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It is of interest to extend classical geometric notions to generalized geometry. Various approaches have been proposed in the recent literature. Employing a class of generalized connections, we describe certain differential complices $(\widetilde{\Omega}^*_{\mathbb{T}}(M), \widetilde{\mathbb{d}}^{\mathbb{T}})$ constructed from $\bigwedge^* \mathbb{T}M$ and study some of their basic properties, where $\mathbb{T}M = TM \oplus T^*M$ is the generalized tangent bundle on M. To illustrate how various constructions fit together from this point of view, we describe within the proposed framework the analogues to the Levi-Civita connection when $\mathbb{T}M$ is endowed with a generalized metric and a structure of exact Courant algebroid, the Chern–Weil homomorphism, a Weitzenböck identity, the Ricci flow as a Lax flow and Ricci soliton, the Hermitian–Einstein equation and the degree of a holomorphic vector bundle.

1. Introduction

In generalized geometry à la Hitchin [24], over a smooth manifold M of real dimension n, the bundle $\mathbb{T}M := TM \oplus T^*M$ is considered the analogue of the classical tangent bundle TM. It fits into the natural exact sequence

$$0 \to T^*M \hookrightarrow \mathbb{T}M \xrightarrow{\pi} TM \to 0$$

and is endowed with the natural pairing

$$\langle x, y\rangle = \langle X+\xi, Y+\eta\rangle := \tfrac{1}{2}(\iota_X\eta + \iota_Y\xi),$$

where $x, y \in C^\infty(\mathbb{T}M)$ and $X, Y \in C^\infty(TM)$, $\xi, \eta \in C^\infty(T^*M)$ are their respective components. The dual of $\mathbb{T}M$ can be identified with itself under the pairing $2\langle\ ,\ \rangle$. Well-known geometric structures on $\mathbb{T}M$ such as generalized complex, Riemannian, Hermitian, and Kähler structures and generalized connections are natural extensions of the corresponding classical notions on TM. There are by now many references in the literature, including the pioneering works by Gualtieri [19; 20; 21; 22] on the subjects. We show that the analogy can be pushed further, with $\mathbb{T}M$ consistently

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taking the role of the tangent bundle, leading to coherent extensions of well-known geometric notions.

One of the motivations of this work is to understand an analogue of the Hermitian–Einstein equation proposed by Hitchin [25, Remark in §3.3] to describe a stability condition on generalized holomorphic bundles. One of the obstacles is that some of the most natural choices for a curvature operator, such as the *naïve* curvature operator (2-10), are in general *not* tensorial, and hence are not directly suitable for such an analogue or defining the corresponding notions of degree, or slope stability.

In the literature, there have been various attempts at extending the notion of curvature tensor and related constructions to generalized geometry. In Streets [35], the generalized Ricci flow (7-13) is put in Lax form, where a generalized Ricci tensor is constructed from the Ricci tensor of $\nabla^{-\phi}$, the metric connection with totally skew torsion $-\phi \in \Omega^3(M)$. Ševera and Valach [37; 38] extended similar constructions to general Courant algebroids. In Garcia-Fernandez [10] and Garcia-Fernandez and Streets [12], many notions related to those discussed in this article were discussed in somewhat different contexts. For instance, the notion of metric compatible generalized connection and their eigendecomposition with respect to the generalized metric $\mathbb{G}$ can be found in [10] (see also Definition 2.4). Based on the generalized torsion in [20] and the notion of *divergence*, Garcia-Fernandez and Streets [10; 12] discussed a notion of generalized Levi-Civita connections associated to a generalized metric $\mathbb{G}$ on a Courant algebroid E. Different from the constructions proposed here (Theorem 2.8), the generalized Levi-Civita connections described in [10; 12] are not uniquely determined by $\mathbb{G}$ and the structure of exact Courant algebroid on $\mathbb{T}M$, but form an affine space modeled on a certain space of 3-tensors defined from the eigenbundles of $\mathbb{G}$ (see the discussion surrounding Proposition 3.15 in [12]). Moreover, Garcia-Fernandez and Streets [10; 12] constructed generalized curvature operators for the Courant algebroid E involving only mixed eigensubbundles of $\mathbb{G}$, and an algebraic Bianchi identity was shown involving these components. The resulting generalized Ricci tensors thus only have components that involve different eigensubbundles of $\mathbb{G}$, which provide a description of the generalized Ricci flow in Lax form, with the Ricci curvature as the Lax operator, as in [10, (5.3)] and [12, Remark 4.8]. Using spinors in generalized geometry, Goto [14; 15] and Wang [39] considered the notion of scalar curvature for generalized Kähler manifolds and related constructions. Besides in [12], the discussion of Ricci soliton in generalized geometry has appeared for example in Apostolov, Streets, and Ustinovskiy [3] and Lee [31]. In Garcia-Fernandez, Jordan, and Streets [13] and Garcia-Fernandez and Molina [11], the Hermitian–Einstein equations are considered respectively in the context of pluriclosed flow and the Hull–Strominger system.

As will be described in more detail below, the framework proposed in this article leads to curvature tensors for a generalized connection on *any* vector bundle V, as a section of $\bigwedge^2 \mathbb{T}M \otimes \mathrm{End}(V)$. Such a curvature tensor provides a natural generalization of the Hermitian–Einstein equation (Definition 6.8) to generalized geometry. It also produces a *generalized Ricci tensor* as a section of the bundle $\bigotimes^2 \mathbb{T}M$, which, analogously to the classical case, is symmetric (Section 4B). Using the generalized Ricci curvature as the Lax operator, the Lax equation recovers again the generalized Ricci flow (7-13). It is interesting to note that the equation in Lax form naturally picks out the mixed components of the generalized Ricci tensor (see Theorem 7.14). Indeed, one of the main advantages of the framework we propose is that the extension to generalized geometry of many classical notions follows closely the classical constructions. Hence, for the benefit of brevity, we often omit computations that are in parallel to the classical situations, such as those in the standard textbooks, e.g., do Carmo [6], Griffiths and Harris [18], Lee [30] and Petersen [33]. A large portion of the article consists of examples illustrating the various extensions. Since this is the first of a series of articles exploring the consequences of the proposed framework, we leave discussion of further consequences to future works.

For simplicity, we will restrict our considerations to compact connected oriented smooth manifolds without boundary, while, aside from cohomology computations, most descriptions are of a local nature. The construction starts with a generalized connection $\mathbb{V}^{\mathbb{T}}$ ([20] or (2-1)) on $\mathbb{T}M$. Suppose that $\mathbb{V}^{\mathbb{T}}$ preserves the pairing $\langle\ ,\ \rangle$ and is TM*-torsion-free* (Definition 2.1). It then induces a differential complex, constructed with $\mathbb{T}M$ in place of TM, as a quotient of $\Omega^*_{\mathbb{T}}(M) := C^\infty\big(\bigwedge^* \mathbb{T}M\big)$.

Theorem 1.1 (Section 2A). *Consider the derivation* $\mathbb{d}^{\mathbb{T}} : \Omega^*_{\mathbb{T}}(M) \to \Omega^{*+1}_{\mathbb{T}}(M)$ *defined by*

$$(\mathbb{d}^{\mathbb{T}}\theta)(x_0, x_1, \dots, x_k) := \sum_i (-1)^i (\mathbb{V}^{\mathbb{T}}_{x_i}\theta)(x_0, x_1, \dots, \hat{x}_i, \dots, x_k),$$

where $\theta \in \Omega^k_{\mathbb{T}}(M)$ *and* $x_j \in C^\infty(\mathbb{T}M)$. *Then* $\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}$ *is tensorial if and only if* $\mathbb{V}^{\mathbb{T}}$ *is* TM*-torsion-free, in which case, the quotient* $\widetilde{\Omega}^*_{\mathbb{T}}(M)$ *of* $\Omega^*_{\mathbb{T}}(M)$ *by the image of* $\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}$ *is a differential complex with the induced derivation* $\widetilde{\mathbb{d}}^{\mathbb{T}}$, *whose cohomology is denoted by* $\widetilde{H}^*_{\mathbb{T}}(M)$.

The differential calculus thus established on $\mathbb{T}M$ leads to a natural definition (3-2) of the tensorial $\mathbb{V}^{\mathbb{T}}$*-curvature* $\mathcal{F}^{\mathbb{T}}(\mathbb{V}) \in \Omega^2_{\mathbb{T}}(\mathrm{End}(V))$ for any generalized connection $\mathbb{V}$ on any vector bundle V. A side effect of this is that the resulting curvature tensor $\mathcal{F}^{\mathbb{T}}$ now depends on the TM-torsion-free generalized connection $\mathbb{V}^{\mathbb{T}}$ on $\mathbb{T}M$. Nonetheless, the differential Bianchi identity holds in the quotient $\widetilde{\Omega}^*_{\mathbb{T}}(\mathrm{End}(V))$ (Lemma 3.2). Passing to a further quotient $\overline{\Omega}^*_{\mathbb{T}}(\mathrm{End}(V))$ (3-11), the Chern–Weil homomorphism naturally extends. The results in Section 3A can be summarized as the following theorem.

Theorem 1.2. *Any invariant polynomial of the $\mathbb{V}^{\mathbb{T}}$-curvature $\mathcal{F}^{\mathbb{T}}(\mathbb{V})$ defines a class in $\overline{H}^*_{\mathbb{T}}(M)$, the reduced $\mathbb{V}^{\mathbb{T}}$-de Rham cohomology, which coincides with the image under $\overline{\pi}^*$ of the corresponding classical characteristic class in $H^*(M)$.*

When $\mathbb{T}M$ is endowed with a generalized Riemannian structure $\mathbb{G}$ ([19; 21] or (2-24)), and a structure of exact Courant algebroid defined by a closed 3-form $\gamma \in \Omega^3(M)$, there exists an analogue to the classical Levi-Civita connection. On the exact Courant algebroid $(\mathbb{T}M, \langle\ ,\ \rangle, \pi, *_\gamma)$, where $*_\gamma$ is the Dorfman bracket (2-36), for lack of better terminology and risking conflicts with [10; 12], the (*generalized*) *Levi-Civita connection* $\mathbb{V}^\phi$ for $\mathbb{G}$ is the *unique* $\mathbb{G}$-adapted connection on $\mathbb{T}M$ that is metric compatible with $*_\gamma$ (Theorem 2.8)

Theorem 1.3 (Theorem 2.8, Section 2C). *Let $\mathbb{G}$ be a generalized metric on $\mathbb{T}M$. The* ***(generalized) Levi-Civita connection*** *$\mathbb{V}^\phi$ is the unique TM-torsion-free $\mathbb{G}$-metric connection on $\mathbb{T}M$ that is metric compatible with the Dorfman bracket $*_\gamma$. The natural map $\widetilde{\pi}^* : H^*(M) \to \widetilde{H}^*_{\phi,\mathbb{G}}(M)$ is injective. Moreover, $\widetilde{H}^{2n}_{\phi,\mathbb{G}}(M) \cong \mathbb{R}$.*

The notion *ϕ-curvature* refers to the generalized curvature defined with $\mathbb{V}^\phi$ for a generalized connection $\mathbb{V}$ on a vector bundle V. The analogue to the Riemannian curvature in this context is the ϕ-curvature for $\mathbb{V}^\phi$ itself, denoted by $\mathcal{R}^\phi$ (Definition 4.1). The *ϕ-Ricci curvature* $\mathcal{R}ic^\phi$ (Definition 4.6) and the corresponding scalar curvature (Section 4D) are defined via the usual contractions of $\mathcal{R}^\phi$. In particular, in close analogy with the classical case, $\mathcal{R}ic^\phi$ is an endomorphism of $\mathbb{T}M$, and the corresponding Ricci tensor is symmetric (Section 4B). To illustrate the natural parallel with the classical situation, we show an analogue to the Weitzenböck identity (Theorem 4.7), i.e., the Bochner and Hodge Laplacians differ by the ϕ-Ricci curvature of $\mathbb{G}$.

We next turn to generalized complex geometry. On a generalized complex manifold $(M, \gamma; \mathbb{J})$ [19; 21], $\mathbb{J}$ is integrable with respect to $*_\gamma$. When a generalized connection $\mathbb{V}^{\mathbb{T}}$ is *$\mathbb{J}$-compatible with $*_\gamma$* (Definition 5.1), $\mathbb{d}^{\mathbb{T}}$ decomposes (Lemma 5.2) according to the types with respect to $\mathbb{J}$. Together with a generalized metric $\mathbb{G}$ commuting with $\mathbb{J}$, the resulting generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$ corresponds classically to an almost bi-Hermitian structure $(M, \gamma; g, I_\pm; b)$, where $b \in \Omega^2(M)$ and g is Hermitian with respect to both almost complex structures $I_\pm$. Letting $\phi = \gamma + db$, the $\mathbb{J}$-compatibility of $\mathbb{V}^\phi$ with $*_\gamma$ (Definition 5.1) is equivalent to a generalized Kähler condition given in [20].

Theorem 1.4 (Theorem 5.7). *On a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, let $\phi = \gamma + db$. Then $(M, \gamma; \mathbb{G}, \mathbb{J})$ is a generalized Kähler manifold if and only if $\mathbb{V}^\phi$ is $\mathbb{J}$-compatible with $*_\gamma$.*

In terms of $I_\pm$, the $\mathbb{J}$-compatibility is equivalent to $\nabla^{\pm\phi} I_\pm = 0$. On a generalized Kähler manifold, $I_\pm$ are integrable. Working with $\mathbb{V}^\phi$, we recover a well-known result obtained via holomorphic reduction in [22], namely, the $I_\pm$-(anti)holomorphic tangent bundles on a generalized Kähler manifold carry natural $I_\mp$-holomorphic structures (Proposition 5.11).

For a $\mathbb{J}$-holomorphic Hermitian vector bundle $(V, \bar{\partial}_{\mathbb{J}}, h)$ [19; 21], the notion of Chern connection extends naturally (6-5). Over a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, there is a natural contraction $\Lambda_{\mathbb{J}_-}$ on $\Omega^2_{\mathbb{T}}(M)$ (Definition 6.7), which leads to the notion of degree (Definition 6.10) for a $\mathbb{J}$-holomorphic Hermitian vector bundle. The degree is independent of the choice of Hermitian metric on V if the generalized Hermitian manifold is $\nabla^{\mathbb{T}}$-$\mathbb{J}$-*Gauduchon* (Definition 6.13). For such manifolds, the notions of slope and slope stability naturally extend (Definition 6.14). Given a γ-$\mathbb{J}$-connection $\nabla^{\mathbb{T}}$ (Definition 5.1) on $\mathbb{T}M$, in analogy with the classical case (Lübke and Teleman [32]), we propose the $\nabla^{\mathbb{T}}$-$\mathbb{J}$-*Hermitian–Einstein equation* (Definition 6.8) for the Hermitian metric h on V. Similarly to the classical situation, one should expect a version of Kobayashi–Hitchin correspondence to hold in this case (see Hu, Moraru, and Seyyedali [28]). On a generalized Kähler manifold, we show (Proposition 6.9) that these notions relate to their classical counterparts, in particular, the $\mathbb{J}$-Hermitian–Einstein equation is equivalent to an equation proposed by Hitchin [25, Remark in §3.3].

Theorem 1.5 (Section 6C). *Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a $\mathbb{J}$-Gauduchon generalized Kähler manifold and $\omega_\pm \in \Omega^2(M)$ be the Kähler forms for $I_\pm$ respectively. Let $(V, \partial_{\mathbb{J}})$ be a $\mathbb{J}$-holomorphic vector bundle. Then the $\mathbb{J}$-Hermitian–Einstein equation is equivalent to*

$$\tfrac{\sqrt{-1}}{2}\left(F^C_+(V) \wedge \omega_+^{m-1} + (-1)^\varepsilon F^C_-(V) \wedge \omega_-^{m-1}\right) = c(m-1)!\,\mathrm{Id}_V\, d\mathrm{vol}_g,$$

where $F^C_\pm$ are the classical Chern curvatures with respect to $I_\pm$, $\varepsilon = 0$ if $I_\pm$ induce the same orientation on TM, and $\varepsilon = 1$ otherwise.

Geometric flows such as the mean curvature flow (Brakke [5]) and the Ricci flow (Hamilton [23]) are very important in understanding smooth manifolds and structures associated to them. In generalized geometry, it is natural to consider flows involving structures on $\mathbb{T}M$ such as the generalized metrics or generalized complex structures, e.g., in [10; 12; 36]. In this context, we generally assume that the flow preserves the structure of Courant algebroid on $\mathbb{T}M$ defined by the Dorfman bracket $*_\gamma$.

We describe a general construction of Lax flows of generalized metrics or generalized complex structures in the proposed framework. A Lax flow can be defined from any $\theta \in \Omega^2_{\mathbb{T}}(M)$ (Lemma 7.6) via the induced map $\theta : \mathbb{T}M \to \mathbb{T}M$. In particular, the Lax flow defined by the ϕ-curvature of a Hermitian line bundle generates the action of generalized symmetries on $\mathbb{T}M$ (Theorem 7.10). Even though the Bianchi identities do not hold for $\mathcal{R}^\phi$ in general (Lemma 4.3), it turns out that the ϕ-Ricci tensor $\mathcal{R}c^{\phi}$ is symmetric (Section 4B). The corresponding Lax flow is the *Ricci Lax flow* (7-11), which exactly recovers the *generalized Ricci flow* in the mathematics and physics literature; see, for instance, [12]. Conformal deformations of the Riemannian metric g can be represented as a Lax flow, where the Lax operators are $\mathbb{G}_t$-conformal $\mathbb{T}M$-forms (Definition 7.7). The *Ricci soliton equation* (Streets [34]) can be obtained as a combination of the geometric Lax flows described so far (Definition 7.16), involving the generalized Ricci curvature, generalized curvature of line bundles and conformal $\mathbb{T}M$-forms. We also see that the classical Kähler–Ricci flow can be recast as a geometric Lax flow (Section 7D).

We expect that many classical constructions should admit natural extensions to $\mathbb{T}M$ via the differential calculus developed here. Spinors, which are behind the notion of $\mathbb{J}_-$-contraction in Definition 6.7, relate the geometry on $\mathbb{T}M$ back to $\Omega^*(M)$, and, in particular, lead to the canonical line of a generalized (almost) complex structure [19; 21] as well as the notion of scalar curvature in generalized Kähler geometry [14; 15; 39]. Functionals involving curvatures, such as the Yang–Mills functional, can be extended (Section 3C) and lead to natural questions on extremal/critical (generalized) connections/metrics with respect to them. Explicit examples such as compact Lie groups ([15]; Hu [26]) could provide further insights into understanding these extensions. It should be worth exploring the interaction of the Riemannian, the complex and the Poisson geometric methods in generalized Hermitian geometry. Equations in Lax form admit geometric interpretations (Griffiths [17]), and it would be interesting to understand if this provides new perspective for the related geometric flows. We plan to come back to these topics in future works.

We briefly summarize the structure of the paper. In Section 2, we set up the differential calculus on $\mathbb{T}M$ and compute in Section 2E the group $\widetilde{H}^*_{\gamma,\mathbb{G}}(G)$ for a compact Lie group G, with the bi-invariant metric and the Cartan 3-form γ. The generalized curvature tensors are introduced in Section 3. The rest of the article applies the constructions in various contexts. The analogue to the Riemann curvature is discussed in Section 4, together with the associated Ricci and scalar curvatures, as well as the generalized Bismut connections [20]. In Section 5, we apply the differential calculus to generalized complex and Hermitian manifolds. The degree, stability and Hermitian–Einstein equation for a generalized holomorphic bundle over a generalized Hermitian manifold are discussed in Section 6. In Section 7, we discuss the notion of geometric Lax flows.

2. Differential calculus on $\mathbb{T}M$

Let $V \to M$ be a vector bundle. Recall that a generalized connection on V is a derivation:

$$\nabla : C^\infty(V) \to C^\infty(\mathbb{T}M \otimes V) \quad \text{such that} \quad \nabla(fv) = df \otimes v + f\nabla v, \tag{2-1}$$

where $\mathbb{T}M = TM \oplus T^*M$, $f \in C^\infty(M)$ and $v \in C^\infty(V)$. It is the *lift* of a classical connection ∇_0 on V if

$$\mathbb{\nabla}_x v = \nabla_{0,\pi(x)} v \tag{2-2}$$

for all $x \in C^\infty(\mathbb{T}M)$ and $v \in C^\infty(V)$. The generalized connections naturally extend to tensor bundles in the standard fashion.

2A. $\mathbb{T}M$-forms. Under the pairing $2\langle\ ,\ \rangle$, sections of $\bigwedge^* \mathbb{T}M$ can be seen as $\mathbb{T}M$-*forms* and the space of such forms will be suggestively denoted by

$$\Omega^*_{\mathbb{T}}(M) := C^\infty\big(\textstyle\bigwedge^* \mathbb{T}M\big).$$

Let $\mathbb{\nabla}^{\mathbb{T}}$ be a generalized connection on $\mathbb{T}M$ preserving $\langle\ ,\ \rangle$. The skew-symmetrization of the covariant derivative by $\mathbb{\nabla}^{\mathbb{T}}$ induces the $\mathbb{\nabla}^{\mathbb{T}}$-*derivation* $\mathbb{d}^{\mathbb{T}}$. Namely, for $\theta \in \Omega^k_{\mathbb{T}}(M)$,

$$(\mathbb{d}^{\mathbb{T}}\theta)(x_0, x_1, \ldots, x_k) := \sum_i (-1)^i (\mathbb{\nabla}^{\mathbb{T}}_{x_i}\theta)(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_k), \tag{2-3}$$

where $x_i \in C^\infty(\mathbb{T}M)$. For $f \in C^\infty(M)$, $\mathbb{d}^{\mathbb{T}}$ coincides with the usual differential:

$$\mathbb{d}^{\mathbb{T}} f := df \in \Omega^1_{\mathbb{T}}(M). \tag{2-4}$$

Moreover, $\mathbb{d}^{\mathbb{T}}$ is a graded derivation on $\Omega^*_{\mathbb{T}}(M)$, that is, for $\theta_1 \in \Omega^k(M)$ and $\theta_2 \in \Omega^*(M)$,

$$\mathbb{d}^{\mathbb{T}}(\theta_1 \wedge \theta_2) = (\mathbb{d}^{\mathbb{T}}\theta_1) \wedge \theta_2 + (-1)^k \theta_1 \wedge (\mathbb{d}^{\mathbb{T}}\theta_2). \tag{2-5}$$

Let the $\mathbb{\nabla}^{\mathbb{T}}$-*diamond bracket* $\diamond_{\mathbb{T}}$ be the skew-symmetrization of $\mathbb{\nabla}^{\mathbb{T}}$:

$$x \diamond_{\mathbb{T}} y := \mathbb{\nabla}^{\mathbb{T}}_x y - \mathbb{\nabla}^{\mathbb{T}}_y x. \tag{2-6}$$

Recall that the notion of *generalized torsion* for $\mathbb{\nabla}^{\mathbb{T}}$ is introduced in [20], in the context of generalized Kähler conditions (see also Remark 5.8). Here, a different notion of *TM-torsion* is more convenient.

Definition 2.1. Let $\mathbb{\nabla}^{\mathbb{T}}$ be a generalized connection on $\mathbb{T}M$. Its *TM-torsion* is

$$\tau_T(x, y) := \pi(x \diamond_{\mathbb{T}} y) - [\pi(x), \pi(y)], \tag{2-7}$$

where $x, y \in C^\infty(\mathbb{T}M)$. Then $\mathbb{\nabla}^{\mathbb{T}}$ is *TM-torsion-free* if its TM-torsion vanishes.

Standard computations yield

$$\begin{aligned}(\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}\theta)&(x_0, x_1, \ldots, x_{k+1}) \\ &= \sum_{i<j} (-1)^{i+j} [\tau_T(x_i, x_j)]\theta(x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1}) \\ &\quad - \sum_{i<j<\ell} (-1)^{i+j+\ell} \theta([x_i \diamond_{\mathbb{T}} x_j \diamond_{\mathbb{T}} x_\ell], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, \hat{x}_\ell, \ldots, x_{k+1}),\end{aligned} \tag{2-8}$$

where $[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z]$ is the *Jacobiator* of the diamond bracket $\diamond_{\mathbb{T}}$ (2-6):

$$[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] := (x \diamond_{\mathbb{T}} y) \diamond_{\mathbb{T}} z + \text{c.p.} \tag{2-9}$$

Let $\mathcal{R}^{\mathbb{T}}$ be the *naïve* curvature operator for $\nabla^{\mathbb{T}}$, which may not be tensorial:

$$\mathcal{R}^{\mathbb{T}}_{x,y} z := \nabla^{\mathbb{T}}_x \nabla^{\mathbb{T}}_y z - \nabla^{\mathbb{T}}_y \nabla^{\mathbb{T}}_x z - \nabla^{\mathbb{T}}_{\nabla^{\mathbb{T}}_x y} z + \nabla^{\mathbb{T}}_{\nabla^{\mathbb{T}}_y x} z, \tag{2-10}$$

where $x, y, z \in C^\infty(\mathbb{T}M)$. Then a straightforward rearrangement gives

$$[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] = -\mathcal{R}^{\mathbb{T}}_{x,y} z - \text{c.p.} \tag{2-11}$$

It follows that $\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}$ is tensorial when $\nabla^{\mathbb{T}}$ is TM-torsion-free, since in this case $\mathcal{R}^{\mathbb{T}}$ is tensorial by (3-4). Furthermore, $\nabla^{\mathbb{T}}$ being TM-torsion-free also implies the Jacobiator for $\diamond_{\mathbb{T}}$ has values in T^*M:

$$\pi[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] = \big[[\pi(x), \pi(y)], \pi(z)\big] + \text{c.p.} = 0. \tag{2-12}$$

Any torsion-free affine connection ∇^T on TM lifts to a TM-torsion-free generalized connection on $\mathbb{T}M$:

$$\nabla^{\mathbb{T}}_x y := \nabla^T_X y = \nabla^T_X Y + \nabla^T_X \eta$$

for $x, y \in C^\infty(\mathbb{T}M)$ with $x = X + \xi$ and $y = Y + \eta$. The affine space $\mathscr{D}(\mathbb{T}M)$ of generalized connections on $\mathbb{T}M$ is modeled on the space of bundle homomorphisms

$$\mathscr{D}(\mathbb{T}M) \cong \{A : \mathbb{T}M \otimes \mathbb{T}M \to \mathbb{T}M\},$$

in which the subspace $\mathscr{D}_\tau(\mathbb{T}M)$ of TM-torsion-free ones is modeled on the subspace of the right-hand side consisting of the ones whose skew-symmetric part lies in $\Omega^2_{\mathbb{T}}(T^*M)$:

$$\mathscr{D}(\mathbb{T}M) \supseteq \mathscr{D}_\tau(\mathbb{T}M) \cong \mathrm{Sym}^2_{\mathbb{T}}(\mathbb{T}M) \oplus \Omega^2_{\mathbb{T}}(T^*M), \tag{2-13}$$

where $\mathrm{Sym}^2_{\mathbb{T}}(\mathbb{T}M)$ is the space of symmetric 2-tensors

$$\mathrm{Sym}^2_{\mathbb{T}}(\mathbb{T}M) := \{A : \mathbb{T}M \odot \mathbb{T}M \to \mathbb{T}M\}.$$

The *contraction* by $x \in C^\infty(\mathbb{T}M)$ is a graded derivation on $\Omega^*_{\mathbb{T}}(M)$ defined by

$$\iota_x y := 2\langle x, y\rangle, \tag{2-14}$$

where $y \in \Omega^1_{\mathbb{T}}(M)$. The *Lie derivative* along $x \in C^\infty(\mathbb{T}M)$ is given by

$$\mathcal{L}^{\mathbb{T}}_x \theta := \iota_x \mathbb{d}^{\mathbb{T}} \theta + \mathbb{d}^{\mathbb{T}} \iota_x \theta, \tag{2-15}$$

where $\theta \in \Omega^*_{\mathbb{T}}(M)$. In particular, for $f \in C^\infty(M)$ and $X = \pi(x) \in C^\infty(TM)$,

$$\mathcal{L}^{\mathbb{T}}_x f = Xf.$$

Suppose $\nabla^{\mathbb{T}}$ is TM-torsion-free. Then the familiar relations among the operators $\mathbb{d}^{\mathbb{T}}$, ι_x and $\mathcal{L}^{\mathbb{T}}_x$ almost hold, up to possible terms involving the Jacobiator, similarly to (2-8).

Proposition 2.2. *Let* $x, y, z, w \in C^\infty(\mathbb{T}M)$, $\theta \in \Omega^k_{\mathbb{T}}(M)$, $\alpha \in \Omega^1(M)$ *and* $X = \pi(x)$. *Suppose that* $\nabla^{\mathbb{T}}$ *is* TM*-torsion-free. Then*:

(1) $d\alpha = 0 \implies \mathbb{d}^{\mathbb{T}}\alpha = 0$.

(2) $[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} z] \in C^\infty(T^*M)$.

(3) $X\langle y, z\rangle = \langle x \diamond_{\mathbb{T}} y, z\rangle + \langle y, \mathcal{L}^{\mathbb{T}}_x z\rangle$.

(4) $\mathcal{L}^{\mathbb{T}}_x \iota_y \theta - \iota_y \mathcal{L}^{\mathbb{T}}_x \theta = \iota_{x \diamond_{\mathbb{T}} y}\theta$.

(5) $\langle \mathcal{L}^{\mathbb{T}}_x y - x \diamond_{\mathbb{T}} y, z\rangle = \langle \nabla^{\mathbb{T}}_y x, z\rangle + \langle \nabla^{\mathbb{T}}_z x, y\rangle$.

(6) $\langle [\mathcal{L}^{\mathbb{T}}_x, \mathcal{L}^{\mathbb{T}}_y] z, w\rangle = \langle \mathcal{L}^{\mathbb{T}}_{x \diamond_{\mathbb{T}} y} z, w\rangle + \iota_{[x \diamond_{\mathbb{T}} y \diamond_{\mathbb{T}} w]} z$.

(7) *For* $x_1, \dots, x_{k+1} \in C^\infty(\mathbb{T}M)$,

$$([\mathbb{d}^{\mathbb{T}}, \mathcal{L}^{\mathbb{T}}_x]\theta)(x_1, \dots, x_{k+1}) = \sum_{i<j} (-1)^{i+j+1} \theta\big([x \diamond_{\mathbb{T}} x_i \diamond_{\mathbb{T}} x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{k+1}\big).$$

Proof. The verification follows from standard computations and is left for the reader. □

If $\nabla^{\mathbb{T}}$ is TM-torsion-free, by item (1) of Proposition 2.2, the cohomology of $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\mathbb{T}})$ is well defined for degrees $k < 2$. Furthermore, the Jacobiator (2-9) defines a degree 2 map $\mathscr{J}^{\mathbb{T}}$ on $\Omega^*_{\mathbb{T}}(M)$, which commutes with $\mathbb{d}^{\mathbb{T}}$,

$$\begin{aligned}(\mathscr{J}^{\mathbb{T}}\theta)(x_0, \dots, x_{k+1}) &:= (\mathbb{d}^{\mathbb{T}} \circ \mathbb{d}^{\mathbb{T}}\theta)(x_0, \dots, x_{k+1}) \\ &= \sum_{i<j<\ell} (-1)^{i+j+\ell+1} \theta\big([x_i \diamond_{\mathbb{T}} x_j \diamond_{\mathbb{T}} x_\ell], x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, \hat{x}_\ell, \dots, x_{k+1}\big),\end{aligned} \tag{2-16}$$

where $\theta \in \Omega^k_{\mathbb{T}}(M)$ and $x_j \in C^\infty(\mathbb{T}M)$. It follows that $\mathbb{d}^{\mathbb{T}}$ induces a differential $\widetilde{\mathbb{d}}^{\mathbb{T}}$ on the quotient space of the $\nabla^{\mathbb{T}}$*-reduced* $\mathbb{T}M$*-forms*:

$$0 \to \operatorname{img} \mathscr{J}^{\mathbb{T}} \to \Omega^*_{\mathbb{T}}(M) \xrightarrow{\mathscr{Q}^{\mathbb{T}}} \widetilde{\Omega}^*_{\mathbb{T}}(M) \to 0, \tag{2-17}$$

where $\mathscr{Q}^{\mathbb{T}}$ denotes the quotient map. In particular,

$$\widetilde{\Omega}^k_{\mathbb{T}}(M) = \Omega^k_{\mathbb{T}}(M) \quad \text{for } k \leqslant 2. \tag{2-18}$$

Definition 2.3. Let $\nabla^{\mathbb{T}}$ be a TM-torsion-free generalized connection on $\mathbb{T}M$. The complex $(\widetilde{\Omega}^*_{\mathbb{T}}(M), \widetilde{\mathbb{d}}^{\mathbb{T}})$ is the $\nabla^{\mathbb{T}}$*-de Rham complex* and its k-th cohomology is the *k-th* $\mathbb{V}^{\mathbb{T}}$*-de Rham cohomology of* M:

$$\widetilde{H}^k_{\mathbb{T}}(M) := \frac{\ker\big(\widetilde{\mathbb{d}}^{\mathbb{T}} : \widetilde{\Omega}^k_{\mathbb{T}}(M) \to \widetilde{\Omega}^{k+1}_{\mathbb{T}}(M)\big)}{\operatorname{img}\big(\widetilde{\mathbb{d}}^{\mathbb{T}} : \widetilde{\Omega}^{k-1}_{\mathbb{T}}(M) \to \widetilde{\Omega}^k_{\mathbb{T}}(M)\big)}. \tag{2-19}$$

Because $\mathscr{J}^{\mathbb{T}}$ is of degree 2, when $k < 2$, $\widetilde{H}^k_{\mathbb{T}}(M)$ computes the corresponding cohomology groups of $\Omega^*_{\mathbb{T}}(M)$. By (2-4), it is evident that $\widetilde{H}^0_{\mathbb{T}}(M) = \mathbb{R} = H^0(M)$, which consists of the constant functions. Item (1) in Proposition 2.2 then gives a natural inclusion

$$H^1(M) \subseteq \widetilde{H}^1_{\mathbb{T}}(M), \tag{2-20}$$

in which the equality may not hold in general (see Proposition 2.12).

In general, the map π induces a natural injection $\pi^* : \Omega^k(M) \hookrightarrow \Omega^k_{\mathbb{T}}(M)$ for all k:

$$(\pi^*\alpha)(x_1, \dots, x_k) := \alpha(\pi(x_1), \dots, \pi(x_k)), \tag{2-21}$$

where $\alpha \in \Omega^k(M)$ and $x_j \in C^\infty(\mathbb{T}M)$. Alternatively, π^* is induced from the inclusion $T^*M \hookrightarrow \mathbb{T}M$. When $\mathbb{V}^{\mathbb{T}}$ is TM-torsion-free, π^* commutes with the derivations

$$\pi^*(d\alpha) = \mathbb{d}^{\mathbb{T}}(\pi^*\alpha). \tag{2-22}$$

Thus π^* defines a morphism of cochain complices after passing to the quotient

$$\widetilde{\pi}^* : \Omega^k(M) \to \widetilde{\Omega}^k_{\mathbb{T}}(M),$$

which induces the corresponding maps on the cohomology groups:

$$\widetilde{\pi}^* : H^k(M) \to \widetilde{H}^k_{\mathbb{T}}(M). \tag{2-23}$$

2B. $\mathbb{G}$***-adapted connections.*** To restrict $\mathbb{V}^{\mathbb{T}}$ further, consider a generalized metric $\mathbb{G}$ [19; 20]. In the *standard splitting* of $\mathbb{T}M$, which is defined by the inclusion of TM as the first factor of $\mathbb{T}M = TM \oplus T^*M$, $\mathbb{G}$ corresponds to a pair (g, b) of Riemannian metric g on M and 2-form $b \in \Omega^2(M)$:

$$\mathbb{G}(x, y) := \tfrac{1}{2}[g(X, Y) + g^{-1}(\xi - \iota_X b, \eta - \iota_Y b)]. \tag{2-24}$$

Let the $\mathbb{G}$*-splitting* of $\mathbb{T}M$ be

$$s_0 : TM \to \mathbb{T}M \quad \text{given by} \quad X \mapsto X + \iota_X b, \tag{2-25}$$

under which $\mathbb{G}$ can be written as

$$\mathbb{G}(x, y) = \tfrac{1}{2}[g(X, Y) + g^{-1}(x - s_0(X), y - s_0(Y))].$$

Definition 2.4. A generalized connection $\mathbb{V}^{\mathbb{T}}$ on $\mathbb{T}M$ is a $\mathbb{G}$-*metric connection* [10; 12] if it preserves both $\mathbb{G}$ and $\langle\ ,\ \rangle$, i.e.,

$$X\langle y,z\rangle = \langle \mathbb{V}^{\mathbb{T}}_x y, z\rangle + \langle y, \mathbb{V}^{\mathbb{T}}_x z\rangle, \quad X\mathbb{G}(y,z) = \mathbb{G}(\mathbb{V}^{\mathbb{T}}_x y, z) + \mathbb{G}(y, \mathbb{V}^{\mathbb{T}}_x z), \tag{2-26}$$

where $x, y, z \in C^\infty(\mathbb{T}M)$ and $X = \pi(x)$.

A $\mathbb{G}$-metric connection $\mathbb{V}^{\mathbb{T}}$ preserves the ± 1-eigenbundles $C_\pm$ of $\mathbb{G}$:

$$C_\pm := \big\{ s_\pm(X) := (s_0 \pm g)(X) = X + (b \pm g)X : X \in TM \big\}. \tag{2-27}$$

Hence, $\mathbb{V}^{\mathbb{T}}$ admits a $\mathbb{G}$-*eigendecomposition* [10; 12; 20] into four metric connections $\nabla^{\bullet}_{\star}$ on TM:

$$\nabla^{\bullet}_{\star,X} Y := \pi(\mathbb{V}^{\mathbb{T}}_{s_\star(X)} s_\bullet(Y)), \tag{2-28}$$

where $\star$ and $\bullet$ respectively stand for $+$ or $-$. Furthermore, the corresponding $\mathbb{d}^{\mathbb{T}}$ admits the induced $\mathbb{G}$-eigendecomposition.

Lemma 2.5. *Let $\mathbb{V}^{\mathbb{T}}$ be a $\mathbb{G}$-metric connection. Then the operator $\mathbb{d}^{\mathbb{T}}$ decomposes into components as follows*:

$$\mathbb{d}^{\mathbb{T}} = \mathbb{d}^{\mathbb{T}}_+ + \mathbb{d}^{\mathbb{T}}_- : \Omega^{p,q}_{\mathbb{G}}(M) \to \Omega^{p+1,q}_{\mathbb{G}}(M) \oplus \Omega^{p,q+1}_{\mathbb{G}}(M), \tag{2-29}$$

where

$$\Omega^{p,q}_{\mathbb{G}}(M) := C^\infty\big(\textstyle\bigwedge^p C_+ \otimes \bigwedge^q C_-\big) \cong \Omega^p(M) \otimes \Omega^q(M). \tag{2-30}$$

Proof. Consider $\theta \in \Omega^{p,0}_{\mathbb{G}}(M)$. For $k > 1$, since $\mathbb{V}^{\mathbb{T}}$ preserves $C_\pm$, it is straightforward to verify that for $x^j_\pm = s_\pm(X_j)$, where $X_j \in C^\infty(TM)$,

$$(\mathbb{d}^{\mathbb{T}}\theta)(x^0_-, x^1_-, \dots, x^{k-1}_-, x^k_+, \dots, x^q_+) = 0.$$

Thus $\mathbb{d}^{\mathbb{T}}\theta$ cannot contain any components in $\Omega^{p-k+1,k}_{\mathbb{G}}(M)$ for $k > 1$. The general situation follows from the analogue for $\theta' \in \Omega^{0,q}_{\mathbb{G}}(M)$ and noticing that $\mathbb{d}^{\mathbb{T}}$ is a derivation (2-5). □

In its $\mathbb{G}$-eigendecomposition, a $\mathbb{G}$-metric connection $\mathbb{V}^{\mathbb{T}}$ is TM-torsion-free if and only if $\nabla^+_+ = \nabla^-_- = \nabla$ is the Levi-Civita connection for g and for all $X, Y \in C^\infty(TM)$,

$$\nabla^-_{+,X} Y - \nabla^+_{-,Y} X - [X, Y] = 0.$$

It follows that a 3-form $\phi \in \Omega^3(M)$, which may not be closed, can be defined by

$$\phi(X, Y, Z) := 2g((\nabla^+_- - \nabla)_X Y, Z) = 2g((\nabla^-_+ - \nabla)_Y X, Z). \tag{2-31}$$

The mixed components in the $\mathbb{G}$-eigendecomposition can be expressed in terms of ϕ, e.g.,

$$\nabla^+_{-,X} Y = \nabla^{+\phi}_X Y := \nabla_X Y + \tfrac{1}{2} g^{-1} \iota_Y \iota_X \phi, \tag{2-32}$$

which is a metric connection on TM with totally skew torsion ϕ. The computations are summarized in the following theorem/definition.

Theorem 2.6 ($\mathbb{G}$-adapted connections). *The TM-torsion-free $\mathbb{G}$-metric connections on $\mathbb{T}M$ are classified by 3-forms $\phi \in \Omega^3(M)$, which are denoted by $\nabla^{\phi,\mathbb{G}}$, and are referred to as the $\mathbb{G}$-**adapted connections**. The notation $\nabla^{\phi,\mathbb{G}}$ is often abbreviated as ∇^{ϕ} if $\mathbb{G}$ is understood. The $\mathbb{G}$-eigendecomposition of ∇^{ϕ} is*

$$(\nabla^+_+, \nabla^+_-, \nabla^-_+, \nabla^-_-) = (\nabla, \nabla^{+\phi}, \nabla^{-\phi}, \nabla). \tag{2-33}$$

In the $\mathbb{G}$-splitting (2-25) of $\mathbb{T}M$, the $\mathbb{G}$-adapted connection takes the form

$$\nabla^{\phi}_x y = s_0\big[\nabla_X Y + \tfrac{1}{4} g^{-1}(\iota_{g^{-1}\eta}\iota_X\phi - \iota_Y\iota_{g^{-1}\xi}\phi)\big] + \nabla_X\eta + \tfrac{1}{4}(\iota_Y\iota_X\phi - \iota_{g^{-1}\eta}\iota_{g^{-1}\xi}\phi), \tag{2-34}$$

where $x = s_0(X) + \xi$ and $y = s_0(Y) + \eta \in C^\infty(\mathbb{T}M)$.

Proof. Under the pairing $2\langle\ ,\ \rangle$, the space $\mathscr{D}(\mathbb{T}M)$ of generalized connections on $\mathbb{T}M$ is modeled on

$$\mathscr{D}(\mathbb{T}M) \cong \{A : \mathbb{T}M \to \mathbb{T}M \otimes \mathbb{T}M\}.$$

The subspace $\mathscr{D}_{\mathbb{G}}(\mathbb{T}M)$ of $\mathbb{G}$-metric connections is then modeled on

$$\mathscr{D}_{\mathbb{G}}(\mathbb{T}M) \cong \big\{A : \mathbb{T}M \to \big(\textstyle\bigwedge^2 C_+\big) \oplus \big(\bigwedge^2 C_-\big) \cong \mathrm{End}(TM, g)^{\oplus 2}\big\}. \tag{2-35}$$

From (2-13) and (2-35), it follows that space of TM-torsion-free $\mathbb{G}$-metric connections on $\mathbb{T}M$ is modeled on the intersection. The identity (2-31) can be seen also as

$$\phi(X, Y, Z) = 2\langle A_{x_-} y_+, z_+\rangle = -2\langle A_{y_+} x_-, z_-\rangle,$$

which shows that the intersection is isomorphic to $\Omega^3(M)$. The rest follows from straightforward computations. □

Let $\gamma \in \Omega^3(M)$ be a closed 3-form and let $*_\gamma$ be the corresponding *Dorfman bracket* on $C^\infty(\mathbb{T}M)$, where for $X, Y \in C^\infty(TM)$ and $\xi, \eta \in C^\infty(T^*M)$,

$$(X+\xi) *_\gamma (Y+\eta) := [X, Y] + \mathcal{L}_X\eta - d\iota_Y\xi + \iota_Y\iota_X\gamma. \tag{2-36}$$

Definition 2.7. A $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$ is *metric compatible with* $*_\gamma$ if the diamond bracket $\diamond_{\mathbb{T}}$ coincides with $*_\gamma$ on mixed $\mathbb{G}$-eigensections, i.e.,

$$x_+ \diamond_{\mathbb{T}} y_- = x_+ *_\gamma y_-, \tag{2-37}$$

where $x_+ \in C^\infty(C_+)$ and $y_- \in C^\infty(C_-)$.

Using the explicit description (2-34), it is straightforward to verify that

$$x \diamond_\phi y = s_0([X, Y]) + \nabla_X \eta - \nabla_Y \xi + \tfrac{1}{2}(\iota_Y \iota_X \phi - \iota_{g^{-1}\eta}\iota_{g^{-1}\xi}\phi).$$

Hence, ∇^ϕ is metric compatible with $*_{\phi-db}$. The space of $\mathbb{G}$-metric connections that are metric compatible with $*_\gamma$ but not necessarily TM-torsion-free is modeled on the following subspace of (2-35):

$$\mathscr{D}_{\mathbb{G},\gamma}(\mathbb{T}M) \cong \{A : C_+ \to \textstyle\bigwedge^2 C_+\} \oplus \{A : C_- \to \textstyle\bigwedge^2 C_-\} \cong \Omega^1(\mathrm{End}(TM, g))^{\oplus 2}.$$

Theorem 2.8. *For a closed $\gamma \in \Omega^3(M)$, let $\phi = \gamma + db$. Then ∇^ϕ (Theorem 2.6) is the unique TM-torsion-free $\mathbb{G}$-metric connection on $\mathbb{T}M$ that is metric compatible with $*_\gamma$. It is called the* ***(generalized) Levi-Civita connection*** *for $\mathbb{G}$ on the Courant algebroid $(\mathbb{T}M, \langle\ ,\ \rangle, \pi, *_\gamma)$.* □

Remark 2.9. In Theorem 2.8, the exact Courant algebroid $(\mathbb{T}M, \langle\ ,\ \rangle, \pi, *_\gamma)$ can be seen as defining *two levels* of differential structures — represented by the Lie bracket $\langle\ ,\ \rangle$ on $C^\infty(TM)$ and the Dorfman bracket $*_\gamma$ on $C^\infty(\mathbb{T}M)$. The notion of TM-torsion-free specifies the compatibility with the differential structure on TM, while metric compatibility with $*_\gamma$ concerns the differential structure on $\mathbb{T}M$. The uniqueness stemming from these two compatibility conditions is in complete analogue with the classical case, and hence the choice of the notion *generalized Levi-Civita connection*. Moreover, the classical Levi-Civita connection are two of the components in its $\mathbb{G}$-eigendecomposition, as in Theorem 2.6. In [10; 12], the notion *generalized Levi-Civita connection* has been used to describe generalized metric connections satisfying another set of natural conditions, which are not uniquely determined, and come in an affine family modeled on a certain subspace of $C^\infty(\bigwedge^3 \mathbb{T}M)$.

2C. ***∇^ϕ-de Rham cohomology.*** The $\mathbb{G}$-eigendecomposition of $\mathbb{T}M$ induces two left inverses of the natural injection π^* in (2-21). There are two obvious projections $p_\pm : \Omega^k_{\mathbb{T}}(M) \to C^\infty(\bigwedge^k C_\pm)$ for each k. Let $\theta \in \Omega^k_{\mathbb{T}}(M)$, $X_j \in C^\infty(TM)$ and $x^j_\pm = s_\pm(X_j)$ for all $j = 1, \dots, k$. Then

$$(p_\pm\theta)(x^1_\pm, \dots, x^k_\pm) := \theta(x^1_\pm, \dots, x^k_\pm).$$

The projection π induces the natural isomorphisms $\pi_* : C^\infty(\bigwedge^* C_\pm) \cong \Omega^*(M)$:

$$(\pi_*\theta_\pm)(X_1, \dots, X_k) := \theta_\pm(x^1_\pm, \dots, x^k_\pm),$$

where $\theta_\pm \in C^\infty(\bigwedge^* C_\pm)$. Then $\pi^\pm_* = \pi_* \circ p_\pm : \Omega^k_{\mathbb{T}}(M) \to \Omega^k(M)$ is given by

$$(\pi^\pm_*\theta)(X_1, \dots, X_k) := \theta(x^1_\pm, \dots, x^k_\pm).$$

It is now straightforward to see that

$$\pi_*^{\pm} \circ \pi^* = \mathrm{Id} : \Omega^*(M) \to \Omega^*(M).$$

Let $\mathbb{d}^\phi$ denote the $\mathbb{V}^\phi$-derivation (2-3) induced by $\mathbb{V}^\phi$. The decomposition (2-29) can be explicitly described by the classical de Rham differential and covariant derivatives. For instance, for $\alpha \in \Omega^k(M)$, let $\alpha_+ \in \Omega_{\mathbb{G}}^{p,0}(M)$ such that $\alpha = \pi_*^+(\alpha_+)$. Then $\mathbb{d}_+^\phi \alpha_+$ is essentially the de Rham differential, that is,

$$\begin{aligned}(\mathbb{d}^\phi \alpha_+)(x_+^0, x_+^1, \ldots, x_+^k) &= \sum_i (-1)^i (\mathbb{V}^\phi_{x_+^i} \alpha_+)(x_+^0, x_+^1, \ldots, \hat{x}_+^i, \ldots, x_+^k) \\ &= \sum_i (-1)^i (\nabla_{X_i} \alpha)_+(x_+^0, x_+^1, \ldots, \hat{x}_+^i, \ldots, x_+^k) \\ &= (d\alpha)_+(x_+^0, x_+^1, \ldots, x_+^k),\end{aligned} \tag{2-38}$$

while the component $\mathbb{d}_-^\phi \alpha_+ \in \Omega_{\mathbb{G}}^{p,1}(M)$ is essentially given by $\nabla^{+\phi}$, that is,

$$(\mathbb{d}_-^\phi \alpha_+)(x_-^0, x_+^1, \ldots, x_+^q) = (\nabla^{+\phi}_{X_0} \alpha)_+(x_+^1, \ldots, x_+^q), \tag{2-39}$$

where $x_\pm^j = s_\pm(X_j)$ for $X_j \in C^\infty(TM)$.

Lemma 2.10. *The Jacobiator for $\diamond_\phi$ is given by*

$$[x_\pm \diamond_\phi y_\pm \diamond_\phi z_\mp] = \pm 2 g(R^{\mp\phi}_{X,Y} Z), \tag{2-40}$$

where $x_\pm = s_\pm(X)$ for $X \in C^\infty(TM)$ and so on, and $R^{\pm\phi}$ are respectively the classical curvature for $\nabla^{\pm\phi}$. All other components in the $\mathbb{G}$-eigendecomposition vanish.

Proof. This follows by lengthy but standard computations from the definitions. □

Proposition 2.11. *The natural map $\widetilde{\pi}^* : H^*(M) \to \widetilde{H}^*_{\phi,\mathbb{G}}(M)$ in* (2-23) *is injective.*

Proof. For $\alpha, \beta \in \Omega^*(M)$, suppose that $\pi^*\alpha = \pi^*(d\beta)$. Then the injectivity of π^* implies that $\alpha = d\beta$. The statement then follows from $\operatorname{img} \mathscr{J}^\phi \cap \operatorname{img} \pi^* = \{0\}$. In fact, suppose $\mathscr{J}^\phi \theta = \pi^*\alpha \in \operatorname{img} \mathscr{J}^\phi \cap \operatorname{img} \pi^*$. Then

$$\alpha = \pi_*^+(\pi^*\alpha) = \pi_*^+(\mathscr{J}^\phi \theta) = 0.$$

The last equality is due to $[x_+ \diamond_\phi y_+ \diamond_\phi z_+] = 0$, which follows from Lemma 2.10. □

Since $\widetilde{\Omega}^k_{\mathbb{T}}(M) \cong \Omega^k_{\mathbb{T}}(M)$ for $k \leqslant 2$ and the derivations coincide for $k < 2$, the groups $\widetilde{H}^k_{\phi,\mathbb{G}}(M)$ for $k < 2$ can be determined. When $k = 0$, it is easy to see that $\widetilde{H}^0_{\phi,\mathbb{G}}(M) \cong \mathbb{R}$ consists of the constant functions.

Proposition 2.12. *Let $P^1_\phi(M)$ be the space of ∇-parallel 1-forms on M which also annihilates ϕ, i.e.,*

$$\xi \in P^1_\phi(M) \quad \Longleftrightarrow \quad \nabla\xi = 0 \quad \textit{and} \quad \iota_{g^{-1}\xi}\phi = 0.$$

Then

$$\widetilde{H}^1_{\phi,\mathbb{G}}(M) \cong H^1(M) \oplus P^1_\phi(M). \tag{2-41}$$

Proof. Let $\theta \in \Omega^1_{\mathbb{T}}(M)$. For $X, Y \in C^\infty(TM)$, let $x_\pm = s_\pm(X)$ and so on, and define

$$\alpha(X) := \theta(x_+) \quad \text{and} \quad \beta(X) := \theta(x_-). \tag{2-42}$$

This then gives the identification

$$Q : \Omega^1_{\mathbb{T}}(M) \cong \Omega^1(M) \oplus \Omega^1(M) \quad \text{given by} \quad \theta \mapsto \tfrac{1}{2}(\alpha+\beta, \alpha-\beta). \tag{2-43}$$

Suppose that $\mathbb{d}^\phi \theta = 0$. Then (2-38) implies that

$$d\alpha = d\beta = 0.$$

Hence

$$0 = (\mathbb{d}^\phi_- \theta)(x_+, y_-) = -[\nabla^{+\phi}_Y(\alpha - \beta)](X),$$

i.e., $\nabla^{+\phi}(\alpha-\beta) = 0$. Set $\xi = \alpha - \beta$. Then $d\xi = 0$ implies that

$$0 = (\nabla^{+\phi}_X \xi)(Y) - (\nabla^{+\phi}_Y \xi)(X) = -\phi(X, Y, g^{-1}\xi)$$

for all $X, Y \in C^\infty(TM)$, from which $\xi \in P^1_\phi(M)$ follows. Thus, on $\Omega^1_{\mathbb{T}}(M)$,

$$Q(\ker \mathbb{d}^\phi) = \ker d \oplus P^1_\phi(M).$$

Then (2-41) follows from $Q(df) = (df, 0)$ for $f \in C^\infty(M)$. □

Corollary 2.13. $\widetilde{H}^1_{\phi,\mathbb{G}}(M) \cong H^1(M)$ *if one of the following holds*:

(1) ϕ *is nondegenerate*, *i.e.*, *the following map is injective*:

$$\iota_\bullet\phi : C^\infty(TM) \to \Omega^2(M) \quad \textit{given by} \quad X \mapsto \iota_X\phi.$$

(2) M *admits no nontrivial* ∇*-parallel vector fields.* □

Example 2.14. Let $M = \mathbb{R}^n/\mathbb{Z}^n$, with the induced flat metric. Suppose that $\phi = 0$, and thus the ∇-parallel forms on M are the constant forms. Then

$$\widetilde{H}^1_{0,\mathbb{G}}(M) \cong H^1(M) \oplus T_0M \cong \mathbb{R}^{2n},$$

where T_0M is the tangent space at $0 \in M$. On the other hand, for $n = 3$, let $\phi = \mathrm{vol}_g$, the volume form of the flat metric on M. Then it is nondegenerate. By Corollary 2.13,

$$\widetilde{H}^1_{\mathrm{vol}_g,\mathbb{G}}(M) = H^1(M) \cong \mathbb{R}^3.$$

Let $n = \dim_{\mathbb{R}} M$. The group $H^{2n}_{\mathbb{T}}(M)$ is always well defined for a TM-torsion-free $\nabla^{\mathbb{T}}$:

$$H^{2n}_{\mathbb{T}}(M) := \frac{\Omega^{2n}_{\mathbb{T}}(M)}{\mathrm{img}\big(\mathbb{d}^{\mathbb{T}} : \Omega^{2n-1}_{\mathbb{T}}(M) \to \Omega^{2n}_{\mathbb{T}}(M)\big)}.$$

When $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^{\phi,\mathbb{G}}$ and $d\phi = 0$, the groups $\widetilde{H}^{2n}_{\phi,\mathbb{G}}(M)$ and $H^{2n}_{\phi,\mathbb{G}}(M)$ can be determined.

Proposition 2.15. *Let $\phi \in \Omega^3(M)$ and $d\phi = 0$. Then $\widetilde{H}^{2n}_{\phi,\mathbb{G}}(M) \cong H^{2n}_{\phi,\mathbb{G}}(M) \cong \mathbb{R}$.*

Proof. The derivation $\mathbb{d}^\phi$ on $\Omega^{2n-1}_{\mathbb{T}}(M)$ splits in the decomposition (2-30) as

$$\mathbb{d}^\phi = \mathbb{d}^\phi_+ \oplus \mathbb{d}^\phi_- : \Omega^{n-1,n}_{\mathbb{T}}(M) \oplus \Omega^{n,n-1}_{\mathbb{T}}(M) \to \Omega^{n,n}_{\mathbb{T}}(M) = \Omega^{2n}_{\mathbb{T}}(M).$$

For instance, $\mathbb{d}^\phi_+ : \Omega^{n-1,n}_{\mathbb{T}}(M) \to \Omega^{n,n}_{\mathbb{T}}(M)$ is given by the lifting of $d : \Omega^{n-1}(M) \to \Omega^n(M)$ as in (2-38). Since img $\mathscr{J}^\phi \cap \Omega^{2n}(M) = \{0\}$ by Proposition 4.5, it follows that $\widetilde{\Omega}^{2n}_{\mathbb{T}}(M) = \Omega^{2n}_{\mathbb{T}}(M)$, which implies the statement. □

Let M and M' be two smooth manifolds. It is straightforward to see that if a generalized diffeomorphism $\widetilde{\lambda} = (\lambda, B) : (M, \phi, \mathbb{G}) \to (M', \phi', \mathbb{G}')$ relates the corresponding data on both manifolds, i.e.,

$$\mathbb{G} = \widetilde{\lambda}^*\mathbb{G}' \quad \text{and} \quad \phi = \widetilde{\lambda}^*\phi' = \lambda^*\phi' + dB,$$

it induces isomorphisms throughout the constructions. In particular, it induces the natural isomorphism of cohomology groups $\widetilde{\lambda}^* : \widetilde{H}^*_{\phi',\mathbb{G}'}(M') \xrightarrow{\cong} \widetilde{H}^*_{\phi,\mathbb{G}}(M)$.

2D. *Laplacians.* Analogously to the classical case, a generalized connection $\mathbb{V}$ on V defines the corresponding *Bochner Laplacian* on $C^\infty(V)$. Let $\{X_i\}$ be a local orthonormal frame on TM for the Riemannian metric g. Then

$$\{e^i_\pm := s_\pm(X_i)\}$$

is a local $\mathbb{G}$-orthonormal frame on $\mathbb{T}M$.

Definition 2.16. Let $\mathbb{V}$ be a generalized connection on V. The *Bochner Laplacian* (*for $\mathbb{V}$ with respect to $\mathbb{V}^\phi$*) is defined by

$$\Delta_{\mathbb{V}}\, v := -\sum_i [(\mathbb{V}_{e^i_+}\mathbb{V}_{e^i_+} - \mathbb{V}_{\mathbb{V}^\phi_{e^i_+} e^i_+})v + (\mathbb{V}_{e^i_-}\mathbb{V}_{e^i_-} - \mathbb{V}_{\mathbb{V}^\phi_{e^i_-} e^i_-})v] \tag{2-44}$$

for $v \in C^\infty(V)$.

The operator defined by (2-44) is independent of the choice of $\{X_i\}$. Because the $\mathbb{V}^\phi_{e^i_\pm} e^i_\pm$ involve only the Levi-Civita connection for g, it is evident that

$$\Delta_{\mathbb{V}}\, v = (\Delta_+ + \Delta_-)v,$$

where $\Delta_\pm$ are the classical Bochner Laplacians on V for $\nabla_\pm$ respectively. Thus $\Delta_{\mathbb{V}}$ is a second-order elliptic operator, which in general depends on $\mathbb{G}$, but not on ϕ. When $\mathbb{V}$ is a lift of a classical connection, $\Delta_{\mathbb{V}}$ reduces to (a constant multiple of) the corresponding classical Bochner Laplacian.

The operator $\mathbb{d}^\phi$ can alternatively be written as

$$\mathbb{d}^\phi\theta = \sum_j (e_+^j \wedge \nabla^\phi_{e_+^j}\theta - e_-^j \wedge \nabla^\phi_{e_-^j}\theta), \tag{2-45}$$

where $\theta \in \Omega^*_{\mathbb{T}}(M)$. Thus the principal symbol of $\mathbb{d}^\phi$ is

$$\sigma(\mathbb{d}^\phi) = 2\sqrt{-1}\xi\wedge : \bigwedge^k \mathbb{T}M \to \bigwedge^{k+1}\mathbb{T}M,$$

where $\xi \in T^*M$. It follows that whenever $\mathbb{d}^\phi$ squares to 0 it defines an elliptic complex. Analogously to the classical case, for $\theta \in \Omega^*_{\mathbb{T}}(M)$, define

$$\mathbb{d}^{\phi *}\theta := -\frac{1}{2}\sum_j (\iota_{e_+^j}\nabla^\phi_{e_+^j}\theta + \iota_{e_-^j}\nabla^\phi_{e_-^j}\theta), \tag{2-46}$$

where the $\frac{1}{2}$ is due to the convention (2-14). The principal symbol of $\mathbb{d}^{\phi *}$ is then

$$\sigma(\mathbb{d}^{\phi *}) = -\sqrt{-1}\iota_{g^{-1}\xi + bg^{-1}\xi} : \bigwedge^k \mathbb{T}M \to \bigwedge^{k-1}\mathbb{T}M,$$

where $\xi \in T^*M$. The operators $\mathbb{d}^\phi$ and $\mathbb{d}^{\phi *}$ are formal adjoints with respect to the pairing $(\ ,\ )_{\mathbb{G}}$ on $\bigwedge^*\mathbb{T}M$ induced by $\mathbb{G}$, for which local orthonormal bases are given by $\{\bigwedge_{i\in I} e_+^i \wedge \bigwedge_{j\in J} e_-^j : I, J \subseteq \{1, 2, \ldots, n\}\}$. Indeed, direct computation shows that for $\theta, \mu \in \Omega^*_{\mathbb{T}}(M)$,

$$(\mathbb{d}^\phi\theta, \mu)_{\mathbb{G}} - (\theta, \mathbb{d}^{\phi *}\mu)_{\mathbb{G}} = \frac{1}{2}\sum_j [X_j(\theta, \iota_{s_0(X_j)}\mu)_{\mathbb{G}} - (\theta, \iota_{s_0(\nabla_{X_j}X_j)}\mu)_{\mathbb{G}}],$$

which is the divergence (with respect to g) of the vector field W defined by

$$g(W, Z) = (\theta, \iota_{s_0(Z)}\mu)_{\mathbb{G}}$$

for all $Z \in C^\infty(TM)$.

Definition 2.17. The ∇^ϕ-*Hodge Laplacian* is the operator on $\Omega^*_{\mathbb{T}}(M)$ given by

$$\Delta^\phi := \mathbb{d}^\phi\mathbb{d}^{\phi *} + \mathbb{d}^{\phi *}\mathbb{d}^\phi. \tag{2-47}$$

Proposition 2.18. *The ∇^ϕ-Hodge Laplacian Δ^ϕ is a second-order elliptic operator.*

Proof. It is clear from the discussion above that the principal symbol of Δ^ϕ is

$$\sigma(\Delta^\phi) = 2\|\xi\|_g^2 : \bigwedge^k \mathbb{T}M \to \bigwedge^k \mathbb{T}M,$$

where $\xi \in T^*M$, from which the statement follows. □

Theorem 2.19. *Let M be a closed manifold. Then the following holds*:

$$\widetilde{H}^1_{\phi,\mathbb{G}}(M) \cong \ker \mathbb{d}^\phi|_{\Omega^1_{\mathbb{T}}(M)} \cap \ker \mathbb{d}^{\phi *}|_{\Omega^1_{\mathbb{T}}(M)} \subseteq \ker \Delta^\phi|_{\Omega^1_{\mathbb{T}}(M)}. \tag{2-48}$$

Proof. Let $\theta \in \Omega^1_{\mathbb{T}}(M)$ and consider $\alpha, \beta \in \Omega^1(M)$ as in (2-42). Then it can be shown that

$$\mathbb{d}^{\phi *}\theta = d^*(\alpha + \beta).$$

Combining the identity above with the proof of Proposition 2.12, the first isomorphism follows from the classical Hodge theory. The last inclusion is obvious. □

Even though $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^\phi)$ is generally not a chain complex, Theorem 2.19 nonetheless hints at the analogue of its "cohomology groups".

Definition 2.20. Let $\mathbb{G}$ be a generalized metric on M and $\phi \in \Omega^3(M)$. The $\mathbb{V}^\phi$-*pseudocohomology groups* $\check{H}^*_{\phi,\mathbb{G}}(M)$ of M consist of the common kernels of $\mathbb{d}^\phi$ and $\mathbb{d}^{\phi *}$:

$$\check{H}^k_{\phi,\mathbb{G}}(M) := \ker \mathbb{d}^\phi|_{\Omega^k_{\mathbb{T}}(M)} \cap \ker \mathbb{d}^{\phi *}|_{\Omega^k_{\mathbb{T}}(M)}.$$

The $\mathbb{V}^\phi$-*Laplacian kernels* $\widehat{H}^k_{\phi,\mathbb{G}}(M)$ are the subspaces

$$\widehat{H}^k_{\phi,\mathbb{G}}(M) := \ker \Delta^\phi|_{\Omega^k_{\mathbb{T}}(M)}.$$

Corollary 2.21. *For a closed manifold M, both $\widehat{H}^*_{\phi,\mathbb{G}}(M)$ and $\check{H}^*_{\phi,\mathbb{G}}(M)$ are finite-dimensional.*

Proof. This follows from $\check{H}^k_{\phi,\mathbb{G}}(M) \subseteq \widehat{H}^k_{\phi,\mathbb{G}}(M)$ and the ellipticity of Δ^ϕ. □

Remark 2.22. The inclusions $\check{H}^k_{\phi,\mathbb{G}}(M) \subseteq \widehat{H}^k_{\phi,\mathbb{G}}(M)$ may not be equalities. For $k = 1$, direct computation using the $\mathbb{G}$-eigendecomposition shows that

$$\theta = \widehat{H}^1_{\phi,\mathbb{G}}(M)$$

if and only if

$$\Delta(\alpha+\beta) + \Delta_{+\phi}(\alpha-\beta) - \frac{1}{2}\sum_{j,k}(d\beta)(X_j, X_k)\iota_{X_k}\iota_{X_j}\phi = 0,$$

$$\Delta(\alpha+\beta) - \Delta_{-\phi}(\alpha-\beta) + \frac{1}{2}\sum_{j,k}(d\alpha)(X_j, X_k)\iota_{X_k}\iota_{X_j}\phi = 0,$$

where α, β are as in (2-42) and $\{X_j\}$ is a local g-orthonormal frame of TM. In the case that $d\alpha = d\beta = 0$, it can be shown that the right-hand side is exactly equivalent to $\theta \in \check{H}^1_{\phi,\mathbb{G}}(M)$. Namely, under the identification (2-43),

$$\check{H}^1_{\phi,\mathbb{G}}(M) = \widehat{H}^1_{\phi,\mathbb{G}}(M) \cap (\ker d \oplus \ker d).$$

2E. *Compact Lie groups.* Manifolds admitting flat metric connections with nontrivial completely skew torsions are known by Cartan and Schouten [8; 7], which are essentially compact Lie groups and S^7 (see also Agricola and Friedrich [1]).

Suppose that G is a real semisimple Lie group, endowed with the bi-invariant Killing metric g and the corresponding bi-invariant Cartan 3-form $\gamma \in \Omega^3(G)$. Suppose that G is simply connected. In this case, as will become clear below, the computations are very much parallel to those for the classical de Rham cohomology of the doubled group $G \times G$.

The metric connections $\nabla^{\pm\gamma}$, with torsions $\pm\gamma$ respectively, are flat. Let $\mathbb{G}$ be given by g and $b=0$ and denote the corresponding Levi-Civita connection as $\mathbb{V}^{\gamma}$. By Lemma 2.10, for all $x, y, z \in C^{\infty}(\mathbb{T}G)$,

$$[x \diamond_{\gamma} y \diamond_{\gamma} z] = 0,$$

which implies that $\mathscr{J}^{\gamma} = \mathbb{d}^{\gamma} \circ \mathbb{d}^{\gamma} = 0$. Hence $(\Omega^*_{\mathbb{T}}(G), \mathbb{d}^{\gamma})$ is the $\mathbb{V}^{\gamma}$-de Rham complex, whose cohomology is the $\mathbb{V}^{\gamma}$-de Rham cohomology $\widetilde{H}^*_{\gamma,g}(G)$.

Let X^l_u denote the left-invariant vector field on G such that $X^l_u(e) = u \in \mathfrak{g} := T_eG$. The corresponding right-invariant vector fields are denoted by X^r_u. The Lie algebra structure on $\mathfrak{g}$ is identified with the Lie algebra of left-invariant vector fields:

$$[X^l_u, X^l_v] = X^l_{[u,v]}.$$

For $u, v \in \mathfrak{g}$, set $\theta^r_u := g(X^r_u)$ and $\theta^l_v := g(X^l_v)$. Then

$$x^+_u = X^r_u + \theta^r_u \in C^{\infty}(C_+) \quad \text{and} \quad x^-_v = X^l_v - \theta^l_v \in C^{\infty}(C_-).$$

It is straightforward to see that

$$\mathbb{V}^{\gamma}_{x^+_u} x^+_v = -\tfrac{1}{2} x^+_{[u,v]}, \quad \mathbb{V}^{\gamma}_{x^-_u} x^-_v = \tfrac{1}{2} x^-_{[u,v]}, \quad \text{and} \quad \mathbb{V}^{\gamma}_{x^+_u} x^-_v = \mathbb{V}^{\gamma}_{x^-_u} x^+_v = 0.$$

Let $\mathbb{u} = (u, u'), \mathbb{v} = (v, v') \in \mathfrak{g} \oplus \mathfrak{g}$ and

$$x_{\mathbb{u}} = -x^+_u + x^-_{u'}, \, x_{\mathbb{v}} = -x^+_v + x^-_{v'} \in C^{\infty}(\mathbb{T}G).$$

Then direct computation leads to

$$\mathbb{V}^{\gamma}_{x_{\mathbb{u}}} x_{\mathbb{v}} = \tfrac{1}{2} x_{[\mathbb{u},\mathbb{v}]} \implies x_{\mathbb{u}} \diamond_{\gamma} x_{\mathbb{v}} = x_{\mathbb{u}} *_{\gamma} x_{\mathbb{v}} = x_{[\mathbb{u},\mathbb{v}]}, \tag{2-49}$$

where the Lie bracket on $\mathfrak{g} \oplus \mathfrak{g}$ is the direct sum of those on each factor. The last equality in (2-49) can be seen also from the *Courant trivialization* of Alekseev, Bursztyn, and Meinrenken [2]. The γ-curvature can then be computed as

$$\mathcal{R}^{\gamma}_{x_{\mathbb{u}}, x_{\mathbb{v}}} x_{\mathbb{w}} = -\tfrac{1}{4}\big[[x_{\mathbb{u}}, x_{\mathbb{v}}], x_{\mathbb{w}}\big] = -\tfrac{1}{4} x_{[[\mathbb{u},\mathbb{v}],\mathbb{w}]},$$

which gives the γ-Ricci tensor

$$\mathcal{R}c^{\gamma}(x_{\mathbb{u}}, x_{\mathbb{v}}) = \tfrac{1}{4}\mathbb{G}(x_{\mathbb{u}}, x_{\mathbb{v}}).$$

In particular, $(G, \gamma; \mathbb{G})$ may be seen as an example of a *γ-Einstein manifold*, where the γ-Ricci curvature is proportional to the generalized metric.

To compute $\widetilde{H}^*_{\gamma,g}(G)$, recall $\mathbb{T}G$ is the dual of itself via $2\langle\ ,\ \rangle$, which leads to

$$(\mathbb{d}^{\gamma} x_{\mathbb{u}})(x_{\mathbb{v}}, x_{\mathbb{w}}) = -x_{\mathbb{u}}(x_{[\mathbb{v},\mathbb{w}]}).$$

Let $\theta \in \Omega^*_{\mathbb{T}}(G)$ be decomposable as the product of k sections of the form $x_{\mathbb{v}}$. Then

$$(\mathbb{d}^{\gamma}\theta)(x_{\mathbb{u}_0}, \ldots, x_{\mathbb{u}_k}) = \sum_{i<j} (-1)^{i+j} \theta(x_{[\mathbb{u}_i,\mathbb{u}_j]}, x_{\mathbb{u}_0}, \ldots, \hat{x}_{\mathbb{u}_i}, \ldots, \hat{x}_{\mathbb{u}_j}, \ldots, x_{\mathbb{u}_k}).$$

Let $f_{\mathbb{u}} \in \mathfrak{g}^* \oplus \mathfrak{g}^*$ be defined by

$$f_{\mathbb{u}}(\mathbb{v}) := x_{\mathbb{u}}(x_{\mathbb{v}}).$$

It induces an inclusion of the Chevalley–Eilenberg complex of $\mathfrak{g} \oplus \mathfrak{g}$ for the trivial module:

$$\left(\bigwedge^*(\mathfrak{g}^* \oplus \mathfrak{g}^*), \delta\right) \hookrightarrow (\Omega^*_{\mathbb{T}}(G), \mathbb{d}^{\gamma}) \quad \text{given by} \quad f_{\mathbb{u}} \mapsto x_{\mathbb{u}}.$$

Similarly to the classical case, this induces an isomorphism on the cohomology:

$$\widetilde{H}^*_{\gamma,g}(G) \cong H^*(\mathfrak{g} \oplus \mathfrak{g}) \cong H^*(G \times G).$$

The isomorphism above in fact is an isomorphism of rings, where on $\widetilde{H}^*_{\gamma,g}(G)$ the product is induced by the wedge product in $\Omega^*_{\mathbb{T}}(M)$.

3. Curvature tensors

Consider a generalized connection $\mathbb{V}$ on V. Let $\mathbb{V}^{\mathbb{T}}$ be any generalized connection on $\mathbb{T}M$. The $\mathbb{V}^{\mathbb{T}}$-derivation $\mathbb{d}^{\mathbb{T}}$ (2-3) extends to $\Omega^k_{\mathbb{T}}(V) := C^\infty\left(\bigwedge^k \mathbb{T}M \otimes V\right)$:

(3-1) $$\mathbb{d}^{\mathbb{T}}_{\mathbb{V}}(\theta \otimes v) := (\mathbb{d}^{\mathbb{T}}\theta) \otimes v + (-1)^k \theta \wedge \mathbb{V}v,$$

where $v \in C^\infty(V)$. The $\mathbb{V}^{\mathbb{T}}$-*curvature operator* $\mathcal{F}^{\mathbb{T}}(\mathbb{V})$ of $\mathbb{V}$ is then given by

(3-2) $$\mathcal{F}^{\mathbb{T}}(\mathbb{V}) := \mathbb{d}^{\mathbb{T}}_{\mathbb{V}} \circ \mathbb{V},$$

which generally is not tensorial in v if $\mathbb{V}^{\mathbb{T}}$ is not TM-torsion-free. When $\mathbb{V}$ is understood, it is often dropped from the notation $\mathcal{F}^{\mathbb{T}}(\mathbb{V})$.

In terms of covariant derivatives, the $\mathbb{V}^{\mathbb{T}}$-curvature operator $\mathcal{F}^{\mathbb{T}}$ is given by

(3-3) $$\mathcal{F}^{\mathbb{T}}_{x,y}(\mathbb{V})v := (\mathbb{V}_x\mathbb{V}_y - \mathbb{V}_y\mathbb{V}_x - \mathbb{V}_{x \diamond_{\mathbb{T}} y})v,$$

where $x, y \in C^\infty(\mathbb{T}M)$ and $v \in C^\infty(V)$. It is tensorial if and only if $\mathbb{V}^{\mathbb{T}}$ is TM-torsion-free, in which case, for any $f \in C^\infty(M)$,

(3-4) $$\mathcal{F}^{\mathbb{T}}_{x,y}(fv) - f\mathcal{F}^{\mathbb{T}}_{x,y}v = ([\pi(x), \pi(y)] - \pi(x \diamond_{\mathbb{T}} y))(f)v = 0.$$

The resulting tensor $\mathcal{F}^{\mathbb{T}} \in \Omega^2_{\mathbb{T}}(\mathrm{End}(V))$ is the $\mathbb{V}^{\mathbb{T}}$-*curvature* of $\mathbb{V}$. Similar to (3-1), let $\widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}}$ be the extension of $\widetilde{\mathbb{d}}^{\mathbb{T}}$ to $\widetilde{\Omega}^*_{\mathbb{T}}(V) := \widetilde{\Omega}^*_{\mathbb{T}}(M) \otimes C^\infty(V)$. By (2-18), $\mathcal{F}^{\mathbb{T}}$ can be seen as an element in $\widetilde{\Omega}^2_{\mathbb{T}}(\mathrm{End}(V))$ and (3-2) can also be rewritten as

(3-5) $$\mathcal{F}^{\mathbb{T}} = \widetilde{\mathbb{d}}^{\mathbb{T}}_{\mathbb{V}} \circ \mathbb{V}.$$

Example 3.1. Let (V, h) be a Hermitian vector bundle on M. A generalized connection $\mathbb{V}$ on V *preserves* h, or is (h-)*unitary* if for $v_j \in C^\infty(V)$ and $x \in \mathbb{T}M$ with $X = \pi(x)$,

$$Xh(v_1, v_2) = h(\mathbb{V}_x v_1, v_2) + h(v_1, \mathbb{V}_x.v_2).$$

Suppose now that V is a Hermitian line bundle and s is a local section of V such that $h(s, s) = 1$. Since $\mathbb{V}$ is unitary, it is determined by a local section $u \in C^\infty(\mathbb{T}M)$, such that for $x \in \mathbb{T}M$,

$$\sqrt{-1}\mathbb{V}_x s = 2\langle x, u\rangle s.$$

In analogy with the classical computation, the $\mathbb{V}^\mathbb{T}$-curvature for the line bundle V is then

$$\sqrt{-1}\mathcal{F}^\mathbb{T}_{x,y} = 2(\langle y, \mathbb{V}^\mathbb{T}_x u\rangle - \langle x, \mathbb{V}^\mathbb{T}_y u\rangle) = (\mathbb{d}^\mathbb{T} u)(x, y). \tag{3-6}$$

3A. *Chern–Weil homomorphism.* Let $\mathbb{V}^\mathbb{T}$ be a TM-torsion-free generalized connection on $\mathbb{T}M$. Since $\mathbb{d}^\mathbb{T}$ generally does not square to 0 (2-8), the Bianchi identity generally does not hold for $\mathcal{F}^\mathbb{T}$. In terms of covariant derivatives, $\mathbb{d}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T}$ can be expanded into

$$\begin{aligned}(\mathbb{d}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T})_{x,y,z} &= \mathbb{V}_x\mathcal{F}^\mathbb{T}_{y,z} - \mathcal{F}^\mathbb{T}_{\mathbb{V}^\mathbb{T}_x y,z} - \mathcal{F}^\mathbb{T}_{y,\mathbb{V}^\mathbb{T}_x z} - \mathcal{F}^\mathbb{T}_{y,z}\mathbb{V}_x + \text{c.p. in } x, y, z\\ &= -\mathbb{V}_{\mathcal{R}^\mathbb{T}_{x,y}z} - \text{c.p. in } x, y, z,\end{aligned} \tag{3-7}$$

where $x, y, z \in C^\infty(\mathbb{T}M)$. By (2-11) and (2-12), this gives

$$(\mathbb{d}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T})_{x,y,z} = \mathbb{V}_{[x\diamond_\mathbb{T} y\diamond_\mathbb{T} z]} = \psi_{[x\diamond_\mathbb{T} y\diamond_\mathbb{T} z]}, \tag{3-8}$$

which leads to the Bianchi identity over $\widetilde{\Omega}^*_\mathbb{T}(M)$.

Lemma 3.2. *Let $\mathcal{F}^\mathbb{T} \in \Omega^2_\mathbb{T}(M) \otimes \mathrm{End}(V)$ be the $\mathbb{V}^\mathbb{T}$-curvature of a generalized connection $\mathbb{V}$ on V. Then*

$$\widetilde{\mathbb{d}}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T} = 0. \tag{3-9}$$

Proof. It follows from (3-8) that

$$\mathbb{d}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T} \in \operatorname{img}\mathscr{J}^\mathbb{T} \otimes \mathrm{End}(V).$$

Thus $\widetilde{\mathbb{d}}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T} = \mathscr{Q}^\mathbb{T}(\mathbb{d}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T}) = 0.$ □

The space $\mathscr{D}(V)$ of generalized connections on V is an affine space modeled on $\Omega^1_\mathbb{T}(\mathrm{End}(V))$, which coincides with $\widetilde{\Omega}^1_\mathbb{T}(\mathrm{End}(V))$. For $A \in \widetilde{\Omega}^1_\mathbb{T}(\mathrm{End}(V))$, a standard computation gives

$$\mathcal{F}^\mathbb{T}(\mathbb{V} + A) - \mathcal{F}^\mathbb{T}(\mathbb{V}) = \mathbb{d}^\mathbb{T}_\mathbb{V} A + A \wedge A.$$

It follows that

$$\mathrm{tr}_V(\mathcal{F}^\mathbb{T}(\mathbb{V} + A)) - \mathrm{tr}_V(\mathcal{F}^\mathbb{T}(\mathbb{V})) = \mathrm{tr}_V(\widetilde{\mathbb{d}}^\mathbb{T}_\mathbb{V} A) = \widetilde{\mathbb{d}}^\mathbb{T}\mathrm{tr}_V(A). \tag{3-10}$$

As in the classical case, (3-9) implies that

$$\widetilde{\mathbb{d}}^\mathbb{T}_\mathbb{V}\,\mathrm{tr}_V\,\mathcal{F}^\mathbb{T} = \mathrm{tr}_V\,\widetilde{\mathbb{d}}^\mathbb{T}_\mathbb{V}\mathcal{F}^\mathbb{T} = 0.$$

The gauge group $\mathrm{Aut}(V)$ acts on $\mathscr{D}(V)$ by pushforward. Namely, for $\lambda \in \mathrm{Aut}(V)$, $x \in \mathbb{T}M$ and $v \in C^\infty(V)$,

$$(\lambda\mathbb{V})_x v := \lambda^{-1}[\mathbb{V}_x(\lambda v)].$$

It induces the action on the curvature by conjugation

$$\mathcal{F}^{\mathbb{T}}(\lambda\mathbb{V}) = \lambda^{-1}\mathcal{F}^{\mathbb{T}}(\mathbb{V})\lambda.$$

Let $\mathscr{I}^{\mathbb{T}} \subset \Omega^*_{\mathbb{T}}(M)$ be the ideal generated by $\operatorname{img} \mathscr{J}^{\mathbb{T}}$,

$$\mathscr{I}^{\mathbb{T}} := \operatorname{img} \mathscr{J}^{\mathbb{T}} \wedge \Omega^*_{\mathbb{T}}(M),$$

and define $\overline{\Omega}^*_{\mathbb{T}}(M)$ as the quotient

$$0 \to \mathscr{I}^{\mathbb{T}} \to \Omega^*_{\mathbb{T}}(M) \xrightarrow{\overline{\mathscr{R}}^{\mathbb{T}}} \overline{\Omega}^*_{\mathbb{T}}(M) \to 0. \tag{3-11}$$

Then $\mathbb{d}^{\mathbb{T}}$ induces a differential on $\overline{\Omega}^*_{\mathbb{T}}(M)$,

$$\overline{\mathbb{d}}^{\mathbb{T}} : \overline{\Omega}^*_{\mathbb{T}}(M) \to \overline{\Omega}^*_{\mathbb{T}}(M),$$

whose cohomology is the *reduced* $\mathbb{V}^{\mathbb{T}}$*-de Rham cohomology*. The exact sequence

$$0 \to \frac{\mathscr{I}^{\mathbb{T}}}{\operatorname{img} \mathscr{J}^{\mathbb{T}}} \to \widetilde{\Omega}^*_{\mathbb{T}}(M) \xrightarrow{\mathscr{P}^{\mathbb{T}}} \overline{\Omega}^*_{\mathbb{T}}(M) \to 0$$

induces the map on the cohomologies

$$\mathscr{P}^{\mathbb{T}}_* : \widetilde{H}^*_{\mathbb{T}}(M) \to \overline{H}^*_{\mathbb{T}}(M).$$

The Chern–Weil homomorphism extends to define characteristic classes for V in $\overline{H}^*_{\mathbb{T}}(M)$.

Definition 3.3. For a Hermitian vector bundle (V, h) over M, its *k-th* $\mathbb{V}^{\mathbb{T}}$*-Chern class* is

$$c^{\mathbb{T}}_k(V) := [\mathrm{tr}_V(\sqrt{-1}\mathcal{F}^{\mathbb{T}}(\mathbb{V}))^k] \in \overline{H}^{2k}_{\mathbb{T}}(M), \tag{3-12}$$

where $\mathbb{V}$ is a generalized connection on V. For real vector bundles, their $\mathbb{V}^{\mathbb{T}}$*-Euler* and $\mathbb{V}^{\mathbb{T}}$*-Pontrjagin* classes can similarly be defined, as elements of $\overline{H}^*_{\mathbb{T}}(M)$ of appropriate degrees.

By (3-10), the $\mathbb{V}^{\mathbb{T}}$-Chern classes do not depend on the choice of $\mathbb{V}$ on V. Let $\mathbb{V}$ be the lift of a classical connection ∇_0 on V, and let F_0 be the classical curvature of ∇_0. Then

$$\mathcal{F}^{\mathbb{T}}_{x,y} = F_{0;\pi(x),\pi(y)}$$

for $x, y \in C^\infty(\mathbb{T}M)$. This relates $c^{\mathbb{T}}_*(V)$ to the classical Chern classes $c_*(V)$.

Proposition 3.4. *Let (V, h) be a Hermitian vector bundle. Then*

$$c_k^{\mathbb{T}}(V) = \overline{\pi}^* c_k(V) := \mathscr{P}_*^{\mathbb{T}}(\widetilde{\pi}^* c_k(V))$$

for all k. In particular, $c_k^{\mathbb{T}}(V) = 0$ for all k such that $2k > n$. Similarly, the $\mathbb{V}^{\mathbb{T}}$-Euler and Pontrjagin classes are the images of the respective classical classes under $\overline{\pi}^*$. □

3B. *ϕ-curvatures.* A generalized Riemannian metric $\mathbb{G}$ induces an eigendecomposition of a generalized connection $\mathbb{V}$ on V. For $X \in C^\infty(TM)$ and $v \in C^\infty(V)$,

$$\nabla_{\pm,X} v := \mathbb{V}_{s_\pm(X)} v. \tag{3-13}$$

The connections $\nabla_\pm$ depend on b, while their difference does not (see [20]):

$$\psi_X := \tfrac{1}{2}(\nabla_{+,X} - \nabla_{-,X}) = \tfrac{1}{2}(\mathbb{V}_{s_+(X)} - \mathbb{V}_{s_-(X)}) = \mathbb{V}_{g(X)}. \tag{3-14}$$

The average of $\nabla_\pm$ gives the $\mathbb{G}$-*neutral connection* of $\mathbb{V}$,

$$\nabla_{0,X} := \mathbb{V}_{s_0(X)}, \tag{3-15}$$

which leads to

$$\nabla_\pm = \nabla_0 \pm \psi. \tag{3-16}$$

When $\psi = 0$, the generalized connection $\mathbb{V}$ is the lift of a classical connection ∇_0 on V, in which case $\nabla_\pm = \nabla_0$ are independent of b as well. The dependence on b of the $\mathbb{G}$-eigendecomposition of $\mathbb{V}$ can be described in terms of ψ.

Proposition 3.5. *Let $\mathbb{G}$ and $\mathbb{G}'$ be two generalized metrics corresponding to (g, b) and (g, b') respectively. Let $a = b' - b \in \Omega^2(M)$, and define j_a by*

$$j_a := g^{-1}a : TM \to TM \quad \textit{given by} \quad X \mapsto g^{-1}(\iota_X a).$$

Let $\nabla_\pm$ and $\nabla'_\pm$ be the respective $\mathbb{G}$-eigendecomposition of $\mathbb{V}$ and $\mathbb{V}'$ on V. Then

$$\nabla'_\pm = \nabla_\pm + \psi_{j_a}.$$ □

Definition 3.6. Let $\mathbb{V}$ be a generalized connection on a vector bundle V over M. Let $\phi \in \Omega^3(M)$ and $\mathbb{G}$ a generalized metric on $\mathbb{T}M$. The ($\mathbb{G}$-*adapted*) *ϕ-curvature* $\mathcal{F}^\phi(\mathbb{V})$ of $\mathbb{V}$ is its $\mathbb{V}^\phi$-curvature (3-2), and is denoted by $\mathcal{F}^\phi$ if $\mathbb{V}$ is understood.

Given the pair of classical connections (∇_+, ∇_-), besides the curvature $F_\pm$ of each of them, there is also a *mixed curvature* $F_{+,-}$ [12; 39]:

$$F_{+,-;X,Y} v := (\nabla_{+,X}\nabla_{-,Y} - \nabla_{-,\nabla_X^{-\phi}Y})v - (\nabla_{-,Y}\nabla_{+,X} - \nabla_{+,\nabla_Y^{+\phi}X})v, \tag{3-17}$$

where $X, Y \in C^\infty(TM)$ and $v \in C^\infty(V)$. It can also be expressed using the tensor ψ:

$$F_{+,-;X,Y} = F_{+,X,Y} - 2(\nabla_{+,X}\psi)_Y - (\iota_{g^{-1}\psi}\phi)(X, Y). \tag{3-18}$$

Let F_0 be the classical curvature for ∇_0 (3-15). It gives another decomposition for the mixed curvature (3-17):

$$F_{+,-;X,Y} = F_{0;X,Y} + (\iota_{g^{-1}\psi}\phi)(X,Y) - [\psi_X, \psi_Y] - [(\nabla_{0,X}\psi)_Y + (\nabla_{0,Y}\psi)_X]. \tag{3-19}$$

Theorem 3.7. *The ϕ-curvature $\mathcal{F}^\phi(\mathbb{V})$ admits $\mathbb{G}$-eigendecomposition in terms of the (mixed) curvatures of the pair (∇_+, ∇_-) of classical connections as follows:*

$$\mathcal{F}^\phi_{x_\pm, y_\pm} v = F_{\pm,X,Y} v \quad \textit{and} \quad \mathcal{F}^\phi_{x_+, y_-} v = F_{+,-;X,Y} v, \tag{3-20}$$

where for $X, Y \in TM$, $x_\pm = s_\pm(X)$, etc.

Proof. Straightforward from the definition, via the $\mathbb{G}$-eigendecomposition. □

Example 3.8. Continue with Example 3.1 for $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^\phi$. In this case, the local section $u \in C^\infty(\mathbb{T}M)$ decomposes into

$$u = \tfrac{1}{2}[(g^{-1}\nu_+ + bg^{-1}\nu_+ + \nu_+) - (g^{-1}\nu_- + bg^{-1}\nu_- - \nu_-)],$$

where $\sqrt{-1}\nu_\pm \in \Omega^1(M)$ are the local 1-forms defining the connections $\nabla_\pm$ respectively. It follows that

$$\sqrt{-1}F_\pm = d\nu_\pm,$$

and the mixed component in $\mathcal{F}^\phi$ is given by

$$\sqrt{-1}F_{+,-;X,Y} = [\nabla_X^{-\phi}\nu_-](Y) - [\nabla_Y^{+\phi}\nu_+](X).$$

Note that $F_{+,-}$ is neither symmetric nor skew-symmetric in X and Y, and decomposes into symmetric and skew-symmetric parts as

$$\sqrt{-1}F_{+,-} = -\mathcal{L}_{g^{-1}\psi} g + (\sqrt{-1}F_0 - \iota_{g^{-1}\psi}\phi),$$

where F_0 is the curvature of the $\mathbb{G}$-neutral connection ∇_0 and $\psi = \frac{1}{2}(\nu_+ - \nu_-)$, in the decomposition (3-16) of $\mathbb{V}$.

Corollary 3.9. *In the $\mathbb{G}$-splitting of $\mathbb{T}M$, the ϕ-curvature is*

$$\begin{aligned}\mathcal{F}^\phi_{x,y} = F_{0;X,Y} + (\nabla_{0,X}\psi)_{g^{-1}\eta} - (\nabla_{0,Y}\psi)_{g^{-1}\xi} + [\psi_{g^{-1}\xi}, \psi_{g^{-1}\eta}] \\ + \tfrac{1}{2}\big[(\iota_{g^{-1}\psi}\phi)(X,Y) - (\iota_{g^{-1}\psi}\phi)(g^{-1}\xi, g^{-1}\eta)\big],\end{aligned} \tag{3-21}$$

where $x = s_0(X) + \xi$ and $y = s_0(Y) + \eta$. □

Remark 3.10. In (3-3), only the term $\mathbb{V}_{x\diamond_\phi y}$ depends on $\mathcal{F}^\phi$ on ϕ and $\mathbb{G}$. Let $\mathbb{V}^{\phi'}$ be the $\mathbb{G}'$-adapted ϕ'-connection. Then

$$\mathbb{V}_{x\diamond_\phi y} - \mathbb{V}_{x\diamond_{\phi'} y} = \psi_{g^{-1}(x\diamond_\phi y - x\diamond_{\phi'} y)}.$$

In the classical expansions, the dependence of $\mathcal{F}^\phi$ on ϕ is completely contained in the last term of (3-18), or the second line in (3-21); while the dependence on g is contained in the last term of the second line in (3-21). The dependence of $\mathcal{F}^\phi$ on b is more complicated. Nonetheless, it can be derived from (3-21) by relatively lengthy computations, noting that ∇_0 as well as ξ and η in the expression all depend on b.

3C. ***Yang–Mills functional.*** As one further example, it is straightforward to extend the Yang–Mills functional to this context. The generalized metric $\mathbb{G}$ induces natural inner product on $\bigwedge^* \mathbb{T}M$. For a Hermitian bundle (V, h), it induces a natural norm on $\bigwedge^* \mathbb{T}M \otimes \mathrm{End}(V)$, denoted by $\| \bullet \|_h$. The $\mathbb{V}^{\mathbb{T}}$-*Yang–Mills functional* on $\mathscr{D}(V)$ is given by

$$\mathcal{YM}_{\mathbb{T}}(\mathbb{V}) := \int_M \|\mathcal{F}^{\mathbb{T}}(\mathbb{V})\|_h^2 \, d\mathrm{vol}_g. \tag{3-22}$$

It is evidently invariant under the gauge action on $\mathscr{D}(V)$. When restricted to the subspace of the lifts of classical connections on V, $\mathcal{YM}_{\mathbb{T}}(\mathbb{V})$ reduces to (a constant multiple of) the classical Yang–Mills functional. It can also be regarded as a functional of the pair $(\mathbb{V}, \mathbb{V}^{\mathbb{T}})$ of generalized connections on V and $\mathbb{T}M$ respectively.

When $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^\phi$, it can be represented as

$$\mathcal{YM}_\phi(\mathbb{V}) = \mathrm{YM}(\nabla_+) + \mathrm{YM}(\nabla_-) + 2\int_M \|F_{+,-}\|_h^2 \, d\mathrm{vol}_g, \tag{3-23}$$

where $\mathrm{YM}(\bullet)$ denotes the classical Yang–Mills functional. The 2-form b affects only the $\mathbb{G}$-eigendecomposition of $\mathbb{V}$. The right-hand side can be seen as a functional for a pair of classical connections (∇_+, ∇_-), where the last term encodes the dependence on $\phi \in \Omega^3(M)$ (Remark 3.10), as well as the interaction within the pair.

4. Curvatures on $\mathbb{T}M$

For a $\mathbb{G}$-metric connection $\mathbb{V}^{\mathbb{T}}$ on $\mathbb{T}M$, its ϕ-curvature is denoted by $\mathcal{R}^{\mathbb{T},\phi}$ and the associated *curvature tensor* is given by

$$\mathcal{R}^{\mathbb{T},\phi}(x, y, z, w) := \mathbb{G}(\mathcal{R}^{\mathbb{T},\phi}_{x,y} z, w), \tag{4-1}$$

where $x, y, z, w \in C^\infty(\mathbb{T}M)$. Similar to the classical situation, it is skew in the first two and the last two entries respectively:

$$\mathcal{R}^{\mathbb{T},\phi}(x, y, z, w) = -\mathcal{R}^{\mathbb{T},\phi}(y, x, z, w) = \mathcal{R}^{\mathbb{T},\phi}(y, x, w, z).$$

Definition 4.1. The *ϕ-Riemannian curvature $\mathcal{R}^\phi$* for (M, g) is the ϕ-curvature for $\mathbb{V}^\phi$, and the corresponding curvature tensor is the *ϕ-Riemann tensor*, which is also denoted by $\mathcal{R}^\phi$.

4A. ***Bianchi identities.*** Since $\mathbb{V}^\phi$ preserves $C_\pm$, $\mathcal{R}^\phi(x, y, z, w)$ vanishes when the last two entries are sections of different $\mathbb{G}$-eigenbundles. The nonvanishing components in the $\mathbb{G}$-eigendecomposition of $\mathcal{R}^\phi$ are given by the classical Riemann tensor R of g as well as the curvature tensors $R^{\pm\phi}$ for $\nabla^{\pm\phi}$.

Proposition 4.2. *Let $X, Y, Z, W \in TM$ and $x_\pm = s_\pm(X) \in C_\pm$, etc. Then*:

(1) $\mathcal{R}^\phi(x_\pm, y_\pm, z_\pm, w_\pm) = R(X, Y, Z, W)$.

(2) $\mathcal{R}^\phi(x_\mp, y_\pm, z_\pm, w_\pm) = R^{\pm\phi}(X, Y, Z, W) \mp \frac{1}{2}(\nabla_X^{\pm\phi}\phi)(Y, Z, W)$.

(3) $\mathcal{R}^\phi(x_\mp, y_\mp, z_\pm, w_\pm) = R^{\pm\phi}(X, Y, Z, W)$.

All other components of $\mathcal{R}^\phi$ vanish.

Proof. Standard computations from the definitions, which is left for the reader. □

By (1) above, the algebraic Bianchi identity for $\mathcal{R}^\phi$ holds when all entries involved are from the same $\mathbb{G}$-eigenbundle (see also [12] Proposition 3.24). The analogues to the algebraic and differential Bianchi identities follow from previous discussion.

Lemma 4.3. *In their respective $\mathbb{G}$-eigendecompositions*:

(1) *For $\mathcal{R}^\phi_{x,y}z + c.p.$,*

$$\mathcal{R}^\phi_{x_\mp, y_\mp} z_\pm + \mathcal{R}^\phi_{y_\mp, z_\pm} x_\mp + \mathcal{R}^\phi_{z_\pm, x_\mp} y_\mp = \pm 2g(R^{\pm\phi}_{X,Y} Z). \tag{4-2}$$

(2) *For $\mathbb{d}^\phi_{\mathbb{V}} \mathcal{F}^\phi$,*

$$(\mathbb{d}^{\phi,\mathbb{G}}_{\mathbb{V}} \mathcal{F}^\phi)_{x_\mp, y_\mp, z_\pm} = \mp 2\mathbb{V}_{g(R^{\pm\phi}_{X,Y} Z)} = \mp 2\psi_{R^{\pm\phi}_{X,Y} Z}. \tag{4-3}$$

All other components vanish.

Proof. The identity (4-2) follows from (2-11) and (2-40). If $d\phi = 0$, (4-2) can also be obtained from the explicit expressions in Proposition 4.2 and the identity below (see [4]):

$$R^{+\phi}(X, Y, Z, W) = R^{-\phi}(Z, W, X, Y) + \tfrac{1}{2}(d\phi)(X, Y, Z, W). \tag{4-4}$$

Then (4-3) follows from (3-8) and (4-2). □

In particular, the differential Bianchi identity holds for the ϕ-curvature when $\psi = 0$, i.e., $\mathbb{V}$ is the lifting of a classical connection on V. Another special case is when $\nabla^{\pm\phi}$ are flat [1; 8; 7], i.e., $R^{\pm\phi} = 0$, and thus both Bianchi identities hold. In this second special case, $\mathcal{R}^\phi$ enjoys all the symmetries of a classical Riemann curvature.

Theorem 4.4. *Suppose that the connections $\nabla^{\pm\phi}$ are flat on TM. Then*:

(1) $\mathcal{R}^\phi(x, y, z, w) = -\mathcal{R}^\phi(y, x, z, w) = \mathcal{R}^\phi(y, x, w, z)$.

(2) $\mathcal{R}^\phi_{x,y}z + \mathcal{R}^\phi_{y,z}x + \mathcal{R}^\phi_{z,x}y = 0$.

(3) $\mathbb{d}^\phi_{\mathbb{V}} \mathcal{R}^\phi = 0$.

(4) $\mathcal{R}^\phi(x, y, z, w) = \mathcal{R}^\phi(z, w, x, y)$.

In this case, $\mathcal{R}^\phi$ defines a symmetric pairing on $\bigwedge^2 \mathbb{T}M$,

$$\mathcal{R}^\phi : \textstyle\bigwedge^2 \mathbb{T}M \otimes \bigwedge^2 \mathbb{T}M \to \mathbb{R} \quad \textit{given by} \quad \mathcal{R}^\phi(x \wedge y, w \wedge z) := \mathcal{R}^\phi(x, y, z, w),$$

which defines the corresponding operator on $\bigwedge^2 \mathbb{T}M$ via $\mathbb{G}$.

Proof. Items (1)–(3) follow from previous discussion and the flatness assumption, while (4) follows from (1)–(3) as in the classical situation. The last statement is a consequence of (1) and (4). □

The following consequence of Lemma 4.3 was used in the proof of Proposition 2.15.

Proposition 4.5. *If $d\phi = 0$, then $\mathbb{d}^\phi$ is a differential at $\Omega^{2n-1}_{\mathbb{T}}(M)$.*

Proof. Let $\{X_j\}$ be a local g-orthonormal frame of TM and $\{e^j_\pm\}$ the induced $\mathbb{G}$-orthonormal frame of $\mathbb{T}M$. Let $\theta \in \Omega^{2n-2}_{\mathbb{T}}(M)$. Then, by (2-11), (2-16) and (4-2),

$$\begin{aligned}
&(\mathbb{d}^\phi \circ \mathbb{d}^\phi \theta)(e^1_+, \dots, e^n_+, e^1_-, \dots, e^n_-) \\
&\quad = -\sum_{i<j,k} (-1)^{i+j+k+n} \theta(-g(R^{-\phi}_{X_i,X_j} X_k), \dots, \hat{e}^i_+, \dots, \hat{e}^j_+, \dots, \hat{e}^k_-, \dots) \\
&\qquad\qquad - \sum_{i,j<k} (-1)^{i+j+k} \theta(g(R^{+\phi}_{X_j,X_k} X_i), \dots, \hat{e}^i_+, \dots, \hat{e}^j_-, \dots, \hat{e}^k_-, \dots) \\
&\quad = \sum_{i,j,k} (-1)^{j+k+n} \big[-R^{-\phi}(X_i, X_j, X_k, X_i) \\
&\qquad\qquad + R^{+\phi}(X_i, X_k, X_j, X_i)\big] \theta(\dots, \hat{e}^j_+, \dots, \hat{e}^k_-, \dots) \\
&\quad = 0.
\end{aligned}$$

Since $d\phi = 0$, the last step above follows from (4-4). □

4B. *Ricci curvature.* The trace of the ϕ-curvature on $\mathbb{T}M$ defines the corresponding ϕ-Ricci curvature.

Definition 4.6. For a $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$, the *ϕ-Ricci curvature $\mathcal{R}ic^{\mathbb{T},\phi}$* : $\mathbb{T}M \to \mathbb{T}M$ for $(M, \mathbb{G})$ is the trace of the ϕ-curvature $\mathcal{R}^{\mathbb{T},\phi}$. For $x, y \in C^\infty(\mathbb{T}M)$, in the local orthonormal frame $\{e^i_+, e^j_-\}$ of $\mathbb{T}M$ induced from a local g-orthonormal frame $\{X_i\}$,

$$\mathcal{R}ic^{\mathbb{T},\phi}(x) := \sum_i [\mathcal{R}^{\mathbb{T},\phi}_{x,e^i_+} e^i_+ + \mathcal{R}^{\mathbb{T},\phi}_{x,e^i_-} e^i_-]. \tag{4-5}$$

The *ϕ-Ricci tensor $\mathcal{R}c^{\mathbb{T},\phi}$* $\in C^\infty(\mathbb{T}M \otimes \mathbb{T}M)$ is

$$\mathcal{R}c^{\mathbb{T},\phi}(x, y) := \mathbb{G}(\mathcal{R}ic^{\mathbb{T},\phi}(x), y). \tag{4-6}$$

For the connection ∇^ϕ, these are denoted by $\mathcal{R}ic^\phi$ and $\mathcal{R}c^\phi$ respectively.

Specialized to the $\mathbb{G}$-adapted connection ∇^ϕ, the $\mathbb{G}$-eigendecomposition of $\mathcal{R}c^\phi$ can be determined from that of $\mathcal{R}^\phi$ as follows:

$$\mathcal{R}c^\phi(x_\pm, y_\pm) = \mathrm{Rc}(X, Y) \quad \text{and} \quad \mathcal{R}c^\phi(x_\pm, y_\mp) = \mathrm{Rc}^{\mp\phi}(X, Y), \tag{4-7}$$

where Rc is the Ricci tensor for ∇, $\mathrm{Rc}^{\pm\phi}$ are the Ricci tensors for $\nabla^{\pm\phi}$ respectively:

$$\mathrm{Rc}^{\pm\phi} = \mathrm{Rc} \mp \tfrac{1}{2} d^*\phi - \tfrac{1}{4}\phi^2, \tag{4-8}$$

where

$$\phi^2(X, Y) := \sum_{i,j} \phi(X, X_i, X_j)\phi(Y, X_i, X_j).$$

It follows that $\mathcal{R}c^\phi$ is symmetric; in other words,

$$\langle \mathcal{R}ic^\phi(x), \mathbb{G}y\rangle = \langle \mathbb{G}\,\mathcal{R}ic^\phi(x), y\rangle = \langle \mathbb{G}\,\mathcal{R}ic^\phi(y), x\rangle = \langle \mathbb{G}x, \mathcal{R}ic^\phi(y)\rangle. \tag{4-9}$$

Constructions of generalized Ricci curvature or tensor in the literature, such as in [10; 12; 37; 38], contain only the mixed components, and are generally in the context of generalized Ricci flows. Indeed, as will become clear in Section 7, only the mixed components contribute to the generalized Ricci flow.

Similar to the classical case, the ϕ-Ricci curvature described here appears in a Weitzenböck identity relating two natural Laplacians on $\bigwedge^* \mathbb{T}M$ described in Section 2D.

Theorem 4.7. *On $\Omega^1_{\mathbb{T}}(M)$, the following Weitzenböck identity holds*:

$$\Delta^\phi = \Delta_{\nabla^\phi} + \mathbb{G}\,\mathcal{R}ic^\phi\,\mathbb{G}, \tag{4-10}$$

where Δ^ϕ is the ∇^ϕ-Hodge Laplacian (2-47) *while Δ_{∇^ϕ} is the Bochner Laplacian* (2-44).

Proof. It can be shown following standard computations that for $\theta \in \Omega^*_{\mathbb{T}}(M)$,

$$\Delta^\phi\,\theta = \Delta_{\nabla^\phi}\,\theta - \frac{1}{2}\sum_{\alpha,\beta} \mathbb{G}(e_\alpha) \wedge \iota_{e_\beta}(\mathcal{R}^\phi_{e_\alpha,e_\beta}\theta),$$

where e_α, e_β run through $\{e^i_+, e^j_-\}$. Set $\theta = \mathbb{G}(x)$ for $x \in \mathbb{T}M$. Then

$$\begin{aligned}\sum_{\alpha,\beta}\big\langle \mathbb{G}(e_\alpha) \wedge \iota_{e_\beta}(\mathcal{R}^\phi_{e_\alpha,e_\beta}\mathbb{G}(x)), y\big\rangle &= 2\sum_{\alpha,\beta}\langle \mathbb{G}(\mathcal{R}^\phi_{e_\alpha,e_\beta}x), e_\beta\rangle \mathbb{G}(e_\alpha, y)\\ &= 2\sum_\beta \mathcal{R}^\phi(y, e_\beta, x, e_\beta)\\ &= -2\,\mathcal{R}c^\phi(y, x) = -2\langle \mathbb{G}\,\mathcal{R}ic^\phi(x), y\rangle,\end{aligned}$$

from which (4-10) follows. □

The symmetry of $\mathcal{R}c^{\phi}$ implies that $\mathbb{d}^{\phi *}$ is a differential at $\Omega^1_{\mathbb{T}}(M)$. For $x_i \in C^\infty(\mathbb{T}M)$, $i = 1, \ldots, k$, set $y_i = \mathbb{G}(x_i)$. Then a straightforward computation gives

$$
\begin{aligned}
(4\text{-}11)\quad & (\mathbb{d}^{\phi *} \circ \mathbb{d}^{\phi *})[y_1 \wedge \cdots \wedge y_k] \\
&= \sum_{i<j} (-1)^{i+j} [\mathcal{R}c^{\phi}(x_j, x_i) - \mathcal{R}c^{\phi}(x_i, x_j)] y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_k \\
&\quad - \sum_{i<j<\ell} (-1)^{i+j+\ell} \mathbb{G}(\mathcal{R}^{\phi}_{x_i, x_j} x_\ell + \text{c.p.}) y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge \hat{y}_\ell \wedge \cdots \wedge y_k.
\end{aligned}
$$

Due to the symmetry of $\mathcal{R}c^{\phi}$, the terms in the second line vanish. Furthermore, when $k = 2$, the terms in the last line vanish as well.

Proposition 4.8. *The operator $\mathbb{d}^{\phi *}$ is a differential at $\Omega^1_{\mathbb{T}}(M)$, i.e.,*

$$\theta \in \Omega^2_{\mathbb{T}}(M) \implies (\mathbb{d}^{\phi *} \circ \mathbb{d}^{\phi *})\theta = 0. \tag{4-12}$$

If the connections $\nabla^{\pm\phi}$ are flat on TM, then $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\phi})$ and $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\phi *})$ are both chain complices.

Proof. For the last statement, that $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\phi})$ is a chain complex follows from (2-8), (2-11) and the algebraic Bianchi identity, which is item (2) in Theorem 4.4. The statement for $(\Omega^*_{\mathbb{T}}(M), \mathbb{d}^{\phi *})$ follows from (4-11) and the algebraic Bianchi identity. □

4C. ***Bismut connection.*** The *generalized Bismut connection* $\mathbb{\nabla}^{\phi,\mathcal{B}}$ introduced by Gualtieri [20] is a $\mathbb{G}$-metric connection and is the lift of a classical connection $\nabla^{\phi,\mathcal{B}}$ on $\mathbb{T}M$:

$$\mathbb{\nabla}^{\phi,\mathcal{B}}_X s_\pm(Y) = s_\pm(\nabla^{\pm\phi}_X Y). \tag{4-13}$$

Note $\mathbb{\nabla}^{\phi,\mathcal{B}}$ is compatible with the (almost) Dorfman bracket $*_{\phi-db}$ (Definition 2.7).

Since $\mathbb{\nabla}^{\phi,\mathcal{B}}$ is a lift of a classical connection, by (4-3), the *ϕ-Bismut curvature* $\mathcal{R}^{\phi,\mathcal{B}}$ of $\mathbb{\nabla}^{\phi,\mathcal{B}}$ satisfies the differential Bianchi identity:

$$\mathbb{d}^{\phi}_{\mathbb{\nabla}} \mathcal{R}^{\phi,\mathcal{B}} = 0.$$

More explicitly, $\mathcal{R}^{\phi,\mathcal{B}}$ is determined by the classical curvature of $\nabla^{\phi,\mathcal{B}}$, which in turn is given by $R^{\pm\phi}$:

$$\mathcal{R}^{\phi,\mathcal{B}}(x, y, s_\pm(Z), s_\pm(W)) = R^{\pm\phi}(X, Y, Z, W). \tag{4-14}$$

The $\mathbb{G}$-eigendecomposition of the corresponding *ϕ-Bismut Ricci tensor* $\mathcal{R}c^{\phi,\mathcal{B}}$ is thus

$$\mathcal{R}c^{\phi,\mathcal{B}}(x, s_+(Y)) = \mathrm{Rc}^{+\phi}(X, Y) \quad \text{and} \quad \mathcal{R}c^{\phi,\mathcal{B}}(x, s_-(Y)) = \mathrm{Rc}^{-\phi}(X, Y). \tag{4-15}$$

4D. ***Scalar curvatures.*** The traces of the ϕ-Ricci curvatures (4-5) give the corresponding *ϕ-scalar curvatures*, which depend on the $\mathbb{G}$-metric connection $\nabla^{\mathbb{T}}$. For instance, the *ϕ-Riemann scalar curvature* $\mathcal{S}^{\phi}$ is the trace of $\mathcal{R}ic^{\phi}$:

$$\mathcal{S}^{\phi} = \sum_j [\mathcal{R}c^{\phi}(e_+^j, e_+^j) + \mathcal{R}c^{\phi}(e_-^j, e_-^j)] = 2S, \tag{4-16}$$

where S is the classical scalar curvature of g. On the other hand, the *ϕ-Bismut scalar curvature* $\mathcal{S}^{\phi,\mathcal{B}}$ is the trace of $\mathcal{R}ic^{\phi,\mathcal{B}}$:

$$\mathcal{S}^{\phi,\mathcal{B}} = \sum_j [\mathcal{R}c^{\phi,\mathcal{B}}(e_+^j, e_+^j) + \mathcal{R}c^{\phi,\mathcal{B}}(e_-^j, e_-^j)] = 2S - 3\|\phi\|_g^2,$$

where $\|\phi\|_g$ is the norm of ϕ with respect to g,

$$\|\phi\|_g^2 = \sum_{i<j<k} \phi(X_i, X_j, X_k)^2.$$

5. Generalized complex manifolds

Let $\mathbb{J}$ be a generalized almost complex structure on M [19; 21; 24]. It induces a polarization of $\mathbb{T}_{\mathbb{C}}M := \mathbb{T}M \otimes_{\mathbb{R}} \mathbb{C}$ as the direct sum of its $\pm\sqrt{-1}$-eigenbundles:

$$\mathbb{T}_{\mathbb{C}}M = \mathbb{T}_{\mathbb{J}}^{1,0}M \oplus \mathbb{T}_{\mathbb{J}}^{0,1}M. \tag{5-1}$$

Here $\mathbb{T}_{\mathbb{J}}^{1,0}M$ denotes the $\sqrt{-1}$-eigenbundle of $\mathbb{J}$, and $\mathbb{T}_{\mathbb{J}}^{0,1}M$ its complex conjugate. They are maximally isotropic and are dual to each other under the pairing $2\langle\ ,\ \rangle$ on $\mathbb{T}_{\mathbb{C}}M$. For instance, the space of (0, 1)*-forms with respect to* $\mathbb{J}$ is identified with the sections of $\mathbb{T}_{\mathbb{J}}^{1,0}M$:

$$\Omega_{\mathbb{J}}^{0,1}(M) := C^{\infty}(\mathbb{T}_{\mathbb{J}}^{1,0}M).$$

Similar to the classical case, the $(0, k)$-forms with respect to $\mathbb{J}$ are sections of $\bigwedge^k \mathbb{T}_{\mathbb{J}}^{1,0}M$:

$$\Omega_{\mathbb{J}}^{0,k}(M) := C^{\infty}\big(\textstyle\bigwedge^k \mathbb{T}_{\mathbb{J}}^{1,0}M\big).$$

In general, the type decomposition of $\Omega_{\mathbb{T}}^*(M)$ with respect to $\mathbb{J}$ is given by

$$\Omega_{\mathbb{T}}^*(M) = \bigoplus_{p,q} \Omega_{\mathbb{J}}^{p,q}(M) := \bigoplus_{p,q} C^{\infty}\big(\textstyle\bigwedge^p \mathbb{T}_{\mathbb{J}}^{0,1}M \otimes \bigwedge^q \mathbb{T}_{\mathbb{J}}^{1,0}M\big). \tag{5-2}$$

For notational convenience, sometimes $\mathbb{T}_{\mathbb{J}}^{1,0}M$ is denoted by L, while $\mathbb{T}_{\mathbb{J}}^{0,1}M$ is denoted by $\bar{L}$.

Definition 5.1. Let $(M, \mathbb{J})$ be a generalized almost complex manifold and let $\gamma \in \Omega^3(M)$ be a closed 3-form. Then a generalized connection $\nabla^{\mathbb{T}}$ on $\mathbb{T}M$ is

$\mathbb{J}$-*compatible with* $*_\gamma$ if $\diamond_{\mathbb{T}}$ coincides with $*_\gamma$ on the sections from the same eigenbundle of $\mathbb{J}$, i.e.,

$$x \diamond_{\mathbb{T}} y = x *_\gamma y \quad \text{and} \quad \bar{x} \diamond_{\mathbb{T}} \bar{y} = \bar{x} *_\gamma \bar{y}, \tag{5-3}$$

where $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M)$. Such generalized connection $\nabla^{\mathbb{T}}$ is a γ-$\mathbb{J}$-*connection* if it furthermore is TM-torsion-free.

It is straightforward to see that (5-3) is equivalent to the following, where $x, y \in C^\infty(\mathbb{T}M)$:

$$(\mathbb{J}x) \diamond_{\mathbb{T}} y + x \diamond_{\mathbb{T}} (\mathbb{J}y) = (\mathbb{J}x) *_\gamma y + x *_\gamma (\mathbb{J}y).$$

Thus, if nonempty, the space $\mathscr{D}_{\mathbb{J},\gamma}(\mathbb{T}M)$ of generalized connections that are $\mathbb{J}$-compatible with $*_\gamma$ is modeled on

$$\mathscr{D}_{\mathbb{J},\gamma}(\mathbb{T}M) \cong \big\{A : \mathbb{T}M \otimes \mathbb{T}M \to \mathbb{T}M \ \text{ such that } \ A_{\mathbb{J}x} y - A_y(\mathbb{J}x) = A_{\mathbb{J}y} x - A_x(\mathbb{J}y)\big\}.$$

It's then evident by (2-13) that the subspace $\mathscr{D}_{\mathbb{J},\gamma,\tau}(\mathbb{T}M)$ of the γ-$\mathbb{J}$-connections, if nonempty, is modeled on

$$\mathscr{D}_{\mathbb{J},\gamma,\tau}(\mathbb{T}M) \cong \mathrm{Sym}^2_{\mathbb{T}}(\mathbb{T}M) \oplus \Omega^{1,1}_{\mathbb{T}}(T^*M), \tag{5-4}$$

where $\Omega^{1,1}_{\mathbb{T}}(T^*M)$ consists of T^*M-valued forms that are compatible with $\mathbb{J}$, i.e.,

$$\theta \in \Omega^{1,1}_{\mathbb{T}}(T^*M) \iff \theta(\mathbb{J}x, \mathbb{J}y) = \theta(x, y).$$

Let $\gamma \in \Omega^3(M)$ be a closed 3-form. Recall that $\mathbb{J}$ is *integrable* (*with respect to* γ) if $\mathbb{T}^{1,0}_{\mathbb{J}} M$ is involutive under the Dorfman bracket $*_\gamma$, i.e.,

$$x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M) \implies x *_\gamma y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M).$$

In this case, $(M, \gamma; \mathbb{J})$ is a *generalized complex manifold* [19; 21; 24].

Lemma 5.2. *Let* $(M, \gamma; \mathbb{J})$ *be a generalized complex manifold with a generalized connection* $\nabla^{\mathbb{T}}$ *on* $\mathbb{T}M$ *that is* $\mathbb{J}$*-compatible with* $*_\gamma$. *Then the operator* $\mathbb{d}^{\mathbb{T}}$ *decomposes into components as follows*:

$$\mathbb{d}^{\mathbb{T}} = \partial^{\mathbb{T}}_{\mathbb{J}} + \bar{\partial}^{\mathbb{T}}_{\mathbb{J}} : \Omega^{p,q}_{\mathbb{J}}(M) \to \Omega^{p+1,q}_{\mathbb{J}}(M) \oplus \Omega^{p,q+1}_{\mathbb{J}}(M). \tag{5-5}$$

Proof. The proof is similar to that of Lemma 2.5, employing (5-3) and the integrability of $\mathbb{J}$. The details are left for the reader. □

The integrability of $\mathbb{J}$ implies that both of its eigenbundles are complex Lie algebroids, with their Lie brackets given by the restriction of $*_\gamma$. The corresponding Lie algebroid de Rham differential for $\bar{L} = \mathbb{T}^{0,1}_{\mathbb{J}} M$ will be denoted by $d_{\bar{L}}$:

$$d_{\bar{L}} : \Omega^{0,k}_{\mathbb{J}}(M) \to \Omega^{0,k+1}_{\mathbb{J}}(M). \tag{5-6}$$

If $\nabla^{\mathbb{T}}$ is $\mathbb{J}$-compatible with $*_\gamma$, then $d_{\bar{L}}$ coincides with the restriction of $\mathbb{d}^{\mathbb{T}}$ on $\Omega^{0,*}_{\mathbb{J}}(M)$. More precisely, for $\theta \in \Omega^{0,q}_{\mathbb{J}}(M)$, direct computation shows that

$$(d_{\bar{L}}\theta)(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_q) = \sum_j (-1)^j (\nabla^{\mathbb{T}}_{\bar{x}_j}\theta)(\bar{x}_0, \ldots, \hat{\bar{x}}_j, \ldots, \bar{x}_q), \tag{5-7}$$

where $x_j \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M)$ for all j, which gives

$$\bar{\partial}^{\mathbb{T}}_{\mathbb{J}}\theta = d_{\bar{L}}\theta \in \Omega^{0,q+1}_{\mathbb{J}}(M). \tag{5-8}$$

The other component $\partial^{\mathbb{T}}_{\mathbb{J}}\theta \in \Omega^{1,q}_{\mathbb{J}}(M)$ is given by

$$(\partial^{\mathbb{T}}_{\mathbb{J}}\theta)(x_0, \bar{x}_1, \ldots, \bar{x}_q) = (\nabla^{\mathbb{T}}_{x_0}\theta)(\bar{x}_1, \ldots, \bar{x}_q) + \sum_j \theta(\bar{x}_1, \ldots, \nabla^{\mathbb{T}}_{\bar{x}_j} x_0, \ldots, \bar{x}_q), \tag{5-9}$$

where $x_j \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}} M)$. Via complex conjugation, the analogous versions of the identities (5-7), (5-8) and (5-9) are valid for $\bar{\theta} \in \Omega^{q,0}_{\mathbb{J}}(M)$ with $L = \mathbb{T}^{1,0}_{\mathbb{J}} M$ in place of $\bar{L}$.

Example 5.3. Suppose that $\nabla^{\mathbb{T}}$ is $\mathbb{J}$-compatible with $*_\gamma$ and consider $f \in C^\infty(M)$. It follows from Proposition 2.2 and Lemma 5.2 that

$$\partial^{\mathbb{T}}_{\mathbb{J}}\partial^{\mathbb{T}}_{\mathbb{J}} f = 0, \; \bar{\partial}^{\mathbb{T}}_{\mathbb{J}}\bar{\partial}^{\mathbb{T}}_{\mathbb{J}} f = 0 \quad \text{and} \quad \partial^{\mathbb{T}}_{\mathbb{J}}\bar{\partial}^{\mathbb{T}}_{\mathbb{J}} f + \bar{\partial}^{\mathbb{T}}_{\mathbb{J}}\partial^{\mathbb{T}}_{\mathbb{J}} f = 0. \tag{5-10}$$

The first two identities in the above can also be seen as the consequences of (5-8) and the corresponding version for d_L.

5A. ***Generalized (almost) Hermitian manifolds.*** Let $(M; \mathbb{G}, \mathbb{J})$ be a *generalized almost Hermitian manifold* [19; 22], i.e., $\mathbb{J}$ and $\mathbb{J}_- := \mathbb{G}\mathbb{J}$ are commuting generalized almost complex structures. The eigenbundles of $\mathbb{J}$ (and $\mathbb{J}_-$) decompose into the common eigenbundles of $\mathbb{G}$ and $\mathbb{J}$. Let

$$\ell_\pm := \mathbb{T}^{1,0}_{\mathbb{J}} M \cap (C_\pm \otimes \mathbb{C}). \tag{5-11}$$

Then, for instance,

$$\mathbb{T}^{1,0}_{\mathbb{J}} M = \ell_+ \oplus \ell_- \quad \text{and} \quad \mathbb{T}^{1,0}_{\mathbb{J}_-} M = \ell_+ \oplus \bar{\ell}_-.$$

The restriction of $\mathbb{J}$ to $C_\pm$ induces two almost complex structures $I_\pm$ on TM:

$$s_\pm(I_\pm X) := \mathbb{J}_+[s_\pm(X)]. \tag{5-12}$$

It follows that $(M; \mathbb{G}, \mathbb{J})$ is equivalent to a pair of almost Hermitian structures $(g, I_\pm)$ together with $b \in \Omega^2(M)$ [19; 22].

Lemma 5.4. *For a generalized almost Hermitian manifold* $(M; \mathbb{G}, \mathbb{J})$, *the space of* $\mathbb{G}$*-metric* γ*-*$\mathbb{J}$*-connections, if nonempty, is modeled on the following subspace of* $\Omega^3(M)$:

$$\{\phi \in \Omega^3(M) : \phi(I_- X, Y, Z) + \phi(X, I_+ Y, Z) = 0 \text{ for all } X, Y, Z \in C^\infty(TM)\}.$$

Proof. This follows from (5-4) and Theorem 2.6. □

Let $\phi = \gamma + db$. Then by Theorem 2.8, $\mathbb{V}^\phi$ is metric compatible with $*_\gamma$. Hence, the $\mathbb{J}$-compatibility of $\mathbb{V}^\phi$ with $*_\gamma$ is equivalent to $\diamond_\phi$ and $*_\gamma$ coincide on sections of the same common eigenbundle of $\mathbb{J}$ and $\mathbb{G}$, e.g., for $x_\pm, y_\pm \in C^\infty(\ell_\pm)$,

$$x_\pm *_\gamma y_\pm = x_\pm \diamond_\phi y_\pm. \tag{5-13}$$

Lemma 5.5. *On a generalized almost Hermitian manifold* $(M; \mathbb{G}, \mathbb{J})$, *let* $\gamma \in \Omega^3(M)$ *be a closed* 3*-form and* $\phi = \gamma + db$. *Then* $\mathbb{V}^\phi$ *is* $\mathbb{J}$*-compatible with* $*_\gamma$ *if and only if* $\nabla^{\pm\phi} I_\pm = 0$.

Proof. For any $X, Y, Z \in C^\infty(TM)$, let $x_\pm = s_\pm(X) \in C^\infty(C_\pm)$ and so on. Then

$$\langle x_\pm *_\gamma y_\pm - x_\pm \diamond_\phi y_\pm, z\rangle = \pm g(\nabla_Z^{\pm\phi} X, Y).$$

Thus (5-13) is equivalent to

$$g(\nabla_Z^{\pm\phi} X_\pm, Y_\pm) = 0$$

for all $Z \in C^\infty(TM)$ and $X_\pm, Y_\pm \in C^\infty(T_{\pm;1,0}M)$ respectively. Hence $\nabla^{\pm\phi}$ preserves $T_{\pm;1,0}M$ respectively, from which the statement follows. □

Remark 5.6. The condition $\nabla^{\pm\phi} I_\pm = 0$ can be rewritten as

$$(\nabla_X I_\pm)Y = \pm\tfrac{1}{2}(I_\pm g^{-1}\iota_Y\iota_X\phi - g^{-1}\iota_{I_\pm Y}\iota_X\phi) \tag{5-14}$$

for $X, Y \in C^\infty(TM)$. Hence, $I_\pm$ are of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ in the classification of almost Hermitian structures by Gray and Hervella [16]. The Nijenhuis tensors of $I_\pm$,

$$N_{I_\pm}(X, Y) := [X, Y] + I_\pm[I_\pm X, Y] + I_\pm[X, I_\pm Y] - [I_\pm X, I_\pm Y],$$

can be expressed in terms of ϕ:

$$g(N_{I_\pm}(X, Y), Z) = \phi(I_\pm X, I_\pm Y, Z) + \phi(I_\pm X, Y, I_\pm Z) + \phi(X, I_\pm Y, I_\pm Z) - \phi(X, Y, Z).$$

This implies that the integrability of $I_\pm$ is equivalent to ϕ being of type $(1, 2)+(2, 1)$ with respect to $I_\pm$ respectively. Furthermore, by Friedrich and Ivanov [9], if almost Hermitian connections for $(g, I_\pm)$ that admit completely skew torsions exist, they must be unique.

In $(M; \mathbb{G}, \mathbb{J})$, when $\mathbb{J}$ is integrable with respect to $\gamma \in \Omega^3(M)$, the structure $(M, \gamma; \mathbb{G}, \mathbb{J})$ defines a *generalized Hermitian manifold*. In this case, the $\mathbb{J}$-compatibility of $\mathbb{V}^\phi$ with $*_\gamma$ is equivalent to the integrability of $\mathbb{J}_-$, i.e., it provides a *generalized Kähler condition*.

Theorem 5.7. *On a generalized Hermitian manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$, *let* $\phi = \gamma + db$. *Then* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *is a generalized Kähler manifold if and only if* ∇^ϕ *is* $\mathbb{J}$*-compatible with* $*_\gamma$.

Proof. Starting with the $\mathbb{J}$-compatibility, then (5-13) and the integrability of $\mathbb{J}$ imply that $\ell_\pm$ and $\bar{\ell}_\pm$ are involutive with respect to $*_\gamma$, which further implies that $I_\pm$ are integrable. Let $x_+ = s_+(X_+) \in C^\infty(\ell_+)$ and $y_- = s_-(Y_-) \in C^\infty(\ell_-)$ for $X_+, Y_- \in C^\infty(T_{\mathbb{C}}M)$. Then, by (2-33) and Lemma 5.5,

$$x_+ *_\gamma \bar{y}_- = [\nabla^{-\phi}_{X_+}\bar{Y}_- + (b-g)\nabla^{-\phi}_{X_+}\bar{Y}_-] - [\nabla^{+\phi}_{\bar{Y}_-}X_+ + (b+g)\nabla^{+\phi}_{\bar{Y}_-}X_+] \in \ell_+ \oplus \bar{\ell}_-.$$

Thus $\mathbb{J}_-$ is integrable. The opposite direction is left to the reader. □

Remark 5.8. The generalized Kähler condition in Theorem 5.7 relates to the condition given in [20] as follows. The condition $\nabla^{\pm\phi} I_\pm = 0$ on a generalized almost Hermitian manifold is equivalent to $\nabla^{\phi,\mathcal{B}}\mathbb{J} = 0$, which implies the equivalence of the integrability of $I_\pm$ to the type condition for the *generalized torsion* as defined in [20] — the integrability of $\mathbb{J}$ then follows. In Theorem 5.7, the integrability of $I_\pm$ follows from the integrability of $\mathbb{J}$ and (5-13), obtaining the type of ϕ with respect to $I_\pm$ as a consequence.

Corollary 5.9 [20, Theorem 6.1]. *On a generalized Hermitian manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$, *let* $\phi = \gamma + db$. *Then* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *is a generalized Kähler manifold if and only if* $\nabla^{\phi,\mathcal{B}}\mathbb{J} = 0$.

Proof. As stated in Remark 5.8, $\nabla^{\phi,\mathcal{B}}\mathbb{J} = 0$ is equivalent to $\nabla^{\pm\phi} I_\pm = 0$, which by Lemma 5.5 is equivalent to ∇^ϕ being $\mathbb{J}$-compatible. □

Let $\nabla^{\mathbb{T}}$ be any $\mathbb{G}$-metric connection on a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$. Its $\mathbb{J}$-*Ricci form* $\rho^{\mathbb{T}}_{\mathbb{J}} := \rho_{\mathbb{J}}(\nabla^{\mathbb{T}}) \in \Omega^2_{\mathbb{T}}(M)$ is defined as

$$\rho^{\mathbb{T}}_{\mathbb{J}}(x, y) := \sum_i [\mathcal{R}^{\mathbb{T}}(x, y, \mathbb{J}e^i_+, e^i_+) + \mathcal{R}^{\mathbb{T}}(x, y, \mathbb{J}e^i_-, e^i_-)].$$

The $\mathbb{J}$-*scalar curvature* for $\nabla^{\mathbb{T}}$ is

$$\mathcal{S}^{\mathbb{T}}_{\mathbb{J}} := \sum_i [\rho^{\mathbb{T}}_{\mathbb{J}}(\mathbb{J}e^i_+, e^i_+) + \rho^{\mathbb{T}}_{\mathbb{J}}(\mathbb{J}e^i_-, e^i_-)].$$

5B. ***Generalized Kähler manifolds.*** Recall that the structure $(M, \gamma; \mathbb{G}, \mathbb{J})$ defines a *generalized Kähler manifold* if both $\mathbb{J}$ and $\mathbb{J}_- = \mathbb{G}\mathbb{J}$ are integrable generalized almost complex structures with respect to γ [22]. Let $\phi = \gamma + db$, Theorem 5.7 indicates that ∇^ϕ is $\mathbb{J}$-compatible with $*_\gamma$. In particular, for $\theta \in \Omega^{p,0}_{\mathbb{J}}(M)$, (5-8) gives

$$\partial^\phi_{\mathbb{J}}\theta = d_L\theta \quad \text{and} \quad \bar{\partial}^\phi_{\mathbb{J}}\bar{\theta} = d_{\bar{L}}\bar{\theta}. \tag{5-15}$$

Example 5.10 (see Example 5.3). Set $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^{\phi}$ in (5-10). Then the components in the third identity can further be rewritten in terms of the $\mathbb{G}$-eigendecomposition. For instance,

$$(\partial^{\phi}_{\mathbb{J}}\bar{\partial}^{\phi}_{\mathbb{J}} f)(x_{\pm}, \bar{y}_{\pm}) = (\partial_{\pm}\bar{\partial}_{\pm} f)(X_{\pm}, \overline{Y}_{\pm}), \tag{5-16}$$

where $\partial_{\pm}$ and $\bar{\partial}_{\pm}$ are the operators associated to the classical complex structures $I_{\pm}$, while $X_{\pm}, Y_{\pm} \in C^{\infty}(T_{\pm;1,0}M)$ are sections of the $I_{\pm}$-holomorphic tangent bundles respectively, and $x_{\pm} = s_{\pm}(X_{\pm})$ and so on.

Since $d^2_{\overline{L}} = 0$, as a consequence of (5-15), the algebraic Bianchi identity (4-2) for $\mathcal{R}^{\phi}$ implies that $\nabla^{+\phi}$ (resp. $\nabla^{-\phi}$) induces a natural I_{-}-holomorphic (resp. I_{+}-holomorphic) structure on the eigenbundles of I_{+} (resp. of I_{-}), providing an alternative proof of this well-known result in [20].

Proposition 5.11. *Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Kähler manifold. Let $\phi = \gamma + db$, let $x, y, z, w \in C^{\infty}(\mathbb{T}^{1,0}_{\mathbb{J}}M)$, and $\bar{x}$, etc. are their complex conjugates. Then*

$$\langle \mathcal{R}^{\phi}_{\bar{x},\bar{y}}\bar{z}, w\rangle + c.p. \text{ in } x, y, z = 0 \quad \text{and} \quad \langle \mathcal{R}^{\phi}_{x,y} z, \bar{w}\rangle + c.p. \text{ in } x, y, z = 0, \tag{5-17}$$

which give rise to

$$R^{-\phi}_{\overline{X}_{+},\overline{Y}_{+}} \overline{Z}_{-} = 0,\ R^{-\phi}_{\overline{X}_{+},\overline{Y}_{+}} W_{-} = 0,\ R^{+\phi}_{\overline{X}_{-},\overline{Y}_{-}} \overline{Z}_{+} = 0, \quad \text{and} \quad R^{+\phi}_{\overline{X}_{-},\overline{Y}_{-}} W_{+} = 0,$$

as well as their complex conjugates, where $X_{\pm}, Y_{\pm}, Z_{\pm}, W_{\pm} \in C^{\infty}(T_{\pm;1,0}M)$.

Proof. To see the first identity in (5-17), note that $w \in \Omega^{0,1}_{\mathbb{J}}(M)$. Then

$$\bar{\partial}^{\phi}_{\mathbb{J}} \circ \bar{\partial}^{\phi}_{\mathbb{J}} w = d^2_{\overline{L}} w = 0.$$

By (2-11) and (2-16), for $x, y, z \in C^{\infty}(\mathbb{T}^{1,0}_{\mathbb{J}}M)$,

$$\begin{aligned} 0 &= (\bar{\partial}^{\phi}_{\mathbb{J}} \circ \bar{\partial}^{\phi}_{\mathbb{J}} w)(\bar{x}, \bar{y}, \bar{z}) = (\mathbb{d}^{\phi} \circ \mathbb{d}^{\phi} w)(\bar{x}, \bar{y}, \bar{z}) \\ &= w([\bar{x} \diamond_{\phi} \bar{y} \diamond_{\phi} \bar{z}]) = -2\langle \mathcal{R}^{\phi}_{\bar{x},\bar{y}}\bar{z}, w\rangle - \text{c.p. in } x, y, z. \end{aligned}$$

The second identity then follows by taking complex conjugation.

To see the classical curvature identities, restrict x, y, z, w in (5-17) further to the $\mathbb{G}$-eigenbundles (5-11). For instance, consider $x, y \in C^{\infty}(\ell_{+})$ and $z, w \in C^{\infty}(\ell_{-})$. Set $X_{+} = \pi(x)$, $Z_{-} = \pi(z)$ and so on. Applying (4-2) to the first identity in (5-17) gives

$$-\langle 2g(R^{-\phi}_{\overline{X}_{+},\overline{Y}_{+}} \overline{Z}_{-}), w_{-}\rangle = g(R^{-\phi}_{\overline{X}_{+},\overline{Y}_{+}} W_{-}, \overline{Z}_{-}) = 0.$$

Since (g, I_{-}) is Hermitian and $\nabla^{-\phi}$ preserves I_{-}, it gives the identities involving $R^{-\phi}$. The rest of the identities are obtained similarly. □

For the generalized Bismut connection $\mathbb{V}^{\phi,\mathcal{B}}$, with $\phi = \gamma + db$, (4-14) implies that $\rho_{\mathbb{J}}^{\phi,\mathcal{B}}$ is computed by the sum of the respective classical Bismut–Ricci forms $\rho_\pm$ for $(g, I_\pm)$:

$$\rho_{\mathbb{J}}^{\phi,\mathcal{B}}(x, y) = \rho_+(X, Y) + \rho_-(X, Y).$$

The corresponding $\mathbb{J}$-scalar curvature also decomposes:

$$\mathcal{S}_{\mathbb{J}}^{\phi,\mathcal{B}} = \sum_i [\rho_{\mathbb{J}}^{\phi,\mathcal{B}}(\mathbb{J}e_+^i, e_+^i) + \rho_{\mathbb{J}}^{\phi,\mathcal{B}}(\mathbb{J}e_-^i, e_-^i)] = S_+ + 2S_{+-} + S_-,$$

where $S_\pm$ are the respective classical Bismut scalar curvatures for $(g, I_\pm)$ and S_{+-} is the *mixed Bismut scalar curvature*:

$$S_{+-} := \sum_{i,j} R^{+\phi}(I_- X_i, X_i, I_+ X_j, X_j) = \sum_{i,j} R^{-\phi}(I_+ X_i, X_i, I_- X_j, X_j).$$

6. $\mathbb{J}$-holomorphic vector bundles

Let (V, h) be a Hermitian vector bundle on a generalized complex manifold $(M, \gamma; \mathbb{J})$. Recall that a $\mathbb{J}$*-holomorphic structure* on V is given by a flat $\mathbb{T}_{\mathbb{J}}^{0,1}M$-connection [19; 25]:

$$\bar\partial_{\mathbb{J}} : C^\infty(V) \to \Omega_{\mathbb{J}}^{0,1}(V) := C^\infty(\mathbb{T}_{\mathbb{J}}^{1,0}M \otimes V) \quad \text{such that} \quad \bar\partial_{\mathbb{J}}(fv) = d_{\bar L} f \otimes v + f\bar\partial_{\mathbb{J}} v$$

for $f \in C^\infty(M)$ and $v \in C^\infty(V)$ and such that

$$\bar\partial_{\mathbb{J}} \circ \bar\partial_{\mathbb{J}} = 0, \tag{6-1}$$

where the extension to $\Omega_{\mathbb{J}}^{0,k}(V)$ is given by

$$\bar\partial_{\mathbb{J}} : \Omega_{\mathbb{J}}^{0,k}(V) \to \Omega_{\mathbb{J}}^{0,k+1}(V) \quad \text{such that} \quad \bar\partial_{\mathbb{J}}(\theta \otimes v) = d_{\bar L}\theta \otimes v + (-1)^k \theta \wedge \bar\partial_{\mathbb{J}} v \tag{6-2}$$

for $\theta \in \Omega_{\mathbb{J}}^{0,k}(M)$ and $v \in C^\infty(V)$.

If $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler, via the restriction to $C_\pm$, the $\mathbb{J}$-holomorphic structure $\bar\partial_{\mathbb{J}}$ induces on V an $I_\pm$-holomorphic structure, which will be denoted by $\bar\partial_\pm$ respectively:

$$\bar\partial_{\pm,\bar X_\pm} v := \bar\partial_{\mathbb{J}, s_\pm(\bar X_\pm)} v, \tag{6-3}$$

where $X_\pm \in T_{\pm;1,0}M$ and $v \in C^\infty(V)$.

6A. ***A connection on*** $\mathbb{T}_{\mathbb{J}}^{1,0}M$. On a generalized Kähler manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, natural $\bar L$-connections can be defined on $\mathbb{T}_{\mathbb{J}}^{1,0}M$ using the diamond bracket $\diamond_{\mathbb{T}}$ associated to any $\mathbb{G}$-metric connection $\mathbb{V}^{\mathbb{T}}$:

$$\bar\partial_\diamond^{\mathbb{T}} : C^\infty(\mathbb{T}_{\mathbb{J}}^{0,1}M) \otimes C^\infty(\mathbb{T}_{\mathbb{J}}^{1,0}M) \to C^\infty(\mathbb{T}_{\mathbb{J}}^{1,0}M), \quad \bar\partial_{\diamond,\bar x}^{\mathbb{T}} y := [\bar x \diamond_{\mathbb{T}} y]_{1,0}, \tag{6-4}$$

where $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$ and $[\bullet]_{1,0}$ denotes taking the $(1,0)$-component with respect to $\mathbb{J}$.

It follows from Proposition 5.11 that $\bar{\partial}^{\phi}_{\diamond}$ defined by $\mathbb{V}^{\phi}$ via (6-4) induces an $I_\pm$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ via its restriction to $\bar{\ell}_\pm \cong T_{\pm;0,1}M$. Moreover, by (2-11) and (5-13),

$$\begin{aligned}(\bar{\partial}^{\phi}_{\diamond} \circ \bar{\partial}^{\phi}_{\diamond})_{\bar{x},\bar{y}} z &= [\bar{x} \diamond_\phi (\bar{y} \diamond_\phi z) - \bar{y} \diamond_\phi (\bar{x} \diamond_\phi z) - (\bar{x} \diamond_\phi \bar{y}) \diamond_\phi z]_{1,0} \\ &= [\mathcal{R}^{\phi}_{\bar{x},\bar{y}} z + \mathcal{R}^{\phi}_{\bar{y},z} \bar{x} + \mathcal{R}^{\phi}_{z,\bar{x}} \bar{y}]_{1,0}.\end{aligned}$$

Thus, from (4-2), $\bar{\partial}^{\phi}_{\diamond}$ defines a $\mathbb{J}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ if and only if

$$[g(R^{-\phi}_{\overline{X}_+, Z_+} \overline{Y}_-)]_{1,0} = [g(R^{+\phi}_{\overline{Y}_-, Z_-} \overline{X}_+)]_{1,0} = 0$$

for all $X_\pm, Y_\pm, Z_\pm \in C^\infty(T_{\pm;1,0}M)$. Since $\nabla^{\pm\phi}$ preserves $I_\pm$, together with (4-4), the above is equivalent to

$$R^{+\phi}(\overline{Y}_-, \overline{W}_+, \overline{X}_+, Z_+) = R^{-\phi}(\overline{X}_+, \overline{W}_-, \overline{Y}_-, Z_-) = 0$$

for all $X_\pm, Y_\pm, Z_\pm, W_\pm \in C^\infty(T_{\pm;1,0}M)$. The computations can be summarized as the following result.

Theorem 6.1. *Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Kähler manifold, and $\phi = \gamma + db$. Let $\bar{\partial}^{\phi}_{\diamond}$ be the $\bar{L}$-connection on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ defined by*

$$\bar{\partial}^{\phi}_{\diamond,\bar{x}} y := [\bar{x} \diamond_\phi y]_{1,0}$$

for $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$. It is a $\mathbb{J}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$ if and only if

$$R^{\pm\phi}_{\overline{X}_\mp, \overline{Y}_\pm} \bar{Z}_\pm = 0$$

for all $X_\pm, Y_\pm, Z_\pm \in C^\infty(T_{\pm;1,0}M)$. In particular, if $\nabla^{\pm\phi}$ are flat on TM, $\bar{\partial}^{\phi}_{\diamond}$ is a $\mathbb{J}$-holomorphic structure on $\mathbb{T}^{1,0}_{\mathbb{J}}M$. □

6B. *Chern curvature.* Analogously to the classical situation, on a $\mathbb{J}$-holomorphic Hermitian bundle (V, h), there exists a unique *generalized Chern connection*:

$$\mathbb{V}^{h,C} := \bar{\partial}_{\mathbb{J}} + \partial_{\mathbb{J}}, \tag{6-5}$$

where $\partial_{\mathbb{J}} : C^\infty(V) \to \Omega^{1,0}_{\mathbb{J}}(V) := C^\infty(\mathbb{T}^{0,1}_{\mathbb{J}}M \otimes V)$ is defined by

$$d_{\bar{L}} h(v_1, v_2) = h(\bar{\partial}_{\mathbb{J}} v_1, v_2) + h(v_1, \partial_{\mathbb{J}} v_2)$$

for all $v_j \in C^\infty(V)$.

Definition 6.2. On a generalized complex manifold $(M, \gamma; \mathbb{J})$, let $\mathbb{V}^{\mathbb{T}}$ be a TM-torsion-free generalized connection on $\mathbb{T}M$. For a $\mathbb{J}$-holomorphic Hermitian bundle

$(V, h, \bar\partial_{\mathbb{J}})$, its $\nabla^{\mathbb{T}}$*-Chern curvature* $\mathcal{F}^{\mathbb{T},C}$ is the $\nabla^{\mathbb{T}}$-curvature of its generalized Chern connection (6-5).

When $\nabla^{\mathbb{T}}$ is $\mathbb{J}$-compatible with $*_\gamma$, (5-8) implies that the extension (3-1) of $\nabla^{\mathbb{T}}$ to $\Omega^*_{\mathbb{T}}(V)$ by $\nabla^{h,C}$ is compatible with the extension (6-2) of $\bar\partial_{\mathbb{J}}$ to $\Omega^{0,*}_{\mathbb{J}}(V)$.

Proposition 6.3. *Let* $\nabla^{\mathbb{T}}$ *be a* γ*-*$\mathbb{J}$*-connection on a generalized complex manifold* $(M, \gamma; \mathbb{J})$. *The corresponding* $\nabla^{\mathbb{T}}$*-Chern curvature* $\mathcal{F}^{\mathbb{T},C}$ *is of type* (1, 1) *with respect to* $\mathbb{J}$, *i.e.*,

$$\mathcal{F}^{\mathbb{T},C} \in \Omega^{1,1}_{\mathbb{J}}(\mathrm{End}(V)) := C^\infty\big(\mathbb{T}^{1,0}_{\mathbb{J}}M \wedge \mathbb{T}^{0,1}_{\mathbb{J}}M \otimes \mathrm{End}(V)\big). \tag{6-6}$$

In particular, over a generalized Kähler manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, *the* ***$\phi$-Chern curvature*** $\mathcal{F}^{\phi,C}$ *defined with* $\nabla^{\mathbb{T}} = \nabla^{\phi}$ *for* $\phi = \gamma + db$ *is of type* (1, 1) *with respect to* $\mathbb{J}$. □

Example 6.4 (see Example 3.1). Consider a $\mathbb{J}$-holomorphic Hermitian line bundle $(V, h, \bar\partial_{\mathbb{J}})$ and set $\mathbb{V} = \nabla^{h,C}$, the generalized Chern connection. Choose a unitary local section $s \in C^\infty(V)$. There is a local section $u^{0,1}_{\mathbb{J}} \in \Omega^{0,1}_{\mathbb{J}}(M)$ such that

$$\bar\partial_{\mathbb{J}} s = u^{0,1}_{\mathbb{J}} \otimes s.$$

Let $\nabla^{\mathbb{T}}$ be a γ-$\mathbb{J}$-connection. Then $\mathcal{F}^{\mathbb{T},C}(\mathbb{V}) = \sqrt{-1}\,\mathbb{d}^{\mathbb{T}}u$, where $u \in \Omega^1_{\mathbb{T}}(M)$ and

$$\sqrt{-1}u = u^{0,1}_{\mathbb{J}} - \overline{u^{0,1}_{\mathbb{J}}}.$$

By (5-8), the flatness of $\bar\partial_{\mathbb{J}}$ implies that

$$\bar\partial^{\mathbb{T}}_{\mathbb{J}} u^{0,1} = d_{\bar L} u^{0,1}_{\mathbb{J}} = 0.$$

This then gives

$$\mathcal{F}^{\mathbb{T},C}(\mathbb{V}) = \partial^{\mathbb{T}}_{\mathbb{J}} u^{0,1} - \bar\partial^{\mathbb{T}}_{\mathbb{J}} \overline{u^{0,1}} \in \Omega^{1,1}_{\mathbb{J}}(M).$$

Over a generalized Kähler manifold, the ϕ-Chern connection and curvature are related to the classical Chern connections and curvatures via the $\mathbb{G}$-eigendecomposition.

Lemma 6.5. *Let* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *be a generalized Kähler manifold and* $V \to M$ *be a* $\mathbb{J}$*-holomorphic vector bundle. Let* $\nabla^{h,C}_{\pm}$ *be the* $\mathbb{G}$*-eigendecomposition* (3-13) *of* $\nabla^{h,C}$ (6-5). *Then* $\nabla^{h,C}_{\pm}$ *are the Chern connections for the induced* $I_\pm$*-holomorphic structures* (6-3) *on* V *respectively. Furthermore, the classical Chern curvatures are components of the* $\mathbb{G}$*-eigendecomposition of the* ϕ*-Chern curvature.*

Proof. The statement about the connections follows from a straightforward verification, while the statement about the curvature follows from Theorem 3.7. □

Example 6.6. Continue from Example 6.4 and let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be generalized Kähler. The $I_\pm$-holomorphic structures induced by $\bar{\partial}_{\mathbb{J}}$ are given locally by $\alpha_\pm \in \Omega^{0,1}_\pm(M)$:

$$u^{0,1}_{\mathbb{J}} = [g^{-1}(\alpha_+) + bg^{-1}(\alpha_+) + \alpha_+] - [g^{-1}(\alpha_-) + bg^{-1}(\alpha_-) - \alpha_-],$$

which satisfies

$$\bar{\partial}_\pm \alpha_\pm = 0 \quad \text{and} \quad (\nabla^{-\phi}_{\bar{X}_+}\alpha_-)(\bar{Y}_-) - (\nabla^{+\phi}_{\bar{Y}_-}\alpha_+)(\bar{X}_+) = 0,$$

where $X_\pm \in C^\infty(T_{\pm;1,0}M)$ and so on. The generalized Chern connection is then

$$\mathbb{V}^{h,C} s = (u^{0,1}_{\mathbb{J}} - \overline{u^{0,1}_{\mathbb{J}}}) \otimes s,$$

while the classical Chern connections $\nabla^C_\pm$ are defined locally by $\nu_\pm = \alpha_\pm - \bar{\alpha}_\pm$ respectively. Then Example 3.8 gives the $\mathbb{G}$-eigendecomposition of the ϕ-Chern curvature.

6C. ***$\mathbb{J}$-Hermitian–Einstein equation.*** Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Hermitian manifold. The analogue to the contraction by the Kähler form is the *$\mathbb{J}_-$-contraction*.

Definition 6.7. The *$\mathbb{J}_-$-contraction* $\Lambda_{\mathbb{J}_-} : \bigwedge^2 \mathbb{T}_{\mathbb{C}}M \to \mathbb{R}$ is given by

$$\Lambda_{\mathbb{J}_-}(x \wedge y) := \langle \mathbb{J}_- x, y\rangle = \mathbb{G}(\mathbb{J}x, y), \tag{6-7}$$

where $x, y \in C^\infty(\mathbb{T}_{\mathbb{C}}M)$.

In terms of the $\mathbb{G}$-eigendecomposition, it corresponds to

$$\Lambda_{\mathbb{J}_-}(s_\pm(X) \wedge s_\pm(Y)) = \omega_\pm(X, Y) \quad \text{and} \quad \Lambda_{\mathbb{J}_-}(s_\pm(X) \wedge s_\mp(Y)) = 0 \tag{6-8}$$

for $X, Y \in TM$, where $\omega_\pm = gI_\pm$.

A version of the Hermitian–Einstein equation can thus be formulated in this context.

Definition 6.8. Let $(M, \gamma; \mathbb{G}, \mathbb{J})$ be a generalized Hermitian manifold and $\mathbb{V}^{\mathbb{T}}$ be a γ-$\mathbb{J}$-connection on $\mathbb{T}M$. A Hermitian metric h on a $\mathbb{J}$-holomorphic vector bundle $V \to M$ is $\mathbb{V}^{\mathbb{T}}$-*$\mathbb{J}$-Hermitian–Einstein* if the corresponding $\mathbb{V}^{\mathbb{T}}$-Chern curvature satisfies the following $\mathbb{V}^{\mathbb{T}}$-*$\mathbb{J}$-Hermitian–Einstein equation*:

$$\sqrt{-1}\Lambda_{\mathbb{J}_-}(\mathcal{F}^{\mathbb{T},C}(V)) = 2c\,\mathrm{Id}_V \tag{6-9}$$

for some $c \in \mathbb{R}$. If $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler and $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^\phi$, equation (6-9) will be simply called the *$\mathbb{J}$-Hermitian–Einstein equation*, a solution of which is called a *$\mathbb{J}$-Hermitian–Einstein metric*.

When $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler and $\nabla^{\mathbb{T}} = \nabla^{\phi}$, by (3-20) and (6-8) the $\mathbb{G}$-eigendecomposition of the left-hand side of (6-9) is given by

$$\Lambda_{\mathbb{J}_-}(\mathcal{F}^{\phi,C}(V)) = \Lambda_+(F_+^C(V)) + \Lambda_-(F_-^C(V)), \tag{6-10}$$

where $F_\pm^C$ are the curvatures of the classical Chern connections $\nabla_\pm^h$ on V, and $\Lambda_\pm$ are the contractions by $\omega_\pm$. It follows that (6-9) is equivalent to an equation first proposed by Hitchin in [25].

Proposition 6.9. *Over a generalized Kähler manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$, *equation* (6-9) *is equivalent to*

$$\tfrac{\sqrt{-1}}{2}\big(F_+^C(V)\wedge\omega_+^{m-1} + (-1)^{\varepsilon} F_-^C(V)\wedge\omega_-^{m-1}\big) = c(m-1)!\,\mathrm{Id}_V\, d\mathrm{vol}_g, \tag{6-11}$$

where $\varepsilon = 0$ *if* $I_\pm$ *induce the same orientation on* TM *and* $\varepsilon = 1$ *otherwise.*

Proof. Suppose that $d\mathrm{vol}_g = \frac{1}{m!}\omega_+^m$. Then

$$mF_+^C(V)\wedge\omega_+^{m-1} = \Lambda_+(F_+^C(V))\omega_+^m = m!\,\Lambda_+(F_+^C(V))\, d\mathrm{vol}_g.$$

Note that $\omega_+^m = (-1)^\varepsilon\omega_-^m$, it follows from (6-10) that (6-11) is equivalent to (6-9). □

The $\mathbb{J}_-$-contraction naturally provides the definition of a degree.

Definition 6.10. On a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$ with a γ-$\mathbb{J}$-connection $\nabla^{\mathbb{T}}$, let $(V, h, \bar\partial_{\mathbb{J}})$ be a $\mathbb{J}$-holomorphic Hermitian vector bundle. The $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*degree* of V is given by

$$\deg_{\mathbb{G}}^{\mathbb{T}}(V, h) := \frac{\sqrt{-1}}{4\pi}\int_M \mathrm{tr}_V[\Lambda_{\mathbb{J}_-}(\mathcal{F}^{\mathbb{T},C}(V))]\, d\mathrm{vol}_g. \tag{6-12}$$

When $(M, \gamma; \mathbb{G}, \mathbb{J})$ is generalized Kähler and $\nabla^{\mathbb{T}} = \nabla^{\phi}$ for $\phi = \gamma + db$, the ∇^{ϕ}-$\mathbb{G}$-degree is simply called the $\mathbb{G}$-*degree* and denoted by $\deg_{\mathbb{G}}(V, h)$.

Recall that the classical degrees of (V, h) with the induced $I_\pm$-holomorphic structure are

$$\deg_\pm(V, h) := \frac{\sqrt{-1}}{2\pi}\int_M \mathrm{tr}_h[\Lambda_\pm(F_\pm^C(V))]\, d\mathrm{vol}_g. \tag{6-13}$$

Theorem 6.11. *Let* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *be a generalized Kähler manifold and* $(V, h, \bar\partial_{\mathbb{J}})$ *be a* $\mathbb{J}$*-holomorphic vector bundle on* M. *Then*

$$\deg_{\mathbb{G}}(V, h) = \tfrac{1}{2}[\deg_+(V, h) + \deg_-(V, h)], \tag{6-14}$$

where $\deg_{\mathbb{G}}$ *and* $\deg_\pm$ *are as given in* (6-12) *and* (6-13) *respectively.*

Proof. This follows from (6-10). □

Example 6.12. Continue from Example 6.4 and work now on a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, with a γ-$\mathbb{J}$-connection $\nabla^{\mathbb{T}}$. For $f \in C^\infty(M)$, let $h_1 = e^{2f} h$ be another Hermitian metric on the line bundle V. An h_1-unitary local section is then given by $s_1 = e^{-f} s \in C^\infty(V)$, which leads to

$$\bar{\partial}_{\mathbb{J}} s_1 = (u_{\mathbb{J}}^{0,1} - d_{\bar{L}} f) \otimes s_1.$$

Let ∇_1 be the corresponding generalized Chern connection, whose $\nabla^{\mathbb{T}}$-Chern curvature is given by

$$\mathcal{F}^{\mathbb{T},C}(\nabla_1) = \mathbb{d}^{\mathbb{T}}(\sqrt{-1}u + d_L f - d_{\bar{L}} f) = \mathcal{F}^{\mathbb{T},C}(\nabla) - 2\partial_{\mathbb{J}}^{\mathbb{T}} d_{\bar{L}} f,$$

where the last step is due to (5-10). Hence

$$\deg_{\mathbb{G}}^{\mathbb{T}}(V, h_1) - \deg_{\mathbb{G}}^{\mathbb{T}}(V, h) = -\frac{\sqrt{-1}}{2\pi} \int_M \Lambda_{\mathbb{J}_-}(\partial_{\mathbb{J}}^{\mathbb{T}} d_{\bar{L}} f)\, d\mathrm{vol}_g. \tag{6-15}$$

It follows that $\deg_{\mathbb{G}}^{\mathbb{T}}$ is independent of the Hermitian metric on V if and only if the right-hand side of (6-15) vanishes for all $f \in C^\infty(M)$.

The integrand in (6-15) gives rise to the second-order operator for $f \in C^\infty(M)$:

$$P_{\mathbb{J}}(f) := -\sqrt{-1}\Lambda_{\mathbb{J}_-}(\partial_{\mathbb{J}}^{\mathbb{T}} d_{\bar{L}} f).$$

Similar to the classical case (e.g., [32]), $P_{\mathbb{J}}$ is elliptic since its principal symbol is given by

$$\sigma(P_{\mathbb{J}}) = 4\sqrt{-1}\Lambda_{\mathbb{J}_-}(\xi_{\mathbb{J}}^{0,1} \wedge \xi_{\mathbb{J}}^{1,0}) = 4\mathbb{G}(\xi_{\mathbb{J}}^{0,1}, \xi_{\mathbb{J}}^{1,0}) = \|\xi\|_g^2,$$

where $\xi_{\mathbb{J}}^{0,1}$ is the projection of $\xi \in T^*M$ to $\mathbb{T}_{\mathbb{J}}^{1,0}M$ and $\xi_{\mathbb{J}}^{1,0}$ is the complex conjugate.

Definition 6.13. For a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, the metric $\mathbb{G}$ is $\nabla^{\mathbb{T}}$-$\mathbb{J}$-*Gauduchon* if the right-hand side of (6-15) vanishes for all $f \in C^\infty(M)$. When the structure is generalized Kähler, a ∇^{ϕ}-$\mathbb{J}$-Gauduchon metric $\mathbb{G}$ is simply said to be $\mathbb{J}$-*Gauduchon*.

As in the classical situation, if $(M, \gamma; \mathbb{G}, \mathbb{J})$ is $\nabla^{\mathbb{T}}$-$\mathbb{J}$-Gauduchon and (V, h) solves (6-9), then the constant c is given by

$$c = \frac{2\pi \deg_{\mathbb{G}}^{\mathbb{T}}(V)}{\mathrm{rank}(V)\mathrm{Vol}_g(M)}.$$

It is then natural to extend the notions of slope and slope stability to $\mathbb{J}$-holomorphic vector bundles over a $\nabla^{\mathbb{T}}$-$\mathbb{J}$-Gauduchon generalized Hermitian manifold. The notion of coherent subsheaf in this context can be adopted from Definition 3.4 of [28].

Definition 6.14. Let $\mathbb{G}$ be a $\nabla^{\mathbb{T}}$-$\mathbb{J}$-Gauduchon metric. The $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*slope* of a $\mathbb{J}$-holomorphic vector bundle $(V, \bar{\partial}_{\mathbb{J}})$ over M is

$$\mu^{\mathbb{T}}_{\mathbb{G}}(V) := \frac{\deg^{\mathbb{T}}_{\mathbb{G}}(V)}{\operatorname{rank}(V)}. \tag{6-16}$$

The bundle V is $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*semistable* if for any coherent $\mathbb{J}$-holomorphic subsheaf W of V:

$$\mu^{\mathbb{T}}_{\mathbb{G}}(W) \leqslant \mu^{\mathbb{T}}_{\mathbb{G}}(V). \tag{6-17}$$

V is said to be $\nabla^{\mathbb{T}}$-$\mathbb{G}$-*stable* if strict inequality holds in (6-17). Over a $\mathbb{J}$-Gauduchon generalized Kähler manifold, the corresponding notions are simply referred to as the $\mathbb{G}$-*slope*, $\mathbb{G}$-*semistable* and $\mathbb{G}$-*semistable* respectively.

Recall that over a Hermitian manifold (M, g, I), the degree of any holomorphic vector bundle V is independent of a Hermitian metric on V if and only if g is Gauduchon, i.e.,

$$\partial\bar{\partial}(\omega^{m-1}) = 0, \tag{6-18}$$

where m is the complex dimension of M. On a generalized Kähler manifold, the $\mathbb{J}$-Gauduchon condition can be expressed in a similar fashion.

Proposition 6.15. *A generalized Kähler manifold* $(M, \gamma; \mathbb{G}, \mathbb{J})$ *is* $\mathbb{J}$-*Gauduchon if and only if*

$$\partial_+\bar{\partial}_+(\omega_+^{m-1}) + (-1)^{\varepsilon}\partial_-\bar{\partial}_-(\omega_-^{m-1}) = 0, \tag{6-19}$$

where $\varepsilon = 0$ *if* $I_\pm$ *induce the same orientation on* TM *and* $\varepsilon = 1$ *otherwise.*

Proof. The degree of a Hermitian vector bundle coincides with that of its determinant line bundle, so it's sufficient to consider line bundles. By (5-16) and (6-8),

$$\int_M \Lambda_{\mathbb{J}_-}(\partial^{\phi}_{\mathbb{J}} d_{\bar{L}} f)\, d\mathrm{vol}_g = \int_M [\Lambda_+(\partial_+\bar{\partial}_+ f) + \Lambda_-(\partial_-\bar{\partial}_- f)]\, d\mathrm{vol}_g.$$

The statement then follows from integration by parts. □

Remark 6.16. Evidently, (6-19) holds for $m < 3$, since the 3-form below is closed:

$$\phi = \gamma + db = \mp d^c_\pm \omega_\pm.$$

For $m \geqslant 3$, (6-19) is equivalent to

$$\phi \wedge d(\omega_+^{m-2} - (-1)^{\varepsilon}\omega_-^{m-2}) = 0.$$

In particular, $(M, \gamma; \mathbb{G}, \mathbb{J})$ is $\mathbb{J}$-Gauduchon if the difference $\omega_+^{m-2} - (-1)^{\varepsilon}\omega_-^{m-2}$ is closed. When $m = 3$, (6-19) can also be rewritten as

$$\phi_+^{(2,1)} \wedge \phi_+^{(1,2)} + (-1)^{\varepsilon}\phi_-^{(2,1)} \wedge \phi_-^{(1,2)} = 0,$$

where, for instance, $\phi_{\pm}^{(2,1)}$ denote the (2, 1)-component of ϕ with respect to $I_{\pm}$ respectively. It is clear that the generalized Kähler manifold is $\mathbb{J}$-Gauduchon if g is Gauduchon with respect to both $I_{\pm}$.

7. Geometric Lax flows

A *Lax pair* [29] consists of two families of operators $\{(L_t, P_t) : t \in I \subseteq \mathbb{R}\}$ such that

$$\frac{d}{dt} P_t = [L_t, P_t], \tag{7-1}$$

where $\{L_t\}$ is the *Lax operator* and it is assumed that $0 \in I$. Equation (7-1) is also said to be in *the Lax form*. Suppose that $\{\Psi_t\}$ is generated by $\{L_t\}$, i.e., it solves the equation

$$\frac{d}{dt}\Psi_t = L_t \Psi_t \quad \text{with } \Psi_0 = \mathrm{Id}\,. \tag{7-2}$$

Then $\{P_t\}$ can be obtained from pushing forward an initial operator P_0 by $\{\Psi_t\}$:

$$P_t = \Psi_t P_0 \Psi_t^{-1}.$$

In particular, $\{P_t\}$ is an isospectral family.

Let $\{A_t\}$ be a smooth family of operators on the same space; then the t-differential of $\{\Psi_t^{-1} A_t \Psi_t\}$ describes the extent to which $\{(L_t, A_t)\}$ fails to be a Lax pair as well.

Definition 7.1. Suppose that $\{(L_t, P_t)\}$ is a Lax pair and $\{A_t\}$ be a smooth family of operators on the same space. The *L_t-differential of A_t* (*along the Lax flow*) is

$$\delta_L A_t := \frac{d}{dt} A_t - [L_t, A_t]. \tag{7-3}$$

It is straightforward to verify that commutativity with P_t is preserved by δ_L.

Lemma 7.2. *If A_t and P_t commute for all t, then $\delta_L A_t$ also commutes with P_t for all t.*

Proof. Notice that

$$\frac{d}{dt}(\Psi_t^{-1} A_t \Psi_t) = \Psi_t^{-1}(\delta_L A_t)\Psi_t,$$

from which the statement follows. □

When a pair of geometric quantities forms a Lax pair, the corresponding equation (7-1) is said to generate a *geometric Lax flow*. Two main classes of examples will be described, where the operators $\{P_t\}$ are either generalized metrics or generalized almost complex structures. Such Lax pairs impose certain necessary conditions on the Lax operator.

Definition 7.3. Let $\mathbb{G}$ be a generalized metric and $\mathbb{J}$ a generalized (almost) complex structure. An operator $\mathbb{L}$ on $C^\infty(\mathbb{T}M)$ is *Lax compatible with* $\mathbb{G}$ if

$$\langle \mathbb{L}x_+, y_-\rangle + \langle x_+, \mathbb{L}y_-\rangle = 0 \tag{7-4}$$

for all $x_+ \in C^\infty(C_+)$ and $y_- \in C^\infty(C_-)$. The operator $\mathbb{L}$ is *Lax compatible with* $\mathbb{J}$ if

$$\langle \mathbb{L}x, y\rangle + \langle x, \mathbb{L}y\rangle = \langle \mathbb{L}\bar{x}, \bar{y}\rangle + \langle \bar{x}, \mathbb{L}\bar{y}\rangle = 0 \tag{7-5}$$

for all $x, y \in C^\infty(\mathbb{T}^{1,0}_{\mathbb{J}}M)$.

Lemma 7.4. *Suppose that* $\{(\mathbb{L}_t, \mathbb{P}_t)\}$ *is a Lax pair of operators on* $C^\infty(\mathbb{T}M)$. *If* $\mathbb{P}_t$ *is a smooth family of generalized metrics or almost complex structures*, *then* $\mathbb{L}_t$ *is Lax compatible with* $\mathbb{P}_t$ *for all* t.

Proof. It follow from the orthogonality of $\mathbb{P}_t$ with respect to the pairing $\langle\ ,\ \rangle$ and that $\mathbb{P}_t^2$ are constant operators. □

The following simplified criteria are useful in practice.

Corollary 7.5. *Let* $\mathbb{L} \in C^\infty(\mathrm{End}(\mathbb{T}M))$ *and* $x, y \in \mathbb{T}M$. *Then*:

(1) (7-4) *holds for* $\mathbb{L}$ *if the bilinear form* $\mathbb{G}(\mathbb{L}x, y)$ *is symmetric*, *i.e.*,

$$\mathbb{G}(\mathbb{L}x, y) = \mathbb{G}(\mathbb{L}y, x).$$

(2) (7-4) *and* (7-5) *hold for* $\mathbb{L}$ *if the bilinear form* $\langle \mathbb{L}x, y\rangle$ *is skew-symmetric*, *i.e.*,

$$\langle \mathbb{L}x, y\rangle + \langle \mathbb{L}y, x\rangle = 0.$$

Proof. Left for the reader. □

7A. $\theta \in \Omega^2_{\mathbb{T}}(M)$ *as the Lax operator.* Let $\{P_t \in C^\infty(T^*M^{\otimes 2})\}$ be a smooth family of 2-tensors on M, whose symmetric and skew-symmetric parts are respectively P_t^s and P_t^a:

$$P_t = P_t^s + P_t^a \quad \text{with} \quad P_t^s(X, Y) = P_t^s(Y, X),\ P_t^a(X, Y) = -P_t^a(Y, X)$$

for $X, Y \in C^\infty(TM)$. Then $\{P_t\}$ defines an initial value problem for a family of generalized metrics $\mathbb{G}_t$ as follows:

$$\begin{cases} \dfrac{d}{dt} g_t = -P_t^s, \\ \dfrac{d}{dt} b_t = P_t^a. \end{cases} \tag{7-6}$$

The system (7-6) can be reformulated into a Lax flow for $\mathbb{G}_t$. For such a family $\mathbb{G}_t$, let $C_\pm^t$ be the eigenbundles and $s_\pm^t : TM \to C_\pm^t$ be the corresponding isomorphisms.

Lemma 7.6. *Let* $\{P_t \in C^\infty(T^*M^{\otimes 2})\}$ *and* $\{\theta_t \in \Omega^2_{\mathbb{T}}(M)\}$ *be a smooth family of* $\mathbb{T}M$*-forms such that*

$$\theta_t(x^t_-, y^t_+) = P_t(X, Y), \tag{7-7}$$

where $x^t_\pm = s^t_\pm(X)$ *for* $X \in C^\infty(TM)$ *and so on. Let* $\theta_t : \mathbb{T}M \to \mathbb{T}M$ *be given by*

$$2\langle \theta_t(x), y\rangle := \theta_t(x, y).$$

Then (7-6) *is equivalent to the* ***2-tensor Lax flow***

$$\frac{d}{dt}\mathbb{G}_t = [\theta_t, \mathbb{G}_t]. \tag{7-8}$$

Proof. Note that θ_t satisfies Corollary 7.5 (2). Since $\mathbb{G}^2 = \mathbb{1}$, the differential of a smooth family of generalized metrics is skew with respect to the generalized metric at each time. Obviously the left-hand side of (7-8) is skew with respect to $\mathbb{G}_t$. Thus only the mixed $\mathbb{G}_t$-eigencomponents of the left-hand side are nontrivial, one of which goes as follows:

$$\begin{aligned} 0 &= \frac{d}{dt}\langle \mathbb{G}_t x^t_-, y^t_+\rangle \\ &= \left\langle \frac{d}{dt}\mathbb{G}_t x^t_-, y^t_+\right\rangle + \left\langle \mathbb{G}_t \frac{d}{dt}(b_t - g_t)X, y^t_+\right\rangle + \left\langle \mathbb{G}_t x^t_-, \frac{d}{dt}(b_t + g_t)Y\right\rangle \\ &= \left\langle \frac{d}{dt}\mathbb{G}_t x^t_-, y^t_+\right\rangle + \frac{d}{dt}(b_t - g_t)(X, Y). \end{aligned}$$

The right-hand side is given by

$$\langle [\theta_t, \mathbb{G}_t] x^t_-, y^t_+\rangle = \langle (\theta_t \mathbb{G}_t - \mathbb{G}_t \theta_t) x^t_-, y^t_+\rangle = -2\langle \theta_t(x^t_-), y^t_+\rangle = -P_t(X, Y).$$

Thus (7-8) gives rise to

$$\frac{d}{dt}(b_t - g_t)(X, Y) = P_t(X, Y),$$

from which (7-6) follows. The other direction is left for the reader. □

A smooth conformal family of metrics $\{g_t\}$ can be seen as a solution to (7-6), by setting $P_t = P^s_t = f_t g_0$, where $\{f_t \in C^\infty(M)\}$ is a smooth family of functions. In this case, $b_t \equiv b_0$ is constant throughout the flow. The corresponding $\mathbb{T}M$-forms as given in Lemma 7.6 can be chosen to be dependent only on the generalized metric $\mathbb{G}$.

Definition 7.7. Let $\mathbb{G}$ be a generalized metric defined by (g, b), where g is a Riemannian metric. A $\mathbb{T}M$-form $\theta \in \Omega^2_{\mathbb{T}}(M)$ is $\mathbb{G}$-*conformal* if there are $r_g, r_b \in C^\infty(M)$ such that

$$\theta(x_-, y_+) = r_g g(X, Y) + r_b b(X, Y) \quad \text{and} \quad \theta(x_\pm, y_\pm) = 0,$$

where $x_\pm = s_\pm(X) \in C^\infty(C_\pm)$, etc. The functions r_g, r_b are called the *conformal weights*.

Remark 7.8. When the conformal weights coincide, i.e., $r_g = r_b$, a family of $\mathbb{G}$-conformal forms generates conformal deformations of the generalized metric $\mathbb{G}$. Otherwise, the metric g and b are deformed by different factors. In particular, when the conformal weight $r_b = 0$, it corresponds to classical conformal deformation of the metric g.

7B. *ϕ-curvature Lax flow.* A special case of (7-8) is when θ_t are $\mathbb{d}^{\phi_t}$-exact, i.e., $\theta_t = \mathbb{d}^{\phi_t} u_t$ for a smooth family of sections $\{u_t \in \Omega^1_{\mathbb{T}}(M)\}$ and 3-forms $\{\phi_t \in \Omega^3(M)\}$, with respect to the metric $\mathbb{G}_t$ at time t. Suppose that $u_t = Z_t + \zeta_t$, and set $x^t_\pm = s^t_\pm(X)$ for $X \in C^\infty(TM)$ and so on. Then

$$
\begin{aligned}
(\mathbb{d}^{\phi_t} u_t)(x^t_-, y^t_+) &= X u_t(y^t_+) - Y u_t(x^t_-) - u_t(x^t_- \diamond_{\phi_t, \mathbb{G}_t} y^t_+) \\
&= (\mathcal{L}_{Z_t} g_t)(X, Y) + [d(\zeta_t - \iota_{Z_t} b_t) - \iota_{Z_t}\phi_t](X, Y).
\end{aligned}
$$

It follows that in this case, the Lax flow (7-8) is equivalent to

$$
\begin{cases}
\dfrac{d}{dt} g_t = -\mathcal{L}_{Z_t} g_t, \\
\dfrac{d}{dt} b_t = -\mathcal{L}_{Z_t} b_t + d\zeta_t - \iota_{Z_t}(\phi_t - db_t).
\end{cases}
\tag{7-9}
$$

When $\gamma = \phi_t - db_t$ is a fixed closed 3-form, (7-9) describes the pushforward of an initial generalized metric by the family of generalized diffeomorphism of $\mathbb{T}M$ generated by u_t.

Proposition 7.9. *Let $\gamma \in \Omega^3(M)$ be a closed 3-form and $u_t = Z_t + \zeta_t \in C^\infty(\mathbb{T}M)$ be a smooth family of sections. Let (λ_t, β_t) be the family of generalized diffeomorphisms generated by $\{u_t\}$ under $*_\gamma$* (recalled below). *Set $\phi_t = \gamma + db_t$. Then* (7-9) *coincides with the infinitesimal action by (λ_t, β_t) on a generalized metric $\mathbb{G}$ via pushforward.*

Proof. It's more straightforward to work with the pullback action on generalized metrics. Recall, e.g., from Hu and Uribe [27], that in (λ_t, β_t), λ_t is the 1-parameter family of diffeomorphisms generated by Z_t, and

$$\beta_t := \int_0^t \lambda_s^*(d\zeta_s - \iota_{Z_s}\gamma)\, ds.$$

The pushforward of $x = X + \xi \in \mathbb{T}M$ by (λ_t, β_t) is given by

$$(\lambda_t, \beta_t)_* x = \lambda_{t*}(x + \iota_X \beta_t) = \lambda_{t*} X + (\lambda_t^{-1})^*(\xi + \iota_X \beta_t)$$

with the corresponding infinitesimal action $x \mapsto -u_t *_\gamma x$.

The pullback of $\mathbb{G}$ by (λ_t, β_t) gives the family of generalized metrics

$$\mathbb{G}_t(x, y) := \mathbb{G}\big((\lambda_t, \beta_t)_* x, (\lambda_t, \beta_t)_* y\big),$$

where $x, y \in C^\infty(\mathbb{T}M)$ are independent of t. Analogously to the computations in the classical case, differentiating the above with respect to t gives, by the left-hand side,

$$\begin{aligned}\Big(\frac{d}{dt}\mathbb{G}_t\Big)(x,y) &= \frac{1}{2}\frac{d}{dt}[g_t(X,Y)+g_t^{-1}(\xi-\iota_X b_t,\eta-\iota_Y b_t)]\\ &= \frac{1}{2}\Big[\Big(\frac{d}{dt}g_t\Big)(X,Y)-\Big(\frac{d}{dt}g_t\Big)(g_t^{-1}\xi_t',g_t^{-1}\eta_t')\\ &\qquad -\Big(\frac{d}{dt}b_t\Big)(Y,g_t^{-1}\xi_t')-\Big(\frac{d}{dt}b_t\Big)(X,g_t^{-1}\eta_t')\Big],\end{aligned}$$

where $\xi_t'=\xi-\iota_X b_t$ and $\eta_t'=\eta-\iota_Y b_t$, and, by the right-hand side,

$$\begin{aligned}\Big(\frac{d}{dt}\mathbb{G}_t\Big)(x,y) &= Z_t\mathbb{G}_t(x,y)+\mathbb{G}_t(-u_t *_\gamma x,y)+\mathbb{G}_t(x,-u_t *_\gamma y)\\ &= \tfrac{1}{2}[(\mathcal{L}_{Z_t}g_t)(X,Y)-(\mathcal{L}_{Z_t}g_t)(g_t^{-1}\xi_t',g_t^{-1}\eta_t')]\\ &\quad+\tfrac{1}{2}[(-\mathcal{L}_{Z_t}b+d\zeta_t-\iota_{Z_t}\gamma)(Y,g_t^{-1}\xi_t')\\ &\qquad+(-\mathcal{L}_{Z_t}b+d\zeta_t-\iota_{Z_t}\gamma)(X,g_t^{-1}\eta_t')].\end{aligned}$$

Comparing these two ways of computing $\big(\frac{d}{dt}\mathbb{G}_t\big)(x,y)$, the pullback action on $\mathbb{G}$ gives

$$\begin{cases}\dfrac{d}{dt}g_t=\mathcal{L}_{Z_t}g_t,\\[2mm] \dfrac{d}{dt}b_t=\mathcal{L}_{Z_t}b_t-(d\zeta_t-\iota_{Z_t}\gamma).\end{cases}$$

The pushforward action reverses the signs on the right-hand side, giving (7-9). □

A $\mathbb{d}^\phi$-exact $\mathbb{T}M$-form in $\Omega^2_{\mathbb{T}}(M)$ can be seen as the ϕ-curvature of a unitary generalized connection on the trivial Hermitian line bundle. In general, let (V,h) be a Hermitian line bundle with a family of unitary generalized connections $\{\mathbb{V}_t\}$. By Example 3.8, the Lax flow (7-8) defined by $\{\theta_t=\sqrt{-1}\mathcal{F}^{\phi_t}(\mathbb{V}_t)\}$ where $\phi_t=\gamma+db_t$ is equivalent to the system

$$\text{(7-10)}\qquad \begin{cases}\dfrac{d}{dt}g_t=-\mathcal{L}_{g_t^{-1}\psi_t}g_t,\\[2mm] \dfrac{d}{dt}b_t=\sqrt{-1}F_0^t-\iota_{g_t^{-1}\psi_t}\phi_t,\end{cases}$$

which reduces to (7-9) when V is trivial, and admits similar interpretation in terms of (not necessarily exact) generalized diffeomorphisms.

Theorem 7.10. *Given a family of unitary generalized connections $\{\mathbb{V}_t\}$ on a line bundle V with Hermitian metrics $\{h_t\}$, the Lax flow (7-8) defined by $\{\theta_t=\sqrt{-1}\mathcal{F}^{\phi_t}(\mathbb{V}_t)\}$, where $\phi_t=\gamma+db_t$, corresponds to the pushforward of an initial generalized metric $\mathbb{G}$ by a family of generalized diffeomorphisms, which may not be* ***exact***, *i.e., is not generated by global sections of $\mathbb{T}M$.*

Proof. In the notation of Example 3.8, the local section defining $\mathbb{V}_t$ is given by $\sqrt{-1}u_t = \sqrt{-1}(Z_t + \zeta_t)$, where Z_t is a global vector field and ζ_t is a locally defined 1-form, with

$$Z_t = g_t^{-1}\psi_t = \tfrac{1}{2}g_t^{-1}(\nu_{t,+} - \nu_{t,-}) \quad \text{and} \quad \zeta_t = \iota_{Z_t} b_t + \tfrac{1}{2}(\nu_{t,+} + \nu_{t,-}).$$

Then ∇_0^t is defined by the local section $\frac{1}{2}(\nu_{t,+} + \nu_{t,-})$, which implies that

$$\sqrt{-1}F_0^t = d(\zeta_t - \iota_{Z_t} b_t) = -\mathcal{L}_{Z_t} b_t + d\zeta_t + \iota_{Z_t} db_t.$$

Locally, the second equation in (7-10) is thus

$$\frac{d}{dt} b_t = -\mathcal{L}_{Z_t} b_t + d\zeta_t - \iota_{Z_t}\gamma,$$

noting that $\phi_t = \gamma + db_t$. By Proposition 7.9, (7-10) corresponds to pushing forward by the local exact generalized diffeomorphisms generated by $\{(Z_t, \zeta_t)\}$. Globally, (7-10) corresponds to pushing forward by possibly nonexact generalized diffeomorphisms. □

Remark 7.11. The diffeomorphism components of the generalized diffeomorphisms in Theorem 7.10 arise from the vector components of $\mathbb{V}_t$ on V, which vanish if they are liftings of classical connections on V.

Example 7.12. Let $(M, \gamma; \mathbb{J})$ be a generalized complex manifold and $\{\theta_t \in \Omega^2_{\mathbb{T}}(M)\}$ be a smooth family of $\mathbb{T}M$-forms. The Lax flow with initial value $\mathbb{J}_0 = \mathbb{J}$,

$$\frac{d}{dt}\mathbb{J}_t = [\theta_t, \mathbb{J}_t],$$

consists of generalized almost complex structures. The flow above preserves $\mathbb{J}$ if and only if $[\theta_t, \mathbb{J}] = 0$ for all t, which is equivalent to $\theta_t \in \Omega^{1,1}_{\mathbb{J}}(M)$. Thus, starting with a generalized Hermitian manifold $(M, \gamma; \mathbb{G}, \mathbb{J})$, the Lax flow (7-8) defined by $\{\theta_t \in \Omega^{1,1}_{\mathbb{J}}(M)\}$ produces a family of generalized Hermitian structures with the same $\mathbb{J}$.

Suppose furthermore that $(M, \gamma; \mathbb{G}, \mathbb{J})$ admits a γ-$\mathbb{J}$-connection $\nabla^{\mathbb{T}}$. Fix a $\mathbb{J}$-holomorphic line bundle V and let $\{\theta_t\}$ be the $\nabla^{\mathbb{T}}$-Chern curvatures for a smooth family of Hermitian metrics $\{h_t\}$ on V. By Proposition 6.3, $\{\theta_t\}$ consists of $(1, 1)$-forms with respect to $\mathbb{J}$. By Theorem 7.10, the corresponding Lax flow (7-8) corresponds to the pushforward of $\mathbb{G}$ by a family of generalized diffeomorphisms $\{(\lambda_t, \beta_t)\}$. Alternatively, it can be seen as a family of generalized Hermitian structures $(M, \gamma; \mathbb{G}, (\lambda_t, \beta_t)^*\mathbb{J})$ with $\mathbb{G}$ fixed.

7C. *Ricci Lax flow.* When the Ricci curvature of a generalized connection $\nabla^{\mathbb{T}}$ satisfies (7-4), it can serve as the Lax operator in a Lax pair involving the generalized metric.

Definition 7.13. A smooth family of pairs $\{(\mathbb{G}_t, \mathbb{V}^{\mathbb{T}_t})\}$ of generalized metrics $\mathbb{G}_t$ and $\mathbb{G}_t$-metric connections is a solution to the $\mathbb{V}^{\mathbb{T}}$-*Ricci Lax flow* if $\{\mathcal{R}ic^{\mathbb{T}_t}\}$ satisfies (7-4) and the pair $\{(\mathcal{R}ic^{\mathbb{T}_t}, \mathbb{G}_t)\}$ form a Lax pair, i.e.,

$$\frac{d}{dt}\mathbb{G}_t = [\mathcal{R}ic^{\mathbb{T}_t}, \mathbb{G}_t]. \tag{7-11}$$

When $\mathbb{V}^{\mathbb{T}_t}$ is prescribed to only depend on $\mathbb{G}_t$, (7-11) becomes an equation for $\mathbb{G}_t$ only, in which case the family of generalized metrics $\mathbb{G}_t$ is said to be *a solution* to (7-11). If $\mathbb{V}^{\mathbb{T}_t} = \mathbb{V}^{\phi_t}$, then the flow (7-11) is simply called the *Ricci Lax flow*.

Since $\mathcal{R}c^{\phi}$ is symmetric, it satisfies the condition in Corollary 7.5. Theorem 7.14 below shows that the Ricci Lax flow is equivalent to

$$\frac{d}{dt}(g_t \mp b_t) = -2\,\mathrm{Rc}_t^{\pm\phi_t} = -2\,\mathrm{Rc}_t + \tfrac{1}{2}\phi_t^2 \pm d^*\phi_t. \tag{7-12}$$

Let $\gamma \in \Omega^3(M)$ such that $d\gamma = 0$ and set $\phi_t = \gamma + db_t$. Then (7-12) coincides with the *generalized Ricci flow* [12; 36] as a system for Riemannian metrics g_t and $b_t \in \Omega^2(M)$:

$$\begin{cases} \dfrac{d}{dt}g_t = -2\,\mathrm{Rc}_t + \tfrac{1}{2}(\gamma + db_t)^2, \\ \dfrac{d}{dt}b_t = -d^*(\gamma + db_t). \end{cases} \tag{7-13}$$

Theorem 7.14. *Under the further constraints that $\mathbb{V}^{\mathbb{T}} = \mathbb{V}^{\phi_t}$ and are $\mathbb{G}_t$-metric compatible with the Dorfman bracket $*_\gamma$* (2-37), *i.e., $\phi_t - db_t = \gamma \in \Omega^3(M)$ for all t, the Ricci Lax flow* (7-11) *is equivalent to the generalized Ricci flow* (7-13).

Proof. It's obvious that (7-13) follows from (7-12), by matching the symmetric and skew-symmetric terms on both sides. To obtain (7-12) from (7-11), fix $X, Y \in TM$ and consider

$$x_-^t = s_-^t(X) \quad \text{and} \quad y_+^t = s_+^t(Y).$$

The left-hand side of (7-12) is computed in Lemma 7.6, while the right-hand side becomes

$$\begin{aligned} \langle [\mathcal{R}ic^{\phi_t}, \mathbb{G}_t] x_-^t, y_+^t \rangle &= \langle (\mathcal{R}ic^{\phi_t}\,\mathbb{G}_t - \mathbb{G}_t\,\mathcal{R}ic^{\phi_t}) x_-^t, y_+^t \rangle \\ &= -2\,\mathcal{R}c^{\phi_t}(x_-^t, y_+^t) = -2\,\mathrm{Rc}^{+\phi_t}(X, Y). \end{aligned}$$

This gives half of (7-12). The other half is equivalent. □

Example 7.15. Consider two 3-dimensional Lie groups: $T = (U(1))^3$ and $G = \mathrm{SU}(2)$, with their invariant metrics g and invariant volume forms ϕ. For both, set $b = 0$ and consider the corresponding ϕ-Ricci curvature $\mathcal{R}ic^{\phi}$.

For T, the invariant metric is flat, thus $Ric = 0$ while $Ric^{\pm\phi} \neq 0$ by (4-8). In the $\mathbb{G}$-eigendecomposition, $\mathcal{R}c^{\phi}$ is of the form

$$\mathcal{R}c^{\phi} = \begin{pmatrix} 0 & R \\ R^T & 0 \end{pmatrix},$$

where $R = \mathbb{1}_3$, the 3×3 identity matrix.

For G, the invariant metric is the standard round metric on S^3 which is not flat, while the connections with torsion $\pm\phi$ are flat, hence $Ric^{\pm\phi} = 0$. In this case, $\mathcal{R}c^{\phi}$ is of the form

$$\mathcal{R}c^{\phi} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix},$$

where R here is the classical Ricci tensor for the round metric (see Section 2E).

Definition 7.7 and Theorem 7.14 lead to the natural generalization of Ricci solitons.

Definition 7.16. Let $\gamma \in \Omega^3(M)$ be a closed 3-form. A smooth family of generalized metrics $\{\mathbb{G}_t\}$ is a *Ricci Lax soliton* if there exists a smooth family of sections $\{u_t \in C^\infty(\mathbb{T}M)\}$ and $\mathbb{G}_t$-conformal forms $\{\theta_t \in \Omega^2_{\mathbb{T}}(M)\}$ with constant conformal weights r_t and s_t, and

$$[\mathcal{R}ic^{\phi_t} - \mathbb{d}^{\phi_t} u_t - \theta_t, \mathbb{G}_t] = 0, \tag{7-14}$$

where $\phi_t = \gamma + db_t$. The family is a *gradient Ricci Lax soliton* if furthermore there is $f \in C^\infty(M)$ such that $\left\{u_t = -\frac{1}{2}\mathbb{G}_t(df)\right\}$.

Notice that $\mathbb{G}_t(df) = \mathrm{grad}^t f + \iota_{\mathrm{grad}^t f} b_t$, where $\mathrm{grad}^t f$ is the gradient of f with respect to g_t. When $s_t = 0$, the gradient Ricci Lax soliton equation is then equivalent to the system

$$\begin{cases} Rc_t - \frac{1}{4}\phi_t^2 + \mathrm{Hess}^t f = r_t g_t, \\ \quad d^*\phi_t + \iota_{\mathrm{grad}^t f}\phi_t = 0, \end{cases} \tag{7-15}$$

where $\mathrm{Hess}^t f$ is the Hessian of f with respect to the Levi-Civita connection of g_t. When $r_t \equiv r$ is a constant function independent of t, the system (7-15) is exactly the *generalized Ricci soliton* equation ([3] and references therein).

7D. *Bismut–Ricci Lax flow.* Even though $\mathcal{R}ic^{\phi,\mathcal{B}}$ is neither symmetric nor skew-symmetric, it satisfies (7-4), and thus can be used to define a Lax flow for generalized metrics. In (7-11), taking $\mathcal{R}ic^{\phi,\mathcal{B}}$ as the Lax operator leads to the *Bismut–Ricci Lax flow*:

$$\frac{d}{dt}\mathbb{G}_t = [\mathcal{R}ic^{\phi_t,\mathcal{B}}, \mathbb{G}_t]. \tag{7-16}$$

Since $\mathcal{R}c^{\phi,\mathcal{B}}$ has the same mixed components as $\mathcal{R}c^{\phi,\mathbb{G}}$, i.e.,

$$[\mathcal{R}ic^{\phi,\mathbb{G}}, \mathbb{G}] = [\mathcal{R}ic^{\phi,\mathcal{B}}, \mathbb{G}]$$

by Theorem 7.14, (7-16) is equivalent to (7-12), so they generate the same flow for the generalized metric [35]. The $\mathcal{R}ic^{\phi,\mathcal{B}}$-differential of $\mathbb{V}^{\phi_t,\mathcal{B}}$ takes a particularly simple form.

Proposition 7.17. *Fix $X, Y, Z \in C^\infty(TM)$ and $x \in C^\infty(\mathbb{T}M)$, with $\pi(x) = X$. Let $Y, Z \in C^\infty(TM)$ and $y_\pm^t = s_\pm^t(Y)$, $z_\pm^t = s_\pm^t(Z)$. Then*

$$\mathbb{G}_t((\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{V}_x^{\phi_t,\mathcal{B}})y_\pm^t, z_\pm^t) = -g_t\Big(Y, \frac{d}{dt}\nabla_X^{\pm\phi_t} Z\Big). \tag{7-17}$$

Proof. Only the case for C_+^t is shown here, and the case for C_-^t is similar:

$$\begin{aligned}
&\mathbb{G}_t\Big(\frac{d}{dt}(\mathbb{V}_x^{\phi_t,\mathcal{B}})y_+^t, z_+^t\Big)\\
&\quad= \frac{d}{dt}[\mathbb{G}_t(\mathbb{V}_x^{\phi_t,\mathcal{B}}y_+^t, z_+^t)] - \mathbb{G}_t\Big(\mathbb{V}_x^{\phi_t,\mathcal{B}}\Big[\frac{d}{dt}(b_t+g_t)Y\Big], z_+^t\Big)\\
&\qquad\qquad - \mathbb{G}_t\Big(\mathbb{V}_x^{\phi_t,\mathcal{B}}y_+^t, \frac{d}{dt}(b_t+g_t)Z\Big)\\
&\quad= g_t\Big(\frac{d}{dt}\nabla_X^{+\phi_t}Y, Z\Big) + [XRc_t^{-\phi}(Y,Z) - \mathrm{Rc}_t^{-\phi}(Y, \nabla_X^{+\phi_t}Z) - \mathrm{Rc}_t^{-\phi}(\nabla_X^{+\phi_t}Y, Z)],
\end{aligned}$$

where the final equality follows from (7-12) together with the fact that $\mathbb{V}^{\phi_t,\mathcal{B}}$ preserves $\mathbb{G}_t$. Next, (4-15) implies that

$$\begin{aligned}
&\mathbb{G}_t([\mathcal{R}ic_t^{\phi_t,\mathcal{B}}, \mathbb{V}_x^{\phi_t,\mathcal{B}}]y_+^t, z_+^t)\\
&\qquad= \mathbb{G}_t(\mathcal{R}ic_t^{\phi_t,\mathcal{B}}\,\mathbb{V}_x^{\phi_t,\mathcal{B}}y_+^t - \mathbb{V}_x^{\phi_t,\mathcal{B}}\,\mathcal{R}ic_t^{\phi,\mathcal{B}}\,y_+^t, z_+^t)\\
&\qquad= \mathcal{R}c_t^{\phi_t,\mathcal{B}}(\mathbb{V}_x^{\phi_t,\mathcal{B}}y_+^t, z_+^t) - X\,\mathcal{R}c_t^{\phi_t,\mathcal{B}}(y_+^t, z_+^t) + \mathcal{R}c_t^{\phi_t,\mathcal{B}}(y_+^t, \mathbb{V}_x^{\phi_t,\mathcal{B}}z_+^t)\\
&\qquad= -X\,\mathrm{Rc}_t^{+\phi_t}(Y,Z) + \mathrm{Rc}_t^{+\phi_t}(Y, \nabla_X^{+\phi_t}Z) + \mathrm{Rc}_t^{+\phi_t}(\nabla_X^{+\phi_t}Y, Z).
\end{aligned}$$

Combining the results above and applying (7-12) again lead to

$$\begin{aligned}
&\mathbb{G}_t((\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{V}_x^{\phi_t,\mathcal{B}})y_\pm^t, z_\pm^t)\\
&\quad= g_t\Big(\frac{d}{dt}\nabla_X^{+\phi_t}Y, Z\Big) - \Big[X\Big(\frac{d}{dt}g_t\Big)\Big(Y, Z\Big) - \Big(\frac{d}{dt}g_t\Big)\Big(Y, \nabla_X^{+\phi_t}Z\Big)\\
&\qquad\qquad - \Big(\frac{d}{dt}g_t\Big)\Big(\nabla_X^{+\phi_t}Y, Z\Big)\Big]\\
&\quad= -g_t\Big(Y, \frac{d}{dt}\nabla_X^{+\phi_t}Z\Big),
\end{aligned}$$

where the last equality follows from the fact that $\nabla^{+\phi_t}$ preserves g_t for all t. □

Consider a smooth family of generalized almost Hermitian structures $(M, \gamma; \mathbb{G}_t, \mathbb{J}^t)$, where $\mathbb{G}_t$ is a solution to the Bismut–Ricci Lax flow (7-16). By Lemma 7.2, the $\mathcal{R}ic^{\phi,\mathcal{B}}$-differential $\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t$ preserves the $\mathbb{G}_t$-eigenbundles. Following computations similar to those in Proposition 7.17 leads to

$$\mathbb{G}_t((\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t)y_\pm, z_\pm) = -g_t\Big(Y, \Big(\frac{d}{dt}I_\pm^t\Big)Z\Big). \tag{7-18}$$

Set $\mathbb{J}_-^t := \mathbb{G}_t\mathbb{J}^t$. Then

$$\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}_-^t = (\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{G}_t)\mathbb{J}^t + \mathbb{G}_t(\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t) = \mathbb{G}_t(\delta_{\mathcal{R}ic^{\phi,\mathcal{B}}}\mathbb{J}^t).$$

This implies that $\mathbb{J}^t$ solves the following Lax flow (7-19), in which case so does $\mathbb{J}_-^t$, if and only if $I_\pm^t$ are constant almost complex structures:

$$\frac{d}{dt}\mathbb{J}^t = [\mathcal{R}ic^{\phi_t,\mathcal{B}}, \mathbb{J}^t]. \tag{7-19}$$

In terms of the 2-forms $\omega_\pm$, (7-19) is equivalent to the following simultaneous almost Hermitian Ricci flows:

$$\Big(\frac{d}{dt}\omega_\pm^t\Big)(X, Y) = 2\,\mathrm{Rc}(X, I_\pm^t Y) - \tfrac{1}{2}\phi_t^2(X, I_\pm^t Y), \tag{7-20}$$

where $X, Y \in C^\infty(TM)$. It has the classical Kähler–Ricci flow as a special case.

Example 7.18. Take a family of classical Kähler structures (g_t, I, ω_t), and set

$$\mathbb{J}^t = \begin{bmatrix} I & 0 \\ 0 & -I^* \end{bmatrix} \quad \text{and} \quad \mathbb{J}_-^t = \begin{bmatrix} 0 & -\omega_t^{-1} \\ \omega_t & 0 \end{bmatrix}.$$

The generalized Bismut connections, as well as the $\mathbb{G}$-adapted connections, are simply the lift of the Levi-Civita connections for g_t. Let $X, Y \in TM$. Since $\phi = 0$, (7-20) becomes

$$\Big(\frac{d}{dt}\omega_t\Big)(X, Y) = 2\,\mathrm{Rc}_t(X, IY) = -2\rho_t(X, Y),$$

where $\rho_t(X, Y) = \mathrm{Rc}_t(IX, Y)$ is the Ricci form. Thus (7-19) recovers exactly the equation for the classical Kähler–Ricci flow.

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