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A NORMAL UNIFORM ALGEBRA THAT FAILS TO BE STRONGLY REGULAR AT A PEAK POINT

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Dedicated to Joel Feinstein

We show that there exists a normal uniform algebra, on a compact metrizable space, that fails to be strongly regular at a peak point. This answers a 32-year-old question of Joel Feinstein. Our example is $R(K)$ for a certain compact planar set K. Furthermore, our example has a totally ordered one-parameter family of closed primary ideals whose hull is a peak point. We establish general results regarding lifting ideals under Cole root extensions. These results are applied to obtain a normal uniform algebra, on a compact metrizable space, with every point a peak point but again having a totally ordered one-parameter family of closed primary ideals.

1. Introduction

This paper is devoted to answering questions in the literature regarding strong regularity of uniform algebras and to establishing general results regarding lifting ideals under Cole root extensions. Our main goal is to answer the following question raised by Joel Feinstein [1992, p. 298]. (For definitions of terminology and notation used in this introduction see Section 2.)

Question 1.1. Does there exist a normal uniform algebra A, on a compact metrizable space, such that A fails to be strongly regular at some peak point for A?

There are later variations on this question in the literature. With general uniform algebras replaced by the special class of uniform algebras of the form $R(K)$ for K a compact planar set, the question appears in the recent book of Garth Dales and Ali Ülger [2024, Section 3.6]. Here, as usual, for K a compact set in the complex plane, $R(K)$ denotes the uniform closure on K of the holomorphic rational functions with poles off K. With uniform algebras replaced by the more general class of Banach

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function algebras, the question appears in Feinstein's paper [1995]. An affirmative answer to that variation was given by David Blecher and Charles Read [2016].

Part of the interest in Question 1.1, in its original form, comes from its connection with the 67-year-old question of I. M. Gelfand [1957] whether every natural uniform algebra on the closed unit interval [0, 1] is trivial. As observed by Feinstein, Donald Wilken's proof [1969] that every strongly regular uniform algebra on [0, 1] is trivial actually shows that every natural uniform algebra on [0, 1] that is strongly regular at a dense set of peak points is trivial. Consequently, a negative answer to Question 1.1 would imply that every normal uniform algebra on [0, 1] is trivial.

Another reason for interest in Question 1.1 is that Feinstein [1992, Theorem 5.2] showed that it is equivalent to this question: Does there exist a normal uniform algebra A, on a compact metrizable space X, such that every point of X is a peak point for A but A fails to be strongly regular? Another closely related question which seems not to be explicitly stated in the literature concerns primary ideals (defined in this context to be those ideals contained in a unique maximal ideal): Does there exist a uniform algebra A, normal or not, such that every point of the maximal ideal space of A is a peak point for A, but A has a closed primary ideal that is not maximal? By a result of Feinstein [2001, Corollary 8], there exist nonnormal uniform algebras such that every point of the maximal ideal space is a peak point.

We answer Question 1.1, and all of the variations on it discussed above, affirmatively by establishing the following theorem.

Theorem 1.2. *There exists a normal uniform algebra* A, *on a compact metrizable space*, *such that* A *fails to be strongly regular at some peak point for* A. *In fact*, A *can be taken to be* $R(K)$ *for a certain compact set* K *in the complex plane.*

As already mentioned, Feinstein [1992, Theorem 5.2] showed that the assertion of the first sentence of this theorem is equivalent to the following assertion.

Corollary 1.3. *There exists a normal uniform algebra* A, *on a compact metrizable space* X, *such that every point of* X *is a peak point for* A *but* A *is not strongly regular.*

The inspiration for our proof of Theorem 1.2, which will give considerably more information than is stated above, comes from the Beurling–Rudin theorem on the closed ideals in the disc algebra [Rudin 1957] (see also [Hoffman 1962, pp. 82–89]). Given a compact planar set K contained in the closed unit disc $\overline{D}$, a point λ in $K \cap \partial D$, and a real number $\rho \geq 0$, we will denote by I_ρ^λ the closed ideal in $R(K)$ generated by the function $(z-\lambda)\exp\bigl(\rho\,\frac{z+\lambda}{z-\lambda}\bigr)$. When $\lambda = 1$ we will write I_ρ in place of I_ρ^λ. When K is the closed unit disc, and hence $R(K)$ is the disc algebra, the ideals I_ρ^λ, $\rho \geq 0$, are precisely the closed primary ideals contained in the maximal ideal M_λ of functions that vanish at the point λ. Furthermore, for $0 \leq \rho_1 < \rho_2$ there is the strict inclusion $I_{\rho_1}^\lambda \supsetneq I_{\rho_2}^\lambda$. Our proof of Theorem 1.2

essentially amounts to showing that, taking $\lambda = 1$ for instance, Robert McKissick's construction of the first nontrivial normal uniform algebra [1963] (see also [Stout 1971, Section 27]) can be refined so as to preserve this strict inclusion of ideals. Using results in [Izzo 2022] we will show that, in addition, the uniform algebra can be chosen in such a way that the point 1 is the only point where strong regularity fails. We will thus obtain the following theorem that contains Theorem 1.2. Here, and throughout the paper, we denote the open unit disc in the complex plane by D, and given a disc Δ, we denote the radius of Δ by $r(\Delta)$.

Theorem 1.4. *For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} r(D_k) < r$, the point 1 is in the set $K = \bar{D} \setminus \bigcup_{k=1}^{\infty} D_k$, and the following conditions hold*:

(i) *$R(K)$ is normal.*

(ii) *$R(K)$ is strongly regular at every point of $K \setminus \{1\}$.*

(iii) *$R(K)$ is not strongly regular at the point* 1.

Furthermore, the discs $\{D_k\}_{k=1}^{\infty}$ can be chosen in such a way that $I_{\rho_1} \supsetneqq I_{\rho_2}$ for every $0 \le \rho_1 < \rho_2$.

A modification of the proof of Theorem 1.4 will yield the next result, which shows, in particular, that a normal uniform algebra can fail to be strongly regular at an uncountable set of peak points.

Theorem 1.5. *For each $r > 0$, there exists a sequence of open discs $\{D_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} r(D_k) < r$ and setting $K = \bar{D} \setminus \bigcup_{k=1}^{\infty} D_k$ the following conditions hold*:

(i) *$R(K)$ is normal.*

(ii) *$R(K)$ is strongly regular at every point of $K \setminus \partial D$.*

(iii) *There is a set $\Lambda \subset \partial D$ whose complement in ∂D has one-dimensional Lebesgue measure less than r such that Λ is contained in K and at each point of Λ, the uniform algebra $R(K)$ fails to be strongly regular.*

Furthermore, the discs $\{D_k\}_{k=1}^{\infty}$ can be chosen in such a way that $I_{\rho_1}^{\lambda} \supsetneqq I_{\rho_2}^{\lambda}$ for every $\lambda \in \Lambda$ and every $0 \le \rho_1 < \rho_2$.

Note that the uniform algebras in Theorems 1.4 and 1.5, in spite of failing to be strongly regular, are strongly regular at every *nonpeak point*. Feinstein and Matthew Heath [2007, Question 5.8] raised the question of whether there exists a compact planar set K such that $R(K)$ is regular and has no nonzero bounded point derivations, but is not strongly regular. Each of Theorems 1.4 and 1.5 answers this question affirmatively since for the set K in each of those theorems there are no nonzero bounded point derivations at the points of $K \setminus \partial D$ because $R(K)$ is strongly regular at those points, and there are no nonzero point derivations at the points of $K \cap \partial D$ since those points are peak points for $R(K)$. In [Izzo 2022] the author effectively raised the same question but without the regularity hypothesis, and he promised to give an example answering the question in a future paper. Thus Theorems 1.4 and 1.5 fulfill that promise.

The next two results show that, as one might expect, Corollary 1.3 can be strengthened in ways analogous to how Theorems 1.4 and 1.5 strengthen Theorem 1.2.

Theorem 1.6. *There exists a normal uniform algebra B, on a compact metrizable space X, such that every point of X is a peak point for B but there is a point $x_0 \in X$ such that there is a one-parameter family $\{H_\rho : 0 \le \rho < \infty\}$ of distinct closed primary ideals contained in the maximal ideal M_{x_0} satisfying $H_{\rho_1} \supsetneq H_{\rho_2}$ for all $0 \le \rho_1 < \rho_2$. Furthermore, B can be taken to have bounded relative units at, and hence be strongly regular at, every point of $X \setminus \{x_0\}$.*

Theorem 1.7. *There exists a normal uniform algebra B, on a compact metrizable space X, such that every point of X is a peak point for B but there is an uncountable subset L of X such that for every $x \in L$ there is a one-parameter family $\{H^x_\rho : 0 \le \rho < \infty\}$ of distinct closed primary ideals contained in the maximal ideal M_x satisfying $H^x_{\rho_1} \supsetneq H^x_{\rho_2}$ for all $0 \le \rho_1 < \rho_2$. Furthermore, B can be taken to have bounded relative units at, and hence be strongly regular at, every point of $X \setminus L$.*

Feinstein's proof that Corollary 1.3 is equivalent to the assertion of the first sentence in Theorem 1.2 used Brian Cole's method of root extensions. Theorems 1.6 and 1.7 will be derived from Theorems 1.4 and 1.5 also using Cole's method of root extensions. However, to do so we will need to prove new results about lifting ideals under root extensions. There are several examples in the literature in which a certain object lifts under a root extension as a result of adjoining square roots to only a restricted collection of functions. For instance in [Feinstein 1992; 1995; 2004; Feinstein and Heath 2007; Izzo 2022; Izzo and Papathanasiou 2021] square roots are adjoined only to functions that vanish on a given closed set (or a neighborhood of the closed set), and consequently a copy of the given closed set is preserved in the extension. In [Ghosh and Izzo 2023] square roots are adjoined only to functions on which a given bounded point derivation vanishes, with the result that the bounded point derivation lifts to the extended uniform algebra. In all these instances, the functions for which square roots are adjoined come from some (proper) closed ideal I of the uniform algebra A. We will prove general results regarding such root extensions. Very roughly, the results say that in this situation, the quotient Banach algebra A/I is preserved by the extension, and consequently, all the ideals in A that contain I lift under the extension.

In the next section we define some terminology and notation already used above. In Section 3 we present some known results that we will need. In Section 4 we prove that for a normal uniform algebra, strong regularity at a point is a local property. Although not strictly necessary for the proofs of Theorems 1.4 and 1.5, which are presented in Section 5, this result greatly simplifies establishing the strong regularity assertion in those theorems. In Section 6 we present the theorems discussed in the previous paragraph regarding lifting ideals under root extensions. Theorems 1.6 and 1.7 are proved in Section 7.

2. Terminology and notation

Those readers well versed in uniform algebra concepts may wish to skip or skim this section and refer back to it as needed.

It is to be understood that all sequences, unions, and sums involving an index extend from 1 to ∞; thus for instance $\{D_k\}$ means $\{D_k\}_{k=1}^{\infty}$, and $\bigcup D_k$ means $\bigcup_{k=1}^{\infty} D_k$. If f is a function whose domain contains a subset L, we denote the restriction of f to L by $f|L$, and if A is a collection of such functions, we denote the collection of restrictions of functions in A to L by $A|L$. The set of positive integers will be denoted by $\mathbb{Z}_+$.

Throughout the paper all spaces will tacitly be required to be Hausdorff. Let X be a compact space. We denote by $C(X)$ the algebra of all continuous complex-valued functions on X equipped with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constants and separates the points of X. A uniform algebra A on X is said to be

(a) *natural* if the maximal ideal space of A is X (under the usual identification of a point of X with the corresponding multiplicative linear functional),

(b) *regular on* X if for each closed set K_0 of X and each point x of $X \setminus K_0$, there exists a function f in A such that $f(x) = 1$ and $f = 0$ on K_0,

(c) *normal on* X if for each pair of disjoint closed sets K_0 and K_1 of X, there exists a function f in A such that $f = 1$ on K_1 and $f = 0$ on K_0.

The uniform algebra A on X is *regular* or *normal* if A is natural and is regular on X or normal on X, respectively. In fact, every regular uniform algebra is normal [Stout 1971, Theorem 27.2]. Also, if a uniform algebra A is normal on X, then A is necessarily natural [Stout 1971, Theorem 27.3].

Let A be a uniform algebra on X, and let $x \in X$. We define the ideals M_x and J_x by

$$M_x = \{f \in A : f(x) = 0\},$$
$$J_x = \{f \in A : f^{-1}(0) \text{ contains a neighborhood of } x \text{ in } X\}.$$

More generally, if E is a closed subset of X, we define the ideals M_E and J_E by

$$M_E = \{f \in A : f|E = 0\},$$
$$J_E = \{f \in A : f^{-1}(0) \text{ contains a neighborhood of } E \text{ in } X\}.$$

When it is necessary to indicate with respect to which algebra the ideals are taken, we will denote the ideals J_x and M_x in the uniform algebra A by $J_x(A)$ and $M_x(A)$.

The uniform algebra A is *strongly regular at x* if $\bar{J}_x = M_x$, and A is *strongly regular* if A is strongly regular at every point of X. It was shown by Wilken [1969, Corollary 1] that every strongly regular uniform algebra is normal.

Let A be a natural uniform algebra and let I be an ideal in A. The *hull* of I is the common zero set of the functions in I and is denoted by hull(I). The ideal I is said to be *local* if $I \supset J(\text{hull}(I))$. The following result is standard [Dales 2000, Proposition 4.1.20(iv)].

Theorem 2.1. *Every ideal in a normal uniform algebra is local.*

Consequently, a normal uniform algebra is strongly regular at a point x if and only if there is no closed primary ideal properly contained in the maximal ideal M_x.

The uniform algebra A has *bounded relative units at x* with bound $C \geq 1$ if for each compact subset K of $X \setminus \{x\}$, there exists $f \in J_x$ such that $f|K = 1$ and $\|f\|_X \leq C$. If A has bounded relative units at every point of X, then A has *bounded relative units*.

The point x is said to be a *peak point* for A if there is a function f in A such that $f(x) = 1$ and $|f(y)| < 1$ for every $y \in X \setminus \{x\}$. The point x is said to be a *generalized peak point* if for every neighborhood U of x there exists a function f in A such that $f(x) = \|f\| = 1$ and $|f(y)| < 1$ for every $y \in X \setminus U$. When the space X is metrizable the notions of peak point and generalized peak point coincide.

For ϕ a point in the maximal ideal space of the uniform algebra A, a *bounded point derivation* on A at ϕ is a bounded linear functional d on A satisfying the identity

$$d(fg) = d(f)\phi(g) + \phi(f)d(g)$$

for all f and g in A. It is standard [Browder 1969, p. 64] that a bounded linear functional d on A is a bounded point derivation at ϕ if and only if d annihilates $\overline{M_\phi^2}$ and the constant functions, and hence there exists a bounded point derivation at ϕ if and only if $\overline{M_\phi^2} \neq M_\phi$.

3. Preliminaries

In this section we collect various known results that we will need. The reader may prefer to skip this section and merely refer back to it when the results are used.

As with McKissick's construction [1963] of the first known nontrivial normal uniform algebra, our proofs of Theorems 1.4 and 1.5 rely on the following lemma. (Recall that given a disc Δ, we denote the radius of Δ by $r(\Delta)$.)

Lemma 3.1. *Let Δ be an open disc in the complex plane with center a and radius $r > 0$, and let $\varepsilon > 0$ be given. Then there exist a sequence of open discs $\{\Delta_k\}_{k=1}^\infty$*

and a sequence of rational functions $\{f_j\}_{j=1}^{\infty}$ *such that*:

(a) $\sum_{k=1}^{\infty} r(\Delta_k) < \varepsilon$.

(b) *The poles of the* f_j *lie in* $\bigcup_{k=1}^{\infty} \Delta_k$.

(c) *The sequence* $\{f_j\}$ *converges uniformly on* $\mathbb{C} \setminus \bigcup_{k=1}^{\infty} \Delta_k$ *to a function that is identically zero outside* Δ *and zero free in* $\Delta \setminus \bigcup_{k=1}^{\infty} \Delta_k$.

(d) $\bigcup_{k=1}^{\infty} \Delta_k \subset \{z : r - \varepsilon < |z - a| < r\}$.

Condition (d) is not part of the lemma as stated by McKissick but is established in the paper of Thomas Körner [1986], where a proof of the lemma simpler than the original one is given.

For the proofs of Theorems 1.4 and 1.5 we will need two recent results [Izzo 2022, Theorem 1.2 and Lemma 4.2] on strong regularity in $R(K)$, which we state here.

Theorem 3.2. *For each* $r > 0$, *there exists a sequence of open discs* $\{D_k\}_{k=1}^{\infty}$ *such that* $\sum_{k=1}^{\infty} r(D_k) < r$ *and such that setting* $K = \overline{D} \setminus \bigcup_{k=1}^{\infty} D_k$, *the uniform algebra* $R(K)$ *is nontrivial and strongly regular.*

Lemma 3.3. *Given compact sets* $L \subset K \subset \mathbb{C}$ *and given a point* $x \in L$, *if* $\overline{J_x(R(K))} \supset M_x(R(K))$, *then* $\overline{J_x(R(L))} \supset M_x(R(L))$.

The following three lemmas will be used to prove condition (v) in Theorem 6.9. The first of these is due to Feinstein and Heath [2007, Lemma 4.3].

Lemma 3.4. *Let* A *be a uniform algebra on* X *and* $x \in X$. *Suppose that, for each compact subset* E *of* $X \setminus \{x\}$, *there exists a neighborhood* U *of* x *and a function* $f \in A$ *such that*

(i) $f|U = 1$,

(ii) $f|E = 0$,

(iii) *for each* $k \in \mathbb{Z}_+$ *there is a function* $g \in A$ *with* $g^{2^k} = f$.

Then A *has bounded relative units at* x.

The next lemma is part of a result of Feinstein [1992, Proposition 1.5].

Lemma 3.5. *Let* A *be a uniform algebra on a compact space* X, *and let* $x \in X$. *If* A *has bounded relative units at* x, *then* x *is a generalized peak point for* A, *and* A *is strongly regular at* x.

The next lemma, whose elementary proof we omit, is a modification of a lemma of Feinstein [1992, Lemma 3.5].

Lemma 3.6. *Let* A *be a normal uniform algebra on a compact metrizable space* X, *and let* F *be a closed subset of* X. *Then there exists a countable subset* $\mathscr{F}$ *of* A *consisting of functions each vanishing identically on a neighborhood of* F *such that for each point* $x \in X \setminus F$, *and for each compact subset* E *of* $X \setminus \{x\}$, *there exists a neighborhood* U *of* x, *and a function* $f \in \mathscr{F}$ *such that* $f|U = 1$ *and* $f|E = 0$.

4. Localness of strong regularity

In this section we prove the localness of strong regularity for normal uniform algebras. The result will be used in the next section to obtain condition (ii) in Theorems 1.4 and 1.5.

Theorem 4.1. *Let A be a normal uniform algebra on a compact space X, and let x_0 be a point of X. If there exists a closed neighborhood N of x_0 in X such that $\overline{A|N}$ is strongly regular at x_0, then A is strongly regular at x_0.*

Proof. Suppose that there exists a closed neighborhood N of x_0 in X such that $\overline{A|N}$ is strongly regular at x_0. Fix $f \in A$ satisfying $f(x_0) = 0$, and fix $\varepsilon > 0$. We are to show that there exists a function $g \in A$ such that $\|f - g\|_X < \varepsilon$ and $g = 0$ on a neighborhood of x_0.

Let L be a closed neighborhood of x_0 contained in the interior N° of N. By the normality of A, there is a function $\varphi \in A$ such that $\varphi = 1$ on L and $\varphi = 0$ on $X \setminus N^\circ$.

By the strong regularity of $\overline{A|N}$, there is a function $h \in \overline{A|N}$ such that $\|f - h\|_N < \varepsilon / \|\varphi\|_N$ and $h = 0$ on some closed neighborhood M of x_0. There is a sequence (h_n) in A such that $h_n|N \to h$ uniformly on N.

For each $n \in \mathbb{Z}_+$, set $g_n = \varphi h_n + (1 - \varphi) f$. Then each g_n is in A. Define a function g on X by

$$g = \begin{cases} \varphi h + (1-\varphi) f & \text{on } N, \\ f & \text{on } X \setminus N. \end{cases}$$

On $X \setminus N$ we have $g_n = f = g$. Moreover, $g_n \to g$ uniformly on X. Thus g is in A. Furthermore,

$$\|f - g\|_X = \|f - g\|_N = \|\varphi(f - h)\|_N \le \|\varphi\|_N \, \|f - h\|_N < \varepsilon.$$

Finally, $g = 0$ on the neighborhood $L \cap M$ of x_0. □

5. Proofs of Theorems 1.4 and 1.5

Recall that given a disc Δ, we denote the radius of Δ by $r(\Delta)$. We will denote the distance from Δ to the point 1 by $s(\Delta)$. Explicitly, $s(\Delta) = \inf\{|z - 1| : z \in \Delta\}$. We will denote the open disc with center a_n and radius r_n by $D(a_n, r_n)$ and the corresponding closed disc by $\overline{D}(a_n, r_n)$.

The following lemma is the key to the proofs of Theorems 1.4 and 1.5.

Lemma 5.1. *Fix real numbers $0 \le \rho_1 < \rho_2$. Let $\{D_k\}_{k=1}^\infty$ be a sequence of discs such that $\sum_{k=1}^\infty r(D_k) \exp(2\rho_2 / s(D_k)) < \infty$. Set $K = \overline{D} \setminus \bigcup_{k=1}^\infty D_k$. Then $I_{\rho_1} \supsetneq I_{\rho_2}$.*

Proof. It is easily shown that the function $(z - 1)\exp\bigl(\rho_2 \frac{z+1}{z-1}\bigr)$ is in I_{ρ_1}, and hence $I_{\rho_1} \supset I_{\rho_2}$. To prove that the inclusion is strict, we will exhibit a measure on K that annihilates I_{ρ_2} but does not annihilate the function $(z - 1)\exp\bigl(\rho_1 \frac{z+1}{z-1}\bigr)$.

For $n \in \mathbb{Z}_+$, let $K_n = \bar{D} \setminus \bigcup_{k=1}^{n} D_k$. The boundary of K_n consists of the union of a finite collection of circular arcs (and possibly some isolated points which can be ignored), and we can define a measure μ_n on ∂K_n by requiring that for every function $f \in C(K_n)$ we have

$$\int f\, d\mu_n = \int_{\partial K_n} f(z) \exp\Bigl(-\rho_2 \frac{z+1}{z-1}\Bigr)\, dz.$$

Let $M = \sum_{k=1}^{\infty} r(D_k) \exp(2\rho_2/s(D_k)) < \infty$. Then $\|\mu_n\| \le 2\pi(M+1)$. Consequently, $\{\mu_k\}_{k=1}^{\infty}$ has a weak*-accumulation point μ. Since $K = \bigcap K_n$, the measure μ is supported on K. If g is a rational function with no poles on K, then for large values of n, the function g has no poles on K_n, and by Cauchy's theorem

$$\int g(z)(z-1) \exp\Bigl(\rho_2 \frac{z+1}{z-1}\Bigr)\, d\mu_n = \int_{\partial K_n} g(z)(z-1)\, dz = 0.$$

Thus

$$\int g(z)(z-1) \exp\Bigl(\rho_2 \frac{z+1}{z-1}\Bigr)\, d\mu = 0$$

for every rational function g with no poles on K. It follows that μ annihilates I_{ρ_2}.

To calculate the integral $\int (z-1) \exp\bigl(\rho_1 \frac{z+1}{z-1}\bigr)\, d\mu$ first note that the function $(z-1) \exp\bigl((\rho_1 - \rho_2) \frac{z+1}{z-1}\bigr)$ has a single isolated singularity at $z = 1$, and the residue there is $2(\rho_1 - \rho_2)^2 \exp(\rho_1 - \rho_2)$ since

$$\begin{aligned}
(z-1) &\exp\Bigl((\rho_1 - \rho_2) \frac{z+1}{z-1}\Bigr) \\
&= (z-1) \exp\Bigl((\rho_1 - \rho_2)\Bigl(1 + \frac{2}{z-1}\Bigr)\Bigr) \\
&= (z-1) \exp\Bigl((\rho_1 - \rho_2) + \Bigl(\frac{2(\rho_1 - \rho_2)}{z-1}\Bigr)\Bigr) \\
&= (z-1)(\exp(\rho_1 - \rho_2))\Bigl(1 + \frac{2(\rho_1 - \rho_2)}{z-1} + \frac{1}{2}\Bigl(\frac{2(\rho_1 - \rho_2)}{z-1}\Bigr)^2 + \cdots\Bigr).
\end{aligned}$$

For each $m, n \in \mathbb{Z}_+$, set $K_n^m = K_n \cup \bar{D}\bigl(1, \frac{1}{m}\bigr)$. To each n there corresponds an $N(n) \in \mathbb{Z}_+$ such that for all $m \ge N(n)$, the discs $\bar{D}_1, \ldots, \bar{D}_n$ are disjoint from the disc $\bar{D}\bigl(1, \frac{1}{m}\bigr)$. Consequently, letting γ_m denote the part of ∂D contained in $\bar{D}\bigl(1, \frac{1}{m}\bigr)$, and letting σ_m denote the part of $\partial \bar{D}\bigl(1, \frac{1}{m}\bigr)$ outside D, we have for all $m \ge N(n)$ that

$$\partial K_n^m = (\partial K_n \setminus \gamma_m) \cup \sigma_m.$$

The integrals

$$\int_{\gamma_m} (z-1) \exp\Bigl((\rho_1 - \rho_2) \frac{z+1}{z-1}\Bigr)\, dz \quad \text{and} \quad \int_{\sigma_m} (z-1) \exp\Bigl((\rho_1 - \rho_2) \frac{z+1}{z-1}\Bigr)\, dz$$

each go to zero as $m \to \infty$ because the lengths of γ_m and σ_m go to zero as $m \to \infty$ and the integrands are bounded in modulus by $\frac{1}{m}$. Therefore, applying the residue theorem gives

$$\begin{aligned}\int (z-1)\exp\Big(\rho_1 \frac{z+1}{z-1}\Big)\, d\mu_n &= \int_{\partial K_n} (z-1)\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\, dz \\ &= \lim_{m\to\infty} \int_{\partial K_n^m} (z-1)\exp\Big((\rho_1-\rho_2)\frac{z+1}{z-1}\Big)\, dz \\ &= 4\pi i(\rho_1-\rho_2)^2 \exp{(\rho_1-\rho_2)}.\end{aligned}$$

Thus

$$\int (z-1)\exp\Big(\rho_1 \frac{z+1}{z-1}\Big)\, d\mu = 4\pi i(\rho_1-\rho_2)^2 \exp(\rho_1-\rho_2) \neq 0. \qquad \square$$

Proof of Theorem 1.4. By the preceding lemma, it suffices to show that discs $\{D_k\}$ can be chosen such that $\sum r(D_k) < r$, such that $\sum r(D_k)\exp(\nu/s(D_k)) < \infty$ for every $\nu > 0$, and such that conditions (i) and (ii) hold. We begin by choosing discs such that these conditions are satisfied with the (possible) exception of condition (ii), and then we choose additional discs to achieve condition (ii) in addition.

Choose a sequence $\{\bar{D}(a_n, r_n)\}_{n=1}^{\infty}$ of closed discs such that:

(a) Each of the discs $\bar{D}(1, 1/j)$ for $j = 1, 2, \ldots$ is in $\{\bar{D}(a_n, r_n)\}$.

(b) The discs $\bar{D}(1, 1/j)$ for $j = 1, 2, \ldots$ are the only discs in $\{\bar{D}(a_n, r_n)\}$ that contain the point 1.

(c) For every $\varepsilon > 0$, every point of $\bar{D}$ lies in an open disc $D(a_n, r_n)$ with $r_n < \varepsilon$.

Then for each $n \in \mathbb{Z}_+$, the annulus $\{z : r_n/2 < |z - a_n| < r_n\}$ is at some positive distance δ_n from the point 1. Set $\varepsilon_n = \min\{2^{-(n+1)}r, 2^{-n}e^{-n/\delta_n}\}$. For each $n \in \mathbb{Z}_+$, choose discs $\{\Delta_k^n\}_{k=1}^{\infty}$ in the annulus $\{z : r_n/2 < |z - a_n| < r_n\}$ as in Lemma 3.1 with $\Delta = D(a_n, r_n)$ and $\varepsilon = \varepsilon_n$. Then $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\Delta_n^k) < r/2$. Now let $\nu > 0$ be arbitrary. For each $n \in \mathbb{Z}_+$ we have

$$\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \varepsilon_n \exp(\nu/\delta_n).$$

Thus, in particular, $\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \infty$. Furthermore, for all $n > \nu$, we have

$$\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \varepsilon_n \exp(n/\delta_n) \leq 2^{-n}.$$

Consequently, $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\Delta_k^n)\exp(\nu/s(\Delta_k^n)) < \infty$.

Set

$$K_1 = \bar{D} \setminus \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty} \Delta_k^n.$$

Then $R(K_1)$ is regular, and hence normal, for given a closed set $L \subset K_1$ and a point $x \in K_1 \setminus L$, there is some $D(a_n, r_n)$ that contains x and is disjoint from L, and hence, by the choice of the discs $\{\Delta_k^n\}$, there is a function in $R(K_1)$ that vanishes on L but not at x.

To achieve the strong regularity at points different from 1 we will use Theorem 3.2 together with Theorem 4.1 on the localness of strong regularity. Choose a countable collection of open discs $\{B_n\}$ that covers $K_1 \setminus \{1\}$ such that none of the closed discs $\bar{B}_n$ contains the point 1. Let α_n denote the distance from $\bar{B}_n$ to the point 1. Set $\widetilde{\varepsilon}_n = \min\{2^{-(n+1)}r, 2^{-n}e^{-n/\alpha_n}\}$. Since in Theorem 3.2 the open unit disc can, of course, be replaced by any open disc, there exists a sequence of open discs $\{\widetilde{\Delta}_k^n\}$ such that $\sum_{k=1}^{\infty} r(\widetilde{\Delta}_k^n) < \widetilde{\varepsilon}_n$ and such that setting

$$K_2^n = \bar{B}_n \setminus \bigcup_{k=1}^{\infty} \widetilde{\Delta}_k^n,$$

the uniform algebra $R(K_2^n)$ is strongly regular. Note that $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\widetilde{\Delta}_k^n) < r/2$ and $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} r(\widetilde{\Delta}_k^n)\exp\big(\nu/s(\widetilde{\Delta}_k^n)\big) < \infty$ for every $\nu > 0$ by a computation that is identical to an earlier one.

Now let $\{D_k\}$ be an enumeration of the collection of discs $\{\Delta_k^n\}_{k,n=1}^{\infty} \cup \{\widetilde{\Delta}_k^n\}_{k,n=1}^{\infty}$, and set $K = \bar{D} \setminus \bigcup D_k$. Of course $\sum r(D_k) < r$, and $\sum r(D_k)\exp(\nu/s(D_k)) < \infty$ for every $\nu > 0$. The uniform algebra $R(K)$ is normal because K is contained in K_1 and $R(K_1)$ is normal. Consider an arbitrary point $x_0 \in K \setminus \{1\}$, and choose a disc B_{n_0}, from the collection $\{B_n\}$, that contains x_0. Then $R(K \cap \bar{B}_{n_0})$ is strongly regular by Lemma 3.3 because $K \cap \bar{B}_{n_0}$ is contained in $K_2^{n_0}$ and $R(K_2^{n_0})$ is strongly regular. Furthermore, $\overline{R(K)|(K \cap \bar{B}_{n_0})} = R(K \cap \bar{B}_{n_0})$. Thus Theorem 4.1 shows that $R(K)$ is strongly regular at x_0. □

Proof of Theorem 1.5. Since the proof is similar to the proof of Theorem 1.4, we merely indicate the modifications needed. For the discs $\{\bar{D}(a_n, r_n)\}$, we discard conditions (a) and (b) and instead require that each disc $\bar{D}(a_n, r_n)$ is either centered at a point of ∂D or else is contained in D, and we retain condition (c). Let Γ denote the set of n such that a_n is in ∂D. For each $n \in \Gamma$, choose a number $\gamma_n > 0$ in such a way that the intersection with ∂D of the union of the annuli $\{z : r_n - \gamma_n < |z - a_n| < r_n + \gamma_n\}$ has one-dimensional Lebesgue measure less than r. Let Λ be the complement of that union in ∂D. Choose, for each $n \in \mathbb{Z}_+$, a positive number r_n' such that $r_n - \gamma_n < r_n' < r_n$. Then the annulus $\{z : r_n' < |z - a_n| < r_n\}$ is at a positive distance δ_n from Λ. To establish everything except condition (ii), we choose discs $\{\Delta_k^n\}$ in the annulus $\{z : r_n' < |z - a_n| < r_n\}$ as in Lemma 3.1 arguing as in the proof of Theorem 1.4, but with distance $s(\Delta_k^n)$ to 1 replaced by distance to Λ.

To get condition (ii), we argue essentially as in the proof of Theorem 1.4 except that for the collection $\{B_n\}$ we take the collection $\{D(0, 1 - \frac{1}{n}) : n = 2, 3, \dots\}$, and we again replace distance to 1 by distance to Λ. □

6. Root extensions and ideals

In this section we prove results about systems of root extensions and ideals which we will use in the next section to prove Theorems 1.6 and 1.7. We present the results in greater generality than we will need because we believe they are of interest in their own right and are likely to have further applications.

Cole's method of root extensions [1968] (see also [Stout 1971, Section 19]) involves an iterative process. We begin by discussing a single step of the iteration.

Let A be a uniform algebra on a compact space X, and let $\mathscr{F}$ be a (nonempty) subset of A. Endow $\mathbb{C}^{\mathscr{F}}$ with the product topology. Let $p_1 : X \times \mathbb{C}^{\mathscr{F}} \to X$ and $p_f : X \times \mathbb{C}^{\mathscr{F}} \to \mathbb{C}$ denote the projections given by $p_1(x, (z_g)_{g\in\mathscr{F}}) = x$ and $p_f(x, (z_g)_{g\in\mathscr{F}}) = z_f$. Define $X_{\mathscr{F}} \subset X \times \mathbb{C}^{\mathscr{F}}$ by

$$X_{\mathscr{F}} = \{y \in X \times \mathbb{C}^{\mathscr{F}} : (p_f(y))^2 = f(p_1(y)) \text{ for all } f \in \mathscr{F}\},$$

and let $A_{\mathscr{F}}$ be the uniform algebra on $X_{\mathscr{F}}$ generated by the set of functions $\{f \circ p_1 : f \in A\} \cup \{p_f : f \in \mathscr{F}\}$. On $X_{\mathscr{F}}$ we have $p_f^2 = f \circ p_1$ for every $f \in \mathscr{F}$. Set $\pi = p_1 | X_{\mathscr{F}}$, and note that π is surjective. There is an isometric embedding $\pi^* : A \to A_{\mathscr{F}}$ given by $\pi^*(f) = f \circ \pi$.

We call the uniform algebra $A_{\mathscr{F}}$ or the pair $(A_{\mathscr{F}}, X_{\mathscr{F}})$, the *$\mathscr{F}$-extension of* A, and we call π the *associated surjection*. Note that if X is metrizable and $\mathscr{F}$ is countable, then $X_{\mathscr{F}}$ is metrizable also. Given $x \in X$, if $\mathscr{F}$ is contained in M_x, then the set $\pi^{-1}(x)$ consists of a single point.

There is an operator $S : A_{\mathscr{F}} \to \pi^*(A)$ given by integrating over the fibers of π using the measure on each fiber that is invariant under the obvious action of $(\mathbb{Z}/2)^{\mathscr{F}}$ on each fiber. See [Cole 1968] or [Stout 1971, pp. 194–195] for details. Rather than working with S, we will use the operator $T : A_{\mathscr{F}} \to A$ obtained from S by identifying $\pi^*(A)$ with A. The following properties of T are almost obvious.

Lemma 6.1.

(i) $\|T\| = 1$.

(ii) *$T \circ \pi^*$ is the identity.*

(iii) *Given distinct functions $f_1, \dots, f_r \in \mathscr{F}$ and a function $f \in A$,*

$$T(\pi^*(f) p_{f_1} \cdots p_{f_r}) = 0.$$

One can iterate the above extension process. This leads to the notion of a system of root extensions, which we next define.

Henceforth, τ will be a fixed infinite ordinal. A *system of root extensions* is a triple of indexed sets $(\{A_\alpha\}, \{X_\alpha\}, \{\pi_{\alpha,\beta}\})$ $(0 \le \alpha \le \beta \le \tau)$ (denoted for brevity by $\{A_\alpha\}_{0\le\alpha\le\tau}$) where each X_α is a compact space, each A_α is a uniform algebra on X_α, and each $\pi_{\alpha,\beta}$ is a continuous surjective map $\pi_{\alpha,\beta} : X_\beta \to X_\alpha$ such that the following conditions hold:

(i) The equation $\pi^*_{\alpha,\beta}(f) = f \circ \pi_{\alpha,\beta}$ defines a homomorphism of A_α into A_β.

(ii) For $\alpha \le \beta \le \gamma$, $\pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$, and $\pi_{\alpha,\alpha}$ is the identity on X_α.

(iii) For $\alpha < \tau$, there is a subset $\mathscr{F}_\alpha$ of A_α such that $A_{\alpha+1}$ is the $\mathscr{F}_\alpha$-extension of A_α and $\pi_{\alpha,\alpha+1}$ is the associated surjection.

(iv) For γ a limit ordinal, the space X_γ is the inverse limit of the inverse system $\{X_\alpha, \pi_{\alpha,\beta}\}_{\alpha \le \beta < \gamma}$, the maps $\pi_{\alpha,\gamma} : X_\gamma \to X_\alpha$ are those associated with the inverse limit, and A_γ is the closure in $C(X_\gamma)$ of $\bigcup_{\alpha<\gamma} \pi^*_{\alpha,\gamma}(A_\alpha)$.

The existence of systems of root extensions is of course proved by transfinite induction. A choice of the subsets $\mathscr{F}_\alpha$ uniquely determines a system of root extensions.

Remark 6.2. It follows trivially from conditions (i) and (ii) that for $\alpha \le \beta \le \gamma$, $\pi^*_{\beta,\gamma} \circ \pi^*_{\alpha,\beta} = \pi^*_{\alpha,\gamma}$, and $\pi^*_{\alpha,\alpha}$ is the identity on A_α.

Given a uniform algebra A on X, a uniform algebra $\widetilde{A}$ on $\widetilde{X}$, and a surjective continuous map $\widetilde{\pi} : \widetilde{X} \to X$, we will say that $\widetilde{A}$ and $\widetilde{\pi}$ are *obtained from A by a system of root extensions* if there exists a system of root extensions $(\{A_\alpha\}, \{X_\alpha\}, \{\pi_{\alpha,\beta}\})$ $(0 \le \alpha \le \beta \le \tau)$ with $A_0 = A$, $A_\tau = \widetilde{A}$, and $\pi_{0,\tau} = \widetilde{\pi}$.

The following is [Feinstein 1992, Corollary 2.9].

Lemma 6.3. *Given a system of root extensions $\{A_\alpha\}_{0\le\alpha\le\tau}$, if A_0 is normal, then A_α is normal for all α.*

For a system of root extensions $\{A_\alpha\}_{0\le\alpha\le\tau}$, Cole introduced certain surjective linear operators $T_\beta : A_\beta \to A_0$. It will be helpful for us to introduce, more generally, operators $T_{\alpha,\beta} : A_\beta \to A_\alpha$ for every $\alpha \le \beta$.

Lemma 6.4. *Given a system of root extensions $\{A_\alpha\}_{0\le\alpha\le\tau}$ there exists a system of surjective linear operators $\{T_{\alpha,\beta} : A_\beta \to A_\alpha\}_{0\le\alpha\le\beta\le\tau}$ such that the following conditions hold for all $0 \le \alpha \le \beta \le \gamma \le \tau$:*

(a) *$T_{\alpha,\alpha}$ is the identity operator on A_α.*

(b) $\|T_{\alpha,\beta}\| = 1$.

(c) $T_{\alpha,\beta} \circ T_{\beta,\gamma} = T_{\alpha,\gamma}$.

(d) *$T_{\alpha,\beta} \circ \pi^*_{\alpha,\beta}$ is the identity on A_α.*

(e) $T_{\alpha,\gamma} \circ \pi^*_{\beta,\gamma} = T_{\alpha,\beta}$.

(f) $T_{\beta,\gamma} \circ \pi^*_{\alpha,\gamma} = \pi^*_{\alpha,\beta}$.

Proof. First note that condition (d) is an immediate consequence of conditions (a) and (e), note that condition (f) is an immediate consequence of condition (d) and Remark 6.2, and note that the surjectivity of the operators $\{T_{\alpha,\beta}\}$ follows from condition (d). Thus it suffices to show that the operators $\{T_{\alpha,\beta}\}$ can be chosen to satisfy conditions (a), (b), (c), and (e). We will apply transfinite induction on β with α fixed to obtain operators $\{T_{\alpha,\beta}\}$ satisfying conditions (a), (b), and (e), and then observe that these operators satisfy condition (c) also.

The operator $T_{\alpha,\alpha}$ is specified. Consider $\alpha \le \beta \le \tau$, and assume as the induction hypothesis that operators $T_{\alpha,\delta}$ have been defined for all $\alpha \le \delta < \beta$ in such a way that conditions (a), (b), and (e) hold. If $\beta = \delta + 1$ for some δ, then A_β is the $\mathscr{F}_\delta$-extension of A_δ. Let $T : A_\beta \to A_\delta$ be the operator discussed in the paragraph immediately preceding Lemma 6.1, set $T_{\alpha,\beta} = T_{\alpha,\delta} \circ T$, and verify that conditions (a), (b), and (e) continue to hold. If β is a limit ordinal, define an operator $\widetilde{T}_{\alpha,\beta}$ on the dense subspace $\bigcup_{\alpha\le\delta<\beta} \pi^*_{\delta,\beta}(A_\delta)$ of A_β by

$$\widetilde{T}_{\alpha,\beta}(\pi^*_{\delta,\beta} f) = T_{\alpha,\delta} f \quad \text{for } f \in A_\delta.$$

Condition (e) ensures that $\widetilde{T}_{\alpha,\beta}$ is well defined, i.e., if $f_1 \in A_{\delta_1}$ and $f_2 \in A_{\delta_2}$ satisfy $\pi^*_{\delta_1,\beta} f_1 = \pi^*_{\delta_2,\beta} f_2$, then $T_{\alpha,\delta_1} f_1 = T_{\alpha,\delta_2} f_2$. Furthermore, $\|\widetilde{T}_{\alpha,\beta}\| = 1$, so $\widetilde{T}_{\alpha,\beta}$ has a unique continuous extension to an operator on A_β, which we declare to be $T_{\alpha,\beta}$. Conditions (a), (b), and (e) continue to hold. Thus the existence of operators $\{T_{\alpha,\beta}\}_{0\le\alpha\le\beta\le\tau}$ satisfying conditions (a), (b), and (e) is established.

To verify that the operators we have defined satisfy condition (c), fix α and β, and apply transfinite induction on γ. □

Lemma 6.5. *If $\pi^{-1}_{\alpha,\beta}(x)$ consists of a single point, then $(T_{\alpha,\beta} f)(x) = f(\pi^{-1}_{\alpha,\beta}(x))$ for each function $f \in A_\beta$.*

Proof. For fixed α, apply transfinite induction on β. □

We will need the following functional analysis lemma, whose elementary proof we omit.

Lemma 6.6. *Let X be a Banach space, let Y be a closed subspace of X, let I be a closed subspace of Y, and let $S : X \to X$ be a norm 1 projection of X onto Y. Then the map $\widetilde{S} : X/S^{-1}(I) \to X/I$ induced by S is an isometry.*

The next lemma is the key to the proofs of our results on root extensions and ideals. Its proof is similar to the proof of [Ghosh and Izzo 2023, Lemma 4.1], which is essentially the special case in which the closed ideal I arises from a bounded point derivation. In the lemma, $T : A_{\mathscr{F}} \to A$ is the operator in Lemma 6.1.

Lemma 6.7. *Let A be a uniform algebra on a compact space X, let I be a closed ideal in A, and let $\phi : A \to A/I$ denote the quotient map. Let $\mathscr{F}$ be a subset of I. Then in the $\mathscr{F}$-extension $A_{\mathscr{F}}$ of A the set $I_{\mathscr{F}} = T^{-1}(I)$ is a closed ideal, and the map $\phi \circ T : A_{\mathscr{F}} \to A/I$ is a Banach algebra homomorphism that induces an isometric Banach algebra isomorphism of $A_{\mathscr{F}}/I_{\mathscr{F}}$ onto A/I.*

Proof. For notational convenience set $\Phi = \phi \circ T$. Then Φ is a linear map with kernel $I_{\mathscr{F}}$. Therefore, if Φ is multiplicative, i.e., if it satisfies

$$\Phi(fg) = \Phi(f)\Phi(g) \tag{1}$$

for all $f, g \in A_{\mathscr{F}}$, then Φ is a Banach algebra homomorphism, $I_{\mathscr{F}}$ is an ideal in $A_{\mathscr{F}}$, and identifying A with the subspace $\pi^*(A)$ of $A_{\mathscr{F}}$ and applying Lemma 6.6 shows that the induced Banach algebra isomorphism of $A_{\mathscr{F}}/I_{\mathscr{F}}$ onto A/I is isometric. Thus it suffices to show that Φ satisfies (1) for all $f, g \in A_{\mathscr{F}}$. Moreover, it is enough to verify (1) for f and g belonging to the dense subalgebra H of $A_{\mathscr{F}}$ that is algebraically generated by $\pi^*(A) \cup \{p_f : f \in \mathscr{F}\}$. Functions f and g in H can be expressed in the form

$$f = \pi^*(f_0) + \sum_{u=1}^{s} \pi^*(f_u)F_u \quad \text{and} \quad g = \pi^*(g_0) + \sum_{v=1}^{t} \pi^*(g_v)G_v,$$

where $f_0, f_1, \dots, f_s, g_0, g_1, \dots, g_t \in A$ and each F_u and each G_v is a nonempty product of distinct functions of the form p_f for $f \in \mathscr{F}$.

By Lemma 6.1, $Tf = f_0$ and $Tg = g_0$, so

$$\Phi(f) = (\phi \circ T)(f) = \phi(f_0) \quad \text{and} \quad \Phi(g) = (\phi \circ T)(g) = \phi(g_0).$$

Since $\phi(f_0 g_0) = \phi(f_0)\phi(g_0)$, the proof will be complete once we show that $\Phi(fg) = \phi(f_0 g_0)$.

View fg as a sum of four terms:

$$fg = \pi^*(f_0 g_0) + \left(\sum_{u=1}^{s} \pi^*(f_u g_0)F_u\right) + \left(\sum_{v=1}^{t} \pi^*(f_0 g_v)G_v\right) + \left(\sum_{u=1}^{s}\sum_{v=1}^{t} \pi^*(f_u g_v)F_u G_v\right).$$

By Lemma 6.1,

$$T(\pi^*(f_0 g_0)) = f_0 g_0, \tag{2}$$

$$T\left(\sum_{u=1}^{s} \pi^*(f_u g_0)F_u\right) = 0, \tag{3}$$

$$T\left(\sum_{v=1}^{t} \pi^*(f_0 g_v)G_v\right) = 0. \tag{4}$$

Now for fixed u and v, consider $T(\pi^*(f_u g_v)F_u G_v)$. We have $F_u = p_{f_1} \cdots p_{f_a}$ and $G_v = p_{g_1} \cdots p_{g_b}$ where $f_1, \dots, f_a$ are distinct elements of $\mathscr{F}$ and $g_1, \dots, g_b$ are also distinct elements of $\mathscr{F}$. Note that each of the sets $\{f_1, \dots, f_a\}$ and $\{g_1, \dots, g_b\}$

is necessarily nonempty. If $\{f_1, \ldots, f_a\} = \{g_1, \ldots, g_b\}$, then $F_u G_v = p_{f_1}^2 \cdots p_{f_a}^2 = \pi^*(f_1 \cdots f_a)$, and hence, by Lemma 6.1(ii),

$$(\phi \circ T)(\pi^*(f_u g_v) F_u G_v) = (\phi \circ T)(\pi^*(f_u g_v f_1 \cdots f_a)) = \phi(f_u g_v f_1 \cdots f_a);$$

the last quantity above is zero because $f_1, \ldots, f_a$ belong to the ideal I. If instead $\{f_1, \ldots, f_a\} \neq \{g_1, \ldots, g_b\}$, then $F_u G_v$ can be expressed as the product of a possibly empty set of elements of $\pi^*(A)$ and a *nonempty* set of functions $p_{h_1}, \ldots, p_{h_c}$ with $h_1, \ldots, h_c \in \{f_1, \ldots, f_a, g_1, \ldots, g_b\}$; consequently, $T(\pi^*(f_u g_v) F_u G_v) = 0$ by Lemma 6.1(iii). We conclude that

$$(\phi \circ T)\Bigl(\sum_{u=1}^{s} \sum_{v=1}^{t} \pi^*(f_u g_v) F_u G_v\Bigr) = 0. \tag{5}$$

Collectively, (2)–(5) yield that

$$\Phi(fg) = (\phi \circ T)(fg) = \phi(f_0 g_0),$$

as desired. □

Finally we come to the theorems of this section.

Theorem 6.8. *Let* $(\{A_\alpha\}, \{X_\alpha\}, \{\pi_{\alpha,\beta}\})$ $(0 \le \alpha \le \beta \le \tau)$ *be a system of root extensions. Let I_0 be a closed ideal in A_0, and set $S_0 = \mathrm{hull}(I_0)$. For every $0 \le \alpha \le \tau$, set $I_\alpha = T_{0,\alpha}^{-1}(I_0)$ and $S_\alpha = \pi_{0,\alpha}^{-1}(S_0)$. Suppose that $I_\alpha \supset \mathscr{F}_\alpha$ for every $0 \le \alpha < \tau$. Then, for every $0 \le \alpha \le \tau$:*

(i) *$\pi_{0,\alpha}$ takes S_α homeomorphically onto S_0.*

(ii) *$\pi_{0,\alpha}^*$ induces in the obvious way an isometric isomorphism of $A_0|S_0$ onto $A_\alpha|S_\alpha$.*

(iii) *I_α is a closed ideal in A_α such that $\mathrm{hull}(I_\alpha) = S_\alpha$, such that $I_\alpha \cap \pi_{0,\alpha}^*(A_0) = \pi_{0,\alpha}^*(I_0)$, and such that A_α/I_α is isometrically isomorphic as a Banach algebra to A_0/I_0. Consequently, there is an order-preserving bijective correspondence between the closed ideals of A_α containing I_α and the closed ideals of A_0 containing I_0.*

Proof. By a simple transfinite induction, one shows simultaneously that the hypotheses imply that every function in I_α, and hence every function in $\mathscr{F}_\alpha$, is zero on S_α, and that $\pi_{0,\alpha}$ takes S_α one-to-one onto S_0. Since $\pi_{0,\alpha}$ is continuous and S_α is compact, condition (i) follows.

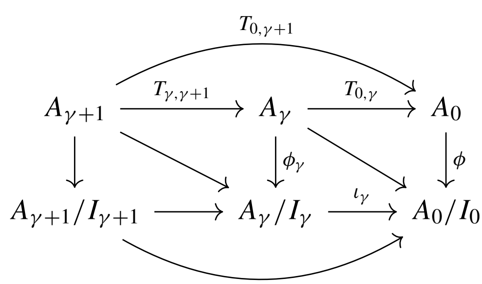
Given $f \in A_0$, the restriction of $\pi_{0,\alpha}^*(f) = f \circ \pi_{0,\alpha}$ to $\pi_{0,\alpha}^{-1}(S_0) = S_\alpha$ depends only on the restriction of f to S_0, so $\pi_{0,\alpha}^*$ induces a map of $A_0|S_0$ into $A_\alpha|S_\alpha$ that is obviously isometric. Given $g \in A_\alpha$, Lemma 6.5 shows that $\pi_{0,\alpha}^*(T_{0,\alpha} g)|S_\alpha = g$, so the map is onto. Thus condition (ii) holds.

We have already noted that every function in I_α is zero on S_α. Since for every point x of $X_\alpha \setminus S_\alpha$ there is a function $f \in I_0$ such that $\pi^*_{0,\alpha}(f)$, a function in I_α, is nonzero at x, this gives that $\mathrm{hull}(I_\alpha) = S_\alpha$.

The equality $I_\alpha \cap \pi^*_{0,\alpha}(A_0) = \pi^*_{0,\alpha}(I_0)$ follows immediately from Lemma 6.4(d).

It remains to show, for each α, that I_α is an ideal in A_α and that A_α/I_α and A_0/I_0 are isometrically isomorphic as Banach algebras. Let $\phi : A_0 \to A_0/I_0$ denote the quotient map. Assume for the moment that the map $\phi \circ T_{0,\alpha} : A_\alpha \to A_0/I_0$ is a Banach algebra homomorphism for every α. Then since $I_\alpha = \ker(\phi \circ T_{0,\alpha})$, it follows that I_α is an ideal in A_α, and that the induced map $A_\alpha/I_\alpha \to A_0/I_0$ is a Banach algebra isomorphism. Identifying A_0 with the subspace $\pi^*_{0,\alpha}(A_0)$ of A_α and applying Lemma 6.6 shows that this isomorphism is an isometry. Thus to complete the proof it suffices to show that the map $\phi \circ T_{0,\alpha} : A_\alpha \to A_0/I_0$ is indeed a Banach algebra homomorphism for every α.

We apply transfinite induction. Consider $0 \le \beta \le \tau$, and assume as the induction hypothesis that $\phi \circ T_{0,\alpha}$ is a Banach algebra homomorphism for every $\alpha < \beta$. When $\beta = 0$, nothing needs to be proved. If β is a limit ordinal, then it is immediate from the induction hypothesis that the restriction of $\phi \circ T_{0,\beta}$ to the dense subset $\bigcup_{\alpha<\beta} \pi^*_{\alpha,\beta}(A_\alpha)$ of A_β is an algebra homomorphism, and hence, $\phi \circ T_{0,\beta}$ is a Banach algebra homomorphism by continuity. Now suppose instead that $\beta = \gamma + 1$ for some γ. The map $\phi \circ T_{0,\gamma}$ is a Banach algebra homomorphism by the induction hypothesis. Consequently, I_γ is a closed ideal in A_γ. Let $\phi_\gamma : A_\gamma \to A_\gamma/I_\gamma$ denote the quotient map, and let $\iota_\gamma : A_\gamma/I_\gamma \to A_0/I_0$ denote the Banach algebra isomorphism induced by $\phi \circ T_{0,\gamma}$. By Lemma 6.7, the map $\phi_\gamma \circ T_{\gamma,\gamma+1} : A_{\gamma+1} \to A_\gamma/I\gamma$ is a Banach algebra homomorphism. Now consider the following commutative diagram:



Observe that the map $\phi \circ T_{0,\gamma+1} : A_{\gamma+1} \to A_0/I_0$ coincides with the composition of the Banach algebra homomorphisms $\phi_\gamma \circ T_{\gamma,\gamma+1}$ and ι_γ and hence is itself a Banach algebra homomorphism, as desired. □

By suitable choice of system of root extensions we will obtain the following as a corollary.

Theorem 6.9. *Let A be a uniform algebra on a compact space X, let I be an ideal in A, and set $S = \mathrm{hull}(I)$. Then there exists a uniform algebra $\widetilde{A}$ on a compact* *space* $\widetilde{X}$ *and a surjective continuous map* $\widetilde{\pi} : \widetilde{X} \to X$, *obtained from* A *by a system of root extensions*, *and there exists an ideal* $\widetilde{I}$ *in* $\widetilde{A}$ *such that setting* $\widetilde{S} = \widetilde{\pi}^{-1}(S)$ *conditions* (i)–(iii) *of Theorem 6.8 hold with* $\widetilde{A}$, $\widetilde{X}$, $\widetilde{I}$, $\widetilde{S}$, *and* $\widetilde{\pi}$ *in place of* A_α, X_α, I_α, S_α, *and* $\pi_{0,\alpha}$, *respectively*, *and such that furthermore*:

(iv) *Every function in* $\widetilde{I}$ *has a square root in* $\widetilde{I}$.

(v) *If* A *is normal*, *then* $\widetilde{A}$ *is normal and has bounded relative units at every point of* $\widetilde{X} \setminus \widetilde{S}$, *and hence every point of* $\widetilde{X} \setminus \widetilde{S}$ *is a generalized peak point for* $\widetilde{A}$.

If X *is metrizable*, *then*, *in addition*, *we can take* $\widetilde{X}$ *to be metrizable provided we replace condition* (iv) *by*:

(iv′) *There is a dense subset* $\mathscr{F}$ *of* $\widetilde{I}$ *such that every function in* $\mathscr{F}$ *has a square root in* $\mathscr{F}$.

Proof. Using Theorem 6.8 and transfinite induction, it is easily shown that there is a system of root extensions satisfying the conditions of Theorem 6.8 with $\tau = \Omega$ (the first uncountable ordinal) and $\mathscr{F}_\alpha = I_\alpha$ for every $0 \le \alpha < \Omega$. Set $\widetilde{A} = A_\Omega$, $\widetilde{X} = X_\Omega$, $\widetilde{I} = I_\Omega$, etc. Then conditions (i)–(iii) hold with $\widetilde{A}$, $\widetilde{X}$, $\widetilde{I}$, $\ldots$ in place of A_α, X_α, I_α, $\ldots$, respectively, by Theorem 6.8.

Given $f \in \widetilde{I} = I_\Omega$, there is some $\alpha < \Omega$ and some $g \in A_\alpha$ such that $f = \pi^*_{\alpha,\Omega} g$. By construction and Lemma 6.1(iii), $\pi^*_{\alpha,\alpha+1} g = h^2$ for some $h \in A_{\alpha+1}$ such that

$$T_{\alpha,\alpha+1} h = 0. \tag{6}$$

Now

$$(\pi^*_{\alpha+1,\Omega} h)^2 = \pi^*_{\alpha+1,\Omega} h^2 = \pi^*_{\alpha+1,\Omega} \pi^*_{\alpha,\alpha+1} g = \pi^*_{\alpha,\Omega} g = f.$$

Furthermore, by Lemma 6.4 and (6),

$$T_{0,\Omega}(\pi^*_{\alpha+1,\Omega} h) = (T_{0,\alpha+1} \circ T_{\alpha+1,\Omega})(\pi^*_{\alpha+1,\Omega} h) = T_{0,\alpha+1} h = T_{0,\alpha} \circ T_{\alpha,\alpha+1} h = 0,$$

so $\pi^*_{\alpha+1,\Omega} h$ is in $\widetilde{I}$. Thus every function in $\widetilde{I}$ has a square root in $\widetilde{I}$.

Now suppose that A is normal. Then, by Lemma 6.3, $\widetilde{A}$ is normal. Consequently, given a point $\tilde{x} \in \widetilde{X} \setminus \widetilde{S}$ and a compact subset E of $\widetilde{X} \setminus \{\tilde{x}\}$, Theorem 2.1 ensures that there is a function in $\widetilde{I}$ that is one on a neighborhood of $\tilde{x}$ and zero on E. Therefore, $\widetilde{A}$ has bounded relative units at $\tilde{x}$ by Lemma 3.4. The final assertion of condition (v) follows by Lemma 3.5.

All that remains is to prove the last sentence of the theorem. From now on suppose that X is metrizable. Using Theorem 6.8 and induction, it is easily shown that there is a system of root extensions satisfying the conditions of Theorem 6.8 with $\tau = \omega$ (the first infinite ordinal) and with the property that, for every $0 \le \alpha < \omega$, the collection $\mathscr{F}_\alpha$ is a countable dense subset of I_α such that for every function $f \in \mathscr{F}_\alpha$ the function $\pi^*_{\alpha,\alpha+1} f$ is the square of a function in $\mathscr{F}_{\alpha+1}$. Furthermore, if A is normal, then Lemma 3.6 and Theorem 2.1 show that we can, and therefore we shall, choose $\mathscr{F}_\alpha$ such that, setting $S_\alpha = \pi_{0,\alpha}^{-1}(S)$, we have that for each point $x \in X_\alpha \setminus S_\alpha$, and for each compact subset E of $X_\alpha \setminus \{x\}$, there exists a neighborhood U of x, and a function $f \in \mathscr{F}_\alpha$ such that $f|U = 1$ and $f|E = 0$. Set $\widetilde{A} = A_\omega$, $\widetilde{X} = X_\omega$, $\widetilde{I} = I_\omega$, etc. Then $\widetilde{X}$ is metrizable. Furthermore, conditions (i)–(iii) hold with $\widetilde{A}$, $\widetilde{X}$, $\widetilde{I}, \ldots$ in place of A_α, X_α, $I_\alpha, \ldots$, respectively, by Theorem 6.8.

We will establish condition (iv′) with $\mathscr{F} = \bigcup_{\alpha<\omega} \pi_{\alpha,\omega}^*(\mathscr{F}_\alpha)$. First we show that every function in $\mathscr{F}$ has a square root in $\mathscr{F}$. Given $f \in \mathscr{F}$, there is some $\alpha < \omega$ and some $g \in \mathscr{F}_\alpha$ such that $f = \pi_{\alpha,\omega}^* g$. By construction, $\pi_{\alpha,\alpha+1}^* g = h^2$ for some $h \in \mathscr{F}_{\alpha+1}$. Then $\pi_{\alpha+1,\omega}^* h$ is in $\mathscr{F}$, and by Remark 6.2

$$(\pi_{\alpha+1,\omega}^* h)^2 = \pi_{\alpha+1,\omega}^* h^2 = \pi_{\alpha+1,\omega}^* \pi_{\alpha,\alpha+1}^* g = \pi_{\alpha,\omega}^* g = f,$$

so f has a square root in $\mathscr{F}$.

Next we show that $\mathscr{F}$ is contained in $\widetilde{I}$. Let f, α, and g be as in the previous paragraph. Then, by Lemma 6.4,

$$T_{0,\omega} f = T_{0,\omega} \pi_{\alpha,\omega}^* g = T_{0,\alpha} T_{\alpha,\omega} \pi_{\alpha,\omega}^* g = T_{0,\alpha} g,$$

and $T_{0,\alpha} g$ is in I because g is in $\mathscr{F}_\alpha \subset I_\alpha = T_{0,\alpha}^{-1}(I)$. Thus f is in $\widetilde{I}$, as desired.

To prove the density of $\mathscr{F}$ in $\widetilde{I}$, first note that $\pi_{\alpha,\omega}^*(\mathscr{F}_\alpha)$ is dense in $\pi_{\alpha,\omega}^*(I_\alpha)$, so it suffices to show that $\bigcup_{\alpha<\omega} \pi_{\alpha,\omega}^*(I_\alpha)$ is dense in $\widetilde{I}$. Fix $f \in \widetilde{I}$ and $\varepsilon > 0$ arbitrary. We will show that $\|\pi_{\alpha,\omega}^*(T_{\alpha,\omega} f) - f\| < \varepsilon$ for some $\alpha < \omega$. Since $T_{0,\alpha}(T_{\alpha,\omega} f) = T_{0,\omega} f$ is in I, the function $T_{\alpha,\omega} f$ is in I_α, so this will establish the desired density.

By the definition of $\widetilde{A} = A_\omega$, there exists $\alpha < \omega$ and $a \in A_\alpha$ such that

$$\|f - \pi_{\alpha,\omega}^* a\| < \varepsilon/2. \tag{7}$$

Then

$$\|(\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(f) - (\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(\pi_{\alpha,\omega}^* a)\| < \varepsilon/2.$$

Since $T_{\alpha,\omega} \circ \pi_{\alpha,\omega}^*$ is the identity, this gives

$$\|(\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(f) - \pi_{\alpha,\omega}^* a\| < \varepsilon/2. \tag{8}$$

From (7) and (8) we get

$$\|(\pi_{\alpha,\omega}^* \circ T_{\alpha,\omega})(f) - f\| < \varepsilon.$$

This concludes the proof of condition (iv′).

If A is normal, then a simple argument shows that our choice of the $\mathscr{F}_\alpha$ ensures that for every point $\tilde{x} \in \widetilde{X} \setminus \widetilde{S}$ and every compact subset E of $\widetilde{X} \setminus \{\tilde{x}\}$, there exists a neighborhood U of $\tilde{x}$ and a function $f \in \mathscr{F}$ such that $f|U = 1$ and $f|E = 0$. Consequently, condition (v) can be proven in the same manner as was done earlier when we took $\mathscr{F}_\alpha = I_\alpha$. □

7. Normal peak point algebras that are not strongly regular

In this section we prove Theorems 1.6 and 1.7.

Proof of Theorem 1.6. Set $A = R(K)$ with K as in Theorem 1.4, and set $I = \bar{J}_1$. Then let B be the uniform algebra $\widetilde{A}$ obtained by applying Theorem 6.9 taking $\widetilde{X}$ to be metrizable. Let x_0 be the unique point of $\widetilde{\pi}^{-1}(1)$. The uniform algebra B is normal. Also B has bounded relative units at every point of $\widetilde{X} \setminus \{x_0\}$, and hence, by Lemma 3.5, B is strongly regular at every point of $\widetilde{X} \setminus \{x_0\}$ and every point of $\widetilde{X} \setminus \{x_0\}$ is a peak point for B. The point x_0 is a peak point for B as well because the function $((1+z)/2) \circ \widetilde{\pi}$ peaks at x_0.

There is an order-preserving bijection between the closed ideals of B containing $\widetilde{I}$ and the closed ideals of $R(K)$ containing I. So the family of ideals $\{I_\rho : 0 \le \rho < \infty\}$ in $R(K)$ yields the family of ideals $\{H_\rho : 0 \le \rho < \infty\}$ in B. □

Proof of Theorem 1.7. The proof is essentially the same as the previous proof except that now we set $A = R(K)$ with K as in Theorem 1.5, let Λ be as in Theorem 1.5, and set $I = \bar{J}_\Lambda$. □

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