{T}^{1,0}_{\mathbb{J}} M \wedge \mathbb{T}^{0,1}_{\mathbb{J}} M \otimes \mathrm{End}(V)\big). \tag{6-6}$$

In particular, over a generalized Kähler manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, *the* ***$\phi$-Chern curvature*** $\mathcal{F}^{\phi,C}$ *defined with* $\nabla^{\mathbb{T}} = \nabla^\phi$ *for* $\phi = \gamma + db$ *is of type* $(1, 1)$ *with respect to* $\mathbb{J}$. □

Example 6.4 (see Example 3.1). Consider a $\mathbb{J}$-holomorphic Hermitian line bundle $(V, h, \bar{\partial}_{\mathbb{J}})$ and set $\nabla = \nabla^{h,C}$, the generalized Chern connection. Choose a unitary local section $s \in C^\infty(V)$. There is a local section $u^{0,1}_{\mathbb{J}} \in \Omega^{0,1}_{\mathbb{J}}(M)$ such that

$$\bar{\partial}_{\mathbb{J}} s = u^{0,1}_{\mathbb{J}} \otimes s.$$

Let $\nabla^{\mathbb{T}}$ be a γ-$\mathbb{J}$-connection. Then $\mathcal{F}^{\mathbb{T},C}(\nabla) = \sqrt{-1}\,\mathbb{d}^{\mathbb{T}} u$, where $u \in \Omega^1_{\mathbb{T}}(M)$ and

$$\sqrt{-1}u = u^{0,1}_{\mathbb{J}} - \overline{u^{0,1}_{\mathbb{J}}}.$$

By (5-8), the flatness of $\bar{\partial}_{\mathbb{J}}$ implies that

$$\bar{\partial}^{\mathbb{T}}_{\mathbb{J}} u^{0,1} = d_{\bar{L}} u^{0,1}_{\mathbb{J}} = 0.$$

This then gives

$$\mathcal{F}^{\mathbb{T},C}(\nabla) = \partial^{\mathbb{T}}_{\mathbb{J}} u^{0,1} - \bar{\partial}^{\mathbb{T}}_{\mathbb{J}} \overline{u^{0,1}} \in \Omega^{1,1}_{\mathbb{J}}(M).$$

Over a generalized Kähler manifold, the ϕ-Chern connection and curvature are related to the classical Chern connections and curvatures via the $\mathbb{G}$-eigendecomposition.

Lemma 6.5. *Let* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *be a generalized Kähler manifold and* $V \to M$ *be a* $\mathbb{J}$-*holomorphic vector bundle. Let* $\nabla^{h,C}_\pm$ *be the* $\mathbb{G}$-*eigendecomposition* (3-13) *of* $\nabla^{h,C}$ (6-5). *Then* $\nabla^{h,C}_\pm$ *are the Chern connections for the induced* $I_\pm$-*holomorphic structures* (6-3) *on* V *respectively. Furthermore, the classical Chern curvatures are components of the* $\mathbb{G}$-*eigendecomposition of the* ϕ-*Chern curvature.*

Proof. The statement about the connections follows from a straightforward verification, while the statement about the curvature follows from Theorem 3.7. □

Example 6.6. Continue from Example 6.4 and let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be generalized Kähler. The $I_\pm$-holomorphic structures induced by $\bar{\partial}_{\mathbb{J}}$ are given locally by $\alpha_\pm \in \Omega^{0,1}_\pm(M)$:

$$u^{0,1}_{\mathbb{J}} = [g^{-1}(\alpha_+) + bg^{-1}(\alpha_+) + \alpha_+] - [g^{-1}(\alpha_-) + bg^{-1}(\alpha_-) - \alpha_-],$$

which satisfies

$$\bar{\partial}_\pm \alpha_\pm = 0 \quad \text{and} \quad (\nabla^{-\phi}_{\bar{X}_+}\alpha_-)(\bar{Y}_-) - (\nabla^{+\phi}_{\bar{Y}_-}\alpha_+)(\bar{X}_+) = 0,$$

where $X_\pm \in C^\infty(T_{\pm;1,0}M)$ and so on. The generalized Chern connection is then

$$\nabla^{h,C} s = (u^{0,1}_{\mathbb{J}} - \overline{u^{0,1}_{\mathbb{J}}}) \otimes s,$$

while the classical Chern connections $\nabla^C_\pm$ are defined locally by $\nu_\pm = \alpha_\pm - \bar{\alpha}_\pm$ respectively. Then Example 3.8 gives the $\mathbb{G}$-eigendecomposition of the ϕ-Chern curvature.

6C. ***$\mathbb{J}$-Hermitian–Einstein equation.*** Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Hermitian manifold. The analogue to the contraction by the Kähler form is the *$\mathbb{J}_-$-contraction*.

Definition 6.7. The *$\mathbb{J}_-$-contraction* $\Lambda_{\mathbb{J}_-} : \bigwedge^2 \mathbb{T}_{\mathbb{C}} M \to \mathbb{R}$ is given by

$$\Lambda_{\mathbb{J}_-}(x \wedge y) := \langle \mathbb{J}_- x, y\rangle = \mathbb{G}(\mathbb{J}x, y), \tag{6-7}$$

where $x, y \in C^\infty(\mathbb{T}_{\mathbb{C}} M)$.

In terms of the $\mathbb{G}$-eigendecomposition, it corresponds to

$$\Lambda_{\mathbb{J}_-}(s_\pm(X) \wedge s_\pm(Y)) = \omega_\pm(X, Y) \quad \text{and} \quad \Lambda_{\mathbb{J}_-}(s_\pm(X) \wedge s_\mp(Y)) = 0 \tag{6-8}$$

for $X, Y \in TM$, where $\omega_\pm = gI_\pm$.

A version of the Hermitian–Einstein equation can thus be formulated in this context.

Definition 6.8. Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Hermitian manifold and $\nabla^{\mathbb{T}}$ be a γ-$\mathbb{J}$-connection on $\mathbb{T}M$. A Hermitian metric h on a $\mathbb{J}$-holomorphic vector bundle $V \to M$ is *$\nabla^{\mathbb{T}}$-$\mathbb{J}$-Hermitian–Einstein* if the corresponding $\nabla^{\mathbb{T}}$-Chern curvature satisfies the following *$\nabla^{\mathbb{T}}$-$\mathbb{J}$-Hermitian–Einstein equation*:

$$\sqrt{-1}\Lambda_{\mathbb{J}_-}(\mathcal{F}^{\mathbb{T},C}(V)) = 2c\,\mathrm{Id}_V \tag{6-9}$$

for some $c \in \mathbb{R}$. If $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler and $\nabla^{\mathbb{T}} = \nabla^\phi$, equation (6-9) will be simply called the *$\mathbb{J}$-Hermitian–Einstein equation*, a solution of which is called a *$\mathbb{J}$-Hermitian–Einstein metric*.

When $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler and $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^{\phi}$, by (3-20) and (6-8) the $\mathbb{G}$-eigendecomposition of the left-hand side of (6-9) is given by

$$\Lambda_{\mathbb{J}_-}(\mathcal{F}^{\phi,C}(V)) = \Lambda_+(F_+^C(V)) + \Lambda_-(F_-^C(V)), \tag{6-10}$$

where $F_\pm^C$ are the curvatures of the classical Chern connections $\nabla_\pm^h$ on V, and $\Lambda_\pm$ are the contractions by $\omega_\pm$. It follows that (6-9) is equivalent to an equation first proposed by Hitchin in [25].

Proposition 6.9. *Over a generalized Kähler manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$, *equation* (6-9) *is equivalent to*

$$\tfrac{\sqrt{-1}}{2}\big(F_+^C(V)\wedge\omega_+^{m-1} + (-1)^\varepsilon F_-^C(V)\wedge\omega_-^{m-1}\big) = c(m-1)!\,\mathrm{Id}_V\, d\mathrm{vol}_g, \tag{6-11}$$

where $\varepsilon = 0$ *if* $I_\pm$ *induce the same orientation on* TM *and* $\varepsilon = 1$ *otherwise.*

Proof. Suppose that $d\mathrm{vol}_g = \frac{1}{m!}\omega_+^m$. Then

$$mF_+^C(V)\wedge\omega_+^{m-1} = \Lambda_+(F_+^C(V))\omega_+^m = m!\,\Lambda_+(F_+^C(V))\,d\mathrm{vol}_g.$$

Note that $\omega_+^m = (-1)^\varepsilon\omega_-^m$, it follows from (6-10) that (6-11) is equivalent to (6-9). □

The $\mathbb{J}_-$-contraction naturally provides the definition of a degree.

Definition 6.10. On a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$ with a γ-$\mathbb{J}$-connection $\mathbb{V}^{\mathbb{T}}$, let $(V, h, \bar\partial_{\mathbb{J}})$ be a $\mathbb{J}$-holomorphic Hermitian vector bundle. The $\mathbb{V}^{\mathbb{T}}$-$\mathbb{G}$-*degree* of V is given by

$$\deg_{\mathbb{G}}^{\mathbb{T}}(V, h) := \frac{\sqrt{-1}}{4\pi}\int_M \mathrm{tr}_V[\Lambda_{\mathbb{J}_-}(\mathcal{F}^{\mathbb{T},C}(V))]\,d\mathrm{vol}_g. \tag{6-12}$$

When $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler and $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^{\phi}$ for $\phi = \gamma + db$, the $\mathbb{V}^{\phi}$-$\mathbb{G}$-degree is simply called the $\mathbb{G}$-*degree* and denoted by $\deg_{\mathbb{G}}(V, h)$.

Recall that the classical degrees of (V, h) with the induced $I_\pm$-holomorphic structure are

$$\deg_\pm(V, h) := \frac{\sqrt{-1}}{2\pi}\int_M \mathrm{tr}_h[\Lambda_\pm(F_\pm^C(V))]\,d\mathrm{vol}_g. \tag{6-13}$$

Theorem 6.11. *Let* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *be a generalized Kähler manifold and* $(V, h, \bar\partial_{\mathbb{J}})$ *be a* $\mathbb{J}$-*holomorphic vector bundle on* M. *Then*

$$\deg_{\mathbb{G}}(V, h) = \tfrac{1}{2}[\deg_+(V, h) + \deg_-(V, h)], \tag{6-14}$$

where $\deg_{\mathbb{G}}$ *and* $\deg_\pm$ *are as given in* (6-12) *and* (6-13) *respectively.*

Proof. This follows from (6-10). □

Example 6.12. Continue from Example 6.4 and work now on a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, with a γ-$\mathbb{J}$-connection $\mathbb{V}^{\mathbb{T}}$. For $f \in C^\infty(M)$, let $h_1 = e^{2f} h$ be another Hermitian metric on the line bundle V. An h_1-unitary local section is then given by $s_1 = e^{-f} s \in C^\infty(V)$, which leads to

$$\bar{\partial}_{\mathbb{J}} s_1 = (u_{\mathbb{J}}^{0,1} - d_{\bar{L}} f) \otimes s_1.$$

Let $\mathbb{V}_1$ be the corresponding generalized Chern connection, whose $\mathbb{V}^{\mathbb{T}}$-Chern curvature is given by

$$\mathcal{F}^{\mathbb{T},C}(\mathbb{V}_1) = \mathbb{d}^{\mathbb{T}}(\sqrt{-1}u + d_L f - d_{\bar{L}} f) = \mathcal{F}^{\mathbb{T},C}(\mathbb{V}) - 2\partial_{\mathbb{J}}^{\mathbb{T}} d_{\bar{L}} f,$$

where the last step is due to (5-10). Hence

$$\deg_{\mathbb{G}}^{\mathbb{T}}(V, h_1) - \deg_{\mathbb{G}}^{\mathbb{T}}(V, h) = -\frac{\sqrt{-1}}{2\pi} \int_M \Lambda_{\mathbb{J}_-}(\partial_{\mathbb{J}}^{\mathbb{T}} d_{\bar{L}} f)\, d\mathrm{vol}_g. \tag{6-15}$$

It follows that $\deg_{\mathbb{G}}^{\mathbb{T}}$ is independent of the Hermitian metric on V if and only if the right-hand side of (6-15) vanishes for all $f \in C^\infty(M)$.

The integrand in (6-15) gives rise to the second-order operator for $f \in C^\infty(M)$:

$$P_{\mathbb{J}}(f) := -\sqrt{-1}\Lambda_{\mathbb{J}_-}(\partial_{\mathbb{J}}^{\mathbb{T}} d_{\bar{L}} f).$$

Similar to the classical case (e.g., [32]), $P_{\mathbb{J}}$ is elliptic since its principal symbol is given by

$$\sigma(P_{\mathbb{J}}) = 4\sqrt{-1}\Lambda_{\mathbb{J}_-}(\xi_{\mathbb{J}}^{0,1} \wedge \xi_{\mathbb{J}}^{1,0}) = 4\mathbb{G}(\xi_{\mathbb{J}}^{0,1}, \xi_{\mathbb{J}}^{1,0}) = \|\xi\|_g^2,$$

where $\xi_{\mathbb{J}}^{0,1}$ is the projection of $\xi \in T^*M$ to $\mathbb{T}_{\mathbb{J}}^{1,0}M$ and $\xi_{\mathbb{J}}^{1,0}$ is the complex conjugate.

Definition 6.13. For a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, the metric $\mathbb{G}$ is $\mathbb{V}^{\mathbb{T}}$-$\mathbb{J}$-*Gauduchon* if the right-hand side of (6-15) vanishes for all $f \in C^\infty(M)$. When the structure is generalized Kähler, a $\mathbb{V}^{\phi}$-$\mathbb{J}$-Gauduchon metric $\mathbb{G}$ is simply said to be $\mathbb{J}$-*Gauduchon*.

As in the classical situation, if $(M, \gamma; \mathbb{G}, \mathbb{J})$ is $\mathbb{V}^{\mathbb{T}}$-$\mathbb{J}$-Gauduchon and (V, h) solves (6-9), then the constant c is given by

$$c = \frac{2\pi \deg_{\mathbb{G}}^{\mathbb{T}}(V)}{\mathrm{rank}(V)\mathrm{Vol}_g(M)}.$$

It is then natural to extend the notions of slope and slope stability to $\mathbb{J}$-holomorphic vector bundles over a $\mathbb{V}^{\mathbb{T}}$-$\mathbb{J}$-Gauduchon generalized Hermitian manifold. The notion of coherent subsheaf in this context can be adopted from Definition 3.4 of [28].

Definition 6.14. Let $\mathbb{G}$ be a $\nabla^{\mathbb{T}}$-$\mathbb{J}$-Gauduchon metric. The $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*slope* of a $\mathbb{J}$-holomorphic vector bundle $(V, \bar{\partial}_{\mathbb{J}})$ over M is

$$\mu^{\mathbb{T}}_{\mathbb{G}}(V) := \frac{\deg^{\mathbb{T}}_{\mathbb{G}}(V)}{\operatorname{rank}(V)}. \tag{6-16}$$

The bundle V is $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*semistable* if for any coherent $\mathbb{J}$-holomorphic subsheaf W of V:

$$\mu^{\mathbb{T}}_{\mathbb{G}}(W) \leqslant \mu^{\mathbb{T}}_{\mathbb{G}}(V). \tag{6-17}$$

V is said to be $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*stable* if strict inequality holds in (6-17). Over a $\mathbb{J}$-Gauduchon generalized Kähler manifold, the corresponding notions are simply referred to as the $\mathbb{G}$-*slope*, $\mathbb{G}$-*semistable* and $\mathbb{G}$-*semistable* respectively.

Recall that over a Hermitian manifold (M, g, I), the degree of any holomorphic vector bundle V is independent of a Hermitian metric on V if and only if g is Gauduchon, i.e.,

$$\partial\bar{\partial}(\omega^{m-1}) = 0, \tag{6-18}$$

where m is the complex dimension of M. On a generalized Kähler manifold, the $\mathbb{J}$-Gauduchon condition can be expressed in a similar fashion.

Proposition 6.15. *A generalized Kähler manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *is* $\mathbb{J}$-*Gauduchon if and only if*

$$\partial_+\bar{\partial}_+(\omega_+^{m-1}) + (-1)^{\varepsilon}\partial_-\bar{\partial}_-(\omega_-^{m-1}) = 0, \tag{6-19}$$

where $\varepsilon = 0$ *if* $I_\pm$ *induce the same orientation on* TM *and* $\varepsilon = 1$ *otherwise.*

Proof. The degree of a Hermitian vector bundle coincides with that of its determinant line bundle, so it's sufficient to consider line bundles. By (5-16) and (6-8),

$$\int_M \Lambda_{\mathbb{J}_-}(\partial^{\phi}_{\mathbb{J}} d_{\bar{L}} f)\, d\mathrm{vol}_g = \int_M [\Lambda_+(\partial_+\bar{\partial}_+ f) + \Lambda_-(\partial_-\bar{\partial}_- f)]\, d\mathrm{vol}_g.$$

The statement then follows from integration by parts. □

Remark 6.16. Evidently, (6-19) holds for $m < 3$, since the 3-form below is closed:

$$\phi = \gamma + db = \mp d^c_\pm \omega_\pm.$$

For $m \geqslant 3$, (6-19) is equivalent to

$$\phi \wedge d(\omega_+^{m-2} - (-1)^{\varepsilon}\omega_-^{m-2}) = 0.$$

In particular, $(M, \gamma; \mathbb{G}, \mathbb{J})$ is $\mathbb{J}$-Gauduchon if the difference $\omega_+^{m-2} - (-1)^{\varepsilon}\omega_-^{m-2}$ is closed. When $m = 3$, (6-19) can also be rewritten as

$$\phi_+^{(2,1)} \wedge \phi_+^{(1,2)} + (-1)^{\varepsilon}\phi_-^{(2,1)} \wedge \phi_-^{(1,2)} = 0,$$

where, for instance, $\phi_{\pm}^{(2,1)}$ denote the (2, 1)-component of ϕ with respect to $I_{\pm}$ respectively. It is clear that the generalized Kähler manifold is $\mathbb{J}$-Gauduchon if g is Gauduchon with respect to both $I_{\pm}$.

7. Geometric Lax flows

A *Lax pair* [29] consists of two families of operators $\{(L_t, P_t) : t \in I \subseteq \mathbb{R}\}$ such that

$$\frac{d}{dt} P_t = [L_t, P_t], \tag{7-1}$$

where $\{L_t\}$ is the *Lax operator* and it is assumed that $0 \in I$. Equation (7-1) is also said to be in *the Lax form*. Suppose that $\{\Psi_t\}$ is generated by $\{L_t\}$, i.e., it solves the equation

$$\frac{d}{dt} \Psi_t = L_t \Psi_t \quad \text{with } \Psi_0 = \mathrm{Id}\,. \tag{7-2}$$

Then $\{P_t\}$ can be obtained from pushing forward an initial operator P_0 by $\{\Psi_t\}$:

$$P_t = \Psi_t P_0 \Psi_t^{-1}.$$

In particular, $\{P_t\}$ is an isospectral family.

Let $\{A_t\}$ be a smooth family of operators on the same space; then the t-differential of $\{\Psi_t^{-1} A_t \Psi_t\}$ describes the extent to which $\{(L_t, A_t)\}$ fails to be a Lax pair as well.

Definition 7.1. Suppose that $\{(L_t, P_t)\}$ is a Lax pair and $\{A_t\}$ be a smooth family of operators on the same space. The *L_t-differential of A_t* (*along the Lax flow*) is

$$\delta_L A_t := \frac{d}{dt} A_t - [L_t, A_t]. \tag{7-3}$$

It is straightforward to verify that commutativity with P_t is preserved by δ_L.

Lemma 7.2. *If A_t and P_t commute for all t, then $\delta_L A_t$ also commutes with P_t for all t.*

Proof. Notice that

$$\frac{d}{dt}(\Psi_t^{-1} A_t \Psi_t) = \Psi_t^{-1} (\delta_L A_t) \Psi_t,$$

from which the statement follows. □

When a pair of geometric quantities forms a Lax pair, the corresponding equation (7-1) is said to generate a *geometric Lax flow*. Two main classes of examples will be described, where the operators $\{P_t\}$ are either generalized metrics or generalized almost complex structures. Such Lax pairs impose certain necessary conditions on the Lax operator.

Definition 7.3. Let $\mathbb{G}$ be a generalized metric and $\mathbb{J}$ a generalized (almost) complex structure. An operator $\mathbb{L}$ on $C^\infty(\mathbb{T}M)$ is *Lax compatible with* $\mathbb{G}$ if

$$\langle \mathbb{L}x_+, y_-\rangle + \langle x_+, \mathbb{L}y_-\rangle = 0 \tag{7-4}$$

for all $x_+ \in C^\infty(C_+)$ and $y_- \in C^\infty(C_-)$. The operator $\mathbb{L}$ is *Lax compatible with* $\mathbb{J}$ if

$$\langle \mathbb{L}x, y\rangle + \langle x, \mathbb{L}y\rangle = \langle \mathbb{L}\bar{x}, \bar{y}\rangle + \langle \bar{x}, \mathbb{L}\bar{y}\rangle = 0 \tag{7-5}$$

for all $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$.

Lemma 7.4. *Suppose that* $\{(\mathbb{L}_t, \mathbb{P}_t)\}$ *is a Lax pair of operators on* $C^\infty(\mathbb{T}M)$. *If* $\mathbb{P}_t$ *is a smooth family of generalized metrics or almost complex structures*, *then* $\mathbb{L}_t$ *is Lax compatible with* $\mathbb{P}_t$ *for all* t.

Proof. It follow from the orthogonality of $\mathbb{P}_t$ with respect to the pairing $\langle\ ,\ \rangle$ and that $\mathbb{P}_t^2$ are constant operators. □

The following simplified criteria are useful in practice.

Corollary 7.5. *Let* $\mathbb{L} \in C^\infty(\mathrm{End}(\mathbb{T}M))$ *and* $x, y \in \mathbb{T}M$. *Then*:

(1) (7-4) *holds for* $\mathbb{L}$ *if the bilinear form* $\mathbb{G}(\mathbb{L}x, y)$ *is symmetric*, *i.e.*,

$$\mathbb{G}(\mathbb{L}x, y) = \mathbb{G}(\mathbb{L}y, x).$$

(2) (7-4) *and* (7-5) *hold for* $\mathbb{L}$ *if the bilinear form* $\langle \mathbb{L}x, y\rangle$ *is skew-symmetric*, *i.e.*,

$$\langle \mathbb{L}x, y\rangle + \langle \mathbb{L}y, x\rangle = 0.$$

Proof. Left for the reader. □

7A. ***$\theta \in \Omega^2_{\mathbb{T}}(M)$ as the Lax operator.*** Let $\{P_t \in C^\infty(T^*M^{\otimes 2})\}$ be a smooth family of 2-tensors on M, whose symmetric and skew-symmetric parts are respectively P_t^s and P_t^a:

$$P_t = P_t^s + P_t^a \quad \text{with } P_t^s(X, Y) = P_t^s(Y, X),\ P_t^a(X, Y) = -P_t^a(Y, X)$$

for $X, Y \in C^\infty(TM)$. Then $\{P_t\}$ defines an initial value problem for a family of generalized metrics $\mathbb{G}_t$ as follows:

$$\begin{cases} \dfrac{d}{dt} g_t = -P_t^s, \\ \dfrac{d}{dt} b_t = P_t^a. \end{cases} \tag{7-6}$$

The system (7-6) can be reformulated into a Lax flow for $\mathbb{G}_t$. For such a family $\mathbb{G}_t$, let $C_\pm^t$ be the eigenbundles and $s_\pm^t : TM \to C_\pm^t$ be the corresponding isomorphisms.

Lemma 7.6. *Let* $\{P_t \in C^\infty(T^*M^{\otimes 2})\}$ *and* $\{\theta_t \in \Omega^2_{\mathbb{T}}(M)\}$ *be a smooth family of* $\mathbb{T}M$*-forms such that*

$$\theta_t(x^t_-, y^t_+) = P_t(X, Y), \tag{7-7}$$

where $x^t_\pm = s^t_\pm(X)$ *for* $X \in C^\infty(TM)$ *and so on. Let* $\theta_t : \mathbb{T}M \to \mathbb{T}M$ *be given by*

$$2\langle \theta_t(x), y\rangle := \theta_t(x, y).$$

Then (7-6) *is equivalent to the* ***2-tensor Lax flow***

$$\frac{d}{dt}\mathbb{G}_t = [\theta_t, \mathbb{G}_t]. \tag{7-8}$$

Proof. Note that θ_t satisfies Corollary 7.5 (2). Since $\mathbb{G}^2 = \mathbb{1}$, the differential of a smooth family of generalized metrics is skew with respect to the generalized metric at each time. Obviously the left-hand side of (7-8) is skew with respect to $\mathbb{G}_t$. Thus only the mixed $\mathbb{G}_t$-eigencomponents of the left-hand side are nontrivial, one of which goes as follows:

$$\begin{aligned} 0 &= \frac{d}{dt}\langle \mathbb{G}_t x^t_-, y^t_+\rangle \\ &= \left\langle \frac{d}{dt}\mathbb{G}_t x^t_-, y^t_+\right\rangle + \left\langle \mathbb{G}_t \frac{d}{dt}(b_t - g_t)X, y^t_+\right\rangle + \left\langle \mathbb{G}_t x^t_-, \frac{d}{dt}(b_t + g_t)Y\right\rangle \\ &= \left\langle \frac{d}{dt}\mathbb{G}_t x^t_-, y^t_+\right\rangle + \frac{d}{dt}(b_t - g_t)(X, Y). \end{aligned}$$

The right-hand side is given by

$$\langle [\theta_t, \mathbb{G}_t] x^t_-, y^t_+\rangle = \langle (\theta_t \mathbb{G}_t - \mathbb{G}_t \theta_t) x^t_-, y^t_+\rangle = -2\langle \theta_t(x^t_-), y^t_+\rangle = -P_t(X, Y).$$

Thus (7-8) gives rise to

$$\frac{d}{dt}(b_t - g_t)(X, Y) = P_t(X, Y),$$

from which (7-6) follows. The other direction is left for the reader. □

A smooth conformal family of metrics $\{g_t\}$ can be seen as a solution to (7-6), by setting $P_t = P^s_t = f_t g_0$, where $\{f_t \in C^\infty(M)\}$ is a smooth family of functions. In this case, $b_t \equiv b_0$ is constant throughout the flow. The corresponding $\mathbb{T}M$-forms as given in Lemma 7.6 can be chosen to be dependent only on the generalized metric $\mathbb{G}$.

Definition 7.7. Let $\mathbb{G}$ be a generalized metric defined by (g, b), where g is a Riemannian metric. A $\mathbb{T}M$-form $\theta \in \Omega^2_{\mathbb{T}}(M)$ is $\mathbb{G}$-*conformal* if there are $r_g, r_b \in C^\infty(M)$ such that

$$\theta(x_-, y_+) = r_g g(X, Y) + r_b b(X, Y) \quad \text{and} \quad \theta(x_\pm, y_\pm) = 0,$$

where $x_\pm = s_\pm(X) \in C^\infty(C_\pm)$, etc. The functions r_g, r_b are called the *conformal weights*.

Remark 7.8. When the conformal weights coincide, i.e., $r_g = r_b$, a family of $\mathbb{G}$-conformal forms generates conformal deformations of the generalized metric $\mathbb{G}$. Otherwise, the metric g and b are deformed by different factors. In particular, when the conformal weight $r_b = 0$, it corresponds to classical conformal deformation of the metric g.

7B. *ϕ-curvature Lax flow.* A special case of (7-8) is when θ_t are $\mathbb{d}^{\phi_t}$-exact, i.e., $\theta_t = \mathbb{d}^{\phi_t} u_t$ for a smooth family of sections $\{u_t \in \Omega^1_{\mathbb{T}}(M)\}$ and 3-forms $\{\phi_t \in \Omega^3(M)\}$, with respect to the metric $\mathbb{G}_t$ at time t. Suppose that $u_t = Z_t + \zeta_t$, and set $x^t_\pm = s^t_\pm(X)$ for $X \in C^\infty(TM)$ and so on. Then

$$\begin{aligned}(\mathbb{d}^{\phi_t} u_t)(x^t_-, y^t_+) &= X u_t(y^t_+) - Y u_t(x^t_-) - u_t(x^t_- \diamond_{\phi_t,\mathbb{G}_t} y^t_+)\\ &= (\mathcal{L}_{Z_t} g_t)(X, Y) + [d(\zeta_t - \iota_{Z_t} b_t) - \iota_{Z_t}\phi_t](X, Y).\end{aligned}$$

It follows that in this case, the Lax flow (7-8) is equivalent to

$$\begin{cases} \dfrac{d}{dt} g_t = -\mathcal{L}_{Z_t} g_t, \\ \dfrac{d}{dt} b_t = -\mathcal{L}_{Z_t} b_t + d\zeta_t - \iota_{Z_t}(\phi_t - db_t). \end{cases} \tag{7-9}$$

When $\gamma = \phi_t - db_t$ is a fixed closed 3-form, (7-9) describes the pushforward of an initial generalized metric by the family of generalized diffeomorphism of $\mathbb{T}M$ generated by u_t.

Proposition 7.9. *Let $\gamma \in \Omega^3(M)$ be a closed* 3*-form and $u_t = Z_t + \zeta_t \in C^\infty(\mathbb{T}M)$ be a smooth family of sections. Let (λ_t, β_t) be the family of generalized diffeomorphisms generated by $\{u_t\}$ under $*_\gamma$* (recalled below). *Set $\phi_t = \gamma + db_t$. Then* (7-9) *coincides with the infinitesimal action by (λ_t, β_t) on a generalized metric $\mathbb{G}$ via pushforward.*

Proof. It's more straightforward to work with the pullback action on generalized metrics. Recall, e.g., from Hu and Uribe [27], that in (λ_t, β_t), λ_t is the 1-parameter family of diffeomorphisms generated by Z_t, and

$$\beta_t := \int_0^t \lambda_s^*(d\zeta_s - \iota_{Z_s}\gamma)\, ds.$$

The pushforward of $x = X + \xi \in \mathbb{T}M$ by (λ_t, β_t) is given by

$$(\lambda_t, \beta_t)_* x = \lambda_{t*}(x + \iota_X \beta_t) = \lambda_{t*} X + (\lambda_t^{-1})^*(\xi + \iota_X \beta_t)$$

with the corresponding infinitesimal action $x \mapsto -u_t *_\gamma x$.

The pullback of $\mathbb{G}$ by (λ_t, β_t) gives the family of generalized metrics

$$\mathbb{G}_t(x, y) := \mathbb{G}\big((\lambda_t, \beta_t)_* x, (\lambda_t, \beta_t)_* y\big),$$

where $x, y \in C^\infty(\mathbb{T}M)$ are independent of t. Analogously to the computations in the classical case, differentiating the above with respect to t gives, by the left-hand side,

$$\begin{aligned}\Big(\frac{d}{dt}\mathbb{G}_t\Big)(x, y) &= \frac{1}{2}\frac{d}{dt}[g_t(X, Y) + g_t^{-1}(\xi - \iota_X b_t, \eta - \iota_Y b_t)]\\ &= \frac{1}{2}\Big[\Big(\frac{d}{dt}g_t\Big)(X, Y) - \Big(\frac{d}{dt}g_t\Big)(g_t^{-1}\xi_t', g_t^{-1}\eta_t')\\ &\qquad - \Big(\frac{d}{dt}b_t\Big)(Y, g_t^{-1}\xi_t') - \Big(\frac{d}{dt}b_t\Big)(X, g_t^{-1}\eta_t')\Big],\end{aligned}$$

where $\xi_t' = \xi - \iota_X b_t$ and $\eta_t' = \eta - \iota_Y b_t$, and, by the right-hand side,

$$\begin{aligned}\Big(\frac{d}{dt}\mathbb{G}_t\Big)(x, y) &= Z_t\mathbb{G}_t(x, y) + \mathbb{G}_t(-u_t *_\gamma x, y) + \mathbb{G}_t(x, -u_t *_\gamma y)\\ &= \tfrac{1}{2}[(\mathcal{L}_{Z_t}g_t)(X, Y) - (\mathcal{L}_{Z_t}g_t)(g_t^{-1}\xi_t', g_t^{-1}\eta_t')]\\ &\quad + \tfrac{1}{2}[(-\mathcal{L}_{Z_t}b + d\zeta_t - \iota_{Z_t}\gamma)(Y, g_t^{-1}\xi_t')\\ &\qquad + (-\mathcal{L}_{Z_t}b + d\zeta_t - \iota_{Z_t}\gamma)(X, g_t^{-1}\eta_t')].\end{aligned}$$

Comparing these two ways of computing $\big(\frac{d}{dt}\mathbb{G}_t\big)(x, y)$, the pullback action on $\mathbb{G}$ gives

$$\begin{cases}\dfrac{d}{dt}g_t = \mathcal{L}_{Z_t}g_t,\\[2mm] \dfrac{d}{dt}b_t = \mathcal{L}_{Z_t}b_t - (d\zeta_t - \iota_{Z_t}\gamma).\end{cases}$$

The pushforward action reverses the signs on the right-hand side, giving (7-9). □

A $\mathbb{d}^\phi$-exact $\mathbb{T}M$-form in $\Omega^2_{\mathbb{T}}(M)$ can be seen as the ϕ-curvature of a unitary generalized connection on the trivial Hermitian line bundle. In general, let (V, h) be a Hermitian line bundle with a family of unitary generalized connections $\{\mathbb{V}_t\}$. By Example 3.8, the Lax flow (7-8) defined by $\{\theta_t = \sqrt{-1}\mathcal{F}^{\phi_t}(\mathbb{V}_t)\}$ where $\phi_t = \gamma + db_t$ is equivalent to the system

$$\begin{cases}\dfrac{d}{dt}g_t = -\mathcal{L}_{g_t^{-1}\psi_t}g_t,\\[2mm] \dfrac{d}{dt}b_t = \sqrt{-1}F_0^t - \iota_{g_t^{-1}\psi_t}\phi_t,\end{cases} \tag{7-10}$$

which reduces to (7-9) when V is trivial, and admits similar interpretation in terms of (not necessarily exact) generalized diffeomorphisms.

Theorem 7.10. *Given a family of unitary generalized connections $\{\mathbb{V}_t\}$ on a line bundle V with Hermitian metrics $\{h_t\}$, the Lax flow* (7-8) *defined by $\{\theta_t = \sqrt{-1}\mathcal{F}^{\phi_t}(\mathbb{V}_t)\}$, where $\phi_t = \gamma + db_t$, corresponds to the pushforward of an initial generalized metric $\mathbb{G}$ by a family of generalized diffeomorphisms, which may not be* ***exact****, i.e., is not generated by global sections of $\mathbb{T}M$.*

Proof. In the notation of Example 3.8, the local section defining $\mathbb{V}_t$ is given by $\sqrt{-1}u_t = \sqrt{-1}(Z_t + \zeta_t)$, where Z_t is a global vector field and ζ_t is a locally defined 1-form, with

$$Z_t = g_t^{-1}\psi_t = \tfrac{1}{2}g_t^{-1}(\nu_{t,+} - \nu_{t,-}) \quad \text{and} \quad \zeta_t = \iota_{Z_t}b_t + \tfrac{1}{2}(\nu_{t,+} + \nu_{t,-}).$$

Then $\mathbb{V}_0^t$ is defined by the local section $\frac{1}{2}(\nu_{t,+} + \nu_{t,-})$, which implies that

$$\sqrt{-1}F_0^t = d(\zeta_t - \iota_{Z_t}b_t) = -\mathcal{L}_{Z_t}b_t + d\zeta_t + \iota_{Z_t}db_t.$$

Locally, the second equation in (7-10) is thus

$$\frac{d}{dt}b_t = -\mathcal{L}_{Z_t}b_t + d\zeta_t - \iota_{Z_t}\gamma,$$

noting that $\phi_t = \gamma + db_t$. By Proposition 7.9, (7-10) corresponds to pushing forward by the local exact generalized diffeomorphisms generated by $\{(Z_t, \zeta_t)\}$. Globally, (7-10) corresponds to pushing forward by possibly nonexact generalized diffeomorphisms. □

Remark 7.11. The diffeomorphism components of the generalized diffeomorphisms in Theorem 7.10 arise from the vector components of $\mathbb{V}_t$ on V, which vanish if they are liftings of classical connections on V.

Example 7.12. Let $(M, \gamma; \mathbb{J})$ be a generalized complex manifold and $\{\theta_t \in \Omega^2_{\mathbb{T}}(M)\}$ be a smooth family of $\mathbb{T}M$-forms. The Lax flow with initial value $\mathbb{J}_0 = \mathbb{J}$,

$$\frac{d}{dt}\mathbb{J}_t = [\theta_t, \mathbb{J}_t],$$

consists of generalized almost complex structures. The flow above preserves $\mathbb{J}$ if and only if $[\theta_t, \mathbb{J}] = 0$ for all t, which is equivalent to $\theta_t \in \Omega^{1,1}_{\mathbb{J}}(M)$. Thus, starting with a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, the Lax flow (7-8) defined by $\{\theta_t \in \Omega^{1,1}_{\mathbb{J}}(M)\}$ produces a family of generalized Hermitian structures with the same $\mathbb{J}$.

Suppose furthermore that $(M, \gamma; \mathbb{G}, \mathbb{J})$ admits a γ-$\mathbb{J}$-connection $\mathbb{V}^{\mathbb{T}}$. Fix a $\mathbb{J}$-holomorphic line bundle V and let $\{\theta_t\}$ be the $\mathbb{V}^{\mathbb{T}}$-Chern curvatures for a smooth family of Hermitian metrics $\{h_t\}$ on V. By Proposition 6.3, $\{\theta_t\}$ consists of (1, 1)-forms with respect to $\mathbb{J}$. By Theorem 7.10, the corresponding Lax flow (7-8) corresponds to the pushforward of $\mathbb{G}$ by a family of generalized diffeomorphisms $\{(\lambda_t, \beta_t)\}$. Alternatively, it can be seen as a family of generalized Hermitian structures $(M, \gamma; \mathbb{G}, (\lambda_t, \beta_t)^*\mathbb{J})$ with $\mathbb{G}$ fixed.

7C. ***Ricci Lax flow.*** When the Ricci curvature of a generalized connection $\mathbb{V}^{\mathbb{T}}$ satisfies (7-4), it can serve as the Lax operator in a Lax pair involving the generalized metric.

Definition 7.13. A smooth family of pairs $\{(\mathbb{G}_t, \nabla^{\mathbb{T}_t})\}$ of generalized metrics $\mathbb{G}_t$ and $\mathbb{G}_t$-metric connections is a solution to the $\nabla^{\mathbb{T}}$-*Ricci Lax flow* if $\{\mathcal{R}ic^{\mathbb{T}_t}\}$ satisfies (7-4) and the pair $\{(\mathcal{R}ic^{\mathbb{T}_t}, \mathbb{G}_t)\}$ form a Lax pair, i.e.,

$$\frac{d}{dt}\mathbb{G}_t = [\mathcal{R}ic^{\mathbb{T}_t}, \mathbb{G}_t]. \tag{7-11}$$

When $\nabla^{\mathbb{T}_t}$ is prescribed to only depend on $\mathbb{G}_t$, (7-11) becomes an equation for $\mathbb{G}_t$ only, in which case the family of generalized metrics $\mathbb{G}_t$ is said to be *a solution* to (7-11). If $\nabla^{\mathbb{T}_t} = \nabla^{\phi_t}$, then the flow (7-11) is simply called the *Ricci Lax flow*.

Since $\mathcal{R}c^{\phi}$ is symmetric, it satisfies the condition in Corollary 7.5. Theorem 7.14 below shows that the Ricci Lax flow is equivalent to

$$\frac{d}{dt}(g_t \mp b_t) = -2\,\mathrm{Rc}_t^{\pm\phi_t} = -2\,\mathrm{Rc}_t + \tfrac{1}{2}\phi_t^2 \pm d^*\phi_t. \tag{7-12}$$

Let $\gamma \in \Omega^3(M)$ such that $d\gamma = 0$ and set $\phi_t = \gamma + db_t$. Then (7-12) coincides with the *generalized Ricci flow* [12; 36] as a system for Riemannian metrics g_t and $b_t \in \Omega^2(M)$:

$$\begin{cases} \dfrac{d}{dt}g_t = -2\,\mathrm{Rc}_t + \tfrac{1}{2}(\gamma + db_t)^2, \\ \dfrac{d}{dt}b_t = -d^*(\gamma + db_t). \end{cases} \tag{7-13}$$

Theorem 7.14. *Under the further constraints that $\nabla^{\mathbb{T}} = \nabla^{\phi_t}$ and are $\mathbb{G}_t$-metric compatible with the Dorfman bracket $*_\gamma$* (2-37), *i.e., $\phi_t - db_t = \gamma \in \Omega^3(M)$ for all t, the Ricci Lax flow* (7-11) *is equivalent to the generalized Ricci flow* (7-13).

Proof. It's obvious that (7-13) follows from (7-12), by matching the symmetric and skew-symmetric terms on both sides. To obtain (7-12) from (7-11), fix $X, Y \in TM$ and consider

$$x_-^t = s_-^t(X) \quad \text{and} \quad y_+^t = s_+^t(Y).$$

The left-hand side of (7-12) is computed in Lemma 7.6, while the right-hand side becomes

$$\begin{aligned} \langle [\mathcal{R}ic^{\phi_t}, \mathbb{G}_t]x_-^t, y_+^t\rangle &= \langle (\mathcal{R}ic^{\phi_t}\,\mathbb{G}_t - \mathbb{G}_t\,\mathcal{R}ic^{\phi_t})x_-^t, y_+^t\rangle \\ &= -2\,\mathcal{R}c^{\phi_t}(x_-^t, y_+^t) = -2\,\mathrm{Rc}^{+\phi_t}(X, Y). \end{aligned}$$

This gives half of (7-12). The other half is equivalent. □

Example 7.15. Consider two 3-dimensional Lie groups: $T = (U(1))^3$ and $G = \mathrm{SU}(2)$, with their invariant metrics g and invariant volume forms ϕ. For both, set $b = 0$ and consider the corresponding ϕ-Ricci curvature $\mathcal{R}ic^{\phi}$.

For T, the invariant metric is flat, thus $Ric = 0$ while $Ric^{\pm\phi} \neq 0$ by (4-8). In the $\mathbb{G}$-eigendecomposition, $\mathcal{R}c^{\phi}$ is of the form

$$\mathcal{R}c^{\phi} = \begin{pmatrix} 0 & R \\ R^T & 0 \end{pmatrix},$$

where $R = \mathbb{1}_3$, the 3×3 identity matrix.

For G, the invariant metric is the standard round metric on S^3 which is not flat, while the connections with torsion $\pm\phi$ are flat, hence $Ric^{\pm\phi} = 0$. In this case, $\mathcal{R}c^{\phi}$ is of the form

$$\mathcal{R}c^{\phi} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix},$$

where R here is the classical Ricci tensor for the round metric (see Section 2E).

Definition 7.7 and Theorem 7.14 lead to the natural generalization of Ricci solitons.

Definition 7.16. Let $\gamma \in \Omega^3(M)$ be a closed 3-form. A smooth family of generalized metrics $\{\mathbb{G}_t\}$ is a *Ricci Lax soliton* if there exists a smooth family of sections $\{u_t \in C^\infty(\mathbb{T}M)\}$ and $\mathbb{G}_t$-conformal forms $\{\theta_t \in \Omega^2_{\mathbb{T}}(M)\}$ with constant conformal weights r_t and s_t, and

$$[\mathcal{R}ic^{\phi_t} - \mathbb{d}^{\phi_t} u_t - \theta_t, \mathbb{G}_t] = 0, \tag{7-14}$$

where $\phi_t = \gamma + db_t$. The family is a *gradient Ricci Lax soliton* if furthermore there is $f \in C^\infty(M)$ such that $\left\{u_t = -\frac{1}{2}\mathbb{G}_t(df)\right\}$.

Notice that $\mathbb{G}_t(df) = \mathrm{grad}^t f + \iota_{\mathrm{grad}^t f} b_t$, where $\mathrm{grad}^t f$ is the gradient of f with respect to g_t. When $s_t = 0$, the gradient Ricci Lax soliton equation is then equivalent to the system

$$\begin{cases} Rc_t - \frac{1}{4}\phi_t^2 + \mathrm{Hess}^t f = r_t g_t, \\ \quad d^*\phi_t + \iota_{\mathrm{grad}^t f}\phi_t = 0, \end{cases} \tag{7-15}$$

where $\mathrm{Hess}^t f$ is the Hessian of f with respect to the Levi-Civita connection of g_t. When $r_t \equiv r$ is a constant function independent of t, the system (7-15) is exactly the *generalized Ricci soliton* equation ([3] and references therein).

7D. *Bismut–Ricci Lax flow.* Even though $\mathcal{R}ic^{\phi,\mathcal{B}}$ is neither symmetric nor skew-symmetric, it satisfies (7-4), and thus can be used to define a Lax flow for generalized metrics. In (7-11), taking $\mathcal{R}ic^{\phi,\mathcal{B}}$ as the Lax operator leads to the *Bismut–Ricci Lax flow*:

$$\frac{d}{dt}\mathbb{G}_t = [\mathcal{R}ic^{\phi_t,\mathcal{B}}, \mathbb{G}_t]. \tag{7-16}$$

Since $\mathcal{R}c^{\phi,\mathcal{B}}$ has the same mixed components as $\mathcal{R}c^{\phi,\mathbb{G}}$, i.e.,

$$[\mathcal{R}ic^{\phi,\mathbb{G}}, \mathbb{G}] = [\mathcal{R}ic^{\phi,\mathcal{B}}, \mathbb{G}]$$

by Theorem 7.14, (7-16) is equivalent to (7-12), so they generate the same flow for the generalized metric [35]. The $\mathcal{R}ic^{\phi,\mathcal{B}}$-differential of $\mathbb{V}^{\phi_t,\mathcal{B}}$ takes a particularly simple form.

Proposition 7.17. *Fix $X, Y, Z \in C^\infty(TM)$ and $x \in C^\infty(\mathbb{T}M)$, with $\pi(x) = X$. Let $Y, Z \in C^\infty(TM)$ and $y^t_\pm = s^t_\pm(Y)$, $z^t_\pm = s^t_\pm(Z)$. Then*

$$\mathbb{G}_t((\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{V}^{\phi_t,\mathcal{B}}_x)y^t_\pm, z^t_\pm) = -g_t\Big(Y, \frac{d}{dt}\nabla^{\pm\phi_t}_X Z\Big). \tag{7-17}$$

Proof. Only the case for C^t_+ is shown here, and the case for C^t_- is similar:

$$\begin{aligned}
&\mathbb{G}_t\Big(\frac{d}{dt}(\mathbb{V}^{\phi_t,\mathcal{B}}_x)y^t_+, z^t_+\Big)\\
&\quad= \frac{d}{dt}[\mathbb{G}_t(\mathbb{V}^{\phi_t,\mathcal{B}}_x y^t_+, z^t_+)] - \mathbb{G}_t\Big(\mathbb{V}^{\phi_t,\mathcal{B}}_x\Big[\frac{d}{dt}(b_t+g_t)Y\Big], z^t_+\Big)\\
&\qquad\qquad - \mathbb{G}_t\Big(\mathbb{V}^{\phi_t,\mathcal{B}}_x y^t_+, \frac{d}{dt}(b_t+g_t)Z\Big)\\
&\quad= g_t\Big(\frac{d}{dt}\nabla^{+\phi_t}_X Y, Z\Big) + [X Rc^{-\phi}_t(Y,Z) - \mathrm{Rc}^{-\phi}_t(Y, \nabla^{+\phi_t}_X Z) - \mathrm{Rc}^{-\phi}_t(\nabla^{+\phi_t}_X Y, Z)],
\end{aligned}$$

where the final equality follows from (7-12) together with the fact that $\mathbb{V}^{\phi_t,\mathcal{B}}$ preserves $\mathbb{G}_t$. Next, (4-15) implies that

$$\begin{aligned}
&\mathbb{G}_t([\mathcal{R}ic^{\phi_t,\mathcal{B}}_t, \mathbb{V}^{\phi_t,\mathcal{B}}_x]y^t_+, z^t_+)\\
&\qquad= \mathbb{G}_t(\mathcal{R}ic^{\phi_t,\mathcal{B}}_t\,\mathbb{V}^{\phi_t,\mathcal{B}}_x y^t_+ - \mathbb{V}^{\phi_t,\mathcal{B}}_x\,\mathcal{R}ic^{\phi,\mathcal{B}}_t\, y^t_+, z^t_+)\\
&\qquad= \mathcal{R}c^{\phi_t,\mathcal{B}}_t(\mathbb{V}^{\phi_t,\mathcal{B}}_x y^t_+, z^t_+) - X\,\mathcal{R}c^{\phi_t,\mathcal{B}}_t(y^t_+, z^t_+) + \mathcal{R}c^{\phi_t,\mathcal{B}}_t(y^t_+, \mathbb{V}^{\phi_t,\mathcal{B}}_x z^t_+)\\
&\qquad= -X\,\mathrm{Rc}^{+\phi_t}_t(Y,Z) + \mathrm{Rc}^{+\phi_t}_t(Y, \nabla^{+\phi_t}_X Z) + \mathrm{Rc}^{+\phi_t}_t(\nabla^{+\phi_t}_X Y, Z).
\end{aligned}$$

Combining the results above and applying (7-12) again lead to

$$\begin{aligned}
&\mathbb{G}_t((\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{V}^{\phi_t,\mathcal{B}}_x)y^t_\pm, z^t_\pm)\\
&\quad= g_t\Big(\frac{d}{dt}\nabla^{+\phi_t}_X Y, Z\Big) - \Big[X\Big(\frac{d}{dt}g_t\Big)\Big(Y,Z\Big) - \Big(\frac{d}{dt}g_t\Big)\Big(Y, \nabla^{+\phi_t}_X Z\Big)\\
&\qquad\qquad - \Big(\frac{d}{dt}g_t\Big)\Big(\nabla^{+\phi_t}_X Y, Z\Big)\Big]\\
&\quad= -g_t\Big(Y, \frac{d}{dt}\nabla^{+\phi_t}_X Z\Big),
\end{aligned}$$

where the last equality follows from the fact that $\nabla^{+\phi_t}$ preserves g_t for all t. □

Consider a smooth family of generalized almost Hermitian structures $(M, \gamma; \mathbb{G}_t, \mathbb{J}^t)$, where $\mathbb{G}_t$ is a solution to the Bismut–Ricci Lax flow (7-16). By Lemma 7.2, the $\mathcal{R}ic^{\phi,\mathcal{B}}$-differential $\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t$ preserves the $\mathbb{G}_t$-eigenbundles. Following computations similar to those in Proposition 7.17 leads to

$$\mathbb{G}_t((\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t)y_\pm, z_\pm) = -g_t\Big(Y, \Big(\frac{d}{dt}I^t_\pm\Big)Z\Big). \tag{7-18}$$

Set $\mathbb{J}^t_- := \mathbb{G}_t\mathbb{J}^t$. Then

$$\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t_- = (\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{G}_t)\mathbb{J}^t + \mathbb{G}_t(\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t) = \mathbb{G}_t(\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t).$$

This implies that $\mathbb{J}^t$ solves the following Lax flow (7-19), in which case so does $\mathbb{J}^t_-$, if and only if $I^t_\pm$ are constant almost complex structures:

$$\frac{d}{dt}\mathbb{J}^t = [\mathcal{R}ic^{\phi_t,\mathcal{B}}, \mathbb{J}^t]. \tag{7-19}$$

In terms of the 2-forms $\omega_\pm$, (7-19) is equivalent to the following simultaneous almost Hermitian Ricci flows:

$$\Big(\frac{d}{dt}\omega^t_\pm\Big)(X, Y) = 2\,\mathrm{Rc}(X, I^t_\pm Y) - \tfrac{1}{2}\phi^2_t(X, I^t_\pm Y), \tag{7-20}$$

where $X, Y \in C^\infty(TM)$. It has the classical Kähler–Ricci flow as a special case.

Example 7.18. Take a family of classical Kähler structures (g_t, I, ω_t), and set

$$\mathbb{J}^t = \begin{bmatrix} I & 0 \\ 0 & -I^* \end{bmatrix} \quad\text{and}\quad \mathbb{J}^t_- = \begin{bmatrix} 0 & -\omega_t^{-1} \\ \omega_t & 0 \end{bmatrix}.$$

The generalized Bismut connections, as well as the $\mathbb{G}$-adapted connections, are simply the lift of the Levi-Civita connections for g_t. Let $X, Y \in TM$. Since $\phi = 0$, (7-20) becomes

$$\Big(\frac{d}{dt}\omega_t\Big)(X, Y) = 2\,\mathrm{Rc}_t(X, IY) = -2\rho_t(X, Y),$$

where $\rho_t(X, Y) = \mathrm{Rc}_t(IX, Y)$ is the Ricci form. Thus (7-19) recovers exactly the equation for the classical Kähler–Ricci flow.

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A NORMAL UNIFORM ALGEBRA THAT FAILS TO BE STRONGLY REGULAR AT A PEAK POINT

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Dedicated to Joel Feinstein

We show that there exists a normal uniform algebra, on a compact metrizable space, that fails to be strongly regular at a peak point. This answers a 32-year-old question of Joel Feinstein. Our example is $R(K)$ for a certain compact planar set K. Furthermore, our example has a totally ordered one-parameter family of closed primary ideals whose hull is a peak point. We establish general results regarding lifting ideals under Cole root extensions. These results are applied to obtain a normal uniform algebra, on a compact metrizable space, with every point a peak point but again having a totally ordered one-parameter family of closed primary ideals.

1. Introduction

This paper is devoted to answering questions in the literature regarding strong regularity of uniform algebras and to establishing general results regarding lifting ideals under Cole root extensions. Our main goal is to answer the following question raised by Joel Feinstein [1992, p. 298]. (For definitions of terminology and notation used in this introduction see Section 2.)

Question 1.1. Does there exist a normal uniform algebra A, on a compact metrizable space, such that A fails to be strongly regular at some peak point for A?

There are later variations on this question in the literature. With general uniform algebras replaced by the special class of uniform algebras of the form $R(K)$ for K a compact planar set, the question appears in the recent book of Garth Dales and Ali Ülger [2024, Section 3.6]. Here, as usual, for K a compact set in the complex plane, $R(K)$ denotes the uniform closure on K of the holomorphic rational functions with poles off K. With uniform algebras replaced by the more general class of Banach

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function algebras, the question appears in Feinstein's paper [1995]. An affirmative answer to that variation was given by David Blecher and Charles Read [2016].

Part of the interest in Question 1.1, in its original form, comes from its connection with the 67-year-old question of I. M. Gelfand [1957] whether every natural uniform algebra on the closed unit interval [0, 1] is trivial. As observed by Feinstein, Donald Wilken's proof [1969] that every strongly regular uniform algebra on [0, 1] is trivial actually shows that every natural uniform algebra on [0, 1] that is strongly regular at a dense set of peak points is trivial. Consequently, a negative answer to Question 1.1 would imply that every normal uniform algebra on [0, 1] is trivial.

Another reason for interest in Question 1.1 is that Feinstein [1992, Theorem 5.2] showed that it is equivalent to this question: Does there exist a normal uniform algebra A, on a compact metrizable space X, such that every point of X is a peak point for A but A fails to be strongly regular? Another closely related question which seems not to be explicitly stated in the literature concerns primary ideals (defined in this context to be those ideals contained in a unique maximal ideal): Does there exist a uniform algebra A, normal or not, such that every point of the maximal ideal space of A is a peak point for A, but A has a closed primary ideal that is not maximal? By a result of Feinstein [2001, Corollary 8], there exist nonnormal uniform algebras such that every point of the maximal ideal space is a peak point.

We answer Question 1.1, and all of the variations on it discussed above, affirmatively by establishing the following theorem.

Theorem 1.2. *There exists a normal uniform algebra A, on a compact metrizable space, such that A fails to be strongly regular at some peak point for A. In fact, A can be taken to be $R(K)$ for a certain compact set K in the complex plane.*

As already mentioned, Feinstein [1992, Theorem 5.2] showed that the assertion of the first sentence of this theorem is equivalent to the following assertion.

Corollary 1.3. *There exists a normal uniform algebra A, on a compact metrizable space X, such that every point of X is a peak point for A but A is not strongly regular.*

The inspiration for our proof of Theorem 1.2, which will give considerably more information than is stated above, comes from the Beurling–Rudin theorem on the closed ideals in the disc algebra [Rudin 1957] (see also [Hoffman 1962, pp. 82–89]). Given a compact planar set K contained in the closed unit disc $\overline{D}$, a point λ in $K \cap \partial D$, and a real number $\rho \geq 0$, we will denote by I_ρ^λ the closed ideal in $R(K)$ generated by the function $(z-\lambda)\exp\bigl(\rho\,\frac{z+\lambda}{z-\lambda}\bigr)$. When $\lambda = 1$ we will write I_ρ in place of I_ρ^λ. When K is the closed unit disc, and hence $R(K)$ is the disc algebra, the ideals I_ρ^λ, $\rho \geq 0$, are precisely the closed primary ideals contained in the maximal ideal M_λ of functions that vanish at the point λ. Furthermore, for $0 \leq \rho_1 < \rho_2$ there is the strict inclusion $I_{\rho_1}^\lambda \supsetneq I_{\rho_2}^\lambda$. Our proof of Theorem 1.2

essentially amounts to showing that, taking $\lambda = 1$ for instance, Robert McKissick's construction of the first nontrivial normal uniform algebra [1963] (see also [Stout 1971, Section 27]) can be refined so as to preserve this strict inclusion of ideals. Using results in [Izzo 2022] we will show that, in addition, the uniform algebra can be chosen in such a way that the point 1 is the only point where strong regularity fails. We will thus obtain the following theorem that contains Theorem 1.2. Here, and throughout the paper, we denote the open unit disc in the complex plane by D, and given a disc Δ, we denote the radius of Δ by $r(\Delta)$.

Theorem 1.4. *For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} r(D_k) < r$, the point 1 is in the set $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$, and the following conditions hold*:

(i) *$R(K)$ is normal.*

(ii) *$R(K)$ is strongly regular at every point of $K \setminus \{1\}$.*

(iii) *$R(K)$ is not strongly regular at the point 1.*

Furthermore, the discs $\{D_k\}_{k=1}^{\infty}$ can be chosen in such a way that $I_{\rho_1} \supsetneq I_{\rho_2}$ for every $0 \le \rho_1 < \rho_2$.

A modification of the proof of Theorem 1.4 will yield the next result, which shows, in particular, that a normal uniform algebra can fail to be strongly regular at an uncountable set of peak points.

Theorem 1.5. *For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} r(D_k) < r$ and setting $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$ the following conditions hold*:

(i) *$R(K)$ is normal.*

(ii) *$R(K)$ is strongly regular at every point of $K \setminus \partial D$.*

(iii) *There is a set $\Lambda \subset \partial D$ whose complement in ∂D has one-dimensional Lebesgue measure less than r such that Λ is contained in K and at each point of Λ, the uniform algebra $R(K)$ fails to be strongly regular.*

Furthermore, the discs $\{D_k\}_{k=1}^{\infty}$ can be chosen in such a way that $I_{\rho_1}^{\lambda} \supsetneq I_{\rho_2}^{\lambda}$ for every $\lambda \in \Lambda$ and every $0 \le \rho_1 < \rho_2$.

Note that the uniform algebras in Theorems 1.4 and 1.5, in spite of failing to be strongly regular, are strongly regular at every *nonpeak point*. Feinstein and Matthew Heath [2007, Question 5.8] raised the question of whether there exists a compact planar set K such that $R(K)$ is regular and has no nonzero bounded point derivations, but is not strongly regular. Each of Theorems 1.4 and 1.5 answers this question affirmatively since for the set K in each of those theorems there are no nonzero bounded point derivations at the points of $K \setminus \partial D$ because $R(K)$ is strongly regular at those points, and there are no nonzero point derivations at the points of $K \cap \partial D$ since those points are peak points for $R(K)$. In [Izzo 2022] the author effectively raised the same question but without the regularity hypothesis, and he promised to give an example answering the question in a future paper. Thus Theorems 1.4 and 1.5 fulfill that promise.

The next two results show that, as one might expect, Corollary 1.3 can be strengthened in ways analogous to how Theorems 1.4 and 1.5 strengthen Theorem 1.2.

Theorem 1.6. *There exists a normal uniform algebra B, on a compact metrizable space X, such that every point of X is a peak point for B but there is a point $x_0 \in X$ such that there is a one-parameter family $\{H_\rho : 0 \leq \rho < \infty\}$ of distinct closed primary ideals contained in the maximal ideal M_{x_0} satisfying $H_{\rho_1} \supsetneq H_{\rho_2}$ for all $0 \leq \rho_1 < \rho_2$. Furthermore, B can be taken to have bounded relative units at, and hence be strongly regular at, every point of $X \setminus \{x_0\}$.*

Theorem 1.7. *There exists a normal uniform algebra B, on a compact metrizable space X, such that every point of X is a peak point for B but there is an uncountable subset L of X such that for every $x \in L$ there is a one-parameter family $\{H^x_\rho : 0 \leq \rho < \infty\}$ of distinct closed primary ideals contained in the maximal ideal M_x satisfying $H^x_{\rho_1} \supsetneq H^x_{\rho_2}$ for all $0 \leq \rho_1 < \rho_2$. Furthermore, B can be taken to have bounded relative units at, and hence be strongly regular at, every point of $X \setminus L$.*

Feinstein's proof that Corollary 1.3 is equivalent to the assertion of the first sentence in Theorem 1.2 used Brian Cole's method of root extensions. Theorems 1.6 and 1.7 will be derived from Theorems 1.4 and 1.5 also using Cole's method of root extensions. However, to do so we will need to prove new results about lifting ideals under root extensions. There are several examples in the literature in which a certain object lifts under a root extension as a result of adjoining square roots to only a restricted collection of functions. For instance in [Feinstein 1992; 1995; 2004; Feinstein and Heath 2007; Izzo 2022; Izzo and Papathanasiou 2021] square roots are adjoined only to functions that vanish on a given closed set (or a neighborhood of the closed set), and consequently a copy of the given closed set is preserved in the extension. In [Ghosh and Izzo 2023] square roots are adjoined only to functions on which a given bounded point derivation vanishes, with the result that the bounded point derivation lifts to the extended uniform algebra. In all these instances, the functions for which square roots are adjoined come from some (proper) closed ideal I of the uniform algebra A. We will prove general results regarding such root extensions. Very roughly, the results say that in this situation, the quotient Banach algebra A/I is preserved by the extension, and consequently, all the ideals in A that contain I lift under the extension.

In the next section we define some terminology and notation already used above. In Section 3 we present some known results that we will need. In Section 4 we prove that for a normal uniform algebra, strong regularity at a point is a local property. Although not strictly necessary for the proofs of Theorems 1.4 and 1.5, which are presented in Section 5, this result greatly simplifies establishing the strong regularity assertion in those theorems. In Section 6 we present the theorems discussed in the previous paragraph regarding lifting ideals under root extensions. Theorems 1.6 and 1.7 are proved in Section 7.

2. Terminology and notation

Those readers well versed in uniform algebra concepts may wish to skip or skim this section and refer back to it as needed.

It is to be understood that all sequences, unions, and sums involving an index extend from 1 to ∞; thus for instance $\{D_k\}$ means $\{D_k\}_{k=1}^{\infty}$, and $\bigcup D_k$ means $\bigcup_{k=1}^{\infty} D_k$. If f is a function whose domain contains a subset L, we denote the restriction of f to L by $f|L$, and if A is a collection of such functions, we denote the collection of restrictions of functions in A to L by $A|L$. The set of positive integers will be denoted by $\mathbb{Z}_+$.

Throughout the paper all spaces will tacitly be required to be Hausdorff. Let X be a compact space. We denote by $C(X)$ the algebra of all continuous complex-valued functions on X equipped with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constants and separates the points of X. A uniform algebra A on X is said to be

(a) *natural* if the maximal ideal space of A is X (under the usual identification of a point of X with the corresponding multiplicative linear functional),

(b) *regular on* X if for each closed set K_0 of X and each point x of $X \setminus K_0$, there exists a function f in A such that $f(x) = 1$ and $f = 0$ on K_0,

(c) *normal on* X if for each pair of disjoint closed sets K_0 and K_1 of X, there exists a function f in A such that $f = 1$ on K_1 and $f = 0$ on K_0.

The uniform algebra A on X is *regular* or *normal* if A is natural and is regular on X or normal on X, respectively. In fact, every regular uniform algebra is normal [Stout 1971, Theorem 27.2]. Also, if a uniform algebra A is normal on X, then A is necessarily natural [Stout 1971, Theorem 27.3].

Let A be a uniform algebra on X, and let $x \in X$. We define the ideals M_x and J_x by

$$M_x = \{f \in A : f(x) = 0\},$$
$$J_x = \{f \in A : f^{-1}(0) \text{ contains a neighborhood of } x \text{ in } X\}.$$

More generally, if E is a closed subset of X, we define the ideals M_E and J_E by

$$M_E = \{f \in A : f|E = 0\},$$
$$J_E = \{f \in A : f^{-1}(0) \text{ contains a neighborhood of } E \text{ in } X\}.$$

When it is necessary to indicate with respect to which algebra the ideals are taken, we will denote the ideals J_x and M_x in the uniform algebra A by $J_x(A)$ and $M_x(A)$.

The uniform algebra A is *strongly regular at* x if $\bar{J}_x = M_x$, and A is *strongly regular* if A is strongly regular at every point of X. It was shown by Wilken [1969, Corollary 1] that every strongly regular uniform algebra is normal.

Let A be a natural uniform algebra and let I be an ideal in A. The *hull* of I is the common zero set of the functions in I and is denoted by $\mathrm{hull}(I)$. The ideal I is said to be *local* if $I \supset J(\mathrm{hull}(I))$. The following result is standard [Dales 2000, Proposition 4.1.20(iv)].

Theorem 2.1. *Every ideal in a normal uniform algebra is local.*

Consequently, a normal uniform algebra is strongly regular at a point x if and only if there is no closed primary ideal properly contained in the maximal ideal M_x.

The uniform algebra A has *bounded relative units at* x with bound $C \geq 1$ if for each compact subset K of $X \setminus \{x\}$, there exists $f \in J_x$ such that $f|K = 1$ and $\|f\|_X \leq C$. If A has bounded relative units at every point of X, then A has *bounded relative units*.

The point x is said to be a *peak point* for A if there is a function f in A such that $f(x) = 1$ and $|f(y)| < 1$ for every $y \in X \setminus \{x\}$. The point x is said to be a *generalized peak point* if for every neighborhood U of x there exists a function f in A such that $f(x) = \|f\| = 1$ and $|f(y)| < 1$ for every $y \in X \setminus U$. When the space X is metrizable the notions of peak point and generalized peak point coincide.

For ϕ a point in the maximal ideal space of the uniform algebra A, a *bounded point derivation* on A at ϕ is a bounded linear functional d on A satisfying the identity

$$d(fg) = d(f)\phi(g) + \phi(f)d(g)$$

for all f and g in A. It is standard [Browder 1969, p. 64] that a bounded linear functional d on A is a bounded point derivation at ϕ if and only if d annihilates $\overline{M_\phi^2}$ and the constant functions, and hence there exists a bounded point derivation at ϕ if and only if $\overline{M_\phi^2} \neq M_\phi$.

3. Preliminaries

In this section we collect various known results that we will need. The reader may prefer to skip this section and merely refer back to it when the results are used.

As with McKissick's construction [1963] of the first known nontrivial normal uniform algebra, our proofs of Theorems 1.4 and 1.5 rely on the following lemma. (Recall that given a disc Δ, we denote the radius of Δ by $r(\Delta)$.)

Lemma 3.1. *Let Δ be an open disc in the complex plane with center a and radius $r > 0$, and let $\varepsilon > 0$ be given. Then there exist a sequence of open discs $\{\Delta_k\}_{k=1}^{\infty}$*

and a sequence of rational functions $\{f_j\}_{j=1}^{\infty}$ *such that*:

(a) $\sum_{k=1}^{\infty} r(\Delta_k) < \varepsilon$.

(b) *The poles of the* f_j *lie in* $\bigcup_{k=1}^{\infty} \Delta_k$.

(c) *The sequence* $\{f_j\}$ *converges uniformly on* $\mathbb{C} \setminus \bigcup_{k=1}^{\infty} \Delta_k$ *to a function that is identically zero outside* Δ *and zero free in* $\Delta \setminus \bigcup_{k=1}^{\infty} \Delta_k$.

(d) $\bigcup_{k=1}^{\infty} \Delta_k \subset \{z : r - \varepsilon < |z - a| < r\}$.

Condition (d) is not part of the lemma as stated by McKissick but is established in the paper of Thomas Körner [1986], where a proof of the lemma simpler than the original one is given.

For the proofs of Theorems 1.4 and 1.5 we will need two recent results [Izzo 2022, Theorem 1.2 and Lemma 4.2] on strong regularity in $R(K)$, which we state here.

Theorem 3.2. *For each* $r > 0$, *there exists a sequence of open discs* $\{D_k\}_{k=1}^{\infty}$ *such that* $\sum_{k=1}^{\infty} r(D_k) < r$ *and such that setting* $K = \bar{D} \setminus \bigcup_{k=1}^{\infty} D_k$, *the uniform algebra* $R(K)$ *is nontrivial and strongly regular.*

Lemma 3.3. *Given compact sets* $L \subset K \subset \mathbb{C}$ *and given a point* $x \in L$, *if* $\overline{J_x(R(K))} \supset M_x(R(K))$, *then* $\overline{J_x(R(L))} \supset M_x(R(L))$.

The following three lemmas will be used to prove condition (v) in Theorem 6.9. The first of these is due to Feinstein and Heath [2007, Lemma 4.3].

Lemma 3.4. *Let* A *be a uniform algebra on* X *and* $x \in X$. *Suppose that, for each compact subset* E *of* $X \setminus \{x\}$, *there exists a neighborhood* U *of* x *and a function* $f \in A$ *such that*

(i) $f|U = 1$,

(ii) $f|E = 0$,

(iii) *for each* $k \in \mathbb{Z}_+$ *there is a function* $g \in A$ *with* $g^{2^k} = f$.

Then A *has bounded relative units at* x.

The next lemma is part of a result of Feinstein [1992, Proposition 1.5].

Lemma 3.5. *Let* A *be a uniform algebra on a compact space* X, *and let* $x \in X$. *If* A *has bounded relative units at* x, *then* x *is a generalized peak point for* A, *and* A *is strongly regular at* x.

The next lemma, whose elementary proof we omit, is a modification of a lemma of Feinstein [1992, Lemma 3.5].

Lemma 3.6. *Let* A *be a normal uniform algebra on a compact metrizable space* X, *and let* F *be a closed subset of* X. *Then there exists a countable subset* $\mathscr{F}$ *of* A *consisting of functions each vanishing identically on a neighborhood of* F *such that for each point* $x \in X \setminus F$, *and for each compact subset* E *of* $X \setminus \{x\}$, *there exists a neighborhood* U *of* x, *and a function* $f \in \mathscr{F}$ *such that* $f|U = 1$ *and* $f|E = 0$.

4. Localness of strong regularity

In this section we prove the localness of strong regularity for normal uniform algebras. The result will be used in the next section to obtain condition (ii) in Theorems 1.4 and 1.5.

Theorem 4.1. *Let A be a normal uniform algebra on a compact space X, and let x_0 be a point of X. If there exists a closed neighborhood N of x_0 in X such that $\overline{A|N}$ is strongly regular at x_0, then A is strongly regular at x_0.*

Proof. Suppose that there exists a closed neighborhood N of x_0 in X such that $\overline{A|N}$ is strongly regular at x_0. Fix $f \in A$ satisfying $f(x_0) = 0$, and fix $\varepsilon > 0$. We are to show that there exists a function $g \in A$ such that $\|f - g\|_X < \varepsilon$ and $g = 0$ on a neighborhood of x_0.

Let L be a closed neighborhood of x_0 contained in the interior N° of N. By the normality of A, there is a function $\varphi \in A$ such that $\varphi = 1$ on L and $\varphi = 0$ on $X \setminus N^\circ$.

By the strong regularity of $\overline{A|N}$, there is a function $h \in \overline{A|N}$ such that $\|f - h\|_N < \varepsilon / \|\varphi\|_N$ and $h = 0$ on some closed neighborhood M of x_0. There is a sequence (h_n) in A such that $h_n|N \to h$ uniformly on N.

For each $n \in \mathbb{Z}_+$, set $g_n = \varphi h_n + (1 - \varphi) f$. Then each g_n is in A. Define a function g on X by

$$g = \begin{cases} \varphi h + (1 - \varphi) f & \text{on } N, \\ f & \text{on } X \setminus N. \end{cases}$$

On $X \setminus N$ we have $g_n = f = g$. Moreover, $g_n \to g$ uniformly on X. Thus g is in A. Furthermore,

$$\|f - g\|_X = \|f - g\|_N = \|\varphi(f - h)\|_N \le \|\varphi\|_N \, \|f - h\|_N < \varepsilon.$$

Finally, $g = 0$ on the neighborhood $L \cap M$ of x_0. □

5. Proofs of Theorems 1.4 and 1.5

Recall that given a disc Δ, we denote the radius of Δ by $r(\Delta)$. We will denote the distance from Δ to the point 1 by $s(\Delta)$. Explicitly, $s(\Delta) = \inf\{|z - 1| : z \in \Delta\}$. We will denote the open disc with center a_n and radius r_n by $D(a_n, r_n)$ and the corresponding closed disc by $\overline{D}(a_n, r_n)$.

The following lemma is the key to the proofs of Theorems 1.4 and 1.5.

Lemma 5.1. *Fix real numbers $0 \le \rho_1 < \rho_2$. Let $\{D_k\}_{k=1}^\infty$ be a sequence of discs such that $\sum_{k=1}^\infty r(D_k) \exp(2\rho_2 / s(D_k)) < \infty$. Set $K = \overline{D} \setminus \bigcup_{k=1}^\infty D_k$. Then $I_{\rho_1} \supsetneqq I_{\rho_2}$.*

Proof. It is easily shown that the function $(z - 1) \exp\bigl(\rho_2 \frac{z+1}{z-1}\bigr)$ is in I_{ρ_1}, and hence $I_{\rho_1} \supset I_{\rho_2}$. To prove that the inclusion is strict, we will exhibit a measure on K that annihilates I_{ρ_2} but does not annihilate the function $(z - 1) \exp\bigl(\rho_1 \frac{z+1}{z-1}\bigr)$.

For $n \in \mathbb{Z}_+$, let $K_n = \bar{D} \setminus \bigcup_{k=1}^n D_k$. The boundary of K_n consists of the union of a finite collection of circular arcs (and possibly some isolated points which can be ignored), and we can define a measure μ_n on ∂K_n by requiring that for every function $f \in C(K_n)$ we have

$$\int f\, d\mu_n = \int_{\partial K_n} f(z) \exp\Big(-\rho_2 \frac{z+1}{z-1}\Big)\, dz.$$

Let $M = \sum_{k=1}^\infty r(D_k) \exp(2\rho_2/s(D_k)) < \infty$. Then $\|\mu_n\| \le 2\pi(M+1)$. Consequently, $\{\mu_k\}_{k=1}^\infty$ has a weak*-accumulation point μ. Since $K = \bigcap K_n$, the measure μ is supported on K. If g is a rational function with no poles on K, then for large values of n, the function g has no poles on K_n, and by Cauchy's theorem

$$\int g(z)(z-1) \exp\Big(\rho_2 \frac{z+1}{z-1}\Big)\, d\mu_n = \int_{\partial K_n} g(z)(z-1)\, dz = 0.$$

Thus

$$\int g(z)(z-1) \exp\Big(\rho_2 \frac{z+1}{z-1}\Big)\, d\mu = 0$$

for every rational function g with no poles on K. It follows that μ annihilates I_{ρ_2}.

To calculate the integral $\int (z-1) \exp\big(\rho_1 \frac{z+1}{z-1}\big)\, d\mu$ first note that the function $(z-1)\exp\big((\rho_1-\rho_2)\frac{z+1}{z-1}\big)$ has a single isolated singularity at $z=1$, and the residue there is $2(\rho_1-\rho_2)^2 \exp(\rho_1-\rho_2)$ since

$$\begin{aligned}
(z-1) &\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\\
&= (z-1)\exp\Big((\rho_1-\rho_2)\Big(1+\frac{2}{z-1}\Big)\Big)\\
&= (z-1)\exp\Big((\rho_1-\rho_2)+\Big(\frac{2(\rho_1-\rho_2)}{z-1}\Big)\Big)\\
&= (z-1)(\exp(\rho_1-\rho_2))\Big(1+\frac{2(\rho_1-\rho_2)}{z-1}+\frac{1}{2}\Big(\frac{2(\rho_1-\rho_2)}{z-1}\Big)^2+\cdots\Big).
\end{aligned}$$

For each $m, n \in \mathbb{Z}_+$, set $K_n^m = K_n \cup \bar{D}\big(1, \frac{1}{m}\big)$. To each n there corresponds an $N(n) \in \mathbb{Z}_+$ such that for all $m \ge N(n)$, the discs $\bar{D}_1, \ldots, \bar{D}_n$ are disjoint from the disc $\bar{D}\big(1, \frac{1}{m}\big)$. Consequently, letting γ_m denote the part of ∂D contained in $\bar{D}\big(1, \frac{1}{m}\big)$, and letting σ_m denote the part of $\partial \bar{D}\big(1, \frac{1}{m}\big)$ outside D, we have for all $m \ge N(n)$ that

$$\partial K_n^m = (\partial K_n \setminus \gamma_m) \cup \sigma_m.$$

The integrals

$$\int_{\gamma_m} (z-1)\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\, dz \quad \text{and} \quad \int_{\sigma_m} (z-1)\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\, dz$$

each go to zero as $m \to \infty$ because the lengths of γ_m and σ_m go to zero as $m \to \infty$ and the integrands are bounded in modulus by $\frac{1}{m}$. Therefore, applying the residue theorem gives

$$\begin{aligned}\int (z-1)\exp\Big(\rho_1 \frac{z+1}{z-1}\Big)\, d\mu_n &= \int_{\partial K_n} (z-1)\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\, dz \\ &= \lim_{m\to\infty} \int_{\partial K_n^m} (z-1)\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\, dz \\ &= 4\pi i(\rho_1-\rho_2)^2 \exp(\rho_1-\rho_2).\end{aligned}$$

Thus

$$\int (z-1)\exp\Big(\rho_1\frac{z+1}{z-1}\Big)\, d\mu = 4\pi i(\rho_1-\rho_2)^2\exp(\rho_1-\rho_2) \neq 0. \qquad \square$$

Proof of Theorem 1.4. By the preceding lemma, it suffices to show that discs $\{D_k\}$ can be chosen such that $\sum r(D_k) < r$, such that $\sum r(D_k)\exp(\nu/s(D_k)) < \infty$ for every $\nu > 0$, and such that conditions (i) and (ii) hold. We begin by choosing discs such that these conditions are satisfied with the (possible) exception of condition (ii), and then we choose additional discs to achieve condition (ii) in addition.

Choose a sequence $\{\overline{D}(a_n, r_n)\}_{n=1}^{\infty}$ of closed discs such that:

(a) Each of the discs $\overline{D}(1, 1/j)$ for $j = 1, 2, \ldots$ is in $\{\overline{D}(a_n, r_n)\}$.

(b) The discs $\overline{D}(1, 1/j)$ for $j = 1, 2, \ldots$ are the only discs in $\{\overline{D}(a_n, r_n)\}$ that contain the point 1.

(c) For every $\varepsilon > 0$, every point of $\overline{D}$ lies in an open disc $D(a_n, r_n)$ with $r_n < \varepsilon$.

Then for each $n \in \mathbb{Z}_+$, the annulus $\{z : r_n/2 < |z - a_n| < r_n\}$ is at some positive distance δ_n from the point 1. Set $\varepsilon_n = \min\{2^{-(n+1)}r, 2^{-n}e^{-n/\delta_n}\}$. For each $n \in \mathbb{Z}_+$, choose discs $\{\Delta_k^n\}_{k=1}^{\infty}$ in the annulus $\{z : r_n/2 < |z - a_n| < r_n\}$ as in Lemma 3.1 with $\Delta = D(a_n, r_n)$ and $\varepsilon = \varepsilon_n$. Then $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\Delta_n^k) < r/2$. Now let $\nu > 0$ be arbitrary. For each $n \in \mathbb{Z}_+$ we have

$$\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \varepsilon_n \exp(\nu/\delta_n).$$

Thus, in particular, $\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \infty$. Furthermore, for all $n > \nu$, we have

$$\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \varepsilon_n \exp(n/\delta_n) \leq 2^{-n}.$$

Consequently, $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \infty$.

Set

$$K_1 = \overline{D} \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} \Delta_k^n.$$

Then $R(K_1)$ is regular, and hence normal, for given a closed set $L \subset K_1$ and a point $x \in K_1 \setminus L$, there is some $D(a_n, r_n)$ that contains x and is disjoint from L, and hence, by the choice of the discs $\{\Delta_k^n\}$, there is a function in $R(K_1)$ that vanishes on L but not at x.

To achieve the strong regularity at points different from 1 we will use Theorem 3.2 together with Theorem 4.1 on the localness of strong regularity. Choose a countable collection of open discs $\{B_n\}$ that covers $K_1 \setminus \{1\}$ such that none of the closed discs $\bar{B}_n$ contains the point 1. Let α_n denote the distance from $\bar{B}_n$ to the point 1. Set $\widetilde{\varepsilon}_n = \min\{2^{-(n+1)}r, 2^{-n}e^{-n/\alpha_n}\}$. Since in Theorem 3.2 the open unit disc can, of course, be replaced by any open disc, there exists a sequence of open discs $\{\widetilde{\Delta}_k^n\}$ such that $\sum_{k=1}^{\infty} r(\widetilde{\Delta}_k^n) < \widetilde{\varepsilon}_n$ and such that setting

$$K_2^n = \bar{B}_n \setminus \bigcup_{k=1}^{\infty} \widetilde{\Delta}_k^n,$$

the uniform algebra $R(K_2^n)$ is strongly regular. Note that $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\widetilde{\Delta}_k^n) < r/2$ and $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\widetilde{\Delta}_k^n)\exp\bigl(\nu/s(\widetilde{\Delta}_k^n)\bigr) < \infty$ for every $\nu > 0$ by a computation that is identical to an earlier one.

Now let $\{D_k\}$ be an enumeration of the collection of discs $\{\Delta_k^n\}_{k,n=1}^{\infty} \cup \{\widetilde{\Delta}_k^n\}_{k,n=1}^{\infty}$, and set $K = \bar{D} \setminus \bigcup D_k$. Of course $\sum r(D_k) < r$, and $\sum r(D_k)\exp(\nu/s(D_k)) < \infty$ for every $\nu > 0$. The uniform algebra $R(K)$ is normal because K is contained in K_1 and $R(K_1)$ is normal. Consider an arbitrary point $x_0 \in K \setminus \{1\}$, and choose a disc B_{n_0}, from the collection $\{B_n\}$, that contains x_0. Then $R(K \cap \bar{B}_{n_0})$ is strongly regular by Lemma 3.3 because $K \cap \bar{B}_{n_0}$ is contained in $K_2^{n_0}$ and $R(K_2^{n_0})$ is strongly regular. Furthermore, $\overline{R(K)|(K \cap \bar{B}_{n_0})} = R(K \cap \bar{B}_{n_0})$. Thus Theorem 4.1 shows that $R(K)$ is strongly regular at x_0. □

Proof of Theorem 1.5. Since the proof is similar to the proof of Theorem 1.4, we merely indicate the modifications needed. For the discs $\{\bar{D}(a_n, r_n)\}$, we discard conditions (a) and (b) and instead require that each disc $\bar{D}(a_n, r_n)$ is either centered at a point of ∂D or else is contained in D, and we retain condition (c). Let Γ denote the set of n such that a_n is in ∂D. For each $n \in \Gamma$, choose a number $\gamma_n > 0$ in such a way that the intersection with ∂D of the union of the annuli $\{z : r_n - \gamma_n < |z - a_n| < r_n + \gamma_n\}$ has one-dimensional Lebesgue measure less than r. Let Λ be the complement of that union in ∂D. Choose, for each $n \in \mathbb{Z}_+$, a positive number r_n' such that $r_n - \gamma_n < r_n' < r_n$. Then the annulus $\{z : r_n' < |z - a_n| < r_n\}$ is at a positive distance δ_n from Λ. To establish everything except condition (ii), we choose discs $\{\Delta_k^n\}$ in the annulus $\{z : r_n' < |z - a_n| < r_n\}$ as in Lemma 3.1 arguing as in the proof of Theorem 1.4, but with distance $s(\Delta_k^n)$ to 1 replaced by distance to Λ.

To get condition (ii), we argue essentially as in the proof of Theorem 1.4 except that for the collection $\{B_n\}$ we take the collection $\bigl\{D\bigl(0, 1 - \frac{1}{n}\bigr) : n = 2, 3, \ldots\bigr\}$, and we again replace distance to 1 by distance to Λ. □

6. Root extensions and ideals

In this section we prove results about systems of root extensions and ideals which we will use in the next section to prove Theorems 1.6 and 1.7. We present the results in greater generality than we will need because we believe they are of interest in their own right and are likely to have further applications.

Cole's method of root extensions [1968] (see also [Stout 1971, Section 19]) involves an iterative process. We begin by discussing a single step of the iteration.

Let A be a uniform algebra on a compact space X, and let $\mathscr{F}$ be a (nonempty) subset of A. Endow $\mathbb{C}^{\mathscr{F}}$ with the product topology. Let $p_1 : X \times \mathbb{C}^{\mathscr{F}} \to X$ and $p_f : X \times \mathbb{C}^{\mathscr{F}} \to \mathbb{C}$ denote the projections given by $p_1(x, (z_g)_{g\in\mathscr{F}}) = x$ and $p_f(x, (z_g)_{g\in\mathscr{F}}) = z_f$. Define $X_{\mathscr{F}} \subset X \times \mathbb{C}^{\mathscr{F}}$ by

$$X_{\mathscr{F}} = \{y \in X \times \mathbb{C}^{\mathscr{F}} : (p_f(y))^2 = f(p_1(y)) \text{ for all } f \in \mathscr{F}\},$$

and let $A_{\mathscr{F}}$ be the uniform algebra on $X_{\mathscr{F}}$ generated by the set of functions $\{f \circ p_1 : f \in A\} \cup \{p_f : f \in \mathscr{F}\}$. On $X_{\mathscr{F}}$ we have $p_f^2 = f \circ p_1$ for every $f \in \mathscr{F}$. Set $\pi = p_1|X_{\mathscr{F}}$, and note that π is surjective. There is an isometric embedding $\pi^* : A \to A_{\mathscr{F}}$ given by $\pi^*(f) = f \circ \pi$.

We call the uniform algebra $A_{\mathscr{F}}$ or the pair $(A_{\mathscr{F}}, X_{\mathscr{F}})$, the *$\mathscr{F}$-extension of A*, and we call π the *associated surjection*. Note that if X is metrizable and $\mathscr{F}$ is countable, then $X_{\mathscr{F}}$ is metrizable also. Given $x \in X$, if $\mathscr{F}$ is contained in M_x, then the set $\pi^{-1}(x)$ consists of a single point.

There is an operator $S : A_{\mathscr{F}} \to \pi^*(A)$ given by integrating over the fibers of π using the measure on each fiber that is invariant under the obvious action of $(\mathbb{Z}/2)^{\mathscr{F}}$ on each fiber. See [Cole 1968] or [Stout 1971, pp. 194–195] for details. Rather than working with S, we will use the operator $T : A_{\mathscr{F}} \to A$ obtained from S by identifying $\pi^*(A)$ with A. The following properties of T are almost obvious.

Lemma 6.1.

(i) $\|T\| = 1$.

(ii) $T \circ \pi^*$ *is the identity.*

(iii) *Given distinct functions* $f_1, \ldots, f_r \in \mathscr{F}$ *and a function* $f \in A$,

$$T(\pi^*(f) p_{f_1} \cdots p_{f_r}) = 0.$$

One can iterate the above extension process. This leads to the notion of a system of root extensions, which we next define.

Henceforth, τ will be a fixed infinite ordinal. A *system of root extensions* is a triple of indexed sets $(\{A_\alpha\}, \{X_\alpha\}, \{\pi_{\alpha,\beta}\})$ $(0 \le \alpha \le \beta \le \tau)$ (denoted for brevity by $\{A_\alpha\}_{0\le\alpha\le\tau}$) where each X_α is a compact space, each A_α is a uniform algebra on X_α, and each $\pi_{\alpha,\beta}$ is a continuous surjective map $\pi_{\alpha,\beta} : X_\beta \to X_\alpha$ such that the following conditions hold:

(i) The equation $\pi^*_{\alpha,\beta}(f) = f \circ \pi_{\alpha,\beta}$ defines a homomorphism of A_α into A_β.

(ii) For $\alpha \le \beta \le \gamma$, $\pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$, and $\pi_{\alpha,\alpha}$ is the identity on X_α.

(iii) For $\alpha < \tau$, there is a subset $\mathscr{F}_\alpha$ of A_α such that $A_{\alpha+1}$ is the $\mathscr{F}_\alpha$-extension of A_α and $\pi_{\alpha,\alpha+1}$ is the associated surjection.

(iv) For γ a limit ordinal, the space X_γ is the inverse limit of the inverse system $\{X_\alpha, \pi_{\alpha,\beta}\}_{\alpha\le\beta<\gamma}$, the maps $\pi_{\alpha,\gamma} : X_\gamma \to X_\alpha$ are those associated with the inverse limit, and A_γ is the closure in $C(X_\gamma)$ of $\bigcup_{\alpha<\gamma} \pi^*_{\alpha,\gamma}(A_\alpha)$.

The existence of systems of root extensions is of course proved by transfinite induction. A choice of the subsets $\mathscr{F}_\alpha$ uniquely determines a system of root extensions.

Remark 6.2. It follows trivially from conditions (i) and (ii) that for $\alpha \le \beta \le \gamma$, $\pi^*_{\beta,\gamma} \circ \pi^*_{\alpha,\beta} = \pi^*_{\alpha,\gamma}$, and $\pi^*_{\alpha,\alpha}$ is the identity on A_α.

Given a uniform algebra A on X, a uniform algebra $\widetilde{A}$ on $\widetilde{X}$, and a surjective continuous map $\widetilde{\pi} : \widetilde{X} \to X$, we will say that $\widetilde{A}$ and $\widetilde{\pi}$ are *obtained from A by a system of root extensions* if there exists a system of root extensions $(\{A_\alpha\}, \{X_\alpha\}, \{\pi_{\alpha,\beta}\})$ $(0 \le \alpha \le \beta \le \tau)$ with $A_0 = A$, $A_\tau = \widetilde{A}$, and $\pi_{0,\tau} = \widetilde{\pi}$.

The following is [Feinstein 1992, Corollary 2.9].

Lemma 6.3. *Given a system of root extensions* $\{A_\alpha\}_{0\le\alpha\le\tau}$, *if* A_0 *is normal*, *then* A_α *is normal for all* α.

For a system of root extensions $\{A_\alpha\}_{0\le\alpha\le\tau}$, Cole introduced certain surjective linear operators $T_\beta : A_\beta \to A_0$. It will be helpful for us to introduce, more generally, operators $T_{\alpha,\beta} : A_\beta \to A_\alpha$ for every $\alpha \le \beta$.

Lemma 6.4. *Given a system of root extensions* $\{A_\alpha\}_{0\le\alpha\le\tau}$ *there exists a system of surjective linear operators* $\{T_{\alpha,\beta} : A_\beta \to A_\alpha\}_{0\le\alpha\le\beta\le\tau}$ *such that the following conditions hold for all* $0 \le \alpha \le \beta \le \gamma \le \tau$:

(a) $T_{\alpha,\alpha}$ *is the identity operator on* A_α.

(b) $\|T_{\alpha,\beta}\| = 1$.

(c) $T_{\alpha,\beta} \circ T_{\beta,\gamma} = T_{\alpha,\gamma}$.

(d) $T_{\alpha,\beta} \circ \pi^*_{\alpha,\beta}$ *is the identity on* A_α.

(e) $T_{\alpha,\gamma} \circ \pi^*_{\beta,\gamma} = T_{\alpha,\beta}$.

(f) $T_{\beta,\gamma} \circ \pi^*_{\alpha,\gamma} = \pi^*_{\alpha,\beta}$.

Proof. First note that condition (d) is an immediate consequence of conditions (a) and (e), note that condition (f) is an immediate consequence of condition (d) and Remark 6.2, and note that the surjectivity of the operators $\{T_{\alpha,\beta}\}$ follows from condition (d). Thus it suffices to show that the operators $\{T_{\alpha,\beta}\}$ can be chosen to satisfy conditions (a), (b), (c), and (e). We will apply transfinite induction on β with α fixed to obtain operators $\{T_{\alpha,\beta}\}$ satisfying conditions (a), (b), and (e), and then observe that these operators satisfy condition (c) also.

The operator $T_{\alpha,\alpha}$ is specified. Consider $\alpha \le \beta \le \tau$, and assume as the induction hypothesis that operators $T_{\alpha,\delta}$ have been defined for all $\alpha \le \delta < \beta$ in such a way that conditions (a), (b), and (e) hold. If $\beta = \delta + 1$ for some δ, then A_β is the $\mathscr{F}_\delta$-extension of A_δ. Let $T : A_\beta \to A_\delta$ be the operator discussed in the paragraph immediately preceding Lemma 6.1, set $T_{\alpha,\beta} = T_{\alpha,\delta} \circ T$, and verify that conditions (a), (b), and (e) continue to hold. If β is a limit ordinal, define an operator $\widetilde{T}_{\alpha,\beta}$ on the dense subspace $\bigcup_{\alpha \le \delta < \beta} \pi^*_{\delta,\beta}(A_\delta)$ of A_β by

$$\widetilde{T}_{\alpha,\beta}(\pi^*_{\delta,\beta} f) = T_{\alpha,\delta} f \quad \text{for } f \in A_\delta.$$

Condition (e) ensures that $\widetilde{T}_{\alpha,\beta}$ is well defined, i.e., if $f_1 \in A_{\delta_1}$ and $f_2 \in A_{\delta_2}$ satisfy $\pi^*_{\delta_1,\beta} f_1 = \pi^*_{\delta_2,\beta} f_2$, then $T_{\alpha,\delta_1} f_1 = T_{\alpha,\delta_2} f_2$. Furthermore, $\|\widetilde{T}_{\alpha,\beta}\| = 1$, so $\widetilde{T}_{\alpha,\beta}$ has a unique continuous extension to an operator on A_β, which we declare to be $T_{\alpha,\beta}$. Conditions (a), (b), and (e) continue to hold. Thus the existence of operators $\{T_{\alpha,\beta}\}_{0 \le \alpha \le \beta \le \tau}$ satisfying conditions (a), (b), and (e) is established.

To verify that the operators we have defined satisfy condition (c), fix α and β, and apply transfinite induction on γ. □

Lemma 6.5. *If $\pi^{-1}_{\alpha,\beta}(x)$ consists of a single point, then $(T_{\alpha,\beta} f)(x) = f(\pi^{-1}_{\alpha,\beta}(x))$ for each function $f \in A_\beta$.*

Proof. For fixed α, apply transfinite induction on β. □

We will need the following functional analysis lemma, whose elementary proof we omit.

Lemma 6.6. *Let X be a Banach space, let Y be a closed subspace of X, let I be a closed subspace of Y, and let $S : X \to X$ be a norm 1 projection of X onto Y. Then the map $\widetilde{S} : X/S^{-1}(I) \to X/I$ induced by S is an isometry.*

The next lemma is the key to the proofs of our results on root extensions and ideals. Its proof is similar to the proof of [Ghosh and Izzo 2023, Lemma 4.1], which is essentially the special case in which the closed ideal I arises from a bounded point derivation. In the lemma, $T : A_{\mathscr{F}} \to A$ is the operator in Lemma 6.1.

Lemma 6.7. *Let A be a uniform algebra on a compact space X, let I be a closed ideal in A, and let $\phi : A \to A/I$ denote the quotient map. Let $\mathscr{F}$ be a subset of I. Then in the $\mathscr{F}$-extension $A_{\mathscr{F}}$ of A the set $I_{\mathscr{F}} = T^{-1}(I)$ is a closed ideal, and the map $\phi \circ T : A_{\mathscr{F}} \to A/I$ is a Banach algebra homomorphism that induces an isometric Banach algebra isomorphism of $A_{\mathscr{F}}/I_{\mathscr{F}}$ onto A/I.*

Proof. For notational convenience set $\Phi = \phi \circ T$. Then Φ is a linear map with kernel $I_{\mathscr{F}}$. Therefore, if Φ is multiplicative, i.e., if it satisfies

$$\Phi(fg) = \Phi(f)\Phi(g) \tag{1}$$

for all $f, g \in A_{\mathscr{F}}$, then Φ is a Banach algebra homomorphism, $I_{\mathscr{F}}$ is an ideal in $A_{\mathscr{F}}$, and identifying A with the subspace $\pi^*(A)$ of $A_{\mathscr{F}}$ and applying Lemma 6.6 shows that the induced Banach algebra isomorphism of $A_{\mathscr{F}}/I_{\mathscr{F}}$ onto A/I is isometric. Thus it suffices to show that Φ satisfies (1) for all $f, g \in A_{\mathscr{F}}$. Moreover, it is enough to verify (1) for f and g belonging to the dense subalgebra H of $A_{\mathscr{F}}$ that is algebraically generated by $\pi^*(A) \cup \{p_f : f \in \mathscr{F}\}$. Functions f and g in H can be expressed in the form

$$f = \pi^*(f_0) + \sum_{u=1}^{s} \pi^*(f_u)F_u \quad \text{and} \quad g = \pi^*(g_0) + \sum_{v=1}^{t} \pi^*(g_v)G_v,$$

where $f_0, f_1, \ldots, f_s, g_0, g_1, \ldots, g_t \in A$ and each F_u and each G_v is a nonempty product of distinct functions of the form p_f for $f \in \mathscr{F}$.

By Lemma 6.1, $Tf = f_0$ and $Tg = g_0$, so

$$\Phi(f) = (\phi \circ T)(f) = \phi(f_0) \quad \text{and} \quad \Phi(g) = (\phi \circ T)(g) = \phi(g_0).$$

Since $\phi(f_0 g_0) = \phi(f_0)\phi(g_0)$, the proof will be complete once we show that $\Phi(fg) = \phi(f_0 g_0)$.

View fg as a sum of four terms:

$$fg = \pi^*(f_0 g_0) + \left(\sum_{u=1}^{s} \pi^*(f_u g_0)F_u \right) + \left(\sum_{v=1}^{t} \pi^*(f_0 g_v)G_v \right) + \left(\sum_{u=1}^{s} \sum_{v=1}^{t} \pi^*(f_u g_v)F_u G_v \right).$$

By Lemma 6.1,

$$T(\pi^*(f_0 g_0)) = f_0 g_0, \tag{2}$$

$$T\left(\sum_{u=1}^{s} \pi^*(f_u g_0)F_u \right) = 0, \tag{3}$$

$$T\left(\sum_{v=1}^{t} \pi^*(f_0 g_v)G_v \right) = 0. \tag{4}$$

Now for fixed u and v, consider $T(\pi^*(f_u g_v)F_u G_v)$. We have $F_u = p_{f_1} \cdots p_{f_a}$ and $G_v = p_{g_1} \cdots p_{g_b}$ where $f_1, \ldots, f_a$ are distinct elements of $\mathscr{F}$ and $g_1, \ldots, g_b$ are also distinct elements of $\mathscr{F}$. Note that each of the sets $\{f_1, \ldots, f_a\}$ and $\{g_1, \ldots, g_b\}$

is necessarily nonempty. If $\{f_1, \dots, f_a\} = \{g_1, \dots, g_b\}$, then $F_u G_v = p_{f_1}^2 \cdots p_{f_a}^2 = \pi^*(f_1 \cdots f_a)$, and hence, by Lemma 6.1(ii),

$$(\phi \circ T)(\pi^*(f_u g_v) F_u G_v) = (\phi \circ T)(\pi^*(f_u g_v f_1 \cdots f_a)) = \phi(f_u g_v f_1 \cdots f_a);$$

the last quantity above is zero because $f_1, \dots, f_a$ belong to the ideal I. If instead $\{f_1, \dots, f_a\} \neq \{g_1, \dots, g_b\}$, then $F_u G_v$ can be expressed as the product of a possibly empty set of elements of $\pi^*(A)$ and a *nonempty* set of functions $p_{h_1}, \dots, p_{h_c}$ with $h_1, \dots, h_c \in \{f_1, \dots, f_a, g_1, \dots, g_b\}$; consequently, $T(\pi^*(f_u g_v) F_u G_v) = 0$ by Lemma 6.1(iii). We conclude that

$$(\phi \circ T)\left(\sum_{u=1}^{s} \sum_{v=1}^{t} \pi^*(f_u g_v) F_u G_v \right) = 0. \tag{5}$$

Collectively, (2)–(5) yield that

$$\Phi(fg) = (\phi \circ T)(fg) = \phi(f_0 g_0),$$

as desired. □

Finally we come to the theorems of this section.

Theorem 6.8. *Let* $(\{A_\alpha\}, \{X_\alpha\}, \{\pi_{\alpha,\beta}\})$ $(0 \le \alpha \le \beta \le \tau)$ *be a system of root extensions. Let* I_0 *be a closed ideal in* A_0, *and set* $S_0 = \mathrm{hull}(I_0)$. *For every* $0 \le \alpha \le \tau$, *set* $I_\alpha = T_{0,\alpha}^{-1}(I_0)$ *and* $S_\alpha = \pi_{0,\alpha}^{-1}(S_0)$. *Suppose that* $I_\alpha \supset \mathscr{F}_\alpha$ *for every* $0 \le \alpha < \tau$. *Then, for every* $0 \le \alpha \le \tau$:

- (i) $\pi_{0,\alpha}$ *takes* S_α *homeomorphically onto* S_0.
- (ii) $\pi_{0,\alpha}^*$ *induces in the obvious way an isometric isomorphism of* $A_0|S_0$ *onto* $A_\alpha|S_\alpha$.
- (iii) I_α *is a closed ideal in* A_α *such that* $\mathrm{hull}(I_\alpha) = S_\alpha$, *such that* $I_\alpha \cap \pi_{0,\alpha}^*(A_0) = \pi_{0,\alpha}^*(I_0)$, *and such that* A_α / I_α *is isometrically isomorphic as a Banach algebra to* A_0/I_0. *Consequently, there is an order-preserving bijective correspondence between the closed ideals of* A_α *containing* I_α *and the closed ideals of* A_0 *containing* I_0.

Proof. By a simple transfinite induction, one shows simultaneously that the hypotheses imply that every function in I_α, and hence every function in $\mathscr{F}_\alpha$, is zero on S_α, and that $\pi_{0,\alpha}$ takes S_α one-to-one onto S_0. Since $\pi_{0,\alpha}$ is continuous and S_α is compact, condition (i) follows.

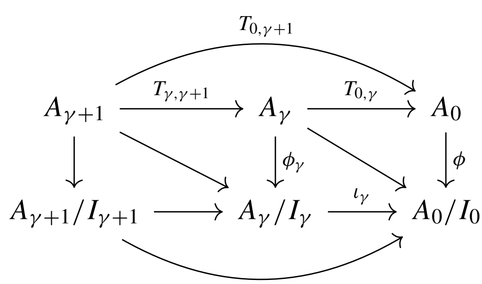
Given $f \in A_0$, the restriction of $\pi_{0,\alpha}^*(f) = f \circ \pi_{0,\alpha}$ to $\pi_{0,\alpha}^{-1}(S_0) = S_\alpha$ depends only on the restriction of f to S_0, so $\pi_{0,\alpha}^*$ induces a map of $A_0|S_0$ into $A_\alpha|S_\alpha$ that is obviously isometric. Given $g \in A_\alpha$, Lemma 6.5 shows that $\pi_{0,\alpha}^*(T_{0,\alpha} g)|S_\alpha = g$, so the map is onto. Thus condition (ii) holds.

We have already noted that every function in I_α is zero on S_α. Since for every point x of $X_\alpha \setminus S_\alpha$ there is a function $f \in I_0$ such that $\pi^*_{0,\alpha}(f)$, a function in I_α, is nonzero at x, this gives that $\text{hull}(I_\alpha) = S_\alpha$.

The equality $I_\alpha \cap \pi^*_{0,\alpha}(A_0) = \pi^*_{0,\alpha}(I_0)$ follows immediately from Lemma 6.4(d).

It remains to show, for each α, that I_α is an ideal in A_α and that A_α/I_α and A_0/I_0 are isometrically isomorphic as Banach algebras. Let $\phi : A_0 \to A_0/I_0$ denote the quotient map. Assume for the moment that the map $\phi \circ T_{0,\alpha} : A_\alpha \to A_0/I_0$ is a Banach algebra homomorphism for every α. Then since $I_\alpha = \ker(\phi \circ T_{0,\alpha})$, it follows that I_α is an ideal in A_α, and that the induced map $A_\alpha/I_\alpha \to A_0/I_0$ is a Banach algebra isomorphism. Identifying A_0 with the subspace $\pi^*_{0,\alpha}(A_0)$ of A_α and applying Lemma 6.6 shows that this isomorphism is an isometry. Thus to complete the proof it suffices to show that the map $\phi \circ T_{0,\alpha} : A_\alpha \to A_0/I_0$ is indeed a Banach algebra homomorphism for every α.

We apply transfinite induction. Consider $0 \le \beta \le \tau$, and assume as the induction hypothesis that $\phi \circ T_{0,\alpha}$ is a Banach algebra homomorphism for every $\alpha < \beta$. When $\beta = 0$, nothing needs to be proved. If β is a limit ordinal, then it is immediate from the induction hypothesis that the restriction of $\phi \circ T_{0,\beta}$ to the dense subset $\bigcup_{\alpha<\beta} \pi^*_{\alpha,\beta}(A_\alpha)$ of A_β is an algebra homomorphism, and hence, $\phi \circ T_{0,\beta}$ is a Banach algebra homomorphism by continuity. Now suppose instead that $\beta = \gamma + 1$ for some γ. The map $\phi \circ T_{0,\gamma}$ is a Banach algebra homomorphism by the induction hypothesis. Consequently, I_γ is a closed ideal in A_γ. Let $\phi_\gamma : A_\gamma \to A_\gamma/I_\gamma$ denote the quotient map, and let $\iota_\gamma : A_\gamma/I_\gamma \to A_0/I_0$ denote the Banach algebra isomorphism induced by $\phi \circ T_{0,\gamma}$. By Lemma 6.7, the map $\phi_\gamma \circ T_{\gamma,\gamma+1} : A_{\gamma+1} \to A_\gamma/I\gamma$ is a Banach algebra homomorphism. Now consider the following commutative diagram:



Observe that the map $\phi \circ T_{0,\gamma+1} : A_{\gamma+1} \to A_0/I_0$ coincides with the composition of the Banach algebra homomorphisms $\phi_\gamma \circ T_{\gamma,\gamma+1}$ and ι_γ and hence is itself a Banach algebra homomorphism, as desired. □

By suitable choice of system of root extensions we will obtain the following as a corollary.

Theorem 6.9. *Let A be a uniform algebra on a compact space X, let I be an ideal in A, and set $S = \text{hull}(I)$. Then there exists a uniform algebra $\widetilde{A}$ on a compact* *space $\widetilde{X}$ and a surjective continuous map $\widetilde{\pi} : \widetilde{X} \to X$, obtained from A by a system of root extensions, and there exists an ideal $\widetilde{I}$ in $\widetilde{A}$ such that setting $\widetilde{S} = \widetilde{\pi}^{-1}(S)$ conditions* (i)–(iii) *of Theorem 6.8 hold with $\widetilde{A}$, $\widetilde{X}$, $\widetilde{I}$, $\widetilde{S}$, and $\widetilde{\pi}$ in place of A_α, X_α, I_α, S_α, and $\pi_{0,\alpha}$, respectively, and such that furthermore*:

(iv) *Every function in $\widetilde{I}$ has a square root in $\widetilde{I}$.*

(v) *If A is normal, then $\widetilde{A}$ is normal and has bounded relative units at every point of $\widetilde{X} \setminus \widetilde{S}$, and hence every point of $\widetilde{X} \setminus \widetilde{S}$ is a generalized peak point for $\widetilde{A}$.*

If X is metrizable, then, in addition, we can take $\widetilde{X}$ to be metrizable provided we replace condition (iv) *by*:

(iv′) *There is a dense subset $\mathscr{F}$ of $\widetilde{I}$ such that every function in $\mathscr{F}$ has a square root in $\mathscr{F}$.*

Proof. Using Theorem 6.8 and transfinite induction, it is easily shown that there is a system of root extensions satisfying the conditions of Theorem 6.8 with $\tau = \Omega$ (the first uncountable ordinal) and $\mathscr{F}_\alpha = I_\alpha$ for every $0 \le \alpha < \Omega$. Set $\widetilde{A} = A_\Omega$, $\widetilde{X} = X_\Omega$, $\widetilde{I} = I_\Omega$, etc. Then conditions (i)–(iii) hold with $\widetilde{A}$, $\widetilde{X}$, $\widetilde{I}, \ldots$ in place of A_α, X_α, $I_\alpha, \ldots$, respectively, by Theorem 6.8.

Given $f \in \widetilde{I} = I_\Omega$, there is some $\alpha < \Omega$ and some $g \in A_\alpha$ such that $f = \pi^*_{\alpha,\Omega} g$. By construction and Lemma 6.1(iii), $\pi^*_{\alpha,\alpha+1} g = h^2$ for some $h \in A_{\alpha+1}$ such that

$$T_{\alpha,\alpha+1} h = 0. \tag{6}$$

Now

$$(\pi^*_{\alpha+1,\Omega} h)^2 = \pi^*_{\alpha+1,\Omega} h^2 = \pi^*_{\alpha+1,\Omega} \pi^*_{\alpha,\alpha+1} g = \pi^*_{\alpha,\Omega} g = f.$$

Furthermore, by Lemma 6.4 and (6),

$$T_{0,\Omega}(\pi^*_{\alpha+1,\Omega} h) = (T_{0,\alpha+1} \circ T_{\alpha+1,\Omega})(\pi^*_{\alpha+1,\Omega} h) = T_{0,\alpha+1} h = T_{0,\alpha} \circ T_{\alpha,\alpha+1} h = 0,$$

so $\pi^*_{\alpha+1,\Omega} h$ is in $\widetilde{I}$. Thus every function in $\widetilde{I}$ has a square root in $\widetilde{I}$.

Now suppose that A is normal. Then, by Lemma 6.3, $\widetilde{A}$ is normal. Consequently, given a point $\tilde{x} \in \widetilde{X} \setminus \widetilde{S}$ and a compact subset E of $\widetilde{X} \setminus \{\tilde{x}\}$, Theorem 2.1 ensures that there is a function in $\widetilde{I}$ that is one on a neighborhood of $\tilde{x}$ and zero on E. Therefore, $\widetilde{A}$ has bounded relative units at $\tilde{x}$ by Lemma 3.4. The final assertion of condition (v) follows by Lemma 3.5.

All that remains is to prove the last sentence of the theorem. From now on suppose that X is metrizable. Using Theorem 6.8 and induction, it is easily shown that there is a system of root extensions satisfying the conditions of Theorem 6.8 with $\tau = \omega$ (the first infinite ordinal) and with the property that, for every $0 \le \alpha < \omega$, the collection $\mathscr{F}_\alpha$ is a countable dense subset of I_α such that for every function $f \in \mathscr{F}_\alpha$ the function $\pi^*_{\alpha,\alpha+1} f$ is the square of a function in $\mathscr{F}_{\alpha+1}$. Furthermore, if A is normal, then Lemma 3.6 and Theorem 2.1 show that we can, and therefore we shall, choose $\mathscr{F}_\alpha$ such that, setting $S_\alpha = \pi_{0,\alpha}^{-1}(S)$, we have that for each point $x \in X_\alpha \setminus S_\alpha$, and for each compact subset E of $X_\alpha \setminus \{x\}$, there exists a neighborhood U of x, and a function $f \in \mathscr{F}_\alpha$ such that $f|U = 1$ and $f|E = 0$. Set $\widetilde{A} = A_\omega$, $\widetilde{X} = X_\omega$, $\widetilde{I} = I_\omega$, etc. Then $\widetilde{X}$ is metrizable. Furthermore, conditions (i)–(iii) hold with $\widetilde{A}$, $\widetilde{X}$, $\widetilde{I}$, ... in place of A_α, X_α, I_α, ..., respectively, by Theorem 6.8.

We will establish condition (iv′) with $\mathscr{F} = \bigcup_{\alpha<\omega} \pi_{\alpha,\omega}^*(\mathscr{F}_\alpha)$. First we show that every function in $\mathscr{F}$ has a square root in $\mathscr{F}$. Given $f \in \mathscr{F}$, there is some $\alpha < \omega$ and some $g \in \mathscr{F}_\alpha$ such that $f = \pi_{\alpha,\omega}^* g$. By construction, $\pi_{\alpha,\alpha+1}^* g = h^2$ for some $h \in \mathscr{F}_{\alpha+1}$. Then $\pi_{\alpha+1,\omega}^* h$ is in $\mathscr{F}$, and by Remark 6.2

$$(\pi_{\alpha+1,\omega}^* h)^2 = \pi_{\alpha+1,\omega}^* h^2 = \pi_{\alpha+1,\omega}^* \pi_{\alpha,\alpha+1}^* g = \pi_{\alpha,\omega}^* g = f,$$

so f has a square root in $\mathscr{F}$.

Next we show that $\mathscr{F}$ is contained in $\widetilde{I}$. Let f, α, and g be as in the previous paragraph. Then, by Lemma 6.4,

$$T_{0,\omega} f = T_{0,\omega} \pi_{\alpha,\omega}^* g = T_{0,\alpha} T_{\alpha,\omega} \pi_{\alpha,\omega}^* g = T_{0,\alpha} g,$$

and $T_{0,\alpha} g$ is in I because g is in $\mathscr{F}_\alpha \subset I_\alpha = T_{0,\alpha}^{-1}(I)$. Thus f is in $\widetilde{I}$, as desired.

To prove the density of $\mathscr{F}$ in $\widetilde{I}$, first note that $\pi_{\alpha,\omega}^*(\mathscr{F}_\alpha)$ is dense in $\pi_{\alpha,\omega}^*(I_\alpha)$, so it suffices to show that $\bigcup_{\alpha<\omega} \pi_{\alpha,\omega}^*(I_\alpha)$ is dense in $\widetilde{I}$. Fix $f \in \widetilde{I}$ and $\varepsilon > 0$ arbitrary. We will show that $\|\pi_{\alpha,\omega}^*(T_{\alpha,\omega} f) - f\| < \varepsilon$ for some $\alpha < \omega$. Since $T_{0,\alpha}(T_{\alpha,\omega} f) = T_{0,\omega} f$ is in I, the function $T_{\alpha,\omega} f$ is in I_α, so this will establish the desired density.

By the definition of $\widetilde{A} = A_\omega$, there exists $\alpha < \omega$ and $a \in A_\alpha$ such that

$$\|f - \pi_{\alpha,\omega}^* a\| < \varepsilon/2. \tag{7}$$

Then

$$\|(\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(f) - (\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(\pi_{\alpha,\omega}^* a)\| < \varepsilon/2.$$

Since $T_{\alpha,\omega} \circ \pi_{\alpha,\omega}^*$ is the identity, this gives

$$\|(\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(f) - \pi_{\alpha,\omega}^* a\| < \varepsilon/2. \tag{8}$$

From (7) and (8) we get

$$\|(\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(f) - f\| < \varepsilon.$$

This concludes the proof of condition (iv′).

If A is normal, then a simple argument shows that our choice of the $\mathscr{F}_\alpha$ ensures that for every point $\tilde{x} \in \widetilde{X} \setminus \widetilde{S}$ and every compact subset E of $\widetilde{X} \setminus \{\tilde{x}\}$, there exists a neighborhood U of $\tilde{x}$ and a function $f \in \mathscr{F}$ such that $f|U = 1$ and $f|E = 0$. Consequently, condition (v) can be proven in the same manner as was done earlier when we took $\mathscr{F}_\alpha = I_\alpha$. □

7. Normal peak point algebras that are not strongly regular

In this section we prove Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Set $A = R(K)$ with K as in Theorem 1.4, and set $I = \bar{J}_1$. Then let B be the uniform algebra $\widetilde{A}$ obtained by applying Theorem 6.9 taking $\widetilde{X}$ to be metrizable. Let x_0 be the unique point of $\widetilde{\pi}^{-1}(1)$. The uniform algebra B is normal. Also B has bounded relative units at every point of $\widetilde{X} \setminus \{x_0\}$, and hence, by Lemma 3.5, B is strongly regular at every point of $\widetilde{X} \setminus \{x_0\}$ and every point of $\widetilde{X} \setminus \{x_0\}$ is a peak point for B. The point x_0 is a peak point for B as well because the function $((1+z)/2) \circ \widetilde{\pi}$ peaks at x_0.

There is an order-preserving bijection between the closed ideals of B containing $\widetilde{I}$ and the closed ideals of $R(K)$ containing I. So the family of ideals $\{I_\rho : 0 \le \rho < \infty\}$ in $R(K)$ yields the family of ideals $\{H_\rho : 0 \le \rho < \infty\}$ in B. □

Proof of Theorem 1.7. The proof is essentially the same as the previous proof except that now we set $A = R(K)$ with K as in Theorem 1.5, let Λ be as in Theorem 1.5, and set $I = \bar{J}_\Lambda$. □

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ULTRAPRODUCT METHODS FOR MIXED q-GAUSSIAN ALGEBRAS

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We provide a unified ultraproduct approach for constructing Wick words in mixed q-Gaussian algebras which are generated by $s_j = a_j + a_j^*$ for $j = 1, \dots, N$, where $a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij}$. Here we also allow equality in $-1 \le q_{ij} = q_{ji} \le 1$. Using the ultraproduct method, we construct an approximate comultiplication of the mixed q-Gaussian algebras. Based on this we prove that these algebras are weakly amenable and strongly solid in the sense of Ozawa and Popa. We also encode Speicher's central limit theorem in the unified ultraproduct method, and show that the Ornstein–Uhlenbeck semigroup is hypercontractive, the Riesz transform associated to the number operator is bounded, and the number operator satisfies the L_p Poincaré inequalities with constants $C\sqrt{p}$.

1. Introduction

Group measure space constructions go back to the original work of Murray and von Neumann [1936]. In the last decades Popa and his collaborators have solved many open problems about fundamental groups and uniqueness of Cartan subalgebras; see, e.g., [Ozawa and Popa 2010a; 2010b; Popa and Vaes 2010; 2014; Houdayer and Shlyakhtenko 2011]. In parallel, von Neumann algebras generated by q-commutation relations (motivated by physics and number theory) were introduced by Bożejko and Speicher [1991], and further investigated by Bożejko, Kümmerer, and Speicher [Bożejko et al. 1997], Shlyakhtenko [2004], Nou [2004], Śniady [2004], Ricard [2005], Kennedy and Nica [2011], and Avsec [2011], among others. More recently, Dabrowski [2014] and Guionnet and Shlyakhtenko [2014] have shown that for small q, the q-Gaussian algebras are isomorphic to free group factors. All these results on factoriality, embeddability in R^ω, and approximation properties face a similar problem: how to derive properties of von Neumann algebras from combinatorial structures given by the original q-commutation relations.

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In this paper we study generalized q-commutation relations: Given a symmetric matrix $Q=(q_{ij})_{i,j=1}^N$, $q_{ij}\in[-1,1]$, Speicher [1993] considered variables satisfying

$$a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij}. \tag{1-1}$$

The mixed q-Gaussian algebra Γ_Q is generated by the self-adjoint variables $s_j = a_j^* + a_j$ and admits a normal faithful tracial state (see Section 3 for more details). Bożejko and Speicher [1994] systematically constructed the Fock space representation of the so-called braid relations, which is more general than (1-1). Then various properties were studied in, e.g., [Nou 2004; Kròlak 2000; 2005]. As for (1-1), Lust-Piquard [1999] showed the L_p boundedness of the Riesz transforms associated to the number operator of the system when $q_{ii} < 1$. Other generalized Gaussian systems related to our investigation have also been studied; see, e.g., [Guţă and Maassen 2002; Guţă 2003].

It is very tempting to believe that mixed q-Gaussian algebras behave in any respect the same way as the q-Gaussian algebras with constant q. Indeed, the L_2 space of such an algebra admits a decomposition

$$L_2(\Gamma_Q) = \bigoplus_{k=0}^{\infty} H_Q^k$$

into finite-dimensional subspaces H_Q^k of dimension N^k, which are eigenspaces of the number operator. For fixed $q_{ij} = q$ the number operator can be defined in a functorial way following Voiculescu's lead [Voiculescu et al. 1992] for $q=0$. Indeed, for every real Hilbert H one finds the q-Gaussian von Neumann algebra $\Gamma_q(H)$ and a group homomorphism $\alpha : O(H) \to \mathrm{Aut}(\Gamma_q(H))$ such that for $o \in O(H)$ and $h \in H$

$$\alpha(o)(s^q(h)) = s^q(o(h)).$$

Here, $s^q(e_j) = s_j$ and (e_j) is an orthonormal basis for the N-dimensional Hilbert space H. Then

$$T_t = E\alpha(o_t)\pi,$$

where $\pi : \Gamma_q(H) \to \Gamma_q(H \oplus H)$ is the natural embedding with conditional expectation E, and

$$o_t = \begin{pmatrix} e^{-t}\,\mathrm{id} & -\sqrt{1-e^{-2t}}\,\mathrm{id} \\ \sqrt{1-e^{-2t}}\,\mathrm{id} & e^{-t}\,\mathrm{id} \end{pmatrix}.$$

For nonconstant $Q = (q_{ij})$ we can no longer refer to functoriality directly. One of the first results in this paper is to provide a unified approach to the Ornstein–Uhlenbeck semigroup for $|q_{ij}| \le 1$ including the classical cases $q = 1$ for bosons and $q = -1$ for fermions. The fact that the dimension of the eigenspace H_Q^k is not more than N^k uniformly for all Q is based on thorough analysis of different forms of Wick words and probabilistic estimates (see Section 3).

Another new feature of these generalized relations comes from studying the operators

$$\Phi(x) = E(s_{N+1} x s_{N+1}),$$

where x is generated by $s_1, \dots, s_N$. For constant $q_{ij} = q$ we find $\Phi(x) = q^{l(x)} x$ can be easily computed in terms of the length function $l(x) = k$ if $x \in H_Q^k$. The formula for general Q is vastly more complicated. However, such expressions are crucial building blocks in proving strong solidity.

Let us recall some notions in operator algebras. We always assume the von Neumann algebras to be finite in this paper. Recall that a von Neumann algebra $\mathcal{M}$ has the weak* completely bounded approximation property (w*CBAP) if there exists a net of normal, completely bounded, finite-rank maps $\phi_\alpha : \mathcal{M} \to \mathcal{M}$ such that $\|\phi_\alpha\|_{\mathrm{cb}} \le C$ for all α, and $\phi_\alpha \to \mathrm{id}$ in the point weak* topology. Here, $\|\cdot\|_{\mathrm{cb}}$ denotes the completely bounded norm. The infimum of such constants C is called the Cowling–Haagerup constant and is denoted by $\Lambda_{\mathrm{cb}}(\mathcal{M})$. Cowling and Haagerup [1989] showed that a discrete group G is weakly amenable if and only if its group von Neumann algebra LG has w*CBAP. Thus, a von Neumann algebra with w*CBAP is also said to be weakly amenable. If $\Lambda_{\mathrm{cb}}(\mathcal{M}) = 1$, $\mathcal{M}$ is said to have the weak* completely contractive approximation property (w*CCAP). See, e.g., [Brown and Ozawa 2008] for more details of the approximation properties. Following Ozawa and Popa [2010a], a von Neumann algebra $\mathcal{M}$ is called strongly solid if the normalizer $\mathcal{N}_{\mathcal{M}}(P) := \{u \in \mathcal{U}(\mathcal{M}) : uPu^* = P\}$ of any diffuse amenable subalgebra $P \subset \mathcal{M}$ generates an amenable von Neumann algebra. Here, $\mathcal{U}(\mathcal{M})$ is the set of unitary operators in $\mathcal{M}$.

Theorem 1.1. *Γ_Q has w*CCAP and is strongly solid provided* $\max_{1 \le i,j \le N} |q_{ij}| < 1$.

These properties extend similar results due to Avsec [2011] for q-Gaussian von Neumann algebras. The w*CCAP for Γ_Q is proved using a transference method based on Avsec's w*CCAP result for the q-Gaussian algebras. Then we show a weak containment result of certain bimodules. These results, together with a modification of Popa's s-malleable deformation estimate, leads to strong solidity using a, by now, standard argument. The method used here follows that of [Houdayer and Shlyakhtenko 2011; Avsec 2011]. However, the techniques are more difficult than the case of q-Gaussian algebras. We have to use some nontrivial tricks to achieve certain results similar to those in [Avsec 2011].

The ultraproduct method plays an essential role in many aspects of this paper. It is well known that CCAP is a stepping stone for proving strong solidity. The transference method mentioned above relies entirely on an embedding of Γ_Q into an ultraproduct of von Neumann algebras which preserves the Wick words. This allows us to transfer the CCAP result of the constant q case of Avsec to the current mixed q

case. The argument can be illustrated using the following commutative diagram:

$$\begin{array}{ccc} \Gamma_Q & \xhookrightarrow{\pi_{\mathcal{U}}} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \mathbin{\overline{\otimes}} \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \\ \Big\downarrow{\scriptstyle \psi_\alpha} & & \Big\downarrow{\scriptstyle \varphi_\alpha(A)\otimes \mathrm{id}} \\ \Gamma_Q & \xhookrightarrow{\pi_{\mathcal{U}}} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \mathbin{\overline{\otimes}} \Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \end{array}$$

The notation will be explained in the proof of Theorem 5.5. The map $\pi_{\mathcal{U}}$ can be understood as an approximate comultiplication. Without the help of the ultraproduct method above, we will have to extend directly the argument for the constant q case to the mixed q case, which may be very hard due to the involved combinatorial structure.

We also prove some analytic properties for Γ_Q following the unified ultraproduct approach. The cornerstone is a Wick word decomposition result, whose proof involves some complicated combinatorial and probabilistic arguments. In this context, the ultraproduct construction provides a natural framework to encode Speicher's central limit theorem; see [Speicher 1992; 1993; Junge 2006]. Furthermore, the Wick words are identified as some special sequences in the ultraproduct of spin matrix models. Once we have the Wick word decomposition, it follows immediately that the Ornstein–Uhlenbeck semigroup $(T_t)_{t\geq 0}$ associated to Γ_Q is hypercontractive: For $1 \leq p, r < \infty$,

$$\|T_t\|_{L_p\to L_r} = 1 \quad \text{if and only if} \quad e^{-2t} \leq \frac{p-1}{r-1}.$$

Here, $L_p = L_p(\Gamma_Q, \tau_Q)$ is the noncommutative L_p space associated to the canonical tracial state τ_Q on Γ_Q. This result is a vast generalization of the work of Biane [1997] and Junge et al. [2015]. Indeed, we obtain hypercontractivity results for free products of q-Gaussian algebras and, in particular, free products of Clifford algebras. More exotic choices may be obtained for general q_{ij}. We also recover and extend the result of Lust-Piquard [1999] on the boundedness of Riesz transforms. Let A be the number operator of Γ_Q, which is also the generator of T_t. Define the gradient form (Meyer's "carré du champ") associated to A as

$$\Gamma(f,g) = \tfrac{1}{2}(A(f^*)g + f^*Ag - A(f^*g))$$

for f, g in the domain of A. We show that

(a) for $p \geq 2$,

$$c_p^{-1}\|A^{1/2}f\|_p \leq \max\bigl\{\|\Gamma(f,f)^{1/2}\|_p, \|\Gamma(f^*,f^*)^{1/2}\|_p\bigr\} \leq K_p\|A^{1/2}f\|_p$$

with $c_p = O(p^2)$ and $K_p = O(p^{3/2})$;

(b) for $1 < p < 2$,

$$K_{p'}^{-1}\|A^{1/2}f\|_p \le \inf_{\substack{\delta(f)=g+h\\ g\in G_p^c, h\in G_p^r}} \big\{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\big\} \le C_p\|A^{1/2}f\|,$$

with $K_{p'} = O(1/(p-1)^{3/2})$ and $C_p = O(1/(p-1)^2)$, where δ is a derivation related to the Riesz transforms and G_p^c and G_p^r are two Gaussian spaces (all will be defined below).

Moreover, we obtain the L_p Poincaré inequalities:

$$\|f - \tau_Q(f)\|_p \le C\sqrt{p}\max\big\{\|\Gamma(f, f)^{1/2}\|_p, \Gamma(f^*, f^*)^{1/2}\|_p\big\} \quad \text{for } p \ge 2.$$

This is an extension of similar results for the Walsh and Fermionic system in [Efraim and Lust-Piquard 2008]. It is known that the constant $C\sqrt{p}$ in such inequalities is crucial for proving concentration and transportation inequalities; see, e.g., [Zeng 2014].

The paper is organized as follows. Some preliminaries and notation are presented in Section 2. We construct the mixed q-Gaussian algebras and the Ornstein–Uhlenbeck semigroup in Section 3, where the Wick word decomposition result is also proved with a lengthy argument. The analytic properties are proved in Section 4, and the strong solidity is proved in Section 5.

2. Preliminaries and notation

2A. ***Notation.*** We write $[N] = \{1, 2, \dots, N\}$ for $N \in \mathbb{N}$. The set of nonnegative integers is denoted by $\mathbb{Z}_+$. For $n \in \mathbb{N}$, we denote by M_n the algebra of $n \times n$ matrices. We will use some notation to analyze combinatorial structures following [Speicher 1992; Junge et al. 2015]. Denote by $P(d)$ the set of all partitions of $[d] = \{1, \dots, d\}$. For $\sigma, \pi \in P(d)$, we write $\sigma \le \pi$ or $\pi \ge \sigma$ if σ is a refinement of π. We denote the integer valued vectors by $\underline{i}$, $\underline{j}$, etc. Given $\underline{i} = (i_1, \dots, i_d) \in [N]^d$, we associate a partition $\sigma(\underline{i})$ to $\underline{i}$ by requiring $k, l \in [d]$ belonging to the same block of $\sigma(\underline{i})$ if and only if $i_k = i_l$.

We denote by $|S|$ or $\#S$ the cardinality of a (finite) set S. If d is an even integer, we define $P_2(d)$ to be the set of pair partitions of $[d]$, i.e., $P_2(d)$ consists of $\pi = \{V_1, \dots, V_{d/2}\}$ such that $|V_k| = 2$ for every block V_k. Write $V_k = \{e_k, z_k\}$ with $e_k < z_k$ and $e_1 < e_2 < \cdots < e_{d/2}$. Given $\pi \in P_2(d)$, the set of crossings of π is denoted by

$$I(\pi) = \big\{\{k, l\} \mid 1 \le k, l \le d/2 \text{ and } e_k < e_l < z_k < z_l\big\}. \tag{2-1}$$

For $d \in \mathbb{N}$, we denote by $P_1(d)$ the one element set of singleton partition of $[d]$, i.e., $P_1(d) = \{\sigma_0\}$ and $\sigma_0 = \{\{1\}, \{2\}, \dots, \{d\}\}$. Let $P_{1,2}(d)$ denote the set of partitions consisting of only singletons and pair blocks, and $P_r(d) = P(d) \setminus P_{1,2}(d)$. Let $\sigma \in P_{1,2}(d)$ be given by

$$\sigma = \{V_1, \dots, V_{s+u}\},$$

where the V_j's are singletons ($V_j = \{e_j = z_j\}$) or pair blocks ($V_j = \{e_j, z_j\}$). Assume there are s singleton blocks and u pair blocks in σ. Let σ_p be a subpartition consisting of the u pair blocks of $\sigma \in P_{1,2}(d)$. Denote by $I_p(\sigma) := I(\sigma_p)$ the set of pair crossings of σ given in (2-1) and define

$$I_{\rm sp}(\sigma) = \big\{\{r, t\} : e_r < e_t = z_t < z_r\big\}$$

to be the set of crossings between pairs and singletons.

Given a discrete group G, the left regular representation is $\lambda : G \to \ell_2(G)$, $\lambda(g)\delta_h = \delta_{gh}$ for $g, h \in G$, and $(\delta_g)_{g\in G}$ is a canonical basis of $\ell_2(G)$. The group von Neumann algebra of G is denoted by LG and the canonical trace by τ_G. The Kronecker delta function is denoted by $\delta_{i,j}$. The use of two δ's will not appear in the same place. It should be clear from the context which one we are using. We let $\mathbb{1}_n$ denote the $n \times n$ matrix with all entries equal to 1.

2B. ***Spin matrix model.*** We consider a general spin matrix model, following [Lust-Piquard 1998; Junge et al. 2015]. Fix a finite integer N. Let $J_{N,m} = [N] \times [m]$ and $J_N = [N] \times \mathbb{N}$. We usually do not specify the dependence on N and simply write $J_m = J_{N,m}$ and $J = J_N$ if there is no ambiguity. We equip J_m with the lexicographical order. Let $\varepsilon : J \times J \to \{-1, 1\}$ be a map satisfying $\varepsilon(x, y) = \varepsilon(y, x)$ and $\varepsilon(x, x) = -1$ for all $x, y \in J$. Consider the complex unital algebra $\mathcal{A}_m = \mathcal{A}_m(N, \varepsilon)$ generated by $(x_i(k))_{(i,k)\in J_m}$, where the $x_i(k)$'s satisfy $x_i(k)^* = x_i(k)$ and

$$x_i(k)x_j(l) - \varepsilon((i, k), (j, l))x_j(l)x_i(k) = 2\delta_{(i,k),(j,l)}$$

for $(i, k), (j, l) \in J_m$. It is well known that the $x_i(k)$'s can be represented as tensor products of Pauli matrices. Thus $\mathcal{A}_m$ can be represented as a matrix subalgebra of $M_{2^{Nm}}$. A generic element of $\mathcal{A}_m$ can be written as a linear combination of words of the form

$$x_B = x_{i_1}(k_1) \cdots x_{i_d}(k_d),$$

where $B = \{(i_1, k_1), \dots, (i_d, k_d)\} \subset J_m$. We say x_B is a reduced word if the $x_{i_r}(k_r)$'s in x_B are pairwise different for $r = 1, \dots, d$. Using the commutation relation, every word ξ can be written in the reduced form, denoted by $\widetilde{\xi}$. There is a canonical normalized trace τ_m on $\mathcal{A}_m$ such that $\tau_m(x_B) = \delta_{B,\varnothing}$ for a reduced word x_B.

2C. ***Pisier's method for multi-index summations.*** Let $\sigma \in P(d)$ be a partition. In the following we need to estimate the L_p norm of

$$\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})\geq\sigma} x_1(k_1) \cdots x_d(k_d),$$

where $x_i(k_i) \in \bigcap_{p<\infty} L_p(\tau)$, and $L_p(\tau)$ is a noncommutative L_p space associated to a trace τ. To this end, we follow Pisier's method [2000]. As illustrated in the proof of [Pisier 2000, Sublemma 3.3], one can find $\xi_1(k_1), \dots, \xi_d(k_d) \in \mathrm{LG}$ such that $\tau_G(\xi_1(k_1)\cdots\xi_d(k_d)) = 1$ if and only if $\sigma(\underline{k}) \geq \sigma$ and $\tau_G(\xi_1(k_1)\cdots\xi_d(k_d)) = 0$ otherwise. Here, G is a suitable product of free groups, LG is the von Neumann algebra of G, and τ_G is the canonical trace on LG.

Let us explain this in more detail using an example. We denote by $\mathbb{F}_m$ the free group with free generators $(g_i)_{i\in[m]}$. Suppose $d = 6$ and $\sigma = \{\{1, 3, 5\}, \{2, 6\}, \{4\}\}$. In this case, $G = \mathbb{F}_m \times \mathbb{F}_m \times \mathbb{F}_m$ and for $i \in [m]$,

$$\begin{aligned} \xi_1(i) &= \lambda(g_i)^* \otimes 1 \otimes 1, & \xi_2(i) &= 1 \otimes \lambda(g_i)^* \otimes 1, \\ \xi_3(i) &= \lambda(g_i) \otimes 1 \otimes \lambda(g_i)^*, & \xi_4(i) &= 1 \otimes 1 \otimes 1, \\ \xi_5(i) &= 1 \otimes 1 \otimes \lambda(g_i), & \xi_6(i) &= 1 \otimes \lambda(g_i) \otimes 1. \end{aligned}$$

Then $\tau_G(\xi_1(k_1)\cdots\xi_6(k_6)) = 1$ if and only if $k_1 = k_3 = k_5$ and $k_2 = k_6$.

Returning to the general setting, consider the algebraic tensor product $\mathrm{LG}\otimes L_p(\tau)$. Since $\tau_G \otimes \mathrm{id}$ extends to contractions on L_p, using Hölder's inequality, we have

$$\begin{aligned} \Big\| \sum_{\underline{k}\in[m]^d:\sigma(\underline{k})\geq\sigma} x_1(k_1)\cdots x_d(k_d) \Big\|_p &= \Big\| \sum_{\underline{k}\in[m]^d} \tau_G(\xi_1(k_1)\cdots\xi_d(k_d)) x_1(k_1)\cdots x_d(k_d) \Big\|_p \\ &\leq \Big\| \sum_{\underline{k}\in[m]^d} \xi_1(k_1)\otimes x_1(k_1)\cdots\xi_d(k_d)\otimes x_d(k_d) \Big\|_p \\ &\leq \prod_{i=1}^d \Big\| \sum_{k_i=1}^m \xi_i(k_i)\otimes x_i(k_i) \Big\|_{pd}. \end{aligned}$$

If i belongs to a singleton block of σ, then $\xi_i(k_i) = 1$ and

$$\Big\| \sum_{k_i=1}^m \xi_i(k_i)\otimes x_i(k_i) \Big\|_{pd} = \Big\| \sum_{k_i=1}^m x_i(k_i) \Big\|_{pd}.$$

If i does not belong to any singleton of σ, then it is well known that

$$\Big\| \sum_{k_i=1}^m \xi_i(k_i)\otimes x_i(k_i) \Big\|_{pd} = \Big\| \sum_{k_i=1}^m \lambda(g_i)\otimes x_i(k_i) \Big\|_{pd}.$$

By [Pisier 2000, Lemma 3.4], we have, for any even integer $p \geq 2$,

$$\begin{aligned} &\Big\| \sum_{k_i=1}^m \lambda(g_i)\otimes x_i(k_i) \Big\|_p \\ &\qquad \leq \frac{3\pi}{4} \max\Big\{ \Big\| \Big(\sum_{k_i} x_i(k_i)^* x_i(k_i) \Big)^{\frac{1}{2}} \Big\|_p, \Big\| \Big(\sum_{k_i} x_i(k_i) x_i(k_i)^* \Big)^{\frac{1}{2}} \Big\|_p \Big\}. \end{aligned}$$

We record this result as follows. Denote by σ_{sing} and σ_{ns} the union of singletons and the union of nonsingleton blocks of σ respectively. Thus we have $\#\sigma_{\text{sing}}+\#\sigma_{ns}=d$.

Proposition 2.1. *Let $\sigma\in P(d)$ be a partition and $x_i(k_i)\in L_p(\tau)$ for $\underline{k}\in[m]^d$ and $i\in[d]$. Then, for any even integer $p\geq 2$,*

$$\Big\|\sum_{\underline{k}:\sigma(\underline{k})\geq\sigma}x_1(k_1)\cdots x_d(k_d)\Big\|_p \leq\Big(\frac{3\pi}{4}\Big)^{\#\sigma_{ns}}\prod_{i\in\sigma_{\text{sing}}}\Big\|\sum_{k_i=1}^m x_i(k_i)\Big\|_{pd}\prod_{i\in\sigma_{ns}}\max\bigg\{\Big\|\Big(\sum_{k_i=1}^m x_i(k_i)^*x_i(k_i)\Big)^{\frac12}\Big\|_{pd}, \Big\|\Big(\sum_{k_i=1}^m x_i(k_i)x_i(k_i)^*\Big)^{\frac12}\Big\|_{pd}\bigg\}.$$

This result will be used in a slightly more general setting. We may have other fixed operators, y_j's, inside the product $x_1(k_1)\cdots x_d(k_d)$. In this case, we may simply attach the y_j's to their adjacent $x_i(k_i)$'s and then invoke Proposition 2.1.

3. Construction and Wick word decomposition

The algebra we study here can be constructed using purely operator algebraic techniques if $\max_{1\leq i,j\leq N}|q_{ij}|<1$ as shown in [Bożejko and Speicher 1994]. However, we use the probabilistic approach due to Speicher [1992; 1993]. This is convenient for studying the analytic properties following Biane's original idea [1997]. The main result of this section is Theorem 3.8. Although the proof is unexpectedly lengthy, the analytic properties are easy consequences of this result. As a byproduct, we also provide an alternative construction of the Fock space representation.

3A. ***Speicher's CLT and von Neumann algebra ultraproducts.*** Let $Q=(q_{ij})_{i,j=1}^N$ be a symmetric matrix where $q_{ij}=q(i,j)\in[-1,1]$. Note that we do not specify the values on the diagonal. Following the notation of Section 2B, we consider a probability space $(\Omega,\mathbb{P})$ and a family of independent random variables $\varepsilon((i,k),(j,l)):\Omega\to\{-1,1\}$ for $(i,k)<(j,l)$ with distribution

$$\begin{aligned}\mathbb{P}\big(\varepsilon((i,k),(j,l))=-1\big)&=\tfrac12(1-q(i,j)),\\ \mathbb{P}\big(\varepsilon((i,k),(j,l))=1\big)&=\tfrac12(1+q(i,j)),\end{aligned}\tag{3-1}$$

so that $\mathbb{E}[\varepsilon((i,k),(j,l))]=q(i,j)$. Here, $(i,k),(j,l)\in[N]\times\mathbb{N}$. Given $\omega\in\Omega$, the commutation/anticommutation relation is fixed. We understand all generators $x_i(k)(\omega)$ to depend on ω. Restricting $k\in[m]$ we get random $\mathcal{A}_m$. Because the dependence on ω should be clear from the context, we will not write ω in the following to simplify notation. Let $\widetilde{x}_i(m)=\frac{1}{\sqrt{m}}\sum_{k=1}^m x_i(k)$.

The following central limit theorem result was due to Speicher [1993] and is a generalization of [Speicher 1992]. We streamline Speicher's proof in the appendix for the reader's convenience. The same strategy will be used repeatedly when we prove Theorem 3.8.

Theorem 3.1. *Let* $\underline{i} \in [N]^s$. *Then*

$$\lim_{m\to\infty} \tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m)) = \delta_{s\in 2\mathbb{Z}} \sum_{\substack{\sigma\in P_2(s)\\ \sigma\le\sigma(\underline{i})}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)) \quad a.s.$$

Here and in what follows, we understand $\prod_{\{i,j\}\in\varnothing} q(i, j)$ *to be equal to* 1.

By Theorem 3.1, we can find a full probability set $\Omega_0 \subset \Omega$ such that the convergence holds for all $\omega \in \Omega_0$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By the well-known ultraproduct construction of von Neumann algebras (see, e.g., [Brown and Ozawa 2008, Appendix A]), we have a finite von Neumann algebra $\mathcal{A}_{\mathcal{U}} := \prod_{m,\mathcal{U}} \mathcal{A}_m$ with normal faithful tracial state $\tau_{\mathcal{U}} = \lim_{m,\mathcal{U}} \tau_m$. Put $\mathcal{A}_{\mathcal{U}}^{\infty} = \bigcap_{p<\infty} L_p(\mathcal{A}_{\mathcal{U}})$. For each $\omega \in \Omega_0$,

$$(\widetilde{x}_i(m)(\omega))^{\bullet} \in \mathcal{A}_{\mathcal{U}}^{\infty}.$$

Here and in what follows, we write $(\widetilde{x}_i(m)(\omega))^{\bullet}$ for the element represented by $(\widetilde{x}_i(m)(\omega))_{m\in\mathbb{N}}$ in the ultraproduct. We have the moment formula

$$\tau_{\mathcal{U}}\big((\widetilde{x}_{i_1}(m)(\omega))^{\bullet}\cdots(\widetilde{x}_{i_s}(m)(\omega))^{\bullet}\big) = \delta_{s\in 2\mathbb{Z}} \sum_{\substack{\sigma\in P_2(s)\\ \sigma\le\sigma(\underline{i})}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)). \tag{3-2}$$

It follows that

$$\tau_{\mathcal{U}}(|(\widetilde{x}_i(m)(\omega))^{\bullet}|^p) \le Cp^p.$$

By the uniqueness argument in [Junge 2006, Section 6], the von Neumann algebras generated by the spectral projections of the $(\widetilde{x}_i(m)(\omega))^{\bullet}$, where $i = 1, \dots, N$, for different $\omega \in \Omega_0$ are isomorphic. We denote by Γ_Q any von Neumann algebra in the isomorphic class with generators $(\widetilde{x}_i(m)(\omega))^{\bullet}$, where $i = 1, \dots, N$. This algebra was introduced by Speicher [1993] and studied in [Bożejko and Speicher 1994; Lust-Piquard 1999]. Note that $(\widetilde{x}_i(m)(\omega))^{\bullet}$ may be an unbounded operator, therefore may not be in Γ_Q. But, by our construction, it belongs to $\Gamma_Q^{\infty} := \bigcap_{p<\infty} L_p(\Gamma_Q, \tau_{\mathcal{U}})$. In the following, whenever we say that the $(\widetilde{x}_i(m)(\omega))^{\bullet}$, where $i = 1, \dots, N$, are generators of Γ_Q, we always mean $(\widetilde{x}_i(m)(\omega))^{\bullet} \in \Gamma_Q^{\infty}$ and Γ_Q is generated by the spectral projections of the $(\widetilde{x}_i(m)(\omega))^{\bullet}$'s. We call Γ_Q *the* mixed q-Gaussian algebra, and Q the structure matrix of Γ_Q. Sometimes we also write $\tau_Q = \tau_{\mathcal{U}}|_{\Gamma_Q}$.

There is another way of constructing Γ_Q. All the $x_i(k)$'s are in fact in $L_\infty(\Omega; \mathcal{A}_m)$ and thus $\widetilde{x}_i(m) \in L_\infty(\Omega; \mathcal{A}_m)$. Here, the trace on $L_\infty(\Omega; \mathcal{A}_m)$ is given by $\mathbb{E}\otimes\tau_m$. By the same CLT argument as for Theorem 3.1, we find the moment formula (A-4) in the limit, which is the same as (3-2). Therefore, as before, the $(\widetilde{x}_i(m))^{\bullet}$ for $i = 1, \dots, N$

generate a von Neumann algebra, denoted by Γ_Q^a. We call it the average model. Using the uniqueness results in [Junge 2006, Section 6], we have that Γ_Q^a is isomorphic to Γ_Q. When we write $(\widetilde{x}_i(m))^\bullet$, it can mean either an element in $\bigcap_{p<\infty} L_p\big(\prod_{m,\mathcal{U}} L_\infty(\Omega;\mathcal{A}_m)\big)$ or simply $(\widetilde{x}_i(m)(\omega))^\bullet$ for some $\omega\in\Omega_0$. It should be clear from the context which one we are using. In fact, we may simply write $x_1,\dots,x_N$ for the generators of Γ_Q if we are not concerned with the construction.

By considering different structure matrix Q, we can construct various examples as special cases of Γ_Q. The same philosophy was used before by Lust-Piquard [1999].

Example 3.2. $\Gamma_q(H)$, *where* $q\in[-1,1]$ *is fixed.* If $q(i,j)=q$ all $1\le i,j\le N$, then we recover the classical q-Gaussian algebra $\Gamma_q(H)$, where H is a real Hilbert space with $\dim H=N$.

Example 3.3. $\ast_{i=1}^n \Gamma_{q_i}(H_i)$, *where* $q_i\in[-1,1]$ *is fixed for* $i=1,\dots,n$. Here, the H_i's are real Hilbert spaces with $\dim H_i=d_i$. Let $N=d_1+\cdots+d_n$. Define Q as follows. For $k=0,\dots,n-1$ and $1\le\alpha,\beta\le d_{k+1}$, put

$$q\Big(\sum_{j=1}^k d_j+\alpha,\ \sum_{j=1}^k d_j+\beta\Big)=q_{k+1},$$

and $q(\alpha,\beta)=0$ otherwise. Then by the moment formula (3-2), we recover $\ast_{i=1}^n \Gamma_{q_i}(H_i)$. The case $q_i=-1$ for all $i=1,\dots,n$ was considered in [Junge et al. 2015].

Example 3.4. $\overline{\bigotimes}_{i=1}^n(\Gamma_{q_i}(H_i)\ast\Gamma_{p_i}(K_i))$, *where* $q_i,p_i\in[-1,1]$ *are fixed.* Here, the H_i's and K_i's are real Hilbert spaces with $\dim H_i=d_i$ and $\dim K_i=d_i'$. Let $N=\sum_{i=1}^n d_i+d_i'$. For $k=0,\dots,n-1$, define

$$\widetilde{q}\Big(\sum_{j=1}^i(d_j+d_j')+\alpha,\ \sum_{j=1}^i(d_j+d_j')+\beta\Big)=\begin{cases} q_i & \text{if } 1\le\alpha,\beta\le d_{i+1},\\ p_i & \text{if } d_{i+1}+1\le\alpha,\beta\le d_{i+1}+d_{i+1}',\\ 0 & \text{if } 1\le\alpha\le d_{i+1}<\beta\le d_{i+1}+d_{i+1}',\end{cases}$$

and $\widetilde{q}(\alpha,\beta)=1$ otherwise. Let $Q=(\widetilde{q}_{\alpha,\beta})_{1\le\alpha,\beta\le N}$. By the moment formula (3-2), this model gives mixed products of q-Gaussian algebras. For example, consider the von Neumann algebra of the integer lattice $L(\mathbb{Z}^n)$. We may identify $L(\mathbb{Z}^n)$ with $\overline{\bigotimes}_{i=1}^n\Gamma_0(\mathbb{R})$ via $\lambda(g_k)\mapsto x_k$, where the g_k's are the generators of $\mathbb{Z}^n$ and the x_k's are generators of $\overline{\bigotimes}_{i=1}^n\Gamma_0(\mathbb{R})$. Alternatively, by extending $\lambda(g_k)\mapsto x_{2k-1}x_{2k}$, we may embed $L(\mathbb{Z}^n)$ into $\overline{\bigotimes}_{i=1}^n\Gamma_{-1}(\mathbb{R})\ast\Gamma_{-1}(\mathbb{R})$.

3B. ***Wick word decomposition.*** For our later development, we need an analogue of Wick word decomposition, i.e., rewriting $(\widetilde{x}_{i_1}(m))^\bullet\cdots(\widetilde{x}_{i_d}(m))^\bullet$ as a linear combination of Wick words (to be defined) so that we can analyze the Ornstein–Uhlenbeck semigroup easily. This procedure is conceptually clear with the help of Fock space representation because $(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet$ belongs to $L_2(\Gamma_Q)$ and $L_2(\Gamma_Q)$ should coincide with the Fock space, which is spanned by Wick products; see [Bożejko et al. 1997; Bożejko and Speicher 1994]. However, we do not know the explicit formula for the decomposition of $(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet$ in terms of matrix models. Moreover, the known Fock space construction usually requires $\max_{i,j}|q_{ij}| < 1$.

Our approach is again probabilistic. We refer the readers to Section 2A for the notation used in the following. By definition

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet = \biggl(\frac{1}{m^{d/2}} \sum_{\underline{k}\in[m]^d} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\biggr)^\bullet.$$

Note that

$$\sum_{\underline{k}\in[m]^d} x_{i_1}(k_1)\cdots x_{i_d}(k_d) = \sum_{\sigma\in P_{1,2}(d)}\sum_{\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d) + \sum_{\sigma\in P_r(d)}\sum_{\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d).$$

We first record a simple algorithm which we will refer to later on.

Proposition 3.5. *Let $\underline{i} \in [N]^d$, $\underline{k} \in [m]^d$, $\sigma(\underline{k}) \le \sigma(\underline{i})$ and $\sigma(\underline{k}) \in P_{1,2}(d)$. Then there is a specific algorithm to interchange $x_{i_\alpha}(k_\alpha)$'s in $x_{i_1}(k_1)\cdots x_{i_d}(k_d)$ such that*

(1) $x_{i_1}(k_1)\cdots x_{i_d}(k_d) = \varepsilon(\underline{i},\underline{k})x_{j_1}(l_1')\cdots x_{j_s}(l_s')\cdots x_{j_d}(l_d')$, *where $\varepsilon(\underline{i},\underline{k})$ is a random sign resulting from interchanging $x_{i_\alpha}(k_\alpha)$'s which is given by*

$$\varepsilon(\underline{i},\underline{k}) = \prod_{\{r,t\}\in I_{\mathrm{sp}}(\sigma(\underline{k}))} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]) \times \prod_{\{r,t\}\in I_p(\sigma(\underline{k}))} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]);$$

(2) $(l_1',\dots,l_s')$ *are pairwise different and maintain their relative positions in $\underline{k}$, i.e., $(l_1',\dots,l_s')$ is obtained from $\underline{k}$ by removing the k_α's which correspond to pair blocks*;

(3) $l_{s+1}' = l_{s+2}', \dots, l_{d-1}' = l_d'$.

Proof. Since $\sigma(\underline{k}) \in P_{1,2}(d)$, for each k_α in $\underline{k}$, there is at most one k_β in $\underline{k}$ equal to k_α. We can find the first k_α corresponding to a singleton in $\sigma(\underline{k})$, and move $x_{i_\alpha}(k_\alpha)$ to the beginning of the word by interchanging it with the $x_{i_\beta}(k_\beta)$'s which are to the left of $x_{i_\alpha}(k_\alpha)$. Rename this $x_{i_\alpha}(k_\alpha)$ to be $x_{j_1}(l_1')$. This process produces a product of random signs of the form $\varepsilon((i_\alpha,k_\alpha),(i_\beta,k_\beta))$, where k_α corresponds to a singleton and k_β corresponds to a pair block in $\sigma(\underline{k})$. Then we repeat this procedure for the second k_α corresponding to a singleton in $\sigma(\underline{k})$, and rename it $x_{j_2}(l_2')$. We continue until all the $x_{i_\alpha}(k_\alpha)$ corresponding to singletons in $\sigma(\underline{k})$

are in front of the remaining $x_{i_\beta}(k_\beta)$'s corresponding to pair blocks in $\sigma(\underline{k})$. In this way, we get $x_{j_1}(l'_1)\cdots x_{j_s}(l'_s)$ and a product of random signs. Afterwards, we rename the variable $x_{i_\alpha}(k_\alpha)$ right-adjacent to $x_{j_s}(l'_s)$ to be $x_{j_{s+1}}(l'_{s+1})$. Then move the other term with the same k_α to the right of $x_{j_{s+1}}(l'_{s+1})$, and call it $x_{j_{s+2}}(l'_{s+2})$. This produces a product of $\varepsilon((i_\alpha,k_\alpha),(i_\beta,k_\beta))$, where k_β and k_α correspond to different pair blocks. Repeat this procedure for the next pair of k_α's. After finitely many steps, the algorithm will stop and we obtain $\varepsilon(\underline{i},\underline{k})x_{j_1}(l'_1)\cdots x_{j_s}(l'_s)\cdots x_{j_d}(l'_d)$ with the desired three properties. □

We write

$$(l'_1,\dots,l'_d)=(k_{\pi(1)},\dots,k_{\pi(d)}), \tag{3-3}$$

where π is a permutation determined by the algorithm. Similarly, $(j_1,\dots,j_d)=(i_{\pi(1)},\dots,i_{\pi(d)})$. Let

$$l_1=l'_1,\quad \dots,\quad l_s=l'_s,\quad l_{s+1}=l'_{s+1}=l'_{s+2},\quad \dots,\quad l_{s+u}=l'_{d-1}=l'_d. \tag{3-4}$$

Here, s and u are the number of singletons and pair blocks of $\sigma(\underline{k})$, respectively.

Lemma 3.6. *Let $\sigma\in P_{1,2}(d)$. Then, for all $2<p<\infty$ and fixed $\omega\in\Omega$,*

$$\lim_{m\to\infty}\Bigg\|\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma}x_{i_1}(k_1)\cdots x_{i_d}(k_d) - \frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma}E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Bigg\|_{L_p(\mathcal{A}_m,\tau_m)}=0.$$

Here, $\mathcal{N}_s(\underline{k})$ denotes the von Neumann algebra generated by all the $x_{i_\alpha}(k_\alpha)$'s, where the k_α's correspond to singleton blocks in $\sigma(\underline{k})$.

Proof. Let s and u denote the number of singletons and pair blocks of σ, respectively. Clearly, $s+2u=d$ and there are

$$m_{(s+u)}:=m(m-1)\cdots(m-s-u+1)$$

vectors $\underline{k}\in[m]^d$ with $\sigma(\underline{k})=\sigma$. Let $\underline{l}$ be given in (3-4). $\underline{l}$ is a vector of length $s+u$. Let $\delta_1,\dots,\delta_m$ be i.i.d. random selectors uniformly distributed on $\{1,2,\dots,s+u\}$ which are independent from $L_\infty(\Omega;\mathcal{A}_m)$. If all the l_α's are pairwise different, then by independence,

$$\mathbb{E}_\delta(1_{[\delta_{l_1}=1]}1_{[\delta_{l_2}=2]}\cdots 1_{[\delta_{l_{s+u}}=s+u]})=(s+u)^{-s-u},$$

where $\mathbb{E}_\delta$ is the expectation with respect to the δ_{l_q}'s. Define random sets B_q for $q=1,\dots,s+u$ by

$$B_q=\{l_q\in[m]:\delta_{l_q}=q\}.$$

Then, for each instance of the δ_{l_q}'s, the B_q's are pairwise disjoint and their union is $[m]$. By (3-3), there is a 1-to-1 correspondence between $\underline{k}$ and $\underline{l}$. We may rewrite

$$\begin{aligned}
&\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\\
&\quad= \sum_{\underline{l}\in[m]^{s+u}:\sigma(\underline{l})\in P_1(s+u)} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\\
&\quad= (s+u)^{s+u}\sum_{\underline{l}:\sigma(\underline{l})\in P_1(s+u)} \mathbb{E}_\delta[1_{[\delta_{l_1}=1]}1_{[\delta_{l_2}=2]}\cdots 1_{[\delta_{l_{s+u}}=s+u]}x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\\
&\quad= (s+u)^{s+u}\mathbb{E}_\delta\biggl(\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\biggr),
\end{aligned}$$

where $\sigma(\underline{l})\in P_1(s+u)$ amounts to saying that all the l_q's are pairwise different. For $q=s,s+1,\dots,s+u$, let $\mathcal{N}_q(\underline{k})$ be the von Neumann algebra generated by

$$\{x_{j_\alpha}(l'_\alpha):\alpha\le s+2(q-s)\}.$$

Recall that $l_{s+u}=k_{\pi(d-1)}=k_{\pi(d)}$. Let

$$w_{\underline{i},\underline{l}}(l_{s+u})=\sum_{l_{s+u-1}\in B_{s+u-1}}\cdots\sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d).$$

Here we only fix l_{s+u} and sum over all the other indices. It is straightforward to check that

$$\{w_{\underline{i},\underline{l}}(l_{s+u})-E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\}_{l_{s+u}\in B_{s+u}}$$

is a sequence of martingale differences. Using the noncommutative Burkholder–Gundy inequality [Pisier and Xu 1997], we have

$$\begin{aligned}
&\biggl\|\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1}\bigl(x_{i_1}(k_1)\cdots x_{i_d}(k_d)-E_{\mathcal{N}_{s+u-1}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\bigr)\biggr\|_p\\
&\quad=\biggl\|\sum_{l_{s+u}\in B_{s+u}}\bigl(w_{\underline{i},\underline{l}}(l_{s+u})-E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\bigr)\biggr\|_p\\
&\quad\le C_p\biggl\|\biggl(\sum_{l_{s+u}\in B_{s+u}}|w_{\underline{i},\underline{l}}(l_{s+u})-E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))|^2\\
&\qquad\qquad+\bigl|\bigl(w_{\underline{i},\underline{l}}(l_{s+u})-E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\bigr)^*\bigr|^2\biggr)^{\frac{1}{2}}\biggr\|_p\\
&\quad=:\Psi.
\end{aligned}$$

By the triangle inequality, we have

$$\begin{aligned}\Psi &\le C_p\sqrt{|B_{s+u}|}\sup_{l_{s+u}\in B_{s+u}}\|w_{\underline{i},\underline{l}}(l_{s+u})-E_{\mathcal{N}_{s+u-1}(\underline{k})}(w_{\underline{i},\underline{l}}(l_{s+u}))\|_p\\ &\le 2C_p\sqrt{|B_{s+u}|}\sup_{l_{s+u}\in B_{s+u}}\|w_{\underline{i},\underline{l}}(l_{s+u})\|_p.\end{aligned}$$

Recall that $k_\alpha = l_{\pi^{-1}(\alpha)}$ if α is a singleton of σ. In this case, $l_\beta \in B_\beta$ if and only if $k_\alpha \in B_{\pi^{-1}(\alpha)}$, where $\pi(\beta)=\alpha$. Replacing p by a larger even integer if necessary, arguing as for Proposition 2.1, or simply adding zeros to apply Proposition 2.1, we find

$$\|w_{\underline{i},\underline{l}}(l_{s+u})\|_p \le \Big(\frac{3\pi}{4}\Big)^{2u}\prod_{\alpha\in\widetilde{\sigma}_{\mathrm{sing}}}\Big\|\sum_{k_\alpha\in B_{\pi^{-1}(\alpha)}} x_{i_\alpha}(k_\alpha)\Big\|_{pd}\prod_{\alpha\in\widetilde{\sigma}_{ns}} m^{1/2}.$$

Here, $\widetilde{\sigma}$ is obtained from σ by erasing one pair block containing $\pi(d)$ so that $\#\widetilde{\sigma}_{ns} = 2(u-1)$. We mention one subtlety here in applying Proposition 2.1. Since l_{s+u} is fixed, the term $x_{i_\alpha}(l_{s+u})$ is regarded to "attach" to its adjacent term. For instance, $x_{i_{j'}}(k_{j'})x_{i_\alpha}(l_{s+u})x_{i_j}(k_j)$ is regarded as a product of two terms, i.e., $[x_{i_{j'}}(k_{j'})x_{i_\alpha}(l_{s+u})]x_{i_j}(k_j)$ or $x_{i_{j'}}(k_{j'})[x_{i_\alpha}(l_{s+u})x_{i_j}(k_j)]$. Using the noncommutative Khintchine inequality [Lust-Piquard 1986; Lust-Piquard and Pisier 1991] or the Burkholder–Gundy inequality [Pisier and Xu 1997], we have, for $\alpha\in\widetilde{\sigma}_{\mathrm{sing}}$,

$$\begin{aligned}\Big\|\sum_{k_\alpha\in B_{\pi^{-1}(\alpha)}} x_{i_\alpha}(k_\alpha)\Big\|_{pd} &\le C_{pd}\max\Big\{\Big\|\sum_{k_\alpha} x_{i_\alpha}(k_\alpha)^* x_{i_\alpha}(k_\alpha)\Big\|_{pd/2}^{1/2}, \Big\|\sum_{k_\alpha} x_{i_\alpha}(k_\alpha)x_{i_\alpha}(k_\alpha)^*\Big\|_{pd/2}^{1/2}\Big\}\\ &\le C_{pd}m^{1/2}.\end{aligned}$$

It follows that $\|w_{\underline{i},\underline{l}}(l_{s+u})\|_p \le C_{p,\sigma}m^{s/2+u-1}$ and thus $\Psi \le C_{p,\sigma}m^{s/2+u-1/2}$. We have shown that

$$\text{(3-5)}\quad \frac{1}{m^{d/2}}\Big\|\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1}\big(x_{i_1}(k_1)\cdots x_{i_d}(k_d) - E_{\mathcal{N}_{s+u-1}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p \le \frac{C_{p,\sigma}}{m^{1/2}}.$$

Repeating the argument $u-1$ times by replacing u with $u-1, u-2, \dots, 1$, we find

$$\frac{1}{m^{q-s/2}}\Big\|\sum_{l_q\in B_q}\cdots\sum_{l_1\in B_1}\big(E_{\mathcal{N}_q(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]-E_{\mathcal{N}_{q-1}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p \le \frac{C_{p,\sigma}}{m^{1/2}},$$

for $q = s+u-1, \dots, s+1$. In this iteration argument, we use the same "attaching" procedure as described above in order to apply Proposition 2.1. By the triangle inequality, we have

$$\frac{1}{m^{d/2}}\Big\| \sum_{l_{s+u}\in B_{s+u}} \cdots \sum_{l_1\in B_1} \big(x_{i_1}(k_1)\cdots x_{i_d}(k_d) - E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p \le \frac{C_{p,\sigma}}{m^{1/2}}.$$

Hence, by Jensen's inequality,

$$\begin{aligned}&\Big\|\frac{1}{m^{d/2}}\sum_{\underline{k}:\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d) - \frac{1}{m^{d/2}}\sum_{\underline{k}:\sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Big\|_{L_p(\mathcal{A}_m,\tau_m)}\\ &\le \frac{(s+u)^{s+u}}{m^{d/2}}\mathbb{E}_\delta \Big\| \sum_{l_{s+u}\in B_{s+u}} \cdots \sum_{l_1\in B_1} \big(x_{i_1}(k_1)\cdots x_{i_d}(k_d) - E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p\\ &\le \frac{C_{p,\sigma}}{m^{1/2}}.\end{aligned}$$

In the last inequality, the upper bound holds for every instance of δ and thus holds for the average. The proof is complete by sending $m \to \infty$. □

Lemma 3.7. *Let $\sigma \in P_r(d)$. Then, for all $p < \infty$ and fixed $\omega \in \Omega$,*

$$\lim_{m\to\infty}\Big\|\frac{1}{m^{d/2}}\sum_{\underline{k}:\sigma(\underline{k})=\sigma} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\Big\|_{L_p(\mathcal{A}_m,\tau_m)} = 0.$$

Proof. We follow the same argument as for Lemma 3.6 and only indicate the differences. For $\sigma \in P_r(d)$, there is at lease one block with more than two elements. Without loss of generality, assume there is only one block in σ with more than two elements. Suppose this block has, say, three elements. We list the running indices $\underline{k}$ in the sum as $\{l_1, \dots, l_s, l_{s+1}, \dots, l_{s+u}, l_{s+u+1}\}$, where there are s singletons, u pairs and one block with three elements in σ. Using the random selectors, it suffices to show that

$$\frac{1}{m^{d/2}}\Big\| \sum_{l_{s+u+1}\in B_{s+u+1}} \cdots \sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\Big\|_p \to 0$$

as $m \to \infty$, where $B_1, \dots, B_{s+u+1}$ are disjoint random sets with union $[m]$. Denote by $\mathcal{N}_{s+u}(\underline{k})$ the von Neumann algebra generated by $x_{i_{\pi(\alpha)}}(l'_\alpha)$ for all $\alpha \le s+2u$, where $\underline{l}'$ is a permutation of $\underline{k}$ so that $l_1 = l'_1, \dots, l_s = l'_s, l_{s+1} = l'_{s+1} = l'_{s+2}$, etc. Then using the noncommutative Burkholder–Gundy inequality, we can show that

$$\begin{aligned}&\frac{1}{m^{d/2}}\Big\| \sum_{l_{s+u+1}\in B_{s+u+1}} \cdots \sum_{l_1\in B_1} \big(x_{i_1}(k_1)\cdots x_{i_d}(k_d) - E_{\mathcal{N}_{s+u}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\big)\Big\|_p\\ &\qquad\le \frac{C_{p,\sigma}m^{s/2+u+1/2}}{m^{s/2+u+3/2}} \to 0\end{aligned}$$

as $m \to \infty$. It remains to show

$$\frac{1}{m^{d/2}}\Big\|\Big(\sum_{l_{s+u+1}\in B_{s+u+1}}\cdots\sum_{l_1\in B_1} E_{\mathcal{N}_{s+u}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Big)\Big\|_p \to 0.$$

Note that

$$\begin{aligned}\frac{1}{m^{d/2}}&\Big\|\Big(\sum_{l_{s+u+1}\in B_{s+u+1}}\cdots\sum_{l_1\in B_1} E_{\mathcal{N}_{s+u}(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Big)\Big\|_p\\ &\le \frac{1}{m}\sum_{l_{s+u+1}\in B_{s+u+1}}\frac{1}{m^{(d-2)/2}}\Big\|\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\Big\|_p.\end{aligned}$$

Now apply Proposition 2.1 with the same "attaching" procedure as above, yielding

$$\Big\|\sum_{l_{s+u}\in B_{s+u}}\cdots\sum_{l_1\in B_1} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\Big\|_p \le C_{p,\sigma} m^{s/2+u},$$

which gives a decay factor and completes the proof. □

Theorem 3.8. *Let* $(\widetilde{x}_j(m))^\bullet \in \bigcap_{p<\infty} L_p\big(\prod_{m,\mathcal{U}} L_\infty(\Omega;\mathcal{A}_m)\big)$ *for* $j = 1,\dots,d$. *Then*

$$(\widetilde{x}_{i_1}(m))^\bullet\cdots(\widetilde{x}_{i_d}(m))^\bullet = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\le\sigma(\underline{i})}} w_\sigma(\underline{i}),$$

where the equality holds for all $\omega\in\Omega$ *and*

$$w_\sigma(\underline{i}) = \Big(\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Big)^\bullet. \tag{3-6}$$

Proof. By Lemmas 3.6 and 3.7, we have

$$(\widetilde{x}_{i_1}(m))^\bullet\cdots(\widetilde{x}_{i_d}(m))^\bullet = \sum_{\sigma\in P_{1,2}(d)}\Big(\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]\Big)^\bullet.$$

By Proposition 3.5, we write

$$(j_1,\dots,j_d) = (i_{\pi(1)},\dots,i_{\pi(d)}) \quad\text{and}\quad (l'_1,\dots,l'_d) = (k_{\pi(1)},\dots,k_{\pi(d)}).$$

It follows that

$$E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)] = \varepsilon(\underline{i},\underline{k})x_{j_1}(l'_1)\cdots x_{j_s}(l'_s)\delta_{j_{s+1},j_{s+2}}\cdots\delta_{j_{d-1},j_d}.$$

Note that $E_{\mathcal{N}_s(\underline{k})}[x_{i_1}(k_1)\cdots x_{i_d}(k_d)]$ is nonzero only if $j_{s+1}=j_{s+2},\dots,j_{d-1}=j_d$. Since $\sigma(\underline{k})=\sigma$, we have $\sigma\le\sigma(\underline{i})$. □

If $\sigma\le\sigma(\underline{i})$, we call the $w_\sigma(\underline{i})$ defined in (3-6) the arbitrary Wick words. By Theorem 3.8,

$$L_2(\Gamma_Q) \subset L_2\text{-}\operatorname{span}\{w_\sigma(\underline{i}) : \underline{i}\in[N]^d, \sigma\in P_{1,2}(d), d\in\mathbb{N}\}.$$

Here and in what follows, L_p- span W means the $L_p(\tau_{\mathcal{U}})$ closure of linear combinations of elements in W. We want to identify $L_2(\Gamma_Q)$ with the span of fewer Wick words. Let $\underline{i} \in [N]^s$ for $s \in \mathbb{N}$. We define the special Wick words

$$w(\underline{i}) = \Big(\frac{1}{m^{s/2}} \sum_{\underline{k}\in[m]^s:\sigma(\underline{k})\in P_1(s)} x_{i_1}(k_1)\cdots x_{i_s}(k_s)\Big)^{\bullet}. \tag{3-7}$$

Let $\underline{i'} \in [N]^{s'}$. In order to understand the inner product of $w(\underline{i})$ and $w(\underline{i'})$, we first introduce some notions. Let $\{2\cdot 1, 2\cdot 2, \dots, 2\cdot s\}$ be a multiset, each element with multiplicity 2. One can regard it as a set of cardinality $2s$ given by $[2s] = \{1, 2, \dots, s, \widetilde{1}, \widetilde{2}, \dots, \widetilde{s}\}$. Let σ^b be a partition of the set $[2s]$. We call it a bipartite pair partition of $[2s]$ if

$$\sigma^b = \big\{\{e_k, z_k\} : e_k = 1, 2, \dots, s, z_k = \widetilde{1}, \widetilde{2}, \dots, \widetilde{s}\big\}.$$

Let $P_2^b(2s)$ denote the set of all bipartite pair partitions. Let $\underline{i}, \underline{i'} \in [m]^s$, where $\underline{i'}$ is understood as a map $\underline{i'} : \{\widetilde{1}, \widetilde{2}, \dots, \widetilde{s}\} \to [m]$. Define the concatenation operation by

$$\underline{i} \sqcup \underline{i'} = (i_1, \dots, i_s, i'_{\widetilde{1}}, \dots, i'_{\widetilde{s}}). \tag{3-8}$$

We denote by $\sigma(\underline{i} \sqcup \underline{i'})$ the partition induced by $\underline{i}$ and $\underline{i'}$ on the multiset $[2s]$. For example, $\{k, l, \widetilde{k}\}$ are in the same block of $\sigma(\underline{i}\sqcup\underline{i'})$ if $i_k = i_l = i'_{\widetilde{k}}$. Given $\sigma^b \in P_2^b(2s)$, define the set of bipartite crossings by

$$I^b(\sigma^b) = \big\{\{k, l\} : 1 \le k, l \le s, e_k < e_l, z_l > z_k\big\}.$$

Recall that $\langle w(\underline{i}), w(\underline{i'})\rangle = \tau_{\mathcal{U}}[w(\underline{i'})^* w(\underline{i})]$.

Proposition 3.9. *Let $w(\underline{i})$ and $w(\underline{i'})$ be special Wick words. Then there exists a full probability set $\Omega_0 \subset \Omega$ such that for all $\omega \in \Omega_0$,*

$$\langle w(\underline{i}), w(\underline{i'})\rangle = \begin{cases} \displaystyle\sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i'})}} \prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r), i(e_t)) & \text{if } \{i_1, \dots, i_s\} = \{i'_1, \dots, i'_{s'}\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\{i_1, \dots, i_s\} = \{i'_1, \dots, i'_{s'}\}$ means that $\underline{i}$ and $\underline{i'}$ are equal as multisets, i.e., both the elements and their multiplicities are the same.

Proof. We follow the same argument as for Theorem 3.1. By definition,

$$\langle w(\underline{i}), w(\underline{i'})\rangle = \lim_{m,\mathcal{U}} \frac{1}{m^{(s+s')/2}} \sum_{\substack{\underline{k},\underline{k'}:\sigma(\underline{k})\in P_1(s)\\ \sigma(\underline{k'})\in P_1(s')}} \tau_m[x_{i'_{s'}}(k'_{s'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)].$$

Since all the k_α's are pairwise different, $\tau_m[x_{i'_{s'}}(k'_{s'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)] = 0$ unless $s = s'$. Moreover, every $x_{i_\alpha}(k_\alpha)$ has to be the same as exactly one $x_{i'_\beta}(k'_\beta)$

to contribute to the sum. This implies $\underline{i}$ and $\underline{i}'$ are equal as multisets. We rewrite

$$\sum_{\substack{\underline{k},\underline{k}':\sigma(\underline{k})\in P_1(s)\\ \sigma(\underline{k}')\in P_1(s')}} \tau_m[x_{i'_{s'}}(k'_{s'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$= \sum_{\sigma(\underline{k}),\sigma(\underline{k}')\in P_1(s)} \tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$= \sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i}')}} \sum_{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b} \tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)].$$

If $\{r,t\}\in I^b(\sigma^b)$, then we have to switch $x_{i(e_r)}(k(e_r))$ and $x_{i(e_t)}(k(e_t))$ to cancel the corresponding $x_{i(z_r)}(k(z_r))$ and $x_{i(z_t)}(k(z_t))$ terms. It follows that

$$\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)] = \prod_{\{r,t\}\in I^b(\sigma^b)} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]).$$

Since $\underline{k}\in P_1(s)$, by independence, we have

$$\frac{1}{m^s}\sum_{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b} \mathbb{E}\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$= \frac{m(m-1)\cdots(m-s+1)}{m^s}\prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r),i(e_t)).$$

Hence, if $\underline{i}=\underline{i}'$ as multisets, then

$$\mathbb{E}\langle w(\underline{i}),w(\underline{i}')\rangle = \sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i}')}} \prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r),i(e_t)).$$

To show almost sure convergence, let

$$X_m = \frac{1}{m^s}\sum_{\substack{\sigma^b\in P_2^b(2s)\\ \sigma^b\le\sigma(\underline{i}\sqcup\underline{i}')}} \sum_{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b} \tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)].$$

Since $\mathbb{P}(\omega: |X_m-\mathbb{E}X_m|>\eta)\le \mathrm{Var}(X_m)/\eta^2$, by the Borel–Cantelli lemma, it suffices to show that $\sum_{m=1}^\infty \mathrm{Var}(X_m)<\infty$. But

$$\mathrm{Var}(X_m) = \frac{1}{m^{2s}}\sum_{\sigma^b,\pi^b\in P_2^b(2s)} \sum_{\substack{\sigma(\underline{k}\sqcup\underline{k}')=\sigma^b\\ \sigma(\underline{\ell}\sqcup\underline{\ell}')=\pi^b}} V_{\underline{k},\underline{\ell}},$$

where

$$V_{\underline{k},\underline{\ell}} = \mathbb{E}(\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)]$$
$$\times\, \tau_m[x_{i'_s}(\ell'_s)\cdots x_{i'_1}(\ell'_1)x_{i_1}(\ell_1)\cdots x_{i_s}(\ell_s)])$$
$$-\, \mathbb{E}(\tau_m[x_{i'_s}(k'_s)\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_s}(k_s)])$$
$$\times\, \mathbb{E}(\tau_m[x_{i'_s}(\ell'_s)\cdots x_{i'_1}(\ell'_1)x_{i_1}(\ell_1)\cdots x_{i_s}(\ell_s)])$$

$$= \mathbb{E}\Bigg[\prod_{\{r,t\}\in I^b(\sigma^b)} \varepsilon([i(e_r), k(e_r)], [i(e_t), k(e_t)]) \times \prod_{\{r',t'\}\in I^b(\pi^b)} \varepsilon([i(e_{r'}), \ell(e_{r'})], [i(e_{t'}), \ell(e_{t'})]) \Bigg] - \prod_{\{r,t\}\in I^b(\sigma^b)} q(i(e_r), i(e_t)) \prod_{\{r',t'\}\in I^b(\pi^b)} q(i(e_{r'}), i(e_{t'})).$$

By independence, $V_{\underline{k},\underline{\ell}}$ is nonzero only if there are two pairs $\{r, t\} \in I^b(\sigma^b)$ and $\{r', t'\} \in I^b(\pi^b)$ such that $\{k(e_r), k(e_t)\} = \{\ell(e_{r'}), \ell(e_{t'})\}$. In this case,

$$\#\big\{\underline{k}, \underline{k}', \underline{\ell}, \underline{\ell}' : \sigma(\underline{k} \sqcup \underline{k}') = \sigma^b \text{ and } \sigma(\underline{\ell} \sqcup \underline{\ell}') = \pi^b\big\} \le m^s m^{s-2} = m^{2s-2}.$$

Since $V_{\underline{k},\underline{\ell}}$ is uniformly bounded and $C(s) := [\#P_2^b(2s)]^2$ is independent from m,

$$\sum_{m=1}^{\infty} \operatorname{Var}(X_m) \le \sum_{m=1}^{\infty} \frac{C(s)}{m^2} < \infty. \qquad \square$$

Recall the notation $I_p(\sigma)$ and $I_{\text{sp}}(\sigma)$ from Section 2A. For $\underline{i} \in [N]^d$ and $\sigma \in P_{1,2}(d)$ with $\sigma \le \sigma(\underline{i})$, put

$$f_\sigma(\underline{i}) = \prod_{\{r,t\}\in I_p(\sigma)} q(i(e_r), i(e_t)) \prod_{\{r,t\}\in I_{\text{sp}}(\sigma)} q(i(e_r), i(e_t)), \tag{3-9}$$

with the convention that the product over an empty index set is 1.

Proposition 3.10. *Let $\sigma \in P_{1,2}(d)$ and $\sigma' \in P_{1,2}(d')$ be partitions. Let $w_\sigma(\underline{i})$ and $w_{\sigma'}(\underline{i}')$ be arbitrary Wick words as defined in* (3-6). *Then, for almost all $\omega \in \Omega$,*

$$\langle w_\sigma(\underline{i}), w_{\sigma'}(\underline{i}')\rangle = \begin{cases} \langle f_\sigma(\underline{i}) w(\underline{i}_{\text{np}}), f_{\sigma'}(\underline{i}') w(\underline{i}'_{\text{np}})\rangle & \text{if } \sigma \le \sigma(\underline{i}) \text{ and } \sigma' \le \sigma(\underline{i}'), \\ 0 & \text{otherwise}. \end{cases} \tag{3-10}$$

Here, $\underline{i}_{\text{np}}$ is the vector obtained by removing coordinates in $\underline{i}$ which correspond to the pair blocks of σ.

Example 3.11. Suppose that $\underline{i} = (2, 4, 7, 4, 7)$ and $\sigma = \{\{1\}, \{2\}, \{4\}, \{3, 5\}\}$. Then $\underline{i}_{\text{np}} = (2, 4, 4)$.

Proof of Proposition 3.10. By definition,

$$\langle w_\sigma(\underline{i}), w_{\sigma'}(\underline{i}')\rangle = \lim_{m,\mathcal{U}} \frac{1}{m^{(d+d')/2}} \sum_{\substack{\underline{k},\underline{k}':\sigma(\underline{k})=\sigma \\ \sigma(\underline{k}')=\sigma'}} \tau_m[x_{i'_{d'}}(k'_{d'}) \cdots x_{i'_1}(k'_1) x_{i_1}(k_1) \cdots x_{i_d}(k_d)].$$

Note that $w_\sigma(\underline{i})$ is nonzero only if $\sigma \le \sigma(\underline{i})$. Then $k_\alpha = k_\beta$ implies $i_\alpha = i_\beta$. By (A-2), we may assume that, in $x_{i'_{d'}}(k'_{d'}) \cdots x_{i'_1}(k'_1) x_{i_1}(k_1) \cdots x_{i_d}(k_d)$, if $k_\alpha = k_\beta$ for $\alpha \ne \beta$, then $k'_\gamma \ne k_\alpha$ for all $\gamma \in [d']$. In other words, $k_\alpha \ne k'_\gamma$ for all $\alpha \in [d]$ and $\gamma \in [d']$ if both of them belong to pair blocks. Applying Proposition 3.5 to $x_{i_1}(k_1)\cdots x_{i_d}(k_d)$ and $x_{i'_1}(k'_1)\cdots x_{i'_{d'}}(k'_{d'})$, we find

$$\tau_m[x_{i'_{d'}}(k'_{d'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_d}(k_d)] \\ = \varepsilon(\underline{i},\underline{k})\varepsilon(\underline{i}',\underline{k}')\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)],$$

where $\underline{\ell}\in P_1(s)$, $\underline{\ell}'\in P_1(s')$, $\underline{j}=\underline{i}_{\rm np}$, $\underline{j}'=\underline{i}'_{\rm np}$, and $\varepsilon(\underline{i},\underline{k})\varepsilon(\underline{i}',\underline{k}')$ is given in Proposition 3.5. By independence, we have

$$\mathbb{E}\tau_m[x_{i'_{d'}}(k'_{d'})\cdots x_{i'_1}(k'_1)x_{i_1}(k_1)\cdots x_{i_d}(k_d)] \\ = \prod_{\{r,t\}\in I_p(\sigma)} q(i(e_r),i(e_t)) \prod_{\{r,t\}\in I_{\rm sp}(\sigma)} q(i(e_r),i(e_t)) \prod_{\{r',t'\}\in I_p(\sigma')} q(i(e_{r'}),i(e_{t'})) \\ \times \prod_{\{r',t'\}\in I_{\rm sp}(\sigma')} q(i(e_{r'}),i(e_{t'}))\mathbb{E}\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)].$$

As shown in Proposition 3.9, $\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)]$ is zero if $\underline{\ell}$ and $\underline{\ell}'$ are not equal as multisets, and there is nothing more to prove. Assume $\underline{\ell}$ and $\underline{\ell}'$ are equal. Let u and u' be the number of pair blocks in σ and σ'. By Proposition 3.9, we find

$$\mathbb{E}\langle w_\sigma(\underline{i}), w_{\sigma'}(\underline{i}')\rangle = \lim_{m,\mathcal{U}} \frac{m\cdots(m-s+1)}{m^s}\cdot\frac{(m-s)\cdots(m-s-u-u'+1)}{m^{u+u'}} \\ \times f_\sigma(\underline{i})f_{\sigma'}(\underline{i}')\mathbb{E}\tau_m[x_{j'_{s'}}(\ell'_{s'})\cdots x_{j'_1}(\ell'_1)x_{j_1}(\ell_1)\cdots x_{j_s}(\ell_s)] \\ = f_\sigma(\underline{i})f_{\sigma'}(\underline{i}')\mathbb{E}\langle w(\underline{i}_{\rm np}), w(\underline{i}'_{\rm np})\rangle.$$

The almost sure convergence follows from the same argument as for Proposition 3.9 using the Borel–Cantelli lemma and independence. □

In the two proofs above, the Borel–Cantelli lemma may be avoided if we use the average model Γ_Q^a; see Section 3A. Note that for $\underline{i}\in[m]^s$, $w_\sigma(\underline{i})=w(\underline{i})$ for any $\sigma\in P_1(s)$.

Corollary 3.12. *Let $\sigma\le\sigma(\underline{i})$. We have $w_\sigma(\underline{i})=f_\sigma(\underline{i})w(\underline{i}_{\rm np})$ for almost all $\omega\in\Omega$.*

Proof. Since $\tau_{\mathcal{U}}$ is faithful on Γ_Q, it suffices to show

$$\tau_{\mathcal{U}}\big((w_\sigma(\underline{i})-f_\sigma(\underline{i})w(\underline{i}_{\rm np}))^*(w_\sigma(\underline{i})-f_\sigma(\underline{i})w(\underline{i}_{\rm np}))\big)=0.$$

But, by Proposition 3.10, we have

$$\langle w(\underline{i}_{\rm np}), w_\sigma(\underline{i})\rangle = f_\sigma(\underline{i})\langle w(\underline{i}_{\rm np}), w(\underline{i}_{\rm np})\rangle.$$

From here the claim follows by linearity. □

This result yields the identification

$$L_2\text{-}\operatorname{span}\{w_\sigma(\underline{i}):\underline{i}\in[N]^d,\sigma\in P_{1,2}(d),d\in\mathbb{Z}_+\}=L_2\text{-}\operatorname{span}\{w(\underline{i}):\underline{i}\in[N]^s,s\in\mathbb{Z}_+\},$$

with the inner product relation given by (3-10).

Proposition 3.13. $L_2(\Gamma_Q) = L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [N]^s, s \in \mathbb{Z}_+\}$.

Proof. Write $H_w := L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [N]^s, s \in \mathbb{Z}_+\}$. By Theorem 3.8 and Corollary 3.12, $L_2(\Gamma_Q) \subset H_w$. It remains to show that $H_w \subset L_2(\Gamma_Q)$. We proceed by induction on the length s of special Wick words $w(\underline{i})$. First observe that if $\sigma(\underline{i}) \in P_1(s)$, then the only partition $\sigma \le \sigma(\underline{i})$ is $\sigma(\underline{i})$ itself. In this case, by Theorem 3.8, we have

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_s}(m))^\bullet = w_{\sigma(\underline{i})}(\underline{i}) = w(\underline{i}) \in L_2(\Gamma_Q), \tag{3-11}$$

since every $(\widetilde{x}_{i_1}(m))^\bullet$ is in $\bigcap_{p<\infty} L_p(\Gamma_Q)$. If $s = 1$,

$$w(\underline{i}) = \left(\frac{1}{\sqrt{m}} \sum_{k_1=1}^{m} x_{i_1}(k_1)\right)^\bullet \in L_2(\Gamma_Q)$$

by definition. If $s = 2$ and $i_1 \ne i_2$, then $w(\underline{i}) \in L_2(\Gamma_Q)$ by (3-11). If $i_1 = i_2$, using Theorem 3.8, we find

$$(\widetilde{x}_{i_1}(m))^\bullet (\widetilde{x}_{i_2}(m))^\bullet = w_{\sigma(\underline{i})}(\underline{i}) + w_{\sigma_0}(\underline{i}) = 1 + w_{\sigma_0}(\underline{i}),$$

where $\sigma_0 \in P_1(2)$. It follows that $w(\underline{i}) = w_{\sigma_0}(\underline{i}) \in L_2(\Gamma_Q)$. Suppose $w(\underline{i}) \in L_2(\Gamma_Q)$ for all $\underline{i}$ with $|\underline{i}| < s$. Consider $\underline{i} \in [N]^s$. We know $w(\underline{i}) \in L_2(\Gamma_Q)$ if $\sigma(\underline{i}) \in P_1(s)$. If $\sigma(\underline{i}) \notin P_1(s)$, by Theorem 3.8, we have

$$(\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_s}(m))^\bullet = w_{\sigma_0}(\underline{i}) + \sum_{\substack{\sigma \in P_{1,2}(s) \\ \sigma \le \sigma(\underline{i}), \sigma \notin P_1(s)}} w_\sigma(\underline{i}), \tag{3-12}$$

where $\sigma_0 \in P_1(s)$. By Corollary 3.12, we have $w_\sigma(\underline{i}) = f_\sigma(\underline{i}) w(\underline{i}_{\text{np}})$, and $\underline{i}_{\text{np}}$ is a vector of dimension at most $s - 2$. By the induction hypothesis, $w_\sigma(\underline{i}) \in L_2(\Gamma_Q)$ for $\sigma \notin P_1(s)$. We deduce from (3-12) that $w(\underline{i}) = w_{\sigma_0}(\underline{i}) \in L_2(\Gamma_Q)$. □

3C. *Fock spaces and mixed q-commutation relations.* From the work in the previous section, we can describe the Fock space and creation/annihilation operators associated to Γ_Q. Given a vector $\underline{i}$, we denote by $|\underline{i}|$ the number of nonzero coordinates in $\underline{i}$. Let $H_Q^s = \operatorname{span}\{w(\underline{i}) : |\underline{i}| = s\}$. We define the mixed Fock space by

$$\mathcal{F}_Q = \bigoplus_{s=0}^{\infty} H_Q^s. \tag{3-13}$$

Clearly, $\mathcal{F}_Q = L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [N]^s, s \in \mathbb{Z}_+\}$, which can be further identified with $L_2(\Gamma_Q)$ by Proposition 3.13.

Proposition 3.14. *Let* $x_j = \left(\frac{1}{\sqrt{m}} \sum_{k=1}^m x_j(k)\right)^\bullet \in \Gamma_Q^\infty$ *for* $j = 1, \dots, N$ *be generators of* Γ_Q *and* $w(\underline{i}) \in H_Q^s$. *Then*

$$x_j w(\underline{i}) = w(j \sqcup \underline{i}) + \sum_{l=1}^{s} \delta_{j,i_l} w(\underline{i} - i_l) \prod_{r=1}^{l-1} q(i_r, i_l).$$

Here, $j \sqcup \underline{i} = (j, i_1, \dots, i_s) \in [N]^{s+1}$ *is the concatenation operation defined in* (3-8), $\underline{i} - i_l = (i_1, \dots, i_{l-1}, i_{l+1}, \dots, i_s)$ *with* $|\underline{i} - i_l| = s - 1$, *and we understand the product over empty index set to be* 1. *Therefore*,

$$x_j = \sum_{s=0}^{\infty} P_{s+1} x_j P_s + \sum_{s=1}^{\infty} P_{s-1} x_j P_s,$$

where $P_s : \mathcal{F}_Q \to H_Q^s$ *is the orthogonal projection.*

Proof. By definition,

$$\begin{aligned} x_j w(\underline{i}) &= \biggl(\frac{1}{\sqrt{m}} \sum_{k_0=1}^{m} x_j(k_0) \frac{1}{m^{s/2}} \sum_{\underline{k} \in [m]^s : \sigma(\underline{k}) \in P_1(s)} x_{i_1}(k_1) \cdots x_{i_s}(k_s) \biggr)^{\bullet} \\ &= \biggl(\frac{1}{m^{(s+1)/2}} \sum_{k_0 \sqcup \underline{k} \in [m]^{s+1} : \sigma(k_0 \sqcup \underline{k}) \in P_1(s+1)} x_j(k_0) x_{i_1}(k_1) \cdots x_{i_s}(k_s) \biggr)^{\bullet} \\ &\qquad + \sum_{l=1}^{s} \biggl(\frac{1}{m^{(s+1)/2}} \sum_{\substack{k_0 \sqcup \underline{k} \in [m]^{s+1} : k_0 = k_l \\ \sigma(\underline{k}) \in P_1(s)}} x_j(k_0) x_{i_1}(k_1) \cdots x_{i_s}(k_s) \biggr)^{\bullet}. \end{aligned}$$

The first term in the above equation is clearly the special Wick word $w(j \sqcup \underline{i})$. To understand the second one, we define $\sigma_l \in P_{1,2}(s+1)$ by $\sigma_l = \sigma(k_0 \sqcup \underline{k})$ for $k_0 = k_l$ and $\underline{k} \in P_1(s)$, i.e.,

$$\sigma_l = \bigl\{ \{1, l+1\}, \{2\}, \dots, \{l\}, \{l+2\}, \dots, \{s+1\} \bigr\}.$$

Using (the proof of) Lemma 3.6, we deduce that the arbitrary Wick word satisfies

$$\biggl(\frac{1}{m^{(s+1)/2}} \sum_{\substack{k_0 \sqcup \underline{k} \in [m]^{s+1} : k_0 = k_l \\ \sigma(\underline{k}) \in P_1(s)}} x_j(k_0) x_{i_1}(k_1) \cdots x_{i_s}(k_s) \biggr)^{\bullet} = w_{\sigma_l}(j \sqcup \underline{i}).$$

Note that $w_{\sigma_l}(j \sqcup \underline{i})$ is nonzero only if $\sigma_l \le \sigma(j \sqcup \underline{i})$ or equivalently $j = i_l$. Using (3-9) and Corollary 3.12, we find

$$w_{\sigma_l}(j \sqcup \underline{i}) = \delta_{j, i_l} \prod_{r=1}^{l-1} q(i_r, i_l) w(\underline{i} - i_l). \qquad \square$$

Define operators c_j and a_j acting on $\mathcal{F}_Q$ by

$$c_j w(\underline{i}) = w(j \sqcup \underline{i}), \quad a_j w(\underline{i}) = \sum_{l=1}^{s} \delta_{j, i_l} w(\underline{i} - i_l) \prod_{r=1}^{l-1} q(i_r, i_l). \tag{3-14}$$

Clearly $x_j = c_j + a_j$, $c_j = \sum_{s=0}^{\infty} P_{s+1} x_j P_s$ and $a_j = \sum_{s=1}^{\infty} P_{s-1} x_j P_s$. Since $x_j = x_j^*$, we have $c_j^* = a_j$. We call c_j and a_j the creation and annihilation operators respectively for $j = 1, \dots, N$. The following result is simply a recapitulation; see, e.g., [Brown and Ozawa 2008] for more about QWEP C*-algebras.

Corollary 3.15. *Let $\widetilde{\Gamma}_Q$ be the von Neumann algebra generated by (the spectral projections of) $c_j + a_j$ for $j = 1, \dots, N$. Then $\Gamma_Q = \widetilde{\Gamma}_Q$. In particular, $\widetilde{\Gamma}_Q$ is QWEP.*

Proposition 3.16. *For $j, k = 1, \dots, N$, c_j and c_j^* satisfy the mixed q-commutation relation*

$$c_k^* c_j - q(j,k) c_j c_k^* = \delta_{j,k} 1. \tag{3-15}$$

Proof. Let $w(\underline{i}) \in H_Q^s$. Then

$$c_j^* c_j w(\underline{i}) = c_j^* w(j \sqcup \underline{i}) = w(\underline{i}) + \sum_{l=1}^{s} \delta_{j,i_l} w(j \sqcup (\underline{i} - i_l)) q(j, i_l) \prod_{r=1}^{l-1} q(i_r, i_l).$$

But

$$c_j c_j^* w(\underline{i}) = \sum_{l=1}^{s} \delta_{j,i_l} w(j \sqcup (\underline{i} - i_l)) \prod_{r=1}^{l-1} q(i_r, i_l).$$

Hence $c_j^* c_j w(\underline{i}) - q(j,j) c_j c_j^* w(\underline{i}) = w(\underline{i})$. If $j \neq k$, then

$$c_k^* c_j w(\underline{i}) = c_k^* w(j \sqcup \underline{i}) = \sum_{l=1}^{s} \delta_{k,i_l} w(j \sqcup (\underline{i} - i_l)) q(j, i_l) \prod_{r=1}^{l-1} q(i_r, i_l),$$

and

$$c_j c_k^* w(\underline{i}) = \sum_{l=1}^{s} \delta_{k,i_l} w(j \sqcup (\underline{i} - i_l)) \prod_{r=1}^{l-1} q(i_r, i_l).$$

Hence $c_k^* c_j w(\underline{i}) - q(j,k) c_j c_k^* w(\underline{i}) = 0$. □

Remark 3.17. The Fock space representation was studied in more general setting by Bożejko and Speicher [1994]. Let (e_i) be an orthonormal basis (o.n.b.) of a Hilbert space H. One can construct the Fock space $\mathcal{F}_Q(H)$ following [Bożejko and Speicher 1994; Lust-Piquard 1999]. Let Ω be the vacuum state and W be the Wick product, i.e.,

$$W(e_{i_1} \otimes \cdots \otimes e_{i_s})\Omega = e_{i_1} \otimes \cdots \otimes e_{i_s}.$$

The Wick product was studied in detail in [Królak 2000]. Suppose $\underline{i} \in [N]^s$ and $\underline{j} \in [N]^{s'}$. We have

$$\langle w(\underline{i}), w(\underline{j}) \rangle = \langle e_{i_1} \otimes \cdots \otimes e_{i_s}, e_{j_1} \otimes \cdots \otimes e_{j_{s'}} \rangle,$$

where the left side is given by Proposition 3.9 and the right side is understood as the inner product in $\mathcal{F}_Q(H)$; see [Bożejko and Speicher 1994; Lust-Piquard 1999]. Our argument shows that one can alternatively implement (3-15) and construct the Fock space using the probabilistic approach (Speicher's CLT) and the von Neumann algebra ultraproduct. If $\sup_{ij} q_{ij} < 1$ and $\underline{i} \in [N]^s$, we can identify our special Wick words $w(\underline{i})$ with $W(e_{i_1}\otimes\cdots\otimes e_{i_s})$. Thus we can also identify H_Q^s with $H^{\otimes s}$. This identification will play an important role when we study the operator algebraic properties of Γ_Q in later parts of this paper.

3D. ***The Ornstein–Uhlenbeck semigroup on*** $\mathbf{\Gamma_Q}$**.** Let T_t^m denote the Ornstein–Uhlenbeck semigroup acting on $\mathcal{A}_m$; see [Biane 1997, Section 2.1]. T_t^m is given by

$$T_t^m x_{i_1}(k_1)\cdots x_{i_d}(k_d) = e^{-td}x_{i_1}(k_1)\cdots x_{i_d}(k_d)$$

if $\sigma(\underline{k})\in P_1(d)$. Let us first recall an elementary fact.

Lemma 3.18. *Let $(\mathcal{N},\tau)$ be noncommutative W^* probability space, where $\mathcal{N}$ is a von Neumann algebra and τ is a normal faithful tracial state. Let $T:\mathcal{N}\to\mathcal{N}$ be a $*$-preserving linear normal map with pre-adjoint map $T_*:\mathcal{N}_*\to\mathcal{N}_*$. Suppose T is self-adjoint on $L_2(\mathcal{N},\tau)$. Then $T=T_*|_{\mathcal{N}}$.*

Proof. Let $x,y\in\mathcal{N}$. Denote the dual pairing between $x_*\in\mathcal{N}_*$ and x by (x_*,x), which can be implemented by $(x_*,x)=\tau(x_*x)$. Since $\mathcal{N}\subset\mathcal{N}_*=L_1(\mathcal{N},\tau)$,

$$(Tx,y)=\tau((Tx)y)=\langle Tx,y^*\rangle_{L_2(\mathcal{N},\tau)}=\tau(x(Ty))=(x,Ty)=(T_*x,y). \qquad \square$$

Let $(T_t^m)_*:L_1(\mathcal{A}_m)\to L_1(\mathcal{A}_m)$ be the pre-adjoint map of T_t. By Lemma 3.18, it coincides with T_t on $\mathcal{A}_m$. Let $\prod_{m,\mathcal{U}}L_1(\mathcal{A}_m)$ be the ultraproduct of Banach spaces $L_1(\mathcal{A}_m)$. Recall that $\mathcal{A}_\mathcal{U}=\prod_{m,\mathcal{U}}\mathcal{A}_m$ is the von Neumann algebra ultraproduct in Section 3A. Note we have the canonical inclusion $L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})\subset\prod_{m,\mathcal{U}}L_1(\mathcal{A}_m,\tau_m)$. Let $((T_t^m)_*)^\bullet$ be the usual ultraproduct of $(T_t^m)_*$. If $(x_m)^\bullet\in\mathcal{A}_\mathcal{U}$, then

$$((T_t^m)_*)^\bullet(x_m)^\bullet=(T_t^m x_m)^\bullet\in\mathcal{A}_\mathcal{U}$$

because $\sup_m\|T_t^m x_m\|\le\sup_m\|x_m\|<\infty$. Hence, $((T_t^m)_*)^\bullet$ leaves $\mathcal{A}_\mathcal{U}$ invariant. We have checked the commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}_\mathcal{U} & \hookrightarrow & L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U}) & \hookrightarrow & \prod_{m,\mathcal{U}}L_1(\mathcal{A}_m,\tau_m)\\
\downarrow{\scriptstyle ((T_t^m)_*)^\bullet|_{\mathcal{A}_\mathcal{U}}} & & \downarrow{\scriptstyle ((T_t^m)_*)^\bullet|_{L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})}} & & \downarrow{\scriptstyle ((T_t^m)_*)^\bullet}\\
\mathcal{A}_\mathcal{U} & \hookrightarrow & L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U}) & \hookrightarrow & \prod_{m,\mathcal{U}}L_1(\mathcal{A}_m,\tau_m)
\end{array}$$

We define $T_t=\big(((T_t^m)_*)^\bullet|_{L_1(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})}\big)^*$. Then, by construction, $T_t:\mathcal{A}_\mathcal{U}\to\mathcal{A}_\mathcal{U}$ is a normal unital completely positive map which is self-adjoint on $L_2(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})$. By Lemma 3.18 again, T_t coincides with $((T_t^m)_*)^\bullet$ on $\mathcal{A}_\mathcal{U}$ and thus on $L_2(\mathcal{A}_\mathcal{U},\tau_\mathcal{U})$. Since $\Gamma_Q\subset\mathcal{A}_\mathcal{U}$ is a von Neumann subalgebra, $L_2(\Gamma_Q)\subset L_2(\mathcal{A}_\mathcal{U})\subset L_1(\mathcal{A}_\mathcal{U})$. Therefore, for $\underline{i}\in[N]^s$ and $w(\underline{i})\in L_2(\Gamma_Q)$,

$$T_t w(\underline{i})=\bigg(\frac{1}{m^{s/2}}\sum_{\underline{k}:\sigma(\underline{k})\in P_1(s)}e^{-ts}x_{i_1}(k_1)\cdots x_{i_s}(k_s)\bigg)^\bullet=e^{-ts}w(\underline{i})\in L_2(\Gamma_Q).$$

Since $L_2(\Gamma_Q) \cap \mathcal{A}_{\mathcal{U}} = \Gamma_Q$, T_t leaves Γ_Q invariant. Also, we see that $(T_t)_{t\geq 0}$ is a strongly continuous semigroup in $L_2(\Gamma_Q)$. Note that in general $(T_t)_{t\geq 0}$ may not be a point σ-weakly continuous semigroup in t, hence may not extend to a strongly continuous semigroup on $L_2(\mathcal{A}_{\mathcal{U}})$. By Theorem 3.8,

$$T_t\big((\widetilde{x}_{i_1}(m))^\bullet \cdots (\widetilde{x}_{i_d}(m))^\bullet\big) = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\leq\sigma(\underline{i})}} e^{-t|\sigma_{\text{sing}}|} w_\sigma(\underline{i}) = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\leq\sigma(\underline{i})}} e^{-t|\underline{i}_{\text{np}}|} f_\sigma(\underline{i}) w(\underline{i}_{\text{np}}),$$

where $|\sigma_{\text{sing}}|$ is the number of singletons in σ, and $|\underline{i}_{\text{np}}|$ is the dimension of $\underline{i}_{\text{np}}$. $f_\sigma(\underline{i})$ and $\underline{i}_{\text{np}}$ are defined in (3-9) and Proposition 3.10. The generator of T_t is the number operator, denoted by A.

4. Analytic properties

Our goal of this section is to prove some analytic properties for Γ_Q. This will be done via a limit procedure, as was used in [Biane 1997; Junge et al. 2015] for proving hypercontractivity.

4A. *Hypercontractivity.* Biane [1997, Theorem 5] proved the Ornstein–Uhlenbeck semigroup acting on $\mathcal{A}_m = \mathcal{A}_m(N, \varepsilon)$ is hypercontractive.

Theorem 4.1. *Let* $1 \leq p, r < \infty$. *Then, for every* $\omega \in \Omega$,

$$\|T_t^m\|_{L_p\to L_r} = 1 \quad \textit{if and only if} \quad e^{-2t} \leq \frac{p-1}{r-1}.$$

With the hard work done in the previous section, it is very easy to prove the following result.

Theorem 4.2. *Let T_t be the Ornstein–Uhlenbeck semigroup on Γ_Q for an arbitrary $N \times N$ symmetric matrix Q with entries in* $[-1, 1]$. *Then, for* $1 \leq p, r < \infty$,

$$\|T_t\|_{L_p\to L_r} = 1 \quad \textit{if and only if} \quad e^{-2t} \leq \frac{p-1}{r-1}.$$

Proof. The "only if" part follows verbatim Biane's argument [1997, p. 461]. For the converse, since the special Wick words span $L_p(\Gamma_Q)$, it suffices to prove that if $e^{-2t} \leq (p-1)/(r-1)$ then

$$\Big\| T_t\Big(\sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i})\Big)\Big\|_r \leq \Big\| \sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i})\Big\|_p,$$

where $\sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i})$ is a finite linear combination of special Wick words.

But, by Theorem 4.1,

$$\Big\|T_t^m\Big(\sum_{\underline{i}}\frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}}\sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))}x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})})\Big)\Big\|_r \le \Big\|\sum_{\underline{i}}\frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}}\sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))}x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})})\Big\|_p.$$

Since there is a canonical inclusion $L_p(\Gamma_Q)\subset\prod_{m,\mathcal{U}}L_p(\mathcal{A},\tau_m)$, we have

$$\Big\|\sum_{\underline{i}}\alpha_{\underline{i}}w(\underline{i})\Big\|_p=\lim_{m,\mathcal{U}}\Big\|\sum_{\underline{i}}\frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}}\sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))}x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})})\Big\|_p.$$

Similarly,

$$\begin{aligned}\Big\|T_t\Big(\sum_{\underline{i}}\alpha_{\underline{i}}w(\underline{i})\Big)\Big\|_r &=\Big\|\sum_{\underline{i}}\alpha_{\underline{i}}e^{-t|\underline{i}|}w(\underline{i})\Big\|_r\\ &=\lim_{m,\mathcal{U}}\Big\|\sum_{\underline{i}}\frac{\alpha_{\underline{i}}e^{-t|\underline{i}|}}{m^{d(\underline{i})/2}}\sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))}x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})})\Big\|_r\\ &=\lim_{m,\mathcal{U}}\Big\|T_t^m\Big(\sum_{\underline{i}}\frac{\alpha_{\underline{i}}}{m^{d(\underline{i})/2}}\sum_{\underline{k}\in[m]^{d(\underline{i})}:\sigma(\underline{k})\in P_1(d(\underline{i}))}x_{i_1}(k_1)\cdots x_{i_{d(\underline{i})}}(k_{d(\underline{i})})\Big)\Big\|_r.\end{aligned}$$

The assertion follows immediately. □

This result in particular implies the hypercontractivity results for $\Gamma_q(H)$ due to Biane [1997] and for the free product of $\Gamma_{-1}(\mathbb{R}^n)$ obtained in [Junge et al. 2015]. See also [Królak 2005] for another generalization with the braid relation. Using the standard argument [Biane 1997], the log-Sobolev inequality follows from optimal hypercontractivity bounds. Recall that A is the number operator associated to Γ_Q.

Corollary 4.3 (log-Sobolev inequality). *For any finite linear combination of special Wick words $f=\sum_{\underline{i}}\alpha_{\underline{i}}w(\underline{i})$,*

$$\tau_Q(|f|^2\ln|f|^2)-\|f\|_2^2\ln\|f\|_2^2\le 2\tau_Q(fAf^*).$$

4B. ***Derivations.*** Given the $N\times N$ matrix $Q=(q_{ij})$, we define a $2N\times 2N$ matrix Q' by

$$Q'=Q\otimes\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}Q&Q\\Q&Q\end{pmatrix}.$$

Recall that $\mathcal{A}_m(N, \varepsilon)$ is the spin matrix system with Nm generators as in Section 2B and $J_{N,m} = [N] \times [m]$. We can extend the function ε to $J_{2N,m} \times J_{2N,m}$ as follows:

$$\varepsilon'((i,k),(j,l)) = \begin{cases} \varepsilon((i,k),(j,l)) & \text{if } (i,k),(j,l) \in J_{N,m}, \\ \varepsilon((i-N,k),(j,l)) & \text{if } 1 \le j < N+1 \le i \le 2N, \\ \varepsilon((i,k),(j-N,l)) & \text{if } 1 \le i < N+1 \le j \le 2N, \\ \varepsilon((i-N,k),(j-N,l)) & \text{if } N+1 \le i, j \le 2N. \end{cases}$$

In other words, $\varepsilon' = \varepsilon \otimes \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. We may write ε for ε' without causing any ambiguity. Now we define a linear map

$$\text{(4-1)} \quad \begin{aligned} &\delta : \mathcal{A}_m(N,\varepsilon) \to \mathcal{A}_m(2N,\varepsilon'), \\ &\quad x_{i_1}(k_1)x_{i_2}(k_2)\cdots x_{i_n}(k_n) \\ &\qquad \mapsto \sum_{\alpha=1}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n), \end{aligned}$$

where $x_{i_1}(k_1)x_{i_2}(k_2)\cdots x_{i_n}(k_n)$ is assumed to be in the reduced form. See also [Lust-Piquard 1999]. It is easy to see that δ is $*$-preserving.

Lemma 4.4. *δ is a derivation, i.e., $\delta(\xi\eta) = \delta(\xi)\eta + \xi\delta(\eta)$ for two words ξ and η.*

Proof. The assertion follows from the fact that δ is the derivative of certain one parameter group of automorphisms; see [Lust-Piquard 1998; 1999; Efraim and Lust-Piquard 2008]. We provide a direct elementary proof here. Note that if ξ and η are two reduced words with no common generators, the derivation property follows easily from (4-1). It remains to verify the derivation property when ξ and η have common generators. Let $\xi = x_{i_1}(k_1)\cdots x_{i_n}(k_n) \in \mathcal{A}_m(N,\varepsilon)$ be a reduced word, and let a be an arbitrary generator. Assume $a = x_{i(\alpha_0)}(k(\alpha_0))$ and write the reduced form of $a\xi$ as $\widetilde{a\xi}$. Then

$$\widetilde{a\xi} = \varepsilon((i_1,k_1),(i_{\alpha_0},k_{\alpha_0}))\cdots\varepsilon((i_{\alpha_0-1},k_{\alpha_0-1}),(i_{\alpha_0},k_{\alpha_0}))x_{i_1}(k_1) \cdots \check{x}_{i_{\alpha_0}}(k_{\alpha_0})\cdots x_{i_n}(k_n),$$

where $\check{x}$ means the generator x is omitted in the expression. We have

$$\delta(\widetilde{a\xi}) = \varepsilon((i_1,k_1),(i_{\alpha_0},k_{\alpha_0}))\cdots\varepsilon((i_{\alpha_0-1},k_{\alpha_0-1}),(i_{\alpha_0},k_{\alpha_0})) \times \sum_{\alpha=1,\alpha\neq\alpha_0}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n).$$

Here we understand that if $\alpha = \alpha_0 - 1$ then $i_{\alpha+1}$ is actually $i_{\alpha+2}$ because $x_{i_{\alpha_0}}(k_{\alpha_0})$ is omitted. Similar remark applies when $\alpha = \alpha_0 + 1$ and we will follow this convention to ease notation in this proof. On the other hand,

$$\begin{aligned}\delta(a)\xi + a\delta(\xi) &= x_{i_{\alpha_0}+N}(k_{\alpha_0})x_{i_1}(k_1)\cdots x_{i_n}(k_n) + a x_{i_1}(k_1)\cdots x_{i_{\alpha_0}+N}(k_{\alpha_0})\cdots x_{i_n}\\ &\quad + \prod_{j=1}^{\alpha_0-1}\varepsilon((i_j,k_j),(i_{\alpha_0},k_{\alpha_0}))\\ &\quad \times \sum_{\alpha=1,\alpha\neq\alpha_0}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n)\\ &= \prod_{j=1}^{\alpha_0-1}\varepsilon((i_j,k_j),(i_{\alpha_0},k_{\alpha_0}))\\ &\quad \times \sum_{\alpha=1,\alpha\neq\alpha_0}^{n} x_{i_1}(k_1)\cdots x_{i_{\alpha-1}}(k_{\alpha-1})x_{i_\alpha+N}(k_\alpha)x_{i_{\alpha+1}}(k_{\alpha+1})\cdots x_{i_n}(k_n).\end{aligned}$$

Here we used the commutation relation given by ε in both equalities. Hence,

$$\delta(\widetilde{a\xi}) = \delta(a)\xi + a\delta(\xi). \tag{4-2}$$

Now assume $\delta(\widetilde{\eta\xi}) = \delta(\eta)\xi + \eta\delta(\xi)$, where both ξ and η are reduced words and the generators of η are all in ξ, i.e., η is a subword of ξ. We want to show that $\delta(\widetilde{a\eta\xi}) = \delta(\widetilde{a\eta})\xi + a\eta\delta(\xi)$, where a is a generator. Note that $\widetilde{a\eta\xi} = \widetilde{a\widetilde{\eta\xi}}$. By (4-2) and the induction hypothesis,

$$\begin{aligned}\delta(\widetilde{a\widetilde{\eta\xi}}) &= \delta(a)\eta\xi + a\delta(\widetilde{\eta\xi})\\ &= \delta(a)\eta\xi + a\delta(\eta)\xi + a\eta\delta(\xi) = \delta(\widetilde{a\eta})\xi + a\eta\delta(\xi).\end{aligned}$$

The derivation property is verified when η is a subword of ξ. For arbitrary reduced words ξ and η, using the commutation relation we can write $\eta = \eta_1\eta_2$ so that η_1 and ξ have no common generators and the generators of η_2 are in ξ. Then

$$\begin{aligned}\delta(\widetilde{\eta\xi}) &= \delta(\eta_1\widetilde{\eta_2\xi}) = \delta(\eta_1)\eta_2\xi + \eta_1\delta(\widetilde{\eta_2\xi}) = \delta(\eta_1)\eta_2\xi + \eta_1\delta(\eta_2)\xi + \eta_1\eta_2\delta(\xi)\\ &= \delta(\eta_1\eta_2)\xi + \eta_1\eta_2\delta(\xi) = \delta(\eta)\xi + \eta\delta(\xi).\end{aligned}$$ □

This lemma implies in particular that $\delta(\xi)$ can be defined by (4-1) and is equal to $\delta(\widetilde{\xi})$ even if ξ is not a reduced word. We will simply write $\delta(\xi)$ for any word ξ in the following. If we denote by A^m the number operator associated to the spin system $\mathcal{A}_m(N,\varepsilon)$, the gradient form is defined as

$$\Gamma^m(f,g) = \tfrac{1}{2}(A^m(f^*)g + f^*A^m(g) - A^m(f^*g))$$

for $f, g \in \mathcal{A}_m(N,\varepsilon)$. The superscript m is used to distinguish the operators from their counterparts defined for the limiting algebra Γ_Q. We may simply omit this superscript if there is no ambiguity.

Lemma 4.5. *Let $f, g \in \mathcal{A}_m(N, \varepsilon)$. Then*

$$\Gamma(f, g) = E(\delta(f)^*\delta(g)),$$

where $E : \mathcal{A}_m(2N, \varepsilon) \to \mathcal{A}_m(N, \varepsilon)$ is the conditional expectation satisfying

$$E(x_B) = \delta_{B\cap(\{N+1,\dots,2N\}\times[m]),\varnothing}\, x_B$$

for a reduced word x_B.

Proof. By linearity, it suffices to check $\Gamma(f, g) = E(\delta(f)^*\delta(g))$ if f and g are reduced words in $\mathcal{A}_m(N, \varepsilon)$. Let $X_B = x_{i_1}(k_1)\cdots x_{i_n}(k_n)$ and $X_C = x_{j_1}(l_1)\cdots x_{j_s}(l_s)$ be two reduced words, where $B, C \subset [N] \times [m]$ consist of (i_α, k_α) and (j_β, l_β) respectively. By the derivation property (4-1),

$$\begin{aligned} &E(\delta(X_B)^*\delta(X_C)) \\ &\quad= \sum_{\alpha=1}^{n}\sum_{\beta=1}^{s} E(x_{i_n}(k_n)\cdots x_{i_\alpha+N}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta+N}(l_\beta)\cdots x_{j_s}(l_s)).\end{aligned}$$

We claim that the only nonzero terms in the above sum are those with $(i_\alpha, k_\alpha) = (j_\beta, l_\beta)$. Indeed, the conditional expectation simply computes the trace of generators with subscript greater than N in the reduced form of

$$x_{i_n}(k_n)\cdots x_{i_\alpha+N}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta+N}(l_\beta)\cdots x_{j_s}(l_s).$$

Thus $x_{i_\alpha+N}(k_\alpha)$ and $x_{j_\beta+N}(l_\beta)$ have to be the same to cancel out in order to contribute to the sum. It follows that

$$\begin{aligned} &E(\delta(X_B)^*\delta(X_C)) \\ &\quad= \sum_{\alpha,\beta:(i_\alpha,k_\alpha)=(j_\beta,l_\beta)} E\big(x_{i_n}(k_n)\cdots x_{i_\alpha+N}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1) \\ &\qquad\qquad\qquad\qquad \cdots x_{j_\beta+N}(l_\beta)\cdots x_{j_s}(l_s)\big) \\ &\quad= \sum_{\alpha,\beta:(i_\alpha,k_\alpha)=(j_\beta,l_\beta)} x_{i_n}(k_n)\cdots x_{i_\alpha}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta}(l_\beta)\cdots x_{j_s}(l_s).\end{aligned}\tag{4-3}$$

Here we used the extended commutation relation on $\mathcal{A}(2N, \varepsilon)$ given by ε in the last equality. Since X_B and X_C are reduced, given $(i_\alpha, k_\alpha) \in B$ there is at most one $(j_\beta, l_\beta) \in C$ such that they are equal, and vice versa. We see that there are $|B \cap C|$ terms in the sum of (4-3). Hence, we find

$$E(\delta(X_B)^*\delta(X_C)) = |B \cap C| X_B^* X_C.$$

On the other hand,

$$\begin{aligned}\Gamma(X_B, X_C) &= \tfrac{1}{2}(A(X_B^*)X_C + X_B^*A(X_C) - A(X_B^*X_C)) \\ &= \tfrac{1}{2}(|B| + |C| - |B\triangle C|) \\ &\qquad\qquad \times x_{i_n}(k_n)\cdots x_{i_\alpha}(k_\alpha)\cdots x_{i_1}(k_1)x_{j_1}(l_1)\cdots x_{j_\beta}(l_\beta)\cdots x_{j_s}(l_s).\end{aligned}$$

Note that we have the same word here as the summand of (4-3). Since $2|B\cap C| = |B|+|C|-|B\triangle C|$, we must have

$$\Gamma(X_B, X_C) = |B\cap C| X_B^* X_C = E(\delta(X_B)^*\delta(X_C)). \qquad \square$$

Let $w(\underline{i}) \in H_Q^s$ be a special Wick word with length $s \in \mathbb{Z}_+$. We define a linear map $\delta : H_Q^s \to H_{Q'}^s$

$$\delta(w(\underline{i})) = \Bigg(\frac{1}{m^{s/2}} \sum_{\underline{k}:\sigma(\underline{k})\in P_1(s)} \delta^m[x_{i_1}(k_1)\cdots x_{i_s}(k_s)]\Bigg)^{\bullet}, \tag{4-4}$$

where δ^m is the derivation defined in (4-1). Here we used Remark A.1 implicitly. Note that δ^m is bounded when acting on words with fixed length s although it is not uniformly (in m) bounded on $\mathcal{A}_m$. Hence $\delta = (\delta^m)^\bullet$ is well-defined on H_Q^s. Since $L_2(\Gamma_Q) = \bigoplus_{s=0}^\infty H_Q^s$, we can define δ on each H_Q^s by (4-4). By definition, δ is densely defined on $\mathcal{F}_Q = L_2(\Gamma_Q)$ and $\mathrm{Dom}(\delta) = \mathrm{Dom}(A)$ can be identified with the linear span of special Wick words with finite length, where A is the number operator on $L_2(\Gamma_Q)$. Since each $w(\underline{i})$ is actually in Γ_Q^∞, $\delta(w(\underline{i}))$ is in $\Gamma_{Q'}^\infty$.

Proposition 4.6. $\delta : L_2(\Gamma_Q) \to L_2(\Gamma_{Q'})$ *is a closed derivation.*

Proof. Let $P_s : L_2(\Gamma_Q) \to H_Q^s$ and $P_s' : L_2(\Gamma_{Q'}) \to H_{Q'}^s$ be the orthogonal projections. Suppose $x_n \in \mathrm{Dom}(\delta)$, $\lim_{n\to\infty} \|x_n\|_2 = 0$ and $\lim_{n\to\infty} \|\delta(x_n) - y\|_2 = 0$. Then $P_s x_n \to 0$ for each $s \in \mathbb{Z}_+$. It follows that

$$P_s'\delta(x_n) = \delta(P_s(x_n)) \to 0 \quad \text{as } n \to \infty.$$

But $P_s'\delta(x_n) \to P_s' y$, we find $P_s' y = 0$ and thus $y = 0$. Hence δ is closed. The derivation property follows from the definition (4-4), (4-1) and Remark A.1. $\square$

Denote by $\mathcal{A}_{\mathcal{U}}(N)$ the von Neumann algebra ultraproduct of $\mathcal{A}_m(N)$. Then $E = (E^m)^\bullet : \mathcal{A}_{\mathcal{U}}(2N) \to \mathcal{A}_{\mathcal{U}}(N)$ is the canonical conditional expectation, where $E^m : \mathcal{A}_m(2N) \to \mathcal{A}_m(N)$ is given in Lemma 4.5. Since $\Gamma_Q \subset \mathcal{A}_{\mathcal{U}}(N)$ as a von Neumann subalgebra, there is a trace-preserving conditional expectation $E : \Gamma_{Q'} \to \Gamma_Q$ which extends to contractions on L_p for $1 \le p < \infty$. Recall that $\Gamma(\cdot,\cdot)$ is the gradient form associated with the number operator A on Γ_Q.

Proposition 4.7. *Let* $f, g \in \mathrm{Dom}(\delta)$. *Then*

$$\Gamma(f, g) = E(\delta(f)^*\delta(g)).$$

Proof. By linearity, it suffices to check the claim for $f = w(\underline{i})$ and $g = w(\underline{i}')$. By the construction of conditional expectation, the proof of Lemma 4.5 and the construction of the Ornstein–Uhlenbeck semigroup on $L_2(\Gamma_Q)$ in Section 3D,

$$\begin{aligned}
&E[\delta(w(\underline{i}))^*\delta(w(\underline{i}'))]\\
&=\Bigg(\frac{1}{m^{(d+d')/2}}E^m\Bigg(\delta^m\Bigg(\sum_{\underline{k}:\sigma(\underline{k})\in P_1(d)} x_{i_1}(k_1)\cdots x_{i_d}(k_d)\Bigg)^*\\
&\qquad\qquad\qquad\qquad\times\delta^m\Bigg(\sum_{\underline{k}':\sigma(\underline{k}')\in P_1(d')} x_{i'_1}(k'_1)\cdots x_{i'_{d'}}(k'_{d'})\Bigg)\Bigg)\Bigg)^{\bullet}\\
&=\tfrac{1}{2}[A(w(\underline{i}^*)w(\underline{i}')+w(\underline{i})^*A(w(\underline{i}'))-A(w(\underline{i})^*w(\underline{i}'))]=\Gamma(w(\underline{i}),w(\underline{i}')),
\end{aligned}$$

where A is the number operator on Γ_Q. □

4C. *Riesz transforms.* Lust-Piquard [1998] showed the boundedness of Riesz transforms for the general spin system. Let $T\in\mathcal{A}_m(N,\varepsilon)$ with $\tau_m(T)=0$. Recall that the Riesz transforms satisfy $R_j(T)=D_j(A^m)^{-1/2}(T)$, where D_j is the annihilation operator and $A^m=\sum_{j=1}^{Nm}D_j^*D_j$ is the number operator for the spin system $\mathcal{A}_m(N,\varepsilon)$. By [Lust-Piquard 1998, Lemma 3.2 and Proposition 1.3], we have

$$\widetilde{K}_{p'}^{-1}\|T\|_p\le\Bigg\|\sum_{j=1}^{Nm}P_jR_j(T)\Bigg\|_p\le\widetilde{K}_p\|T\|_p\quad\text{for } 1<p<\infty,\tag{4-5}$$

where $\widetilde{K}_p=O(p^3/(p-1)^{3/2})$, $1/p+1/p'=1$, and P_j is a certain tensor of Pauli matrices in the general spin system; see [Lust-Piquard 1998, Definition 2.1]. It is known that $\|\sum_{j=1}^{Nm}P_jR_j(T)\|_p=\|\delta^m(A^m)^{-1/2}(T)\|_p$ (see [Lust-Piquard 1999, p. 547]), where δ^m is the derivation defined in (4-1). By considering $T=(A^m)^{1/2}f$, (4-5) can be rewritten as

$$\widetilde{K}_{p'}^{-1}\|(A^m)^{1/2}f\|_p\le\|\delta^m(f)\|_p\le\widetilde{K}_p\|(A^m)^{1/2}f\|_p.\tag{4-6}$$

Now it is easy to recover the main result in [Lust-Piquard 1999]. Recall that A is the number operator on Γ_Q.

Theorem 4.8 (Lust-Piquard). *Let* $1<p<\infty$ *and* $1/p+1/p'=1$. *Let* δ *be defined by* (4-4). *Then, for any* $f\in\mathrm{Dom}(\delta)$,

$$\widetilde{K}_{p'}^{-1}\|A^{1/2}f\|_p\le\|\delta(f)\|_p\le\widetilde{K}_p\|A^{1/2}f\|_p,$$

where $\widetilde{K}_p=O(p^3/(p-1)^{3/2})$.

Proof. We may assume without loss of generality that $f=\sum_{\underline{i}}\alpha_{\underline{i}}w(\underline{i})$ is a finite linear combination of special Wick words. Write $w(\underline{i})=(X(\underline{i},m))^{\bullet}$. Then

$$\|\delta(f)\|_p=\Bigg\|\sum_{\underline{i}}\alpha_{\underline{i}}\delta(w(\underline{i}))\Bigg\|_p=\lim_{m,\mathcal{U}}\Bigg\|\sum_{\underline{i}}\alpha_{\underline{i}}\delta^m(X(\underline{i},m))\Bigg\|_p.$$

Similarly,

$$\|A^{1/2}f\|_p = \lim_{m,\mathcal{U}} \Big\| \sum_{\underline{i}} \alpha_{\underline{i}} \sqrt{|\underline{i}|} X(\underline{i}, m) \Big\|_p = \lim_{m,\mathcal{U}} \|(A^m)^{1/2} f\|_p.$$

The assertion follows from (4-6) with a limiting procedure. □

In fact, we can give more precise estimates using the gradient form. Let

$$G_p = L_p\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [2N]^s,\ s \in \mathbb{N},\ 1 \le i_k \le N \text{ for all but at most one } k\}.$$

Since $L_p(\Gamma_Q) \subset G_p \subset L_p(\Gamma_{Q'})$, we have $E : G_p \to L_p(\Gamma_Q)$ given by the restriction of the conditional expectation $E : \Gamma_{Q'} \to \Gamma_Q$. If $f \in \Gamma_{Q'}$, we define $\|f\|_{L_p^c(E)} = \|E(f^*f)^{1/2}\|_p$ and $\|f\|_{L_p^r(E)} = \|f^*\|_{L_p^c(E)}$. The conditional $L_p(\Gamma_{Q'})$ space is

$$L_p^{rc}(E) = \begin{cases} L_p^r(E) + L_p^c(E) & \text{if } 1 \le p \le 2, \\ L_p^r(E) \cap L_p^c(E) & \text{if } 2 \le p \le \infty. \end{cases}$$

Define G_p^r (resp. G_p^c) as the space of G_p with the norm inherited from $L_p^r(E)$ (resp. $L_p^c(E)$). Now we follow [Junge et al. 2014] to derive a Khintchine-type inequality. First, since $E : \Gamma_{Q'} \to \Gamma_Q$ extends to contractions on L_p for $1 \le p < \infty$, we have, for $f \in L_p(\Gamma_{Q'})$ and $2 \le p < \infty$,

(4-7) $$\max\big\{\|E(f^*f)^{1/2}\|_p, \|E(ff^*)^{1/2}\|_p\big\} \le \|f\|_{L_p}.$$

This means that $L_p(\Gamma_{Q'}) \subset L_p^{rc}(E)$ contractively for $2 \le p < \infty$.

Lemma 4.9. *Let $E : G_p \to L_p(\Gamma_Q)$ be as above. Then, for $2 \le p < \infty$,*

$$\|f\|_{G_p} \le C\sqrt{p} \max\big\{\|E(f^*f)^{1/2}\|_p, \|E(ff^*)^{1/2}\|_p\big\} \le C\sqrt{p}\|f\|_{G_p},$$

and for $1 < p \le 2$,

$$\|f\|_{G_p} \le \inf_{\substack{f=g+h \\ g\in G_p^c, h\in G_p^r}} \big\{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\big\} \le C\sqrt{\frac{p}{p-1}}\|f\|_{G_p}.$$

Proof. Let $2 \le p < \infty$. The right inequality is a special case of (4-7). For the left inequality, let $n \in \mathbb{N}$ and $\underline{i} \in [2N]^s$. For $j = 1, \dots, n$, define

$$\phi_j : [2N]^s \to [2N]^{ns}, \quad \phi_j(\underline{i}) = \underline{0} \sqcup \cdots \underline{0} \sqcup \underline{i} \sqcup \underline{0} \cdots \sqcup \underline{0},$$

where $\underline{i}$ occurs in the j-th position. Put $\widetilde{\pi}_j(w(\underline{i})) = w(\phi_j(\underline{i}))$, where $w(\phi_j(\underline{i}))$ is the special Wick word associated to $\phi_j(\underline{i})$. Define

$$\pi_n : \Gamma_{Q'} \to \Gamma_{Q'\otimes\mathbb{1}_n}, \quad \pi_n(w(\underline{i})) = \frac{1}{\sqrt{n}} \sum_{j=1}^n w(\phi_j(\underline{i})).$$

Here, $\mathbb{1}_n$ is the $n \times n$ matrix with all entries equal to 1. The map π_n extends to a trace-preserving $*$-homomorphism. Alternatively, one may define π_n via the second quantization functor as in [Lust-Piquard 1999]. It is crucial to observe that the $\widetilde{\pi}_j(w(\underline{i}))$, where $j = 1, \dots, n$, are fully independent over $\Gamma_{Q\otimes\mathbb{1}_n}$ (see [Junge and Zeng 2013]) if $w(\underline{i}) \in G_p$, as can be checked from the definition of $E_{\Gamma_Q} : \Gamma_{Q'} \to \Gamma_Q$. We may assume f is a finite linear combination of special Wick words in G_p. By the noncommutative Rosenthal inequality [Junge and Xu 2008; Junge and Zeng 2013],

$$\begin{aligned}\|\pi_n(f)\|_p &\\ &\le \frac{Cp}{\sqrt{n}}\Big(\sum_{j=1}^n \|\widetilde{\pi}_j(f)\|_p^p\Big)^{\frac{1}{p}}\\ &\quad + \frac{C\sqrt{p}}{\sqrt{n}}\max\Big\{\Big\|\Big(\sum_{j=1}^n E[\widetilde{\pi}_j(f)^*\widetilde{\pi}_j(f)]\Big)^{\frac12}\Big\|_p, \Big\|\Big(\sum_{j=1}^n E[\widetilde{\pi}_j(f)\widetilde{\pi}_j(f)^*]\Big)^{\frac12}\Big\|_p\Big\}.\end{aligned}$$

We have extended the conditional expectation $E : \Gamma_{Q'} \to \Gamma_Q$ to $E : \Gamma_{Q'\otimes\mathbb{1}_n} \to \Gamma_{Q\otimes\mathbb{1}_n}$. Note that $E[\widetilde{\pi}_j(f)^*\widetilde{\pi}_j(f)] = \widetilde{\pi}_j[E(f^*f)]$ and $\|\widetilde{\pi}_j(f)\|_p = \|f\|_p$. Sending $n\to\infty$, for $2 < p < \infty$, we have

$$\|f\|_{G_p} \le C\sqrt{p}\max\big\{\|E(f^*f)^{1/2}\|_p, \|E(ff^*)^{1/2}\|_p\big\}. \tag{4-8}$$

For the case $1 < p < 2$, we argue by duality. Define the orthogonal projection $P : L_2^{rc}(E) \to G_2 \cap L_2^{rc}(E)$. By orthogonality, for $g \in \Gamma_{Q'}$,

$$E(g^*g) = E(Pg^*Pg) + E(P^\perp g^*P^\perp g) \ge E(Pg^*Pg).$$

Similarly, $E(gg^*) \ge E(PgPg^*)$. Since

$$\max\big\{\|E(Pg^*Pg)^{1/2}\|_p, \|E(PgPg^*)^{1/2}\|_p\big\} \le \max\big\{\|E(g^*g)^{1/2}\|_p, \|E(gg^*)^{1/2}\|_p\big\},$$

we deduce from (4-8) that P extends to a bounded projection with norm

$$\|P : L_p^{rc}(E) \to L_p(\Gamma_{Q'})\| \le C\sqrt{p}$$

for $2 \le p < \infty$. For $1 < p \le 2$ and $f \in G_2$, since $P^* = P$, we have by duality

$$\|f\|_{L_p^{rc}(E)} = \|Pf\|_{L_p^{rc}(E)} \le C\sqrt{p'}\|f\|_{L_p(\Gamma_{Q'})},$$

where $1/p + 1/p' = 1$. By density, this inequality extends to $f \in G_p$. It suffices to consider the decomposition of $f \in G_p$ in $G_p^c + G_p^r$ when we compute $\|f\|_{L_p^{rc}(E)}$. This gives the right inequality. The left inequality follows from duality and (4-7). □

Remark 4.10. In fact, the above argument also shows that G_p is complemented in $L_p(\Gamma_{Q'})$. Morally speaking, G_p is a Γ_Q–Γ_Q bimodule corresponding to differential forms of order one.

Corollary 4.11. (a) *Let $2 \le p < \infty$. Then, for every $f \in \mathrm{Dom}(A)$,*

$$c_p^{-1}\|A^{1/2}f\|_p \le \max\big\{\|\Gamma(f,f)^{1/2}\|_p, \|\Gamma(f^*,f^*)^{1/2}\|_p\big\} \le K_p\|A^{1/2}f\|_p,$$

where $c_p = O(p^2)$ and $K_p = O(p^{3/2})$.

(b) *Let* $1 < p \le 2$. *Then, for every* $f \in \mathrm{Dom}(A)$,

$$K_{p'}^{-1}\|A^{1/2}f\|_p \le \inf_{\substack{\delta(f)=g+h\\ g\in G_p^c, h\in G_p^r}} \left\{\|E(g^*g)^{1/2}\|_p + \|E(hh^*)^{1/2}\|_p\right\} \le C_p\|A^{1/2}f\|,$$

where $K_{p'} = O(1/(p-1)^{3/2})$ *and* $C_p = O(1/(p-1)^2)$.

Proof. Note that $\delta(f) \in G_p$ if $f \in \mathrm{Dom}(A)$. Since $E(\delta(f)^*\delta(f)) = \Gamma(f, f)$, using Lemma 4.9 for $2 \le p < \infty$, we have

$$\|\delta(f)\|_p \le C\sqrt{p}\max\left\{\|\Gamma(f, f)^{1/2}\|_p, \|\Gamma(f^*, f^*)^{1/2}\|_p\right\} \le C\sqrt{p}\|\delta(f)\|_p.$$

Now apply Theorem 4.8 to conclude (a). For the constants, $K_p = O(p^{3/2})$ is trivial. Since $\widetilde{K}_{p'} = O(p'^3/(p'-1)^{3/2}) = O(p^{3/2})$, we have $c_p \le O(p^2)$. Assertion (b) follows similarly using Lemma 4.9 and Theorem 4.8. □

Compared with Theorem 4.8, which was proved in [Lust-Piquard 1999], this result is closer to Lust-Piquard's original formulation of the Riesz transforms on the Walsh system and the fermions given in [Lust-Piquard 1998]. In particular, we get the exact order of constants as in [Lust-Piquard 1998].

4D. *L_p Poincaré inequalities.* Efraim and Lust-Piquard [2008] proved that the L_p Poincaré inequalities ($2 \le p < \infty$),

$$\|f - \tau_m(f)\|_p \le C\sqrt{p}\max\left\{\|\Gamma^m(f, f)^{1/2}\|_p, \Gamma^m(f^*, f^*)^{1/2}\|_p\right\}, \tag{4-9}$$

hold for Walsh systems and CAR algebras. In fact, the same proof also works for the general spin matrix system $\mathcal{A}_m$ with some technical variants as shown in [Lust-Piquard 1998]. Indeed, Lemmas 6.2–6.5 in [Efraim and Lust-Piquard 2008] hold for the general spin systems, from which (4-9) follows. Recall that we denote by A the number operator on Γ_Q.

Theorem 4.12. *Let* $2 \le p < \infty$. *Then, for every* $f \in \mathrm{Dom}(A)$,

$$\|f - \tau_Q(f)\|_p \le C\sqrt{p}\max\left\{\|\Gamma(f, f)^{1/2}\|_p, \Gamma(f^*, f^*)^{1/2}\|_p\right\}.$$

Proof. Assume without loss of generality that $f = \sum_{\underline{i}} \alpha_{\underline{i}} w(\underline{i}) = (f^m)^\bullet$ is a finite linear combination of special Wick words. Note that $E(\delta(f)^*\delta(f)) = (E^m[\delta^m(f^m)\delta^m(f^m)])^\bullet$. Then the assertion follows from (4-9) and a limiting procedure as for Theorems 4.2 and 4.8 with the help of Lemma 4.5 and Proposition 4.7. □

5. Strong solidity

5A. *CCAP.* Let $\Gamma_q(H)$ be the q-Gaussian von Neumann algebra associated to a real Hilbert space H with $\dim H \ge 2$; see, e.g., [Bożejko et al. 1997] for more information on $\Gamma_q(H)$. Avsec showed that $\Gamma_q(H)$ for $-1 < q < 1$ has the weak* completely contractive approximation property (w*CCAP) in [Avsec 2011]. In particular, $\Gamma_q(H)$ is weakly amenable. Our goal here is to prove that Γ_Q also has w*CCAP if $\max_{1\le i,j\le N}|q_{ij}| < 1$. Our argument is based on Avsec's result.

Assume that $\max_{i,j}|q_{ij}| < 1$. We may find q such that $\max_{i,j}|q_{ij}| < q < 1$. Let $Q = q\widetilde{Q}$, where $\widetilde{Q} = (\widetilde{q}_{ij})$ satisfies $\max_{i,j}|\widetilde{q}_{ij}| < 1$. For $h \in H$, let $c^q(h)$ and $(c^q)^*(h)$ be the creation and annihilation operators, respectively, acting on the q-Fock space $\mathcal{F}_q(H)$, where $\dim H = N$. We write the q-Gaussian variables as $s^q(h) = c^q(h) + (c^q)^*(h)$. In particular, for an orthonormal basis (o.n.b.) (e_j) of H, we write $s_j^q = s^q(e_j)$. Similarly, we write $s^Q(h) = c^Q(h) + (c^Q)^*(h)$ for the mixed q-Gaussian variables of Γ_Q; see [Lust-Piquard 1999]. In particular, $s_j^Q = s^Q(e_j)$. We write $x_{i,j} = s^{\widetilde{Q}\otimes\mathbb{1}_n}(f_i \otimes e_j)$, where (f_i) is an o.n.b. of ℓ_2^N, and (e_j) is an o.n.b. of ℓ_2^n. Clearly, the $x_{i,j}$'s generate $\Gamma_{\widetilde{Q}\otimes\mathbb{1}_n}$. We first construct an "approximate comultiplication" for Γ_Q.

Proposition 5.1. *Let $\pi_{\mathcal{U}} : \Gamma_Q \to \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m) \mathbin{\overline{\otimes}} \Gamma_{\widetilde{Q}\otimes\mathbb{1}_m}$ be a $*$-homomorphism given by*

$$\pi_{\mathcal{U}}(s_i^Q) = \left(\frac{1}{\sqrt{m}}\sum_{k=1}^m s_k^q \otimes x_{i,k}\right)^{\bullet}.$$

Then $\pi_{\mathcal{U}}$ is trace-preserving. Therefore, Γ_Q is isomorphic to the von Neumann algebra generated by $\pi_{\mathcal{U}}(s_i^Q)$.

Proof. Let d be an even integer. By the moment formula (3-2),

$$\begin{aligned}\sum_{\underline{k}\in[m]^d} \tau_q \otimes \tau_{\widetilde{Q}\otimes\mathbb{1}_m}[(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})]\\ &= \sum_{\sigma\in P_2(d),\sigma\le\sigma(\underline{i})}\ \sum_{\sigma(\underline{k})=\sigma} q^{\#I(\sigma)} \prod_{\{r,t\}\in I(\sigma)} \widetilde{q}(i(e_r), i(e_t))\\ &= \sum_{\sigma\in P_2(d),\sigma\le\sigma(\underline{i})}\ \sum_{\sigma(\underline{k})=\sigma} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)),\end{aligned}$$

where $I(\sigma)$ is the set of inversions for the partition σ. Counting the number of $\underline{k}$ with $\sigma(\underline{k}) = \sigma$, we have

$$\tau_{\mathcal{U}}\left(\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d}(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})\right)^{\bullet} = \sum_{\substack{\sigma\in P_2(d)\\ \sigma\le\sigma(\underline{i})}}\ \prod_{\{r,t\}\in I(\sigma)} q(i(e_r), i(e_t)).$$

This coincides with $\tau_Q(s_{i_1}^Q\cdots s_{i_d}^Q)$ given by (3-2). □

Now we want to understand the image of Wick words of Γ_Q under $\pi_{\mathcal{U}}$. We need a Wick word decomposition result similar to Theorem 3.8. For $\underline{i} \in [N]^d$, we define

$$w^s(\underline{i}) = \left(\frac{1}{m^{d/2}}\sum_{\underline{k}:\sigma(\underline{k})\in P_1(d)}(s_{k_1}^q\cdots s_{k_d}^q)\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})\right)^{\bullet}. \tag{5-1}$$

Proposition 5.2. *Following the notation of Proposition 5.1, we have*

$$\pi_{\mathcal{U}}(s^{Q}_{i_1}\cdots s^{Q}_{i_d}) = \sum_{\substack{\sigma\in P_{1,2}(d)\\ \sigma\le\sigma(\underline{i})}} w^{s}_{\sigma}(\underline{i}).$$

Here, $w^{s}_{\sigma}(\underline{i}) = f_{\sigma}(\underline{i})w^{s}(\underline{i}_{\mathrm{np}})$, $f_{\sigma}(\underline{i})$ *and* $\underline{i}_{\mathrm{np}}$ *are the same as those in Proposition 3.10.*

Proof. Following verbatim the argument for Theorem 3.8, we have

$$\pi_{\mathcal{U}}(s^{Q}_{i_1}\cdots s^{Q}_{i_d}) = \sum_{\sigma\in P_{1,2}(d)} w^{s}_{\sigma}(\underline{i}).$$

Here we have

$$w^{s}_{\sigma}(\underline{i}) = \Bigg(\frac{1}{m^{d/2}}\sum_{\underline{k}\in[m]^d:\sigma(\underline{k})=\sigma} E_{\mathcal{N}_s(\underline{k})}[(s^{q}_{k_1}\cdots s^{q}_{k_d})\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})]\Bigg)^{\bullet},$$

and $\mathcal{N}_s(\underline{k})$ is the von Neumann algebra generated by all the $s^{q}_{k_\alpha}\otimes x_{i_\alpha,k_\alpha}$'s, where the k_α's correspond to singleton blocks in $\underline{k}$. To simplify the conditional expectation in the ultraproduct, we denote by $\mathcal{N}^1_s(\underline{k})$ and $\mathcal{N}^2_s(\underline{k})$ the von Neumann algebras generated by the $s^{q}_{k_\alpha}\otimes\Gamma_{\widetilde{Q}\otimes\mathbb{1}_m}$'s and $\Gamma_q(\ell^m_2)\otimes x_{i_\alpha,k_\alpha}$'s, respectively, where the k_α's correspond to singleton blocks in $\underline{k}$. Clearly, $\mathcal{N}_s(\underline{k})\subset\mathcal{N}^1_s(\underline{k})\cap\mathcal{N}^2_s(\underline{k})$. We claim

$$\text{(5-2)}\quad E_{\mathcal{N}^2_s(\underline{k})}(1\otimes(x_{i_1,k_1}\cdots x_{i_d,k_d})) = \begin{cases} f_{\sigma,\widetilde{Q}}(\underline{i})1\otimes(x_{j_1,l_1}\cdots x_{j_s,l_s}) & \text{if } \sigma=\sigma(\underline{k})\le\sigma(\underline{i}),\\ 0 & \text{otherwise,}\end{cases}$$

where $(l_1,\dots,l_s)$ is obtained by deleting pair blocks in $\underline{k}$, which also gives the corresponding $(j_1,\dots,j_s)$, and

$$f_{\sigma,\widetilde{Q}}(\underline{i}) = \prod_{\{r,t\}\in I_p(\sigma)}\widetilde{q}(i(e_r),i(e_t))\prod_{\{r,t\}\in I_{\mathrm{sp}}(\sigma)}\widetilde{q}(i(e_r),i(e_t)).$$

Unlike in the matrix models, the x_{i_α,k_α}'s do not have commutation relations. We check (5-2) by calculating the inner product of $E_{\mathcal{N}^2_s(\underline{k})}(1\otimes x_{i_1,k_1}\cdots x_{i_d,k_d})$ and monomials generated by the $1\otimes x_{i_\alpha,k_\alpha}$'s in $\mathcal{N}^2_s(\underline{k})$. Let $1\otimes x_{i'_1,k'_1}\cdots x_{i'_n,k'_n}\in\mathcal{N}^2_s(\underline{k})$ be a monomial. Since $E_{\mathcal{N}^2_s(\underline{k})}$ is trace-preserving, by the moment formula (3-2) for mixed q-Gaussian algebras,

$$\tau_{\widetilde{Q}\otimes\mathbb{1}_m}[x_{i'_n,k'_n}\cdots x_{i'_1,k'_1}E_{\mathcal{N}^2_s(\underline{k})}(x_{i_1,k_1}\cdots x_{i_d,k_d})] = \begin{cases} f_{\sigma,\widetilde{Q}}(\underline{i})\tau_{\widetilde{Q}\otimes\mathbb{1}_m}(x_{i'_n,k'_n}\cdots x_{i'_1,k'_1}x_{j_1,l_1}\cdots x_{j_s,l_s}) & \text{if } \sigma=\sigma(\underline{k})\le\sigma(\underline{i}),\\ 0 & \text{otherwise.}\end{cases}$$

Hence (5-2) is verified. Similarly, it can be checked that

$$E_{\mathcal{N}^1_s(\underline{k})}((s^{q}_{k_1}\cdots s^{q}_{k_d})\otimes 1) = q^{\#I_p(\sigma)+\#I_{\mathrm{sp}}(\sigma)}s^{q}_{l_1}\cdots s^{q}_{l_s}.$$

Note that $f_\sigma(\underline{i}) = q^{\#I_p(\sigma)+\#I_{\mathrm{sp}}(\sigma)}f_{\sigma,\widetilde{Q}}(\underline{i})$. The assertion follows from the fact that $E_{\mathcal{N}_s(\underline{k})} = E_{\mathcal{N}_s(\underline{k})}E_{\mathcal{N}^1_s(\underline{k})}E_{\mathcal{N}^2_s(\underline{k})}$. □

Proposition 5.3. *$\pi_{\mathcal{U}}$ extends to an isomorphism*

$$L_2(\Gamma_Q) \cong L_2\text{-}\operatorname{span}\{w^s(\underline{i}) : \underline{i} \in [N]^d, d \in \mathbb{Z}_+\}.$$

Proof. Put $H_W = L_2\text{-}\operatorname{span}\{w^s(\underline{i}) : \underline{i} \in [N]^d, d \in \mathbb{Z}_+\}$. By Proposition 5.2, we know that $\pi_{\mathcal{U}}(L_2(\Gamma_Q)) \subset H_W$. The converse containment follows from the same induction argument as for Proposition 3.13. □

Remark 5.4. In fact, one can prove that $\pi_{\mathcal{U}}(w(\underline{i})) = w^s(\underline{i})$ using the Fock space representation. Since we do not need this fact, we leave it to the reader.

Now we are ready for the first main result of this section.

Theorem 5.5. *Γ_Q has the weak* completely contractive approximation property for all Q with* $\max_{1\le i,j\le N}|q_{ij}| < 1$.

Proof. Let H be a real Hilbert space and $-1 < q < 1$. In [Avsec 2011], Avsec proved that there exists a net of finite-rank maps $\varphi_\alpha(A)$ which converges to the identity map on $\Gamma_q(H)$ in the point-weak* topology and such that $\|\varphi_\alpha(A)\|_{\mathrm{cb}} \le 1+\varepsilon$ for some prescribed ε. Here, $\|\cdot\|_{\mathrm{cb}}$ is the completely bounded norm and $\varphi_\alpha(A)$ only depends on the number operator A on $\Gamma_q(H)$. Let $Q = q\widetilde{Q}$ as above. Consider the diagram

$$\begin{array}{ccc}
\Gamma_Q & \overset{\pi_{\mathcal{U}}}{\hookrightarrow} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m)\,\overline{\otimes}\,\Gamma_{\widetilde{Q}\otimes \mathbb{1}_m} \\
\Big\downarrow \psi_\alpha & & \Big\downarrow \varphi_\alpha(A)\otimes \mathrm{id} \\
\Gamma_Q & \overset{\pi_{\mathcal{U}}}{\hookrightarrow} & \prod_{m,\mathcal{U}} \Gamma_q(\ell_2^m)\,\overline{\otimes}\,\Gamma_{\widetilde{Q}\otimes \mathbb{1}_m}
\end{array}$$

where we define $\psi_\alpha = \pi_{\mathcal{U}}^{-1} \circ (\varphi_\alpha(A)\otimes \mathrm{id}) \circ \pi_{\mathcal{U}}$. Here, $\varphi_\alpha(A)\otimes \mathrm{id}$ is well-defined on the ultraproduct of von Neumann algebras because it is uniformly bounded in each $\Gamma_q(\ell_2^m)\,\overline{\otimes}\,\Gamma_{\widetilde{Q}\otimes \mathbb{1}_m}$. By an argument similar to that in Section 3D, $\varphi_\alpha(A)\otimes \mathrm{id}$ is a normal map. Note that ψ_α is well-defined because $\pi_{\mathcal{U}}$ is injective and $\varphi_\alpha(A)\otimes \mathrm{id}$ acts as a multiplier. We claim that ψ_α is the desirable completely contractive approximation of identity. By construction, the only nontrivial thing to check is that ψ_α is of finite rank. To this end, it suffices to show that $\varphi(A)_\alpha \otimes \mathrm{id}$ restricted to

$$\pi_{\mathcal{U}}(L_2(\Gamma_Q)) = L_2\text{-}\operatorname{span}\{w^s(\underline{i}) : \underline{i} \in [N]^d, d \in \mathbb{N}\}$$

is of finite rank thanks to Proposition 5.1 and 5.3. Since $\varphi_\alpha(A)$ is of finite rank, suppose its range is $\operatorname{span}\big\{s_{k_1}^q \cdots s_{k_n}^q : \sigma(\underline{k}) \in \bigcup_{n\in\mathbb{N}} P_1(n), \underline{k} \in B\big\}$ for some finite set B. Then the range of $\varphi_\alpha(A)\otimes \mathrm{id}\,|_{\pi_{\mathcal{U}}(L_2(\Gamma_Q))}$ is

$$\operatorname{span}\Big\{s_{k_1}^q \cdots s_{k_n}^q \otimes x_{i_1,k_1}\cdots x_{i_n,k_n} : \sigma(\underline{k}) \in \bigcup_{n\in\mathbb{N}} P_1(n), \underline{k} \in B\Big\}.$$

Therefore $\varphi_\alpha(A)\otimes \mathrm{id}\,|_{\pi_{\mathcal{U}}(L_2(\Gamma_Q))}$ is a finite-rank map. □

5B. ***Strong solidity.*** We follow closely the argument in [Avsec 2011; Houdayer and Shlyakhtenko 2011]. The strategy is to first prove a weak containment result of bimodules and then use it to prove strong solidity of Γ_Q. See, e.g., [Brown and Ozawa 2008; Avsec 2011] for more details on bimodules and weak containment. For simplicity, we write $Q' = Q \otimes \mathbb{1}_2 = Q \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as in Section 4B. Here we assume

$$\max_{1\le i,j\le N} |q_{ij}| < q^2 < q < 1. \tag{5-3}$$

Recall that $L_2^0(\Gamma_{Q'})$ denotes the subspace of $L_2(\Gamma_{Q'})$ which consists of mean zero elements. Define the following subspaces of $L_2^0(\Gamma_{Q'})$:

$$F_m = L_2\text{-}\operatorname{span}\{w(\underline{i}) : \underline{i} \in [2N]^s, s \in \mathbb{N}, s \ge m, \exists i_1, \dots, i_m \in \{N+1, \dots, 2N\}\},$$
$$E_m = \bigoplus_{k=0}^{m} F_k.$$

Clearly, $E_m^\perp$ is a Γ_Q–Γ_Q-subbimodule of $L_2^0(\Gamma_{Q'})$. We want to show that $E_m^\perp$ is weakly contained in the coarse bimodule $L_2(\Gamma_Q) \otimes L_2(\Gamma_Q)$ for m large enough. By Proposition 3.13, we may identify $L_2(\Gamma_{Q'})$ with the Fock space $\mathcal{F}_{Q'}$. For $\xi, \eta \in L_2^0(\Gamma_{Q'})$, define $\Phi_{\xi,\eta} : L_2(\Gamma_Q) \to L_2(\Gamma_Q)$ by

$$\Phi_{\xi,\eta}(x) = E_{\Gamma_Q}(\xi x \eta).$$

To distinguish the left action and the right action of $\Gamma_{Q'}$ on $L_2(\Gamma_{Q'})$, we write $l(h_i)$ (resp. $r(h_i)$) as the left (resp. right) creation operator associated to h_i acting on the Fock space $\mathcal{F}_{Q'}$, i.e.,

$$l(h_i)(h_{j_1} \otimes \cdots \otimes h_{j_n}) = h_i \otimes h_{j_1} \otimes \cdots \otimes h_{j_n},$$
$$r(h_i)(h_{j_1} \otimes \cdots \otimes h_{j_n}) = h_{j_1} \otimes \cdots \otimes h_{j_n} \otimes h_i.$$

Here, the h_i's are elements in $\mathbb{C}^{2N} = \mathbb{C}^N \oplus \mathbb{C}^N$. We write $l(h_i)^*$ (resp. $r(h_i)^*$) as the left (resp. right) annihilation operator acting on the Fock space $\mathcal{F}_{Q'}$. See more details for these operations in [Bożejko and Speicher 1994; Lust-Piquard 1999]. One can also define them following Section 3C after choosing an o.n.b. Write

$$H'^s = \operatorname{span}\{w(\underline{i}) \in L_2(\Gamma_{Q'}) : \underline{i} \in [2N]^s\} \quad \text{and} \quad H^s = \operatorname{span}\{w(\underline{i}) \in L_2(\Gamma_Q) : \underline{i} \in [N]^s\}.$$

Lemma 5.6. *Assume* (5-3). *Let $(e_i)_{i=1}^{2N}$ be an o.n.b. of $\mathbb{C}^{2N}$. Suppose $\underline{i} \in [2N]^{n_1}$ and $\underline{j} \in [2N]^{n_2}$. If $\{N+1, \dots, 2N\}$ contains exactly n elements of $\{i_{r+1}, \dots, i_{n_1}\}$, then*

$$\big\| E_{\Gamma_Q}\big[l(e_{i_1}) \cdots l(e_{i_r}) l(e_{i_{r+1}})^* \cdots l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_s}) r(e_{j_{s+1}})^* \cdots r(e_{j_{n_2}})^*(x)\big]\big\|_2 \le C_{q,n_1,n_2} q^{(\alpha-(n_2-s)-(n_1-r-n))n} \|x\|_2$$

for all $x \in H^\alpha$.

Proof. Note that among all possible configurations, the assertion is nontrivial only if

$$i_1, \dots, i_r, j_{s+1}, \dots, j_{n_2} \le N.$$

By [Bożejko and Speicher 1994, Theorem 3.1],

$$\|r(e_{j_s})^*\| \le \frac{1}{\sqrt{1-q}}, \quad \|l(e_{i_r})\| \le \frac{1}{\sqrt{1-q}}. \tag{5-4}$$

We may assume without loss of generality that $r = 0$ and $s = n_2$ and estimate the norm of $l(e_{i_1})^* \cdots l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}})$. The idea is that all the e_{i_r}'s with $i_r > N$ have to pair with the e_{j_s}'s to cancel out, and moving across the element x will yield a power of q. Let us assume $i_{n_1} > N$ to illustrate the argument. Note that by Remark 3.17, H^α can be identified with $(\mathbb{C}^N \oplus 0)^{\otimes \alpha}$ via

$$w(\underline{i}) \mapsto W(e_{i_1} \otimes \cdots \otimes e_{i_\alpha}) \mapsto e_{i_1} \otimes \cdots \otimes e_{i_\alpha}.$$

First assume $x = e_{k_1} \otimes \cdots \otimes e_{k_\alpha}$. Using (3-14) (or the formula on p. 109 of [Bożejko and Speicher 1994]), we find

$$\begin{aligned} l(e_{i_{n_1}})^* r(e_{j_1}) \cdots r(e_{j_{n_2}}) x \\ &= \sum_{m=1}^{n_2} \delta_{i_{n_1}, j_m} \prod_{s=1}^{\alpha} q_{i_{n_1}, k_s} \prod_{r=m+1}^{n_2} q_{i_{n_1}, j_r} x \otimes e_{j_{n_2}} \otimes \cdots \otimes \check{e}_{j_m} \otimes \cdots \otimes e_{j_1} \\ &= \sum_{m=1}^{n_2} \delta_{i_{n_1}, j_m} \prod_{s=1}^{\alpha} q_{i_{n_1}, k_s} \prod_{r=m+1}^{n_2} q_{i_{n_1}, j_r} r(e_{j_1}) \cdots \check{r}(e_{j_m}) \cdots r(e_{j_{n_2}}) x, \end{aligned}$$

where $\check{e}_{j_m}$ and $\check{r}(e_{j_m})$ mean that e_{j_m} and $r(e_{j_m})$ are omitted in the expression. The difficulty is that the coefficient in front of x depends on x. In order to extend the above equation to arbitrary $x \in H^\alpha$, we will find a linear operator for any fixed m via deformation and enlargement of the algebra. Define $\widetilde{q}_{ij} = q_{ij}/q$ for $1 \le i, j \le N$, $\widetilde{Q} = (\widetilde{q}_{ij})$, and

$$P = \begin{pmatrix} Q \otimes \mathbb{1}_2 & \widetilde{Q} \otimes \mathbb{1}_2 \\ \widetilde{Q} \otimes \mathbb{1}_2 & Q \otimes \mathbb{1}_2 \end{pmatrix}.$$

Note that (5-3) implies that $\max_{ij} |p_{ij}| < q$. We can construct new von Neumann algebras Γ_P and $\Gamma_{P \otimes \mathbb{1}_{n_2+1}}$. Clearly, we have the relation

$$\Gamma_Q \hookrightarrow \Gamma_{Q'} \hookrightarrow \Gamma_P \hookrightarrow \Gamma_{P \otimes \mathbb{1}_{n_2+1}}.$$

We continue to denote by $E_{\Gamma_Q} : \Gamma_{P \otimes \mathbb{1}_{n_2+1}} \to \Gamma_Q$ the conditional expectation. Let $\hat{i}_{n_1} = i_{n_1} + 2N$, and let

$$\hat{j}_r = \begin{cases} j_r + 2N & \text{if } j_r > N, \\ j_r & \text{otherwise.} \end{cases}$$

For fixed m, let

$$\widetilde{i}_{n_1}=\hat{i}_{n_1}+4mN=i_{n_1}+2N+4mN,\quad \widetilde{j}_m=j_m+4mN,$$

and let $\widetilde{j}_r=\hat{j}_r$ for $r\neq m$. In $L_2(\Gamma_{P\otimes\mathbb{1}_{n_2+1}})$, observing the repetition pattern in the matrix P, we have

$$\begin{aligned}
&r(e_{\widetilde{j}_1})\cdots r(e_{\widetilde{j}_{m-1}})[l(e_{\widetilde{i}_{n_1}})^*r(e_{\widetilde{j}_m})]r(e_{\widetilde{j}_{m+1}})\cdots r(e_{\widetilde{j}_{n_2}})x\\
&\quad=\sum_{u=m}^{n_2}\delta_{\widetilde{i}_{n_1},\widetilde{j}_u}\prod_{s=1}^{\alpha}\widetilde{q}_{i_{n_1},k_s}\prod_{v=u+1}^{n_2}p_{\hat{i}_{n_1},\hat{j}_v}r(e_{\widetilde{j}_1})\cdots r(e_{\widetilde{j}_{m-1}})\cdots\check{r}(e_{\widetilde{j}_u})\cdots r(e_{\widetilde{j}_{n_2}})x\\
&\quad=\prod_{s=1}^{\alpha}\widetilde{q}_{i_{n_1},k_s}\prod_{v=m+1}^{n_2}p_{\hat{i}_{n_1},\hat{j}_v}r(e_{\widetilde{j}_1})\cdots\check{r}(e_{\widetilde{j}_m})\cdots r(e_{\widetilde{j}_{n_2}})x,
\end{aligned}$$

where $p_{\hat{i}_{n_1},\hat{j}_v}=q_{i_{n_1},j_v}$ if $j_v\in\{N+1,\dots,2N\}$ and $p_{\hat{i}_{n_1},\hat{j}_v}=\widetilde{q}_{i_{n_1},j_v}$ otherwise. Note that the term $4mN$ is used to guarantee that $l(e_{\widetilde{i}_{n_1}})^*$ only annihilates $e_{\widetilde{j}_m}$. Let

$$I(m)=\big\{j_v:v\in\{m+1,\dots,n_2\},\ j_v\leq N\big\}.$$

Then

$$\begin{aligned}
(5\text{-}5)\quad &E_{\Gamma_Q}[l(e_{i_{n_1}})^*r(e_{j_1})\cdots r(e_{j_{n_2}})x]\\
&=q^\alpha\sum_{m=1}^{n_2}\delta_{i_{n_1},j_m}\prod_{s=1}^{\alpha}\widetilde{q}_{i_{n_1},k_s}\prod_{r=m+1}^{n_2}q_{i_{n_1},j_r}E_{\Gamma_Q}[r(e_{j_1})\cdots\check{r}(e_{j_m})\cdots r(e_{j_{n_2}})x]\\
&=q^\alpha\sum_{m=1}^{n_2}\delta_{i_{n_1},j_m}q^{\#I(m)}\\
&\qquad\times\prod_{s=1}^{\alpha}\widetilde{q}_{i_{n_1},k_s}\prod_{v=m+1}^{n_2}p_{\hat{i}_{n_1},\hat{j}_v}E_{\Gamma_Q}[r(e_{\widetilde{j}_1})\cdots\check{r}(e_{\widetilde{j}_m})\cdots r(e_{\widetilde{j}_{n_2}})x]\\
&=q^\alpha\sum_{m=1}^{n_2}\delta_{i_{n_1},j_m}q^{\#I(m)}\\
&\qquad\times E_{\Gamma_Q}\big[r(e_{\widetilde{j}_1})\cdots r(e_{\widetilde{j}_{m-1}})[l(e_{\widetilde{i}_{n_1}})^*r(e_{\widetilde{j}_m})]r(e_{\widetilde{j}_{m+1}})\cdots r(e_{\widetilde{j}_{n_2}})x\big].
\end{aligned}$$

Here, the conditional expectation is used in the second equality so that the change in $\underline{i}$ and $\underline{j}$ will not affect the resultant value in $L_2(\Gamma_Q)$. Note that the summand in (5-5) does not depend on x for each fixed m. By linearity, (5-5) holds for any $x\in H^\alpha$. We deduce from (5-4) and the triangle inequality that

$$\big\|E_{\Gamma_Q}[l(e_{i_{n_1}})^*r(e_{j_1})\cdots r(e_{j_{n_2}})x]\big\|_2\leq C_{q,n_2}q^\alpha\|x\|_2$$

for all $x\in H^\alpha$. Since $l(e_{i_{n_1}})^*r(e_{j_1})\cdots r(e_{j_{n_2}})x$ is a linear combination of words with fixed length, the above argument can be easily extended to handle more than one annihilator. To get a norm estimate on $E_{\Gamma_Q}[l(e_{i_1})^*\cdots l(e_{i_{n_1}})^*r(e_{j_1})\cdots r(e_{j_{n_2}})(x)]$, it suffices to consider the configuration yielding the minimal power of q. This occurs if

$$i_1, \dots, i_n, j_{n_2-n+1}, \dots, j_{n_2} \in \{N+1, \dots, 2N\}.$$

In this situation, $l(e_{i_1})^*, \dots, l(e_{i_n})^*$ need to cross at least $\alpha - (n_1 - n)$ terms to cancel with the e_{j_s}'s. This gives $q^{[\alpha-(n_1-n)]n}$. Using (5-4) to estimate the norm of $l(e_{i_{n+1}})^* \cdots l(e_{i_{n_1}})^*$ gives a constant C_{q,n_1}. Proceeding like so finishes the proof. □

We will use the normal form theorem of Wick products [Bożejko et al. 1997; Królak 2000] to estimate the norm of $\Phi_{\xi,\eta}$. We achieve this via the following result.

Lemma 5.7. *Assume* (5-3). *Let $\xi \in H'^{n_1} \cap F_n$ and $\eta \in H'^{n_2} \cap F_n$. Then, for $\alpha > 2(n_1+n_2)$ and $x \in H^\alpha$, we have*

$$\|\Phi_{\xi,\eta}(x)\|_2 \le C_{q,\xi,\eta} q^{n\alpha/2} \|x\|_2.$$

Moreover, $\Phi_{\xi,\eta}(x) \in \bigoplus_{\beta=\alpha-n_1-n_2+2n}^{\alpha+n_1+n_2-2n} H^\beta$.

Proof. First we assume $\xi = w(\underline{i})$, $\eta = w(\underline{j})$ and identify x as a vector in $(\mathbb{C}^N \oplus 0)^\alpha$. By the normal form theorem of Wick products [Bożejko et al. 1997] and [Królak 2000, Theorem 1], we have

$$\begin{aligned} w(\underline{i}) &= W(e_{i_1} \otimes \cdots \otimes e_{i_{n_1}}) \\ &= \sum_{r=0}^{n_1} \sum_{\sigma \in S_{n_1}/(S_r \times S_{n_1-r})} K(Q,\sigma) l(e_{\sigma(i_1)}) \cdots l(e_{\sigma(i_r)}) l(e_{\sigma(i_{r+1})})^* \cdots l(e_{\sigma(i_{n_1})})^*, \end{aligned}$$

where $\sigma(i_r) = i_{\sigma^{-1}(r)}$, and $K(Q,\sigma)$ is a product of certain entries of Q and only depending on Q and σ. The precise value of $K(Q,\sigma)$ is irrelevant here. We only need the fact that $|K(Q,\sigma)| \le C_{q,n_1}$ for some constant C_{q,n_1} depending on q and n_1. We have a similar formula for $w(\underline{j})$. It follows that

$$\begin{aligned} \text{(5-6)}\quad \Phi_{\xi,\eta}(x) = \sum_{r=0}^{n_1}\sum_{s=0}^{n_2} \sum_{\substack{\sigma \in S_{n_1}/(S_r \times S_{n_1-r}) \\ \pi \in S_{n_2}/(S_s \times S_{n_2-s})}} & K(Q,\sigma)K(Q,\pi) E_{\Gamma_Q}\big[l(e_{\sigma(i_1)}) \cdots l(e_{\sigma(i_r)}) \\ & \cdot l(e_{\sigma(i_{r+1})})^* \cdots l(e_{\sigma(i_{n_1})})^* r(e_{\pi(j_1)}) \cdots r(e_{\pi(j_s)}) r(e_{\pi(j_{s+1})})^* \cdots r(e_{\pi(j_{n_2})})^*(x)\big]. \end{aligned}$$

By Lemma 5.6,

$$\begin{aligned} \big\| E_{\Gamma_Q}\big[l(e_{\sigma(i_1)}) \cdots l(e_{\sigma(i_r)}) l(e_{\sigma(i_{r+1})})^* \cdots l(e_{\sigma(i_{n_1})})^* r(e_{\pi(j_1)}) \cdots r(e_{\pi(j_s)}) \\ r(e_{\pi(j_{s+1})})^* \cdots r(e_{\pi(j_{n_2})})^*(x)\big]\big\|_2 \\ \le C_{q,n_1,n_2} q^{(\alpha-(n_2-s)-(n_1-r-n))n} \|x\|_2. \end{aligned}$$

Since $\alpha - n_1 - n_2 + s + r + n \ge \frac{1}{2}\alpha$, it follows from the triangle inequality that

$$\|\Phi_{\xi,\eta}(x)\|_2 \le C_{q,n_1,n_2} q^{n\alpha/2} \|x\|_2.$$

Now suppose ξ, η are linear combinations of special Wick words. Using the triangle inequality again, we have proved the first assertion. As for the range of $\Phi_{\xi,\eta}$, a moment of thought shows that the summand in (5-6) is of length $\alpha - n_1 - n_2 + 2s + 2r$ and that $0 \le r \le n_1 - n, n \le s \le n_2$ because we must have $\sigma(i_1), \dots, \sigma(i_r), \pi(j_{s+1}), \dots, \pi(j_{n_2}) \le N$ so that the right-hand side of (5-6) is nonzero. This gives the "moreover" part of the lemma. □

Lemma 5.8. *Let $K = \bigoplus_{n=0}^{\infty} K_n$ and $T : K \to K$ be an operator such that*

(i) $\dim(K_n) \le d^n$;

(ii) $\|T|_{K_n}\| \le C\alpha^n$ *for* $n \ge n_0$;

(iii) $\alpha^2 d < 1$.

Then T is Hilbert–Schmidt.

Proof. Let $P_n : K \to K_n$ be the orthogonal projection. Then

$$\operatorname{tr}(T^*T) = \sum_n \operatorname{tr}((TP_n)^*TP_n) \le \sum_n \|TP_n\|^2 d^n \le C \sum_n \alpha^{2n} d^n.$$

Since the series is absolutely convergent the assertion follows immediately. □

Lemma 5.9. *Let $\xi, \eta \in F_n$ and $n > -\ln N / \ln q$. Then $\Phi_{\xi,\eta} : L_2(\Gamma_Q) \to L_2(\Gamma_Q)$ is Hilbert–Schmidt.*

Proof. Write $L_2(\Gamma_Q) = \bigoplus_{s=0}^{\infty} H^s$. Then $\dim(H^\alpha) \le N^\alpha$ and $q^n N < 1$. By Lemma 5.7, we have

$$\left\| \Phi_{\xi,\eta}|_{H^\alpha} \right\| \le C_{q,\xi,\eta}(q^{n/2})^\alpha.$$

The assertion follows from Lemma 5.8. □

Proposition 5.10. *Let $n > -\ln N / \ln q$. Then $E_{n-1}^{\perp}$ is weakly contained in the coarse bimodule $L_2(\Gamma_Q) \otimes L_2(\Gamma_Q)$.*

Proof. The proof is given in [Avsec 2011, Proposition 4.1] using Lemma 5.9. □

Let $R_t : \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N$ be the orthogonal transform

$$R_t = \begin{pmatrix} e^{-t}\,\mathrm{id} & -\sqrt{1-e^{-2t}}\,\mathrm{id} \\ \sqrt{1-e^{-2t}}\,\mathrm{id} & e^{-t}\,\mathrm{id} \end{pmatrix},$$

where $\mathrm{id} : \mathbb{R}^N \to \mathbb{R}^N$ is the identity operator and we understand the canonical o.n.b. in $0 \oplus \mathbb{R}^N$ to be $\{e_{N+1}, \dots, e_{2N}\}$. Recall from [Lust-Piquard 1999, Lemma 3.1] that there is a second quantization functor Γ_Q which sends the category of Hilbert spaces to the category of mixed q-Gaussian algebras. Let $\alpha_t = \Gamma_Q(R_t)$. Then α_t is a trace-preserving $*$-automorphism on $\Gamma_{Q'}$ and extends to an isometry on $L_2(\Gamma_{Q'})$. It is easy to check that $T_t = E_{\Gamma_Q} \circ \alpha_t$ coincides with the Ornstein–Uhlenbeck semigroup on Γ_Q defined in Section 3D. The following is a modification of Popa's s-malleable deformation estimate [Popa 2008, Lemma 2.1]. The proof modifies slightly that of [Avsec 2011, Proposition 5.1]. We provide the difference here for the reader's convenience. Recall that $L_2^0(\Gamma_{Q'}) = \bigoplus_{m=1}^{\infty} H'^m$.

Proposition 5.11. *Let $P_k : L_2^0(\Gamma_{Q'}) \to E_k^{\perp}$ be the orthogonal projection. Then, for $k \geq 1$, we have*

$$\|(\alpha_{t^k} - \mathrm{id})(x)\|_2 \leq C_k \|P_{k-1}\alpha_t(x)\|_2$$

for $x \in \bigoplus_{n=k}^{\infty} H^n \subset L_2(\Gamma_Q)$ and $t < 2^{-k}$.

Proof. Note $\alpha_{t^k} - \mathrm{id}$ and $P_{k-1}\alpha_t$ preserve the length of n-tensors for $n \geq k$ and $t > 0$. It suffices to prove the assertion for $x \in H^n$ with $n \geq k$. Identify H^n with $(\mathbb{C}^N \oplus 0)^{\otimes n}$. Let $x = e_{i_1} \otimes \cdots \otimes e_{i_n}$ and $y = e_{j_1} \otimes \cdots \otimes e_{j_n}$. Then

$$\langle P_{k-1}\alpha_t(x), P_{k-1}\alpha_t(y)\rangle = \sum_{m=k}^{n} \langle P_{F_m}\alpha_t(x), P_{F_m}\alpha_t(y)\rangle,$$

where the inner product is given by Proposition 3.9, and $P_{F_m} : L_2^0(\Gamma_{Q'}) \to F_m$ is the orthogonal projection. By the second quantization [Lust-Piquard 1999, Lemma 3.1],

$$\alpha_t(e_{i_1} \otimes \cdots \otimes e_{i_n}) = (e^{-t}e_{i_1} + \sqrt{1 - e^{-2t}}e_{N+i_1}) \otimes \cdots \otimes (e^{-t}e_{i_n} + \sqrt{1 - e^{-2t}}e_{N+i_n}).$$

It follows that

$$P_{F_m}\alpha_t(x) = \sum_{B \subset \{1,\dots,n\}, |B|=m} (1 - e^{-2t})^{m/2} e^{-t(n-m)} e_{\pi_B(i_1)} \otimes \cdots \otimes e_{\pi_B(i_n)},$$

where $\pi_B(i_k) = N + i_k$ for $k \in B$ and $\pi_B(i_k) = i_k$ otherwise. Similarly, we get

$$P_{F_m}\alpha_t(y) = \sum_{C \subset \{1,\dots,n\}, |C|=m} (1 - e^{-2t})^{m/2} e^{-t(n-m)} e_{\pi_C(j_1)} \otimes \cdots \otimes e_{\pi_C(j_n)},$$

where

$$\pi_C(j_k) = N + j_k \quad \text{for } k \in C \qquad \text{and} \qquad \pi_C(j_k) = j_k \quad \text{otherwise.}$$

By Proposition 3.9, $\langle P_{F_m}\alpha_t(x), P_{F_m}\alpha_t(y)\rangle$ is nonzero only if $\{\pi_B(i_1), \dots, \pi_B(i_n)\}$ and $\{\pi_C(j_1), \dots, \pi_C(j_n)\}$ are equal as multisets. Hence, the indices in B have to be paired with the indices in C when we compute $\langle e_{\pi_B(i_1)} \otimes \cdots \otimes e_{\pi_B(i_n)}, e_{\pi_C(j_1)} \otimes \cdots \otimes e_{\pi_C(j_n)}\rangle$ using Proposition 3.9. For every fixed B, pairing all the possible C with B and the corresponding C^c with B^c gives all the bipartite partitions of $\underline{i} \sqcup \underline{j}$. Using Proposition 3.9 again, we see that

$$\langle P_{F_m}\alpha_t(x), P_{F_m}\alpha_t(y)\rangle = (1 - e^{-2t})^m e^{-2t(n-m)} \sum_{B \subset \{1,\dots,n\}, |B|=m} \langle x, y\rangle.$$

By linearity, this identity holds for arbitrary $x, y \in H^n$. Hence,

$$\langle P_{k-1}\alpha_t(x), P_{k-1}\alpha_t(y)\rangle = \sum_{m=k}^{n} (1 - e^{-2t})^m e^{-2t(n-m)} \binom{n}{m} \langle x, y\rangle.$$

Since the Ornstein–Uhlenbeck semigroup T_t is self-adjoint on $L_2(\Gamma_Q)$ and α_t is trace-preserving, we have, for $x, y \in (\mathbb{C}^N \oplus 0)^{\otimes n}$,

$$\big\langle(\alpha_{t^k} - \mathrm{id})(x), (\alpha_{t^k} - \mathrm{id})(y)\big\rangle = 2\big(\langle x, y\rangle - \langle x, T_{t^k}(y)\rangle\big) = 2(1 - e^{-nt^k})\langle x, y\rangle.$$

The rest of the proof is just numerical estimate, which is provided in the proof of [Avsec 2011, Proposition 5.1]. □

The following is the main result of this section.

Theorem 5.12. *Let Q be a real symmetric $N \times N$ matrix with* $\max_{1\le i,j\le N}|q_{ij}| < 1$ *and $N < \infty$. Then Γ_Q is strongly solid.*

Proof. The proof is the same as that of [Avsec 2011, Theorem B], with the help of Theorem 5.5 and Propositions 5.10 and 5.11. The argument in [Avsec 2011] follows literally the same strategy as that of [Houdayer and Shlyakhtenko 2011, Theorem 3.5], which in turn is a suitable modification of [Ozawa and Popa 2010a; 2010b]. □

Appendix: Speicher's central limit theorem

Proof of Theorem 3.1. The proof is rephrased from [Speicher 1993] and also follows [Junge et al. 2015]. We first show that the convergence holds on average, and then prove almost sure convergence using the Borel–Cantelli lemma. We write

$$\text{(A-1)}\qquad \begin{aligned}\tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m)) &= \frac{1}{m^{s/2}}\sum_{\underline{k}\in[m]^s}\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))\\ &= \frac{1}{m^{s/2}}\sum_{\sigma\in P(s)}\sum_{\substack{\underline{k}\in[m]^s\\ \sigma(\underline{k})=\sigma}}\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))\\ &=: \frac{1}{m^{s/2}}\sum_{\sigma\in P(s)}\Delta_\sigma.\end{aligned}$$

By the commutation/anticommutation relation, $\Delta_\sigma = 0$ if σ contains a singleton. Note $|\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))| \le 1$. If σ has r blocks, then $\Delta_\sigma \le m(m-1)\cdots(m-r+1)$. Hence,

$$\text{(A-2)}\qquad \lim_{m\to\infty}\frac{1}{m^{s/2}}\Delta_\sigma = 0$$

for $r < s/2$ and thus for $\sigma \in P(s)\setminus P_2(s)$ since the singleton case is automatically true. Our argument so far is independent of $\omega \in \Omega$, so that (A-2) holds for all $\omega \in \Omega$. The theorem follows immediately from (A-2) if s is odd. Thus, we only need to consider $\sigma \in P_2(s)$ in (A-1). To this end, we write $\sigma = \{\{e_1, z_1\}, \dots, \{e_{s/2}, z_{s/2}\}\}$. Since $\sigma(\underline{k}) = \sigma$ is a pair partition, if $k_j = k_l$, then $i_j = i_l$ in order for $x_{i_j}(k_j)$ and $x_{i_l}(k_l)$ to cancel out. Hence we may assume $\sigma \le \sigma(\underline{i})$. In this case, if $\{r, t\} \in I(\sigma)$, then we have to switch $x_{i(e_r)}(k(e_r))$ and $x_{i(e_t)}(k(e_t))$ to cancel the corresponding $x_{i(z_r)}(k(z_r))$ and $x_{i(z_t)}(k(z_t))$ terms, which yields

$$\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s)) = \prod_{\{r,t\}\in I(\sigma)} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)]). \tag{A-3}$$

By independence and counting the elements in $\{\underline{k}\in[m]^s \mid \sigma(\underline{k})=\sigma\}$, we find

$$\begin{aligned}\mathbb{E}(\Delta_\sigma) &= \sum_{\substack{\underline{k}\in[m]^s\\ \sigma(\underline{k})=\sigma}} \prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t))\\ &= m(m-1)\cdots(m-s/2+1)\prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t)).\end{aligned}$$

Combining these, we have

$$\lim_{m\to\infty}\mathbb{E}\big(\tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m))\big) = \sum_{\substack{\sigma\in P_2(s)\\ \sigma\le\sigma(\underline{i})}}\prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t)). \tag{A-4}$$

It remains to prove the almost sure convergence. Put $X_m=\tau_m(\widetilde{x}_{i_1}(m)\cdots\widetilde{x}_{i_s}(m))$ and $E_m(\alpha)=\{\omega : |X_m-\mathbb{E}X_m|\ge\alpha\}$. Then we only need to show $\mathbb{P}(\limsup_m E_m(\alpha))=0$. By the Borel–Cantelli lemma and Chebyshev's inequality, it suffices to show that

$$\sum_{m=1}^\infty \mathbb{P}(E_m(\alpha)) \le \frac{1}{\alpha^2}\sum_{m=1}^\infty \operatorname{Var}(X_m) < \infty \quad \text{for any } \alpha>0,$$

where $\operatorname{Var}(X_m)$ is the variance of X_m. Decompose X_m as $X_m=Y_m+Z_m$, where Y_m corresponds to sum over all pair partitions in (A-1) and $Z_m=X_m-Y_m$. Since (A-2) holds for $\sigma\in P(s)\setminus P_2(s)$, we have $\lim_{m\to\infty} X_m-Y_m=0$ for all $\omega\in\Omega$. But Z_m is uniformly bounded, so $\lim_{m\to\infty}\operatorname{Var}(Z_m)=0$. Therefore, we only need to show that $\sum_{m=1}^\infty \operatorname{Var}(Y_m)<\infty$. Write

$$\operatorname{Var}(Y_m) = \frac{1}{m^s}\sum_{\sigma,\pi\in P_2(s)}\sum_{\substack{\underline{k}:\sigma(\underline{k})=\sigma\\ \underline{l}:\sigma(\underline{l})=\pi}} V_{\underline{k},\underline{l}},$$

where

$$\begin{aligned} V_{\underline{k},\underline{l}} &= \mathbb{E}[\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))\tau_m(x_{i_1}(l_1)\cdots x_{i_s}(l_s))]\\ &\qquad - \mathbb{E}[\tau_m(x_{i_1}(k_1)\cdots x_{i_s}(k_s))]\mathbb{E}[\tau_m(x_{i_1}(l_1)\cdots x_{i_s}(l_s))]\\ &= \mathbb{E}\bigg(\prod_{\{r,t\}\in I(\sigma)} \varepsilon([i(e_r),k(e_r)],[i(e_t),k(e_t)])\\ &\qquad\times \prod_{\{r',t'\}\in I(\pi)} \varepsilon([i(e_{r'}),l(e_{r'})],[i(e_{t'}),l(e_{t'})])\bigg)\\ &\qquad - \prod_{\{r,t\}\in I(\sigma)} q(i(e_r),i(e_t)) \prod_{\{r',t'\}\in I(\pi)} q(i(e_{r'}),i(e_{t'})).\end{aligned} \tag{A-5}$$

Let us analyze the product in the third and fourth lines of (A-5). If $\{k(e_r), k(e_t)\} \neq \{l(e_{r'}), l(e_{t'})\}$ for all $\{r, t\} \in I(\sigma)$ and $\{r', t'\} \in I(\pi)$, then $V_{\underline{k},\underline{l}} = 0$. In order to contribute for $\mathrm{Var}(Y_m)$, there exists at least one pair $\{r, t\} \in I(\sigma)$ and one pair $\{r', t'\} \in I(\pi)$ such that $\{k(e_r), k(e_t)\} = \{l(e_{r'}), l(e_{t'})\}$. In this case, we have

$$\#\{\underline{k}, \underline{l} : \sigma(\underline{k}) = \sigma, \sigma(\underline{l}) = \pi\} \leq m^{s/2} m^{s/2-2} = m^{s-2}.$$

Note that $|V_{\underline{k},\underline{l}}| \leq 1$ and $C(s) := [\#P_2(s)]^2$ does not depend on m. It follows that

$$\sum_{m=1}^{\infty} \mathrm{Var}(Y_m) \leq \sum_{m=1}^{\infty} \frac{C(s)}{m^2} < \infty. \qquad \square$$

Remark A.1. In the above argument, we assumed that the $\varepsilon((i, k), (j, l))$'s are independent for different indices. However, the independence assumption can be weakened if the structure matrix is of the form $Q \otimes \mathbb{1}_n$, where Q is an $N \times N$ symmetric matrix with entries in $[-1, 1]$. In this case we require that the $\varepsilon((i, k), (j, l))$'s be independent (up to symmetric assumption) with (3-1) for $(i, k), (j, l) \in [N] \times \mathbb{N}$ and then

$$\varepsilon((i + \alpha N, k), (j + \beta N, l)) = \varepsilon((i, k), (j, l)) \tag{A-6}$$

for $\alpha, \beta = 1, \dots, n-1$. In other words, $\varepsilon = \varepsilon|_{1 \leq i, j \leq N} \otimes \mathbb{1}_n$. To verify the claim, we only need to show the dependence introduced in (A-6) will not destroy the proof of Theorem 3.1. Indeed, by (A-2) it suffices to consider pair partitions. Suppose $i_\beta = i_\alpha + N$. Then α and β are not in the same pair block of $\sigma(\underline{i})$. It follows that $k_\alpha \neq k_\beta$ since $\sigma(\underline{k}) \leq \sigma(\underline{i})$. (If $k_\alpha = k_\beta$, then $i_\alpha = i_\beta$ in order for $x_{i_\alpha}(k_\alpha)$ and $x_{i_\beta}(k_\beta)$ to cancel.) Hence, the random signs in (A-3) are pairwise different. Note that unlike the case in the proof of Theorem 3.1, now we may have

$$\varepsilon((i_\alpha, k_\alpha), (i_\gamma, k_\gamma)) \quad \text{and} \quad \varepsilon((i_\beta, k_\beta), (i_\gamma, k_\gamma)) = \varepsilon((i_\alpha, k_\beta), (i_\gamma, k_\gamma))$$

in (A-3), but the two random signs are not equal because $k_\alpha \neq k_\beta$. In other words, the second coordinates (k_α, k_γ) in $\varepsilon((i_\alpha, k_\alpha), (i_\gamma, k_\gamma))$ are never the same for random signs in (A-3) even under the weaker condition (A-6) so that (i_α, i_γ) may be the same for different random signs. The point is that the independence structure in the proof of Theorem 3.1 is given via the second coordinates' k_α's. The rest of the argument is the same as for Theorem 3.1. We invite the interested reader to consider the simplest case $Q = q\mathbb{1}_N$. In this case we can take $\varepsilon((i, k), (j, l)) = \varepsilon((i, k), (i, l))$ and require $\varepsilon((i, k), (i, l))$ to be independent for different k and l up to symmetry.

By this remark, the moment formula (3-2) remains valid with the weaker condition (A-6). The same discussion applies in other parts of the paper when the CLT argument is invoked with (A-6). This subtlety is crucial for our limiting argument in Section 4.

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EQUIVARIANT MIN-MAX HYPERSURFACE IN G-MANIFOLDS WITH POSITIVE RICCI CURVATURE

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We consider a connected orientable closed Riemannian manifold M^{n+1} with positive Ricci curvature. Suppose G is a compact Lie group acting by isometries on M with $3 \leq \operatorname{codim}(G \cdot p) \leq 7$ for all $p \in M$. Then we show the equivariant min-max G-hypersurface Σ corresponding to one-parameter G-sweepouts (of boundary-type) is a multiplicity one minimal G-hypersurface with a G-invariant unit normal and G-equivariant index one. As an application, we are able to establish a genus bound for Σ, a control on the singular points of Σ/G, and an upper bound for the (first) G-width of M provided $n+1=3$ and the actions of G are orientation preserving.

1. Introduction

Given a connected orientable closed Riemannian manifold (M^{n+1}, g_M), minimizing the area within a nontrivial homology class is a natural way to construct minimal hypersurfaces (see [12; 36]). However, if M has positive Ricci curvature, it follows from the stability inequality that this minimization method cannot be applied. In the 1960s, Almgren [1; 2] proposed the *min-max theory* to find minimal submanifolds in the most general situation. Subsequently, the regularity for min-max hypersurfaces was improved by Pitts [30] ($n \leq 5$) and Schoen and Simon [34] ($n = 6$). Indeed, for $n \geq 7$, they showed the min-max minimal hypersurface is smooth embedded except for a singular set of codimension 7.

Due to the generality and abstractness of Almgren–Pitts min-max theory, many of the geometric properties of min-max hypersurfaces have not been understood until recently. For instance, in a closed manifold with positive Ricci curvature, a series of studies were set out to characterize the min-max hypersurfaces generated from one-parameter families. Specifically, using the Heegaard splitting, Marques and Neves [20] studied the index and genus of the min-max surface in certain 3-manifolds. They also obtained sharp estimates for the width and rigidity results. In a higher-dimensional manifold M^{n+1} with positive Ricci curvature, Zhou determined

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the Morse index and multiplicity of the min-max hypersurface for $3 \leq n+1 \leq 7$ in [43] and for $n \geq 7$ in [44]. Subsequently, Ketover, Marques, and Neves [17] refined Zhou's results in dimension $3 \leq n+1 \leq 7$ by showing the orientability of the min-max hypersurface using the *catenoid estimates*. In particular, the min-max hypersurface is an orientable closed minimal hypersurface of Morse index one and has the least area among all orientable closed minimal hypersurfaces. Furthermore, without any curvature assumption, the constructions in [20; 43] were also employed by Mazet and Rosenberg [25] to show the least area minimal hypersurface is either stable or a min-max hypersurface of Morse index one.

Given a 3-manifold M with a finite group G acting by isometries, Pitts and Rubinstein [31; 32] first asserted the existence of a G-invariant minimal surface with estimates on its index and genus. The existence and regularity of minimal G-invariant surfaces (abbreviated as *G-surfaces*) were recently confirmed by Ketover [15] (for finite G of orientation preserving isometries) using the equivariant min-max under the smooth setting. More generally, suppose M^{n+1} is a closed Riemannian manifold with a compact Lie group G acting by isometries so that $3 \leq \operatorname{codim}(G \cdot p) \leq 7$ for all $p \in M$. The equivariant min-max theory was also extended to this general scenario by Liu [19] (for connected G with $\min_{p\in M} \operatorname{codim}(G\cdot p) \neq 0, 2$) in the smooth setting and by Wang [39; 40] in the Almgren–Pitts setting. In particular, Wang [39, Theorem 9] showed an isomorphism between $H_{n+1}(M; \mathbb{Z}_2)$ and $\pi_1(\mathcal{Z}_n^G(M; \mathbb{Z}_2))$, where $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is the space of G-invariant n-cycles (of boundary-type, see geometric measure theory). Then it is similar to the constructions of Almgren–Pitts (see [30]) that the fundamental class $[M] \in H_{n+1}(M; \mathbb{Z}_2)$ corresponds to the (first) *equivariant min-max width* $W^G(M) > 0$ of M defined with one-parameter G-sweepouts (see Definition 2.7 and [30, Corollary 4.7]), which can be realized by the area of some minimal G-invariant hypersurfaces (abbreviated as *G-hypersurfaces*) with multiplicities. Therefore, it now seems reasonable to investigate the geometric features of the equivariant min-max hypersurface, such as its area, multiplicity, index, and topology.

In this paper, our main result generalizes the characterization of the min-max hypersurface into an equivariant version (see Theorem 5.1).

Theorem 1.1. *Let (M^{n+1}, g_M) be a connected orientable closed Riemannian manifold with positive Ricci curvature, and G be a compact Lie group acting by isometries on M so that $3 \leq \operatorname{codim}(G\cdot p) \leq 7$ for all $p \in M$. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M] \in H_{n+1}(M; \mathbb{Z}_2)$ is a multiplicity one minimal G-hypersurface so that*:

(i) *Σ has a G-invariant unit normal vector field.*

(ii) *The equivariant Morse index of* Σ (Definition 4.1) *is one.*

(iii) *Σ has the least area among all closed embedded minimal G-hypersurfaces with G-invariant unit normal vector fields.*

Remark 1.2. We make some remarks about the above theorem:

(i) If M has connected components $\{M_i\}_{i=1}^m$, then we can take a component M_i and the Lie subgroup $G_i := \{g \in G : g \cdot M_i = M_i\}$. By applying the above theorem to M_i and G_i, we obtain a minimal G_i-invariant hypersurface Σ_i of multiplicity one. Additionally, one easily verifies that $G \cdot \Sigma_i \subset G \cdot M_i$ is a minimal G-hypersurface satisfying (i)–(iii) in Theorem 1.1 with $G \cdot M_i$ in place of M.

(ii) Without the positive Ricci curvature assumption, we can combine the proof of Theorem 1.1 and the constructions in [25] to show the existence of a minimal G-hypersurface of the least area (counted with multiplicity) among all minimal G-hypersurfaces. The details will be discussed in an upcoming paper.

Equivariant vs nonequivariant. Note that Theorem 1.1 is an equivariant generalization of the results in [17; 43] where $G = \{id\}$. Nevertheless, due to the equivariant constraints, the equivariant min-max hypersurface exhibits slightly stronger properties (e.g., the unit normal not only exists but also it is G-invariant). Additionally, it should be noted that the equivariant constraints generally have a significant impact on the min-max outcomes. Indeed, if we denote by $W(M) = W^{\{id\}}(M)$ (resp. $W^G(M)$) and Σ (resp. Σ_G) the first (resp. equivariant) min-max width and the corresponding first (resp. equivariant) min-max hypersurface, then we generally have $W(M) \le W^G(M)$ without the equality. Moreover, even if Σ is G-invariant and $W(M) = W^G(M)$, Σ may *not* necessarily be the equivariant min-max hypersurface corresponding to $W^G(M)$. One can easily observe these phenomena from the following examples.

Example 1.3 ($W(M) \le W^G(M)$ without equality). Let $M = \mathbb{S}^3$ be the unit sphere with the standard round metric. Then $W(\mathbb{S}^3) = 4\pi$ is realized by the area of the equator $\Sigma = \mathbb{S}^2$ [29]. Next, take $G = \mathbb{Z}_2$ acting on $\mathbb{S}^3$ by the antipodal map so that $\pi : M = \mathbb{S}^3 \to M/G = \mathbb{RP}^3$ is a (locally isometric) double cover. Hence, $\pi(\Sigma_G)$ is the first min-max hypersurface in $\mathbb{RP}^3$ corresponding to $W(\mathbb{RP}^3)$. Therefore, although Σ is G-invariant, it can *not* be Σ_G, because $\pi(\Sigma) = \mathbb{RP}^2$ is 1-sided, while $\pi(\Sigma_G) \subset \mathbb{RP}^3$ must be 2-sided [17; 43]. Indeed, it follows from [3] that $W^{\mathbb{Z}_2}(\mathbb{S}^3) = 2W(\mathbb{RP}^3) = 2\pi^2$ is realized by the area of the Clifford torus.

Example 1.4 ($\Sigma \neq \Sigma_G$ even if $W = W^G$). Let $M = \mathbb{S}^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$, and $G = \mathbb{Z}_2$ act by the reflection $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$. Then we have $M/G = \mathbb{S}^3_+ = \{x \in \mathbb{S}^3 : x_1 \geq 0\}$. Note the $\mathbb{Z}_2$-equivariant minimal hypersurfaces and $\mathbb{Z}_2$-sweepouts in $\mathbb{S}^3$ correspond one-to-one to the free boundary minimal hypersurfaces and (relative) sweepouts in $\mathbb{S}^3_+$. Thus, Σ_G/G is the first min-max free boundary minimal hypersurface in $\mathbb{S}^3_+$ corresponding to $W(\mathbb{S}^3_+)$. Therefore, $W^{\mathbb{Z}_2}(\mathbb{S}^3) = 2W(\mathbb{S}^3_+) = 4\pi = W(\mathbb{S}^3)$ realized by the area of a great 2-sphere $\Sigma_G = \mathbb{S}^2$ perpendicular to $\{x_1 = 0\}$. (As an example, take the $\mathbb{Z}_2$-sweepout

$\{\mathbb{S}^3 \cap \{x_2 = t\}\}_{t\in[-1,1]}$.) Meanwhile, we notice $\Sigma = \mathbb{S}^3 \cap \{x_1 = 0\}$ is G-invariant and is also a min-max hypersurface corresponding to $W(\mathbb{S}^3)$. However, since $\Sigma/G = \mathbb{S}^2$ is not the free boundary min-max hypersurface corresponding to $W(\mathbb{S}^3_+)$, we have $\Sigma \neq \Sigma_G$ in this case.

One should also notice that in the above examples, Σ_G admits a G-invariant unit normal, while the unit normal of Σ is not G-invariant. Intuitively, this is because our equivariant sweepouts are formed by the boundaries of G-invariant (Caccioppoli) sets admitting a (measure-theoretic) inward G-invariant unit normal. Hence, if a G-invariant min-max hypersurface Σ does not have a G-invariant unit normal, then Σ cannot be a boundary of a G-invariant (Caccioppoli) set, and the min-max sequence $|\partial\Omega_{t_i}|$ must converge to Σ with even multiplicities (Theorem 3.8) so that the constructions in [17; 43] can be generalized to derive Theorem 1.1.

Remark 1.5. To ensure $W^G(M)$ is well defined for any M and G, we only use the boundaries of G-invariant (Caccioppoli) sets in the equivariant min-max constructions in this paper. Note, for some specific choices of M, G, one may construct the equivariant min-max using "G-hypersurfaces without G-invariant unit normal", and Theorem 1.1(i) may fail in this case (see, e.g., [16]). Similarly, the results in [43] may not be applicable for nonboundary-type min-max constructions (without equivariance).

Further discussions and applications. We will now delve deeper into some inspirations and potential applications of Theorem 1.1.

Firstly, one notices that the existence of a *G-invariant* unit normal can help to distinguish the min-max G-hypersurface Σ and the fixed points set under certain $\mathbb{Z}_2$ actions. For instance, consider a positive Ricci curvature 3-ellipsoid M with its major axis (on x_1) sufficiently long and the other principal axes bounded by 2. Then the classical min-max theory shall provide the equator $\Gamma = \{x_1 = 0\} \cap M$ on the major axis as the min-max hypersurface. Although Γ is also invariant under the $\mathbb{Z}_2$-reflections $(x_1, x') \mapsto (-x_1, x')$, it cannot be the min-max $\mathbb{Z}_2$-hypersurface since its unit normal is not $\mathbb{Z}_2$-invariant. An interesting question is what exactly is the min-max $\mathbb{Z}_2$-hypersurface in this case, and how does it relate to the 2-min-max minimal hypersurfaces?

In addition, we see that the characterizations of the Morse index and multiplicity for min-max hypersurfaces are crucial in the study of min-max theory. For instance, a key part in the proof of the Willmore conjecture by Marques and Neves [21] is to show the minimal surface in $\mathbb{S}^3$ constructed by the five-parameter families of min-max has Morse index 5. Additionally, by specifying generically the multiplicity [45] and index [22; 24] of min-max hypersurfaces, the multiparameter min-max theory was used to establish the Morse theory for the area functional. In the equivariant case, Wang [41] also proved general upper bounds for the G-index (Definition 4.1) of equivariant min-max hypersurfaces generated from multiparameter families. Therefore, in light of Theorem 1.1 and Zhou [45], we conjecture that for a generic G-invariant Riemannian metric, the minimal G-hypersurface constructed from k-parameter families of equivariant min-max shall have multiplicity one, G-index k, and a G-invariant unit normal.

Moreover, it has been discovered in numerous studies that the Morse index of a minimal surface is related to its topology. For instance, in a closed 3-manifold with positive Ricci curvature, Choi and Schoen [8] proved the area of a closed minimal surface can be bounded by its genus. Therefore, by Ejiri and Micallef [11, Theorem 4.3], the index of a such minimal surface is also bounded by its genus. Additionally, using the *conformal volume*, Yau (see [35, Chapter VIII, Section 4]) obtained a genus bound for index one minimal surfaces in positive Ricci curvature manifolds. More generally, in an orientable 3-manifold with nonnegative Ricci curvature, it follows from the sharp estimate of Ros [33, Theorem 15] that a closed orientable minimal surface of index one must have genus ≤ 3. Recently, Song [37] showed that the total Betti number of a closed minimal hypersurface in M^{n+1}, $3 \leq n+1 \leq 7$, can be bounded by its index and a constant depending only on n, g_M, and its area, which further indicates a quantified relation [37, Corollary 3] between the genus and index of a minimal surface in M^3. For a complete two-sided minimal surface in $\mathbb{R}^3$, Chodosh and Maximo [6] showed that its genus and the number of ends give a lower bound on its index. We refer to [7; 26] for more related research.

Hence, as an application, we use the conformal volume initiated by Li and Yau [18] in the orbit space to show a general genus bound of the equivariant min-max surface in a 3-manifold with positive Ricci curvature, which further indicates an upper bound of the G-width and a bound for the singular points of Σ/G (see Theorem 5.2).

Theorem 1.6. *Let (M^3, g_M) be a closed connected oriented Riemannian 3-manifold with positive Ricci curvature, and G be a finite group acting on M by orientation preserving isometries. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M]$ is a connected minimal G-hypersurface of multiplicity one with*

$$\operatorname{genus}(\Sigma) \leq 4K, \quad W^G(M) = \operatorname{Area}(\Sigma) \leq \frac{8\pi K}{c_M},$$

where $K := \max_{p\in M} \#G\cdot p \leq \#G$ is the number of points in a principal orbit of M, and $\operatorname{Ric}_M \geq c_M > 0$. Additionally, the quotient space $\pi(\Sigma) = \Sigma/G$ is an orientable surface with finite cone singular points of order $\{n_i\}_{i=1}^k$ so that

$$\sum_{i=1}^{k}\Big(1-\frac{1}{n_i}\Big) < 4 \quad \textit{and} \quad \operatorname{genus}(\pi(\Sigma)) \leq 3.$$

In particular, if Σ/G has no singularity, then $\operatorname{genus}(\Sigma) \leq 1 + 2K$.

Remark 1.7. To generalize Li–Yau's [18] theory to the orbit spaces, G-actions in Theorem 1.6 are assumed to be orientation preserving isometries so that M/G and Σ/G induce orientable orbifolds without boundary.

The conformal method has been employed in many studies for the *volume spectrum*, i.e., the multiparameter version of width. For the first width $W(M)$ in the volume spectrum, Glynn-Adey and Liokumovich [13] gave an upper bound using the min-conformal volume of the ambient manifold. In particular, if M is a closed surface, they showed the first width $W(M)$ can be bounded by the genus and area of M. Also, the conformal upper bounds for the volume spectrum were proved in [38].

Main ideas and outline. The main idea for Theorem 1.1 is as follows. For the closed manifold M and the Lie group G in Theorem 1.1, we can take any closed embedded minimal G-hypersurface Σ in M and use the variation of its first eigenvector field to foliate a G-neighborhood of Σ. Using a half-space version of the equivariant min-max theory (Theorem 3.11), we argue by contradiction to show this local G-equivariant foliation can be extended to a continuous G-sweepout of M with mass no more than Area(Σ) (if Σ has a G-invariant unit normal) or $2\,\text{Area}(\Sigma)$. Therefore, it follows from the equivariant min-max theory [39, Theorem 8] (see also [40, Theorem 4.20]) that the equivariant min-max hypersurface is the minimal G-hypersurface of *least area* in the sense of (5-1). Additionally, if the equivariant min-max hypersurface does not admit a G-invariant unit normal, it must have even multiplicity by the constructions of equivariant min-max (Theorem 3.8). However, in this case, we can further use the catenoid estimates of Ketover et al. [17] to add small G-invariant cylinders in the G-sweepouts (Proposition 4.7), which will strictly decrease the mass and give a contradiction.

The above idea shares the same spirit as in [43]. However, since the equivariant min-max theory was already established in a continuous version [39, Theorem 8], we do not need to invoke the smooth setting of min-max (see [10]) as in [43, Section 2], but give a more self-contained equivariant min-max construction in half spaces (Theorem 3.11). Meanwhile, instead of using the discretization theorem as in [43, Theorem 5.8], we can more easily determine that the extension of the G-equivariant foliation is a G-sweepout.

The article is organized as follows. In Section 2, we collect some notations and definitions of Lie group actions and geometric measure theory. In particular, we introduce the G-equivariant sweepouts and G-width of M in a continuous version using the isomorphic map between $\pi_1(\mathcal{Z}_n^G(M;\mathbb{Z}_2))$ and $H_{n+1}(M;\mathbb{Z}_2)$. Then we introduce in Section 3 the equivariant min-max theory developed by Wang [39; 40] under the Almgren–Pitts setting with some modifications. In Section 4, we will generate a continuous G-sweepout with good properties from a given minimal G-hypersurface. The proof of the main theorem and its applications are given in Section 5.

2. Preliminary

Let $(M^{n+1}, g_{_M})$ be an orientable connected compact Riemannian $(n+1)$-manifold and G be a compact Lie group acting isometrically on M. Denote by μ a biinvariant Haar measure on G normalized to $\mu(G)=1$. For the case that $\partial M \neq \varnothing$, it follows from [40, Lemma A.1] that M can be equivariantly and isometrically extended to a closed Riemannian manifold $(N, g_{_N})$ with G acting on N by isometries. Therefore, we can assume M is a compact domain of a closed Riemannian G-manifold N.

Note that although our main results only involve closed minimal G-hypersurfaces in closed G-manifolds, we also need a half-space version of equivariant min-max to insert any closed embedded minimal G-hypersurface into a good G-sweepout (see main ideas and outline). Hence, we also include some terminologies and results in this paper concerning G-equivariant min-max in compact G-manifolds with nonempty boundary.

Lie group actions. To begin with, we gather some definitions of Lie group actions, most of which are referred from [4; 5].

It follows from [28] that there is an orthogonal representation $\rho : G \to O(L)$ and an isometric embedding $i : M \hookrightarrow \mathbb{R}^L$ for some $L \in \mathbb{N}$ so that i is equivariant, i.e., $i \circ g = \rho(g) \circ i$. For simplicity, we regard M as a subset of $\mathbb{R}^L$ and denote the orthogonal action of $g \in G$ on $x \in \mathbb{R}^L$ as $g \cdot x$. We say a subset (hypersurface) $A \subset M$ is a G-subset (G-hypersurface) if $g \cdot A = A$ for all $g \in G$.

For any $p \in M$, let $G \cdot p := \{g \cdot p : g \in G\}$ be the orbit containing p and $G_p := \{g \in G : g \cdot p = p\}$ be the isotropy group of p. Note $G \cdot p$ is a closed submanifold of M and G_p is a Lie subgroup of G. We then say p has (G_p) *orbit-type*, where (G_p) is the conjugacy class of G_p in G. By [4, Proposition 2.2.4], there is a (unique) minimal conjugacy class (P) of isotropy groups so that $M^{\text{prin}} = M_{(P)} := \{p \in M : (G_p) = (P)\}$ is an open dense G-subset of M. We call any $G \cdot p \subset M^{\text{prin}}$ a *principal orbit* of M and denote by $\operatorname{Cohom}(G)$ the codimension of a principal orbit, which is known as the *cohomogeneity* of the actions of G.

Let M/G be the quotient space, i.e., the *orbit space*, and π be the projection $\pi : M \to M/G,\ p \mapsto [p]$. It is well known that M/G is a Hausdorff metric space with induced metric $\operatorname{dist}_{M/G}([p],[q]) := \operatorname{dist}_M(G \cdot p, G \cdot q)$.

Denote by $B_r(p)$, $B_r([p])$, and $\mathbb{B}^k_r(p)$ the geodesic ball in M (or in N if $\partial M \neq \varnothing$), the metric ball in M/G, and the Euclidean ball in $\mathbb{R}^k$ respectively. Then we use the following notations:

- $\mathfrak{X}(M)$, $\mathfrak{X}(U)$: the space of smooth vector fields compact supported in M or $U \subset M$.
- $\mathfrak{X}^G(M)$, $\mathfrak{X}^G(U)$: the space of G-*vector fields* X in M or U, ($g_* X = X$ for all $g \in G$).

- $B^G_\rho(p)$: the open geodesic tube with radius ρ around the orbit $G\cdot p$ in M (or in N if $\partial M \neq \varnothing$).
- $\mathrm{An}^G(p,s,t)$: the open tube $B^G_t(p)\setminus \bar{B}^G_s(p)$.

For any closed G-hypersurface $\Sigma\subset M$, denote by $\boldsymbol{N}\Sigma$ its normal bundle with G acting on it by $g\cdot v := g_* v$ for all $g\in G$, $v\in \boldsymbol{N}\Sigma$. Let $\exp^\perp_\Sigma : \boldsymbol{N}\Sigma\to M$ be the normal exponential map of Σ. Note $\exp^\perp_\Sigma$ is a G-equivariant diffeomorphism in a neighborhood of Σ.

Geometric measure theory. We refer to [12; 30; 36] for the following definitions:

- $\boldsymbol{I}_k(M;\mathbb{Z}_2)$: the space of k-dimensional mod 2 flat chains in $\mathbb{R}^L$ with support contained in M.
- $\mathcal{Z}_n(M;\mathbb{Z}_2)$: the space of $T\in \boldsymbol{I}_n(M;\mathbb{Z}_2)$ with $T=\partial U$ for $U\in \boldsymbol{I}_{n+1}(M;\mathbb{Z}_2)$, i.e., the boundary-type mod 2 n-cycles.
- $\mathcal{V}_k(M)$: the weak topological closure of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in M.

Let $\mathcal{F}$ and $\boldsymbol{M}$ be the *flat* (*semi*)*norm* and the *mass* norm in $\boldsymbol{I}_k(M;\mathbb{Z}_2)$ [12, 4.2.26]. Define the $\boldsymbol{F}$*-metric* on $\mathcal{V}_k(M)$ as in [30, p. 66]. Then $\boldsymbol{F}$ induces the weak topology on any mass bounded subset $\{V\in\mathcal{V}_k(M) : \|V\|(M)\le C\}$, where $C>0$ and $\|V\|$ is the Radon measure on M induced by V.

For any $T\in \boldsymbol{I}_k(M;\mathbb{Z}_2)$, we denote $|T|$ and $\|T\|$ as the integral varifold and the Radon measure induced by T. Then we define the $\boldsymbol{F}$*-metric* on $\boldsymbol{I}_k(M;\mathbb{Z}_2)$ by

$$\boldsymbol{F}(S,T) := \mathcal{F}(S-T)+\boldsymbol{F}(|S|,|T|)\quad \text{for all } S,T\in \boldsymbol{I}_k(M;\mathbb{Z}_2).$$

It follows from [30, p. 68] that for any T, $\{T_i\}_{i\in\mathbb{N}}\subset \mathcal{Z}_n(M;\mathbb{Z}_2)$,

$$\text{(2-1)}\quad \lim_{i\to\infty}\boldsymbol{F}(T_i,T)=0 \quad\Leftrightarrow\quad \lim_{i\to\infty}\mathcal{F}(T_i,T)=0 \ \text{ and } \ \lim_{i\to\infty}\boldsymbol{M}(T_i)=\boldsymbol{M}(T).$$

For $\boldsymbol{v}\in\{\boldsymbol{M},\boldsymbol{F},\mathcal{F}\}$, let $\boldsymbol{I}_k(M;\boldsymbol{v};\mathbb{Z}_2)$ and $\mathcal{Z}_n(M;\boldsymbol{v};\mathbb{Z}_2)$ be the spaces with topology induced by $\boldsymbol{v}$. Additionally, we denote by $[\![\Gamma]\!]$ the element in $\boldsymbol{I}_k(M;\mathbb{Z}_2)$ induced by a k-submanifold $\Gamma\subset M$.

We say $T\in \boldsymbol{I}_k(M;\mathbb{Z}_2)$ (or $V\in\mathcal{V}_k(M)$) is G*-invariant* if $g_\# T=T$ ($g_\# V=V$) for all $g\in G$. Then we have the following subspaces of G-invariant elements:

- $\boldsymbol{I}^G_k(M;\mathbb{Z}_2) := \{T\in \boldsymbol{I}_k(M;\mathbb{Z}_2) : g_\# T = T \text{ for all } g\in G\}$.
- $\mathcal{Z}^G_n(M;\mathbb{Z}_2) := \{T\in \mathcal{Z}_n(M;\mathbb{Z}_2) : T=\partial U \text{ for some } U\in \boldsymbol{I}^G_{n+1}(M;\mathbb{Z}_2)\}$.
- $\mathcal{V}^G_k(M) := \{V\in\mathcal{V}_k(M) : g_\# V = V \text{ for all } g\in G\}$.

Remark 2.1. Note $\mathcal{Z}^G_n(M;\mathbb{Z}_2)\subsetneq\{T\in\mathcal{Z}_n(M;\mathbb{Z}_2) : g_\# T=T \text{ for all } g\in G\}$ in general, and intuitively, $T\in\mathcal{Z}^G_n(M;\mathbb{Z}_2)$ is not only a boundary that is G-invariant but also "bounds a G-invariant region". This is essential to derive Theorem 1.1(i) as explained in Remark 1.5.

Since G acts by isometries, $\boldsymbol{I}_k^G(M;\mathbb{Z}_2)$, $\mathcal{Z}_n^G(M;\mathbb{Z}_2)$, and $\mathcal{V}_k^G(M)$ are closed subspaces with induced metrics $\boldsymbol{M}$, $\mathcal{F}$, $\boldsymbol{F}$. Moreover, we have the following isoperimetric lemma (see [39, Lemma 5]), which is also valid when $\partial M \neq \varnothing$.

Lemma 2.2. *There are $\epsilon_M > 0$, $C_M > 1$ such that for any $T_1, T_2 \in \boldsymbol{I}_n^G(M;\mathbb{Z}_2)$ with $\partial T_1 = \partial T_2 = 0$, and*

$$\mathcal{F}(T_1 - T_2) < \epsilon_M,$$

there is a unique $Q \in \boldsymbol{I}_{n+1}^G(M;\mathbb{Z}_2)$, called ***the isoperimetric choice of T_1, T_2****, satisfying*

(i) $\partial Q = T_1 - T_2$,

(ii) $\boldsymbol{M}(Q) \leq C_M \cdot \mathcal{F}(T_1 - T_2)$.

For any $V \in \mathcal{V}_n(M)$ and $X \in \mathfrak{X}(M)$, the first variation of V along X is given by

$$\delta V(X) := \frac{d}{dt}\Big|_{t=0} \|(F_t)_\# V\|(M) = \int_{G_n(M)} \operatorname{div}_S(X)(p)\, dV(p,S),$$

where $\{F_t\}$ are the diffeomorphisms generated by X, and $G_n(M)$ is the Grassmannian bundle of unoriented n-planes over M. Suppose $V \in \mathcal{V}_n^G(M)$ is G-invariant and $U \subset M$ is an open G-subset, then we say:

- V is *stationary* in U if $\delta V(X) = 0$ for all $X \in \mathfrak{X}(U)$.
- V is *G-stationary* in U if $\delta V(X) = 0$ for all $X \in \mathfrak{X}^G(U)$.

Clearly, a stationary G-varifold must be G-stationary. Meanwhile, let

$$X_G := \int_G (g^{-1})_* X \, d\mu(g) \quad \text{for all } X \in \mathfrak{X}(U). \tag{2-2}$$

A direct computation shows $X_G \in \mathfrak{X}^G(U)$ and $\delta V(X) = \delta V(X_G)$ for any $V \in \mathcal{V}_n^G(M)$ (see [19, Lemma 2.2]). Hence, we have:

(2-3) $\quad V \in \mathcal{V}_n^G(M)$ is stationary in U if and only if it is G-stationary in U.

G-Sweepouts and G-width. To define the equivariant sweepouts and width, we need to introduce a technical assumption:

Definition 2.3. For any $\mathcal{F}$-continuous map $\Phi : [0,1] \to \mathcal{Z}_n^G(M;\mathbb{Z}_2)$, define

$$\boldsymbol{m}^G(\Phi, r) := \sup\{\|\Phi(x)\|(B_r^G(p)) : x \in [0,1],\ p \in M\},$$

where $B_r^G(p)$ is the geodesic r-neighborhood of $G \cdot p$ in M (or in N if $M \subset N$ has nonempty boundary). Then we say Φ has *no concentration of mass on orbits* if $\lim_{r\to 0} \boldsymbol{m}^G(\Phi, r) = 0$.

By (2-1) and a continuous argument, we have the following lemma (see [39, Lemma 8]), which is quite useful in Section 3.

Lemma 2.4. *If* $\Phi : [0, 1] \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ *is* $\boldsymbol{F}$*-continuous, then* Φ *has no concentration of mass on orbits and* $\sup_{x\in[0,1]} \boldsymbol{M}(\Phi(x)) < \infty$.

Closed G*-manifolds.* In this case, $\partial M = \varnothing$. Then for any $\mathcal{F}$-continuous closed curve $\Phi : [0, 1] \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, $\Phi(0) = \Phi(1)$, we can take $a_j = j/3^k$, $j = 0, 1, \dots, 3^k$ with $k \in \mathbb{N}$ large enough so that

$$\mathcal{F}(\Phi(x) - \Phi(y)) \le \epsilon_M \quad \text{for all } x, y \in [a_j, a_{j+1}], \tag{2-4}$$

where $\epsilon_M > 0$ is given by Lemma 2.2. By Lemma 2.2, there is $Q_j \in \boldsymbol{I}_{n+1}^G(M; \mathbb{Z}_2)$ with $\partial Q_j = \Phi(a_{j+1}) - \Phi(a_j)$ and $\boldsymbol{M}(Q_j) \le C_M \mathcal{F}(\Phi(a_{j+1}) - \Phi(a_j))$, where $j = 0, 1, \dots, 3^k - 1$. Therefore, $Q := \sum_{j=0}^{3^k-1} Q_j \in \boldsymbol{I}_{n+1}^G(M; \mathbb{Z}_2)$ satisfies $\partial Q = 0$, which indicates $Q = [\![M]\!]$ or 0 by the constancy theorem [36, 26.27]. Hence, we can correspond Φ to a homology class:

$$F_M(\Phi) := [Q] \in H_{n+1}(M^{n+1}; \mathbb{Z}_2). \tag{2-5}$$

By the constancy theorem, $F_M(\Phi)$ does not depend on the choice of k. Moreover, by [39, Remark 2] and the arguments in [1], we have $F_M(\Phi) = F_M(\Phi')$ for any closed curve Φ' that is homotopic to Φ in $\mathcal{Z}_n^G(M; \mathcal{F}; \mathbb{Z}_2)$, and F_M induces an isomorphism [39, Theorem 9]:

$$F_M : \pi_1(\mathcal{Z}_n^G(M; \mathbb{Z}_2)) \to H_{n+1}(M; \mathbb{Z}_2).$$

In the above, we do not need to specify the base point of $\pi_1(\mathcal{Z}_n^G(M; \mathbb{Z}_2))$. This is because $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is the $\mathcal{F}$-path connected component of $\boldsymbol{I}_n^G(M; \mathbb{Z}_2) \cap \mathcal{Z}_n(M; \mathbb{Z}_2)$ containing 0 (by Lemma 2.2 and the contraction approach in [24, Claim 5.3]).

Definition 2.5 (G-sweepout). A closed $\mathcal{F}$-continuous curve $\Phi : S^1 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is said to be a *G-sweepout of M* if $F_M(\Phi) = [M] \ne 0$.

Remark 2.6. Since $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is $\mathcal{F}$-path connected, every two G-sweepouts are homotopic to each other in $\mathcal{Z}_n^G(M; \mathcal{F}; \mathbb{Z}_2)$. Hence, the set of G-sweepouts of M is exactly the nontrivial homotopy class of closed curves in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$.

Next, we introduce the min-max G-width of M, which can be regarded as a critical value for the area functional with respect to *all* variations by (2-3).

Definition 2.7 (G-width). Let $\mathcal{P}^G(M)$ be the set of G-sweepouts of M with no concentration of mass on orbits. Then we define the *G-width* of M by

$$W^G(M) := \inf_{\Phi \in \mathcal{P}^G(M)} \sup_{x \in S^1} \boldsymbol{M}(\Phi(x)).$$

Compact G-manifolds with boundary. Now we consider the case that $\partial M \neq \varnothing$, and regard M as a compact domain of a closed Riemannian G-manifold N. Let F_N be given by (2-5), and $\nu_{\partial M}$ be the unit normal of ∂M pointing inward M. Then for $\eta > 0$ small enough, define

$$M_\eta := M \setminus \exp^\perp_{\partial M}([0,\eta)\cdot \nu_{\partial M}) = \{p \in M : \mathrm{dist}_M(p, \partial M) \geq \eta\}. \tag{2-6}$$

Let $\Phi_i : [0,1] \to \mathcal{Z}^G_n(M;\mathbb{Z}_2)$, $i = 1, 2$, be two $\mathcal{F}$-continuous curve so that $\Phi_i(0) = [\![\partial M]\!]$ and $\Phi_i(1) = 0$. As the constructions in (2-4), we can associate Φ_i to $Q_i \in \boldsymbol{I}^G_{n+1}(M;\mathbb{Z}_2)$ with $\partial Q_i = [\![\partial M]\!]$. Then the constancy theorem implies $Q_i = [\![M]\!]$. Therefore, the curves product, i.e., joint curve, $\Phi_2^{-1}\cdot\Phi_1$ satisfies $F_N(\Phi_2^{-1}\cdot\Phi_1) = 0$, and thus $\Phi_2^{-1}\cdot\Phi_1$ is homotopic to 0 in $\mathcal{Z}^G_n(N;\mathcal{F};\mathbb{Z}_2)$. Since $\mathrm{spt}(\Phi_i(x)) \subset M$ for all $x \in [0,1]$ and $i = 1, 2$, we can apply the double cover argument in [24, Theorem 5.1] with Lemma 2.2 in place of [1, Corollary 1.14], and see the homotopy map between $\Phi_2^{-1}\cdot\Phi_1$ and 0 can be taken in $\mathcal{Z}^G_n(M;\mathcal{F};\mathbb{Z}_2)$. Thus, Φ_1 and Φ_2 are homotopic to each other in $\mathcal{Z}^G_n(M;\mathcal{F};\mathbb{Z}_2)$.

Next, we introduce the following definition for G-manifold with boundary, which is generalized from the smooth min-max setting [43, Definitions 2.1, 2.5].

Definition 2.8. Suppose M is a compact Riemannian G-manifold with boundary $\partial M \neq \varnothing$. Then we call a $\mathcal{F}$-continuous curve $\Phi : [0,1] \to \mathcal{Z}^G_n(M;\mathbb{Z}_2)$ a *G-sweepout of* $(M, \partial M)$, if:

(i) $\Phi(0) = [\![\partial M]\!]$, $\Phi(1) = 0$.

(ii) There exist $\epsilon > 0$ and a smooth G-invariant function $w : [0,\epsilon] \times \partial M \to [0,\infty)$ with $w(0,\cdot) \equiv 0$ and $\frac{\partial}{\partial x} w(0,\cdot) > 0$, so that $\Phi(x)$, $x \in [0,\epsilon]$, is induced by the smooth G-hypersurface $\exp^\perp_{\partial M}(w(x,\cdot)\,\nu_{\partial M})$.

(iii) For any $x_0 \in (0,1]$, there exists $\eta > 0$ so that $\mathrm{spt}(\Phi(x)) \Subset M_\eta$ for all $x \in [x_0, 1]$.

Denote by $\mathcal{P}^G(M, \partial M)$ the set of G-sweepouts of $(M, \partial M)$ with no concentration of mass on orbits. Then we define the *G-width of* $(M, \partial M)$ by

$$W^G(M, \partial M) := \inf_{\Phi \in \mathcal{P}^G(M,\partial M)} \sup_{x\in[0,1]} \boldsymbol{M}(\Phi(x)).$$

Remark 2.9. As we mentioned before, any two G-sweepouts Φ_1, Φ_2 of $(M, \partial M)$ are homotopic to each other in $\mathcal{Z}^G_n(M;\mathcal{F};\mathbb{Z}_2)$. Moreover, by reparametrization, the foliation parts of Φ_i, $i = 1, 2$, are homotopic through $v_t := (1-t)\,w_1 + t w_2$, where $t \in [0,1]$ and $w_1, w_2 : \left[0, \frac{1}{3}\right] \times \partial M \to [0,\infty)$ are given by Definition 2.8(ii). The nonfoliation parts $\Phi_i \llcorner \left[\frac{1}{3}, 1\right]$ and $\exp^\perp_{\partial M}\left(v_t\left(\frac{1}{3},\cdot\right)\nu_{\partial M}\right)$ are all in M_η for some $\eta > 0$, and thus the homotopy between these parts can be taken in $\mathcal{Z}^G_n(M_\eta;\mathcal{F};\mathbb{Z}_2)$ (see the constructions in [24, Theorem 5.1] with Lemma 2.2). Therefore, we can take a homotopy map $H : [0,1]\times[0,1] \to \mathcal{Z}^G_n(M;\mathcal{F};\mathbb{Z}_2)$ so that $H(0,\cdot) = \Phi_1$, $H(1,\cdot) = \Phi_2$, and for every $t \in [0,1]$, $H(t,\cdot)$ is a G-sweepout of $(M, \partial M)$.

3. Equivariant min-max theory

In this section, we introduce the equivariant min-max constructions in [39] (see [40; 41] for modified versions). Then main purpose is to find an integral G-varifold $V \in \mathcal{V}_n^G(M)$ induced by a smooth embedded minimal G-hypersurface so that $\|V\|(M) = W^G(M)$ (or $W^G(M, \partial M)$ if $\partial M \neq \varnothing$). Since our definitions differ slightly from those in [40; 41], we shall outline the essential steps for the sake of completeness.

Throughout this section, let $\mathcal{P}^G = \mathcal{P}^G(M)$ or $\mathcal{P}^G(M, \partial M)$, $W^G = W^G(M)$ or $W^G(M, \partial M)$ depending on whether ∂M is empty. By reparametrization, we always assume the domain of $\Phi \in \mathcal{P}^G$ is $I = [0, 1]$, and if $\partial M \neq \varnothing$, then $\Phi \llcorner \big[0, \frac{1}{3}\big]$ are smooth G-hypersurfaces as in Definition 2.8(ii).

For any sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G$, define the *width* of $\{\Phi_i\}_{i\in\mathbb{N}}$ by

$$\boldsymbol{L}(\{\Phi_i\}_{i\in\mathbb{N}}) := \limsup_{i\to\infty} \sup_{x\in I} \boldsymbol{M}(\Phi_i(x)).$$

Then we say $\{\Phi_i\}_{i\in\mathbb{N}}$ is a *min-max* sequence if

$$\boldsymbol{L}(\{\Phi_i\}_{i\in\mathbb{N}}) = W^G.$$

The *image set* of $\{\Phi_i\}_{i\in\mathbb{N}}$ is defined by

$$\boldsymbol{\Lambda}(\{\Phi_i\}_{i\in\mathbb{N}}) := \Big\{ V \in \mathcal{V}_n^G(M) : V = \lim_{j\to\infty} |\Phi_{i_j}(x_{i_j})| \text{ for some } i_j \to \infty,\ x_{i_j} \in I \Big\}.$$

Moreover, we define the *critical set* of $\{\Phi_i\}_{i\in\mathbb{N}}$ by

$$\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) := \{V \in \boldsymbol{\Lambda}(\{\Phi_i\}_{i\in\mathbb{N}}) : \|V\|(M) = \boldsymbol{L}(\{\Phi_i\}_{i\in\mathbb{N}})\}.$$

Discrete min-max settings. To apply the equivariant min-max constructions in [39; 40], we need the following discrete notations. Since we only consider curves in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$, we will restrict the notations to the 1-parameter case.

Denote by $I := [0, 1]$. For any $j \in \mathbb{N}$, let $I(1, j)$ be the cube complex on I with 1-cells and 0-cells (vertices) given by

$$I(1, j)_1 := \{[0, 3^{-j}], [3^{-j}, 2\cdot 3^{-j}], \dots, [1 - 3^{-j}, 1]\},$$
$$I(1, j)_0 := \{[0], [3^{-j}], \dots, [1]\}.$$

The boundary homeomorphism ∂ is defined by $\partial[a, b] = [b] - [a]$. Then we denote by $I(2, j) = I(1, j) \otimes I(1, j)$ the cell complex on $I^2 = I \times I$. For any $\alpha = \alpha_1 \otimes \alpha_2 \in I(2, j)$ and $p \in \{0, 1, 2\}$, we say α is a *p-cell*, if $\dim(\alpha_1) + \dim(\alpha_2) = p$. Then the set of p-cells of $I(2, j)$ is denoted by $I(2, j)_p$, and the set of p-cells in $\alpha \in I(i, j)_q$ is denoted by α_p.

Let $J := \left[\frac{1}{3}, 1\right]$. Then we denote by $J(1, j)$ the cubical subcomplex containing all the cells of $I(1, j)$ supported in J. Similarly, the set of p-cells of $J(1, j)$ is denoted by $J(1, j)_p$ for $p \in \{0, 1, 2\}$.

Let $m \in \{1, 2\}$ and two vertices $x, y \in I(m, j)_0$, define $\boldsymbol{d}(x, y) := 3^j \sum_{i=1}^m |x_i - y_i|$. For any map $\phi : I(1, j)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, the *$\boldsymbol{M}$-fineness* of ϕ is defined by

$$\boldsymbol{f_M}(\phi) := \sup\{\boldsymbol{M}(\phi(x) - \phi(y)) : \boldsymbol{d}(x, y) = 1, \ x, y \in I(1, j)_0\}.$$

Suppose $S = \{\varphi_i\}_{i \in \mathbb{N}}$ is a sequence of maps $\varphi_i : I(1, k_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ such that $k_i \to \infty$ and $\boldsymbol{f_M}(\varphi_i) \to 0$ as $i \to \infty$. Then we use the following notations:

$$\boldsymbol{L}(S) := \limsup_{i \to \infty} \max_{x \in I(1, k_i)_0} \boldsymbol{M}(\varphi_i(x)),$$

$$\boldsymbol{\Lambda}(S) := \left\{ V \in \mathcal{V}_n^G(M) : V = \lim_{j \to \infty} |\varphi_{i_j}(x_{i_j})| \text{ for some } i_j \to \infty, \ x_{i_j} \in I(1, k_i)_0 \right\},$$

$$\boldsymbol{C}(S) := \{V \in \boldsymbol{\Lambda}(S) : \|V\|(M) = \boldsymbol{L}(S)\}.$$

For any $i, j \in \mathbb{N}$, let $\boldsymbol{n}(i, j) : I(1, i)_0 \to I(1, j)_0$ be the nearest projection, i.e.,

$$\boldsymbol{d}(x, \boldsymbol{n}(i, j)(x)) = \inf\{\boldsymbol{d}(x, y) : y \in I(m, j)_0\}.$$

Then we define the discrete homotopy:

Definition 3.1. Given $\phi_i : I(1, k_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, $i = 1, 2$, we say ϕ_1 and ϕ_2 are 1-*homotopic in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness* δ if there exists a map

$$\psi : I(1, k)_0 \times I(1, k)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$$

for some $k \geq \max\{k_1, k_2\}$ such that $\boldsymbol{f_M}(\psi) < \delta$ and $\psi([i-1], x) = \phi_i(\boldsymbol{n}(k, k_i)(x))$ for $i \in \{1, 2\}$ and $x \in I(1, k)_0$.

Definition 3.2. A sequence of mappings $S = \{\phi_i\}_{i \in \mathbb{N}}$, $\phi_i : I(1, k_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, is said to be a

$$(1, \boldsymbol{M})\textit{-homotopy sequence of mappings into } \mathcal{Z}_n^G(M; \mathbb{Z}_2)$$

if ϕ_i and ϕ_{i+1} are 1-homotopic in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness δ_i such that

(i) $\lim_{i \to \infty} \delta_i = 0$,

(ii) $\sup\{\boldsymbol{M}(\phi_i(x)) : x \in I(1, k_i)_0, \ i \in \mathbb{N}\} < +\infty$.

Definition 3.3. Let $S^j = \{\phi_i^j\}_{i \in \mathbb{N}}$, $j = 1, 2$, be two $(1, \boldsymbol{M})$-homotopy sequences of mappings into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$. Then S^1 and S^2 are *homotopic in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$* if there exists a sequence $\{\delta_i\}_{i \in \mathbb{N}}$ such that

(i) ϕ_i^1 is 1-homotopic to ϕ_i^2 in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness δ_i,

(ii) $\lim_{i \to \infty} \delta_i = 0$.

By the following discretization theorem from [39, Theorem 2], we can generate a $(1, \boldsymbol{M})$-homotopy sequence of mappings into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ from any $\Phi \in \mathcal{P}^G$.

Theorem 3.4 (discretization theorem). *Let $\Phi : I \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ be a continuous map in the flat topology so that $\sup_{x\in I} \boldsymbol{M}(\Phi(x)) < \infty$ and Φ has no concentration of mass on orbits. Then there exists a sequence of maps*

$$\phi_i : I(1, j_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2),$$

with $j_i < j_{i+1}$, and a sequence $\{\delta_i > 0\}_{i\in\mathbb{N}}$ converging to zero such that:

(i) *$S = \{\phi_i\}_{i\in\mathbb{N}}$ is a $(1, \boldsymbol{M})$-homotopy sequence of mappings into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness $\boldsymbol{f_M}(\phi_i) < \delta_i$.*

(ii) *There exists some sequence $k_i \to +\infty$ such that for all $x \in I(1, j_i)_0$,*

$$\boldsymbol{M}(\phi_i(x)) \le \sup\{\boldsymbol{M}(\Phi(y)) : \alpha \in I(1, k_i)_1, x, y \in \alpha\} + \delta_i,$$

which implies $\boldsymbol{L}(S) \le \sup_{x\in I} \boldsymbol{M}(\Phi(x))$.

(iii) $\sup\{\mathcal{F}(\phi_i(x) - \Phi(x)) : x \in I(1, j_i)_0\} \le \delta_i$.

(iv) $\Phi(0) = \phi_i([0]) = \psi_i(\,\cdot\,, [0])$ *and* $\Phi(1) = \phi_i([1]) = \psi(\,\cdot\,, [1])$, *where ψ_i is the discrete homotopy map of ϕ_i and ϕ_{i+1} with $\psi_i([0], \boldsymbol{n}(\,\cdot\,)) = \phi_i$ and $\psi_i([1], \boldsymbol{n}(\,\cdot\,)) = \phi_{i+1}$.*

Moreover, let $K \subset M$ be a compact G-invariant domain with smooth boundary. Then for any $j \in \mathbb{N}$ and $\alpha \in I(1, j)_1$, if $\operatorname{spt}(\Phi(x)) \subset K$ *for all $x \in \alpha$, then we can further make* $\operatorname{spt}(\phi_i(x)) \subset K$ *for all $x \in \alpha \cap I(1, j_i)_0$.*

Proof. The statements in (i)–(iii) follow directly from [39, Theorem 2]. Note that the proof of [39, Theorem 2] is basically the combinatorial approach in [21, Theorem 13.1] with Lemma 2.2 in place of [1, Corollary 1.14] and $\operatorname{dist}(G \cdot p, \cdot)$ in place of $\operatorname{dist}(p, \cdot)$. Meanwhile, since the maps are defined on the 1-dimensional cubical complex, statement (iv) follows from [21, Proposition 13.5(ii)] and the combinatorial constructions of [21, Theorem 13.1(iv)]. Moreover, these cut-and-paste and combinatorial arguments would also carry over in the case $\partial M \neq \varnothing$ by restricting in the compact domain $M \subset N$, and thus (i)–(iv) are still valid when M has boundary. Finally, if K and $\alpha \in I(1, j)$ are given as in the last statement. Then we can apply the above discretization result to $\Phi \llcorner \alpha$ in K and $\Phi \llcorner (I \setminus \operatorname{int}(\alpha))$ in M respectively. Note the boundary values are unchanged by (iv). Hence, the discrete maps defined in α and $I \setminus \operatorname{int}(\alpha)$ can be connected together, which gives the last statement. □

The following interpolation theorem (see [39, Theorem 3]) indicates that a $\boldsymbol{M}$-continuous map into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ can be generated from a discrete map with small $\boldsymbol{M}$-fineness.

Theorem 3.5 (interpolation theorem). *For $m = 1, 2$, there exists a positive constant $C_0 = C_0(M, G, m)$ so that if $\phi : I(m,k)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ has $\boldsymbol{f_M}(\phi) < \epsilon_M$ with $\epsilon_M > 0$ given in Lemma 2.2, then there exists a map*

$$\Phi : I^m \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$$

continuous in the $\boldsymbol{M}$-topology satisfying:

(i) $\Phi(x) = \phi(x)$ *for all* $x \in I(m,k)_0$.

(ii) *If α is some j-cell in $I(m,k)$, then Φ restricted to α depends only on the values of ϕ assumed on the vertices of α.*

(iii) $\sup\{\boldsymbol{M}(\Phi(x) - \Phi(y)) : x, y$ *lie in a common cell of* $I(m,k)\} \le C_0\, \boldsymbol{f_M}(\phi)$.

(iv) *For any $\alpha \in I(m,k)_j$, if $\phi \llcorner \alpha_0 \equiv T \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is a constant, then $\Phi \llcorner \alpha \equiv T$.*

We call the map Φ in Theorem 3.5 the *Almgren G-extension* of ϕ.

Proof. The statements in (i)–(iii) follow directly from [39, Theorem 3]. If $\partial M \neq \varnothing$, then the constructions in [40, Theorem 4.13] would carry over with $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ and Lemma 2.2 in place of $\mathcal{Z}_n^G(M, \partial M; \mathbb{Z}_2)$ and [40, Lemma 3.10]. If $\phi \llcorner \alpha_0 \equiv T \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is a constant for some j-cell α, then for any 1-cell $\gamma_1 = [a, b] \in \alpha_1$, the isoperimetric choice $Q(\gamma_1)$ of $\phi(a)$ and $\phi(b)$ (Lemma 2.2) must be 0. Hence, for any cell $\beta \subset \alpha$, the map h_β constructed in [40, Theorem 4.13] is 0 implying $\Phi \llcorner \alpha \equiv T$ [1, 4.5]. □

Using the discretization/interpolation Theorems 3.4 and 3.5, we have the following corollary (see [39, Corollary 1]):

Corollary 3.6. *Let $\Phi : I \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ be a $\mathcal{F}$-continuous map with no concentration of mass on orbits and* $\sup_{x \in I} \boldsymbol{M}(\Phi(x)) < \infty$. *Suppose $S = \{\phi_i\}_{i \in \mathbb{N}}$ is given by* Theorem 3.4 *applied to Φ, and Φ_i is the Almgren G-extension of ϕ_i given by* Theorem 3.5 *for every i sufficiently large. Then*:

(i) *For each i large enough, a* ***relative*** *homotopy map $H_i : I^2 \to \mathcal{Z}_n^G(M; \mathcal{F}; \mathbb{Z}_2)$ exists with $H_i(0, \cdot) = \Phi$, $H_i(1, \cdot) = \Phi_i$, $H_i(\cdot, 0) \equiv \Phi(0) = \Phi_i(0)$, and $H_i(\cdot, 1) \equiv \Phi(1) = \Phi_i(1)$.*

(ii) $\boldsymbol{L}(\{\Phi_i\}_{i \in \mathbb{N}}) = \boldsymbol{L}(S) \le \sup_{x \in I} \boldsymbol{M}(\Phi(x))$.

Proof. Using Theorem 3.5 and the arguments in [1], we see that [1, Theorem 8.2] is valid in our G-invariant settings (even if ∂M may be nonempty). Hence, the proof of [23, Corollary 3.9] would carry over with Theorems 3.4 and 3.5 in place of [23, Theorem 3.6]. Thus, Φ_i is homotopic to Φ in $\mathcal{Z}_n^G(M; \mathcal{F}; \mathbb{Z}_2)$ for i-large, and (ii) is valid. Also, by (iv) in Theorems 3.4 and 3.5, we have $\Phi(0) = \phi_i([0]) = \Phi_i(0)$ and $\Phi(1) = \phi_i([1]) = \Phi_i(1)$ for all i-large. So, combining (iv) in Theorems 3.4 and 3.5 with the homotopy constructions in [23, Propositions 3.3, 3.8], one easily verifies that the homotopy map H_i of Φ and Φ_i is relative to the boundary values. □

Let $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G$ be any min-max sequence. If $\partial M = \varnothing$, then we can apply Corollary 3.6 to each Φ_i and obtain a sequence of $\boldsymbol{M}$-continuous curves $\{\Phi^i_j\}_{j\in\mathbb{N}}$ relative homotopic to Φ_i in $\mathcal{Z}^G_n(M;\mathcal{F};\mathbb{Z}_2)$ and $\boldsymbol{L}(\{\Phi^i_j\}_{j\in\mathbb{N}}) \le \sup_{x\in I}\boldsymbol{M}(\Phi_i(x))$. Choose $j(i)$ sufficiently large so that $\sup_{x\in I}\boldsymbol{M}(\Phi^i_{j(i)}(x)) \le \sup_{x\in I}\boldsymbol{M}(\Phi_i(x)) + \frac{1}{i}$. Hence, we have $\{\Phi^i_{j(i)}\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M)$ is a min-max sequence continuous in the $\boldsymbol{M}$-topology and so in the $\boldsymbol{F}$-topology.

For the case $\partial M \neq \varnothing$, we can apply the above arguments to each $\Phi_i \llcorner J$, where $J := \big[\frac{1}{3}, 1\big]$, in a G-submanifold M_{η_i} given by Definition 2.8(iii) with $x_0 = \frac{1}{3}$, and get $\Phi^i_{j(i)} : J \to \mathcal{Z}^G_n(M_{\eta_i};\boldsymbol{M};\mathbb{Z}_2)$ satisfying

- $\Phi^i_{j(i)}$ is relative homotopic to $\Phi_i \llcorner J$ in $\mathcal{Z}^G_n(M_{\eta_i};\mathcal{F};\mathbb{Z}_2)$,
- $\sup_{x\in J}\boldsymbol{M}(\Phi^i_{j(i)}(x)) \le \sup_{x\in J}\boldsymbol{M}(\Phi_i(x)) + \frac{1}{i}$.

Since the homotopy map of $\Phi^i_{j(i)}$ and $\Phi_i \llcorner J$ is relative to the boundary values, we can define $\Phi^i_{j(i)} \llcorner \big[0, \frac{1}{3}\big] = \Phi_i \llcorner \big[0, \frac{1}{3}\big]$, and see that $\{\Phi^i_{j(i)}\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ is an $\boldsymbol{F}$-continuous min-max sequence.

Therefore, the above arguments give the following corollary, which implies we only need to consider the $\boldsymbol{F}$-continuous G-sweepouts.

Corollary 3.7. *The G-width defined in Definitions 2.7 and 2.8 satisfies*

$$W^G = \inf\left\{\sup_{x\in I}\boldsymbol{M}(\Phi(x)) : \Phi \in \mathcal{P}^G \textit{ is } \boldsymbol{F}\textit{-continuous}\right\}.$$

Min-max theorems. We now use the min-max method to construct a minimal G-hypersurface (with multiplicity) so that the width W^G is realized by its area.

Closed G-manifolds. For the case that M is closed, it follows from Remark 2.6 and Corollary 3.7 that $\boldsymbol{\Pi} := \{\Phi \in \mathcal{P}^G(M) \text{ is } \boldsymbol{F}\text{-continuous}\}$ is a *continuous G-homotopy class* in the sense of [39, Definition 5], and $W^G(M) = \boldsymbol{L}(\boldsymbol{\Pi})$ in the sense of [39, Definition 6]. Hence, we have the following min-max theorem by [39, Theorem 8]. (Note that the assumptions on $M \setminus M^{\text{prin}}$ in [39, Theorem 8] can be removed by the modifications in [40], and the dimension assumption is modified in [41, Theorem 5.1].)

Theorem 3.8. *Suppose M is closed, i.e., $\partial M = \varnothing$, and $3 \le \operatorname{codim}(G\cdot p) \le 7$ for all $p \in M$. Then there exists an integral G-varifold $V \in \mathcal{V}^G_n(M)$ so that*

$$\|V\|(M) = W^G(M) \quad \text{and} \quad V = \sum_{i=1}^m n_i|\Sigma_i|,$$

where $m, n_i \in \mathbb{N}$, $\{\Sigma_i\}_{i=1}^m$ are disjoint G-connected (Definition 4.4) *smooth embedded closed minimal G-hypersurfaces. Moreover, if Σ_i does not admit a G-invariant unit normal vector field, then n_i is an even number.*

Proof. We only need to show the last statement since the existence and regularity of V are given by [39, Theorem 8] (see also [41, Theorem 5.1]). Note that the min-max varifold V is $(G, \mathbb{Z}_2)$-almost minimizing in annuli of *boundary-type* in the sense of [39, Definitions 10, 11]. Hence, for each Σ_i, we can take a small G-tube $B^G_{2r}(p)$ with center $G \cdot p \subset \Sigma_i$ and $r \in \big(0, \frac{1}{2}\mathrm{inj}(G \cdot p)\big)$ so that:

- V is $(G, \mathbb{Z}_2)$-almost minimizing of boundary-type in $B^G_{2r}(p)$.
- $B^G_t(p)$ has mean convex boundary for all $t \in (0, 2r)$.
- $B^G_{2r}(p) \cap \mathrm{spt}(\|V\|) \subset \Sigma_i$, and $\partial B^G_r(p)$ is transversal to Σ_i.

Then by the constructions [39, Proposition 2, 3] and the consistency [41, Proposition 4.19] of G-replacements, there exists a sequence $\{T_j\}_{j\in\mathbb{N}} \subset \mathcal{Z}^G_n(M; \mathbb{Z}_2)$ so that:

(1) $T_j = \partial Q_j$ is locally mass minimizing in $B^G_r(p)$ with $Q_j \in \boldsymbol{I}^G_{n+1}(M; \mathbb{Z}_2)$.

(2) $|T_j| \to V$ in the sense of varifolds.

By compactness, let $T_j \to T = \partial Q$ in the flat topology with $Q \in \boldsymbol{I}^G_{n+1}(M; \mathbb{Z}_2)$. Thus, we have $\mathrm{spt}(T) \subset \mathrm{spt}(\|V\|) = \cup_{i=1}^m \Sigma_i$, which implies $T = \sum_{i=1}^m n'_i [\![\Sigma_i]\!]$ for some $n'_i \in \mathbb{Z}_2$ by the Constancy Theorem. By regarding $Q \in \boldsymbol{I}^G_{n+1}(M; \mathbb{Z}_2)$ as a G-invariant Caccioppoli set whose boundary is induced by smooth G-hypersurfaces $\{\Sigma_i : 1 \leq i \leq m, n'_i = 1\}$, we see ∂Q admits an inward unit normal that is also G-invariant. Hence, $n'_i = 0$ provided that Σ_i does not admit a G-invariant unit normal. Now we can use the slicing theory [36, 28.5] to take $s \in \big(\frac{r}{2}, r\big)$ so that $\boldsymbol{M}\big(\partial(T_j \llcorner B^G_s(p))\big)$ are uniformly bounded, and thus $T_j \llcorner B^G_s(p)$ converges up to a subsequence. Finally, by (1) we know [42, Theorem 1.1] indicates that $n_i \equiv n'_i$ mod 2, and thus the multiplicity n_i must be even for Σ_i without a G-invariant unit normal. □

Compact G-manifolds with boundary. Now we consider the case that $\partial M \neq \varnothing$. In this case, we make the assumption that

$$H_{\partial M} > 0 \quad \text{and} \quad W^G(M, \partial M) > \mathrm{Area}(\partial M), \tag{3-1}$$

where $H_{\partial M}$ is the mean curvature of ∂M with respect to the inward unit normal $\nu_{\partial M}$. By Corollary 3.7, we can take a min-max sequence $\{\Phi^*_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ that are continuous in the $\boldsymbol{F}$-topology. The strategy is to use the following proposition to deform $\{\Phi^*_i\}_{i\in\mathbb{N}}$ into a new $\boldsymbol{F}$-continuous min-max sequence so that every $V \in \boldsymbol{C}(\{\Phi^*_i\}_{i\in\mathbb{N}})$ is supported in a G-invariant subdomain $M_a \Subset M$. With this benefit, the min-max constructions can be restricted in the interior of M to build a closed minimal G-hypersurface realizing the width $W^G(M, \partial M)$. This deformation approach is based on the idea of [20, Lemma 2.2] and we list the details here for the sake of completeness.

Proposition 3.9. *Let $\partial M \neq \varnothing$ satisfy* (3-1). *Then there exist a constant $a > 0$ and a min-max sequence $\{\Phi_i^*\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ continuous in the $\boldsymbol{F}$-topology so that*

$$\operatorname{spt}(\Phi_i^*(x)) \Subset M_a := \{p \in M : \operatorname{dist}_M(p, \partial M) \geq a\} \quad \text{for any } x \in I,$$

with $\boldsymbol{M}(\Phi_i^*(x)) \geq W^G(M, \partial M) - \delta$ and $\delta = \frac{1}{4}(W^G(M, \partial M) - \operatorname{Area}(\partial M))$.

Proof. Let $a > 0$ be small enough so that $d := \operatorname{dist}_M(\partial M, \cdot\,)$ is a G-invariant smooth function in a $4a$-neighborhood of ∂M. By (3-1), we can set $a > 0$ even smaller so that for any $r \in [0, 3a]$, $\partial M_r = d^{-1}(r)$ has positive mean curvature H_r with respect to the inner unit normal ∇d. Denote by A_r the second fundamental form of ∂M_r, and $c = \sup_{r\in[0,3a],\, p\in\partial M_r} |A_r|(p)$. Then we take the function $\phi \geq 0$ as in [20, Lemma 2.2] so that

$$\phi' + c\phi \leq 0, \qquad \phi(r) > 0 \quad \text{for } r < 2a, \qquad \phi(r) = 0 \quad \text{for } r \geq 2a.$$

For any $p \in \operatorname{int}(M) \setminus M_{3a}$ and n-subspace $S \subset T_p M$, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of S, and $P : T_p M \to T_p \partial M_{d(p)}$ be the projection. Since we have that $\dim(S \cap T_p \partial M_{d(p)}) \geq n-1$, we can assume $\{e_i\}_{i=1}^{n-1} \cup \{e^*\}$ gives an orthonormal basis of $T_p \partial M_{d(p)}$, where e^* satisfies $\langle e^*, P(e_n)\rangle = |P(e_n)|$. Noting $\nabla d \perp T_p \partial M_{d(p)}$ and $\nabla_{\nabla d} \nabla d = 0$, we have

$$\begin{aligned}
\operatorname{div}_S(\phi\nabla d) &= \phi'(d(p)) \cdot \langle e_n, \nabla d\rangle^2 + \phi(d(p)) \cdot \sum_{i=1}^n \langle \nabla_{e_i} \nabla d, e_i\rangle \\
&= \phi' \langle e_n, \nabla d\rangle^2 - \phi \sum_{i=1}^n A_{d(p)}(P(e_i), P(e_i)) \\
&= (\phi' + \phi A_{d(p)}(e^*, e^*))\langle e_n, \nabla d\rangle^2 - \phi H_{d(p)} \\
&\leq (\phi' + c\phi)\langle e_n, \nabla d\rangle^2 - \phi H_{d(p)} \\
&\leq 0.
\end{aligned} \tag{3-2}$$

We can take any $\boldsymbol{F}$-continuous min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ by Corollary 3.7. Then for each Φ_i, there exist $\epsilon_i > 0$ and $\eta_i \in \left(0, \frac{a}{8}\right)$ so that:

(1) $\Phi_i \llcorner [0, 4\epsilon_i]$ are smooth G-hypersurfaces with $\boldsymbol{M}(\Phi_i(x)) \leq \operatorname{Area}(\partial M) + \delta$ for all $x \in [0, 4\epsilon_i]$.

(2) $\operatorname{spt}(\Phi_i(x)) \Subset M_{2\eta_i}$ for all $x \in [\epsilon_i, 1]$.

Let κ_i be a cut-off function so that $\kappa_i(r) = 0$ for $r \leq \eta_i$ and $\kappa_i(r) = 1$ for $r \geq 2\eta_i$. Then the G-vector field $X_i := \kappa_i(d)\,\phi(d)\nabla d$ generates G-equivariant diffeomorphisms $\{F_t^i\}$. By (2) and (3-2), for any $x \in [\epsilon_i, 1]$ and $t_0 \geq 0$, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=t_0} \boldsymbol{M}((F_t^i)_\# \Phi_i(x)) &= \frac{d}{dt}\Big|_{t=0} \|(F_t^i)_\#(F_{t_0}^i)_\# \Phi_i(x)\|(M) \\
&= \int \operatorname{div}_S(X_i)\, dV_{t_0,x} = \int \operatorname{div}_S(\phi\nabla d)\, dV_{t_0,x} \leq 0,
\end{aligned}$$

where $V_{t_0,x} := |(F^i_{t_0})_\# \Phi_i(x)| \in \mathcal{V}^G_n(M_{2\eta_i})$. Therefore,

$$\boldsymbol{M}((F^i_t)_\# \Phi_i(x)) \le \boldsymbol{M}(\Phi_i(x)) \quad \text{for all } x \in [\epsilon_i, 1],\ t \ge 0. \tag{3-3}$$

Since $M_{2\eta_i} \setminus M_{2a} \subset \operatorname{spt}(X_i) \subset M_{\eta_i} \setminus \operatorname{int}(M_{2a})$, we see $\lim_{t\to\infty} F^i_t(p) \in \partial M_{2a}$ for any $p \in M_{2\eta_i} \setminus M_{2a}$, and thus $F^i_{T_i}(M_{2\eta_i}) \subset M_a$ for some $T_i > 0$. Choose a smooth function $h_i : [0, 1] \to [0, T_i]$ with $h_i \llcorner [0, \epsilon_i] = 0$, $h_i \llcorner [2\epsilon_i, 1] = T_i$. Then $\Phi^*_i(x) := (F^i_{h_i(x)})_\# \Phi_i(x)$ satisfies:

(a) $\Phi^*_i(x) = \Phi_i(x)$ for $x \in [0, \epsilon_i]$ (since $h_i = 0$).

(b) $\boldsymbol{M}(\Phi^*_i(x)) \le \boldsymbol{M}(\Phi_i(x))$ for all $x \in [\epsilon_i, 1]$ (by (3-3)).

(c) $\operatorname{spt}(\Phi^*_i(x)) \Subset M_a$ for all $x \in [2\epsilon_i, 1]$ (by (2) and the definitions of T_i, h_i).

Clearly, $\{\Phi^*_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ is also an $\boldsymbol{F}$-continuous min-max sequence. Additionally, if $\boldsymbol{M}(\Phi^*_i(x)) \ge W^G(M, \partial M) - \delta \ge \operatorname{Area}(\partial M) + \delta$, then $x \in (4\epsilon_i, 1]$ by (1), (a) and (b), and thus $\operatorname{spt}(\Phi^*_i(x)) \Subset M_a$ by (c). □

Next, we use the pull-tight argument to make every $V \in \boldsymbol{C}(\{\Phi^*_i\}_{i\in\mathbb{N}})$ stationary in M. By Proposition 3.9, the pull-tight procedure can be restricted in $\operatorname{int}(M_a)$.

Proposition 3.10. *Suppose that $\partial M \ne \varnothing$ satisfies the inequalities* (3-1) *and that $\delta := \frac{1}{4}(W^G(M, \partial M) - \operatorname{Area}(\partial M))$. Suppose $a > 0$ and $\{\Phi^*_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ are given by Proposition 3.9. Then there is an $\boldsymbol{F}$-continuous min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ with*:

(i) $\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \boldsymbol{C}(\{\Phi^*_i\}_{i\in\mathbb{N}}) \cap \mathcal{V}^G_n(M_a)$.

(ii) *Every G-varifold $V \in \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ is stationary in M.*

(iii) *If $\boldsymbol{M}(\Phi_i(x)) \ge W^G(M, \partial M) - \delta$, then* $\operatorname{spt}(\Phi_i(x)) \Subset M_{a/2}$.

Proof. Let $C := \sup_{i\in\mathbb{N}} \sup_{x\in I} \boldsymbol{M}(\Phi^*_i(x)) < \infty$ and $\mathring{M}_{a/2} := \operatorname{int}(M_{a/2})$ be a G-invariant open set of M. Define then $A := \{V \in \mathcal{V}^G_n(M) : \|V\|(M) \le C\}$ and

$$A_0 := \{V \in A : V \text{ is stationary in } \mathring{M}_{a/2}\}.$$

Since G acts by isometries, A and A_0 are compact subset of $\mathcal{V}^G_n(M)$. Additionally, for any $V \in A$, it follows from (2-2) that $V \in A_0$ if and only if $\delta V(X) = 0$ for all $X \in \mathfrak{X}^G(\mathring{M}_{a/2})$. Hence, we can follow [21, p. 765] (or [30, p. 153]) with $\mathfrak{X}^G(\mathring{M}_{a/2})$ in place of $\mathfrak{X}(M)$ to define a continuous map $X : A \to \mathfrak{X}^G(\mathring{M}_{a/2})$ and a continuous function $\eta : A \to [0, 1]$ satisfying:

- $X(V) = 0$ and $\eta(V) = 0$ if $V \in A_0$.
- $\delta V(X(V)) < 0$ and $\eta(V) > 0$ if $V \in A \setminus A_0$.
- $\|(f^{X(V)}_t)_\# V\|(M) < \|(f^{X(V)}_s)_\# V\|(M)$ for all $V \in A$ and $0 \le s < t \le \eta(V)$,

where $\{f_t^{X(V)}\}$ are the equivariant diffeomorphisms generated by $X(V)$. Define

$$H : I \times \{T \in \mathcal{Z}_n^G(M; \boldsymbol{F}; \mathbb{Z}_2) : \boldsymbol{M}(T) \le C\} \to \{T \in \mathcal{Z}_n^G(M; \boldsymbol{F}; \mathbb{Z}_2) : \boldsymbol{M}(T) \le C\},$$
$$H(t, T) := (f_{\eta(|T|)t}^{X(|T|)})_\# T.$$

One easily verifies $H(0, T) = T$ for all $T \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}(T) \le C$, and that:

- If $|T|$ is stationary in $\mathring{M}_{a/2}$, then $H(t, T) = T$ for all $t \in [0, 1]$.
- If $|T|$ is not stationary in $\mathring{M}_{a/2}$, then $\boldsymbol{M}(H(1, T)) < \boldsymbol{M}(T)$.

Let $\Phi_i := H(1, \Phi_i^*)$. Note $X(V)$ is supported in $\mathring{M}_{a/2}$ and $f_t^{X(V)} \llcorner (M \setminus \mathring{M}_{a/2}) = id$. Hence, Φ_i is also a G-sweepout of $(M, \partial M)$. Additionally, by the above constructions, one easily verifies that $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M; \partial M)$ is a min-max sequence continuous in the $\boldsymbol{F}$-topology, and $\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \boldsymbol{C}(\{\Phi_i^*\}_{i\in\mathbb{N}}) \cap A_0$. Moreover, it follows from Proposition 3.9 that $\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \mathcal{V}_n^G(M_a) \cap A_0$, which implies every $V \in \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ is stationary in M. Finally, since the deformations $f_t^{X(V)}$ are restricted in $\mathring{M}_{a/2}$, the last bullet follows directly from Proposition 3.9 and the above constructions. □

Finally, we can now show the equivariant min-max theorem for compact G-manifold M with boundary ∂M satisfying (3-1). The proof is generally the approach in [22, Theorem 3.8], and we list some necessary modifications.

Theorem 3.11. *Suppose $\partial M \neq \varnothing$ satisfies inequality* (3-1), *and* $3 \le \operatorname{codim}(G\cdot p) \le 7$ *for all $p \in M$. Then there exists an integral G-varifold $V \in \mathcal{V}_n^G(M)$ so that $\|V\|(M) = W^G(M, \partial M)$ and $V = \sum_{i=1}^m n_i|\Sigma_i|$, where $m, n_i \in \mathbb{N}$, $\{\Sigma_i\}_{i=1}^m$ are disjoint smooth embedded closed minimal G-hypersurfaces in the interior of M.*

Proof. Let $a > 0$ and $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ be given by Proposition 3.10 so that every $V \in \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ is stationary in M and compactly supported in $\operatorname{int}(M_{a_0})$ for $a_0 = \frac{a}{2}$. Let $\delta = \frac{1}{4}(W^G(M, \partial M) - \operatorname{Area}(\partial M)) > 0$. Then by reparametrization, we assume $\Phi_i \llcorner \big[0, \frac{1}{3}\big]$ foliates a neighborhood of ∂M so that

$$\boldsymbol{M}(\Phi_i(x)) \le \operatorname{Area}(\partial M) + \delta = W^G(M, \partial M) - 3\delta \quad \text{for all } x \in \big[0, \tfrac{1}{3}\big]. \tag{3-4}$$

Recall that $J = \big[\frac{1}{3}, 1\big]$. Denote by

$$\Phi_i' := \Phi_i \llcorner J.$$

By Definition 2.8, there exists $\eta_i \in (0, a_0)$ satisfying $\operatorname{spt}(\Phi_i'(x)) \Subset M_{\eta_i}$ for all $x \in J$. Additionally, since the map $x \mapsto \boldsymbol{M}(\Phi_i'(x))$ is continuous (by (2-1)), we can take $k_i \in \mathbb{N}$ large enough so that $|\boldsymbol{M}(\Phi_i'(x)) - \boldsymbol{M}(\Phi_i'(y))| \le \frac{\delta}{4}$ for all x, y in a common 1-cell of $J(1, k_i)$. Denote by U_i the union of 1-cells $\alpha \in J(1, k_i)_1$ with

$\boldsymbol{M}(\Phi'_i(x)) \leq W^G(M, \partial M) - \frac{3\delta}{4}$ for all $x \in \alpha$, and $V_i := J \setminus U_i$. Therefore, by Proposition 3.10(iii), we have

$$\boldsymbol{M}(\Phi'_i(x)) \geq W^G(M, \partial M) - \delta \quad \text{and} \quad \operatorname{spt}(\Phi'_i(x)) \subset M_{a_0} \qquad \text{for all } x \in V_i.$$

By Lemma 2.4, we can apply Theorem 3.4 to each Φ'_i in the G-submanifold M_{η_i} and obtain a sequence of maps $\phi^i_j : J(1, k^i_j)_0 \to \mathcal{Z}^G_n(M_{\eta_i}; \mathbb{Z}_2)$ with $k^i_j < k^i_{j+1}$, $j \in \mathbb{N}$. The last statement in Theorem 3.4 indicates $\{\phi^i_j\}_{j\in\mathbb{N}}$ can be chosen to satisfy $\operatorname{spt}(\phi^i_j(x)) \subset M_{a_0}$ for all $x \in V_i \cap J(1, k^i_j)_0$. Moreover, we claim that:

Claim 1. *For j large enough*, $\operatorname{spt}(\phi^i_j(x)) \subset M_{a_0}$ *if* $\boldsymbol{M}(\phi^i_j(x)) \geq W^G(M, \partial M) - \frac{\delta}{2}$.

Proof of Claim 1. By the continuity of $x \mapsto \boldsymbol{M}(\Phi'_i(x))$ and Theorem 3.4(ii), if $\boldsymbol{M}(\phi^i_j(x)) \geq W^G(M, \partial M) - \frac{\delta}{2}$, then we have $\boldsymbol{M}(\Phi'_i(x)) > W^G(M, \partial M) - \frac{3\delta}{4}$ for j large enough. Thus, such vertex x must be in V_i, so $\operatorname{spt}(\phi^i_j(x)) \subset M_{a_0}$. □

Additionally, we also have the following equality due to the lower semicontinuity of mass, the continuity of $x \mapsto \boldsymbol{M}(\Phi'_i(x))$ and Theorem 3.4(ii)-(iii):

$$\lim_{j\to\infty} \sup\{\boldsymbol{F}(\phi^i_j(x), \Phi'_i(x)) : x \in J(1, k^i_j)_0\} = 0. \tag{3-5}$$

Let $\Phi^i_j : J \to \mathcal{Z}^G_n(M_{\eta_i}; \boldsymbol{M}; \mathbb{Z}_2)$ be the Almgren G-extension of ϕ^i_j given by Theorem 3.5 for j-large. By Corollary 3.6, Φ^i_j and Φ'_i are *relative* homotopic in $\mathcal{Z}^G_n(M_{\eta_i}; \mathcal{F}; \mathbb{Z}_2)$. Therefore,

$$\tilde{\Phi}^j_i(x) := \begin{cases} \Phi_i(x), & x \in \left[0, \frac{1}{3}\right], \\ \Phi^j_i(x), & x \in J = \left[\frac{1}{3}, 1\right] \end{cases}$$

is a well-defined $\boldsymbol{F}$-continuous G-sweepout of $(M, \partial M)$ for each $i \in \mathbb{N}$ and j-large, and thus

$$\begin{aligned} W^G(M, \partial M) \leq \boldsymbol{L}(\{\tilde{\Phi}^i_j\}_{j\in\mathbb{N}}) &= \boldsymbol{L}(\{\Phi^i_j\}_{j\in\mathbb{N}}) \\ &= \boldsymbol{L}(\{\phi^i_j\}_{j\in\mathbb{N}}) \\ &\leq \sup\{\boldsymbol{M}(\Phi_i(x)) : x \in I\} \to W^G(M, \partial M) \end{aligned} \tag{3-6}$$

by (3-4) and Corollary 3.6.

Now, we take a subsequence $j(i) \to \infty$ and define $\tilde{\Phi}_i = \tilde{\Phi}^i_{j(i)}$, $S = \{\varphi_i\}_{i\in\mathbb{N}}$, $\varphi_i := \phi^i_{j(i)}$ so that $\boldsymbol{f_M}(\varphi_i) \to 0$ and that:

1. $C_i \boldsymbol{f_M}(\varphi_i) \to 0$ as $i \to \infty$, where $C_i = C_0(M_{\eta_i}, G, 1)$ is given by Theorem 3.5.
2. If $\boldsymbol{M}(\varphi_i(x)) \geq W^G(M, \partial M) - \frac{\delta}{2}$ then $\operatorname{spt}(\varphi_i(x)) \subset M_{a_0}$ (by Claim 1).
3. $W^G(M, \partial M) = \boldsymbol{L}(\{\varphi_i\}_{i\in\mathbb{N}})$ (by (3-6)).
4. $\lim_{i\to\infty} \sup\{\boldsymbol{F}(\varphi_i(x), \Phi_i(x)) : x \in J(1, k^i_{j(i)})_0\} = 0$ (by (3-5)).
5. $\lim_{i\to\infty} \sup\{\boldsymbol{F}(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in I(1, k^i_{j(i)})\} = 0$ (by the $\boldsymbol{F}$-continuity).

Combining (3), (4), and (5) with (3-4), we have $\boldsymbol{C}(S)=\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})\subset\mathcal{V}_n^G(M_{2a_0})$ and every $V\in\boldsymbol{C}(S)$ is stationary in M.

Claim 2. *There exists $V\in\boldsymbol{C}(S)$ that is $(G,\mathbb{Z}_2)$-almost minimizing in annuli (of boundary-type) in the sense of* [39, Definition 11].

Proof of Claim 2. Suppose none of $V\in\boldsymbol{C}(S)$ is $(G,\mathbb{Z}_2)$-almost minimizing in annuli in the sense of [39, Definition 11]. Then there is a new sequence $S^*=\{\varphi_i^*\}_{i\in\mathbb{N}}$ of mappings $\varphi_i^*:J(1,l_i)_0\to\mathcal{Z}_n^G(M_{\eta_i};\mathbb{Z}_2)$ for some $l_i\geq k^i_{j(i)}\to\infty$ as $i\to\infty$, such that:

(i) $\boldsymbol{L}(S^*)<\boldsymbol{L}(S)=W^G(M,\partial M)$.

(ii) φ_i and φ_i^* are 1-homotopic in $\mathcal{Z}_n^G(M_{\eta_i};\mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness tending to zero, (Specifically, there is a map $\psi_i:I(1,l_i)_0\times J(1,l_i)_0\to\mathcal{Z}_n^G(M_{\eta_i};\mathbb{Z}_2)$ so that $\boldsymbol{f_M}(\psi_i)\to 0$ as $i\to\infty$, $\psi_i([0],\cdot)=\varphi_i\circ\boldsymbol{n}_i$, and $\psi_i([1],\cdot)=\varphi_i^*$, where $\boldsymbol{n}_i=\boldsymbol{n}(l_i,k^i_{j(i)})$).

(iii) $\operatorname{spt}(\psi_i(t,x)-\varphi_i\circ\boldsymbol{n}_i(x))\Subset M_{a_0}$ for any $t\in I(1,l_i)_0$ and $x\in J(1,l_i)_0$.

(iv) For any $x\in J(1,l_i)_0$, if $\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x))<W^G(M,\partial M)-\frac{\delta}{4}$, then we have that $\psi_i(\cdot,x)\equiv\varphi_i\circ\boldsymbol{n}_i(x)$ is a constant discrete homotopy at x.

Indeed, since each $V\in\boldsymbol{C}(S)$ is supported in M_{2a_0}, we can take G-annuli

$$\{\operatorname{An}^G(p(V),r_i-s_i,r_i+s_i)\}_{i=1}^{27}$$

in M_{a_0} as in [30, Theorem 4.10, Part 1], which implies all the deformations will be restricted in M_{a_0}. Using [40, Theorem 3.14] and $\operatorname{dist}_M(G\cdot p,\cdot)$, we can make the constructions in [30, Theorem 4.10, Parts 2–9] with G-invariant objects. Then the rest parts in [30, Theorem 4.10] are purely combinatorial, which would carry over with M_{a_0} in place of M. This gives (i)–(iii). Moreover, by taking the constant ϵ_2 in [30, Theorem 4.10, Part 3] smaller than $\frac{\delta}{8}$, we have $\psi_i(\cdot,x)\equiv\varphi_i\circ\boldsymbol{n}_i(x)$ provided $\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x))<W^G(M,\partial M)-\frac{\delta}{4}$ (see Parts 10(c), 14 and 18 in [30, Theorem 4.10]).

Next, we can extend φ_i^* (for i-large) to an $\boldsymbol{F}$-continuous map $\tilde{\Phi}_i^*\in\mathcal{P}^G(M,\partial M)$ so that $\tilde{\Phi}_i^*\llcorner\big[0,\frac{1}{3}\big]=\tilde{\Phi}_i\llcorner\big[0,\frac{1}{3}\big]=\Phi_i\llcorner\big[0,\frac{1}{3}\big]$. Take any 1-cell $\alpha=[x_0,x_1]\in J(1,l_i)_1$, we will construct the extension $\tilde{\Phi}_i^*\llcorner\alpha$ separately in two cases.

Case 1: $\max\{\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x_0)),\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x_1))\}<W^G(M,\partial M)-\frac{\delta}{4}$.

By (ii)–(iv), we can define $\tilde{\Phi}_i^*\llcorner\alpha:=\tilde{\Phi}_i\circ f_\alpha$ as the extension of $\varphi_i^*\llcorner\alpha_0$, where $f_\alpha:\alpha=[x_0,x_1]\to[\boldsymbol{n}_i(x_0),\boldsymbol{n}_i(x_1)]$ is an affine transformation. Hence, in this case, we have

$$\tilde{\Phi}_i^*\llcorner\alpha\subset\mathcal{Z}_n^G(M_{\eta_i};\mathbb{Z}_2)\quad\text{and}\quad\tilde{\Phi}_i^*(x)=\tilde{\Phi}_i(\boldsymbol{n}_i(x))\text{ for all }x\in\alpha_0=\{x_0,x_1\}.\tag{3-7}$$

In particular, $\tilde{\Phi}_i^*(1)=\tilde{\Phi}_i(1)=0$ provided $\boldsymbol{f_M}(\psi_i)<W^G(M,\partial M)-\frac{\delta}{4}$, which holds for i-large. Additionally, it follows from (1), Theorem 3.5(i)–(iii), and the choice of α that

$$\begin{aligned}(3\text{-}8)\qquad \sup\{\boldsymbol{M}(\tilde{\Phi}_i^*(x)) : x \in \alpha\} &= \sup\{\boldsymbol{M}(\tilde{\Phi}_i(x)) : x \in f_\alpha(\alpha)\}\\ &\le \sup\{\boldsymbol{M}(\varphi_i(x)) : x \in \partial f_\alpha(\alpha)\} + C_i\, \boldsymbol{f_M}(\varphi_i)\\ &< W^G(M, \partial M) - \tfrac{\delta}{4} + C_i\, \boldsymbol{f_M}(\varphi_i)\\ &\le W^G(M, \partial M) - \tfrac{\delta}{5}\end{aligned}$$

for i-large, where $C_i = C_0(M_{\eta_i}, G, 1)$ is given by Theorem 3.5.

Case 2: $\max\{\boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x_0)), \boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x_1))\} \ge W^G(M, \partial M) - \frac{\delta}{4}$.

Let $A_i \subset J = \left[\frac{1}{3}, 1\right]$ be union of all 1-cells of this case in $J(1, l_i)_1$. Take i sufficiently large so that $\boldsymbol{f_M}(\psi_i) < \frac{\delta}{4}$ (by (ii)). Then $\boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x)) \ge W^G(M, \partial M) - \frac{\delta}{2}$ for all $x \in J(1, l_i)_0 \cap A_i$. By (2) and (iii), we have that $\varphi_i^*(x) = \psi_i^*([1], x)$ is supported in M_{a_0} for all $x \in J(1, l_i)_0 \cap A_i$. Applying Theorem 3.5 to $\varphi_i^* \llcorner [J(1, l_i)_0 \cap A_i]$ in M_{a_0} (for i-large) will give an $\boldsymbol{M}$-continuous extension $\tilde{\Phi}_i^* : A_i \to \mathcal{Z}_n^G(M_{a_0}; \mathbb{Z}_2)$ so that

$$(3\text{-}9)\quad \sup\{\boldsymbol{M}(\tilde{\Phi}_i^*(x)) : x \in A_i\} \le \sup\{\boldsymbol{M}(\varphi_i^*(x)) : x \in J(1, l_i)_0 \cap A_i\} + C_0\, \boldsymbol{f_M}(\psi_i),$$

where $C_0 = C_0(M_{a_0}, G, 1) \ge 1$ is a uniform constant. Note for any $x \in \partial A_i$, we must have $\boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x)) < W^G(M, \partial M) - \frac{\delta}{4}$. Hence, by (iv) and Theorem 3.5(i),

$$(3\text{-}10)\qquad \tilde{\Phi}_i^*(x) = \varphi_i^*(x) = \varphi_i \circ \boldsymbol{n}_i(x) = \tilde{\Phi}_i(\boldsymbol{n}_i(x)) \quad \text{for all } x \in \partial A_i.$$

It now follows from (3-7)–(3-10) that $\tilde{\Phi}_i^* : I \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is a well defined $\boldsymbol{F}$-continuous map so that $\tilde{\Phi}_i^* \llcorner \left[0, \frac{1}{3}\right] = \tilde{\Phi}_i \llcorner \left[0, \frac{1}{3}\right] = \Phi_i \llcorner \left[0, \frac{1}{3}\right]$, $\tilde{\Phi}_i^*(1) = 0$, and $\tilde{\Phi}_i^* \llcorner J \subset \mathcal{Z}_n^G(M_{\eta_i}; \mathbb{Z}_2)$, which implies $\tilde{\Phi}_i^* \in \mathcal{P}^G(M, \partial M)$. Therefore, by equations (3-4), (3-8), (3-9), and statements (i)-(ii),

$$W^G(M, \partial M) \le \boldsymbol{L}(\{\tilde{\Phi}_i^*\}_{i\in\mathbb{N}}) \le \max\left\{W^G(M, \partial M) - \tfrac{\delta}{5}, \boldsymbol{L}(\{\varphi_i^*\}_{i\in\mathbb{N}})\right\} < W^G(M, \partial M),$$

which is a contradiction. □

Thus, there must exist $V \in \boldsymbol{C}(S)$ that is $(G, \mathbb{Z}_2)$-almost minimizing in annuli (of boundary-type) in the sense of [39, Definition 11]. Since $\boldsymbol{C}(S) \subset \mathcal{V}_n^G(M_{2a_0})$, the interior regularity result [39, Theorem 7] (modified in [41, Theorem 4.18]) indicates that V is an integral G-varifold induced by closed smooth embedded minimal G-hypersurfaces. □

4. G-sweepouts in positive Ricci curvature G-manifolds

Throughout this section we assume that (M^{n+1}, g_M) is a closed connected orientable Riemannian manifold with positive Ricci curvature $\mathrm{Ric}_M > 0$, and G is a compact Lie group acting on M isometrically so that $3 \le \mathrm{codim}(G \cdot p) \le 7$ for all $p \in M$. Our goal is to associate an $\boldsymbol{F}$-continuous G-sweepout to each closed minimal G-hypersurface in M.

To begin with, we collect some notations and classical results for minimal hypersurfaces. Let $\Sigma \subset M$ be a closed smooth embedded minimal hypersurface. Recall the second variation of Σ for the area functional is given by

$$\delta^2\Sigma(X) := \frac{d^2}{d^2t}\Big|_{t=0}\text{Area}(F_t(\Sigma)) = -\int_\Sigma \langle L_\Sigma(X^\perp), X^\perp\rangle, \tag{4-1}$$

where $L_\Sigma : \mathfrak{X}^\perp(\Sigma) \to \mathfrak{X}^\perp(\Sigma)$ is the Jacobi operator of Σ, and $\{F_t\}$ are diffeomorphisms generated by $X \in \mathfrak{X}(M)$. Then we denote:

- $\text{Index}(\Sigma)$: the *Morse index* of Σ, i.e., the number of the negative eigenvalues (counted with multiplicities) of L_Σ.
- $\mu_1(\Sigma)$: the first eigenvalue of L_Σ.

If $\text{Index}(\Sigma) = 0$ or equivalently $\mu_1(\Sigma) \geq 0$, then we say Σ is *stable*.

For $\Sigma \subset M$ as a G-invariant minimal hypersurface, we have $L_\Sigma(X) \in \mathfrak{X}^{\perp,G}(\Sigma)$ for all $X \in \mathfrak{X}^{\perp,G}(\Sigma)$, where $\mathfrak{X}^{\perp,G}(\Sigma)$ is the space of normal G-vector fields on Σ. By restricting L_Σ to $\mathfrak{X}^{\perp,G}(\Sigma)$, we make the following definition.

Definition 4.1. Let $\Sigma \subset M$ be a closed smooth embedded minimal G-hypersurface. The *equivariant Morse index* (or *G-index* for simplicity) $\text{Index}_G(\Sigma)$ is defined by the number of the negative eigenvalues (counted with multiplicities) of $L_\Sigma \llcorner \mathfrak{X}^{\perp,G}(\Sigma)$. Additionally, we denote $\mu_1^G(\Sigma)$ as the first eigenvalue of $L_\Sigma \llcorner \mathfrak{X}^{\perp,G}(\Sigma)$.

Suppose Σ is a closed minimal G-hypersurface with a G-invariant unit normal ν, and u_1 is the first eigenfunction of L_Σ. Then for any $g \in G$, the G-invariance of Σ and ν indicates $u_1 \circ g$ is also the first eigenfunction of L_Σ. It is well-known that $\mu_1(\Sigma)$ has multiplicity one and the first eigenfunction u_1 does not change sign. Hence, $u_1 \circ g = u_1$ for all $g \in G$, which implies that $u_1\nu \in \mathfrak{X}^{\perp,G}(\Sigma)$ and that:

Lemma 4.2 [39, Lemma 7]. *If Σ is a closed minimal G-hypersurface with a G-invariant unit normal ν, then the first eigenfunction $u_1 > 0$ of L_Σ is G-invariant and $\mu_1(\Sigma) = \mu_1^G(\Sigma)$.*

Since we mainly consider the ambient manifolds with positive Ricci curvature, we collect the following useful results, which are well known to experts (see [44, Section 2]).

Lemma 4.3. *Suppose (M^{n+1}, g_M) is a closed connected orientable Riemannian manifold. Let $\Sigma, \Sigma_1, \Sigma_2 \subset M$ be closed embedded hypersurfaces. Then we have*:

(i) *If Σ is connected*, *then Σ is orientable if and only if it is 2-sided* (*i.e.*, *Σ has a unit normal vector field*).

(ii) *If Σ is connected and separates M*, *i.e.*, *$M \setminus \Sigma$ has two connected components*, *then Σ is orientable.*

Moreover, *suppose M has positive Ricci curvature*, *then we have*:

(iii) *If* Σ *is connected and orientable*, *then* Σ *separates* M.

(iv) *If* Σ *is minimal and* 2*-sided*, *then it cannot be stable*, *i.e.*, $\mu_1(\Sigma) < 0$.

(v) *If* Σ_1, Σ_2 *are minimal hypersurfaces*, *then* $\Sigma_1 \cap \Sigma_2 \neq \varnothing$.

After involving the actions of G, a connected component of some G-hypersurface Σ may not be G-invariant. Hence, we introduce the following notions of equivariant connectivity.

Definition 4.4. Let $U \subset M$ be a G-invariant subset with connected components $\{U_i\}_{i=1}^m$. Then we say U is *G-connected* if for any $i, j \in \{1, \cdots, m\}$, there exists $g_{ij} \in G$ so that $g_{ij} \cdot U_j = U_i$. Additionally, we say $U' \subset U$ is a *G-connected component* (or *G-component* for simplicity) of U, if U' has the form of $\cup_{j=1}^l U_{i(j)}$ and is G-connected.

Note that any G-subset U of M can be separated into some G-components. Additionally, by the above lemma, it is easy to show the following result.

Lemma 4.5. *Suppose* $(M^{n+1}, g_{_M})$ *is a closed connected orientable Riemannian manifold with positive Ricci curvature*, *and* G *is a compact Lie group acting on* M *isometrically. Let* $\Sigma \subset M$ *be a closed embedded minimal* G*-hypersurface. Then* Σ *is connected*, *and*:

- *If* Σ *has a* G*-invariant unit normal, then* Σ *separates* M *into two* G*-components.*
- *If* Σ *does not admit a* G*-invariant unit normal, then* $M \setminus \Sigma$ *is* G*-connected.*

Proof. It follows from Lemma 4.3(v) that Σ is connected. If Σ has a G-invariant unit normal ν, then by Lemma 4.3(i)–(iii), $M \setminus \Sigma$ has two connected components M_1, M_2, with ν pointing inward M_1. Since ν and $M_1 \cup M_2$ are G-invariant, we have $g_*\nu = \nu$ and $g \cdot M_i = M_i$ for all $g \in G$ and $i \in \{1, 2\}$, which indicates each M_i is G-connected. If the unit normal ν exists but is not G-invariant, then there exists $g \in G$ so that $g_*\nu = -\nu$ pointing inward M_2, which implies $g \cdot M_1 = M_2$, and thus $M_1 \cup M_2$ is G-connected. If Σ does not admit a unit normal, then $M \setminus \Sigma$ has only one component, which is also G-connected. □

Recall that, Zhou [43] constructed sweepouts of M by separating orientable and nonorientable minimal hypersurfaces. It follows from Lemma 4.3 that the orientability of a connected closed hypersurface is equivalent to the nonconnectivity of its unit normal bundle. Hence, after involving the actions of G, we shall separate the constructions by the G-connectivity (Definition 4.4) of the unit normal bundle for minimal G-hypersurfaces.

Therefore, we denote

$$\mathcal{S}^G(M) := \left\{ \Sigma^n \subset M^{n+1} \;\middle|\; \begin{array}{l} \Sigma \text{ is a closed smooth embedded} \\ \text{minimal } G\text{-hypersurface in } (M, g_{_M}) \end{array} \right\}. \tag{4-2}$$

By Theorem 3.8, $\mathcal{S}^G(M) \neq \varnothing$ provided $3 \leq \operatorname{codim}(G \cdot p) \leq 7$ for all $p \in M$. Define

$$\mathcal{S}^G_+(M) := \{\Sigma \in \mathcal{S}^G(M) : \Sigma \text{ has a } G\text{-invariant unit normal}\}$$

and $\mathcal{S}^G_-(M) := \mathcal{S}^G(M) \setminus \mathcal{S}^G_+(M)$. It follows directly from Lemma 4.5 that

$$\Sigma \in \mathcal{S}^G_-(M) \quad \Leftrightarrow \quad S\Sigma \text{ is } G\text{-connected} \quad \Leftrightarrow \quad M \setminus \Sigma \text{ is } G\text{-connected},$$

where $S\Sigma = \{v \in \boldsymbol{N}\Sigma : |v| = 1\}$ is the unit normal bundle of Σ.

Moreover, for any $\Sigma \in \mathcal{S}^G_-(M)$, we can cut M along Σ to obtain a new manifold $\tilde{M}$ so that $\tilde{M}$ is locally isometric to M, G acts on $\tilde{M}$ by isometries, and $\partial\tilde{M} \in \mathcal{S}^G_+(\tilde{M})$ is a G-invariant double cover of Σ. Specifically, let $r > 0$ be small enough so that the normal exponential map $\exp^\perp_\Sigma : \boldsymbol{N}\Sigma \to M$ is a G-equivariant diffeomorphism on $B_{2r}(\Sigma) := \{p \in M : \operatorname{dist}_M(\Sigma, p) < 2r\}$. Hence, we have

$$E : S\Sigma \times (-2r, 2r) \to B_{2r}(\Sigma), \quad E(v, t) := \exp^\perp_\Sigma(t \cdot v) \tag{4-3}$$

which is a double cover of $B_{2r}(\Sigma)$. Define the action of G on $S\Sigma \times (-2r, 2r)$ by $g \cdot (v, t) := (g_* v, t)$ for any $v \in S\Sigma$ and $t \in (-2r, 2r)$, which indicates E is G-equivariant and

$$\tilde{\Sigma} = S\Sigma \times \{0\} \tag{4-4}$$

is a G-equivariant double cover of Σ. Let $E_r := E \llcorner (S\Sigma \times (r, 2r))$ be a G-equivariant diffeomorphism on $B_{2r}(\Sigma) \setminus \operatorname{Clos}(B_r(\Sigma))$. Then by gluing $M \setminus \operatorname{Clos}(B_r(\Sigma))$ and $S\Sigma \times [0, 2r)$ on $B_{2r}(\Sigma) \setminus \operatorname{Clos}(B_r(\Sigma))$ with E_r, we can define

$$\tilde{M} := \big(M \setminus \operatorname{Clos}(B_r(\Sigma))\big) \cup_{E_r} \big(S\Sigma \times [0, 2r)\big) \tag{4-5}$$

as a compact manifold with boundary $\partial\tilde{M} = \tilde{\Sigma}$. Then we have

$$F : \tilde{M} \to M, \qquad F := \begin{cases} id & \text{in } M \setminus \operatorname{Clos}(B_r(\Sigma)), \\ E & \text{in } S\Sigma \times [0, 2r) \end{cases} \tag{4-6}$$

is a G-equivariant smooth map so that $F \llcorner (\tilde{M} \setminus \tilde{\Sigma})$ gives a diffeomorphism to $M \setminus \Sigma$, and $F \llcorner \tilde{\Sigma}$ gives a double cover of Σ. Using F, we can pull back the metric g_M from M to $\tilde{M}$ so that F is a local isometry and G acts on $\tilde{M}$ by isometries. Thus, $\tilde{\Sigma}$ is a minimal G-hypersurface in $\tilde{M}$ with an inward pointing G-invariant unit normal. In particular, $\Sigma \in \mathcal{S}^G_-(M)$ implies $S\Sigma$ and $M \setminus \Sigma$ are both G-connected, and thus $\tilde{M}$ is G-connected.

G-sweepouts correspond to $\boldsymbol{\Sigma \in \mathcal{S}^G(M)}$.

Proposition 4.6. *Given any $\Sigma \in \mathcal{S}^G_+(M)$, there exists an $\boldsymbol{F}$-continuous G-sweepout $\Phi : [-1, 1] \to \mathcal{Z}^G_n(M; \mathbb{Z}_2)$ of M so that*:

(i) $\Phi(0) = [\![\Sigma]\!]$, $\Phi(-1) = \Phi(1) = 0$.

(ii) $\boldsymbol{M}(\Phi(x)) \leq \operatorname{Area}(\Sigma)$ *with equality only if $x = 0$.*

Proof. By Lemma 4.5, $M \setminus \Sigma$ has two G-components M_1 and M_2 so that the unit normal ν of Σ pointing inward M_1. Additionally, it follows from Lemmas 4.2 and 4.3 that the first eigenfunction $u_1 > 0$ of L_Σ is a G-invariant function satisfying $L_\Sigma u_1 = -\mu_1(\Sigma)\, u_1 > 0$.

Denote by $d_\pm$ the signed distance function to Σ so that $d_\pm = \operatorname{dist}_M(\Sigma, \cdot)$ in M_1, and $d_\pm = -\operatorname{dist}_M(\Sigma, \cdot)$ in M_2. Suppose $X \in \mathfrak{X}^G(M)$ is a G-vector field with $X = (u_1 \circ n_\Sigma) \cdot \nabla d_\pm$ in a neighborhood of Σ, where n_Σ is the nearest projection (in M) to Σ. Then we consider the G-equivariant variation $\{\Sigma_t := F_t(\Sigma)\}_{t\in[-r,r]}$ of Σ, where $\{F_t\}$ are the G-equivariant diffeomorphisms generated by X. By the second variation formula (4-1), we have

$$\delta^2\Sigma(X) = \frac{d^2}{dt^2}\Big|_{t=0} \operatorname{Area}(\Sigma_t) = -\int_\Sigma u_1 L_\Sigma u_1 < 0, \qquad \frac{d}{dt}\Big|_{t=0} \langle \vec{H}_{\Sigma_t}, \nabla d_\pm\rangle = L_\Sigma u_1 > 0,$$

where $\vec{H}_{\Sigma_t}$ is the mean curvature vector field of Σ_t. Thus, for $r > 0$ small enough,

$$\operatorname{Area}(\Sigma_t) < \operatorname{Area}(\Sigma), \quad \langle \vec{H}_{\Sigma_t}, \nabla \operatorname{dist}_M(\Sigma, \cdot)\rangle > 0 \qquad \text{for all } t \in [-r, 0) \cup (0, r].$$

Define $\Phi(x) := [\![\Sigma_x]\!] = (F_x)_\# [\![\Sigma]\!] \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ for $x \in [-r, r]$, which is $\boldsymbol{F}$-continuous.

Since $u_1 > 0$, $\{\Sigma_t\}_{t\in[-r,r]}$ is a smooth foliation of a G-neighborhood of Σ, and $\Sigma_t \subset M_1$ for $t > 0$ and $\Sigma_t \subset M_2$ for $t < 0$. We now consider the compact manifolds $M_1' := M_1 \setminus \{\Sigma_t\}_{t\in[0,r)}$ and $M_2' := M_2 \setminus \{\Sigma_t\}_{t\in(-r,0]}$, whose boundary $\partial M_i' = \Sigma_{r_i}$ ($i \in \{1, 2\}$, $r_1 = r$, $r_2 = -r$), is a G-hypersurface with positive mean curvature pointing inward M_i'.

Suppose $W^G(M_i', \partial M_i') > \operatorname{Area}(\Sigma_{r_i})$ for $i \in \{1, 2\}$. Then by Theorem 3.11, there exists a closed minimal G-hypersurface Σ' in the interior of M_i'. Noting $\Sigma \cap \Sigma' = \varnothing$, we get a contradiction from Lemma 4.3(v). Therefore, $W^G(M_i', \partial M_i') \le \operatorname{Area}(\Sigma_{r_i})$. By Definition 2.8 and Corollary 3.7, there exist $\epsilon > 0$ small enough and an $\boldsymbol{F}$-continuous G-sweepout $\Phi_i : [0, 1] \to \mathcal{Z}_n^G(M_i; \mathbb{Z}_2)$ so that $\Phi_i(0) = [\![\Sigma_{r_i}]\!]$, $\Phi_i(1) = 0$, and

$$\sup\{\boldsymbol{M}(\Phi_i(x)) : x \in [0, 1]\} \le W^G(M_i', \partial M_i') + \epsilon \le \operatorname{Area}(\Sigma_{r_i}) + \epsilon < \operatorname{Area}(\Sigma).$$

Now, by reparametrization, we have a well-defined map $\Phi : [-1, 1] \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$,

$$\Phi(x) := \begin{cases} \Phi_2\big(-\frac{3}{2}x - \frac{1}{2}\big), & x \in \big[-1, -\frac{1}{3}\big], \\ (F_{3rx})_\# [\![\Sigma]\!], & x \in \big[-\frac{1}{3}, \frac{1}{3}\big], \\ \Phi_1\big(\frac{3}{2}x - \frac{1}{2}\big), & x \in \big[\frac{1}{3}, 1\big] \end{cases}$$

continuous in the $\boldsymbol{F}$-topology satisfying (i) and (ii). Additionally, the arguments before Definitions 2.5 and 2.8 indicate $F_M(\Phi) = [\![M_2]\!] + [\![M_1]\!] = [\![M]\!]$, where F_M is given by (2-5). Hence, we have $\Phi \in \mathcal{P}^G(M)$. $\square$

Proposition 4.7. *Given any $\Sigma \in \mathcal{S}^G_-(M)$, there exists an $\mathcal{F}$-continuous G-sweepout $\Phi : [0, 1] \to \mathcal{Z}^G_n(M; \mathbb{Z}_2)$ of M with no concentration of mass on orbits so that*:

(i) $\Phi(0) = \Phi(1) = 0$.

(ii) $\sup\{\boldsymbol{M}(\Phi(x)) : x \in [0, 1]\} < 2\,\mathrm{Area}(\Sigma)$.

Proof. Let $\tilde{\Sigma} = S\Sigma \times \{0\}$ and $\tilde{M}$ be given by (4-4) and (4-5). Then $\tilde{M}$ is G-connected, $\mathrm{Area}(\tilde{\Sigma}) = 2\,\mathrm{Area}(\Sigma)$, and $\tilde{\Sigma}$ has a G-invariant unit normal $\tilde{\nu}$ pointing inward $\tilde{M}$. Let $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ be the isometric involution, i.e., $\tau(v, 0) = (-v, 0)$ for $v \in S\Sigma$.

Using the constructions in Proposition 4.6 with $\tilde{M}$ in place of M_1, we get an $\boldsymbol{F}$-continuous G-sweepout $\tilde{\Phi} : [0, 1] \to \mathcal{Z}^G_n(\tilde{M}; \mathbb{Z}_2)$ so that $\tilde{\Phi}(0) = [\![\tilde{\Sigma}]\!]$, $\tilde{\Phi}(1) = 0$, and $\boldsymbol{M}(\tilde{\Phi}(x)) \le 2\,\mathrm{Area}(\Sigma)$ for all $x \in [0, 1]$ with equality only at $x = 0$. Additionally, for $t \in \left[0, \frac{1}{3}\right]$, $\tilde{\Phi}(t) = \tilde{\Sigma}_t := [\![\exp^{\perp}_{\tilde{\Sigma}}(t\tilde{u}\tilde{\nu})]\!]$, where $\tilde{u} = 3tr\tilde{u}_1$ and $\tilde{u}_1 : \tilde{\Sigma} \to \mathbb{R}^+$ is the G-invariant first eigenfunction of $L_{\tilde{\Sigma}}$ with eigenvalue $\mu_1(\tilde{\Sigma}) = \mu^G_1(\tilde{\Sigma}) < 0$.

Now, by the second variation formula (4-1), there are $\delta_0 \in \left(0, \frac{1}{3}\right)$, $C_0 > 0$ so that

$$\boldsymbol{M}(\tilde{\Phi}(t)) = \mathcal{H}^n(\tilde{\Sigma}_t) = \mathcal{H}^n(\tilde{\Sigma}) - \frac{t^2}{2}\int_{\tilde{\Sigma}} \langle L_{\tilde{\Sigma}}\tilde{u}\tilde{\nu}, \tilde{u}\tilde{\nu}\rangle + O(t^3) \le \mathcal{H}^n(\tilde{\Sigma}) - C_0 t^2 \tag{4-7}$$

for all $t \in (0, \delta_0)$. For any $\delta \in (0, \delta_0)$ (will be specified later), the $\boldsymbol{F}$-continuity of $\tilde{\Phi}$ and Proposition 4.6(ii) imply the existence of $\epsilon > 0$ with

$$\boldsymbol{M}(\tilde{\Phi}(t)) \le \mathcal{H}^n(\tilde{\Sigma}) - \epsilon \quad \text{for all } t \in [\delta, 1]. \tag{4-8}$$

Now, we will open up $\tilde{\Sigma}_t$, $t \in [0, \delta]$, at some orbit to decrease the area.

Specifically, let $G \cdot \tilde{p} \subset \tilde{\Sigma}^{\mathrm{prin}}$ be any principal orbit of $\tilde{\Sigma}$. Then by the G-invariance of $\tilde{\nu}$ and [4, Corollary 2.2.2], $G \cdot \tilde{p} \subset \tilde{M}^{\mathrm{prin}}$ is also a principal orbit in $\tilde{M}$. Note either $G \cdot \tilde{p} = G \cdot \tau(\tilde{p})$ or $G \cdot \tilde{p} \cap G \cdot \tau(\tilde{p}) = \varnothing$. Thus, we can define $P := G \cdot \tilde{p} \cup G \cdot \tau(\tilde{p})$ as a G-invariant submanifold in $\tilde{\Sigma}$ with dimension $n - l$. By assumptions, $3 \le \mathrm{codim}(G \cdot \tilde{p}) = l + 1 \le 7$.

Case 1: $3 \le l \le 6$.

For any $r > 0$, $t \in [0, \delta]$, define the following G-invariant sets:

$$\begin{aligned}
\tilde{B}_r(P) &:= \{\tilde{q} \in \tilde{\Sigma} : \mathrm{dist}_{\tilde{\Sigma}}(\tilde{q}, P) < r\} \subset \tilde{\Sigma},\\
\tilde{B}_{r,t}(P) &:= \{\exp^{\perp}_{\tilde{\Sigma}}((t\tilde{u}\tilde{\nu})(\tilde{q})) : \tilde{q} \in \tilde{B}_r(P)\} \subset \tilde{\Sigma}_t,\\
\tilde{C}_{r,t}(P) &:= \{\exp^{\perp}_{\tilde{\Sigma}}((s\tilde{u}\tilde{\nu})(\tilde{q})) : \tilde{q} \in \partial\tilde{B}_r(P), s \in [0, t]\}.
\end{aligned}$$

For $R, \delta > 0$ small enough, it follows from the integral formula in [40, (C.4)] that

$$ctr^{l-1} \le \mathcal{H}^n(\tilde{C}_{r,t}(P)) \le Ctr^{l-1} \quad \text{and} \quad cr^l \le \mathcal{H}^n(\tilde{B}_{r,t}(P)) \le Cr^l \tag{4-9}$$

for all $r \in [0, R]$, $t \in [0, \delta]$, where $c, C > 0$ are constants depending on $\tilde{\Sigma}, \tilde{M}, P$. Define

$$\tilde{\Sigma}_{r,t} := (\tilde{\Sigma}_t \setminus \tilde{B}_{r,t}(P)) \cup \tilde{C}_{r,t}(P) \cup \tilde{B}_r(P), \quad r \in [0, R],\ t \in [0, \delta].$$

By (4-7)–(4-9), $\|\tilde{\Sigma}_{r,t}\|(\tilde{M}\setminus\tilde{\Sigma}) \le \mathcal{H}^n(\tilde{\Sigma}) - C_0 t^2 - cr^l + Ctr^{l-1}$.

Note, in this case, that

$$Ctr^{l-1} \le \frac{C_0}{2}t^2 + \frac{C^2}{2C_0}r^{2l-2}, \quad l \ge 3.$$

We can take $R > 0$ small enough so that $\frac{C^2}{2C_0}R^{l-2} < \frac{c}{2}$. Hence,

$$\|\tilde{\Sigma}_{r,t}\|(\tilde{M}\setminus\tilde{\Sigma}) \le \mathcal{H}^n(\tilde{\Sigma}) - \frac{C_0}{2}t^2 - \frac{c}{2}r^l$$

for all $t \in [0,\delta]$, $r \in [0,R]$, and thus

$$\tilde{\Sigma}'_t := \begin{cases} \tilde{\Sigma}_{R,2t}, & t \in \left[0, \frac{\delta}{2}\right], \\ \tilde{\Sigma}_{2R(1-\frac{t}{\delta}),\delta}, & t \in \left[\frac{\delta}{2}, \delta\right] \end{cases}$$

satisfies

$$\|\tilde{\Sigma}'_t\|(\tilde{M}\setminus\tilde{\Sigma}) \le \begin{cases} \mathcal{H}^n(\tilde{\Sigma}) - \frac{c}{2}R^l, & t \in \left[0, \frac{\delta}{2}\right], \\ \mathcal{H}^n(\tilde{\Sigma}) - \frac{C_0}{2}\delta^2, & t \in \left[\frac{\delta}{2}, \delta\right]. \end{cases}$$

Set $\epsilon' := \min\left\{\epsilon, \frac{cR^l}{2}, \frac{C_0\delta^2}{2}\right\}$ and define $\tilde{\Phi}'(t) := [\![\tilde{\Sigma}'_t]\!]$ for $t \in [0,\delta]$ in this case.

Case 2: $l = 2$.

For $R > r > 0$ small enough, let $\eta_{r,R} : \tilde{\Sigma} \to [0,1]$ be the G-invariant logarithmic cut-off function defined by

$$\eta_{r,R}(\tilde{q}) := \begin{cases} 1, & \tilde{q} \notin \tilde{B}_R(P), \\ \frac{\log r - \log(\mathrm{dist}_{\tilde{\Sigma}}(\tilde{q},P))}{\log r - \log R}, & \tilde{q} \in \tilde{B}_R(P)\setminus\tilde{B}_r(P), \\ 0, & \tilde{q} \in \tilde{B}_r(P), \end{cases}$$

which is also τ-invariant. Consider

$$\tilde{\Sigma}_{r,R,t} := \exp^{\perp}_{\tilde{\Sigma}}(t\eta_{r,R}\,\tilde{u}\tilde{\nu}).$$

By [17, Proposition 2.5] and [40, (C.4)], we can take $R, \delta > 0$ small enough so that

$$\begin{aligned} \|\tilde{\Sigma}_{r,R,t}\|(\tilde{M}\setminus\tilde{\Sigma}) &\le \mathcal{H}^n(\tilde{\Sigma}\setminus\tilde{B}_r(P)) + \frac{t^2}{2}\int_{\tilde{\Sigma}} |\nabla(\eta_{r,R}\,\tilde{u})|^2 - (\mathrm{Ric}(\tilde{\nu},\tilde{\nu}) + |A|^2)(\eta_{r,R}\,\tilde{u})^2 \\ &\quad + Ct^3\int_{\tilde{\Sigma}} 1 + |\nabla(\eta_{r,R}\,\tilde{u})|^2 \\ &\le \mathcal{H}^n(\tilde{\Sigma}\setminus\tilde{B}_r(P)) - C_1t^2 + C_2t^2\int_{\tilde{\Sigma}}|\nabla\eta_{r,R}|^2 + t^2\int_{\tilde{B}_R(P)} \tilde{u}\eta_{r,R}\nabla\tilde{u}\nabla\eta_{r,R} \\ &\quad + Ct^3\int_{\tilde{\Sigma}} 1 + 2\eta_{r,R}^2|\nabla\tilde{u}|^2 + 2\tilde{u}^2|\nabla\eta_{r,R}|^2 \\ &\le \mathcal{H}^n(\tilde{\Sigma}) - cr^2 - C_1t^2 + \frac{C_3}{\log\left(\frac{R}{r}\right)}t^2 + C_4R^2t^2 + C_5t^3 + \frac{C_6}{\log\left(\frac{R}{r}\right)}t^3 \end{aligned}$$

for all $r \in (0,R)$, $t \in [0,\delta]$, where $c, C, C_i > 0$ are uniform constants depending on $\tilde{\Sigma}, \tilde{M}, P$. Set $R, \delta > 0$ even smaller so that $C_4R^2 < \frac{C_1}{4}$, $C_5\delta < \frac{C_1}{4}$, and $C_6\delta < C_3$.

Then choose $r > 0$ small enough with $\frac{2C_3}{\log(R/r)} < \frac{C_1}{4}$. Thus,

$$\|\tilde{\Sigma}_{r,R,t}\|(\tilde{M} \setminus \tilde{\Sigma}) \leq \mathcal{H}^n(\tilde{\Sigma}) - cr^2 - \frac{C_1}{2}t^2 + \frac{2C_3}{\log(R/r)}t^2 \leq \mathcal{H}^n(\tilde{\Sigma}) - cr^2 - \frac{C_1}{4}t^2$$

for all $t \in [0, \delta]$, and

$$\tilde{\Sigma}'_t := \begin{cases} \tilde{\Sigma}_{r,R,2t}, & t \in \left[0, \frac{\delta}{2}\right], \\ \tilde{\Sigma}_{2r(1-\frac{t}{\delta}),2R(1-\frac{t}{\delta}),\delta}, & t \in \left[\frac{\delta}{2}, \delta\right], \end{cases}$$

satisfies

$$\|\tilde{\Sigma}'_t\|(\tilde{M} \setminus \tilde{\Sigma}) \leq \begin{cases} \mathcal{H}^n(\tilde{\Sigma}) - cr^2, & t \in \left[0, \frac{\delta}{2}\right], \\ \mathcal{H}^n(\tilde{\Sigma}) - \frac{C_1}{4}\delta^2, & t \in \left[\frac{\delta}{2}, \delta\right]. \end{cases}$$

In this case, set $\epsilon' := \min\left\{\epsilon, cr^2, \frac{C_1\delta^2}{4}\right\}$ and define $\tilde{\Phi}'(t) := [\![\tilde{\Sigma}'_t]\!]$ for $t \in [0, \delta]$.

In both cases, we define $\tilde{\Phi}' \llcorner [\delta, 1] = \tilde{\Phi} \llcorner [\delta, 1]$ and see

$$\sup\{\|\tilde{\Phi}'(t)\|(\tilde{M} \setminus \tilde{\Sigma}) : t \in [0, 1]\} \leq \mathcal{H}^n(\tilde{\Sigma}) - \epsilon'. \tag{4-10}$$

Additionally, by (2-1), $\tilde{\Phi}'$ is still an $\boldsymbol{F}$-continuous map with $\tilde{\Phi}' = [\![\tilde{\Sigma}]\!]$, $\tilde{\Phi}'(1) = 0$.

Finally, we define $\Phi(x) := F_\# \tilde{\Phi}'(x)$ for all $x \in [0, 1]$, where $F : \tilde{M} \to M$ is the equivariant local isometry given by (4-6). Because $F : \tilde{M} \setminus \tilde{\Sigma} \to M \setminus \Sigma$ is an equivariant isometry, the arguments before Definitions 2.5 and 2.8 indicate $F_M(\Phi) = F_\#([\![\tilde{M}]\!]) = M$, where F_M is given by (2-5). Additionally, note $F : \tilde{\Sigma} \to \Sigma$ is a double cover and $\tilde{\Sigma}'_t \cap \tilde{\Sigma}$ is τ-invariant in both cases. Hence, by $\mathbb{Z}_2$-coefficients and (4-10), we have $\Phi(0) = F_\# [\![\tilde{\Sigma}]\!] = 0$ and

$$\boldsymbol{M}(\Phi(x)) = \|\tilde{\Phi}'(t)\|(\tilde{M} \setminus \tilde{\Sigma}) \leq \mathcal{H}^n(\tilde{\Sigma}) - \epsilon' = 2\,\mathrm{Area}(\Sigma) - \epsilon'.$$

At last, noting that $\|\Phi(x)\|(B^G_r(p)) \leq \|\tilde{\Phi}'(x)\|\big(F^{-1}(B^G_r(p))\big) \leq 2\boldsymbol{m}^G(\Phi', r)$ for all $x \in [0, 1]$ and $p \in M$ by the definition of $\boldsymbol{m}^G$ (Definition 2.3), we see that $\boldsymbol{m}^G(\Phi, r) \leq 2\boldsymbol{m}^G(\tilde{\Phi}', r)$ and Φ has no concentration of mass on orbits. □

5. Proof of the main theorems

Let $\mathcal{S}^G(M)$ be given in (4-2). Then we define

$$\mathcal{A}^G(M) := \inf_{\Sigma \in \mathcal{S}^G(M)} \begin{cases} \mathrm{Area}(\Sigma) & \text{if } \Sigma \in \mathcal{S}^G_+(M), \\ 2\,\mathrm{Area}(\Sigma) & \text{if } \Sigma \in \mathcal{S}^G_-(M). \end{cases} \tag{5-1}$$

Theorem 5.1. *Let (M^{n+1}, g_M) be a closed connected orientable Riemannian manifold with positive Ricci curvature, and G be a compact Lie group acting on M isometrically so that $3 \leq \mathrm{codim}(G \cdot p) \leq 7$ for all $p \in M$. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M]$ is a connected minimal G-hypersurface of multiplicity one with a G-invariant unit normal vector field so that*

$$\mathrm{Index}_G(\Sigma) = 1 \quad \text{and} \quad \mathrm{Area}(\Sigma) = W^G(M) = \mathcal{A}^G(M).$$

Proof. By Theorem 3.8, which is a min-max theorem, there exists an integral G-varifold $V \in \mathcal{V}_n^G(M)$ induced by a smooth embedded closed minimal G-hypersurface $\Sigma \in \mathcal{S}^G(M)$ so that $\|V\|(M) = W^G(M)$. Since M has positive Ricci curvature, Lemma 4.3(v) indicates that Σ is connected, and thus $V = m|\Sigma|$ for some $m \in \{1, 2, \dots\}$. Suppose $\Sigma \in \mathcal{S}_-^G(M)$, then it follows from the last statement in Theorem 3.8 that m must be even, so $m \geq 2$. However, we have a contradiction $W^G(M) < 2\,\mathrm{Area}(\Sigma) \leq \|V\|(M) = W^G(M)$ by Proposition 4.7. Therefore, $\Sigma \in \mathcal{S}_+^G(M)$. By Proposition 4.6, we see $W^G(M) \leq \mathrm{Area}(\Sigma) \leq \|V\|(M) = W^G(M)$, and thus $m = 1$. Additionally, by the definition of $\mathcal{A}^G(M)$ and Propositions 4.6, 4.7,

$$\mathcal{A}^G(M) \leq \mathrm{Area}(\Sigma) = \|V\|(M) = W^G(M) \leq \mathcal{A}^G(M).$$

Now, it is sufficient to show $\mathrm{Index}_G(\Sigma) = 1$. Suppose $\mathrm{Index}_G(\Sigma) \geq 2$, and u_1, u_2 are the first two L^2-orthonormal G-invariant eigenfunctions of $L_\Sigma \llcorner \mathfrak{X}^{\perp, G}(\Sigma)$ with negative eigenvalues. Let $u_2\nu$ be a G-invariant normal vector field on Σ, which extends to a smooth vector field $X \in \mathfrak{X}(M)$. Then $X_2 := \int_G (g^{-1})_* X \, d\mu(g) \in \mathfrak{X}^G(M)$ gives an equivariant extension of $u_2\nu$. Consider the equivariant diffeomorphisms $\{F_s^2\}$ generated by X_G, and define $\Phi_s(t) := (F_s^2)_\# \Phi(t)$ for $t \in [-1, 1]$, where $\Phi \in \mathcal{P}^G(M)$ is the $\boldsymbol{F}$-continuous sweepout given by Proposition 4.6. Recall that in the proof of Proposition 4.6, $\Phi(t) = [\![\Sigma_t]\!] = [\![F_t^1(\Sigma)]\!]$ for $t \in \left[-\frac{1}{3}, \frac{1}{3}\right]$, where $\{F_t^1\}$ are the equivariant diffeomorphism generated by $X_1 \in \mathfrak{X}^G(M)$ with $X_1 \llcorner \Sigma = 3ru_1\nu$ for some $r > 0$. Hence, for the smooth family $\{F_s^2(\Sigma_t)\}_{s \in [-\sigma, \sigma], t \in [-1/3, 1/3]}$, the area function $A(s, t) := \mathrm{Area}(F_s^2(\Sigma_t)) = \boldsymbol{M}(\Phi_s(t))$ satisfies that:

- $\nabla A(0, 0) = 0$ since Σ is minimal.
- $\frac{\partial^2}{\partial t^2} A(0, 0) = -9r^2 \int_\Sigma u_1 L_\Sigma u_1 < 0$ and $\frac{\partial^2}{\partial s^2} A(0, 0) = -\int_\Sigma u_2 L_\Sigma u_2 < 0$.
- $\frac{\partial^2}{\partial s \, \partial t} A(0, 0) = -3r \int_\Sigma u_2 L_\Sigma u_1 = 3r\mu_1(\Sigma) \int_\Sigma u_1 u_2 = 0$.

Therefore, we can set $\sigma, \delta > 0$ sufficiently small so that

$$\boldsymbol{M}(\Phi_s(t)) = \mathrm{Area}(F_s^2(\Sigma_t)) < \mathrm{Area}(\Sigma) \quad \text{for all } t \in [-\delta, \delta],\ s \in (0, \sigma].$$

Moreover, there exists $\epsilon > 0$ so that by Proposition 4.6(ii), $\boldsymbol{M}(\Phi(t)) \leq \mathrm{Area}(\Sigma) - \epsilon$ for all $t \in [-1, -\delta] \cup [\delta, 1]$. Hence, by setting $\sigma > 0$ even smaller, we have $\boldsymbol{M}(\Phi_\sigma(t)) = \boldsymbol{M}((F_\sigma^2)_\# \Phi(t)) < \mathrm{Area}(\Sigma)$ for all $t \in [-1, 1]$. Note Φ_σ is an $\boldsymbol{F}$-continuous curve homotopic to Φ in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$. Thus,

$$W^G(M) \leq \sup\{\boldsymbol{M}(\Phi_\sigma(t)) : t \in [-1, 1]\} < \mathrm{Area}(\Sigma) = W^G(M),$$

which is a contradiction. □

As an application, we use the conformal volume to show a genus bound for the equivariant min-max minimal G-hypersurface Σ in Theorem 5.1 provided that $\dim(M) = 3$ and the actions of G are orientation preserving.

Theorem 5.2. *Let (M^3, g_M) be a closed connected oriented Riemannian 3-manifold with positive Ricci curvature, and G be a finite group acting on M by orientation preserving isometries. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M]$ is a connected closed minimal G-surface of multiplicity one satisfying*

$$\operatorname{genus}(\Sigma) \le 4K \quad \text{and} \quad W^G(M) = \operatorname{Area}(\Sigma) \le \frac{8\pi K}{\inf_{|v|=1} \operatorname{Ric}_M(v, v)},$$

where $K := \max_{p\in M} \#G \cdot p \le \#G$ is the number of points in a principal orbit of M. Additionally, $\pi(\Sigma) = \Sigma/G$ is an orientable surface with finite cone singular points of order $\{n_i\}_{i=1}^k$ (i.e., locally modeled by $\mathbb{B}^2_1(0)$ quotient a cyclic rotation group $\mathbb{Z}_{n_i}$), so that

$$\sum_{i=1}^k \Big(1 - \frac{1}{n_i}\Big) < 4 \quad \text{and} \quad \operatorname{genus}(\pi(\Sigma)) \le 3.$$

In particular, if $\Sigma \subset M^{prin}$, i.e., $k = 0$, then $\operatorname{genus}(\Sigma) \le 1 + 2K$.

Proof. By Theorem 5.1, Σ is a closed embedded connected minimal G-surface with a G-invariant unit normal ν so that $\operatorname{Area}(\Sigma) = W^G(M)$ and $\operatorname{Index}_G(\Sigma) = 1$. By Lemma 4.3, Σ has an induced orientation. Additionally, since the unit normal ν is G-invariant, the actions of G on Σ are also orientation preserving. Therefore, the orbifold $\underline{\Sigma}$ induced by (Σ, G) is an orientable closed 2-orbifold whose underlying space is the quotient distance space $(\pi(\Sigma), \operatorname{dist}_{\Sigma/G})$.

Let Σ^{prin} be the union of principal orbits for the G-action on Σ, and $\underline{\Sigma}^{\text{prin}}$ be the orbifold induced by $(\Sigma^{\text{prin}}, G)$. Denote by $\boldsymbol{N}^{\Sigma}_p G \cdot p$ and $\boldsymbol{N}_p G \cdot p$ the normal vector spaces of $G \cdot p$ at p in Σ and M respectively. Note an orbit $G \cdot p$ is principal in Σ (resp. M) if and only if the slice representation of G_p on $\boldsymbol{N}^{\Sigma}_p G \cdot p$ (resp. $\boldsymbol{N}_p G \cdot p$) is trivial (see [4, Corollary 2.2.2]). Additionally, we also notice that G_p acts trivially on $\operatorname{span}(\nu(p))$ for any $p \in \Sigma$ by the G-invariance of ν. Hence, combining these with the fact that $\boldsymbol{N}^{\Sigma}_p G \cdot p \oplus \operatorname{span}(\nu(p)) = \boldsymbol{N}_p G \cdot p$, we see $\Sigma^{\text{prin}} \subset M^{\text{prin}}$, and thus $K = \#G \cdot p = \#G \cdot q$ for all $p \in \Sigma^{\text{prin}}$ and $q \in M^{\text{prin}}$. Next, it follows from [5, Chapter IV, Theorem 3.3] that there is an induced Riemannian metric $g_{\underline{\Sigma}}$ on $\underline{\Sigma}^{\text{prin}}$ so that $\pi : \Sigma^{\text{prin}} \to \underline{\Sigma}^{\text{prin}}$ is an Riemannian submersion. Moreover, since G acts on Σ by orientation preserving isometries, the singular points $\underline{\Sigma} \setminus \underline{\Sigma}^{\text{prin}}$ are a finite number of cone points $\{[p_i]\}_{i=1}^k$ of orders $n_1, \dots, n_k$. By the orbifold version of Gauss–Bonnet theorem (see [9, Proposition 2.17]), we have

$$\int_{\pi(\Sigma)} K_{\underline{\Sigma}}\, dA_{g_{\underline{\Sigma}}} = 2\pi(\chi(\underline{\Sigma})) = 2\pi\Big(2 - 2\operatorname{genus}(\pi(\Sigma)) - \sum_{i=1}^k \Big(1 - \frac{1}{n_i}\Big)\Big), \tag{5-2}$$

where $K_{\underline{\Sigma}}$ is the Gauss curvature of $(\underline{\Sigma}^{\text{prin}}, g_{\underline{\Sigma}})$, and the integral is taken over $\underline{\Sigma}^{\text{prin}}$.

For any $r > 0$ small enough, let $\Sigma_r := \Sigma \setminus \cup_{i=1}^k B^G_r(p_i)$, and η_r be the G-invariant logarithmic cut-off function on Σ given by

$$\eta_r(p) := \begin{cases} 0, & d(p) \in [0, r], \\ 2 - \frac{2\log d(p)}{\log r}, & d(p) \in (r, \sqrt{r}], \\ 1, & d(p) \in (\sqrt{r}, \infty), \end{cases}$$

where

$$d(p) := \operatorname{dist}_\Sigma(p, \Sigma \setminus \Sigma^{\text{prin}}) = \operatorname{dist}_\Sigma(p, \cup_{i=1}^k G \cdot p_i).$$

Define then $\underline{\Sigma}_r := \underline{\Sigma} \setminus \cup_{i=1}^k B_r([p_i])$. Note $(\underline{\Sigma}_r, g_{\underline{\Sigma}})$ is a smooth Riemannian manifold (with boundary). We can take any conformal immersion $\phi : \underline{\Sigma}_r \to \mathbb{S}^m$, $m \geq 2$, and define $P : \operatorname{Conf}(\mathbb{S}^m) \to \mathbb{B}_1^{m+1}(0)$ by

$$P(h) := \frac{1}{\int_\Sigma \eta_r u_1}\Big(\int_\Sigma (\eta_r u_1)(h_1 \circ \phi \circ \pi), \ldots, \int_\Sigma (\eta_r u_1)(h_{m+1} \circ \phi \circ \pi)\Big),$$

where $h = (h_1, \cdots, h_{m+1}) \in \operatorname{Conf}(\mathbb{S}^m)$ is any conformal diffeomorphism of $\mathbb{S}^m$ (under the standard metric), and $u_1 : \Sigma \to \mathbb{R}_+$ is the first (G-invariant) eigenfunction of L_Σ. Since $u_1 > 0$ and $\sum_{j=1}^{m+1} h_j^2 = 1$, one easily verifies that P is well defined. Meanwhile, for each $x \in \mathbb{B}^{m+1}$, define $h_x \in \operatorname{Conf}(\mathbb{S}^m)$ as in [27, (1.1)] by

$$h_x(y) := \frac{y + (\mu\langle x, y\rangle + \lambda)\, x}{\lambda(\langle x, y\rangle + 1)}, \quad \text{with } \lambda := (1 - |x|^2)^{-1/2},\ \mu := (\lambda - 1)|x|^{-2}.$$

Then we have a continuous map $f : \mathbb{B}_1^{m+1}(0) \to \mathbb{B}_1^{m+1}(0)$ given by $f(x) = P(h_x)$, which can be continuously extended to $\partial\mathbb{B}_1^{m+1}(0) = \mathbb{S}^m$ by the identity map. Note $\operatorname{Clos}(\mathbb{B}_1^{m+1}(0))$ is homotopic to $f\big(\operatorname{Clos}(\mathbb{B}_1^{m+1}(0))\big)$, and $\operatorname{Clos}(\mathbb{B}_1^{m+1}(0)) \setminus \{x\}$ is homotopic to $\mathbb{S}^m$ for any $x \in \mathbb{B}_1^{m+1}(0)$. Hence, f must be surjective. In particular, there exists $h = (h_1, \ldots, h_{m+1}) \in \operatorname{Conf}(\mathbb{S}^m)$ so that $P(h) = 0$. Thus, we have that $\{\tilde{h}_j := h_j \circ \phi \circ \pi\}_{j=1}^{m+1}$ are G-invariant smooth functions on Σ_r so that

$$\sum_{j=1}^{m+1} \tilde{h}_j^2 = 1 \quad \text{and} \quad \int_\Sigma u_1 \cdot (\eta_r \tilde{h}_j) = 0 \qquad \text{for all } j = 1, \ldots, m+1.$$

Since $\operatorname{Index}_G(\Sigma) = 1$, we see $\delta^2\Sigma(\eta_r \tilde{h}_j \nu) \geq 0$ for all $j = 1, \ldots, m+1$, and

$$\begin{aligned}
\int_{\Sigma_{\sqrt{r}}} \operatorname{Ric}_M(\nu, \nu) + |A|^2 &\leq \int_\Sigma (\operatorname{Ric}_M(\nu, \nu) + |A|^2)\,\eta_r^2 \\
&= \int_\Sigma (\operatorname{Ric}_M(\nu, \nu) + |A|^2) \sum_{j=1}^{m+1} (\eta_r \tilde{h}_j)^2 \\
&\leq \int_\Sigma \sum_{j=1}^{m+1} |\nabla(\eta_r \tilde{h}_j)|^2 \\
&\leq \int_\Sigma \sum_{j=1}^{m+1} \Big[(1+\epsilon)|\nabla \tilde{h}_j|^2 \eta_r^2 + \Big(1 + \frac{1}{\epsilon}\Big)|\nabla \eta_r|^2 \tilde{h}_j^2\Big] \\
&\leq (1+\epsilon) K \cdot \int_{\underline{\Sigma}_r} \sum_{j=1}^{m+1} |\nabla h_j \circ \phi|^2 + \Big(1 + \frac{1}{\epsilon}\Big)\int_\Sigma |\nabla \eta_r|^2 \\
&= 2(1+\epsilon) K \cdot \operatorname{Area}(\underline{\Sigma}_r; (h \circ \phi)^* g_{\mathbb{S}^{m+1}}) + \Big(1 + \frac{1}{\epsilon}\Big)\int_\Sigma |\nabla \eta_r|^2,
\end{aligned}$$

where $\epsilon > 0$ is any constant, $\mathrm{Area}(\underline{\Sigma}_r; (h\circ\phi)^* g_{\mathbb{S}^{m+1}})$ is the area of $\underline{\Sigma}_r$ under the conformal metric $(h\circ\phi)^* g_{\mathbb{S}^{m+1}}$, and the coarea formula is used in the last inequality. Let $A_c(m, \underline{\Sigma}_r)$ be the *m-conformal area* of $\underline{\Sigma}_r$ defined as in [18]:

$$A_c(m, \underline{\Sigma}_r) := \inf_{\phi} \sup_{h\in \mathrm{Conf}(\mathbb{S}^m)} \mathrm{Area}(\underline{\Sigma}_r; (h\circ\phi)^* g_{\mathbb{S}^m}),$$

where the infimum is taken over all nondegenerated conformal map ϕ of $\underline{\Sigma}_r$ into $\mathbb{S}^m$. Since $\phi : \underline{\Sigma}_r \to \mathbb{S}^m$ is arbitrary conformal immersion in the above computation, we have

$$\int_{\Sigma_{\sqrt{r}}} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 2(1+\epsilon) K \cdot A_c(m, \underline{\Sigma}_r) + \left(1+\frac{1}{\epsilon}\right)\int_{\Sigma} |\nabla \eta_r|^2.$$

By [14, Chapter IV, Remark 5.5.1], every closed orientable surface can be conformally branched over $\mathbb{S}^2$ with degree $\lfloor \frac{\mathrm{genus}+3}{2} \rfloor$, where $\lfloor a \rfloor$ is the integer part of $a \in \mathbb{R}_+$. It then follows from [18, Facts 1, 5] that $A_c(m, \underline{\Sigma}_r) \le 4\pi \lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \rfloor$, and thus

$$\int_{\Sigma_{\sqrt{r}}} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 4\pi(1+\epsilon) K \cdot 2 \left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor + \left(1+\frac{1}{\epsilon}\right)\int_{\Sigma} |\nabla \eta_r|^2.$$

Since $\int_{\Sigma} |\nabla \eta_r|^2 \to 0$ as $r \to 0$, we first take $r \to 0$ and then let $\epsilon \to 0$, which gives

$$\int_{\Sigma} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 4\pi K \cdot 2 \left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor.$$

Denote by $\{e_i\}_{i=1}^2$ a local orthonormal basis on Σ. Since $\mathrm{Ric}_M > 0$, we have

$$\mathrm{Ric}_M(\nu,\nu) + |A|^2 = \sum_{i=1}^{2} \mathrm{Ric}_M(e_i, e_i) - 2K_\Sigma > -2K_\Sigma$$

on Σ^{prin}, where K_Σ is the Gauss curvature of Σ. Therefore, by the coarea formula,

$$-2K \int_{\underline{\Sigma}} K_{\underline{\Sigma}} = -2\int_{\Sigma} K_\Sigma < \int_{\Sigma} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 4\pi K \cdot 2 \left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor.$$

Then, it follows from the above *strict* inequality and the Gauss–Bonnet formula (5-2) that $\mathrm{genus}(\pi(\Sigma)) + \sum_{i=1}^k \bigl(1 - \frac{1}{n_i}\bigr) < 5$ and

$$\mathrm{genus}(\Sigma) = 1 + K\left[\mathrm{genus}(\pi(\Sigma)) - 1 + \sum_{i=1}^{k}\left(1 - \frac{1}{n_i}\right)\right] < 1 + 4K.$$

Moreover, one notices that $2\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \rfloor = \mathrm{genus}(\pi(\Sigma)) + 3$ if $\mathrm{genus}(\pi(\Sigma)) \ge 1$ is odd, and $2\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \rfloor = \mathrm{genus}(\pi(\Sigma)) + 2$ if $\mathrm{genus}(\pi(\Sigma)) \ge 0$ is even. Hence, the above computation actually shows

- $\mathrm{genus}(\pi(\Sigma)) + \sum_{i=1}^k \bigl(1 - \frac{1}{n_i}\bigr) < 5$ if $\mathrm{genus}(\pi(\Sigma)) \ge 1$ is odd, and
- $\mathrm{genus}(\pi(\Sigma)) + \sum_{i=1}^k \bigl(1 - \frac{1}{n_i}\bigr) < 4$ if $\mathrm{genus}(\pi(\Sigma)) \ge 0$ is even,

which further implies that $\mathrm{genus}(\pi(\Sigma)) \leq 3$ and $\sum_{i=1}^{k}(1-\frac{1}{n_i}) < 4$. In particular, if $\Sigma \subset M^{\mathrm{prin}}$, then $\sum_{i=1}^{k}\bigl(1-\frac{1}{n_i}\bigr) = 0$ and

$$\mathrm{genus}(\Sigma) = 1 + K(\mathrm{genus}(\pi(\Sigma)) - 1) \leq 1 + 2K.$$

Finally, we see that

$$\begin{aligned}
2c_M W^G(M) &\leq \int_\Sigma \sum_{i=1}^{2} \mathrm{Ric}_M(e_i, e_i) \\
&\leq 4\pi K \cdot 2 \left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor + 2K \int_{\underline{\Sigma}} K_{\underline{\Sigma}} \\
&= 4\pi K \cdot \left(2 - 2\,\mathrm{genus}(\pi(\Sigma)) - \sum_{i=1}^{k} \left(1 - \frac{1}{n_i}\right) + 2 \left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor \right) \\
&\leq 16\pi K,
\end{aligned}$$

where $\mathrm{Ric}_M \geq c_M > 0$. □

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