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EQUIVARIANT MIN-MAX HYPERSURFACE IN G-MANIFOLDS WITH POSITIVE RICCI CURVATURE

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We consider a connected orientable closed Riemannian manifold M^{n+1} with positive Ricci curvature. Suppose G is a compact Lie group acting by isometries on M with $3 \leq \operatorname{codim}(G \cdot p) \leq 7$ for all $p \in M$. Then we show the equivariant min-max G-hypersurface Σ corresponding to one-parameter G-sweepouts (of boundary-type) is a multiplicity one minimal G-hypersurface with a G-invariant unit normal and G-equivariant index one. As an application, we are able to establish a genus bound for Σ, a control on the singular points of Σ/G, and an upper bound for the (first) G-width of M provided $n+1=3$ and the actions of G are orientation preserving.

1. Introduction

Given a connected orientable closed Riemannian manifold (M^{n+1}, g_M), minimizing the area within a nontrivial homology class is a natural way to construct minimal hypersurfaces (see [12; 36]). However, if M has positive Ricci curvature, it follows from the stability inequality that this minimization method cannot be applied. In the 1960s, Almgren [1; 2] proposed the *min-max theory* to find minimal submanifolds in the most general situation. Subsequently, the regularity for min-max hypersurfaces was improved by Pitts [30] ($n \leq 5$) and Schoen and Simon [34] ($n = 6$). Indeed, for $n \geq 7$, they showed the min-max minimal hypersurface is smooth embedded except for a singular set of codimension 7.

Due to the generality and abstractness of Almgren–Pitts min-max theory, many of the geometric properties of min-max hypersurfaces have not been understood until recently. For instance, in a closed manifold with positive Ricci curvature, a series of studies were set out to characterize the min-max hypersurfaces generated from one-parameter families. Specifically, using the Heegaard splitting, Marques and Neves [20] studied the index and genus of the min-max surface in certain 3-manifolds. They also obtained sharp estimates for the width and rigidity results. In a higher-dimensional manifold M^{n+1} with positive Ricci curvature, Zhou determined

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the Morse index and multiplicity of the min-max hypersurface for $3 \le n+1 \le 7$ in [43] and for $n \ge 7$ in [44]. Subsequently, Ketover, Marques, and Neves [17] refined Zhou's results in dimension $3 \le n+1 \le 7$ by showing the orientability of the min-max hypersurface using the *catenoid estimates*. In particular, the min-max hypersurface is an orientable closed minimal hypersurface of Morse index one and has the least area among all orientable closed minimal hypersurfaces. Furthermore, without any curvature assumption, the constructions in [20; 43] were also employed by Mazet and Rosenberg [25] to show the least area minimal hypersurface is either stable or a min-max hypersurface of Morse index one.

Given a 3-manifold M with a finite group G acting by isometries, Pitts and Rubinstein [31; 32] first asserted the existence of a G-invariant minimal surface with estimates on its index and genus. The existence and regularity of minimal G-invariant surfaces (abbreviated as *G-surfaces*) were recently confirmed by Ketover [15] (for finite G of orientation preserving isometries) using the equivariant min-max under the smooth setting. More generally, suppose M^{n+1} is a closed Riemannian manifold with a compact Lie group G acting by isometries so that $3 \le \operatorname{codim}(G \cdot p) \le 7$ for all $p \in M$. The equivariant min-max theory was also extended to this general scenario by Liu [19] (for connected G with $\min_{p\in M} \operatorname{codim}(G\cdot p) \neq 0, 2$) in the smooth setting and by Wang [39; 40] in the Almgren–Pitts setting. In particular, Wang [39, Theorem 9] showed an isomorphism between $H_{n+1}(M; \mathbb{Z}_2)$ and $\pi_1(\mathcal{Z}_n^G(M; \mathbb{Z}_2))$, where $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is the space of G-invariant n-cycles (of boundary-type, see geometric measure theory). Then it is similar to the constructions of Almgren–Pitts (see [30]) that the fundamental class $[M] \in H_{n+1}(M; \mathbb{Z}_2)$ corresponds to the (first) *equivariant min-max width* $W^G(M) > 0$ of M defined with one-parameter G-sweepouts (see Definition 2.7 and [30, Corollary 4.7]), which can be realized by the area of some minimal G-invariant hypersurfaces (abbreviated as *G-hypersurfaces*) with multiplicities. Therefore, it now seems reasonable to investigate the geometric features of the equivariant min-max hypersurface, such as its area, multiplicity, index, and topology.

In this paper, our main result generalizes the characterization of the min-max hypersurface into an equivariant version (see Theorem 5.1).

Theorem 1.1. *Let (M^{n+1}, g_M) be a connected orientable closed Riemannian manifold with positive Ricci curvature, and G be a compact Lie group acting by isometries on M so that $3 \le \operatorname{codim}(G\cdot p) \le 7$ for all $p \in M$. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M] \in H_{n+1}(M; \mathbb{Z}_2)$ is a multiplicity one minimal G-hypersurface so that*:

(i) *Σ has a G-invariant unit normal vector field.*

(ii) *The equivariant Morse index of Σ* (Definition 4.1) *is one.*

(iii) *Σ has the least area among all closed embedded minimal G-hypersurfaces with G-invariant unit normal vector fields.*

Remark 1.2. We make some remarks about the above theorem:

(i) If M has connected components $\{M_i\}_{i=1}^m$, then we can take a component M_i and the Lie subgroup $G_i := \{g \in G : g \cdot M_i = M_i\}$. By applying the above theorem to M_i and G_i, we obtain a minimal G_i-invariant hypersurface Σ_i of multiplicity one. Additionally, one easily verifies that $G \cdot \Sigma_i \subset G \cdot M_i$ is a minimal G-hypersurface satisfying (i)–(iii) in Theorem 1.1 with $G \cdot M_i$ in place of M.

(ii) Without the positive Ricci curvature assumption, we can combine the proof of Theorem 1.1 and the constructions in [25] to show the existence of a minimal G-hypersurface of the least area (counted with multiplicity) among all minimal G-hypersurfaces. The details will be discussed in an upcoming paper.

Equivariant vs nonequivariant. Note that Theorem 1.1 is an equivariant generalization of the results in [17; 43] where $G = \{id\}$. Nevertheless, due to the equivariant constraints, the equivariant min-max hypersurface exhibits slightly stronger properties (e.g., the unit normal not only exists but also it is G-invariant). Additionally, it should be noted that the equivariant constraints generally have a significant impact on the min-max outcomes. Indeed, if we denote by $W(M) = W^{\{id\}}(M)$ (resp. $W^G(M)$) and Σ (resp. Σ_G) the first (resp. equivariant) min-max width and the corresponding first (resp. equivariant) min-max hypersurface, then we generally have $W(M) \le W^G(M)$ without the equality. Moreover, even if Σ is G-invariant and $W(M) = W^G(M)$, Σ may *not* necessarily be the equivariant min-max hypersurface corresponding to $W^G(M)$. One can easily observe these phenomena from the following examples.

Example 1.3 ($W(M) \le W^G(M)$ without equality). Let $M = \mathbb{S}^3$ be the unit sphere with the standard round metric. Then $W(\mathbb{S}^3) = 4\pi$ is realized by the area of the equator $\Sigma = \mathbb{S}^2$ [29]. Next, take $G = \mathbb{Z}_2$ acting on $\mathbb{S}^3$ by the antipodal map so that $\pi : M = \mathbb{S}^3 \to M/G = \mathbb{RP}^3$ is a (locally isometric) double cover. Hence, $\pi(\Sigma_G)$ is the first min-max hypersurface in $\mathbb{RP}^3$ corresponding to $W(\mathbb{RP}^3)$. Therefore, although Σ is G-invariant, it can *not* be Σ_G, because $\pi(\Sigma) = \mathbb{RP}^2$ is 1-sided, while $\pi(\Sigma_G) \subset \mathbb{RP}^3$ must be 2-sided [17; 43]. Indeed, it follows from [3] that $W^{\mathbb{Z}_2}(\mathbb{S}^3) = 2W(\mathbb{RP}^3) = 2\pi^2$ is realized by the area of the Clifford torus.

Example 1.4 ($\Sigma \neq \Sigma_G$ even if $W = W^G$). Let $M = \mathbb{S}^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$, and $G = \mathbb{Z}_2$ act by the reflection $(x_1, x_2, x_3, x_4) \mapsto (-x_1, x_2, x_3, x_4)$. Then we have $M/G = \mathbb{S}^3_+ = \{x \in \mathbb{S}^3 : x_1 \ge 0\}$. Note the $\mathbb{Z}_2$-equivariant minimal hypersurfaces and $\mathbb{Z}_2$-sweepouts in $\mathbb{S}^3$ correspond one-to-one to the free boundary minimal hypersurfaces and (relative) sweepouts in $\mathbb{S}^3_+$. Thus, Σ_G/G is the first min-max free boundary minimal hypersurface in $\mathbb{S}^3_+$ corresponding to $W(\mathbb{S}^3_+)$. Therefore, $W^{\mathbb{Z}_2}(\mathbb{S}^3) = 2W(\mathbb{S}^3_+) = 4\pi = W(\mathbb{S}^3)$ realized by the area of a great 2-sphere $\Sigma_G = \mathbb{S}^2$ perpendicular to $\{x_1 = 0\}$. (As an example, take the $\mathbb{Z}_2$-sweepout

$\{\mathbb{S}^3 \cap \{x_2 = t\}\}_{t\in[-1,1]}$.) Meanwhile, we notice $\Sigma = \mathbb{S}^3 \cap \{x_1 = 0\}$ is G-invariant and is also a min-max hypersurface corresponding to $W(\mathbb{S}^3)$. However, since $\Sigma/G = \mathbb{S}^2$ is not the free boundary min-max hypersurface corresponding to $W(\mathbb{S}^3_+)$, we have $\Sigma \neq \Sigma_G$ in this case.

One should also notice that in the above examples, Σ_G admits a G-invariant unit normal, while the unit normal of Σ is not G-invariant. Intuitively, this is because our equivariant sweepouts are formed by the boundaries of G-invariant (Caccioppoli) sets admitting a (measure-theoretic) inward G-invariant unit normal. Hence, if a G-invariant min-max hypersurface Σ does not have a G-invariant unit normal, then Σ cannot be a boundary of a G-invariant (Caccioppoli) set, and the min-max sequence $|\partial\Omega_{t_i}|$ must converge to Σ with even multiplicities (Theorem 3.8) so that the constructions in [17; 43] can be generalized to derive Theorem 1.1.

Remark 1.5. To ensure $W^G(M)$ is well defined for any M and G, we only use the boundaries of G-invariant (Caccioppoli) sets in the equivariant min-max constructions in this paper. Note, for some specific choices of M, G, one may construct the equivariant min-max using "G-hypersurfaces without G-invariant unit normal", and Theorem 1.1(i) may fail in this case (see, e.g., [16]). Similarly, the results in [43] may not be applicable for nonboundary-type min-max constructions (without equivariance).

Further discussions and applications. We will now delve deeper into some inspirations and potential applications of Theorem 1.1.

Firstly, one notices that the existence of a *G-invariant* unit normal can help to distinguish the min-max G-hypersurface Σ and the fixed points set under certain $\mathbb{Z}_2$ actions. For instance, consider a positive Ricci curvature 3-ellipsoid M with its major axis (on x_1) sufficiently long and the other principal axes bounded by 2. Then the classical min-max theory shall provide the equator $\Gamma = \{x_1 = 0\} \cap M$ on the major axis as the min-max hypersurface. Although Γ is also invariant under the $\mathbb{Z}_2$-reflections $(x_1, x') \mapsto (-x_1, x')$, it cannot be the min-max $\mathbb{Z}_2$-hypersurface since its unit normal is not $\mathbb{Z}_2$-invariant. An interesting question is what exactly is the min-max $\mathbb{Z}_2$-hypersurface in this case, and how does it relate to the 2-min-max minimal hypersurfaces?

In addition, we see that the characterizations of the Morse index and multiplicity for min-max hypersurfaces are crucial in the study of min-max theory. For instance, a key part in the proof of the Willmore conjecture by Marques and Neves [21] is to show the minimal surface in $\mathbb{S}^3$ constructed by the five-parameter families of min-max has Morse index 5. Additionally, by specifying generically the multiplicity [45] and index [22; 24] of min-max hypersurfaces, the multiparameter min-max theory was used to establish the Morse theory for the area functional. In the equivariant case, Wang [41] also proved general upper bounds for the G-index (Definition 4.1) of equivariant min-max hypersurfaces generated from multiparameter families. Therefore, in light of Theorem 1.1 and Zhou [45], we conjecture that for a generic G-invariant Riemannian metric, the minimal G-hypersurface constructed from k-parameter families of equivariant min-max shall have multiplicity one, G-index k, and a G-invariant unit normal.

Moreover, it has been discovered in numerous studies that the Morse index of a minimal surface is related to its topology. For instance, in a closed 3-manifold with positive Ricci curvature, Choi and Schoen [8] proved the area of a closed minimal surface can be bounded by its genus. Therefore, by Ejiri and Micallef [11, Theorem 4.3], the index of a such minimal surface is also bounded by its genus. Additionally, using the *conformal volume*, Yau (see [35, Chapter VIII, Section 4]) obtained a genus bound for index one minimal surfaces in positive Ricci curvature manifolds. More generally, in an orientable 3-manifold with nonnegative Ricci curvature, it follows from the sharp estimate of Ros [33, Theorem 15] that a closed orientable minimal surface of index one must have genus ≤ 3. Recently, Song [37] showed that the total Betti number of a closed minimal hypersurface in M^{n+1}, $3 \leq n+1 \leq 7$, can be bounded by its index and a constant depending only on n, g_M, and its area, which further indicates a quantified relation [37, Corollary 3] between the genus and index of a minimal surface in M^3. For a complete two-sided minimal surface in $\mathbb{R}^3$, Chodosh and Maximo [6] showed that its genus and the number of ends give a lower bound on its index. We refer to [7; 26] for more related research.

Hence, as an application, we use the conformal volume initiated by Li and Yau [18] in the orbit space to show a general genus bound of the equivariant min-max surface in a 3-manifold with positive Ricci curvature, which further indicates an upper bound of the G-width and a bound for the singular points of Σ/G (see Theorem 5.2).

Theorem 1.6. *Let (M^3, g_M) be a closed connected oriented Riemannian 3-manifold with positive Ricci curvature, and G be a finite group acting on M by orientation preserving isometries. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M]$ is a connected minimal G-hypersurface of multiplicity one with*

$$\operatorname{genus}(\Sigma) \leq 4K, \quad W^G(M) = \operatorname{Area}(\Sigma) \leq \frac{8\pi K}{c_M},$$

where $K := \max_{p\in M} \#G \cdot p \leq \#G$ is the number of points in a principal orbit of M, and $\operatorname{Ric}_M \geq c_M > 0$. Additionally, the quotient space $\pi(\Sigma) = \Sigma/G$ is an orientable surface with finite cone singular points of order $\{n_i\}_{i=1}^k$ so that

$$\sum_{i=1}^{k}\left(1-\frac{1}{n_i}\right) < 4 \quad \textit{and} \quad \operatorname{genus}(\pi(\Sigma)) \leq 3.$$

In particular, if Σ/G has no singularity, then $\operatorname{genus}(\Sigma) \leq 1+2K$.

Remark 1.7. To generalize Li–Yau's [18] theory to the orbit spaces, G-actions in Theorem 1.6 are assumed to be orientation preserving isometries so that M/G and Σ/G induce orientable orbifolds without boundary.

The conformal method has been employed in many studies for the *volume spectrum*, i.e., the multiparameter version of width. For the first width $W(M)$ in the volume spectrum, Glynn-Adey and Liokumovich [13] gave an upper bound using the min-conformal volume of the ambient manifold. In particular, if M is a closed surface, they showed the first width $W(M)$ can be bounded by the genus and area of M. Also, the conformal upper bounds for the volume spectrum were proved in [38].

Main ideas and outline. The main idea for Theorem 1.1 is as follows. For the closed manifold M and the Lie group G in Theorem 1.1, we can take any closed embedded minimal G-hypersurface Σ in M and use the variation of its first eigenvector field to foliate a G-neighborhood of Σ. Using a half-space version of the equivariant min-max theory (Theorem 3.11), we argue by contradiction to show this local G-equivariant foliation can be extended to a continuous G-sweepout of M with mass no more than Area(Σ) (if Σ has a G-invariant unit normal) or $2\,$Area(Σ). Therefore, it follows from the equivariant min-max theory [39, Theorem 8] (see also [40, Theorem 4.20]) that the equivariant min-max hypersurface is the minimal G-hypersurface of *least area* in the sense of (5-1). Additionally, if the equivariant min-max hypersurface does not admit a G-invariant unit normal, it must have even multiplicity by the constructions of equivariant min-max (Theorem 3.8). However, in this case, we can further use the catenoid estimates of Ketover et al. [17] to add small G-invariant cylinders in the G-sweepouts (Proposition 4.7), which will strictly decrease the mass and give a contradiction.

The above idea shares the same spirit as in [43]. However, since the equivariant min-max theory was already established in a continuous version [39, Theorem 8], we do not need to invoke the smooth setting of min-max (see [10]) as in [43, Section 2], but give a more self-contained equivariant min-max construction in half spaces (Theorem 3.11). Meanwhile, instead of using the discretization theorem as in [43, Theorem 5.8], we can more easily determine that the extension of the G-equivariant foliation is a G-sweepout.

The article is organized as follows. In Section 2, we collect some notations and definitions of Lie group actions and geometric measure theory. In particular, we introduce the G-equivariant sweepouts and G-width of M in a continuous version using the isomorphic map between $\pi_1(\mathcal{Z}_n^G(M;\mathbb{Z}_2))$ and $H_{n+1}(M;\mathbb{Z}_2)$. Then we introduce in Section 3 the equivariant min-max theory developed by Wang [39; 40] under the Almgren–Pitts setting with some modifications. In Section 4, we will generate a continuous G-sweepout with good properties from a given minimal G-hypersurface. The proof of the main theorem and its applications are given in Section 5.

2. Preliminary

Let (M^{n+1}, g_M) be an orientable connected compact Riemannian $(n+1)$-manifold and G be a compact Lie group acting isometrically on M. Denote by μ a biinvariant Haar measure on G normalized to $\mu(G) = 1$. For the case that $\partial M \neq \emptyset$, it follows from [40, Lemma A.1] that M can be equivariantly and isometrically extended to a closed Riemannian manifold (N, g_N) with G acting on N by isometries. Therefore, we can assume M is a compact domain of a closed Riemannian G-manifold N.

Note that although our main results only involve closed minimal G-hypersurfaces in closed G-manifolds, we also need a half-space version of equivariant min-max to insert any closed embedded minimal G-hypersurface into a good G-sweepout (see main ideas and outline). Hence, we also include some terminologies and results in this paper concerning G-equivariant min-max in compact G-manifolds with nonempty boundary.

Lie group actions. To begin with, we gather some definitions of Lie group actions, most of which are referred from [4; 5].

It follows from [28] that there is an orthogonal representation $\rho : G \to O(L)$ and an isometric embedding $i : M \hookrightarrow \mathbb{R}^L$ for some $L \in \mathbb{N}$ so that i is equivariant, i.e., $i \circ g = \rho(g) \circ i$. For simplicity, we regard M as a subset of $\mathbb{R}^L$ and denote the orthogonal action of $g \in G$ on $x \in \mathbb{R}^L$ as $g \cdot x$. We say a subset (hypersurface) $A \subset M$ is a G-subset (G-hypersurface) if $g \cdot A = A$ for all $g \in G$.

For any $p \in M$, let $G \cdot p := \{g \cdot p : g \in G\}$ be the orbit containing p and $G_p := \{g \in G : g \cdot p = p\}$ be the isotropy group of p. Note $G \cdot p$ is a closed submanifold of M and G_p is a Lie subgroup of G. We then say p has (G_p) *orbit-type*, where (G_p) is the conjugacy class of G_p in G. By [4, Proposition 2.2.4], there is a (unique) minimal conjugacy class (P) of isotropy groups so that $M^{\text{prin}} = M_{(P)} := \{p \in M : (G_p) = (P)\}$ is an open dense G-subset of M. We call any $G \cdot p \subset M^{\text{prin}}$ a *principal orbit* of M and denote by $\text{Cohom}(G)$ the codimension of a principal orbit, which is known as the *cohomogeneity* of the actions of G.

Let M/G be the quotient space, i.e., the *orbit space*, and π be the projection $\pi : M \to M/G,\ p \mapsto [p]$. It is well known that M/G is a Hausdorff metric space with induced metric $\text{dist}_{M/G}([p], [q]) := \text{dist}_M(G \cdot p, G \cdot q)$.

Denote by $B_r(p)$, $B_r([p])$, and $\mathbb{B}^k_r(p)$ the geodesic ball in M (or in N if $\partial M \neq \emptyset$), the metric ball in M/G, and the Euclidean ball in $\mathbb{R}^k$ respectively. Then we use the following notations:

- $\mathfrak{X}(M)$, $\mathfrak{X}(U)$: the space of smooth vector fields compact supported in M or $U \subset M$.
- $\mathfrak{X}^G(M)$, $\mathfrak{X}^G(U)$: the space of G-*vector fields* X in M or U, ($g_* X = X$ for all $g \in G$).

- $B^G_\rho(p)$: the open geodesic tube with radius ρ around the orbit $G \cdot p$ in M (or in N if $\partial M \neq \varnothing$).
- $\mathrm{An}^G(p, s, t)$: the open tube $B^G_t(p) \setminus \bar{B}^G_s(p)$.

For any closed G-hypersurface $\Sigma \subset M$, denote by $\boldsymbol{N}\Sigma$ its normal bundle with G acting on it by $g \cdot v := g_* v$ for all $g \in G$, $v \in \boldsymbol{N}\Sigma$. Let $\exp^\perp_\Sigma : \boldsymbol{N}\Sigma \to M$ be the normal exponential map of Σ. Note $\exp^\perp_\Sigma$ is a G-equivariant diffeomorphism in a neighborhood of Σ.

Geometric measure theory. We refer to [12; 30; 36] for the following definitions:

- $\boldsymbol{I}_k(M; \mathbb{Z}_2)$: the space of k-dimensional mod 2 flat chains in $\mathbb{R}^L$ with support contained in M.
- $\mathcal{Z}_n(M; \mathbb{Z}_2)$: the space of $T \in \boldsymbol{I}_n(M; \mathbb{Z}_2)$ with $T = \partial U$ for $U \in \boldsymbol{I}_{n+1}(M; \mathbb{Z}_2)$, i.e., the boundary-type mod 2 n-cycles.
- $\mathcal{V}_k(M)$: the weak topological closure of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in M.

Let $\mathcal{F}$ and $\boldsymbol{M}$ be the *flat (semi)norm* and the *mass* norm in $\boldsymbol{I}_k(M; \mathbb{Z}_2)$ [12, 4.2.26]. Define the $\boldsymbol{F}$*-metric* on $\mathcal{V}_k(M)$ as in [30, p. 66]. Then $\boldsymbol{F}$ induces the weak topology on any mass bounded subset $\{V \in \mathcal{V}_k(M) : \|V\|(M) \le C\}$, where $C > 0$ and $\|V\|$ is the Radon measure on M induced by V.

For any $T \in \boldsymbol{I}_k(M; \mathbb{Z}_2)$, we denote $|T|$ and $\|T\|$ as the integral varifold and the Radon measure induced by T. Then we define the $\boldsymbol{F}$*-metric* on $\boldsymbol{I}_k(M; \mathbb{Z}_2)$ by

$$\boldsymbol{F}(S, T) := \mathcal{F}(S - T) + \boldsymbol{F}(|S|, |T|) \quad \text{for all } S, T \in \boldsymbol{I}_k(M; \mathbb{Z}_2).$$

It follows from [30, p. 68] that for any $T, \{T_i\}_{i\in\mathbb{N}} \subset \mathcal{Z}_n(M; \mathbb{Z}_2)$,

$$\text{(2-1)} \quad \lim_{i\to\infty} \boldsymbol{F}(T_i, T) = 0 \quad \Leftrightarrow \quad \lim_{i\to\infty} \mathcal{F}(T_i, T) = 0 \;\text{ and }\; \lim_{i\to\infty} \boldsymbol{M}(T_i) = \boldsymbol{M}(T).$$

For $\boldsymbol{v} \in \{\boldsymbol{M}, \boldsymbol{F}, \mathcal{F}\}$, let $\boldsymbol{I}_k(M; \boldsymbol{v}; \mathbb{Z}_2)$ and $\mathcal{Z}_n(M; \boldsymbol{v}; \mathbb{Z}_2)$ be the spaces with topology induced by $\boldsymbol{v}$. Additionally, we denote by $[\![\Gamma]\!]$ the element in $\boldsymbol{I}_k(M; \mathbb{Z}_2)$ induced by a k-submanifold $\Gamma \subset M$.

We say $T \in \boldsymbol{I}_k(M; \mathbb{Z}_2)$ (or $V \in \mathcal{V}_k(M)$) is G*-invariant* if $g_\# T = T$ ($g_\# V = V$) for all $g \in G$. Then we have the following subspaces of G-invariant elements:

- $\boldsymbol{I}^G_k(M; \mathbb{Z}_2) := \{T \in \boldsymbol{I}_k(M; \mathbb{Z}_2) : g_\# T = T \text{ for all } g \in G\}$.
- $\mathcal{Z}^G_n(M; \mathbb{Z}_2) := \{T \in \mathcal{Z}_n(M; \mathbb{Z}_2) : T = \partial U \text{ for some } U \in \boldsymbol{I}^G_{n+1}(M; \mathbb{Z}_2)\}$.
- $\mathcal{V}^G_k(M) := \{V \in \mathcal{V}_k(M) : g_\# V = V \text{ for all } g \in G\}$.

Remark 2.1. Note $\mathcal{Z}^G_n(M; \mathbb{Z}_2) \subsetneq \{T \in \mathcal{Z}_n(M; \mathbb{Z}_2) : g_\# T = T \text{ for all } g \in G\}$ in general, and intuitively, $T \in \mathcal{Z}^G_n(M; \mathbb{Z}_2)$ is not only a boundary that is G-invariant but also "bounds a G-invariant region". This is essential to derive Theorem 1.1(i) as explained in Remark 1.5.

Since G acts by isometries, $\boldsymbol{I}_k^G(M;\mathbb{Z}_2)$, $\mathcal{Z}_n^G(M;\mathbb{Z}_2)$, and $\mathcal{V}_k^G(M)$ are closed subspaces with induced metrics $\boldsymbol{M}$, $\mathcal{F}$, $\boldsymbol{F}$. Moreover, we have the following isoperimetric lemma (see [39, Lemma 5]), which is also valid when $\partial M \neq \varnothing$.

Lemma 2.2. *There are $\epsilon_M > 0$, $C_M > 1$ such that for any $T_1, T_2 \in \boldsymbol{I}_n^G(M;\mathbb{Z}_2)$ with $\partial T_1 = \partial T_2 = 0$, and*

$$\mathcal{F}(T_1 - T_2) < \epsilon_M,$$

*there is a unique $Q \in \boldsymbol{I}_{n+1}^G(M;\mathbb{Z}_2)$, called **the isoperimetric choice of T_1, T_2**, satisfying*

(i) $\partial Q = T_1 - T_2$,

(ii) $\boldsymbol{M}(Q) \leq C_M \cdot \mathcal{F}(T_1 - T_2)$.

For any $V \in \mathcal{V}_n(M)$ and $X \in \mathfrak{X}(M)$, the first variation of V along X is given by

$$\delta V(X) := \frac{d}{dt}\Big|_{t=0} \|(F_t)_\# V\|(M) = \int_{G_n(M)} \operatorname{div}_S(X)(p)\, dV(p,S),$$

where $\{F_t\}$ are the diffeomorphisms generated by X, and $G_n(M)$ is the Grassmannian bundle of unoriented n-planes over M. Suppose $V \in \mathcal{V}_n^G(M)$ is G-invariant and $U \subset M$ is an open G-subset, then we say:

- V is *stationary* in U if $\delta V(X) = 0$ for all $X \in \mathfrak{X}(U)$.
- V is *G-stationary* in U if $\delta V(X) = 0$ for all $X \in \mathfrak{X}^G(U)$.

Clearly, a stationary G-varifold must be G-stationary. Meanwhile, let

$$X_G := \int_G (g^{-1})_* X \, d\mu(g) \quad \text{for all } X \in \mathfrak{X}(U). \tag{2-2}$$

A direct computation shows $X_G \in \mathfrak{X}^G(U)$ and $\delta V(X) = \delta V(X_G)$ for any $V \in \mathcal{V}_n^G(M)$ (see [19, Lemma 2.2]). Hence, we have:

(2-3) $\quad V \in \mathcal{V}_n^G(M)$ is stationary in U if and only if it is G-stationary in U.

G-Sweepouts and G-width. To define the equivariant sweepouts and width, we need to introduce a technical assumption:

Definition 2.3. For any $\mathcal{F}$-continuous map $\Phi : [0,1] \to \mathcal{Z}_n^G(M;\mathbb{Z}_2)$, define

$$\boldsymbol{m}^G(\Phi, r) := \sup\{\|\Phi(x)\|(B_r^G(p)) : x \in [0,1],\, p \in M\},$$

where $B_r^G(p)$ is the geodesic r-neighborhood of $G \cdot p$ in M (or in N if $M \subset N$ has nonempty boundary). Then we say Φ has *no concentration of mass on orbits* if $\lim_{r\to 0} \boldsymbol{m}^G(\Phi, r) = 0$.

By (2-1) and a continuous argument, we have the following lemma (see [39, Lemma 8]), which is quite useful in Section 3.

Lemma 2.4. *If* $\Phi : [0,1] \to \mathcal{Z}_n^G(M;\mathbb{Z}_2)$ *is* $\boldsymbol{F}$*-continuous, then* Φ *has no concentration of mass on orbits and* $\sup_{x\in[0,1]} \boldsymbol{M}(\Phi(x)) < \infty$.

Closed G-manifolds. In this case, $\partial M = \varnothing$. Then for any $\mathcal{F}$-continuous closed curve $\Phi : [0,1] \to \mathcal{Z}_n^G(M;\mathbb{Z}_2)$, $\Phi(0) = \Phi(1)$, we can take $a_j = j/3^k$, $j = 0, 1, \dots, 3^k$ with $k \in \mathbb{N}$ large enough so that

$$\mathcal{F}(\Phi(x) - \Phi(y)) \le \epsilon_M \quad \text{for all } x, y \in [a_j, a_{j+1}], \tag{2-4}$$

where $\epsilon_M > 0$ is given by Lemma 2.2. By Lemma 2.2, there is $Q_j \in \boldsymbol{I}_{n+1}^G(M;\mathbb{Z}_2)$ with $\partial Q_j = \Phi(a_{j+1}) - \Phi(a_j)$ and $\boldsymbol{M}(Q_j) \le C_M \mathcal{F}(\Phi(a_{j+1}) - \Phi(a_j))$, where $j = 0, 1, \dots, 3^k - 1$. Therefore, $Q := \sum_{j=0}^{3^k-1} Q_j \in \boldsymbol{I}_{n+1}^G(M;\mathbb{Z}_2)$ satisfies $\partial Q = 0$, which indicates $Q = [\![M]\!]$ or 0 by the constancy theorem [36, 26.27]. Hence, we can correspond Φ to a homology class:

$$F_M(\Phi) := [Q] \in H_{n+1}(M^{n+1};\mathbb{Z}_2). \tag{2-5}$$

By the constancy theorem, $F_M(\Phi)$ does not depend on the choice of k. Moreover, by [39, Remark 2] and the arguments in [1], we have $F_M(\Phi) = F_M(\Phi')$ for any closed curve Φ' that is homotopic to Φ in $\mathcal{Z}_n^G(M;\mathcal{F};\mathbb{Z}_2)$, and F_M induces an isomorphism [39, Theorem 9]:

$$F_M : \pi_1(\mathcal{Z}_n^G(M;\mathbb{Z}_2)) \to H_{n+1}(M;\mathbb{Z}_2).$$

In the above, we do not need to specify the base point of $\pi_1(\mathcal{Z}_n^G(M;\mathbb{Z}_2))$. This is because $\mathcal{Z}_n^G(M;\mathbb{Z}_2)$ is the $\mathcal{F}$-path connected component of $\boldsymbol{I}_n^G(M;\mathbb{Z}_2)\cap\mathcal{Z}_n(M;\mathbb{Z}_2)$ containing 0 (by Lemma 2.2 and the contraction approach in [24, Claim 5.3]).

Definition 2.5 (G-sweepout). A closed $\mathcal{F}$-continuous curve $\Phi : S^1 \to \mathcal{Z}_n^G(M;\mathbb{Z}_2)$ is said to be a *G-sweepout of M* if $F_M(\Phi) = [M] \neq 0$.

Remark 2.6. Since $\mathcal{Z}_n^G(M;\mathbb{Z}_2)$ is $\mathcal{F}$-path connected, every two G-sweepouts are homotopic to each other in $\mathcal{Z}_n^G(M;\mathcal{F};\mathbb{Z}_2)$. Hence, the set of G-sweepouts of M is exactly the nontrivial homotopy class of closed curves in $\mathcal{Z}_n^G(M;\mathbb{Z}_2)$.

Next, we introduce the min-max G-width of M, which can be regarded as a critical value for the area functional with respect to *all* variations by (2-3).

Definition 2.7 (G-width). Let $\mathcal{P}^G(M)$ be the set of G-sweepouts of M with no concentration of mass on orbits. Then we define the *G-width* of M by

$$W^G(M) := \inf_{\Phi\in\mathcal{P}^G(M)} \sup_{x\in S^1} \boldsymbol{M}(\Phi(x)).$$

Compact G-manifolds with boundary. Now we consider the case that $\partial M \neq \varnothing$, and regard M as a compact domain of a closed Riemannian G-manifold N. Let F_N be given by (2-5), and $\nu_{\partial M}$ be the unit normal of ∂M pointing inward M. Then for $\eta > 0$ small enough, define

$$M_\eta := M \setminus \exp^\perp_{\partial M}([0, \eta) \cdot \nu_{\partial M}) = \{p \in M : \mathrm{dist}_M(p, \partial M) \geq \eta\}. \tag{2-6}$$

Let $\Phi_i : [0, 1] \to \mathcal{Z}^G_n(M; \mathbb{Z}_2)$, $i = 1, 2$, be two $\mathcal{F}$-continuous curve so that $\Phi_i(0) = [\![\partial M]\!]$ and $\Phi_i(1) = 0$. As the constructions in (2-4), we can associate Φ_i to $Q_i \in \boldsymbol{I}^G_{n+1}(M; \mathbb{Z}_2)$ with $\partial Q_i = [\![\partial M]\!]$. Then the constancy theorem implies $Q_i = [\![M]\!]$. Therefore, the curves product, i.e., joint curve, $\Phi_2^{-1} \cdot \Phi_1$ satisfies $F_N(\Phi_2^{-1} \cdot \Phi_1) = 0$, and thus $\Phi_2^{-1} \cdot \Phi_1$ is homotopic to 0 in $\mathcal{Z}^G_n(N; \mathcal{F}; \mathbb{Z}_2)$. Since $\mathrm{spt}(\Phi_i(x)) \subset M$ for all $x \in [0, 1]$ and $i = 1, 2$, we can apply the double cover argument in [24, Theorem 5.1] with Lemma 2.2 in place of [1, Corollary 1.14], and see the homotopy map between $\Phi_2^{-1} \cdot \Phi_1$ and 0 can be taken in $\mathcal{Z}^G_n(M; \mathcal{F}; \mathbb{Z}_2)$. Thus, Φ_1 and Φ_2 are homotopic to each other in $\mathcal{Z}^G_n(M; \mathcal{F}; \mathbb{Z}_2)$.

Next, we introduce the following definition for G-manifold with boundary, which is generalized from the smooth min-max setting [43, Definitions 2.1, 2.5].

Definition 2.8. Suppose M is a compact Riemannian G-manifold with boundary $\partial M \neq \varnothing$. Then we call a $\mathcal{F}$-continuous curve $\Phi : [0, 1] \to \mathcal{Z}^G_n(M; \mathbb{Z}_2)$ a *G-sweepout of* $(M, \partial M)$, if:

(i) $\Phi(0) = [\![\partial M]\!]$, $\Phi(1) = 0$.

(ii) There exist $\epsilon > 0$ and a smooth G-invariant function $w : [0, \epsilon] \times \partial M \to [0, \infty)$ with $w(0, \cdot) \equiv 0$ and $\frac{\partial}{\partial x} w(0, \cdot) > 0$, so that $\Phi(x)$, $x \in [0, \epsilon]$, is induced by the smooth G-hypersurface $\exp^\perp_{\partial M}(w(x, \cdot)\, \nu_{\partial M})$.

(iii) For any $x_0 \in (0, 1]$, there exists $\eta > 0$ so that $\mathrm{spt}(\Phi(x)) \Subset M_\eta$ for all $x \in [x_0, 1]$.

Denote by $\mathcal{P}^G(M, \partial M)$ the set of G-sweepouts of $(M, \partial M)$ with no concentration of mass on orbits. Then we define the *G-width of* $(M, \partial M)$ by

$$W^G(M, \partial M) := \inf_{\Phi \in \mathcal{P}^G(M, \partial M)} \sup_{x \in [0,1]} \boldsymbol{M}(\Phi(x)).$$

Remark 2.9. As we mentioned before, any two G-sweepouts Φ_1, Φ_2 of $(M, \partial M)$ are homotopic to each other in $\mathcal{Z}^G_n(M; \mathcal{F}; \mathbb{Z}_2)$. Moreover, by reparametrization, the foliation parts of Φ_i, $i = 1, 2$, are homotopic through $v_t := (1 - t) w_1 + t w_2$, where $t \in [0, 1]$ and $w_1, w_2 : \left[0, \frac{1}{3}\right] \times \partial M \to [0, \infty)$ are given by Definition 2.8(ii). The nonfoliation parts $\Phi_i \llcorner \left[\frac{1}{3}, 1\right]$ and $\exp^\perp_{\partial M}\left(v_t\left(\frac{1}{3}, \cdot\right) \nu_{\partial M}\right)$ are all in M_η for some $\eta > 0$, and thus the homotopy between these parts can be taken in $\mathcal{Z}^G_n(M_\eta; \mathcal{F}; \mathbb{Z}_2)$ (see the constructions in [24, Theorem 5.1] with Lemma 2.2). Therefore, we can take a homotopy map $H : [0, 1] \times [0, 1] \to \mathcal{Z}^G_n(M; \mathcal{F}; \mathbb{Z}_2)$ so that $H(0, \cdot) = \Phi_1$, $H(1, \cdot) = \Phi_2$, and for every $t \in [0, 1]$, $H(t, \cdot)$ is a G-sweepout of $(M, \partial M)$.

3. Equivariant min-max theory

In this section, we introduce the equivariant min-max constructions in [39] (see [40; 41] for modified versions). Then main purpose is to find an integral G-varifold $V \in \mathcal{V}_n^G(M)$ induced by a smooth embedded minimal G-hypersurface so that $\|V\|(M) = W^G(M)$ (or $W^G(M, \partial M)$ if $\partial M \neq \varnothing$). Since our definitions differ slightly from those in [40; 41], we shall outline the essential steps for the sake of completeness.

Throughout this section, let $\mathcal{P}^G = \mathcal{P}^G(M)$ or $\mathcal{P}^G(M, \partial M)$, $W^G = W^G(M)$ or $W^G(M, \partial M)$ depending on whether ∂M is empty. By reparametrization, we always assume the domain of $\Phi \in \mathcal{P}^G$ is $I = [0, 1]$, and if $\partial M \neq \varnothing$, then $\Phi \llcorner \big[0, \frac{1}{3}\big]$ are smooth G-hypersurfaces as in Definition 2.8(ii).

For any sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G$, define the *width* of $\{\Phi_i\}_{i\in\mathbb{N}}$ by

$$\boldsymbol{L}(\{\Phi_i\}_{i\in\mathbb{N}}) := \limsup_{i\to\infty} \sup_{x\in I} \boldsymbol{M}(\Phi_i(x)).$$

Then we say $\{\Phi_i\}_{i\in\mathbb{N}}$ is a *min-max* sequence if

$$\boldsymbol{L}(\{\Phi_i\}_{i\in\mathbb{N}}) = W^G.$$

The *image set* of $\{\Phi_i\}_{i\in\mathbb{N}}$ is defined by

$$\boldsymbol{\Lambda}(\{\Phi_i\}_{i\in\mathbb{N}}) := \Big\{ V \in \mathcal{V}_n^G(M) : V = \lim_{j\to\infty} |\Phi_{i_j}(x_{i_j})| \text{ for some } i_j \to \infty,\ x_{i_j} \in I \Big\}.$$

Moreover, we define the *critical set* of $\{\Phi_i\}_{i\in\mathbb{N}}$ by

$$\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) := \{V \in \boldsymbol{\Lambda}(\{\Phi_i\}_{i\in\mathbb{N}}) : \|V\|(M) = \boldsymbol{L}(\{\Phi_i\}_{i\in\mathbb{N}})\}.$$

Discrete min-max settings. To apply the equivariant min-max constructions in [39; 40], we need the following discrete notations. Since we only consider curves in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$, we will restrict the notations to the 1-parameter case.

Denote by $I := [0, 1]$. For any $j \in \mathbb{N}$, let $I(1, j)$ be the cube complex on I with 1-cells and 0-cells (vertices) given by

$$I(1, j)_1 := \{[0, 3^{-j}], [3^{-j}, 2\cdot 3^{-j}], \ldots, [1 - 3^{-j}, 1]\},$$
$$I(1, j)_0 := \{[0], [3^{-j}], \ldots, [1]\}.$$

The boundary homeomorphism ∂ is defined by $\partial[a, b] = [b] - [a]$. Then we denote by $I(2, j) = I(1, j) \otimes I(1, j)$ the cell complex on $I^2 = I \times I$. For any $\alpha = \alpha_1 \otimes \alpha_2 \in I(2, j)$ and $p \in \{0, 1, 2\}$, we say α is a *p-cell*, if $\dim(\alpha_1) + \dim(\alpha_2) = p$. Then the set of p-cells of $I(2, j)$ is denoted by $I(2, j)_p$, and the set of p-cells in $\alpha \in I(i, j)_q$ is denoted by α_p.

Let $J := \left[\frac{1}{3}, 1\right]$. Then we denote by $J(1, j)$ the cubical subcomplex containing all the cells of $I(1, j)$ supported in J. Similarly, the set of p-cells of $J(1, j)$ is denoted by $J(1, j)_p$ for $p \in \{0, 1, 2\}$.

Let $m \in \{1, 2\}$ and two vertices $x, y \in I(m, j)_0$, define $\boldsymbol{d}(x, y) := 3^j \sum_{i=1}^m |x_i - y_i|$. For any map $\phi : I(1, j)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, the ***M**-fineness* of ϕ is defined by

$$\boldsymbol{f_M}(\phi) := \sup\{\boldsymbol{M}(\phi(x) - \phi(y)) : \boldsymbol{d}(x, y) = 1,\ x, y \in I(1, j)_0\}.$$

Suppose $S = \{\varphi_i\}_{i \in \mathbb{N}}$ is a sequence of maps $\varphi_i : I(1, k_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ such that $k_i \to \infty$ and $\boldsymbol{f_M}(\varphi_i) \to 0$ as $i \to \infty$. Then we use the following notations:

$$\boldsymbol{L}(S) := \limsup_{i \to \infty} \max_{x \in I(1, k_i)_0} \boldsymbol{M}(\varphi_i(x)),$$

$$\boldsymbol{\Lambda}(S) := \left\{ V \in \mathcal{V}_n^G(M) : V = \lim_{j \to \infty} |\varphi_{i_j}(x_{i_j})| \text{ for some } i_j \to \infty,\ x_{i_j} \in I(1, k_i)_0 \right\},$$

$$\boldsymbol{C}(S) := \{V \in \boldsymbol{\Lambda}(S) : \|V\|(M) = \boldsymbol{L}(S)\}.$$

For any $i, j \in \mathbb{N}$, let $\boldsymbol{n}(i, j) : I(1, i)_0 \to I(1, j)_0$ be the nearest projection, i.e.,

$$\boldsymbol{d}(x, \boldsymbol{n}(i, j)(x)) = \inf\{\boldsymbol{d}(x, y) : y \in I(m, j)_0\}.$$

Then we define the discrete homotopy:

Definition 3.1. Given $\phi_i : I(1, k_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, $i = 1, 2$, we say ϕ_1 and ϕ_2 are 1*-homotopic in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with **M**-fineness* δ if there exists a map

$$\psi : I(1, k)_0 \times I(1, k)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$$

for some $k \geq \max\{k_1, k_2\}$ such that $\boldsymbol{f_M}(\psi) < \delta$ and $\psi([i-1], x) = \phi_i(\boldsymbol{n}(k, k_i)(x))$ for $i \in \{1, 2\}$ and $x \in I(1, k)_0$.

Definition 3.2. A sequence of mappings $S = \{\phi_i\}_{i \in \mathbb{N}}$, $\phi_i : I(1, k_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$, is said to be a

$$(1, \boldsymbol{M})\text{-}\textit{homotopy sequence of mappings into } \mathcal{Z}_n^G(M; \mathbb{Z}_2)$$

if ϕ_i and ϕ_{i+1} are 1-homotopic in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness δ_i such that

(i) $\lim_{i \to \infty} \delta_i = 0$,

(ii) $\sup\{\boldsymbol{M}(\phi_i(x)) : x \in I(1, k_i)_0,\ i \in \mathbb{N}\} < +\infty$.

Definition 3.3. Let $S^j = \{\phi_i^j\}_{i \in \mathbb{N}}$, $j = 1, 2$, be two $(1, \boldsymbol{M})$-homotopy sequences of mappings into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$. Then S^1 and S^2 are *homotopic in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$* if there exists a sequence $\{\delta_i\}_{i \in \mathbb{N}}$ such that

(i) ϕ_i^1 is 1-homotopic to ϕ_i^2 in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness δ_i,

(ii) $\lim_{i \to \infty} \delta_i = 0$.

By the following discretization theorem from [39, Theorem 2], we can generate a $(1, \boldsymbol{M})$-homotopy sequence of mappings into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ from any $\Phi \in \mathcal{P}^G$.

Theorem 3.4 (discretization theorem). *Let $\Phi : I \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ be a continuous map in the flat topology so that $\sup_{x \in I} \boldsymbol{M}(\Phi(x)) < \infty$ and Φ has no concentration of mass on orbits. Then there exists a sequence of maps*

$$\phi_i : I(1, j_i)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2),$$

with $j_i < j_{i+1}$, and a sequence $\{\delta_i > 0\}_{i \in \mathbb{N}}$ converging to zero such that:

(i) *$S = \{\phi_i\}_{i \in \mathbb{N}}$ is a $(1, \boldsymbol{M})$-homotopy sequence of mappings into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness $\boldsymbol{f_M}(\phi_i) < \delta_i$.*

(ii) *There exists some sequence $k_i \to +\infty$ such that for all $x \in I(1, j_i)_0$,*

$$\boldsymbol{M}(\phi_i(x)) \le \sup\{\boldsymbol{M}(\Phi(y)) : \alpha \in I(1, k_i)_1, x, y \in \alpha\} + \delta_i,$$

which implies $\boldsymbol{L}(S) \le \sup_{x \in I} \boldsymbol{M}(\Phi(x))$.

(iii) $\sup\{\mathcal{F}(\phi_i(x) - \Phi(x)) : x \in I(1, j_i)_0\} \le \delta_i$.

(iv) $\Phi(0) = \phi_i([0]) = \psi_i(\cdot, [0])$ *and* $\Phi(1) = \phi_i([1]) = \psi(\cdot, [1])$, *where ψ_i is the discrete homotopy map of ϕ_i and ϕ_{i+1} with $\psi_i([0], \boldsymbol{n}(\cdot)) = \phi_i$ and $\psi_i([1], \boldsymbol{n}(\cdot)) = \phi_{i+1}$.*

Moreover, let $K \subset M$ be a compact G-invariant domain with smooth boundary. Then for any $j \in \mathbb{N}$ and $\alpha \in I(1, j)_1$, if $\operatorname{spt}(\Phi(x)) \subset K$ *for all $x \in \alpha$, then we can further make* $\operatorname{spt}(\phi_i(x)) \subset K$ *for all $x \in \alpha \cap I(1, j_i)_0$.*

Proof. The statements in (i)–(iii) follow directly from [39, Theorem 2]. Note that the proof of [39, Theorem 2] is basically the combinatorial approach in [21, Theorem 13.1] with Lemma 2.2 in place of [1, Corollary 1.14] and $\operatorname{dist}(G \cdot p, \cdot)$ in place of $\operatorname{dist}(p, \cdot)$. Meanwhile, since the maps are defined on the 1-dimensional cubical complex, statement (iv) follows from [21, Proposition 13.5(ii)] and the combinatorial constructions of [21, Theorem 13.1(iv)]. Moreover, these cut-and-paste and combinatorial arguments would also carry over in the case $\partial M \neq \varnothing$ by restricting in the compact domain $M \subset N$, and thus (i)–(iv) are still valid when M has boundary. Finally, if K and $\alpha \in I(1, j)$ are given as in the last statement. Then we can apply the above discretization result to $\Phi \llcorner \alpha$ in K and $\Phi \llcorner (I \setminus \operatorname{int}(\alpha))$ in M respectively. Note the boundary values are unchanged by (iv). Hence, the discrete maps defined in α and $I \setminus \operatorname{int}(\alpha)$ can be connected together, which gives the last statement. □

The following interpolation theorem (see [39, Theorem 3]) indicates that a $\boldsymbol{M}$-continuous map into $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ can be generated from a discrete map with small $\boldsymbol{M}$-fineness.

Theorem 3.5 (interpolation theorem). *For $m = 1, 2$, there exists a positive constant $C_0 = C_0(M, G, m)$ so that if $\phi : I(m, k)_0 \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ has $\boldsymbol{f_M}(\phi) < \epsilon_M$ with $\epsilon_M > 0$ given in Lemma 2.2, then there exists a map*

$$\Phi : I^m \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$$

continuous in the $\boldsymbol{M}$-topology satisfying:

(i) $\Phi(x) = \phi(x)$ *for all* $x \in I(m, k)_0$.

(ii) *If α is some j-cell in $I(m, k)$, then Φ restricted to α depends only on the values of ϕ assumed on the vertices of α.*

(iii) $\sup\{\boldsymbol{M}(\Phi(x) - \Phi(y)) : x, y$ *lie in a common cell of* $I(m, k)\} \le C_0\, \boldsymbol{f_M}(\phi)$.

(iv) *For any $\alpha \in I(m, k)_j$, if $\phi \llcorner \alpha_0 \equiv T \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is a constant, then $\Phi \llcorner \alpha \equiv T$.*

We call the map Φ in Theorem 3.5 the *Almgren G-extension* of ϕ.

Proof. The statements in (i)–(iii) follow directly from [39, Theorem 3]. If $\partial M \neq \varnothing$, then the constructions in [40, Theorem 4.13] would carry over with $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$ and Lemma 2.2 in place of $\mathcal{Z}_n^G(M, \partial M; \mathbb{Z}_2)$ and [40, Lemma 3.10]. If $\phi \llcorner \alpha_0 \equiv T \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ is a constant for some j-cell α, then for any 1-cell $\gamma_1 = [a, b] \in \alpha_1$, the isoperimetric choice $Q(\gamma_1)$ of $\phi(a)$ and $\phi(b)$ (Lemma 2.2) must be 0. Hence, for any cell $\beta \subset \alpha$, the map h_β constructed in [40, Theorem 4.13] is 0 implying $\Phi \llcorner \alpha \equiv T$ [1, 4.5]. □

Using the discretization/interpolation Theorems 3.4 and 3.5, we have the following corollary (see [39, Corollary 1]):

Corollary 3.6. *Let $\Phi : I \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ be a $\mathcal{F}$-continuous map with no concentration of mass on orbits and $\sup_{x \in I} \boldsymbol{M}(\Phi(x)) < \infty$. Suppose $S = \{\phi_i\}_{i \in \mathbb{N}}$ is given by Theorem 3.4 applied to Φ, and Φ_i is the Almgren G-extension of ϕ_i given by Theorem 3.5 for every i sufficiently large. Then*:

(i) *For each i large enough, a* ***relative*** *homotopy map $H_i : I^2 \to \mathcal{Z}_n^G(M; \mathcal{F}; \mathbb{Z}_2)$ exists with $H_i(0, \cdot) = \Phi$, $H_i(1, \cdot) = \Phi_i$, $H_i(\cdot, 0) \equiv \Phi(0) = \Phi_i(0)$, and $H_i(\cdot, 1) \equiv \Phi(1) = \Phi_i(1)$.*

(ii) $\boldsymbol{L}(\{\Phi_i\}_{i \in \mathbb{N}}) = \boldsymbol{L}(S) \le \sup_{x \in I} \boldsymbol{M}(\Phi(x))$.

Proof. Using Theorem 3.5 and the arguments in [1], we see that [1, Theorem 8.2] is valid in our G-invariant settings (even if ∂M may be nonempty). Hence, the proof of [23, Corollary 3.9] would carry over with Theorems 3.4 and 3.5 in place of [23, Theorem 3.6]. Thus, Φ_i is homotopic to Φ in $\mathcal{Z}_n^G(M; \mathcal{F}; \mathbb{Z}_2)$ for i-large, and (ii) is valid. Also, by (iv) in Theorems 3.4 and 3.5, we have $\Phi(0) = \phi_i([0]) = \Phi_i(0)$ and $\Phi(1) = \phi_i([1]) = \Phi_i(1)$ for all i-large. So, combining (iv) in Theorems 3.4 and 3.5 with the homotopy constructions in [23, Propositions 3.3, 3.8], one easily verifies that the homotopy map H_i of Φ and Φ_i is relative to the boundary values. □

Let $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G$ be any min-max sequence. If $\partial M = \varnothing$, then we can apply Corollary 3.6 to each Φ_i and obtain a sequence of $\boldsymbol{M}$-continuous curves $\{\Phi^i_j\}_{j\in\mathbb{N}}$ relative homotopic to Φ_i in $\mathcal{Z}^G_n(M;\mathcal{F};\mathbb{Z}_2)$ and $\boldsymbol{L}(\{\Phi^i_j\}_{j\in\mathbb{N}}) \le \sup_{x\in I}\boldsymbol{M}(\Phi_i(x))$. Choose $j(i)$ sufficiently large so that $\sup_{x\in I}\boldsymbol{M}(\Phi^i_{j(i)}(x)) \le \sup_{x\in I}\boldsymbol{M}(\Phi_i(x)) + \frac{1}{i}$. Hence, we have $\{\Phi^i_{j(i)}\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M)$ is a min-max sequence continuous in the $\boldsymbol{M}$-topology and so in the $\boldsymbol{F}$-topology.

For the case $\partial M \neq \varnothing$, we can apply the above arguments to each $\Phi_i \llcorner J$, where $J := \left[\frac{1}{3}, 1\right]$, in a G-submanifold M_{η_i} given by Definition 2.8(iii) with $x_0 = \frac{1}{3}$, and get $\Phi^i_{j(i)} : J \to \mathcal{Z}^G_n(M_{\eta_i};\boldsymbol{M};\mathbb{Z}_2)$ satisfying

- $\Phi^i_{j(i)}$ is relative homotopic to $\Phi_i \llcorner J$ in $\mathcal{Z}^G_n(M_{\eta_i};\mathcal{F};\mathbb{Z}_2)$,
- $\sup_{x\in J}\boldsymbol{M}(\Phi^i_{j(i)}(x)) \le \sup_{x\in J}\boldsymbol{M}(\Phi_i(x)) + \frac{1}{i}$.

Since the homotopy map of $\Phi^i_{j(i)}$ and $\Phi_i \llcorner J$ is relative to the boundary values, we can define $\Phi^i_{j(i)} \llcorner \left[0, \frac{1}{3}\right] = \Phi_i \llcorner \left[0, \frac{1}{3}\right]$, and see that $\{\Phi^i_{j(i)}\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ is an $\boldsymbol{F}$-continuous min-max sequence.

Therefore, the above arguments give the following corollary, which implies we only need to consider the $\boldsymbol{F}$-continuous G-sweepouts.

Corollary 3.7. *The G-width defined in Definitions 2.7 and 2.8 satisfies*

$$W^G = \inf\left\{\sup_{x\in I}\boldsymbol{M}(\Phi(x)) : \Phi \in \mathcal{P}^G \textit{ is } \boldsymbol{F}\textit{-continuous}\right\}.$$

Min-max theorems. We now use the min-max method to construct a minimal G-hypersurface (with multiplicity) so that the width W^G is realized by its area.

Closed G-manifolds. For the case that M is closed, it follows from Remark 2.6 and Corollary 3.7 that $\boldsymbol{\Pi} := \{\Phi \in \mathcal{P}^G(M)$ is $\boldsymbol{F}$-continuous$\}$ is a *continuous G-homotopy class* in the sense of [39, Definition 5], and $W^G(M) = \boldsymbol{L}(\boldsymbol{\Pi})$ in the sense of [39, Definition 6]. Hence, we have the following min-max theorem by [39, Theorem 8]. (Note that the assumptions on $M \setminus M^{\text{prin}}$ in [39, Theorem 8] can be removed by the modifications in [40], and the dimension assumption is modified in [41, Theorem 5.1].)

Theorem 3.8. *Suppose M is closed, i.e., $\partial M = \varnothing$, and $3 \le \operatorname{codim}(G\cdot p) \le 7$ for all $p \in M$. Then there exists an integral G-varifold $V \in \mathcal{V}^G_n(M)$ so that*

$$\|V\|(M) = W^G(M) \quad \text{and} \quad V = \sum_{i=1}^m n_i|\Sigma_i|,$$

where $m, n_i \in \mathbb{N}$, $\{\Sigma_i\}_{i=1}^m$ are disjoint G-connected (Definition 4.4) smooth embedded closed minimal G-hypersurfaces. Moreover, if Σ_i does not admit a G-invariant unit normal vector field, then n_i is an even number.

Proof. We only need to show the last statement since the existence and regularity of V are given by [39, Theorem 8] (see also [41, Theorem 5.1]). Note that the min-max varifold V is $(G, \mathbb{Z}_2)$-almost minimizing in annuli of *boundary-type* in the sense of [39, Definitions 10, 11]. Hence, for each Σ_i, we can take a small G-tube $B^G_{2r}(p)$ with center $G\cdot p\subset\Sigma_i$ and $r\in\big(0,\frac{1}{2}\mathrm{inj}(G\cdot p)\big)$ so that:

- V is $(G,\mathbb{Z}_2)$-almost minimizing of boundary-type in $B^G_{2r}(p)$.
- $B^G_t(p)$ has mean convex boundary for all $t\in(0,2r)$.
- $B^G_{2r}(p)\cap\mathrm{spt}(\|V\|)\subset\Sigma_i$, and $\partial B^G_r(p)$ is transversal to Σ_i.

Then by the constructions [39, Proposition 2, 3] and the consistency [41, Proposition 4.19] of G-replacements, there exists a sequence $\{T_j\}_{j\in\mathbb{N}}\subset\mathcal{Z}^G_n(M;\mathbb{Z}_2)$ so that:

(1) $T_j=\partial Q_j$ is locally mass minimizing in $B^G_r(p)$ with $Q_j\in\boldsymbol{I}^G_{n+1}(M;\mathbb{Z}_2)$.

(2) $|T_j|\to V$ in the sense of varifolds.

By compactness, let $T_j\to T=\partial Q$ in the flat topology with $Q\in\boldsymbol{I}^G_{n+1}(M;\mathbb{Z}_2)$. Thus, we have $\mathrm{spt}(T)\subset\mathrm{spt}(\|V\|)=\cup_{i=1}^m\Sigma_i$, which implies $T=\sum_{i=1}^m n_i'[\![\Sigma_i]\!]$ for some $n_i'\in\mathbb{Z}_2$ by the Constancy Theorem. By regarding $Q\in\boldsymbol{I}^G_{n+1}(M;\mathbb{Z}_2)$ as a G-invariant Caccioppoli set whose boundary is induced by smooth G-hypersurfaces $\{\Sigma_i:1\le i\le m, n_i'=1\}$, we see ∂Q admits an inward unit normal that is also G-invariant. Hence, $n_i'=0$ provided that Σ_i does not admit a G-invariant unit normal. Now we can use the slicing theory [36, 28.5] to take $s\in\big(\frac{r}{2},r\big)$ so that $\boldsymbol{M}\big(\partial(T_j\llcorner B^G_s(p))\big)$ are uniformly bounded, and thus $T_j\llcorner B^G_s(p)$ converges up to a subsequence. Finally, by (1) we know [42, Theorem 1.1] indicates that $n_i\equiv n_i'$ mod 2, and thus the multiplicity n_i must be even for Σ_i without a G-invariant unit normal. □

Compact G-manifolds with boundary. Now we consider the case that $\partial M\neq\varnothing$. In this case, we make the assumption that

$$H_{\partial M}>0\quad\text{and}\quad W^G(M,\partial M)>\mathrm{Area}(\partial M), \tag{3-1}$$

where $H_{\partial M}$ is the mean curvature of ∂M with respect to the inward unit normal $\nu_{\partial M}$. By Corollary 3.7, we can take a min-max sequence $\{\Phi_i^*\}_{i\in\mathbb{N}}\subset\mathcal{P}^G(M,\partial M)$ that are continuous in the $\boldsymbol{F}$-topology. The strategy is to use the following proposition to deform $\{\Phi_i^*\}_{i\in\mathbb{N}}$ into a new $\boldsymbol{F}$-continuous min-max sequence so that every $V\in\boldsymbol{C}(\{\Phi_i^*\}_{i\in\mathbb{N}})$ is supported in a G-invariant subdomain $M_a\Subset M$. With this benefit, the min-max constructions can be restricted in the interior of M to build a closed minimal G-hypersurface realizing the width $W^G(M,\partial M)$. This deformation approach is based on the idea of [20, Lemma 2.2] and we list the details here for the sake of completeness.

Proposition 3.9. *Let $\partial M \neq \varnothing$ satisfy (3-1). Then there exist a constant $a > 0$ and a min-max sequence $\{\Phi_i^*\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ continuous in the $\boldsymbol{F}$-topology so that*

$$\operatorname{spt}(\Phi_i^*(x)) \Subset M_a := \{p \in M : \operatorname{dist}_M(p, \partial M) \geq a\} \quad \text{for any } x \in I,$$

with $\boldsymbol{M}(\Phi_i^*(x)) \geq W^G(M, \partial M) - \delta$ and $\delta = \frac{1}{4}(W^G(M, \partial M) - \operatorname{Area}(\partial M))$.

Proof. Let $a > 0$ be small enough so that $d := \operatorname{dist}_M(\partial M, \cdot)$ is a G-invariant smooth function in a $4a$-neighborhood of ∂M. By (3-1), we can set $a > 0$ even smaller so that for any $r \in [0, 3a]$, $\partial M_r = d^{-1}(r)$ has positive mean curvature H_r with respect to the inner unit normal ∇d. Denote by A_r the second fundamental form of ∂M_r, and $c = \sup_{r\in[0,3a],\, p\in\partial M_r} |A_r|(p)$. Then we take the function $\phi \geq 0$ as in [20, Lemma 2.2] so that

$$\phi' + c\phi \leq 0, \qquad \phi(r) > 0 \quad \text{for } r < 2a, \qquad \phi(r) = 0 \quad \text{for } r \geq 2a.$$

For any $p \in \operatorname{int}(M) \setminus M_{3a}$ and n-subspace $S \subset T_p M$, let $\{e_i\}_{i=1}^n$ be an orthonormal basis of S, and $P : T_p M \to T_p \partial M_{d(p)}$ be the projection. Since we have that $\dim(S \cap T_p \partial M_{d(p)}) \geq n-1$, we can assume $\{e_i\}_{i=1}^{n-1} \cup \{e^*\}$ gives an orthonormal basis of $T_p \partial M_{d(p)}$, where e^* satisfies $\langle e^*, P(e_n)\rangle = |P(e_n)|$. Noting $\nabla d \perp T_p \partial M_{d(p)}$ and $\nabla_{\nabla d} \nabla d = 0$, we have

$$\begin{aligned}
\text{(3-2)} \qquad \operatorname{div}_S(\phi \nabla d) &= \phi'(d(p)) \cdot \langle e_n, \nabla d\rangle^2 + \phi(d(p)) \cdot \sum_{i=1}^n \langle \nabla_{e_i} \nabla d, e_i\rangle \\
&= \phi' \langle e_n, \nabla d\rangle^2 - \phi \sum_{i=1}^n A_{d(p)}(P(e_i), P(e_i)) \\
&= (\phi' + \phi A_{d(p)}(e^*, e^*))\langle e_n, \nabla d\rangle^2 - \phi H_{d(p)} \\
&\leq (\phi' + c\phi)\langle e_n, \nabla d\rangle^2 - \phi H_{d(p)} \\
&\leq 0.
\end{aligned}$$

We can take any $\boldsymbol{F}$-continuous min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ by Corollary 3.7. Then for each Φ_i, there exist $\epsilon_i > 0$ and $\eta_i \in \left(0, \frac{a}{8}\right)$ so that:

(1) $\Phi_i \llcorner [0, 4\epsilon_i]$ are smooth G-hypersurfaces with $\boldsymbol{M}(\Phi_i(x)) \leq \operatorname{Area}(\partial M) + \delta$ for all $x \in [0, 4\epsilon_i]$.

(2) $\operatorname{spt}(\Phi_i(x)) \Subset M_{2\eta_i}$ for all $x \in [\epsilon_i, 1]$.

Let κ_i be a cut-off function so that $\kappa_i(r) = 0$ for $r \leq \eta_i$ and $\kappa_i(r) = 1$ for $r \geq 2\eta_i$. Then the G-vector field $X_i := \kappa_i(d)\,\phi(d)\nabla d$ generates G-equivariant diffeomorphisms $\{F_t^i\}$. By (2) and (3-2), for any $x \in [\epsilon_i, 1]$ and $t_0 \geq 0$, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=t_0} \boldsymbol{M}((F_t^i)_\# \Phi_i(x)) &= \frac{d}{dt}\Big|_{t=0} \|(F_t^i)_\#(F_{t_0}^i)_\# \Phi_i(x)\|(M) \\
&= \int \operatorname{div}_S(X_i)\, dV_{t_0,x} = \int \operatorname{div}_S(\phi\nabla d)\, dV_{t_0,x} \leq 0,
\end{aligned}$$

where $V_{t_0,x} := |(F^i_{t_0})_\# \Phi_i(x)| \in \mathcal{V}^G_n(M_{2\eta_i})$. Therefore,

$$\boldsymbol{M}((F^i_t)_\# \Phi_i(x)) \le \boldsymbol{M}(\Phi_i(x)) \quad \text{for all } x \in [\epsilon_i, 1],\ t \ge 0. \tag{3-3}$$

Since $M_{2\eta_i} \setminus M_{2a} \subset \operatorname{spt}(X_i) \subset M_{\eta_i} \setminus \operatorname{int}(M_{2a})$, we see $\lim_{t\to\infty} F^i_t(p) \in \partial M_{2a}$ for any $p \in M_{2\eta_i} \setminus M_{2a}$, and thus $F^i_{T_i}(M_{2\eta_i}) \subset M_a$ for some $T_i > 0$. Choose a smooth function $h_i : [0,1] \to [0, T_i]$ with $h_i \llcorner [0, \epsilon_i] = 0$, $h_i \llcorner [2\epsilon_i, 1] = T_i$. Then $\Phi^*_i(x) := (F^i_{h_i(x)})_\# \Phi_i(x)$ satisfies:

(a) $\Phi^*_i(x) = \Phi_i(x)$ for $x \in [0, \epsilon_i]$ (since $h_i = 0$).

(b) $\boldsymbol{M}(\Phi^*_i(x)) \le \boldsymbol{M}(\Phi_i(x))$ for all $x \in [\epsilon_i, 1]$ (by (3-3)).

(c) $\operatorname{spt}(\Phi^*_i(x)) \Subset M_a$ for all $x \in [2\epsilon_i, 1]$ (by (2) and the definitions of T_i, h_i).

Clearly, $\{\Phi^*_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ is also an $\boldsymbol{F}$-continuous min-max sequence. Additionally, if $\boldsymbol{M}(\Phi^*_i(x)) \ge W^G(M, \partial M) - \delta \ge \operatorname{Area}(\partial M) + \delta$, then $x \in (4\epsilon_i, 1]$ by (1), (a) and (b), and thus $\operatorname{spt}(\Phi^*_i(x)) \Subset M_a$ by (c). □

Next, we use the pull-tight argument to make every $V \in \boldsymbol{C}(\{\Phi^*_i\}_{i\in\mathbb{N}})$ stationary in M. By Proposition 3.9, the pull-tight procedure can be restricted in $\operatorname{int}(M_a)$.

Proposition 3.10. *Suppose that $\partial M \ne \varnothing$ satisfies the inequalities (3-1) and that $\delta := \frac{1}{4}(W^G(M, \partial M) - \operatorname{Area}(\partial M))$. Suppose $a > 0$ and $\{\Phi^*_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ are given by Proposition 3.9. Then there is an $\boldsymbol{F}$-continuous min-max sequence $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ with:*

(i) $\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \boldsymbol{C}(\{\Phi^*_i\}_{i\in\mathbb{N}}) \cap \mathcal{V}^G_n(M_a)$.

(ii) *Every G-varifold $V \in \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ is stationary in M.*

(iii) *If $\boldsymbol{M}(\Phi_i(x)) \ge W^G(M, \partial M) - \delta$, then* $\operatorname{spt}(\Phi_i(x)) \Subset M_{a/2}$.

Proof. Let $C := \sup_{i\in\mathbb{N}} \sup_{x\in I} \boldsymbol{M}(\Phi^*_i(x)) < \infty$ and $\mathring{M}_{a/2} := \operatorname{int}(M_{a/2})$ be a G-invariant open set of M. Define then $A := \{V \in \mathcal{V}^G_n(M) : \|V\|(M) \le C\}$ and

$$A_0 := \{V \in A : V \text{ is stationary in } \mathring{M}_{a/2}\}.$$

Since G acts by isometries, A and A_0 are compact subset of $\mathcal{V}^G_n(M)$. Additionally, for any $V \in A$, it follows from (2-2) that $V \in A_0$ if and only if $\delta V(X) = 0$ for all $X \in \mathfrak{X}^G(\mathring{M}_{a/2})$. Hence, we can follow [21, p. 765] (or [30, p. 153]) with $\mathfrak{X}^G(\mathring{M}_{a/2})$ in place of $\mathfrak{X}(M)$ to define a continuous map $X : A \to \mathfrak{X}^G(\mathring{M}_{a/2})$ and a continuous function $\eta : A \to [0, 1]$ satisfying:

- $X(V) = 0$ and $\eta(V) = 0$ if $V \in A_0$.
- $\delta V(X(V)) < 0$ and $\eta(V) > 0$ if $V \in A \setminus A_0$.
- $\|(f^{X(V)}_t)_\# V\|(M) < \|(f^{X(V)}_s)_\# V\|(M)$ for all $V \in A$ and $0 \le s < t \le \eta(V)$,

where $\{f_t^{X(V)}\}$ are the equivariant diffeomorphisms generated by $X(V)$. Define

$$H : I \times \{T \in \mathcal{Z}_n^G(M; \boldsymbol{F}; \mathbb{Z}_2) : \boldsymbol{M}(T) \leq C\} \to \{T \in \mathcal{Z}_n^G(M; \boldsymbol{F}; \mathbb{Z}_2) : \boldsymbol{M}(T) \leq C\},$$
$$H(t, T) := (f^{X(|T|)}_{\eta(|T|)t})_{\#} T.$$

One easily verifies $H(0, T) = T$ for all $T \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ with $\boldsymbol{M}(T) \leq C$, and that:

- If $|T|$ is stationary in $\mathring{M}_{a/2}$, then $H(t, T) = T$ for all $t \in [0, 1]$.
- If $|T|$ is not stationary in $\mathring{M}_{a/2}$, then $\boldsymbol{M}(H(1, T)) < \boldsymbol{M}(T)$.

Let $\Phi_i := H(1, \Phi_i^*)$. Note $X(V)$ is supported in $\mathring{M}_{a/2}$ and $f_t^{X(V)} \llcorner (M \setminus \mathring{M}_{a/2}) = id$. Hence, Φ_i is also a G-sweepout of $(M, \partial M)$. Additionally, by the above constructions, one easily verifies that $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M; \partial M)$ is a min-max sequence continuous in the $\boldsymbol{F}$-topology, and $\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \boldsymbol{C}(\{\Phi_i^*\}_{i\in\mathbb{N}}) \cap A_0$. Moreover, it follows from Proposition 3.9 that $\boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \mathcal{V}_n^G(M_a) \cap A_0$, which implies every $V \in \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ is stationary in M. Finally, since the deformations $f_t^{X(V)}$ are restricted in $\mathring{M}_{a/2}$, the last bullet follows directly from Proposition 3.9 and the above constructions. □

Finally, we can now show the equivariant min-max theorem for compact G-manifold M with boundary ∂M satisfying (3-1). The proof is generally the approach in [22, Theorem 3.8], and we list some necessary modifications.

Theorem 3.11. *Suppose $\partial M \neq \varnothing$ satisfies inequality (3-1), and $3 \leq \operatorname{codim}(G\cdot p) \leq 7$ for all $p \in M$. Then there exists an integral G-varifold $V \in \mathcal{V}_n^G(M)$ so that $\|V\|(M) = W^G(M, \partial M)$ and $V = \sum_{i=1}^m n_i |\Sigma_i|$, where $m, n_i \in \mathbb{N}$, $\{\Sigma_i\}_{i=1}^m$ are disjoint smooth embedded closed minimal G-hypersurfaces in the interior of M.*

Proof. Let $a > 0$ and $\{\Phi_i\}_{i\in\mathbb{N}} \subset \mathcal{P}^G(M, \partial M)$ be given by Proposition 3.10 so that every $V \in \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ is stationary in M and compactly supported in $\operatorname{int}(M_{a_0})$ for $a_0 = \frac{a}{2}$. Let $\delta = \frac{1}{4}(W^G(M, \partial M) - \operatorname{Area}(\partial M)) > 0$. Then by reparametrization, we assume $\Phi_i \llcorner \left[0, \frac{1}{3}\right]$ foliates a neighborhood of ∂M so that

$$\boldsymbol{M}(\Phi_i(x)) \leq \operatorname{Area}(\partial M) + \delta = W^G(M, \partial M) - 3\delta \quad \text{for all } x \in \left[0, \tfrac{1}{3}\right]. \tag{3-4}$$

Recall that $J = \left[\frac{1}{3}, 1\right]$. Denote by

$$\Phi_i' := \Phi_i \llcorner J.$$

By Definition 2.8, there exists $\eta_i \in (0, a_0)$ satisfying $\operatorname{spt}(\Phi_i'(x)) \Subset M_{\eta_i}$ for all $x \in J$. Additionally, since the map $x \mapsto \boldsymbol{M}(\Phi_i'(x))$ is continuous (by (2-1)), we can take $k_i \in \mathbb{N}$ large enough so that $|\boldsymbol{M}(\Phi_i'(x)) - \boldsymbol{M}(\Phi_i'(y))| \leq \frac{\delta}{4}$ for all x, y in a common 1-cell of $J(1, k_i)$. Denote by U_i the union of 1-cells $\alpha \in J(1, k_i)_1$ with

$\boldsymbol{M}(\Phi_i'(x)) \leq W^G(M, \partial M) - \frac{3\delta}{4}$ for all $x \in \alpha$, and $V_i := J \setminus U_i$. Therefore, by Proposition 3.10(iii), we have

$$\boldsymbol{M}(\Phi_i'(x)) \geq W^G(M, \partial M) - \delta \quad \text{and} \quad \operatorname{spt}(\Phi_i'(x)) \subset M_{a_0} \qquad \text{for all } x \in V_i.$$

By Lemma 2.4, we can apply Theorem 3.4 to each Φ_i' in the G-submanifold M_{η_i} and obtain a sequence of maps $\phi_j^i : J(1, k_j^i)_0 \to \mathcal{Z}_n^G(M_{\eta_i}; \mathbb{Z}_2)$ with $k_j^i < k_{j+1}^i$, $j \in \mathbb{N}$. The last statement in Theorem 3.4 indicates $\{\phi_j^i\}_{j\in\mathbb{N}}$ can be chosen to satisfy $\operatorname{spt}(\phi_j^i(x)) \subset M_{a_0}$ for all $x \in V_i \cap J(1, k_j^i)_0$. Moreover, we claim that:

Claim 1. *For j large enough*, $\operatorname{spt}(\phi_j^i(x)) \subset M_{a_0}$ *if* $\boldsymbol{M}(\phi_j^i(x)) \geq W^G(M, \partial M) - \frac{\delta}{2}$.

Proof of Claim 1. By the continuity of $x \mapsto \boldsymbol{M}(\Phi_i'(x))$ and Theorem 3.4(ii), if $\boldsymbol{M}(\phi_j^i(x)) \geq W^G(M, \partial M) - \frac{\delta}{2}$, then we have $\boldsymbol{M}(\Phi_i'(x)) > W^G(M, \partial M) - \frac{3\delta}{4}$ for j large enough. Thus, such vertex x must be in V_i, so $\operatorname{spt}(\phi_j^i(x)) \subset M_{a_0}$. □

Additionally, we also have the following equality due to the lower semicontinuity of mass, the continuity of $x \mapsto \boldsymbol{M}(\Phi_i'(x))$ and Theorem 3.4(ii)-(iii):

$$\lim_{j\to\infty} \sup\{\boldsymbol{F}(\phi_j^i(x), \Phi_i'(x)) : x \in J(1, k_j^i)_0\} = 0. \tag{3-5}$$

Let $\Phi_j^i : J \to \mathcal{Z}_n^G(M_{\eta_i}; \boldsymbol{M}; \mathbb{Z}_2)$ be the Almgren G-extension of ϕ_j^i given by Theorem 3.5 for j-large. By Corollary 3.6, Φ_j^i and Φ_i' are *relative* homotopic in $\mathcal{Z}_n^G(M_{\eta_i}; \mathcal{F}; \mathbb{Z}_2)$. Therefore,

$$\tilde{\Phi}_i^j(x) := \begin{cases} \Phi_i(x), & x \in \left[0, \frac{1}{3}\right], \\ \Phi_i^j(x), & x \in J = \left[\frac{1}{3}, 1\right] \end{cases}$$

is a well-defined $\boldsymbol{F}$-continuous G-sweepout of $(M, \partial M)$ for each $i \in \mathbb{N}$ and j-large, and thus

$$\begin{aligned} W^G(M, \partial M) \leq \boldsymbol{L}(\{\tilde{\Phi}_j^i\}_{j\in\mathbb{N}}) &= \boldsymbol{L}(\{\Phi_j^i\}_{j\in\mathbb{N}}) \\ &= \boldsymbol{L}(\{\phi_j^i\}_{j\in\mathbb{N}}) \\ &\leq \sup\{\boldsymbol{M}(\Phi_i(x)) : x \in I\} \to W^G(M, \partial M) \end{aligned} \tag{3-6}$$

by (3-4) and Corollary 3.6.

Now, we take a subsequence $j(i) \to \infty$ and define $\tilde{\Phi}_i = \tilde{\Phi}_{j(i)}^i$, $S = \{\varphi_i\}_{i\in\mathbb{N}}$, $\varphi_i := \phi_{j(i)}^i$ so that $\boldsymbol{f_M}(\varphi_i) \to 0$ and that:

(1) $C_i \boldsymbol{f_M}(\varphi_i) \to 0$ as $i \to \infty$, where $C_i = C_0(M_{\eta_i}, G, 1)$ is given by Theorem 3.5.

(2) If $\boldsymbol{M}(\varphi_i(x)) \geq W^G(M, \partial M) - \frac{\delta}{2}$ then $\operatorname{spt}(\varphi_i(x)) \subset M_{a_0}$ (by Claim 1).

(3) $W^G(M, \partial M) = \boldsymbol{L}(\{\varphi_i\}_{i\in\mathbb{N}})$ (by (3-6)).

(4) $\lim_{i\to\infty} \sup\{\boldsymbol{F}(\varphi_i(x), \Phi_i(x)) : x \in J(1, k_{j(i)}^i)_0\} = 0$ (by (3-5)).

(5) $\lim_{i\to\infty} \sup\{\boldsymbol{F}(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in I(1, k_{j(i)}^i)\} = 0$ (by the $\boldsymbol{F}$-continuity).

Combining (3), (4), and (5) with (3-4), we have $\boldsymbol{C}(S) = \boldsymbol{C}(\{\Phi_i\}_{i\in\mathbb{N}}) \subset \mathcal{V}_n^G(M_{2a_0})$ and every $V \in \boldsymbol{C}(S)$ is stationary in M.

Claim 2. *There exists $V \in \boldsymbol{C}(S)$ that is $(G, \mathbb{Z}_2)$-almost minimizing in annuli (of boundary-type) in the sense of* [39, Definition 11].

Proof of Claim 2. Suppose none of $V \in \boldsymbol{C}(S)$ is $(G, \mathbb{Z}_2)$-almost minimizing in annuli in the sense of [39, Definition 11]. Then there is a new sequence $S^* = \{\varphi_i^*\}_{i\in\mathbb{N}}$ of mappings $\varphi_i^* : J(1, l_i)_0 \to \mathcal{Z}_n^G(M_{\eta_i}; \mathbb{Z}_2)$ for some $l_i \geq k^i_{j(i)} \to \infty$ as $i \to \infty$, such that:

(i) $\boldsymbol{L}(S^*) < \boldsymbol{L}(S) = W^G(M, \partial M)$.

(ii) φ_i and φ_i^* are 1-homotopic in $\mathcal{Z}_n^G(M_{\eta_i}; \mathbb{Z}_2)$ with $\boldsymbol{M}$-fineness tending to zero, (Specifically, there is a map $\psi_i : I(1, l_i)_0 \times J(1, l_i)_0 \to \mathcal{Z}_n^G(M_{\eta_i}; \mathbb{Z}_2)$ so that $\boldsymbol{f_M}(\psi_i) \to 0$ as $i \to \infty$, $\psi_i([0], \cdot) = \varphi_i \circ \boldsymbol{n}_i$, and $\psi_i([1], \cdot) = \varphi_i^*$, where $\boldsymbol{n}_i = \boldsymbol{n}(l_i, k^i_{j(i)})$).

(iii) $\operatorname{spt}(\psi_i(t, x) - \varphi_i \circ \boldsymbol{n}_i(x)) \Subset M_{a_0}$ for any $t \in I(1, l_i)_0$ and $x \in J(1, l_i)_0$.

(iv) For any $x \in J(1, l_i)_0$, if $\boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x)) < W^G(M, \partial M) - \frac{\delta}{4}$, then we have that $\psi_i(\cdot, x) \equiv \varphi_i \circ \boldsymbol{n}_i(x)$ is a constant discrete homotopy at x.

Indeed, since each $V \in \boldsymbol{C}(S)$ is supported in M_{2a_0}, we can take G-annuli

$$\{\operatorname{An}^G(p(V), r_i - s_i, r_i + s_i)\}_{i=1}^{27}$$

in M_{a_0} as in [30, Theorem 4.10, Part 1], which implies all the deformations will be restricted in M_{a_0}. Using [40, Theorem 3.14] and $\operatorname{dist}_M(G \cdot p, \cdot)$, we can make the constructions in [30, Theorem 4.10, Parts 2–9] with G-invariant objects. Then the rest parts in [30, Theorem 4.10] are purely combinatorial, which would carry over with M_{a_0} in place of M. This gives (i)–(iii). Moreover, by taking the constant ϵ_2 in [30, Theorem 4.10, Part 3] smaller than $\frac{\delta}{8}$, we have $\psi_i(\cdot, x) \equiv \varphi_i \circ \boldsymbol{n}_i(x)$ provided $\boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x)) < W^G(M, \partial M) - \frac{\delta}{4}$ (see Parts 10(c), 14 and 18 in [30, Theorem 4.10]).

Next, we can extend φ_i^* (for i-large) to an $\boldsymbol{F}$-continuous map $\tilde{\Phi}_i^* \in \mathcal{P}^G(M, \partial M)$ so that $\tilde{\Phi}_i^* \llcorner \big[0, \frac{1}{3}\big] = \tilde{\Phi}_i \llcorner \big[0, \frac{1}{3}\big] = \Phi_i \llcorner \big[0, \frac{1}{3}\big]$. Take any 1-cell $\alpha = [x_0, x_1] \in J(1, l_i)_1$, we will construct the extension $\tilde{\Phi}_i^* \llcorner \alpha$ separately in two cases.

Case 1: $\max\{\boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x_0)), \boldsymbol{M}(\varphi_i \circ \boldsymbol{n}_i(x_1))\} < W^G(M, \partial M) - \frac{\delta}{4}$.

By (ii)–(iv), we can define $\tilde{\Phi}_i^* \llcorner \alpha := \tilde{\Phi}_i \circ f_\alpha$ as the extension of $\varphi_i^* \llcorner \alpha_0$, where $f_\alpha : \alpha = [x_0, x_1] \to [\boldsymbol{n}_i(x_0), \boldsymbol{n}_i(x_1)]$ is an affine transformation. Hence, in this case, we have

(3-7) $\tilde{\Phi}_i^* \llcorner \alpha \subset \mathcal{Z}_n^G(M_{\eta_i}; \mathbb{Z}_2)$ and $\tilde{\Phi}_i^*(x) = \tilde{\Phi}_i(\boldsymbol{n}_i(x))$ for all $x \in \alpha_0 = \{x_0, x_1\}$.

In particular, $\tilde{\Phi}_i^*(1) = \tilde{\Phi}_i(1) = 0$ provided $\boldsymbol{f_M}(\psi_i) < W^G(M, \partial M) - \frac{\delta}{4}$, which holds for i-large. Additionally, it follows from (1), Theorem 3.5(i)–(iii), and the choice of α that

$$\begin{aligned}\text{(3-8)}\qquad \sup\{\boldsymbol{M}(\tilde\Phi_i^*(x)) : x\in\alpha\} &= \sup\{\boldsymbol{M}(\tilde\Phi_i(x)) : x\in f_\alpha(\alpha)\}\\ &\le \sup\{\boldsymbol{M}(\varphi_i(x)) : x\in\partial f_\alpha(\alpha)\} + C_i\,\boldsymbol{f_M}(\varphi_i)\\ &< W^G(M,\partial M) - \tfrac{\delta}{4} + C_i\,\boldsymbol{f_M}(\varphi_i)\\ &\le W^G(M,\partial M) - \tfrac{\delta}{5}\end{aligned}$$

for i-large, where $C_i = C_0(M_{\eta_i}, G, 1)$ is given by Theorem 3.5.

Case 2: $\max\{\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x_0)), \boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x_1))\} \ge W^G(M,\partial M) - \frac{\delta}{4}$.

Let $A_i \subset J = \left[\frac{1}{3}, 1\right]$ be union of all 1-cells of this case in $J(1,l_i)_1$. Take i sufficiently large so that $\boldsymbol{f_M}(\psi_i) < \frac{\delta}{4}$ (by (ii)). Then $\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x)) \ge W^G(M,\partial M) - \frac{\delta}{2}$ for all $x\in J(1,l_i)_0\cap A_i$. By (2) and (iii), we have that $\varphi_i^*(x) = \psi_i^*([1], x)$ is supported in M_{a_0} for all $x\in J(1,l_i)_0\cap A_i$. Applying Theorem 3.5 to $\varphi_i^*\llcorner[J(1,l_i)_0\cap A_i]$ in M_{a_0} (for i-large) will give an $\boldsymbol{M}$-continuous extension $\tilde\Phi_i^* : A_i \to \mathcal{Z}_n^G(M_{a_0};\mathbb{Z}_2)$ so that

$$\text{(3-9)}\quad \sup\{\boldsymbol{M}(\tilde\Phi_i^*(x)) : x\in A_i\} \le \sup\{\boldsymbol{M}(\varphi_i^*(x)) : x\in J(1,l_i)_0\cap A_i\} + C_0\,\boldsymbol{f_M}(\psi_i),$$

where $C_0 = C_0(M_{a_0}, G, 1) \ge 1$ is a uniform constant. Note for any $x\in\partial A_i$, we must have $\boldsymbol{M}(\varphi_i\circ\boldsymbol{n}_i(x)) < W^G(M,\partial M) - \frac{\delta}{4}$. Hence, by (iv) and Theorem 3.5(i),

$$\text{(3-10)}\qquad \tilde\Phi_i^*(x) = \varphi_i^*(x) = \varphi_i\circ\boldsymbol{n}_i(x) = \tilde\Phi_i(\boldsymbol{n}_i(x)) \quad \text{for all } x\in\partial A_i.$$

It now follows from (3-7)–(3-10) that $\tilde\Phi_i^* : I \to \mathcal{Z}_n^G(M;\mathbb{Z}_2)$ is a well defined $\boldsymbol{F}$-continuous map so that $\tilde\Phi_i^*\llcorner\left[0,\frac{1}{3}\right] = \tilde\Phi_i\llcorner\left[0,\frac{1}{3}\right] = \Phi_i\llcorner\left[0,\frac{1}{3}\right]$, $\tilde\Phi_i^*(1) = 0$, and $\tilde\Phi_i^*\llcorner J \subset \mathcal{Z}_n^G(M_{\eta_i};\mathbb{Z}_2)$, which implies $\tilde\Phi_i^* \in \mathcal{P}^G(M,\partial M)$. Therefore, by equations (3-4), (3-8), (3-9), and statements (i)-(ii),

$$W^G(M,\partial M) \le \boldsymbol{L}(\{\tilde\Phi_i^*\}_{i\in\mathbb{N}}) \le \max\left\{W^G(M,\partial M) - \tfrac{\delta}{5}, \boldsymbol{L}(\{\varphi_i^*\}_{i\in\mathbb{N}})\right\} < W^G(M,\partial M),$$

which is a contradiction. □

Thus, there must exist $V\in\boldsymbol{C}(S)$ that is $(G,\mathbb{Z}_2)$-almost minimizing in annuli (of boundary-type) in the sense of [39, Definition 11]. Since $\boldsymbol{C}(S)\subset\mathcal{V}_n^G(M_{2a_0})$, the interior regularity result [39, Theorem 7] (modified in [41, Theorem 4.18]) indicates that V is an integral G-varifold induced by closed smooth embedded minimal G-hypersurfaces. □

4. *G*-sweepouts in positive Ricci curvature *G*-manifolds

Throughout this section we assume that (M^{n+1}, g_M) is a closed connected orientable Riemannian manifold with positive Ricci curvature $\mathrm{Ric}_M > 0$, and G is a compact Lie group acting on M isometrically so that $3 \le \operatorname{codim}(G\cdot p) \le 7$ for all $p\in M$. Our goal is to associate an $\boldsymbol{F}$-continuous G-sweepout to each closed minimal G-hypersurface in M.

To begin with, we collect some notations and classical results for minimal hypersurfaces. Let $\Sigma \subset M$ be a closed smooth embedded minimal hypersurface. Recall the second variation of Σ for the area functional is given by

$$\delta^2\Sigma(X) := \frac{d^2}{d^2t}\Big|_{t=0}\operatorname{Area}(F_t(\Sigma)) = -\int_\Sigma \langle L_\Sigma(X^\perp), X^\perp\rangle, \tag{4-1}$$

where $L_\Sigma : \mathfrak{X}^\perp(\Sigma) \to \mathfrak{X}^\perp(\Sigma)$ is the Jacobi operator of Σ, and $\{F_t\}$ are diffeomorphisms generated by $X \in \mathfrak{X}(M)$. Then we denote:

- $\operatorname{Index}(\Sigma)$: the *Morse index* of Σ, i.e., the number of the negative eigenvalues (counted with multiplicities) of L_Σ.
- $\mu_1(\Sigma)$: the first eigenvalue of L_Σ.

If $\operatorname{Index}(\Sigma) = 0$ or equivalently $\mu_1(\Sigma) \geq 0$, then we say Σ is *stable*.

For $\Sigma \subset M$ as a G-invariant minimal hypersurface, we have $L_\Sigma(X) \in \mathfrak{X}^{\perp,G}(\Sigma)$ for all $X \in \mathfrak{X}^{\perp,G}(\Sigma)$, where $\mathfrak{X}^{\perp,G}(\Sigma)$ is the space of normal G-vector fields on Σ. By restricting L_Σ to $\mathfrak{X}^{\perp,G}(\Sigma)$, we make the following definition.

Definition 4.1. Let $\Sigma \subset M$ be a closed smooth embedded minimal G-hypersurface. The *equivariant Morse index* (or *G-index* for simplicity) $\operatorname{Index}_G(\Sigma)$ is defined by the number of the negative eigenvalues (counted with multiplicities) of $L_\Sigma \llcorner \mathfrak{X}^{\perp,G}(\Sigma)$. Additionally, we denote $\mu_1^G(\Sigma)$ as the first eigenvalue of $L_\Sigma \llcorner \mathfrak{X}^{\perp,G}(\Sigma)$.

Suppose Σ is a closed minimal G-hypersurface with a G-invariant unit normal ν, and u_1 is the first eigenfunction of L_Σ. Then for any $g \in G$, the G-invariance of Σ and ν indicates $u_1 \circ g$ is also the first eigenfunction of L_Σ. It is well-known that $\mu_1(\Sigma)$ has multiplicity one and the first eigenfunction u_1 does not change sign. Hence, $u_1 \circ g = u_1$ for all $g \in G$, which implies that $u_1\nu \in \mathfrak{X}^{\perp,G}(\Sigma)$ and that:

Lemma 4.2 [39, Lemma 7]. *If Σ is a closed minimal G-hypersurface with a G-invariant unit normal ν, then the first eigenfunction $u_1 > 0$ of L_Σ is G-invariant and $\mu_1(\Sigma) = \mu_1^G(\Sigma)$.*

Since we mainly consider the ambient manifolds with positive Ricci curvature, we collect the following useful results, which are well known to experts (see [44, Section 2]).

Lemma 4.3. *Suppose (M^{n+1}, g_M) is a closed connected orientable Riemannian manifold. Let $\Sigma, \Sigma_1, \Sigma_2 \subset M$ be closed embedded hypersurfaces. Then we have*:

(i) *If Σ is connected*, *then Σ is orientable if and only if it is* 2*-sided* (*i.e.*, *Σ has a unit normal vector field*).

(ii) *If Σ is connected and separates M*, *i.e.*, *$M \setminus \Sigma$ has two connected components*, *then Σ is orientable.*

Moreover, *suppose M has positive Ricci curvature*, *then we have*:

(iii) *If* Σ *is connected and orientable, then* Σ *separates* M.

(iv) *If* Σ *is minimal and* 2*-sided, then it cannot be stable, i.e.,* $\mu_1(\Sigma) < 0$.

(v) *If* Σ_1, Σ_2 *are minimal hypersurfaces, then* $\Sigma_1 \cap \Sigma_2 \neq \varnothing$.

After involving the actions of G, a connected component of some G-hypersurface Σ may not be G-invariant. Hence, we introduce the following notions of equivariant connectivity.

Definition 4.4. Let $U \subset M$ be a G-invariant subset with connected components $\{U_i\}_{i=1}^m$. Then we say U is *G-connected* if for any $i, j \in \{1, \cdots, m\}$, there exists $g_{ij} \in G$ so that $g_{ij} \cdot U_j = U_i$. Additionally, we say $U' \subset U$ is a *G-connected component* (or *G-component* for simplicity) of U, if U' has the form of $\cup_{j=1}^l U_{i(j)}$ and is G-connected.

Note that any G-subset U of M can be separated into some G-components. Additionally, by the above lemma, it is easy to show the following result.

Lemma 4.5. *Suppose* $(M^{n+1}, g_{_M})$ *is a closed connected orientable Riemannian manifold with positive Ricci curvature, and G is a compact Lie group acting on M isometrically. Let* $\Sigma \subset M$ *be a closed embedded minimal G-hypersurface. Then* Σ *is connected, and:*

- *If* Σ *has a G-invariant unit normal, then* Σ *separates M into two G-components.*
- *If* Σ *does not admit a G-invariant unit normal, then* $M \setminus \Sigma$ *is G-connected.*

Proof. It follows from Lemma 4.3(v) that Σ is connected. If Σ has a G-invariant unit normal ν, then by Lemma 4.3(i)–(iii), $M \setminus \Sigma$ has two connected components M_1, M_2, with ν pointing inward M_1. Since ν and $M_1 \cup M_2$ are G-invariant, we have $g_* \nu = \nu$ and $g \cdot M_i = M_i$ for all $g \in G$ and $i \in \{1, 2\}$, which indicates each M_i is G-connected. If the unit normal ν exists but is not G-invariant, then there exists $g \in G$ so that $g_* \nu = -\nu$ pointing inward M_2, which implies $g \cdot M_1 = M_2$, and thus $M_1 \cup M_2$ is G-connected. If Σ does not admit a unit normal, then $M \setminus \Sigma$ has only one component, which is also G-connected. □

Recall that, Zhou [43] constructed sweepouts of M by separating orientable and nonorientable minimal hypersurfaces. It follows from Lemma 4.3 that the orientability of a connected closed hypersurface is equivalent to the nonconnectivity of its unit normal bundle. Hence, after involving the actions of G, we shall separate the constructions by the G-connectivity (Definition 4.4) of the unit normal bundle for minimal G-hypersurfaces.

Therefore, we denote

$$\mathcal{S}^G(M) := \left\{ \Sigma^n \subset M^{n+1} \;\middle|\; \begin{array}{l} \Sigma \text{ is a closed smooth embedded} \\ \text{minimal } G\text{-hypersurface in } (M, g_{_M}) \end{array} \right\}. \tag{4-2}$$

By Theorem 3.8, $\mathcal{S}^G(M) \neq \varnothing$ provided $3 \le \operatorname{codim}(G \cdot p) \le 7$ for all $p \in M$. Define

$$\mathcal{S}^G_+(M) := \{\Sigma \in \mathcal{S}^G(M) : \Sigma \text{ has a } G\text{-invariant unit normal}\}$$

and $\mathcal{S}^G_-(M) := \mathcal{S}^G(M) \setminus \mathcal{S}^G_+(M)$. It follows directly from Lemma 4.5 that

$$\Sigma \in \mathcal{S}^G_-(M) \quad \Leftrightarrow \quad S\Sigma \text{ is } G\text{-connected} \quad \Leftrightarrow \quad M \setminus \Sigma \text{ is } G\text{-connected},$$

where $S\Sigma = \{v \in \boldsymbol{N}\Sigma : |v| = 1\}$ is the unit normal bundle of Σ.

Moreover, for any $\Sigma \in \mathcal{S}^G_-(M)$, we can cut M along Σ to obtain a new manifold $\tilde{M}$ so that $\tilde{M}$ is locally isometric to M, G acts on $\tilde{M}$ by isometries, and $\partial\tilde{M} \in \mathcal{S}^G_+(\tilde{M})$ is a G-invariant double cover of Σ. Specifically, let $r > 0$ be small enough so that the normal exponential map $\exp^\perp_\Sigma : \boldsymbol{N}\Sigma \to M$ is a G-equivariant diffeomorphism on $B_{2r}(\Sigma) := \{p \in M : \operatorname{dist}_M(\Sigma, p) < 2r\}$. Hence, we have

$$E : S\Sigma \times (-2r, 2r) \to B_{2r}(\Sigma), \quad E(v, t) := \exp^\perp_\Sigma(t \cdot v) \tag{4-3}$$

which is a double cover of $B_{2r}(\Sigma)$. Define the action of G on $S\Sigma \times (-2r, 2r)$ by $g \cdot (v, t) := (g_* v, t)$ for any $v \in S\Sigma$ and $t \in (-2r, 2r)$, which indicates E is G-equivariant and

$$\tilde{\Sigma} = S\Sigma \times \{0\} \tag{4-4}$$

is a G-equivariant double cover of Σ. Let $E_r := E\llcorner(S\Sigma \times (r, 2r))$ be a G-equivariant diffeomorphism on $B_{2r}(\Sigma) \setminus \operatorname{Clos}(B_r(\Sigma))$. Then by gluing $M \setminus \operatorname{Clos}(B_r(\Sigma))$ and $S\Sigma \times [0, 2r)$ on $B_{2r}(\Sigma) \setminus \operatorname{Clos}(B_r(\Sigma))$ with E_r, we can define

$$\tilde{M} := \big(M \setminus \operatorname{Clos}(B_r(\Sigma))\big) \cup_{E_r} \big(S\Sigma \times [0, 2r)\big) \tag{4-5}$$

as a compact manifold with boundary $\partial\tilde{M} = \tilde{\Sigma}$. Then we have

$$F : \tilde{M} \to M, \qquad F := \begin{cases} id & \text{in } M \setminus \operatorname{Clos}(B_r(\Sigma)), \\ E & \text{in } S\Sigma \times [0, 2r) \end{cases} \tag{4-6}$$

is a G-equivariant smooth map so that $F\llcorner(\tilde{M} \setminus \tilde{\Sigma})$ gives a diffeomorphism to $M \setminus \Sigma$, and $F\llcorner\tilde{\Sigma}$ gives a double cover of Σ. Using F, we can pull back the metric g_M from M to $\tilde{M}$ so that F is a local isometry and G acts on $\tilde{M}$ by isometries. Thus, $\tilde{\Sigma}$ is a minimal G-hypersurface in $\tilde{M}$ with an inward pointing G-invariant unit normal. In particular, $\Sigma \in \mathcal{S}^G_-(M)$ implies $S\Sigma$ and $M \setminus \Sigma$ are both G-connected, and thus $\tilde{M}$ is G-connected.

***G*-sweepouts correspond to $\Sigma \in \mathcal{S}^G(M)$.**

Proposition 4.6. *Given any $\Sigma \in \mathcal{S}^G_+(M)$, there exists an $\boldsymbol{F}$-continuous G-sweepout $\Phi : [-1, 1] \to \mathcal{Z}^G_n(M; \mathbb{Z}_2)$ of M so that*:

(i) $\Phi(0) = [\![\Sigma]\!]$, $\Phi(-1) = \Phi(1) = 0$.

(ii) $\boldsymbol{M}(\Phi(x)) \le \operatorname{Area}(\Sigma)$ *with equality only if $x = 0$.*

Proof. By Lemma 4.5, $M \setminus \Sigma$ has two G-components M_1 and M_2 so that the unit normal ν of Σ pointing inward M_1. Additionally, it follows from Lemmas 4.2 and 4.3 that the first eigenfunction $u_1 > 0$ of L_Σ is a G-invariant function satisfying $L_\Sigma u_1 = -\mu_1(\Sigma)\, u_1 > 0$.

Denote by $d_\pm$ the signed distance function to Σ so that $d_\pm = \mathrm{dist}_M(\Sigma, \cdot)$ in M_1, and $d_\pm = -\,\mathrm{dist}_M(\Sigma, \cdot)$ in M_2. Suppose $X \in \mathfrak{X}^G(M)$ is a G-vector field with $X = (u_1 \circ n_\Sigma) \cdot \nabla d_\pm$ in a neighborhood of Σ, where n_Σ is the nearest projection (in M) to Σ. Then we consider the G-equivariant variation $\{\Sigma_t := F_t(\Sigma)\}_{t\in[-r,r]}$ of Σ, where $\{F_t\}$ are the G-equivariant diffeomorphisms generated by X. By the second variation formula (4-1), we have

$$\delta^2\Sigma(X) = \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Area}(\Sigma_t) = -\int_\Sigma u_1 L_\Sigma u_1 < 0, \qquad \frac{d}{dt}\Big|_{t=0} \langle \vec{H}_{\Sigma_t}, \nabla d_\pm\rangle = L_\Sigma u_1 > 0,$$

where $\vec{H}_{\Sigma_t}$ is the mean curvature vector field of Σ_t. Thus, for $r > 0$ small enough,

$$\mathrm{Area}(\Sigma_t) < \mathrm{Area}(\Sigma), \quad \langle \vec{H}_{\Sigma_t}, \nabla\, \mathrm{dist}_M(\Sigma, \cdot)\rangle > 0 \qquad \text{for all } t \in [-r, 0) \cup (0, r].$$

Define $\Phi(x) := [\![\Sigma_x]\!] = (F_x)_\# [\![\Sigma]\!] \in \mathcal{Z}_n^G(M; \mathbb{Z}_2)$ for $x \in [-r, r]$, which is $\boldsymbol{F}$-continuous.

Since $u_1 > 0$, $\{\Sigma_t\}_{t\in[-r,r]}$ is a smooth foliation of a G-neighborhood of Σ, and $\Sigma_t \subset M_1$ for $t > 0$ and $\Sigma_t \subset M_2$ for $t < 0$. We now consider the compact manifolds $M_1' := M_1 \setminus \{\Sigma_t\}_{t\in[0,r)}$ and $M_2' := M_2 \setminus \{\Sigma_t\}_{t\in(-r,0]}$, whose boundary $\partial M_i' = \Sigma_{r_i}$ ($i \in \{1, 2\},\ r_1 = r,\ r_2 = -r$), is a G-hypersurface with positive mean curvature pointing inward M_i'.

Suppose $W^G(M_i', \partial M_i') > \mathrm{Area}(\Sigma_{r_i})$ for $i \in \{1, 2\}$. Then by Theorem 3.11, there exists a closed minimal G-hypersurface Σ' in the interior of M_i'. Noting $\Sigma \cap \Sigma' = \varnothing$, we get a contradiction from Lemma 4.3(v). Therefore, $W^G(M_i', \partial M_i') \le \mathrm{Area}(\Sigma_{r_i})$. By Definition 2.8 and Corollary 3.7, there exist $\epsilon > 0$ small enough and an $\boldsymbol{F}$-continuous G-sweepout $\Phi_i : [0, 1] \to \mathcal{Z}_n^G(M_i; \mathbb{Z}_2)$ so that $\Phi_i(0) = [\![\Sigma_{r_i}]\!]$, $\Phi_i(1) = 0$, and

$$\sup\{\boldsymbol{M}(\Phi_i(x)) : x \in [0, 1]\} \le W^G(M_i', \partial M_i') + \epsilon \le \mathrm{Area}(\Sigma_{r_i}) + \epsilon < \mathrm{Area}(\Sigma).$$

Now, by reparametrization, we have a well-defined map $\Phi : [-1, 1] \to \mathcal{Z}_n^G(M; \mathbb{Z}_2)$,

$$\Phi(x) := \begin{cases} \Phi_2\big(-\tfrac{3}{2}x - \tfrac{1}{2}\big), & x \in \big[-1, -\tfrac{1}{3}\big], \\ (F_{3rx})_\# [\![\Sigma]\!], & x \in \big[-\tfrac{1}{3}, \tfrac{1}{3}\big], \\ \Phi_1\big(\tfrac{3}{2}x - \tfrac{1}{2}\big), & x \in \big[\tfrac{1}{3}, 1\big] \end{cases}$$

continuous in the $\boldsymbol{F}$-topology satisfying (i) and (ii). Additionally, the arguments before Definitions 2.5 and 2.8 indicate $F_M(\Phi) = [\![M_2]\!] + [\![M_1]\!] = [\![M]\!]$, where F_M is given by (2-5). Hence, we have $\Phi \in \mathcal{P}^G(M)$. □

Proposition 4.7. *Given any* $\Sigma \in \mathcal{S}^G_-(M)$, *there exists an* $\mathcal{F}$*-continuous* G*-sweepout* $\Phi : [0,1] \to \mathcal{Z}^G_n(M;\mathbb{Z}_2)$ *of* M *with no concentration of mass on orbits so that*:

(i) $\Phi(0) = \Phi(1) = 0$.

(ii) $\sup\{\boldsymbol{M}(\Phi(x)) : x \in [0,1]\} < 2\,\mathrm{Area}(\Sigma)$.

Proof. Let $\tilde{\Sigma} = S\Sigma \times \{0\}$ and $\tilde{M}$ be given by (4-4) and (4-5). Then $\tilde{M}$ is G-connected, $\mathrm{Area}(\tilde{\Sigma}) = 2\,\mathrm{Area}(\Sigma)$, and $\tilde{\Sigma}$ has a G-invariant unit normal $\tilde{\nu}$ pointing inward $\tilde{M}$. Let $\tau : \tilde{\Sigma} \to \tilde{\Sigma}$ be the isometric involution, i.e., $\tau(v,0) = (-v,0)$ for $v \in S\Sigma$.

Using the constructions in Proposition 4.6 with $\tilde{M}$ in place of M_1, we get an $\boldsymbol{F}$-continuous G-sweepout $\tilde{\Phi} : [0,1] \to \mathcal{Z}^G_n(\tilde{M};\mathbb{Z}_2)$ so that $\tilde{\Phi}(0) = [\![\tilde{\Sigma}]\!]$, $\tilde{\Phi}(1) = 0$, and $\boldsymbol{M}(\tilde{\Phi}(x)) \le 2\,\mathrm{Area}(\Sigma)$ for all $x \in [0,1]$ with equality only at $x = 0$. Additionally, for $t \in \left[0, \frac{1}{3}\right]$, $\tilde{\Phi}(t) = \tilde{\Sigma}_t := [\![\exp^{\perp}_{\tilde{\Sigma}}(t\tilde{u}\tilde{\nu})]\!]$, where $\tilde{u} = 3tr\tilde{u}_1$ and $\tilde{u}_1 : \tilde{\Sigma} \to \mathbb{R}^+$ is the G-invariant first eigenfunction of $L_{\tilde{\Sigma}}$ with eigenvalue $\mu_1(\tilde{\Sigma}) = \mu_1^G(\tilde{\Sigma}) < 0$.

Now, by the second variation formula (4-1), there are $\delta_0 \in \left(0, \frac{1}{3}\right)$, $C_0 > 0$ so that

$$\boldsymbol{M}(\tilde{\Phi}(t)) = \mathcal{H}^n(\tilde{\Sigma}_t) = \mathcal{H}^n(\tilde{\Sigma}) - \frac{t^2}{2}\int_{\tilde{\Sigma}} \langle L_{\tilde{\Sigma}}\tilde{u}\tilde{\nu}, \tilde{u}\tilde{\nu}\rangle + O(t^3) \le \mathcal{H}^n(\tilde{\Sigma}) - C_0\, t^2 \tag{4-7}$$

for all $t \in (0, \delta_0)$. For any $\delta \in (0, \delta_0)$ (will be specified later), the $\boldsymbol{F}$-continuity of $\tilde{\Phi}$ and Proposition 4.6(ii) imply the existence of $\epsilon > 0$ with

$$\boldsymbol{M}(\tilde{\Phi}(t)) \le \mathcal{H}^n(\tilde{\Sigma}) - \epsilon \quad \text{for all } t \in [\delta, 1]. \tag{4-8}$$

Now, we will open up $\tilde{\Sigma}_t$, $t \in [0, \delta]$, at some orbit to decrease the area.

Specifically, let $G \cdot \tilde{p} \subset \tilde{\Sigma}^{\mathrm{prin}}$ be any principal orbit of $\tilde{\Sigma}$. Then by the G-invariance of $\tilde{\nu}$ and [4, Corollary 2.2.2], $G \cdot \tilde{p} \subset \tilde{M}^{\mathrm{prin}}$ is also a principal orbit in $\tilde{M}$. Note either $G \cdot \tilde{p} = G \cdot \tau(\tilde{p})$ or $G \cdot \tilde{p} \cap G \cdot \tau(\tilde{p}) = \varnothing$. Thus, we can define $P := G \cdot \tilde{p} \cup G \cdot \tau(\tilde{p})$ as a G-invariant submanifold in $\tilde{\Sigma}$ with dimension $n - l$. By assumptions, $3 \le \mathrm{codim}(G \cdot \tilde{p}) = l + 1 \le 7$.

Case 1: $3 \le l \le 6$.

For any $r > 0$, $t \in [0, \delta]$, define the following G-invariant sets:

$$\begin{aligned}
\tilde{B}_r(P) &:= \{\tilde{q} \in \tilde{\Sigma} : \mathrm{dist}_{\tilde{\Sigma}}(\tilde{q}, P) < r\} \subset \tilde{\Sigma},\\
\tilde{B}_{r,t}(P) &:= \{\exp^{\perp}_{\tilde{\Sigma}}((t\tilde{u}\tilde{\nu})(\tilde{q})) : \tilde{q} \in \tilde{B}_r(P)\} \subset \tilde{\Sigma}_t,\\
\tilde{C}_{r,t}(P) &:= \{\exp^{\perp}_{\tilde{\Sigma}}((s\tilde{u}\tilde{\nu})(\tilde{q})) : \tilde{q} \in \partial\tilde{B}_r(P),\ s \in [0,t]\}.
\end{aligned}$$

For $R, \delta > 0$ small enough, it follows from the integral formula in [40, (C.4)] that

$$ctr^{l-1} \le \mathcal{H}^n(\tilde{C}_{r,t}(P)) \le Ctr^{l-1} \quad \text{and} \quad cr^l \le \mathcal{H}^n(\tilde{B}_{r,t}(P)) \le Cr^l \tag{4-9}$$

for all $r \in [0, R]$, $t \in [0, \delta]$, where $c, C > 0$ are constants depending on $\tilde{\Sigma}, \tilde{M}, P$. Define

$$\tilde{\Sigma}_{r,t} := (\tilde{\Sigma}_t \setminus \tilde{B}_{r,t}(P)) \cup \tilde{C}_{r,t}(P) \cup \tilde{B}_r(P), \quad r \in [0, R],\ t \in [0, \delta].$$

By (4-7)–(4-9), $\|\tilde{\Sigma}_{r,t}\|(\tilde{M}\setminus\tilde{\Sigma}) \le \mathcal{H}^n(\tilde{\Sigma}) - C_0 t^2 - cr^l + Ctr^{l-1}$.

Note, in this case, that

$$Ctr^{l-1} \le \frac{C_0}{2}t^2 + \frac{C^2}{2C_0}r^{2l-2}, \quad l \ge 3.$$

We can take $R > 0$ small enough so that $\frac{C^2}{2C_0}R^{l-2} < \frac{c}{2}$. Hence,

$$\|\tilde{\Sigma}_{r,t}\|(\tilde{M}\setminus\tilde{\Sigma}) \le \mathcal{H}^n(\tilde{\Sigma}) - \frac{C_0}{2}t^2 - \frac{c}{2}r^l$$

for all $t \in [0, \delta],\ r \in [0, R]$, and thus

$$\tilde{\Sigma}'_t := \begin{cases} \tilde{\Sigma}_{R,2t}, & t \in \left[0, \frac{\delta}{2}\right], \\ \tilde{\Sigma}_{2R(1-\frac{t}{\delta}),\delta}, & t \in \left[\frac{\delta}{2}, \delta\right] \end{cases}$$

satisfies

$$\|\tilde{\Sigma}'_t\|(\tilde{M}\setminus\tilde{\Sigma}) \le \begin{cases} \mathcal{H}^n(\tilde{\Sigma}) - \frac{c}{2}R^l, & t \in \left[0, \frac{\delta}{2}\right], \\ \mathcal{H}^n(\tilde{\Sigma}) - \frac{C_0}{2}\delta^2, & t \in \left[\frac{\delta}{2}, \delta\right]. \end{cases}$$

Set $\epsilon' := \min\left\{\epsilon, \frac{cR^l}{2}, \frac{C_0\delta^2}{2}\right\}$ and define $\tilde{\Phi}'(t) := [\![\tilde{\Sigma}'_t]\!]$ for $t \in [0, \delta]$ in this case.

Case 2: $l = 2$.

For $R > r > 0$ small enough, let $\eta_{r,R} : \tilde{\Sigma} \to [0, 1]$ be the G-invariant logarithmic cut-off function defined by

$$\eta_{r,R}(\tilde{q}) := \begin{cases} 1, & \tilde{q} \notin \tilde{B}_R(P), \\ \frac{\log r - \log(\operatorname{dist}_{\tilde{\Sigma}}(\tilde{q}, P))}{\log r - \log R}, & \tilde{q} \in \tilde{B}_R(P)\setminus\tilde{B}_r(P), \\ 0, & \tilde{q} \in \tilde{B}_r(P), \end{cases}$$

which is also τ-invariant. Consider

$$\tilde{\Sigma}_{r,R,t} := \exp^{\perp}_{\tilde{\Sigma}}(t\eta_{r,R}\,\tilde{u}\tilde{\nu}).$$

By [17, Proposition 2.5] and [40, (C.4)], we can take $R, \delta > 0$ small enough so that

$$\begin{aligned}\|\tilde{\Sigma}_{r,R,t}\|(\tilde{M}\setminus\tilde{\Sigma}) &\le \mathcal{H}^n(\tilde{\Sigma}\setminus\tilde{B}_r(P)) + \frac{t^2}{2}\int_{\tilde{\Sigma}} |\nabla(\eta_{r,R}\,\tilde{u})|^2 - (\operatorname{Ric}(\tilde{\nu},\tilde{\nu}) + |A|^2)(\eta_{r,R}\,\tilde{u})^2 \\ &\quad + Ct^3\int_{\tilde{\Sigma}} 1 + |\nabla(\eta_{r,R}\,\tilde{u})|^2 \\ &\le \mathcal{H}^n(\tilde{\Sigma}\setminus\tilde{B}_r(P)) - C_1t^2 + C_2t^2\int_{\tilde{\Sigma}}|\nabla\eta_{r,R}|^2 + t^2\int_{\tilde{B}_R(P)}\tilde{u}\eta_{r,R}\nabla\tilde{u}\nabla\eta_{r,R} \\ &\quad + Ct^3\int_{\tilde{\Sigma}} 1 + 2\eta_{r,R}^2|\nabla\tilde{u}|^2 + 2\tilde{u}^2|\nabla\eta_{r,R}|^2 \\ &\le \mathcal{H}^n(\tilde{\Sigma}) - cr^2 - C_1t^2 + \frac{C_3}{\log\left(\frac{R}{r}\right)}t^2 + C_4R^2t^2 + C_5t^3 + \frac{C_6}{\log\left(\frac{R}{r}\right)}t^3\end{aligned}$$

for all $r \in (0, R),\ t \in [0, \delta]$, where $c, C, C_i > 0$ are uniform constants depending on $\tilde{\Sigma}, \tilde{M}, P$. Set $R, \delta > 0$ even smaller so that $C_4R^2 < \frac{C_1}{4},\ C_5\delta < \frac{C_1}{4}$, and $C_6\delta < C_3$.

Then choose $r > 0$ small enough with $\frac{2C_3}{\log(R/r)} < \frac{C_1}{4}$. Thus,

$$\|\tilde{\Sigma}_{r,R,t}\|(\tilde{M} \setminus \tilde{\Sigma}) \leq \mathcal{H}^n(\tilde{\Sigma}) - cr^2 - \frac{C_1}{2}t^2 + \frac{2C_3}{\log(R/r)}t^2 \leq \mathcal{H}^n(\tilde{\Sigma}) - cr^2 - \frac{C_1}{4}t^2$$

for all $t \in [0, \delta]$, and

$$\tilde{\Sigma}'_t := \begin{cases} \tilde{\Sigma}_{r,R,2t}, & t \in \left[0, \frac{\delta}{2}\right], \\ \tilde{\Sigma}_{2r(1-\frac{t}{\delta}),2R(1-\frac{t}{\delta}),\delta}, & t \in \left[\frac{\delta}{2}, \delta\right], \end{cases}$$

satisfies

$$\|\tilde{\Sigma}'_t\|(\tilde{M} \setminus \tilde{\Sigma}) \leq \begin{cases} \mathcal{H}^n(\tilde{\Sigma}) - cr^2, & t \in \left[0, \frac{\delta}{2}\right], \\ \mathcal{H}^n(\tilde{\Sigma}) - \frac{C_1}{4}\delta^2, & t \in \left[\frac{\delta}{2}, \delta\right]. \end{cases}$$

In this case, set $\epsilon' := \min\left\{\epsilon, cr^2, \frac{C_1\delta^2}{4}\right\}$ and define $\tilde{\Phi}'(t) := [\![\tilde{\Sigma}'_t]\!]$ for $t \in [0, \delta]$.

In both cases, we define $\tilde{\Phi}' \llcorner [\delta, 1] = \tilde{\Phi} \llcorner [\delta, 1]$ and see

$$\sup\{\|\tilde{\Phi}'(t)\|(\tilde{M} \setminus \tilde{\Sigma}) : t \in [0, 1]\} \leq \mathcal{H}^n(\tilde{\Sigma}) - \epsilon'. \tag{4-10}$$

Additionally, by (2-1), $\tilde{\Phi}'$ is still an $\boldsymbol{F}$-continuous map with $\tilde{\Phi}' = [\![\tilde{\Sigma}]\!]$, $\tilde{\Phi}'(1) = 0$.

Finally, we define $\Phi(x) := F_\# \tilde{\Phi}'(x)$ for all $x \in [0, 1]$, where $F : \tilde{M} \to M$ is the equivariant local isometry given by (4-6). Because $F : \tilde{M} \setminus \tilde{\Sigma} \to M \setminus \Sigma$ is an equivariant isometry, the arguments before Definitions 2.5 and 2.8 indicate $F_M(\Phi) = F_\#([\![\tilde{M}]\!]) = M$, where F_M is given by (2-5). Additionally, note $F : \tilde{\Sigma} \to \Sigma$ is a double cover and $\tilde{\Sigma}'_t \cap \tilde{\Sigma}$ is τ-invariant in both cases. Hence, by $\mathbb{Z}_2$-coefficients and (4-10), we have $\Phi(0) = F_\# [\![\tilde{\Sigma}]\!] = 0$ and

$$\boldsymbol{M}(\Phi(x)) = \|\tilde{\Phi}'(t)\|(\tilde{M} \setminus \tilde{\Sigma}) \leq \mathcal{H}^n(\tilde{\Sigma}) - \epsilon' = 2\,\mathrm{Area}(\Sigma) - \epsilon'.$$

At last, noting that $\|\Phi(x)\|(B^G_r(p)) \leq \|\tilde{\Phi}'(x)\|\big(F^{-1}(B^G_r(p))\big) \leq 2\boldsymbol{m}^G(\Phi', r)$ for all $x \in [0, 1]$ and $p \in M$ by the definition of $\boldsymbol{m}^G$ (Definition 2.3), we see that $\boldsymbol{m}^G(\Phi, r) \leq 2\boldsymbol{m}^G(\tilde{\Phi}', r)$ and Φ has no concentration of mass on orbits. □

5. Proof of the main theorems

Let $\mathcal{S}^G(M)$ be given in (4-2). Then we define

$$\mathcal{A}^G(M) := \inf_{\Sigma \in \mathcal{S}^G(M)} \begin{cases} \mathrm{Area}(\Sigma) & \text{if } \Sigma \in \mathcal{S}^G_+(M), \\ 2\,\mathrm{Area}(\Sigma) & \text{if } \Sigma \in \mathcal{S}^G_-(M). \end{cases} \tag{5-1}$$

Theorem 5.1. *Let (M^{n+1}, g_M) be a closed connected orientable Riemannian manifold with positive Ricci curvature, and G be a compact Lie group acting on M isometrically so that $3 \leq \mathrm{codim}(G \cdot p) \leq 7$ for all $p \in M$. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M]$ is a connected minimal G-hypersurface of multiplicity one with a G-invariant unit normal vector field so that*

$$\mathrm{Index}_G(\Sigma) = 1 \quad \text{and} \quad \mathrm{Area}(\Sigma) = W^G(M) = \mathcal{A}^G(M).$$

Proof. By Theorem 3.8, which is a min-max theorem, there exists an integral G-varifold $V \in \mathcal{V}_n^G(M)$ induced by a smooth embedded closed minimal G-hypersurface $\Sigma \in \mathcal{S}^G(M)$ so that $\|V\|(M) = W^G(M)$. Since M has positive Ricci curvature, Lemma 4.3(v) indicates that Σ is connected, and thus $V = m|\Sigma|$ for some $m \in \{1, 2, \dots\}$. Suppose $\Sigma \in \mathcal{S}_-^G(M)$, then it follows from the last statement in Theorem 3.8 that m must be even, so $m \geq 2$. However, we have a contradiction $W^G(M) < 2\,\mathrm{Area}(\Sigma) \leq \|V\|(M) = W^G(M)$ by Proposition 4.7. Therefore, $\Sigma \in \mathcal{S}_+^G(M)$. By Proposition 4.6, we see $W^G(M) \leq \mathrm{Area}(\Sigma) \leq \|V\|(M) = W^G(M)$, and thus $m = 1$. Additionally, by the definition of $\mathcal{A}^G(M)$ and Propositions 4.6, 4.7,

$$\mathcal{A}^G(M) \leq \mathrm{Area}(\Sigma) = \|V\|(M) = W^G(M) \leq \mathcal{A}^G(M).$$

Now, it is sufficient to show $\mathrm{Index}_G(\Sigma) = 1$. Suppose $\mathrm{Index}_G(\Sigma) \geq 2$, and u_1, u_2 are the first two L^2-orthonormal G-invariant eigenfunctions of $L_\Sigma \llcorner \mathfrak{X}^{\perp,G}(\Sigma)$ with negative eigenvalues. Let $u_2\nu$ be a G-invariant normal vector field on Σ, which extends to a smooth vector field $X \in \mathfrak{X}(M)$. Then $X_2 := \int_G (g^{-1})_* X \, d\mu(g) \in \mathfrak{X}^G(M)$ gives an equivariant extension of $u_2\nu$. Consider the equivariant diffeomorphisms $\{F_s^2\}$ generated by X_G, and define $\Phi_s(t) := (F_s^2)_\# \Phi(t)$ for $t \in [-1, 1]$, where $\Phi \in \mathcal{P}^G(M)$ is the $\boldsymbol{F}$-continuous sweepout given by Proposition 4.6. Recall that in the proof of Proposition 4.6, $\Phi(t) = [\![\Sigma_t]\!] = [\![F_t^1(\Sigma)]\!]$ for $t \in \left[-\frac{1}{3}, \frac{1}{3}\right]$, where $\{F_t^1\}$ are the equivariant diffeomorphism generated by $X_1 \in \mathfrak{X}^G(M)$ with $X_1 \llcorner \Sigma = 3ru_1\nu$ for some $r > 0$. Hence, for the smooth family $\{F_s^2(\Sigma_t)\}_{s \in [-\sigma,\sigma], t \in [-1/3, 1/3]}$, the area function $A(s,t) := \mathrm{Area}(F_s^2(\Sigma_t)) = \boldsymbol{M}(\Phi_s(t))$ satisfies that:

- $\nabla A(0,0) = 0$ since Σ is minimal.
- $\frac{\partial^2}{\partial t^2} A(0,0) = -9r^2 \int_\Sigma u_1 L_\Sigma u_1 < 0$ and $\frac{\partial^2}{\partial s^2} A(0,0) = -\int_\Sigma u_2 L_\Sigma u_2 < 0$.
- $\frac{\partial^2}{\partial s\,\partial t} A(0,0) = -3r \int_\Sigma u_2 L_\Sigma u_1 = 3r\mu_1(\Sigma) \int_\Sigma u_1 u_2 = 0$.

Therefore, we can set $\sigma, \delta > 0$ sufficiently small so that

$$\boldsymbol{M}(\Phi_s(t)) = \mathrm{Area}(F_s^2(\Sigma_t)) < \mathrm{Area}(\Sigma) \quad \text{for all } t \in [-\delta, \delta],\ s \in (0, \sigma].$$

Moreover, there exists $\epsilon > 0$ so that by Proposition 4.6(ii), $\boldsymbol{M}(\Phi(t)) \leq \mathrm{Area}(\Sigma) - \epsilon$ for all $t \in [-1, -\delta] \cup [\delta, 1]$. Hence, by setting $\sigma > 0$ even smaller, we have $\boldsymbol{M}(\Phi_\sigma(t)) = \boldsymbol{M}((F_\sigma^2)_\# \Phi(t)) < \mathrm{Area}(\Sigma)$ for all $t \in [-1, 1]$. Note Φ_σ is an $\boldsymbol{F}$-continuous curve homotopic to Φ in $\mathcal{Z}_n^G(M; \mathbb{Z}_2)$. Thus,

$$W^G(M) \leq \sup\{\boldsymbol{M}(\Phi_\sigma(t)) : t \in [-1, 1]\} < \mathrm{Area}(\Sigma) = W^G(M),$$

which is a contradiction. □

As an application, we use the conformal volume to show a genus bound for the equivariant min-max minimal G-hypersurface Σ in Theorem 5.1 provided that $\dim(M) = 3$ and the actions of G are orientation preserving.

Theorem 5.2. *Let (M^3, g_M) be a closed connected oriented Riemannian 3-manifold with positive Ricci curvature, and G be a finite group acting on M by orientation preserving isometries. Then the equivariant min-max hypersurface Σ corresponding to the fundamental class $[M]$ is a connected closed minimal G-surface of multiplicity one satisfying*

$$\mathrm{genus}(\Sigma) \le 4K \quad \text{and} \quad W^G(M) = \mathrm{Area}(\Sigma) \le \frac{8\pi K}{\inf_{|v|=1} \mathrm{Ric}_M(v, v)},$$

where $K := \max_{p\in M} \#G\cdot p \le \#G$ is the number of points in a principal orbit of M. Additionally, $\pi(\Sigma) = \Sigma/G$ is an orientable surface with finite cone singular points of order $\{n_i\}_{i=1}^k$ (i.e., locally modeled by $\mathbb{B}^2_1(0)$ quotient a cyclic rotation group $\mathbb{Z}_{n_i}$), so that

$$\sum_{i=1}^{k}\Big(1 - \frac{1}{n_i}\Big) < 4 \quad \text{and} \quad \mathrm{genus}(\pi(\Sigma)) \le 3.$$

In particular, if $\Sigma \subset M^{prin}$, i.e., $k = 0$, then $\mathrm{genus}(\Sigma) \le 1 + 2K$.

Proof. By Theorem 5.1, Σ is a closed embedded connected minimal G-surface with a G-invariant unit normal ν so that $\mathrm{Area}(\Sigma) = W^G(M)$ and $\mathrm{Index}_G(\Sigma) = 1$. By Lemma 4.3, Σ has an induced orientation. Additionally, since the unit normal ν is G-invariant, the actions of G on Σ are also orientation preserving. Therefore, the orbifold $\underline{\Sigma}$ induced by (Σ, G) is an orientable closed 2-orbifold whose underlying space is the quotient distance space $(\pi(\Sigma), \mathrm{dist}_{\Sigma/G})$.

Let Σ^{prin} be the union of principal orbits for the G-action on Σ, and $\underline{\Sigma}^{\mathrm{prin}}$ be the orbifold induced by $(\Sigma^{\mathrm{prin}}, G)$. Denote by $\boldsymbol{N}^{\Sigma}_p G\cdot p$ and $\boldsymbol{N}_p G\cdot p$ the normal vector spaces of $G\cdot p$ at p in Σ and M respectively. Note an orbit $G\cdot p$ is principal in Σ (resp. M) if and only if the slice representation of G_p on $\boldsymbol{N}^{\Sigma}_p G\cdot p$ (resp. $\boldsymbol{N}_p G\cdot p$) is trivial (see [4, Corollary 2.2.2]). Additionally, we also notice that G_p acts trivially on $\mathrm{span}(\nu(p))$ for any $p\in\Sigma$ by the G-invariance of ν. Hence, combining these with the fact that $\boldsymbol{N}^{\Sigma}_p G\cdot p \oplus \mathrm{span}(\nu(p)) = \boldsymbol{N}_p G\cdot p$, we see $\Sigma^{\mathrm{prin}} \subset M^{\mathrm{prin}}$, and thus $K = \#G\cdot p = \#G\cdot q$ for all $p\in\Sigma^{\mathrm{prin}}$ and $q\in M^{\mathrm{prin}}$. Next, it follows from [5, Chapter IV, Theorem 3.3] that there is an induced Riemannian metric $g_{\underline{\Sigma}}$ on $\underline{\Sigma}^{\mathrm{prin}}$ so that $\pi : \Sigma^{\mathrm{prin}} \to \underline{\Sigma}^{\mathrm{prin}}$ is an Riemannian submersion. Moreover, since G acts on Σ by orientation preserving isometries, the singular points $\underline{\Sigma}\setminus\underline{\Sigma}^{\mathrm{prin}}$ are a finite number of cone points $\{[p_i]\}_{i=1}^k$ of orders $n_1,\dots,n_k$. By the orbifold version of Gauss–Bonnet theorem (see [9, Proposition 2.17]), we have

$$\int_{\pi(\Sigma)} K_{\underline{\Sigma}}\, dA_{g_{\underline{\Sigma}}} = 2\pi(\chi(\underline{\Sigma})) = 2\pi\Big(2 - 2\,\mathrm{genus}(\pi(\Sigma)) - \sum_{i=1}^{k}\Big(1-\frac{1}{n_i}\Big)\Big), \tag{5-2}$$

where $K_{\underline{\Sigma}}$ is the Gauss curvature of $(\underline{\Sigma}^{\mathrm{prin}}, g_{\underline{\Sigma}})$, and the integral is taken over $\underline{\Sigma}^{\mathrm{prin}}$.

For any $r > 0$ small enough, let $\Sigma_r := \Sigma\setminus\cup_{i=1}^k B^G_r(p_i)$, and η_r be the G-invariant logarithmic cut-off function on Σ given by

$$\eta_r(p) := \begin{cases} 0, & d(p) \in [0, r], \\ 2 - \frac{2\log d(p)}{\log r}, & d(p) \in (r, \sqrt{r}], \\ 1, & d(p) \in (\sqrt{r}, \infty), \end{cases}$$

where

$$d(p) := \mathrm{dist}_\Sigma(p, \Sigma \setminus \Sigma^{\mathrm{prin}}) = \mathrm{dist}_\Sigma(p, \cup_{i=1}^k G \cdot p_i).$$

Define then $\underline{\Sigma}_r := \underline{\Sigma} \setminus \cup_{i=1}^k B_r([p_i])$. Note $(\underline{\Sigma}_r, g_{\underline{\Sigma}})$ is a smooth Riemannian manifold (with boundary). We can take any conformal immersion $\phi : \underline{\Sigma}_r \to \mathbb{S}^m$, $m \geq 2$, and define $P : \mathrm{Conf}(\mathbb{S}^m) \to \mathbb{B}_1^{m+1}(0)$ by

$$P(h) := \frac{1}{\int_\Sigma \eta_r u_1}\Big(\int_\Sigma (\eta_r u_1)(h_1 \circ \phi \circ \pi), \ldots, \int_\Sigma (\eta_r u_1)(h_{m+1} \circ \phi \circ \pi)\Big),$$

where $h = (h_1, \cdots, h_{m+1}) \in \mathrm{Conf}(\mathbb{S}^m)$ is any conformal diffeomorphism of $\mathbb{S}^m$ (under the standard metric), and $u_1 : \Sigma \to \mathbb{R}_+$ is the first (G-invariant) eigenfunction of L_Σ. Since $u_1 > 0$ and $\sum_{j=1}^{m+1} h_j^2 = 1$, one easily verifies that P is well defined. Meanwhile, for each $x \in \mathbb{B}^{m+1}$, define $h_x \in \mathrm{Conf}(\mathbb{S}^m)$ as in [27, (1.1)] by

$$h_x(y) := \frac{y + (\mu\langle x, y\rangle + \lambda)\, x}{\lambda(\langle x, y\rangle + 1)}, \quad \text{with } \lambda := (1 - |x|^2)^{-1/2},\ \mu := (\lambda - 1)|x|^{-2}.$$

Then we have a continuous map $f : \mathbb{B}_1^{m+1}(0) \to \mathbb{B}_1^{m+1}(0)$ given by $f(x) = P(h_x)$, which can be continuously extended to $\partial\mathbb{B}_1^{m+1}(0) = \mathbb{S}^m$ by the identity map. Note $\mathrm{Clos}(\mathbb{B}_1^{m+1}(0))$ is homotopic to $f\big(\mathrm{Clos}(\mathbb{B}_1^{m+1}(0))\big)$, and $\mathrm{Clos}(\mathbb{B}_1^{m+1}(0)) \setminus \{x\}$ is homotopic to $\mathbb{S}^m$ for any $x \in \mathbb{B}_1^{m+1}(0)$. Hence, f must be surjective. In particular, there exists $h = (h_1, \ldots, h_{m+1}) \in \mathrm{Conf}(\mathbb{S}^m)$ so that $P(h) = 0$. Thus, we have that $\{\tilde{h}_j := h_j \circ \phi \circ \pi\}_{j=1}^{m+1}$ are G-invariant smooth functions on Σ_r so that

$$\sum_{j=1}^{m+1} \tilde{h}_j^2 = 1 \quad \text{and} \quad \int_\Sigma u_1 \cdot (\eta_r \tilde{h}_j) = 0 \qquad \text{for all } j = 1, \ldots, m+1.$$

Since $\mathrm{Index}_G(\Sigma) = 1$, we see $\delta^2\Sigma(\eta_r \tilde{h}_j \nu) \geq 0$ for all $j = 1, \ldots, m+1$, and

$$\begin{aligned}
\int_{\Sigma_{\sqrt{r}}} \mathrm{Ric}_M(\nu, \nu) + |A|^2 &\leq \int_\Sigma (\mathrm{Ric}_M(\nu, \nu) + |A|^2)\, \eta_r^2 \\
&= \int_\Sigma (\mathrm{Ric}_M(\nu, \nu) + |A|^2) \sum_{j=1}^{m+1} (\eta_r \tilde{h}_j)^2 \\
&\leq \int_\Sigma \sum_{j=1}^{m+1} |\nabla(\eta_r \tilde{h}_j)|^2 \\
&\leq \int_\Sigma \sum_{j=1}^{m+1} \Big[(1+\epsilon)|\nabla \tilde{h}_j|^2 \eta_r^2 + \Big(1 + \frac{1}{\epsilon}\Big)|\nabla \eta_r|^2 \tilde{h}_j^2\Big] \\
&\leq (1+\epsilon)\, K \cdot \int_{\underline{\Sigma}_r} \sum_{j=1}^{m+1} |\nabla h_j \circ \phi|^2 + \Big(1 + \frac{1}{\epsilon}\Big)\int_\Sigma |\nabla \eta_r|^2 \\
&= 2(1+\epsilon) K \cdot \mathrm{Area}(\underline{\Sigma}_r;\, (h \circ \phi)^* g_{\mathbb{S}^{m+1}}) + \Big(1 + \frac{1}{\epsilon}\Big)\int_\Sigma |\nabla \eta_r|^2,
\end{aligned}$$

where $\epsilon > 0$ is any constant, $\mathrm{Area}(\underline{\Sigma}_r; (h\circ\phi)^* g_{\mathbb{S}^{m+1}})$ is the area of $\underline{\Sigma}_r$ under the conformal metric $(h\circ\phi)^* g_{\mathbb{S}^{m+1}}$, and the coarea formula is used in the last inequality. Let $A_c(m, \underline{\Sigma}_r)$ be the *m-conformal area* of $\underline{\Sigma}_r$ defined as in [18]:

$$A_c(m, \underline{\Sigma}_r) := \inf_{\phi} \sup_{h\in \mathrm{Conf}(\mathbb{S}^m)} \mathrm{Area}(\underline{\Sigma}_r; (h\circ\phi)^* g_{\mathbb{S}^m}),$$

where the infimum is taken over all nondegenerated conformal map ϕ of $\underline{\Sigma}_r$ into $\mathbb{S}^m$. Since $\phi : \underline{\Sigma}_r \to \mathbb{S}^m$ is arbitrary conformal immersion in the above computation, we have

$$\int_{\Sigma_{\sqrt{r}}} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 2(1+\epsilon) K \cdot A_c(m, \underline{\Sigma}_r) + \left(1+\frac{1}{\epsilon}\right)\int_{\Sigma} |\nabla\eta_r|^2.$$

By [14, Chapter IV, Remark 5.5.1], every closed orientable surface can be conformally branched over $\mathbb{S}^2$ with degree $\lfloor \frac{\text{genus}+3}{2} \rfloor$, where $\lfloor a \rfloor$ is the integer part of $a \in \mathbb{R}_+$. It then follows from [18, Facts 1, 5] that $A_c(m, \underline{\Sigma}_r) \le 4\pi \lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \rfloor$, and thus

$$\int_{\Sigma_{\sqrt{r}}} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 4\pi(1+\epsilon) K \cdot 2\left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor + \left(1+\frac{1}{\epsilon}\right)\int_{\Sigma} |\nabla\eta_r|^2.$$

Since $\int_\Sigma |\nabla\eta_r|^2 \to 0$ as $r \to 0$, we first take $r \to 0$ and then let $\epsilon \to 0$, which gives

$$\int_{\Sigma} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 4\pi K \cdot 2\left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor.$$

Denote by $\{e_i\}_{i=1}^2$ a local orthonormal basis on Σ. Since $\mathrm{Ric}_M > 0$, we have

$$\mathrm{Ric}_M(\nu,\nu) + |A|^2 = \sum_{i=1}^{2} \mathrm{Ric}_M(e_i, e_i) - 2K_\Sigma > -2K_\Sigma$$

on Σ^{prin}, where K_Σ is the Gauss curvature of Σ. Therefore, by the coarea formula,

$$-2K \int_{\underline{\Sigma}} K_{\underline{\Sigma}} = -2\int_{\Sigma} K_\Sigma < \int_{\Sigma} \mathrm{Ric}_M(\nu,\nu) + |A|^2 \le 4\pi K \cdot 2\left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor.$$

Then, it follows from the above *strict* inequality and the Gauss–Bonnet formula (5-2) that $\mathrm{genus}(\pi(\Sigma)) + \sum_{i=1}^k \left(1-\frac{1}{n_i}\right) < 5$ and

$$\mathrm{genus}(\Sigma) = 1 + K\left[\mathrm{genus}(\pi(\Sigma)) - 1 + \sum_{i=1}^{k}\left(1-\frac{1}{n_i}\right)\right] < 1 + 4K.$$

Moreover, one notices that $2\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \rfloor = \mathrm{genus}(\pi(\Sigma)) + 3$ if $\mathrm{genus}(\pi(\Sigma)) \ge 1$ is odd, and $2\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \rfloor = \mathrm{genus}(\pi(\Sigma)) + 2$ if $\mathrm{genus}(\pi(\Sigma)) \ge 0$ is even. Hence, the above computation actually shows

- $\mathrm{genus}(\pi(\Sigma)) + \sum_{i=1}^k \left(1-\frac{1}{n_i}\right) < 5$ if $\mathrm{genus}(\pi(\Sigma)) \ge 1$ is odd, and
- $\mathrm{genus}(\pi(\Sigma)) + \sum_{i=1}^k \left(1-\frac{1}{n_i}\right) < 4$ if $\mathrm{genus}(\pi(\Sigma)) \ge 0$ is even,

which further implies that $\mathrm{genus}(\pi(\Sigma)) \le 3$ and $\sum_{i=1}^k (1-\frac{1}{n_i}) < 4$. In particular, if $\Sigma \subset M^{\mathrm{prin}}$, then $\sum_{i=1}^k (1-\frac{1}{n_i}) = 0$ and

$$\mathrm{genus}(\Sigma) = 1 + K(\mathrm{genus}(\pi(\Sigma)) - 1) \le 1 + 2K.$$

Finally, we see that

$$\begin{aligned} 2c_M W^G(M) &\le \int_\Sigma \sum_{i=1}^2 \mathrm{Ric}_M(e_i, e_i) \\ &\le 4\pi K \cdot 2\left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor + 2K \int_{\underline{\Sigma}} K_{\underline{\Sigma}} \\ &= 4\pi K \cdot \left(2 - 2\,\mathrm{genus}(\pi(\Sigma)) - \sum_{i=1}^k \left(1 - \frac{1}{n_i}\right) + 2\left\lfloor \frac{\mathrm{genus}(\pi(\Sigma))+3}{2} \right\rfloor \right) \\ &\le 16\pi K, \end{aligned}$$

where $\mathrm{Ric}_M \ge c_M > 0$. □

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