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LONG-TIME BEHAVIOR OF AWESOME HOMOGENEOUS RICCI FLOWS

ROBERTO ARAUJO

We show that the set of *awesome* homogeneous metrics on noncompact manifolds is Ricci flow invariant and that if the universal cover of an awesome homogeneous space is not contractible, the Ricci flow has finite extinction time, confirming the dynamical Alekseevskii conjecture in this case. We also analyze the long-time limits of awesome homogeneous Ricci flows.

1. Introduction

The Ricci flow is the geometric evolution equation given by

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{ric}(g(t)), \quad g(0) = g_0,$$

where $\mathrm{ric}(g)$ is the Ricci $(2,0)$-tensor of the Riemannian manifold (M, g).

Hamilton [1982] introduced the Ricci flow and proved short-time existence and uniqueness when M is compact. Then Chen and Zhu [2006] proved the uniqueness of the flow within the class of complete and bounded curvature Riemannian manifolds.

A maximal Ricci flow solution $g(t)$, $t \in [0, T)$, is called *immortal* if $T = +\infty$, otherwise we say that the flow has *finite extinction time*.

A Riemannian manifold (M, g) is called *homogeneous* if its isometry group acts transitively on it. From the uniqueness of a Ricci flow solution it follows immediately that the isometries are preserved along the flow; thus a solution $g(t)$ from a homogeneous initial metric g_0 would remain homogeneous for the same isometric action. Hence, the Ricci flow equation given above becomes an autonomous nonlinear ordinary differential equation.

In the homogeneous case, the scalar curvature is increasing along the flow (see [Lafuente 2015]). Furthermore, if the scalar curvature is positive at some point along the flow, then it must blow up in finite time, and hence, the solution is not immortal.

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Lafuente [2015] has shown that actually a homogeneous Ricci flow solution has finite extinction time if and only if the scalar curvature blows up in finite time, or equivalently, if and only if the scalar curvature ever becomes positive along the flow. Bérard-Bergery [1978] has shown that a manifold admits a homogeneous Riemannian metric of positive scalar curvature if and only if its universal cover is not diffeomorphic to Euclidean space.

Böhm and Lafuente [2018] then proposed the problem of showing whether the converse is also true, namely they asked whether the universal cover of an immortal homogeneous Ricci flow solution is always diffeomorphic to $\mathbb{R}^n$. This got established later as the *dynamical Alekseevskii conjecture* [Naber et al. 2022].

Böhm [2015, Theorem 3.2] showed that the conjecture is true for the case of compact homogeneous manifolds. However, in the noncompact case not much is known in the direction of the dynamical Alekseevskii conjecture other than in low-dimensions. Isenberg and Jackson [1992] thoroughly studied the 3-dimensional homogeneous Ricci flow, and in [Isenberg et al. 2006] the authors studied a large set of metrics on dimension 4. Indeed, up to dimension 4 the conjecture is true (see [Araujo 2024]). In that article, it was shown that the conjecture is true if the isometry group of the homogeneous Riemannian manifold is, up to a covering, a Lie group product with a compact semisimple factor; which generalizes [Böhm 2015, Theorem 3.2].

We study the long-time behavior of the homogeneous Ricci flow solutions on semisimple homogeneous spaces on a special family of Ricci flow invariant metrics, called *awesome metrics*.

Let G be a semisimple Lie group and G/H a homogeneous Riemannian manifold. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ the Lie algebra of H. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$, K the integral subgroup corresponding to $\mathfrak{k}$, and $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}$ a reductive complement to $\mathfrak{h}$. Then a G-invariant metric g on G/H is *awesome* if $g(\mathfrak{l}, \mathfrak{p}) = 0$.

The set of awesome homogeneous metrics was introduced by Nikonorov [2000], where he proved that it contains no Einstein metric. Semisimple homogeneous spaces with inequivalent irreducible summands in its isotropy representation supply the simplest examples of homogeneous spaces G/H such that every G-invariant metric is awesome.

Our first main result is a generalization of [Nikonorov 2000, Theorem 1] to the dynamical setting, giving a partial positive answer to the conjecture.

Theorem A. *Let $(G/H, g_0)$ be a homogeneous Riemannian manifold such that the universal cover is not diffeomorphic to $\mathbb{R}^n$ and G is semisimple. If g_0 is an awesome G-invariant metric, then the Ricci flow solution starting at g_0 has finite extinction time.*

Dotti and Leite [1982] have shown that $\mathrm{SL}(n, \mathbb{R})$ for $n \geq 3$ admit left-invariant Ricci negative metrics. Later, Dotti, Leite and Miatello [Dotti et al. 1984] were able to extend this result by showing that all but a finite collection of noncompact simple Lie groups admit a Ricci negative left-invariant metric. All those metrics are awesome. This shows that for these spaces such that the universal cover $\tilde{M}$ is not contractible the confirmation of the dynamical Alekseevskii conjecture implies a change of regime of the Ricci flow: from one in which the manifold expands in all directions to one such that for some direction it shrinks in finite time. Böhm's proof [2015, Theorem 3.2] of the finite extinction time of the Ricci flow on nontoral compact homogeneous manifolds works by showing an explicit preferred direction in which the curvature is Ricci positive (the same approach is followed in [Araujo 2024]), but Dotti, Leite and Miatello's results indicate that we cannot directly do the same here.

Indeed, in order to prove Theorem A we need to first prove some scale-invariant pinching estimates (Proposition 4.4) that will eventually lead to the existence of a Ricci positive direction given by the nontoral compact fibers as in [Böhm 2015]. The estimates obtained can be exploited further to prove the following two convergence results.

Theorem B. *Let $M = G/H$ be a homogeneous manifold, such that the universal cover is not diffeomorphic to $\mathbb{R}^n$ and G is semisimple. Let $(M, g(t))$, $t \in [0, T)$, be an awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $R(g)$ be the scalar curvature of the metric g. For any sequence $(t_a)_{a\in\mathbb{N}}$, $t_a \to T$, there exists a subsequence such that $(M, R(g(t_{\hat{a}})) \cdot g(t_{\hat{a}}))$ converges in pointed C^∞-topology to the Riemannian product*

$$E_\infty \times \mathbb{E}^d,$$

where E_∞ is a compact homogeneous Einstein manifold with positive scalar curvature and $\mathbb{E}^d$ is the d-dimensional (flat) Euclidean space with $d \geq \dim \mathfrak{p}$.

The geometry of E_∞ just depends on the subsequence of Riemannian submanifolds $(K/H, R(g(t_{\hat{a}})) \cdot g(t_{\hat{a}}))$.

Theorem C. *Let $\tilde{M} = G/H$ be a homogeneous manifold diffeomorphic to $\mathbb{R}^n$ with G semisimple. Let $(\tilde{M}, g(t))$, $t \in [1, \infty)$, be an awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the parabolic rescaling $(\tilde{M}, t^{-1} g(t))$ converges in pointed C^∞-topology to the Riemannian product*

$$\Sigma_\infty \times \mathbb{E}^{\dim \mathfrak{l}},$$

where $\Sigma_\infty = (G/K, B|_{\mathfrak{p}\times\mathfrak{p}})$ is the noncompact Einstein symmetric space defined by the pair $(\mathfrak{g}, \mathfrak{k})$ and $\mathbb{E}^{\dim \mathfrak{l}}$ is the $\dim \mathfrak{l}$-dimensional (flat) Euclidean space.

Theorem B shows that in order to understand the blow-up limits of the Ricci flow on the awesome metrics we can reduce the investigation to the corresponding blow-up of the compact homogeneous fibers given by the Cartan decomposition. Such analysis was done, for example, in [Böhm 2015].

Also since every left-invariant metric on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is awesome, Theorem C is a generalization of the result by Lott [2007] which states that the parabolic blow-down of any left-invariant metric in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ converges to the Riemannian product $\mathbb{H}^2\times\mathbb{R}$.

The structure of this article is the following: In Section 2, we give a quick overview of the homogeneous Ricci flow and show that the space of awesome metrics is Ricci flow invariant. In Section 3, we mainly establish a priori algebraic bounds that exploit the compatibility of the Cartan decomposition and the metric in the awesome case. In Section 4, we use these algebraic bounds to get control quantities to our dynamics which allows us to prove Theorem A. Finally, in Section 5 we conclude with the analysis of the long-time limits. In particular, under the hypothesis of Theorem A, we show in Theorem B a rigidity result for the possible limit geometries as the solution approaches the singularity. We finish by showing in Theorem C which is the limit geometry at infinity for the case when $g(t)$ is an immortal awesome Ricci flow. This generalizes the work on the Ricci flow of left-invariant metrics on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ done in [Lott 2007] to $\mathbb{R}^d$-bundles over Hermitian symmetric spaces.

2. Homogeneous Ricci flow of awesome metrics

A Riemannian manifold (M^n, g) is said to be homogeneous if its isometry group $I(M, g)$ acts transitively on M. If M is connected (which we will assume from here onward unless otherwise stated), then each transitive closed Lie subgroup $G < I(M, g)$ gives rise to a presentation of (M, g) as a homogeneous space with a G-invariant metric $(G/H, g)$, where H is the isotropy subgroup of G fixing some point $p \in M$.

Let us denote the Lie algebra of G by $\mathfrak{g}$. The G-action induces a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$ assigning to each $X \in \mathfrak{g}$ a Killing field on (M, g), also denoted by X, and given by

$$X(q) := \left(\frac{d}{dt}\exp(tX)\cdot q\right)\bigg|_{t=0}, \quad q \in M.$$

If $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup $H < G$ fixing $p \in M$, then it can be characterized as those $X \in \mathfrak{g}$ such that $X(p) = 0$. Given that, we can take a complementary $\mathrm{Ad}(H)$-module $\mathfrak{m}$ to $\mathfrak{h}$ in $\mathfrak{g}$ and identify $\mathfrak{m} \cong T_pM$ via the above infinitesimal correspondence.

A homogeneous space G/H is called *reductive* if there exists such a complementary vector space $\mathfrak{m}$ such that for the respective Lie algebras of G and H

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}.$$

This is always possible in the case of homogeneous Riemannian manifolds. This is due to a classic result on Riemannian geometry [do Carmo 1992, Chapter VIII, Lemma 4.2], which states that an isometry is uniquely determined by the image of the point p and its derivative at p; hence the isotropy subgroup H is a closed subgroup of $\mathrm{SO}(T_pM)$, and in particular it is compact. Indeed, if $\mathrm{Ad}(H)$ is compact, then one can average over an arbitrary inner product over $\mathfrak{g}$ to make it $\mathrm{Ad}(H)$-invariant and hence take $\mathfrak{m} := \mathfrak{h}^\perp$. Given that, one can identify $\mathfrak{m} \cong T_{eH}G/H$ once and for all and with this identification there is a one-to-one correspondence between homogeneous metrics in $M := G/H$, $p \cong eH$, and $\mathrm{Ad}(H)$-invariant inner products in $\mathfrak{m}$.

In full generality, the Ricci flow is a nonlinear partial differential equation. As mentioned in the introduction, in the case where M is compact, Hamilton [1982] proved short-time existence and uniqueness for the Ricci flow. Then Chen and Zhu [2006] proved the uniqueness of the flow within the class of complete and bounded curvature Riemannian manifolds, which includes the class of homogeneous manifolds. From its uniqueness it follows immediately that the Ricci flow preserves isometries. Thus a solution $g(t)$ from a G-invariant initial metric g_0 would remain G-invariant, and hence it is quite natural to consider a *homogeneous Ricci flow*. We then get an autonomous nonlinear ordinary differential equation,

$$\frac{dg(t)}{dt} = -2\,\mathrm{ric}(g(t)), \quad g(0) = g_0, \tag{2-1}$$

where the Ricci tensor can being seen as the smooth map

$$\mathrm{ric} : (\mathrm{Sym}^2(\mathfrak{m}))_+^{\mathrm{Ad}(H)} \to (\mathrm{Sym}^2(\mathfrak{m}))^{\mathrm{Ad}(H)}.$$

Here $(\mathrm{Sym}^2(\mathfrak{m}))^{\mathrm{Ad}(H)}$ is the nontrivial vector space of $\mathrm{Ad}(H)$-invariant symmetric bilinear forms in $\mathfrak{m}$ and $(\mathrm{Sym}^2(\mathfrak{m}))_+^{\mathrm{Ad}(H)}$ the open set of positive definite ones. By classical ODE theory, given an initial G-invariant metric g_0 corresponding to an initial $\mathrm{Ad}(H)$-invariant inner product, there is a unique $\mathrm{Ad}(H)$-invariant inner product solution corresponding to a unique family of G-invariant metrics $g(t)$ in M.

The general formula for the Ricci curvature of a homogeneous Riemannian manifold $(G/H, g)$ [Besse 1987, Corollary 7.38] is given by

$$\mathrm{ric}_g(X, X) = -\frac{1}{2}B(X, X) - \frac{1}{2}\sum_i \|[X, X_i]_\mathfrak{m}\|_g^2 + \frac{1}{4}\sum_{i,j} g([X_i, X_j]_\mathfrak{m}, X)^2 - g([H_g, X]_\mathfrak{m}, X), \tag{2-2}$$

where B is the Killing form, $\{X_i\}_{i=1}^n$ is a g-orthonormal basis of $\mathfrak{m}$ and H_g is the mean curvature vector defined by $g(H_g, X) := \mathrm{Tr}(\mathrm{ad}_X)$. Immediately it follows that $H_g = 0$ if and only if $\mathfrak{g}$ is unimodular.

Let us now consider $\mathfrak{g}$ to be a noncompact semisimple Lie algebra. By classical structure theory on semisimple Lie algebras [Hilgert and Neeb 2012, Chapter 13], $\mathfrak{g}$ can be described in terms of its *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is a compactly embedded Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is a $\mathfrak{k}$-submodule such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

The Killing form B of $\mathfrak{g}$ is such that

$$B(\mathfrak{k}, \mathfrak{p}) = 0, \quad B|_{\mathfrak{k}\times\mathfrak{k}} < 0, \quad B|_{\mathfrak{p}\times\mathfrak{p}} > 0,$$

and $-B|_{\mathfrak{k}\times\mathfrak{k}} + B|_{\mathfrak{p}\times\mathfrak{p}}$ is an inner product on $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{k})$ are skew-symmetric maps and $\mathrm{ad}(\mathfrak{p})$ are symmetric maps [Hilgert and Neeb 2012, Lemma 13.1.3].

Since the flow only depends on the Lie algebra $\mathfrak{g}$, we can take without loss of generality any G connected with such Lie algebra. So for a $M = G/H$, with G a semisimple noncompact Lie group, we can fix a background Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

such that the integral subgroup K of $\mathfrak{k}$ is a maximal connected compact subgroup of G with $H \subset K$ [Hilgert and Neeb 2012, Theorem 14.1.3]. We call a homogeneous manifold G/H, with G semisimple and $\mathrm{Ad}(H)$ compact, a *semisimple homogeneous space*.

Consider the orthogonal complement $\mathfrak{l} := \mathfrak{h}^{\perp}$ of $\mathfrak{h}$ in $\mathfrak{k}$ with respect to the Killing form B. And let us do the identification

$$T_{eH}G/H \cong \mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}. \tag{2-3}$$

We will call then this reductive complement $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}$ *adapted* to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Definition 2.1 (awesome metric). Let G/H be a homogeneous space with G semisimple and $\mathrm{Ad}(H)$ compact. An $\mathrm{Ad}(H)$-invariant inner product g on the reductive complement $\mathfrak{m}$ is called *awesome* if for some Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for which $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}$ is adapted, we have that $g(\mathfrak{l}, \mathfrak{p}) = 0$. In this case, we say that the awesome metric g is adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{l}$.

As it was mentioned in the introduction, this nonempty set of metrics was introduced by Nikonorov [2000], where he proved that the set contains no Einstein metric. We will see that this set is actually Ricci flow invariant, proving to be a good test ground concerning the dynamical Alekseevskii conjecture. Semisimple homogeneous spaces G/H such that the isotropy representation of H on $\mathfrak{m}$ have inequivalent irreducible summands only admit awesome G-invariant metrics, and as such this set of metrics has an obvious spotlight in the literature.

On the other hand, for example, in the case of a Lie group with dimension larger than 3, the set of awesome metrics is a meager subset of the left-invariant metrics, and its dynamical properties under the phase space of the Ricci flow are largely unknown. Nikonorov [2000, Theorem 2] also gave a necessary and sufficient algebraic condition for a semisimple homogeneous space G/H to be such that every G-invariant metric is awesome.

Via (2-3), we have a one-to-one correspondence between the set of awesome G-invariant metrics on G/H and the open subset of positive definite $\mathrm{Ad}(H)$-invariant symmetric bilinear forms in $\mathfrak{m}$ such that $\mathfrak{l} \perp \mathfrak{p}$, which in turn is a linear subspace of $(\mathrm{Sym}^2(\mathfrak{m}))^{\mathrm{Ad}(H)}$.

Manipulating the Ricci tensor formula (2-2) on the awesome case we can directly prove the following lemma.

Lemma 2.2. *Let G/H be a semisimple homogeneous space. Then the set of G-invariant awesome metrics in G/H is Ricci flow invariant.*

Proof. Let G/H be a semisimple homogeneous space. We need to show that the Ricci operator $\mathrm{ric} : (\mathrm{Sym}^2(\mathfrak{m}))_+^{\mathrm{Ad}(H)} \to (\mathrm{Sym}^2(\mathfrak{m}))^{\mathrm{Ad}(H)}$ takes an element g such that $g(\mathfrak{l}, \mathfrak{p}) = 0$ to a symmetric bilinear form ric_g such that $\mathrm{ric}_g(\mathfrak{l}, \mathfrak{p}) = 0$. By polarizing the formula for the Ricci tensor (2-2) on a homogeneous manifold G/H with G unimodular, we get

$$2\,\mathrm{ric}_g(X, Y) = \\ -B(X, Y) - \sum_i g([X, X_i]_{\mathfrak{m}}, [Y, X_i]_{\mathfrak{m}}) + \frac{1}{2} \sum_{i,j} g([X_i, X_j]_{\mathfrak{m}}, X) g([X_i, X_j]_{\mathfrak{m}}, Y).$$

In our case

$$\{X_1, \ldots, X_{n+m}\} = \{X_1^{\mathfrak{l}}, \ldots, X_n^{\mathfrak{l}}, X_1^{\mathfrak{p}}, \ldots, X_m^{\mathfrak{p}}\},$$

where $\{X_i^{\mathfrak{l}}\}_{i=1}^n$ and $\{X_i^{\mathfrak{p}}\}_{i=1}^m$ are g-orthonormal bases for $\mathfrak{l}$ and $\mathfrak{p}$, respectively. We then get that for $X^{\mathfrak{l}} \in \mathfrak{l}$ and $X^{\mathfrak{p}} \in \mathfrak{p}$

$$\begin{aligned} 2\,\mathrm{ric}_g(X^{\mathfrak{l}}, X^{\mathfrak{p}}) &= -B(X^{\mathfrak{l}}, X^{\mathfrak{p}}) - \sum_i g([X^{\mathfrak{l}}, X_i]_{\mathfrak{m}}, [X^{\mathfrak{p}}, X_i]_{\mathfrak{m}}) \\ &\qquad + \frac{1}{2} \sum_{i,j} g([X_i, X_j]_{\mathfrak{m}}, X^{\mathfrak{l}}) g([X_i, X_j]_{\mathfrak{m}}, X^{\mathfrak{p}}) \\ &= -\sum_i g([X^{\mathfrak{l}}, X_i^{\mathfrak{l}}]_{\mathfrak{m}}, [X^{\mathfrak{p}}, X_i^{\mathfrak{l}}]_{\mathfrak{m}}) - \sum_i g([X^{\mathfrak{l}}, X_i^{\mathfrak{p}}]_{\mathfrak{m}}, [X^{\mathfrak{p}}, X_i^{\mathfrak{p}}]_{\mathfrak{m}}) \\ &\qquad + \frac{1}{2} \sum_{i,j} g([X_i, X_j]_{\mathfrak{m}}, X^{\mathfrak{l}}) g([X_i, X_j]_{\mathfrak{m}}, X^{\mathfrak{p}}) \\ &= \frac{1}{2} \sum_{i,j} g([X_i, X_j]_{\mathfrak{m}}, X^{\mathfrak{l}}) g([X_i, X_j]_{\mathfrak{m}}, X^{\mathfrak{p}}). \end{aligned}$$

We used that $B(\mathfrak{l}, \mathfrak{p}) = 0$ in the second equality and in the third equality we used both the Cartan decomposition relations $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and that

$g(\mathfrak{l},\mathfrak{p})=0$. Moreover, by the same reason, we have that

$$\begin{aligned}\sum_{i,j} g([X_i,X_j]_{\mathfrak{m}},X^{\mathfrak{l}})g([X_i,X_j]_{\mathfrak{m}},X^{\mathfrak{p}}) &= \sum_{i,j} g([X_i^{\mathfrak{l}},X_j^{\mathfrak{l}}]_{\mathfrak{m}},X^{\mathfrak{l}})g([X_i^{\mathfrak{l}},X_j^{\mathfrak{l}}]_{\mathfrak{m}},X^{\mathfrak{p}})\\ &\quad+\sum_{i,j} g([X_i^{\mathfrak{p}},X_j^{\mathfrak{p}}]_{\mathfrak{m}},X^{\mathfrak{l}})g([X_i^{\mathfrak{p}},X_j^{\mathfrak{p}}]_{\mathfrak{m}},X^{\mathfrak{p}})\\ &\quad+2\sum_{i,j} g([X_i^{\mathfrak{l}},X_j^{\mathfrak{p}}]_{\mathfrak{m}},X^{\mathfrak{l}})g([X_i^{\mathfrak{l}},X_j^{\mathfrak{p}}]_{\mathfrak{m}},X^{\mathfrak{p}})\\ &=0.\end{aligned}$$

This means that the set of awesome metrics $\{g\in(\mathrm{Sym}^2(\mathfrak{m}))_+^{\mathrm{Ad}(H)}\mid g(\mathfrak{l},\mathfrak{p})=0\}$ is an invariant subset for the Ricci flow equation (2-1). □

Remark 2.3. Nikonorov [2000, Example 1] had already argued that $\mathrm{ric}_g(\mathfrak{l},\mathfrak{p})=0$ for g awesome, in the particular case of the homogeneous space $\mathrm{SO}(n,2)/\mathrm{SO}(n)$, $n\geq 2$. The isotropy representation of $\mathrm{SO}(n,2)/\mathrm{SO}(n)$, $n\geq 2$, has three summands $\mathfrak{l}_1\subset\mathfrak{k}$, $\mathfrak{p}_1\subset\mathfrak{p}$, and $\mathfrak{p}_2\subset\mathfrak{p}$, and moreover $\mathfrak{l}_1$ is not isomorphic to $\mathfrak{p}_1$ or $\mathfrak{p}_2$; thus any $\mathrm{SO}(n,2)$-invariant metric is awesome. He works this example out in more detail in order to show that $\mathrm{SO}(n,2)/\mathrm{SO}(n)$ admits $\mathrm{SO}(n,2)$-invariant Ricci negative metrics but no Einstein metric.

3. Algebraic bounds for the Ricci curvature

We want to understand the long-time behavior of an awesome metric under the homogeneous Ricci flow. In order to do that we want to compare an arbitrary awesome metric g to a highly symmetric background metric.

Let us fix $Q:=-B|_{\mathfrak{l}\times\mathfrak{l}}+B|_{\mathfrak{p}\times\mathfrak{p}}$ as a background metric. For a given $\mathrm{Ad}(H)$-invariant inner product g on $\mathfrak{m}$, by Schur's lemma, we can decompose it on Q-orthogonal irreducible $\mathfrak{h}$-modules $\mathfrak{m}=\bigoplus_{i=1}^N\mathfrak{m}_i$ such that

$$g=x_1\cdot Q|_{\mathfrak{m}_1\times\mathfrak{m}_1}\perp\cdots\perp x_N\cdot Q|_{\mathfrak{m}_N\times\mathfrak{m}_N},$$

for some positive numbers $x_1,\dots,x_N\in\mathbb{R}$. Note that this decomposition is not necessarily unique, except in the case where all irreducible modules are pairwise inequivalent. Also by Schur's lemma, in each irreducible summand $\mathfrak{m}_i$ the Ricci tensor is given by $\mathrm{ric}_g|_{\mathfrak{m}_i\times\mathfrak{m}_i}=r_i\cdot g|_{\mathfrak{m}_i\times\mathfrak{m}_i}$, for $r_1,\dots,r_N\in\mathbb{R}$. Observe that, in general, the mixed terms $\mathrm{ric}_g(\mathfrak{m}_i,\mathfrak{m}_j)$ for $i\neq j$ are not zero when $\mathfrak{m}_i$ is equivalent to $\mathfrak{m}_j$ as $\mathfrak{h}$-modules.

Now let g be an awesome metric. Then there is a Cartan decomposition such that $g(\mathfrak{l},\mathfrak{p})=0$; hence we can adapt the above decomposition so that

$$g=l_1\cdot Q|_{\mathfrak{l}_1\times\mathfrak{l}_1}\perp\cdots\perp l_n\cdot Q|_{\mathfrak{l}_v\times\mathfrak{l}_v}\perp p_1\cdot Q|_{\mathfrak{p}_1\times\mathfrak{p}_1}\perp\cdots\perp p_m\cdot Q|_{\mathfrak{p}_m\times\mathfrak{p}_m},\tag{3-1}$$

where $(\mathfrak{l}_1, \dots, \mathfrak{l}_n, \mathfrak{p}_1, \dots, \mathfrak{p}_m) = (\mathfrak{m}_1, \dots, \mathfrak{m}_{n+m})$ with

$$\mathfrak{l} = \bigoplus_{i=1}^{n} \mathfrak{l}_i \quad \text{and} \quad \mathfrak{p} = \bigoplus_{i=1}^{m} \mathfrak{p}_i,$$

and $(l_1, \dots, l_n, p_1, \dots, p_m) = (x_1, \dots, x_{n+m})$.

Let us establish the notation $I_\mathfrak{l} := \{1, \dots, n\}$ and $I_\mathfrak{p} := \{n+1, \dots, n+m\}$, and let d_i denote the dimension of $\mathfrak{m}_i$ for all $i \in \{1, \dots, n+m\}$. To simplify notation we are going to write $\hat{i} := n+i$. Finally, let us define

$$r_i^\mathfrak{l} \cdot g|_{\mathfrak{l}_i \times \mathfrak{l}_i} = \mathrm{ric}_g|_{\mathfrak{l}_i \times \mathfrak{l}_i} \quad \text{for } i = 1, \dots, n \tag{3-2}$$

and

$$r_i^\mathfrak{p} \cdot g|_{\mathfrak{p}_i \times \mathfrak{p}_i} = \mathrm{ric}_g|_{\mathfrak{p}_i \times \mathfrak{p}_i} \quad \text{for } i = 1, \dots, m, \tag{3-3}$$

where $(r_1^\mathfrak{l}, \dots, r_n^\mathfrak{l}, r_1^\mathfrak{p}, \dots, r_m^\mathfrak{p}) = (r_1, \dots, r_{n+m})$.

Let us take the following Q-orthonormal basis on $\mathfrak{g}$, $\{E_\alpha^0\}$ for $1 \le \alpha \le n$ on $\mathfrak{h}$, and $\{E_\alpha^i\}$ for $1 \le \alpha \le d_i$ on each $\mathfrak{m}_i$, $i = 1, \dots, n+m$. Then we can define the brackets coefficients

$$[ijk] := \sum_{\alpha,\beta,\gamma} Q([E_\alpha^i, E_\beta^j], E_\gamma^k)^2.$$

By the Cartan decomposition we get that $\mathrm{ad}(\mathfrak{k})$ are skew-symmetric and $\mathrm{ad}(\mathfrak{p})$ are symmetric [Hilgert and Neeb 2012, Lemma 13.1.3]; therefore the coefficient $[ijk]$ is invariant under permutations of the symbols i, j, k.

By Schur's lemma we have that the Casimir operator of the $\mathfrak{h}$ action on the irreducible module $\mathfrak{m}_i$, $C_{\mathfrak{m}_i,\mathfrak{h}} := -\sum_\alpha \mathrm{ad}(E_\alpha^0) \circ \mathrm{ad}(E_\alpha^0)|_{\mathfrak{m}_i}$, is given by

$$c_i \cdot \mathrm{Id}_{\mathfrak{m}_i} = -\sum_\alpha \mathrm{ad}(E_\alpha^0) \circ \mathrm{ad}(E_\alpha^0)|_{\mathfrak{m}_i}, \tag{3-4}$$

with $c_i \ge 0$.

By [Wang and Ziller 1986, Lemma 1.5] (also [Nikonorov 2000, Lemma 1]), we have that, for $i = 1, \dots, n+m$,

$$0 \le \sum_{j,k} [ijk] = d_i(1 - 2c_i) \le d_i. \tag{3-5}$$

Indeed, a direct computation yields

$$\begin{aligned}\sum_{j,k} [ijk] &= \sum_{\substack{\alpha,\beta,\gamma \\ 1 \le j,k \le n+m}} Q([E_\alpha^i, E_\beta^j], E_\gamma^k)^2 \\ &= \sum_{\substack{\alpha,\beta,\gamma \\ 0 \le j,k \le n+m}} Q([E_\alpha^i, E_\beta^j], E_\gamma^k)^2 - 2 \sum_{\alpha,\beta,\gamma} Q([E_\alpha^i, E_\beta^0], E_\gamma^i)^2\end{aligned}$$

$$= \sum_{\substack{\alpha,\beta\\ 0\le j\le n+m}} Q([E^i_\alpha, E^j_\beta],[E^i_\alpha, E^j_\beta]) - 2\sum_{\alpha,\beta} Q([E^0_\beta, E^i_\alpha],[E^0_\beta, E^i_\alpha])$$

$$= \sum_{\substack{\alpha,\beta,\gamma\\ 0\le j\le n+m}} \left|Q\big(E^j_\beta,[E^i_\alpha,[E^i_\alpha, E^j_\beta]]\big)\right| + 2\sum_{\alpha,\beta} Q\big(E^i_\alpha,[E^0_\beta,[E^0_\beta, E^i_\alpha]]\big)$$

$$= \sum_{\alpha} |B(E^i_\alpha, E^i_\alpha)| - 2\operatorname{Tr} C_{\mathfrak{m}_i,\mathfrak{h}} = d_i - 2c_i d_i \le d_i$$

Using the above orthonormal basis, we have, for the Ricci curvature r_i on $\mathfrak{m}_i$, $i = 1, \dots, n+m$, of an awesome metric on $\mathfrak{m}$ (see [Nikonorov 2000, Lemma 2]), the formula

$$r_i = \frac{b_i}{2x_i} + \frac{1}{4d_i}\sum_{j,k}[ijk]\left(\frac{x_i}{x_k x_j} - \frac{x_k}{x_i x_j} - \frac{x_j}{x_k x_i}\right), \tag{3-6}$$

where $b_i = 1$ if $i \in I_{\mathfrak{l}}$, and $b_i = -1$ if $i \in I_{\mathfrak{p}}$. Let us order $\{x_1, \dots, x_n\} = \{l_1, \dots, l_n\}$ as

$$0 < l_1 \le \cdots \le l_n,$$

and $\{x_{n+1}, \dots, x_{n+m}\} = \{p_1, \dots, p_m\}$ as

$$0 < p_1 \le \cdots \le p_m.$$

In [Nikonorov 2000] there are the following estimates for $r^{\mathfrak{p}}_1$ and $r^{\mathfrak{p}}_m$. First, since $\frac{p_1}{p_j} - \frac{p_j}{p_1} \le 0$ for $1 \le j \le m$,

$$\begin{aligned} r^{\mathfrak{p}}_1 &= -\frac{1}{2p_1} + \frac{1}{4d_{\hat 1}}\sum_{j,k}[\hat 1 jk]\left(\frac{p_1}{x_k x_j} - \frac{x_k}{p_1 x_j} - \frac{x_j}{x_k p_1}\right) \\ &= -\frac{1}{2p_1} + \frac{1}{2d_{\hat 1}}\sum_{\substack{\hat j\in I_{\mathfrak{p}}\\ k\in I_{\mathfrak{l}}}}[\hat 1\hat j k]\left(\left(\frac{p_1}{p_j} - \frac{p_j}{p_1}\right)\frac{1}{l_k} - \frac{l_k}{p_j p_1}\right) \le -\frac{1}{2p_1} < 0, \end{aligned} \tag{3-7}$$

and since $\frac{p_m}{p_j} - \frac{p_j}{p_m} \ge 0$ for $1 \le j \le m$,

$$\begin{aligned} r^{\mathfrak{p}}_m &= -\frac{1}{2p_m} + \frac{1}{2d_{\hat m}}\sum_{\substack{\hat j\in I_{\mathfrak{p}}\\ k\in I_{\mathfrak{l}}}}[\hat m\hat j k]\left(\left(\frac{p_m}{p_j} - \frac{p_j}{p_m}\right)\frac{1}{l_k} - \frac{l_k}{p_j p_m}\right) \\ &\ge -\frac{1}{2p_m} - \frac{1}{2d_{\hat m}}\sum_{\substack{\hat j\in I_{\mathfrak{p}}\\ k\in I_{\mathfrak{l}}}}[\hat m\hat j k]\frac{l_k}{p_j p_m} \ge -\frac{1}{2p_m} - \frac{l_n}{4p_1 p_m}, \end{aligned} \tag{3-8}$$

where in the second inequality we used (3-5).

We now observe that for an awesome metric g the Ricci tensor restricted to the tangent space T_pK/H, which can be identified with $\mathfrak{l}$ via (2-3), splits nicely in terms of $\mathfrak{l}$ and $\mathfrak{p}$. Namely, for any $X \in \mathfrak{l}$, the Ricci tensor formula (2-2) for an awesome metric gives us

$$\begin{aligned}\text{(3-9)}\quad \mathrm{ric}_g(X,X) = \mathrm{ric}_{K/H}(X,X) - \frac{1}{2}\operatorname{Tr}(\mathrm{ad}(X)\circ \mathrm{ad}(X)|_{\mathfrak{p}}) \\ -\frac{1}{2}\sum_i \|[X, X_i^{\mathfrak{p}}]_{\mathfrak{m}}\|_g^2 + \frac{1}{4}\sum_{i,j} g([X_i^{\mathfrak{p}}, X_j^{\mathfrak{p}}]_{\mathfrak{m}}, X)^2,\end{aligned}$$

where $\{X_i^{\mathfrak{p}}\}$ is a g-orthonormal basis for $\mathfrak{p}$ and $\mathrm{ric}_{K/H}$ is the Ricci tensor on $K/H = K\cdot p$. In particular, we have the following lemma.

Lemma 3.1. *Let $(G/H, g)$ be a semisimple homogeneous space with an awesome G-invariant metric g. Then the Ricci curvature $r_n^{\mathfrak{l}}$ in the largest $\mathfrak{l}$-eigendirection of g with respect to the background metric Q satisfies*

$$\text{(3-10)}\quad r_n^{\mathfrak{l}} \geq \frac{1}{4d_n}\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n\hat{j}\hat{k}]\left(\frac{2}{l_n} + \frac{l_n}{p_j p_k} - \frac{p_j}{l_n p_k} - \frac{p_k}{p_j l_n}\right) + \frac{1}{4d_n l_n}\sum_{j,k\in I_{\mathfrak{l}}}[njk].$$

Proof. Observe that as computed in [Nikonorov 2000, Theorem 1] and in [Böhm 2015, Theorem 3.1] for $i > j \in I_{\mathfrak{l}}$

$$\begin{aligned} l_j^2 - 2l_il_j + l_i^2 = (l_j - l_i)^2 \leq (l_n - l_j)(l_i - l_j) = l_nl_i - l_il_j - l_nl_j + l_j^2 \\ \leq l_n^2 - l_il_j - l_nl_j + l_jl_n = l_n^2 - l_il_j.\end{aligned}$$

Hence $l_n^2 - l_j^2 - l_i^2 + l_jl_i \geq 0$ and the formula for $r_n^{\mathfrak{l}}$ with the Cartan decomposition relations then yields

$$\begin{aligned} r_n^{\mathfrak{l}} &= \frac{1}{2l_n} + \frac{1}{4d_n}\sum_{j,k}[njk]\left(\frac{l_n}{x_kx_j} - \frac{x_j}{l_nx_j} - \frac{x_j}{x_kl_n}\right) \\ &= \frac{d_n}{2l_nd_n} + \frac{1}{4d_n}\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n\hat{j}\hat{k}]\left(\frac{l_n}{p_kp_j} - \frac{p_i}{l_np_j} - \frac{p_j}{p_kl_n}\right) \\ &\qquad + \frac{1}{4d_n}\sum_{j,k\in I_{\mathfrak{l}}}[njk]\left(\frac{l_n}{l_kl_j} - \frac{l_k}{l_nl_j} - \frac{l_j}{l_kl_n}\right) \\ &\geq \frac{1}{4d_n}\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n\hat{j}\hat{k}]\left(\frac{2}{l_n} + \frac{l_n}{p_kp_j} - \frac{p_k}{l_np_j} - \frac{p_j}{p_kl_n}\right) \\ &\qquad + \frac{1}{4d_n}\sum_{j,k\in I_{\mathfrak{l}}}[njk]\left(\frac{2}{l_n} + \frac{l_n}{l_kl_j} - \frac{l_k}{l_nl_j} - \frac{l_j}{l_kl_n}\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4d_n} \sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}} [n\hat{j}\hat{k}]\left(\frac{2}{l_n} + \frac{l_n}{p_k p_j} - \frac{p_k}{l_n p_j} - \frac{p_j}{p_k l_n}\right) \\
&\qquad\qquad + \frac{1}{4d_n l_n} \sum_{j,k\in I_{\mathfrak{l}}} [njk]\left(1 + \frac{l_n^2 - l_j^2 - l_k^2 + l_j l_k}{l_k l_j}\right) \\
&\ge \frac{1}{4d_n} \sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}} [n\hat{j}\hat{k}]\left(\frac{2}{l_n} + \frac{l_n}{p_k p_j} - \frac{p_k}{l_n p_j} - \frac{p_j}{p_k l_n}\right) + \frac{1}{4d_n l_n} \sum_{j,k\in I_{\mathfrak{l}}} [njk],
\end{aligned}
$$

where in the first inequality we used (3-5). □

Remark 3.2. Observe that in general $\sum_{j,k\in I_{\mathfrak{l}}} [njk]$ may be equal to zero. But in the case when K/H is not a torus, $[\mathfrak{l}, \mathfrak{l}]_{\mathfrak{m}} \neq 0$ and we can consider the largest eigenvalue $l_{n'}$ such that $[\mathfrak{l}_{n'}, \mathfrak{l}]_{\mathfrak{m}} \neq 0$, for which $\sum_{j,k\in I_{\mathfrak{l}}} [n'jk] > 0$.

We have also the following alternative for the values of $r_n^{\mathfrak{l}}$ and $r_m^{\mathfrak{p}}$.

Lemma 3.3. *Let $(G/H, g)$ be a semisimple homogeneous space with an awesome G-invariant metric g. We have the following dichotomy for the Ricci curvature of g in the largest $\mathfrak{p}$-eigendirection (respectively $\mathfrak{l}$-eigendirection) of g with respect to Q:*

(1) *If $p_m - p_1 \ge l_n$, then*

$$
r_m^{\mathfrak{p}} \ge -\frac{1}{4p_m} - \frac{1}{4p_1} \quad \textit{and} \quad r_n^{\mathfrak{l}} \ge \frac{1}{4l_n}\left(2 - \frac{p_m}{p_1} - \frac{p_1}{p_m}\right). \tag{3-11}
$$

(2) *If $p_m - p_1 \le l_n$, then*

$$
r_m^{\mathfrak{p}} \ge -\frac{1}{2p_m} - \frac{l_n}{4p_1 p_m} \quad \textit{and} \quad r_n^{\mathfrak{l}} \ge 0. \tag{3-12}
$$

Proof. In the first case, i.e., if $p_m - p_1 \ge l_n$, we have

$$
\begin{aligned}
r_m^{\mathfrak{p}} &\ge -\frac{1}{2p_m} - \frac{l_n}{4p_1 p_m} \\
&\ge -\frac{1}{2p_m} + \frac{p_1 - p_m}{4p_1 p_m} \\
&= -\frac{1}{4p_m} - \frac{1}{4p_1},
\end{aligned}
$$

where in the first inequality we used (3-8). Moreover, by Lemma 3.1 we have that

$$
\begin{aligned}
r_n^{\mathfrak{l}} &\ge \frac{1}{4d_n} \sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}} [n\hat{j}\hat{k}]\left(\frac{2}{l_n} + \frac{l_n}{p_k p_j} - \frac{p_k}{l_n p_j} - \frac{p_j}{p_k l_n}\right) + \frac{1}{4d_n l_n} \sum_{j,k\in \mathfrak{l}} [njk] \\
&\ge \frac{1}{4l_n}\left(2 - \frac{p_m}{p_1} - \frac{p_1}{p_m}\right),
\end{aligned}
$$

and we get (3-11).

In the second case, as in [Nikonorov 2000], observe that

$$\begin{aligned} p_m - p_1 \le l_n &\iff |p_j - p_k| \le l_n, \forall j, k \in I_{\mathfrak{p}} \\ &\iff (p_j^2 - 2p_j p_k + p_k^2) \le l_n^2 \\ &\iff \left(\frac{p_j}{p_k l_n} - \frac{2}{l_n} + \frac{p_k}{p_j l_n}\right) \le \frac{l_n}{p_j p_k} \\ &\iff -\frac{2}{l_n} \le \frac{l_n}{p_j p_k} - \frac{p_j}{p_k l_n} - \frac{p_k}{p_j l_n}, \end{aligned}$$

and therefore, substituting this in the estimate for $r_n^{\mathfrak{l}}$ in (3-10) we get that $r_n^{\mathfrak{l}} \ge 0$. Together with the estimate (3-8) for $r_m^{\mathfrak{p}}$, we get (3-12). □

We can combine the estimates above to get a first scale-invariant estimate that will be essential for us in the dynamical analysis to come in Section 4.

Lemma 3.4. *Let* $(G/H, g)$ *be a semisimple homogeneous space with an awesome* G*-invariant metric* g*. Let us consider the eigenvalues of* g *with respect to* Q *as established in* (3-1), (3-2), *and* (3-3)*. Then*

$$2(p_m r_m^{\mathfrak{p}} + l_n r_n^{\mathfrak{l}}) \ge -\frac{p_m + l_n}{p_1}. \tag{3-13}$$

Proof. We know that if $p_m - p_1 \ge l_n$, then

$$\begin{aligned} 2(p_m r_m^{\mathfrak{p}} + l_n r_n^{\mathfrak{l}}) &\ge -\frac{1}{2} - \frac{p_m}{2p_1} + \frac{1}{2}\left(2 - \frac{p_m}{p_1} - \frac{p_1}{p_m}\right) \\ &= \frac{1}{2} - \frac{p_m}{p_1} - \frac{p_1}{2p_m} = -\frac{p_m}{p_1} + \frac{p_m - p_1}{2p_m} \\ &\ge -\frac{p_m}{p_1} \ge -\frac{p_m + l_n}{p_1}, \end{aligned}$$

where in the first inequality we used (3-11). And if $p_m - p_1 \le l_n$, then

$$2(p_m r_m^{\mathfrak{p}} + l_n r_n^{\mathfrak{l}}) \ge -1 - \frac{l_n}{2p_1} \ge -\frac{2p_1 + 2l_n}{2p_1} \ge -\frac{p_m + l_n}{p_1},$$

where in the first inequality we used (3-12). □

We can observe in the above computations that the advantage of working with awesome metrics is that the Ricci curvature in the eigenvectors of the metric tangent to $K/H \subset G/H$ splits nicely, since the algebraic and the metric decompositions are compatible. Lemmas 3.3 and 3.4 give us our first estimates on the Ricci curvature which we will be able to exploit in the next section in order to examine the long-time behavior of the Ricci flow on awesome metrics.

4. The dynamical Alekseevskii conjecture for awesome metrics

We will prove Theorem A. In order to do that, we must use the algebraic estimates we got in the last section in order to get dynamical estimates for the eigenvalues of the Ricci flow solution $g(t)$ (which from here onward we will also denote by g_t) with respect to our background metric $Q := -B|_{\mathfrak{l}\times\mathfrak{l}} + B|_{\mathfrak{p}\times\mathfrak{p}}$.

The next lemma is equivalent to the one in [Chow et al. 2006, Lemma B.40], but here we use for better convenience upper left-hand Dini derivatives.

Lemma 4.1 [Chow et al. 2006, Lemma B.40]. *Let C be a compact metric space, I an interval of $\mathbb{R}$, and $g : I \times C \to \mathbb{R}$ a function such that g and $\frac{\partial g}{\partial t}$ are continuous. Define $\phi : I \to \mathbb{R}$ by*

$$\phi(t) := \sup_{x\in C} g(t, x)$$

and its upper left-hand Dini derivative by

$$\frac{d^-\phi(t)}{dt} := \limsup_{h\to 0^+} \frac{\phi(t) - \phi(t-h)}{h}.$$

Let $C_t := \{x \in C \mid \phi(t) = g(t, x)\}$. We have that ϕ is continuous and that for any $t \in I$

$$\frac{d^-\phi(t)}{dt} = \min_{x\in C_t} \frac{\partial g}{\partial t}(t, x).$$

At first, we will apply Lemma 4.1 to

$$\begin{aligned} -p_1(t) &= \max\{-g_t(X, X) \mid X \in \mathfrak{p},\ \|X\|_Q = 1\}, \\ p_m(t) &= \max\{g_t(X, X) \mid X \in \mathfrak{p},\ \|X\|_Q = 1\}, \\ l_n(t) &= \max\{g_t(X, X) \mid X \in \mathfrak{l},\ \|X\|_Q = 1\}, \end{aligned}$$

and

$$\begin{aligned} &\log(p_m + l_n)(t) \\ &\quad= \log\max\{(g_t(X, X) + g_t(Y, Y)) \mid X \in \mathfrak{p},\ Y \in \mathfrak{l},\ \|X\|_Q = 1,\ \|Y\|_Q = 1\} \\ &\quad= \max\{\log(g_t(X, X) + g_t(Y, Y)) \mid X \in \mathfrak{p},\ Y \in \mathfrak{l},\ \|X\|_Q = 1,\ \|Y\|_Q = 1\}. \end{aligned}$$

Lemma 4.1 will be fundamental to us when combined with the following elementary real analysis result.

Lemma 4.2. *Let $[a, b]$ be a closed interval of $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ a continuous function such that $\frac{d^- f}{dt} \le 0$ and $f(a) = 0$. Then $f(t) \le 0$.*

With Lemmas 4.1 and 4.2 at hand, we can obtain our first main estimate for analyzing the long-time behavior of the Ricci flow on an awesome metric. Indeed, using the algebraic estimates in the previous section we get that:

Lemma 4.3. *Let $(G/H, g(t))$, $t \in [0, T)$, be an awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then we have, for the growth of $p_m(t)$ and $l_n(t)$, the upper bounds*

$$t + p_1(0) \leq p_1(t) \quad \textit{and} \quad (p_m + l_n)(t) \leq (t + p_1(0)) \frac{(p_m + l_n)(0)}{p_1(0)}. \tag{4-1}$$

Proof. Let us get first the estimate for $p_1(t)$. Using Lemma 4.1 and the estimate (3-7) we get that

$$\frac{d^-(-p_1)}{dt} \leq 2 \operatorname{ric}_t(E^{\hat{1}}, E^{\hat{1}}) = 2r_1^{\mathfrak{p}} p_1 \leq -1, \tag{4-2}$$

where in the second inequality we used (3-7). Hence by Lemma 4.2

$$t + p_1(0) \leq p_1(t) \tag{4-3}$$

Using Lemma 3.3 we get the estimate

$$\begin{aligned} \frac{d^- \log(p_m + l_n)}{dt} &\leq \frac{d}{dt} \log(g_t(E^{\hat{m}}, E^{\hat{m}}) + g_t(E^n, E^n)) \\ &= \frac{g_t'(E^{\hat{m}}, E^{\hat{m}}) + g_t'(E^n, E^n)}{g_t(E^{\hat{m}}, E^{\hat{m}}) + g_t(E^n, E^n)} = \frac{-2r_m^{\mathfrak{p}} p_m - 2r_n^{\mathfrak{l}} l_n}{p_m + l_n} \\ &\leq \frac{1}{p_1} \leq \frac{1}{t + p_1(0)}, \end{aligned}$$

where in the first, second, and third inequalities we used Lemma 4.1, (3-13), and (4-3), respectively. Hence $(p_m + l_n)(t) \leq (t + p_1(0)) \frac{(p_m + l_n)(0)}{p_1(0)}$. □

We see then that the maximum eigenvalue of $g(t)$ with respect to Q can grow at most as $O(t)$. In particular, $p_m(t) \leq (1 + c_0)(t + p_1(0))$ for some given positive constant $c_0 > 0$ that only depends on the initial metric g_0.

Therefore, using (4-3) we get a pinching estimate along the flow for the metric $g(t)|_{\mathfrak{p} \times \mathfrak{p}}$, namely

$$\frac{p_m(t)}{p_1(t)} \leq 1 + c_0. \tag{4-4}$$

We can then reuse this estimate to get the following proposition (for the sake of simplicity, from now on we will mostly omit that we are using the Lemmas 4.1 and 4.2 above).

Proposition 4.4. *Let $(G/H, g(t))$, $t \in [0, T)$, be an awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let us consider the eigenvalues $p_1(t)$, $p_m(t)$, and $l_n(t)$ of g_t with respect to Q. Then for all $t \geq t_0$ in the maximal interval of the*

Ricci flow, there is an explicit constant $c_0 > 0$, *which only depends on the initial conditions at* t_0, *such that*

$$p_m(t) \leq t + (p_1(t_0) - t_0) + c_0\sqrt{t + (p_1(t_0) - t_0)} \leq p_1(t) + c_0\sqrt{p_1(t)} \tag{4-5}$$

and

$$l_n(t) \leq c_0\sqrt{(t - t_0 + p_1(t_0))}. \tag{4-6}$$

Proof. By rescaling the initial metric $g(t_0)$ we may assume without loss of generality that $p_1(t_0) = 1$. Let $t \geq t_0$. We will show by induction that, for all $N \in \mathbb{N}$,

$$p_m(t) \leq \left(1 + \frac{c_0}{2^N}\right)(t - t_0 + 1) + c_0 \sum_{k=0}^{N-1} \frac{(\log(t - t_0 + 1))^k}{2^k k!}$$

and

$$l_n(t) \leq \frac{c_0}{2^N}(t - t_0 + 1) + c_0 \sum_{k=0}^{N-1} \frac{(\log(t - t_0 + 1))^k}{2^k k!},$$

where $c_0 := p_m(t_0) + l_n(t_0) - 1$.

By Lemma 4.3, we have already seen that $p_m(t) \leq (1 + c_0)(t - t_0 + 1)$. Moreover, by the discussion following Lemma 3.3 we get

$$\frac{d^- l_n(t)}{dt} \leq -2r_n(t) l_n(t) \leq \frac{1}{2}\left(\frac{p_m(t)}{p_1(t)} + \frac{p_1(t)}{p_m(t)} - 2\right) \leq \frac{1}{2}\left(\frac{p_m(t)}{p_1(t)} - 1\right) \leq \frac{c_0}{2},$$

where in the last inequality we used (4-4). Hence, for $t \geq t_0$,

$$l_n(t) \leq \frac{c_0}{2}(t - t_0 + 1) - \frac{c_0}{2} + l_n(t_0) \leq \frac{c_0}{2}(t - t_0 + 1) + c_0. \tag{4-7}$$

Moreover,

$$\begin{aligned}\frac{d^- p_m(t)}{dt} &\leq -2r_m(t) p_m(t) \leq 1 + \frac{l_n(t)}{2p_1(t)} \\ &\leq \left(1 + \frac{c_0}{2^2}\right)\frac{(t - t_0 + 1)}{p_1(t)} + \frac{c_0}{2p_1(t)} \leq \left(1 + \frac{c_0}{2^2}\right) + \frac{c_0}{2(t - t_0 + 1)},\end{aligned}$$

where in the second and third inequalities we used (3-8) and (4-7), respectively. Hence, for $t \geq t_0$,

$$p_m(t) \leq \left(1 + \frac{c_0}{2^2}\right)(t - t_0 + 1) + \frac{c_0}{2}\log(t - t_0 + 1) + c_0.$$

This establishes the basis of induction. Suppose that, for $N \geq 2$ and $t \geq t_0$,

$$p_m(t) \leq \left(1 + \frac{c_0}{2^N}\right)(t - t_0 + 1) + c_0 \sum_{k=0}^{N-1} \frac{(\log(t - t_0 + 1))^k}{2^k k!}$$

and

$$l_n(t) \le \frac{c_0}{2^N}(t-t_0+1) + c_0 \sum_{k=0}^{N-1} \frac{(\log(t-t_0+1))^k}{2^k k!}.$$

Then we can reuse this to get better estimates for $l_n(t)$ and $p_m(t)$. We have that

$$\frac{d^- l_n(t)}{dt} \le \frac{1}{2}\left(\frac{p_m(t)}{p_1(t)} - 1\right) \le \frac{1}{2}\left(\frac{p_m(t)}{(t-t_0+1)} - 1\right) \le \frac{c_0}{2^{N+1}} + c_0 \sum_{k=0}^{N-1} \frac{(\log(t-t_0+1))^k}{(t-t_0+1)2^{k+1}k!};$$

hence

$$l_n(t) \le \frac{c_0}{2^{N+1}}(t-t_0+1) + c_0 \sum_{k=0}^{N} \frac{(\log(t-t_0+1))^k}{2^k k!},$$

and

$$\frac{d^- p_m(t)}{dt} \le 1 + \frac{l_n(t)}{2p_1(t)} \le \left(1 + \frac{c_0}{2^{N+1}}\right) + c_0 \sum_{k=0}^{N-1} \frac{(\log(t-t_0+1))^k}{(t-t_0+1)2^{k+1}k!}.$$

Therefore,

$$p_m(t) \le \left(1 + \frac{c_0}{2^{N+1}}\right)(t-t_0+1) + c_0 \sum_{k=0}^{N} \frac{(\log(t-t_0+1))^k}{2^k k!}.$$

Now this is valid for arbitrary $t \ge t_0$ in the maximal interval of the dynamics. Hence, taking the limit at $N \to \infty$ for the right-hand side, we get

$$p_m(t) \le (t-t_0+1) + c_0\sqrt{(t-t_0+1)} \le p_1(t) + c_0\sqrt{p_1(t)}$$

and

$$l_n(t) \le c_0\sqrt{(t-t_0+p_1(t_0))}. \qquad \square$$

The next corollary follows immediately from the estimate (4-5) obtained in Proposition 4.4 above.

Corollary 4.5. *Let* $(G/H, g(t))$, $t \in [t_0, \infty)$, *be an immortal awesome Ricci flow adapted to the Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. *Then*

$$\lim_{t\to\infty} \frac{p_m(t)}{p_1(t)} = 1.$$

Remark 4.6. Observe that the background metric Q given by $-B|_{\mathfrak{k}\times\mathfrak{k}} + B|_{\mathfrak{p}\times\mathfrak{p}}$ corresponds to a minimum for the moment map of the $\mathrm{SL}(\mathfrak{g}, \mathbb{R})$-action of determinant-one matrices in the space of Lie brackets on $\mathfrak{g}$ [Lauret 2003, Proposition 8.1]. Thus, Corollary 4.5 reinforces the relation between geometric invariant theory and the geometry of G/H by telling us that in the noncompact part $\mathfrak{p}$ the awesome metric under the Ricci flow approximates $-B|_{\mathfrak{p}\times\mathfrak{p}}$ up to scaling and G-equivariant isometry. In Section 5 we will prove a convergence result in this direction (see Theorem 5.2).

The next corollary of Proposition 4.4, about the asymptotic behavior of the metric $g(t)$ restricted to the tangent space of K/H, will also be important for the long-time analysis of immortal awesome homogeneous Ricci flows in Section 5.

Corollary 4.7. *Let $(G/H, g(t))$, $t \in [t_0, \infty)$, be an immortal awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then the rescaled metric $\tilde{g}_t := t^{-1} g(t)$ is such that*

$$\lim_{t\to\infty} \tilde{g}_t|_{\mathfrak{l}\times\mathfrak{l}} = 0.$$

Proof. This follows immediately from the estimate (4-6) for $l_n(t)$, which shows that it may grow at most sublinearly. Hence, $\lim_{t\to\infty} \frac{l_n(t)}{t} = 0$. □

We can now prove our first main result.

Theorem 4.8. *Let $(M^d = G/H, g_0)$ be a semisimple homogeneous space such that the universal cover is not diffeomorphic to $\mathbb{R}^d$. If g_0 is an awesome metric, then the Ricci flow solution starting at g_0 has finite extinction time.*

Proof. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Let us consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{h} \subset \mathfrak{k}$ and let us fix as a background metric $Q := -B|_{\mathfrak{l}\times\mathfrak{l}} + B|_{\mathfrak{p}\times\mathfrak{p}}$, where B is the Killing form in $\mathfrak{g}$. We have then the canonical reductive identification $T_{eH} G/H \cong \mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}$, with $\mathfrak{l} := \mathfrak{h}^{\perp_Q} \cap \mathfrak{k}$. First, observe that by hypothesis the universal cover of M is not diffeomorphic to $\mathbb{R}^d$, which implies that K/H is not a torus; hence $[\mathfrak{k}, \mathfrak{k}] \not\subset \mathfrak{h}$. Moreover, since $[\mathfrak{k}, \mathfrak{k}] \perp_Q \mathfrak{z}(\mathfrak{k})$, where $\mathfrak{z}(\mathfrak{k})$ is the center of $\mathfrak{k}$, the condition $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$ is equivalent to $\mathfrak{l} \subset \mathfrak{z}(\mathfrak{k})$, which in turn is equivalent to $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{l}] = 0$.

So in terms of the irreducible representations decomposition (3-1), $[\mathfrak{k}, \mathfrak{k}] \not\subset \mathfrak{h}$ is equivalent to say that for at least one $i \in I_{\mathfrak{l}}$ either $[\mathfrak{l}, \mathfrak{l}] \not\subset \mathfrak{h}$ and

$$\sum_{j,k\in I_{\mathfrak{l}}} [ijk] > 0;$$

or $[\mathfrak{h}, \mathfrak{l}] \neq 0$ and, using (3-5) in the equality,

$$\sum_{j,k} [ijk] = d_i(1 - 2c_i) < d_i,$$

since then the Casimir operator $C_{\mathfrak{l}_i,\mathfrak{h}}$ given in (3-4) is not zero.

Therefore, there is a constant $\lambda > 0$ such that for any of the irreducible $\mathfrak{h}$-modules $\mathfrak{l}_i$ with $[\mathfrak{l}_i, \mathfrak{k}] \not\subset \mathfrak{h}$, either

$$\sum_{j,k\in I_{\mathfrak{l}}} [ijk] \geq 2\lambda > 0 \quad \text{or} \quad \sum_{j,k} [ijk] \leq d_i - \lambda < d_i.$$

Now let g_t, $t \in [0, T)$, be the Ricci flow (2-1) solution starting at g_0. By Lemma 2.2 we know that g_t is an awesome homogeneous Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{l}$.

Let us consider the diagonalization of g_t with regard to Q as in (3-1),

$$g_t = l_1(t)\cdot Q|_{\mathfrak{l}_1\times\mathfrak{l}_1} \perp \cdots \perp l_n(t)\cdot Q|_{\mathfrak{l}_n\times\mathfrak{l}_n} \perp p_1(t)\cdot Q|_{\mathfrak{p}_1\times\mathfrak{p}_1} \perp \cdots \perp p_m(t)\cdot Q|_{\mathfrak{p}_m\times\mathfrak{p}_m}.$$

Let us define the Ricci curvatures, as in (3-2) and (3-3),

$$r_i^{\mathfrak{l}}\cdot g(t)|_{\mathfrak{l}_i\times\mathfrak{l}_i} = \mathrm{ric}_{g(t)}|_{\mathfrak{l}_i\times\mathfrak{l}_i} \quad \text{for } i=1,\dots,n$$

and

$$r_i^{\mathfrak{p}}\cdot g(t)|_{\mathfrak{p}_i\times\mathfrak{p}_i} = \mathrm{ric}_{g(t)}|_{\mathfrak{p}_i\times\mathfrak{p}_i} \quad \text{for } i=1,\dots,m.$$

Define the $\mathfrak{h}$-submodules $V=\{X\in W^{\perp_Q}\mid [\mathfrak{l},X]\in\mathfrak{h}\}$ and $W=\{X\in\mathfrak{l}\mid[\mathfrak{h},X]=0\}$, and let

$$L(t) := \max\{g_t(X,X)\mid X\in V^{\perp_Q}\cup W^{\perp_Q},\ \|X\|_Q=1\}.$$

Note that $L(t)$ is the largest eigenvalue $l_{n'}(t)$ of g_t such that the corresponding irreducible $\mathfrak{h}$-module $\mathfrak{l}_{n'}$ satisfies either $[\mathfrak{l}_n,\mathfrak{l}]\not\subset\mathfrak{h}$ or $[\mathfrak{h},\mathfrak{l}_n]\neq 0$, which we already know is a nonempty condition. Consider $p_1(t):=\min\{g_t(X,X)\mid X\in\mathfrak{p},\ \|X\|_Q=1\}$ and $p_m(t):=\max\{g_t(X,X)\mid X\in\mathfrak{p},\ \|X\|_Q=1\}$. Then using all the estimates we got so far and the fact that for any $j\in I_{\mathfrak{l}}$, if $\sum_{k\in I_{\mathfrak{l}}}[n'jk]\neq 0$, then $l_{n'}\geq l_j$, we get, for $r_{n'}^{\mathfrak{l}}$, the lower bound

$$\begin{aligned}
r_{n'}^{\mathfrak{l}} &= \frac{1}{2l_{n'}} + \frac{1}{4d_{n'}}\sum_{j,k}[n'jk]\left(\frac{l_{n'}}{x_kx_j} - \frac{x_j}{l_{n'}x_j} - \frac{x_j}{x_kl_{n'}}\right)\\
&= \frac{d_{n'}}{2l_{n'}d_{n'}} + \frac{1}{4d_{n'}}\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n'\hat{j}\hat{k}]\left(\frac{l_{n'}}{p_kp_j} - \frac{p_i}{l_{n'}p_j} - \frac{p_j}{p_kl_{n'}}\right)\\
&\qquad\qquad + \frac{1}{4d_{n'}}\sum_{j,k\in I_{\mathfrak{l}}}[n'jk]\left(\frac{l_{n'}}{l_kl_j} - \frac{l_k}{l_{n'}l_j} - \frac{l_j}{l_kl_{n'}}\right)\\
&\geq \frac{d_{n'}}{2l_{n'}d_{n'}} + \frac{1}{4d_{n'}}\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n'\hat{j}\hat{k}]\left(-\frac{p_i}{l_{n'}p_j} - \frac{p_j}{p_kl_{n'}}\right)\\
&\qquad\qquad + \frac{1}{4d_{n'}l_{n'}}\sum_{j,k\in I_{\mathfrak{l}}}[n'jk]\left(\frac{l_{n'}^2-l_j^2-l_k^2}{l_kl_j}\right)\\
&\geq \frac{d_{n'}}{2l_{n'}d_{n'}} - \frac{1}{2d_{n'}l_{n'}}\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n'\hat{j}\hat{k}]\frac{p_m}{p_1} - \frac{1}{4d_{n'}l_{n'}}\sum_{j,k\in I_{\mathfrak{l}}}[n'jk],
\end{aligned}$$

where in the first equality we used (3-6). Therefore,

$$\frac{d^-l_{n'}(t)}{dt} \leq -2r_{n'}^{\mathfrak{l}}(t)l_{n'}(t) \leq \frac{1}{d_{n'}}\left(-d_{n'} + \frac{1}{2}\sum_{j,k\in I_{\mathfrak{l}}}[n'jk] + \sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n'\hat{j}\hat{k}]\frac{p_m(t)}{p_1(t)}\right).$$

Let us assume by contradiction that g_t is immortal. Then by Corollary 4.5, given $\epsilon > 0$ we can assume that t is big enough such that

$$\frac{p_m(t)}{p_1(t)} \leq 1+\epsilon.$$

Therefore we get that

$$\begin{aligned}\frac{d^- l_{n'}(t)}{dt} &\leq \frac{1}{d_{n'}}\biggl(-d_{n'}+\frac{1}{2}\sum_{j,k\in I_{\mathfrak{l}}}[n'jk]+\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n'\hat{j}\hat{k}]+\epsilon\sum_{\hat{j},\hat{k}\in I_{\mathfrak{p}}}[n'\hat{j}\hat{k}]\biggr)\\ &= \frac{1}{d_{n'}}\biggl((1+\epsilon)\sum_{j,k}[n'jk]-d_{n'}-\Bigl(\frac{1}{2}+\epsilon\Bigr)\sum_{j,k\in I_{\mathfrak{l}}}[n'jk]\biggr),\end{aligned}$$

and since either $[\mathfrak{h}, \mathfrak{l}_{n'}] \neq 0$ or $[\mathfrak{l}_n, \mathfrak{l}] \not\subset \mathfrak{h}$, there exists a constant independent of t, $\lambda > 0$, such that either $\sum_{j,k}[n'jk] = d_{n'} - \lambda < d_{n'}$ or $\sum_{j,k\in I_{\mathfrak{l}}}[n'jk] \geq 2\lambda > 0$. Therefore, if $d = \dim M$, then either

$$\begin{aligned}\frac{d^- l_{n'}(t)}{dt} &= \frac{1}{d_{n'}}\biggl((1+\epsilon)\sum_{j,k}[n'jk]-d_{n'}-\Bigl(\frac{1}{2}+\epsilon\Bigr)\sum_{j,k\in I_{\mathfrak{l}}}[n'jk]\biggr)\\ &= \frac{1}{d_{n'}}((1+\epsilon)(d_{n'}-\lambda)-d_{n'}) \leq -\frac{\lambda}{d}+\epsilon\end{aligned}$$

or

$$\begin{aligned}\frac{d^- l_{n'}(t)}{dt} &= \frac{1}{d_{n'}}\biggl((1+\epsilon)\sum_{j,k}[n'jk]-d_{n'}-\Bigl(\frac{1}{2}+\epsilon\Bigr)\sum_{j,k\in I_{\mathfrak{l}}}[n'jk]\biggr)\\ &\leq \frac{1}{d_{n'}}((1+\epsilon)(d_{n'})-d_{n'})-\frac{\lambda}{d} \leq -\frac{\lambda}{d}+\epsilon.\end{aligned}$$

Finally, choosing ϵ small enough we get that $-\frac{\lambda}{d}+\epsilon < 0$, and then for big enough t we get that

$$l_{n'}(t) \leq \Bigl(-\frac{\lambda}{d}+\epsilon\Bigr)t + l_{n'}(0),$$

which means that $l_{n'}(t)$ converges to zero in finite time and the Ricci flow is not immortal. □

5. Convergence results for awesome metrics

We will further our long-time behavior analysis of awesome Ricci flows by examining the long-time limit solitons we obtain by appropriately rescaling the Ricci flow solution $g(t)$. We first investigate the case where the universal cover of G/H is not contractible, i.e., it is not diffeomorphic to $\mathbb{R}^n$ so that by Theorem 4.8 the extinction time is finite and later in Section 5.2, the contractible case, when G/H is diffeomorphic to $\mathbb{R}^n$, where the flow is immortal.

5.1. ***The noncontractible case.*** Let us consider the following formula for the scalar curvature $R(g)$ of an awesome metric g which easily follows from the Ricci curvature formula (3-6):

$$R(g) = \sum_{i \in I_{\mathfrak{l}}} \frac{d_i}{2l_i} - \sum_{\hat{j} \in I_{\mathfrak{p}}} \frac{d_{\hat{j}}}{2p_j} - \sum_{i,j,k} [ijk] \frac{x_i}{x_j x_k}. \tag{5-1}$$

We immediately see from the equation above that

$$R(g) \leq \frac{\dim \mathfrak{l}}{2l_1}, \tag{5-2}$$

which in turn means that if the scalar curvature blows-up, then the smallest eigenvalue in the $\mathfrak{l}$-direction goes to zero.

Böhm [2015, Theorem 2.1] showed that every homogeneous Ricci flow with finite extinction time develops a type-I singularity, namely, that there is a constant $C(g_0)$ that only depends on the initial metric g_0 such that we have for the norm of the Riemann tensor $\mathrm{Rm}(g)$ along the Ricci flow the upper bound

$$\|\mathrm{Rm}(g(t))\|_{g(t)} \leq \frac{C(g_0)}{T-t},$$

for $t \in [0, T)$, where T is the maximal time for the flow. Even more, if we assume $R(g_0)$ is positive, he showed that along a finite extinction time homogeneous Ricci flow the Riemann tensor is controlled by the scalar curvature [Böhm 2015, Remark 2.2] and that there are constants $c(g_0)$ and $C(g_0)$ only depending on the initial metric g_0 such that

$$\frac{c(g_0)}{T-t} \leq R(g(t)) \leq \frac{C(g_0)}{T-t},$$

for $t \in [0, T)$.

This gives us a natural scaling parameter for a blow-up analysis of the Ricci flow solution $g(t)$. By [Enders et al. 2011], we can extract a nonflat limit shrinking soliton from such a blow-up. In the following theorem, which is our second main result, we show that these limits only depend on the induced geometry on the compact fiber K/H.

Theorem 5.1. *Let $M = G/H$ be a semisimple homogeneous space such that the universal cover is not contractible. Let $(M, g(s))$, $s \in [0, T)$, be an awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $R(g)$ be the scalar curvature of the metric g. For any sequence $(s_a)_{a \in \mathbb{N}}$, $s_a \to T$, there exists a subsequence such that $(M, R(g(s_{\hat{a}})) \cdot g(s_{\hat{a}}))$ converges in pointed C^∞-topology to the Riemannian product*

$$E_\infty \times \mathbb{E}^d,$$

where E_∞ is a compact homogeneous Einstein manifold with positive scalar curvature and $\mathbb{E}^d$ is the d-dimensional (*flat*) *Euclidean space with* $d \geq \dim \mathfrak{p}$.

The geometry of E_∞ just depends on the subsequence of Riemannian submanifolds $(K/H,\, R(g(s_{\hat{a}})) \cdot g(s_{\hat{a}}))$.

Proof. By Theorem 4.8 we know that such a solution $g(s)$ has finite extinction time; hence by [Böhm 2015, Theorem 2.1] we know that it is a type-I flow. By [Lafuente 2015, Theorem 1.1] we can assume without loss of generality that $R(g(0)) > 0$, so let us define the parabolic rescaled metric $\tilde{g}_s := R(g(s)) \cdot g(s)$. By the work of Enders, Müller, and Topping on type-I singularities of the Ricci flow [Enders et al. 2011, Theorem 1.1], it follows via Hamilton's compactness theorem [1982] that along any sequence of times converging to the singularity time T, these scalar curvature normalized parabolic rescalings will subconverge to a nonflat homogeneous gradient shrinking soliton.

Moreover, by work of Petersen and Wylie [2009], we have that a homogeneous gradient shrinking soliton is rigid, in the sense that it is a Riemannian product of an Euclidean factor and a positive scalar curvature homogeneous Einstein manifold.

We have already seen in (3-7) that in the direction of the smallest eigenvalue of $g|_{\mathfrak{p}\times\mathfrak{p}}$ the Ricci curvature is negative, i.e., $r_1^{\mathfrak{p}} < 0$. Since any limit gradient shrinking soliton is Ricci nonnegative, this implies that

$$\lim_{s\to T}\left(\frac{1}{R(g(s))}\cdot r_1^{\mathfrak{p}}(s)\right) = 0.$$

In particular, for any $\hat{j} \in I_{\mathfrak{p}}$ and $k \in I_{\mathfrak{l}}$,

$$\lim_{s\to T}\left([\hat{1}\hat{j}k]_s\left(\frac{p_1(s)}{p_j(s)} - \frac{p_j(s)}{p_1(s)}\right)\frac{1}{R(g(s))l_k(s)}\right) = 0.$$

This in turn implies that, for the second largest eigenvalue in the $\mathfrak{p}$-direction p_2, $\lim r_2^{\mathfrak{p}}(s) \leq 0$ as t approaches the singular time T. Hence, once again we can conclude that, for any $\hat{j} \in I_{\mathfrak{p}}$ and $k \in I_{\mathfrak{l}}$,

$$\lim_{s\to T}\left([\hat{2}\hat{j}k]_s\left(\frac{p_2(s)}{p_j(s)} - \frac{p_j(s)}{p_2(s)}\right)\frac{1}{R(g(s))l_k(s)}\right) = 0.$$

Arguing recursively, we can then conclude that, for any $\hat{i}$, $\hat{j} \in I_{\mathfrak{p}}$ and $k \in I_{\mathfrak{l}}$,

$$\lim_{s\to T}\left([\hat{i}\hat{j}k]_s\left(\frac{p_i(s)}{p_j(s)} - \frac{p_j(s)}{p_i(s)}\right)\frac{1}{R(g(s))l_k(s)}\right) = 0. \tag{5-3}$$

Therefore, in the limit geometry we find at least as many linear independent directions as $\dim \mathfrak{p}$ such that the Ricci curvature is 0. This can only be the case if the Euclidean factor in the limit has dimension at least as large as $\dim \mathfrak{p}$. Furthermore, by [Böhm 2015, Theorem 5.2] we know that this dimension does not depend on the subsequence taken.

Observe that this in particular implies

$$\sum_{\alpha,\beta,\gamma}\big(\tilde g_s(\tilde E_\alpha^{\mathfrak p_i},[\tilde E_\beta^{\mathfrak p_j},\tilde E_\gamma^{\mathfrak l_k}])-Q(E_\alpha^{\mathfrak p_i},[E_\beta^{\mathfrak p_j},\tilde E_\gamma^{\mathfrak l_k}])\big)\to 0, \tag{5-4}$$

where

$$\tilde E_\alpha^{\mathfrak p_j}=\frac{E_\alpha^{\mathfrak p_j}}{\sqrt{R(g(s))\cdot p_j(s)}}\quad\left(\text{respectively } \tilde E_\gamma^{\mathfrak l_k}=\frac{E_\gamma^{\mathfrak l_k}}{\sqrt{R(g(s))\cdot l_i(s)}}\right)$$

and $\{E_\alpha^{\mathfrak p_j}\}$ (respectively $\{E_\gamma^{\mathfrak l_i}\}$) is a $g(s)$-diagonalizing basis of $\mathfrak p$ (respectively of $\mathfrak l$) with respect to the $\mathrm{Ad}(K)$-invariant background metric $Q:=-B|_{\mathfrak l\times\mathfrak l}+B|_{\mathfrak p\times\mathfrak p}$.

Let $X\in\mathfrak l$ and $x_s:=\|X\|^2_{\tilde g_s}$. The rescaled Ricci curvature on the compact Riemannian submanifold $N_s=(K/H,\tilde g_s)$ in the direction $\tilde X_s=X/\sqrt{x_s}$, as remarked in (3-9), is given by

$$\begin{aligned}\mathrm{ric}_{\tilde g_s}(\tilde X_s,\tilde X_s)=\ &\mathrm{ric}_{N_s}(\tilde X_s,\tilde X_s)-\frac12\operatorname{Tr}(\mathrm{ad}(\tilde X_s)\circ\mathrm{ad}(\tilde X_s)|_{\mathfrak p})\\ &-\frac12\sum_{\substack{\hat\imath\in I_{\mathfrak p}\\ \alpha}}\|[\tilde X_s,\tilde E_\alpha^{\mathfrak p_i}]\|^2_{\tilde g_s}+\frac14\sum_{\substack{\hat\imath,\hat\jmath\in I_{\mathfrak p}\\ \alpha,\beta}}\tilde g_s([\tilde E_\alpha^{\mathfrak p_i},\tilde E_\beta^{\mathfrak p_j}],\tilde X_s)^2.\end{aligned}$$

This means that

$$\begin{aligned}&|\mathrm{ric}_{\tilde g_s}(\tilde X_s,\tilde X_s)-\mathrm{ric}_{N_s}(\tilde X_s,\tilde X_s)|\\ &\quad\le\frac12\left|\sum_{\substack{\hat\imath,\hat\jmath\in I_{\mathfrak p}\\ \alpha,\beta}}Q(E_\alpha^{\mathfrak p_i},[E_\beta^{\mathfrak p_j},\tilde X_s])^2-\sum_{\substack{\hat\imath,\hat\jmath\in I_{\mathfrak p}\\ \alpha,\beta}}\tilde g_s(\tilde E_\alpha^{\mathfrak p_i},[\tilde E_\beta^{\mathfrak p_j},\tilde X_s])^2\right|\\ &\qquad+\frac14\sum_{\substack{\hat\jmath,\hat k\in I_{\mathfrak p}\\ \alpha,\beta}}\tilde g_s([\tilde E_\beta^{\mathfrak p_j},\tilde E_\alpha^{\mathfrak p_k}],\tilde X_s)^2.\end{aligned}$$

For the first term on the right-hand side we have that

$$\tilde g_s(\tilde E_\alpha^{\mathfrak p_i},[\tilde E_\beta^{\mathfrak p_j},\tilde X_s])^2=\left(\sum_{\substack{k\in I_{\mathfrak l}\\ \gamma}}\tilde g_s(\tilde X_s,\tilde E_\gamma^{\mathfrak l_k})\tilde g_s(\tilde E_\alpha^{\mathfrak p_i},[\tilde E_\beta^{\mathfrak p_j},\tilde E_\gamma^{\mathfrak l_k}])\right)^2$$

and by what we observed in (5-4) this can be approximated by

$$\left(\sum_{\substack{k\in I_{\mathfrak l}\\ \gamma}}\tilde g_s(\tilde X_s,\tilde E_\gamma^{\mathfrak l_k})\tilde g_s(\tilde E_\alpha^{\mathfrak p_i},[\tilde E_\beta^{\mathfrak p_j},\tilde E_\gamma^{\mathfrak l_k}])\right)^2=\left(\sum_{\substack{k\in I_{\mathfrak l}\\ \gamma}}\tilde g_s(\tilde X_s,\tilde E_\gamma^{\mathfrak l_k})Q(E_\alpha^{\mathfrak p_i},[E_\beta^{\mathfrak p_j},\tilde E_\gamma^{\mathfrak l_k}])\right)^2+\epsilon(s),$$

where $\epsilon(s)\to 0$ as $s\to T$.

As for the second term on the right-hand side, observe that, by Lemma 4.3, $l_n(s)$ grows at most linearly and $p_1(s)$ grows at least linearly, which implies that, for any $\hat{i}, \hat{j} \in I_{\mathfrak{p}}$ and $k \in I_{\mathfrak{l}}$,

$$\lim_{t\to T}\left(\frac{1}{R(g(s))}\cdot\frac{l_k(s)}{p_i(s)p_j(s)}\right)=0.$$

Hence, by Cauchy–Schwarz we get that

$$\tilde{g}_s([\tilde{E}_\alpha^{\mathfrak{p}_i}, \tilde{E}_\beta^{\mathfrak{p}_j}], \tilde{X}_s)^2 \\ \leq \|[\tilde{E}_\alpha^{\mathfrak{p}_i}, \tilde{E}_\beta^{\mathfrak{p}_j}]\|_{\tilde{g}_s}^2 = \sum_{\substack{k\in I_{\mathfrak{l}}\\ \gamma}} Q([E_\alpha^{\mathfrak{p}_i}, E_\beta^{\mathfrak{p}_j}], E_\gamma^{\mathfrak{l}_k})^2 \cdot \frac{1}{R(g(s))}\cdot\frac{l_k(s)}{p_i(s)p_j(s)} \to 0.$$

Therefore $|\mathrm{ric}_{\tilde{g}_s}(\tilde{X}_s, \tilde{X}_s) - \mathrm{ric}_{N_s}(\tilde{X}_s, \tilde{X}_s)| \to 0$ and we conclude that the Ricci $(1,1)$-tensor restricted to $\mathfrak{l}$ approximates the Ricci tensor given by the induced metric on $(K/H, \tilde{g}_s)$ as s approaches the singularity time T.

Let us now consider the rescaled Ricci flow solution

$$\tilde{g}_s(t) := R(g(s))\cdot g\left(s+\frac{t}{R(g(s))}\right)$$

restricted to the submanifold K/H. The argument above could be carried out by taking $\tilde{g}_s(t)$ instead of $\tilde{g}_s(0)$. So we have shown that the family of Ricci flow solutions $(K/H, \tilde{g}_s(t))$ is equivalent, as s approaches the singularity time T, to the Ricci flow solution $(K/H, \hat{g}_s(t))$, where $\hat{g}_s(t)$ is the Ricci flow on K/H with initial metric $\tilde{g}_s(0)|_{K/H}$. In particular, that means that the limit Einstein factor E_∞ only depends on a convergent subsequence of the submanifold geometry of $(K/H, \tilde{g}_s(0)|_{K/H})$. □

5.2. ***The contractible case.*** For the sake of completeness, we want now to understand the limit geometry in the case when the universal cover of our semisimple homogeneous space is contractible.

A homogeneous Riemannian manifold $\tilde{M}$ diffeomorphic to $\mathbb{R}^n$ must be a Riemannian product of a noncompact symmetric space and an $\mathbb{R}^d$-bundle over a *Hermitian symmetric space* (see [Böhm and Lafuente 2022, Proposition 3.1]). Hermitian symmetric spaces are a special class of noncompact symmetric spaces which are also Hermitian manifolds. Irreducible ones correspond to irreducible symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ with $\dim \mathfrak{z}(\mathfrak{k}) = 1$. In particular, if we write $\mathfrak{k} = \mathfrak{k}_{ss} \oplus \mathfrak{z}(\mathfrak{k})$ and consider the integral subgroup $K_{ss} \subset G$ with Lie algebra $\mathfrak{k}_{ss}$, then the homogeneous space G/K_{ss} is a homogeneous line bundle over the irreducible Hermitian symmetric space G/K (for more on Hermitian symmetric spaces, see [Helgason 1978, Chapter VIII, Theorem 6.1] and [Besse 1987, 7.104]).

An example is the product $\widetilde{(\mathrm{SL}(2,\mathbb{R}))^k}$ with left-invariant metrics, which is an $\mathbb{R}^k$-bundle over the product of hyperbolic planes $(\mathbb{H}^2)^k$. It is worth mentioning that semisimple homogeneous $\mathbb{R}$-bundles over irreducible Hermitian symmetric spaces only admit awesome homogeneous metrics (see [Böhm and Lafuente 2022, Remark 3.2]), which is the case, for example, of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Let $\mathfrak{g}$ be semisimple of noncompact type with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{h}$ a compactly embedded subalgebra of $\mathfrak{k}$. Let G, K and H be Lie groups for the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$, respectively, such that G/H is simply connected. We know that $G/H = K/H \times \mathbb{R}^{\dim \mathfrak{p}}$ is diffeomorphic to $\mathbb{R}^n$ if and only if $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$. Given the reductive decomposition $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, this is equivalent to

$$[\mathfrak{h}, \mathfrak{l}] = 0 \quad \text{and} \quad [\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{h}.$$

This implies that, for any G-invariant metric g on G/H, $(K/H, g|_{K/H})$ is isometric to a flat Euclidean space. If g is awesome, then by Remark 3.2 this implies that

$$r_i^{\mathfrak{l}} = \frac{1}{4d_i} \sum_{\hat{j},\hat{k} \in I_{\mathfrak{p}}} [i\hat{j}\hat{k}] \left(\frac{2}{l_i} + \frac{l_i}{p_j p_k} - \frac{p_j}{l_i p_k} - \frac{p_k}{l_i p_j} \right).$$

We already know, by Corollary 4.5, that on an immortal awesome Ricci flow, $g(t)|_{\mathfrak{p}\times\mathfrak{p}}$ approximates $t \cdot B|_{\mathfrak{p}\times\mathfrak{p}}$ and, by Corollary 4.7, that the blow-down of $g(t)$ in the $\mathfrak{l}$-direction goes to 0. These estimates are enough to have the following convergence result, which is our third main result.

Theorem 5.2. *Let $\tilde{M} = G/H$ be a contractible semisimple homogeneous space. Let $(\tilde{M}, g(s))$, $s \in [1, \infty)$, be an immortal awesome Ricci flow adapted to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{l}$. Then the parabolic rescaling $(\tilde{M}, s^{-1} g(s))$ converges in pointed C^∞-topology to the Riemannian product*

$$\Sigma_\infty \times \mathbb{E}^{\dim \mathfrak{l}},$$

where Σ_∞ is the noncompact Einstein symmetric space $(G/K, B|_{\mathfrak{p}\times\mathfrak{p}})$ and $\mathbb{E}^{\dim \mathfrak{l}}$ is the $\dim \mathfrak{l}$*-dimensional (flat) Euclidean space.*

Proof. Let us fix the background metric $Q := -B|_{\mathfrak{l}\times\mathfrak{l}} + B|_{\mathfrak{p}\times\mathfrak{p}}$ and let $g(s) = Q(P(s)\,\cdot\,,\cdot)$. Let us define $\tilde{g}_s := s^{-1} g(s)$. Since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$ without loss of generality we can assume $\mathfrak{l} \subset \mathfrak{z}(\mathfrak{k})$ (just take $\mathfrak{l} \perp_B \mathfrak{h}$). By [Hilgert and Neeb 2012, Theorem 13.1.7] we have the diffeomorphism

$$\phi : \mathfrak{p} \times K/H \xrightarrow{\sim} G/H, \quad (x, kH) \mapsto \exp(x) \cdot kH.$$

Moreover, since $K/H = \mathbb{R}^{\dim \mathfrak{l}}$ is an abelian group, $K/H = \exp(\mathfrak{l})H$.

Let us then define the 1-parameter family of diffeomorphisms

$$\phi_s : \mathfrak{p} \times \mathfrak{l} \xrightarrow{\sim} G/H, \quad (x, u) \mapsto \alpha(x) \cdot \beta_s(u) H,$$

with $\alpha(x) = \exp(x)$ and $\beta_s(u) = \exp\big(\sqrt{s(P^{\mathfrak{l}}(s))^{-1}}u\big)$, where $P^{\mathfrak{l}}(s)$ is the positive definite matrix defined by $g(s)|_{\mathfrak{l}\times\mathfrak{l}} = Q(P^{\mathfrak{l}}(s)\cdot, \cdot)$.

Let

$$\frac{\partial\phi_s}{\partial E_i^{\mathfrak{l}}}(x, u) = \frac{d}{dt}\alpha(x)\cdot\beta_s(u + tE_i^{\mathfrak{l}})H|_{t=0}$$

and notice that since K/H is abelian, we get that

$$\begin{aligned}(L_{(\alpha(x)\cdot\beta_s(u))^{-1}})_*\frac{\partial\phi_s}{\partial E_i^{\mathfrak{l}}}(x, u) &= (L_{\beta_s(-u)})_*(L_{\alpha(-x)})_*(L_{\alpha(x)})_*\frac{d}{dt}\beta_s(u + tE_i^{\mathfrak{l}})H|_{t=0} \\ &= (L_{\beta_s(-u)})_*\frac{d}{dt}\beta_s(u + tE_i^{\mathfrak{l}})H|_{t=0} \\ &= \frac{d}{dt}\beta_s(tE_i^{\mathfrak{l}})H|_{t=0}.\end{aligned}$$

This implies

$$\begin{aligned}\phi_s^*\tilde{g}_s(E_i^{\mathfrak{l}}, E_j^{\mathfrak{l}})_{(x,u)} &= \tilde{g}_s\left(\frac{\partial\phi_s^\beta}{\partial i}(x, u), \frac{\partial\phi_s^\beta}{\partial j}(x, u)\right)_{\phi_s(x,u)} \\ &= \tilde{g}_s\left(\frac{d}{dt}\beta_s(tE_i^{\mathfrak{l}})|_{t=0}, \frac{d}{dt}\beta_s(tE_i^{\mathfrak{l}})|_{t=0}\right)_{\phi_s(0,0)} \\ &= s^{-1}Q\big(P^{\mathfrak{l}}(s)\cdot\sqrt{s(P^{\mathfrak{l}}(s))^{-1}}E_i^{\mathfrak{l}}, \sqrt{s(P^{\mathfrak{l}}(s))^{-1}}E_j^{\mathfrak{l}}\big) \\ &= Q(E_i^{\mathfrak{l}}, E_j^{\mathfrak{l}}).\end{aligned}$$

Let

$$\frac{\partial\phi_s}{\partial E_i^{\mathfrak{p}}}(x, u) = \frac{d}{dt}\alpha(x + tE_i^{\mathfrak{p}})\cdot\beta_s(u)|_{t=0} \text{ and } \mathcal{A}_s^i(x, u) := (L_{(\alpha(x)\cdot\beta_s(u))^{-1}})_*\frac{\partial\phi_s}{\partial E_i^{\mathfrak{p}}}(x, u).$$

Observe that since K belongs to the normalizer of H in G, it acts on the right on G/H. Hence, using that $a^{-1}\exp(x)a = \exp(\mathrm{Ad}(a)x)$ [Hilgert and Neeb 2012, Proposition 9.2.10], we get that

$$\begin{aligned}\mathcal{A}_s^i(x, u) &= (L_{\beta_s(-u)})_*(L_{\alpha(-x)})_*(R_{\beta_s(u)})_*\frac{d}{dt}\alpha(x + tE_i^{\mathfrak{p}})|_{t=0} \\ &= (L_{\beta_s(-u)})_*(R_{\beta_s(u)})_*(L_{\alpha(-x)})_*\frac{d}{dt}\alpha(x + tE_i^{\mathfrak{p}})|_{t=0} \\ &= \mathrm{Ad}(\beta_s(u))(L_{\exp(-x)})_*\frac{d\exp}{dt}(x + tE_i^{\mathfrak{p}})|_{t=0}\end{aligned}$$

and that $\mathcal{A}_s^i(x, 0) = \mathcal{A}^i(x)$ does not depend on s.

Now observe that by [Hilgert and Neeb 2012, Theorem 13.1.5] $\mathrm{Ad}(K)$ is compact. Hence, there is a constant C, such that

$$\|\mathrm{Ad}(\beta_s(u))\|_Q \leq C,$$

in the operator norm with respect to Q. By Corollary 4.7, we have that

$$\begin{aligned}
|\phi_s^*\tilde{g}_s(E_i^{\mathfrak{p}}, E_j^{\mathfrak{l}})_{(x,u)}| &= \left|\tilde{g}_s\left(\frac{\partial\phi_s}{\partial E_i^{\mathfrak{p}}}(x,u), \frac{\partial\phi_s}{\partial E_j^{\mathfrak{l}}}(x,u)\right)\right|_{\phi_s(x,u)} \\
&= \left|\tilde{g}_s\left(\mathcal{A}_s^i(x,u), \frac{d}{dt}\beta_s(tE_j^{\mathfrak{l}})|_{t=0}\right)\right|_{\phi_s(0,0)} \\
&= s^{-1}\big|Q\big(\mathcal{A}_s^i(x,u), P^{\mathfrak{l}}(s)\cdot\sqrt{s(P^{\mathfrak{l}}(s))^{-1}}E_j^{\mathfrak{l}}\big)\big| \\
&= \sqrt{s^{-1}}\big|Q\big(\mathcal{A}_s^i(x,u), \sqrt{P^{\mathfrak{l}}(s)}E_j^{\mathfrak{l}}\big)\big| \\
&\le C\sqrt{\frac{\|P^{\mathfrak{l}}(s)\|_Q}{s}}\cdot\|\mathcal{A}^i(x)\|_Q\|E_j^{\mathfrak{l}}\|_Q \to 0,
\end{aligned}$$

uniformly on compact sets of $\mathfrak{p}\times\mathfrak{l}$.

Finally, we have that

$$\begin{aligned}
\phi_s^*\tilde{g}_s(E_i^{\mathfrak{p}}, E_j^{\mathfrak{p}})_{(x,u)} &= \tilde{g}_s\left(\frac{\partial\phi_s}{\partial E_i^{\mathfrak{p}}}(x,u), \frac{\partial\phi_s}{\partial E_j^{\mathfrak{p}}}(x,u)\right)_{\phi_s(x,u)} \\
&= \tilde{g}_s(\mathcal{A}_s^i(x,u), \mathcal{A}_s^j(x,u))_{\phi_s(0,0)} \\
&= \tilde{g}_s|_{\mathfrak{l}\times\mathfrak{l}}(\mathcal{A}_s^i(x,u), \mathcal{A}_s^j(x,u)) + \tilde{g}_s|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}_s^i(x,u), \mathcal{A}_s^j(x,u)).
\end{aligned}$$

Again by the fact that $\mathrm{Ad}(K)$ is compact and by Corollary 4.7, we have that

$$\big|\tilde{g}_s|_{\mathfrak{l}\times\mathfrak{l}}(\mathcal{A}_s^i(x,u), \mathcal{A}_s^j(x,u))\big| \le C^2\frac{\|P^{\mathfrak{l}}(s)\|_Q}{s}\cdot\|\mathcal{A}^i(x)\|_Q\|\mathcal{A}^j(x)\|_Q \to 0,$$

uniformly on compact subsets of $\mathfrak{p}\times\mathfrak{l}$. Moreover, by Corollary 4.5 we know that $\tilde{g}_s|_{\mathfrak{p}\times\mathfrak{p}} \to B|_{\mathfrak{p}\times\mathfrak{p}}$ and since B is $\mathrm{Ad}(K)$-invariant, we get that

$$\tilde{g}_s|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}_s^i(x,u), \mathcal{A}_s^j(x,u)) \to Q|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}^i(x), \mathcal{A}^j(x)),$$

uniformly on compact subsets of $\mathfrak{p}\times\mathfrak{l}$. Indeed, assume the contrary, then there exists $\epsilon > 0$ and sequences $s_n \to \infty$ and $(x_n, u_n) \to (x_\infty, u_\infty)$ such that

$$\big|\tilde{g}_{s_n}|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}_{s_n}^i(x_n,u_n), \mathcal{A}_{s_n}^j(x_n,u_n)) - Q|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}^i(x_n), \mathcal{A}^j(x_n))\big| \ge \epsilon. \tag{5-5}$$

By the compactness of $\mathrm{Ad}(K/H)$, we can extract a convergent subsequence, $\mathrm{Ad}(\beta_{s_n}(u_n)) \to \mathrm{Ad}(\exp(u'_\infty))$, $u'_\infty \in \mathfrak{k}$. Therefore, taking the limit on (5-5) as $n \to \infty$, we get that

$$\begin{aligned}
\big|Q|_{\mathfrak{p}\times\mathfrak{p}}\big(&\mathrm{Ad}(\exp(u'_\infty))\cdot\mathcal{A}^i(x_\infty), \mathrm{Ad}(\exp(u'_\infty))\cdot\mathcal{A}^j(x_\infty)\big) \\
&- Q|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}^i(x_\infty), \mathcal{A}^j(x_\infty))\big| = 0,
\end{aligned}$$

since Q is $\mathrm{Ad}(K)$-invariant.

Observe that $Q|_{\mathfrak{p}\times\mathfrak{p}}(\mathcal{A}^i(x), \mathcal{A}^j(x))$ is the pullback by the diffeomorphism

$$\exp : \mathfrak{p} \to G/K$$

of the Ricci negative Einstein symmetric metric of $\Sigma_\infty := (G/K, B|_{\mathfrak{p}\times\mathfrak{p}})$.

Hence, we have proven that $\phi_s^* \tilde{g}_s$ converges in the C^∞-topology to the Riemannian product $\Sigma_\infty \times \mathbb{E}^{\dim \mathfrak{l}}$. □

Remark 5.3. Since every immortal homogeneous Ricci flow is of type III [Böhm 2015, Theorem 4.1], the proof above actually shows that the parabolic rescaled flow $\tilde{g}_s(t) := s^{-1} g(st)$, $t \in (0, \infty)$, converges in Cheeger–Gromov sense to the expanding Ricci soliton given by the Riemannian product of the Ricci negative Einstein metric in Σ_∞ and the flat Euclidean factor $\mathbb{E}^{\dim \mathfrak{l}}$. This generalizes the 3-dimensional case $\widetilde{\mathrm{SL}(2, \mathbb{R})} \to \mathbb{H}^2 \times \mathbb{R}$ (see [Lott 2007, Case 3.3.5]) to every semisimple homogeneous $\mathbb{R}$-bundle over irreducible Hermitian symmetric spaces G/H, since for those every G-invariant metric is awesome [Böhm and Lafuente 2022, Remark 3.2].

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SOBOLEV NORMS OF L^2-SOLUTIONS TO THE NONLINEAR SCHRÖDINGER EQUATION

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We apply inverse spectral theory to study Sobolev norms of solutions to the nonlinear Schrödinger equation. For initial datum $q_0 \in L^2(\mathbb{R})$ and $s \in [-1, 0]$, we prove that there exists a conserved quantity which is equivalent to $H^s(\mathbb{R})$-norm of the solution.

1. Introduction

In the last two decades, the theory of orthogonal polynomials on the unit circle (OPUC) has been used to obtain some of the strongest results in the spectral theory (see, e.g., [9; 17; 18]). Bessonov and Denisov [3] have applied OPUC techniques to characterize existence and completeness of wave operators for the Dirac evolution on the half-line. One area where scattering theory for Dirac systems finds applications is the so-called inverse scattering approach to the nonlinear Schrödinger equation (NLSE). Below, we develop a general framework that enables one to use the theory of Krein systems (a continuous analog of OPUC [13]) in the context of NLSE. To illustrate our approach, we study the Sobolev norms of solutions to NLSE adding to the area which attracted much attention in recent years [5; 6; 7; 15; 19; 20; 21; 23]. Our Theorem 1.2 stated below is not new and can be deduced from the results of Koch and Tataru [21] or by using an alternative method of Killip, Vişan and Zhang [19]. However, we have developed a new and promising approach to that problem which adapts the technique from [3] to the setting of NLSE and shows, in particular, that the sharp regularity class used to characterize scattering in the Dirac system can be studied in the context of Sobolev spaces. Then, we employ our analysis to obtain Theorem 1.2 which represents the first step in applying methods of [3] to NLSE. In the current paper, we also develop a convenient language which we hope can be used by the spectral theory community to further study NLSE dynamics.

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Turning to the actual content of the paper, consider the classical defocusing nonlinear Schrödinger equation [14; 26; 30] on the real line,

$$\begin{cases} i\frac{\partial q}{\partial t} = -\frac{\partial^2 q}{\partial \xi^2} + 2|q|^2 q, \\ q|_{t=0} = q_0, \end{cases} \qquad \xi \in \mathbb{R},\ t \in \mathbb{R}. \tag{1-1}$$

It is known that for sufficiently regular initial datum q_0 the unique classical solution $q = q(\xi, t)$ exists globally in time. For example, if q_0 lies in the Schwartz class $\mathcal{S}(\mathbb{R})$, then $q(\,\cdot\,, t) \in \mathcal{S}(\mathbb{R})$ for all $t \in \mathbb{R}$. The long-time asymptotics of q is known [10; 11; 29]. For less regular initial datum q_0, one can define the solution by an approximation argument (see, e.g., [28]):

Theorem 1.1. *Let $q_0 \in L^2(\mathbb{R})$ and let $q_{0,n} \in \mathcal{S}(\mathbb{R})$ converge to q_0 in $L^2(\mathbb{R})$. Denote by $q_n(\xi, t)$ the solution of* (1-1) *corresponding to $q_{0,n}$. We have*

$$\lim_{n\to+\infty} \|q_n(\,\cdot\,, t) - q(\,\cdot\,, t)\|_{L^2(\mathbb{R})} = 0, \quad t \in \mathbb{R}$$

for some function $q(\xi, t) : \mathbb{R}^2 \to \mathbb{R}$ that does not depend on the choice of the sequence $q_{0,n}$.

The function q in Theorem 1.1 is called the L^2-solution of (1-1) corresponding to the initial datum $q_0 \in L^2(\mathbb{R})$. It is clear that such a solution is unique. The total energy of the solution is its $L^2(\mathbb{R})$-norm and it is conserved in time:

$$\|q(\,\cdot\,, t)\|_{L^2(\mathbb{R})} = \|q_0\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}.$$

By Plancherel's formula, it is equal to $\|(\mathcal{F}q)(\,\cdot\,, t)\|_{L^2(\mathbb{R})}$ where $\mathcal{F}$ stands for the Fourier transform. In this paper, we work with Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$. The $H^s(\mathbb{R})$-norm of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\|f\|_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1 + |\eta|^2)^s |(\mathcal{F}f)(\eta)|^2 \, d\eta\right)^{1/2}. \tag{1-2}$$

The space $H^s(\mathbb{R})$ is the completion of $\mathcal{S}(\mathbb{R})$ with respect to this norm. Equivalently, one can define it by

$$H^s(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) : (1 + |\eta|^2)^{s/2} \mathcal{F}f \in L^2(\mathbb{R})\},$$

where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distribution.

In contrast to the linear Schrödinger equation for which all Sobolev norms are conserved, the solutions of NLSE can exhibit inflation of Sobolev norm $H^s(\mathbb{R})$ for $s \leqslant -\frac{1}{2}$ (see, e.g., [8; 20] for details). Specifically, given an arbitrarily small positive ε and $s \leqslant -\frac{1}{2}$, there exists a solution q to (1-1) that satisfies

$$q_0 \in \mathcal{S}(\mathbb{R}), \quad \|q_0\|_{H^s(\mathbb{R})} \leqslant \varepsilon, \quad \|q(\,\cdot\,, \varepsilon)\|_{H^s(\mathbb{R})} \geqslant \varepsilon^{-1}, \tag{1-3}$$

see [8] for that construction. This result is related to the "high-to-low frequency cascade". It occurs when for initial datum $q_0 \in \mathcal{S}(\mathbb{R})$, a part of $L^2(\mathbb{R})$-norm of q, when written on the Fourier side, moves from high to low frequencies as time increases. The Sobolev norms with negative index s can be used to capture this phenomenon. Indeed, since $\|q(\,\cdot\,,t)\|_{L^2(\mathbb{R})}$ is time-invariant and the weight $(1+\eta^2)^s$ in (1-2) vanishes at infinity when $s<0$, the transfer of L^2-norm from high to low values of frequency η makes the $H^s(\mathbb{R})$-norm grow.

For NLSE, the inflation of $H^s(\mathbb{R})$-norm cannot happen for $s>-\frac{1}{2}$. Koch and Tataru [21] discovered the set of conserved quantities which agree with $H^s(\mathbb{R})$-norm up to a quadratic term for a small value of $\|q_0\|_{H^s(\mathbb{R})}$ and $s>-\frac{1}{2}$. As a corollary, they obtained the bounds on $\|q(\,\cdot\,,t)\|_{H^s(\mathbb{R})}$ that are uniform in time:

$$\text{(1-4)}\qquad \|q(\,\cdot\,,t)\|_{H^s(\mathbb{R})} \leqslant C(s)\begin{cases}\mathcal{R}+\mathcal{R}^{1+2s}, & s>0,\\ \mathcal{R}+\mathcal{R}^{\frac{1+4s}{1+2s}}, & s\in\left(-\frac{1}{2},0\right),\end{cases}\qquad \mathcal{R}:=\|q_0\|_{H^s(\mathbb{R})}.$$

Killip, Vişan, and Zhang [19] proved a similar estimate using a different method. The estimates on the growth of $H^s(\mathbb{R})$-norms are related to questions of well-posedness and ill-posedness of NLSE in Sobolev classes which have been extensively studied previously, see, e.g., [5; 6; 7; 15; 19; 20; 21; 23].

We use some recent results in the inverse spectral theory [1; 2; 3] to show that there are conserved quantities of NLSE which agree with $H^s(\mathbb{R})$-norm provided that $s\in[-1,0]$ and the value of $\|q_0\|_{L^2(\mathbb{R})}$ is under control. We apply our analysis to prove the following theorem.

Theorem 1.2. *Let $q_0\in L^2(\mathbb{R})$ and let $q=q(\xi,t)$ be the solution of* (1-1) *corresponding to q_0. Then,*

$$\text{(1-5)}\qquad C_1(1+\|q_0\|_{L^2(\mathbb{R})})^{2s}\|q_0\|_{H^s(\mathbb{R})} \leqslant \|q(\,\cdot\,,t)\|_{H^s(\mathbb{R})} \leqslant C_2(1+\|q_0\|_{L^2(\mathbb{R})})^{-2s}\|q_0\|_{H^s(\mathbb{R})},$$

where $t\in\mathbb{R}$, $s\in[-1,0]$, and C_1 and C_2 are two positive absolute constants.

This result shows, in particular, that for a given function $q_0:\|q_0\|_{L^2(\mathbb{R})}=1$ whose $L^2(\mathbb{R})$-norm is concentrated on high frequencies, we will never see a significant part of $L^2(\mathbb{R})$-norm of the solution q moving to the low frequencies. That limits the "high-to-low frequency cascade" we discussed above. The close inspection of construction used in [8] shows that the function q_0 in (1-3) has $H^s(\mathbb{R})$-norm smaller than ε but its $L^2(\mathbb{R})$-norm is large when ε is small. Hence, the bounds in Theorem 1.2 do not contradict the estimates in (1-3) when $s\in\left[-1,-\frac{1}{2}\right]$. We do not know whether Theorem 1.2 holds for $s<-1$.

The main idea of the proof of Theorem 1.2 is based on the analysis of the conserved quantity $a(z)$, $\operatorname{Im} z>0$, which is a coefficient in the transition matrix for the Dirac equation with potential $q=q(\,\cdot\,,t)$. We take $z=i$ and show that $\log|a(i)|$

is related to a certain quantity $\widetilde{\mathcal{K}}_Q$ (see Lemma 3.4 below) that characterizes both size and oscillation of q. Use of $\widetilde{\mathcal{K}}_Q$ in the context of NLSE is the main novelty of our work. We study $\widetilde{\mathcal{K}}_Q$ and show that it is equivalent to $H^{-1}(\mathbb{R})$ norm of q with constants that depend on its $L^2(\mathbb{R})$-norm. That gives the estimate (1-5) for $s = -1$ and the intermediate range of $s \in (-1, 0)$ is handled by interpolation. Our analysis relies heavily on the recent results [1; 2; 3] that characterize Krein–de Branges canonical systems and the Dirac operators whose spectral measures belong to the Szegő class on the real line. We also establish the framework that allows working with NLSE in the context of well-studied Krein systems.

Notation.

- The symbol I stands for 2×2 identity matrix $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and symbol J stands for $J = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. Constant matrices σ_3, $\sigma_\pm$, σ are defined in (2-2).
- For a measurable set $S \subset \mathbb{R}$, we say that $f \in L^1_{\mathrm{loc}}(S)$ is $f \in L^1(K)$ for every compact $K \subset S$.
- The Fourier transform of a function f is defined by
$$(\mathcal{F}f)(\eta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x)\, e^{-i\eta x}\, dx.$$
- The symbol C, unless we specify explicitly, denotes the positive absolute constant which can change its value in different formulas. If we write, e.g., $C(\alpha)$, this defines a positive function of parameter α.
- For two nonnegative functions f_1 and f_2, we write $f_1 \lesssim f_2$ if there is an absolute constant C such that $f_1 \leqslant C f_2$ for all values of the arguments of f_1 and f_2. We define $\gtrsim$ similarly and say that $f_1 \sim f_2$ if $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ simultaneously. If $|f_3| \lesssim f_4$, we will write $f_3 = O(f_4)$.
- Symbols $\{e_j\}$ are reserved for the standard basis in $\mathbb{C}^2$: $e_1 = \left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\ 1\end{smallmatrix}\right)$.
- For matrix A, the symbol $\|A\|_{\mathrm{HS}}$ denotes its Hilbert–Schmidt norm such that $\|A\|_{\mathrm{HS}} = (\mathrm{tr}(A^*A))^{1/2}$.

2. Preliminaries

Our proof of Theorem 1.1 uses complete integrability of (1-1). In that framework, (1-1) can be solved by using the method of inverse scattering which we discuss following [14].

2.1. ***The inverse scattering approach to NLSE.*** Given a complex-valued function $q \in \mathcal{S}(\mathbb{R})$, define the differential operator

$$L_q = i\sigma_3 \frac{d}{d\xi} + i(q\sigma_- - \bar{q}\sigma_+), \tag{2-1}$$

where we borrow notation for constant matrices σ_3, $\sigma_\pm$ from [14]:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma = \sigma_- + \sigma_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2-2}$$

The expression L_q is one of the forms in which the Dirac operator can be written. In Section 3, we will introduce another form and will show how the two are related. Let us also define

$$E(\xi, \lambda) = e^{\frac{\lambda}{2i}\xi\sigma_3} = \begin{pmatrix} e^{\frac{\lambda}{2i}\xi} & 0 \\ 0 & e^{-\frac{\lambda}{2i}\xi} \end{pmatrix},$$

as in [14]. In the free case when $q = 0$, the matrix-function E solves $L_0 E = \frac{\lambda}{2} E$, $E(0, \lambda) = I$. Since $q \in \mathcal{S}(\mathbb{R})$, it decays at infinity fast and therefore one can find two solutions $T_\pm = T_\pm(\xi, \lambda)$ such that

$$L_q T_\pm = \frac{\lambda}{2} T_\pm, \quad T_\pm = E(\xi, \lambda) + o(1), \quad \xi \to \pm\infty \tag{2-3}$$

for every $\lambda \in \mathbb{R}$. These solutions are called the *Jost solutions* for L_q. Since both T_+ and T_- solve the same ODE, they must satisfy

$$T_-(\xi, \lambda) = T_+(\xi, \lambda) T(\lambda), \quad \xi \in \mathbb{R}, \ \lambda \in \mathbb{R}, \tag{2-4}$$

where the matrix $T = T(\lambda)$ does not depend on $\xi \in \mathbb{R}$. One can show that it has the form

$$T(\lambda) = \begin{pmatrix} a(\lambda) & \overline{b(\lambda)} \\ b(\lambda) & \overline{a(\lambda)} \end{pmatrix}, \quad \det T = |a|^2 - |b|^2 = 1. \tag{2-5}$$

The matrix T is called the *reduced transition matrix* for L_q, and the ratio $\boldsymbol{r}_q = b/a$ is called the *reflection coefficient* for L_q. One can obtain T in a different way: let $Z_q = Z_q(\xi, \lambda)$, $\xi \in \mathbb{R}$, $\lambda \in \mathbb{C}$ be the fundamental matrix for L_q, that is,

$$L_q Z_q = \frac{\lambda}{2} Z_q, \quad Z_q(0, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2-6}$$

Then, we have $Z_q(\xi, \lambda) = T_\pm(\xi, \lambda) T_\pm^{-1}(0, \lambda)$ and the pointwise limits

$$T_\pm^{-1}(0, \lambda) = \lim_{\xi \to \pm\infty} E^{-1}(\xi, \lambda)\, Z_q(\xi, \lambda) \tag{2-7}$$

exist for every $\lambda \in \mathbb{R}$. Moreover, we have $T(\lambda) = T_+^{-1}(0, \lambda) T_-(0, \lambda)$ on $\mathbb{R}$.

The coefficients a, b, and $\boldsymbol{r}_q$ were defined for $\lambda \in \mathbb{R}$ and they satisfy $|a|^2 = 1 + |b|^2$, $1 - |\boldsymbol{r}_q|^2 = |a|^{-2}$ for these λ. However, one can show that $a(\lambda)$ is the boundary value of the outer function defined in $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by the formula (see (6.22) in [14])

$$a(z) = \exp\left(\frac{1}{\pi i} \int_{\mathbb{R}} \frac{1}{\lambda - z} \log|a(\lambda)|\, d\lambda \right), \quad z \in \mathbb{C}_+,$$

which, in view of identity $1-|\boldsymbol{r}_q|^2=|a|^{-2}$ on $\mathbb{R}$, can be written as

$$a(z)=\exp\Bigl(-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{1}{\lambda-z}\log(1-|\boldsymbol{r}_q(\lambda)|^2)\,d\lambda\Bigr). \tag{2-8}$$

That shows, in particular, that b defines both a and $\boldsymbol{r}_q$, and $\boldsymbol{r}_q$ defines a and b.

The map $q\mapsto\boldsymbol{r}_q$ is called the direct scattering transform and its inverse is called the inverse scattering transform. These maps are well studied when $q\in\mathcal{S}(\mathbb{R})$. In particular, we have the following result (see [14] for the proof).

Theorem 2.1. *The map $q\mapsto\boldsymbol{r}_q$ is a bijection from $\mathcal{S}(\mathbb{R})$ onto the set of complex-valued functions $\{\boldsymbol{r}\in\mathcal{S}(\mathbb{R}),\ \|\boldsymbol{r}\|_{L^\infty(\mathbb{R})}<1\}$.*

The scattering transform has some symmetries:

Lemma 2.2. *If $q\in\mathcal{S}(\mathbb{R})$ and $\lambda\in\mathbb{R}$, then*

(dilation)	$\boldsymbol{r}_{\alpha q(\alpha\xi)}(\lambda)=\boldsymbol{r}_{q(\xi)}(\alpha^{-1}\lambda)$,	$\alpha>0$,
(conjugation)	$\boldsymbol{r}_{\bar q(\xi)}(\lambda)=\overline{\boldsymbol{r}_{q(\xi)}(-\lambda)}$,	
(translation)	$\boldsymbol{r}_{q(\xi-\ell)}(\lambda)=\boldsymbol{r}_{q(\xi)}(\lambda)\,e^{-i\lambda\ell}$,	$\ell\in\mathbb{R}$,
(modulation)	$\boldsymbol{r}_{e^{-i\beta\xi}q(\xi)}(\lambda)=\boldsymbol{r}_{q(\xi)}(\lambda+\beta)$,	$\beta\in\mathbb{R}$.
(rotation)	$\boldsymbol{r}_{\mu q(\xi)}(\lambda)=\mu\boldsymbol{r}_{q(\xi)}(\lambda)$,	$\mu\in\mathbb{C},\ \lvert\mu\rvert=1$.

Proof. Indeed, the direct substitution into (2-3) shows that if $T_\pm(\xi,\lambda)$ are Jost solutions for $q(\xi)$, then:

(a) $T_\pm(\alpha\xi,\alpha^{-1}\lambda)$ are the Jost solutions for $\alpha q(\alpha\xi)$.

(b) $\overline{T}_\pm(\xi,-\lambda)$ are the Jost solutions for $\overline{q(\xi)}$.

(c) $T_\pm(\xi-\ell,\lambda)\,E(\ell,\lambda)$ are the Jost solutions for $q(\xi-\ell)$.

(d) $E(-\xi,\beta)\,T_\pm(\xi,\lambda+\beta)$ are the Jost solutions for $e^{-i\beta\xi}q(\xi)$.

(e) $\left(\begin{smallmatrix}1&0\\0&\mu\end{smallmatrix}\right)T_\pm(\xi,\lambda)\left(\begin{smallmatrix}1&0\\0&\bar\mu\end{smallmatrix}\right)$ are the Jost solutions for $\mu q(\xi)$, $|\mu|=1$.

Now, it is left to use the formula (2-4) which defines T. A computation using (2-5) shows how a and b change under symmetries (a)–(e). For example, the translation does not change a and it multiplies b by $e^{-i\lambda l}$. The modulation $e^{-i\beta\xi}q(\xi)$, however, gives $a_{e^{-i\beta\xi}q(\xi)}(\lambda)=a_{q(\xi)}(\lambda+\beta)$. Then, the claim follows from the definition of the reflection coefficient $\boldsymbol{r}_q=b/a$. □

The next result (see formula (7.5) in [14]), along with the previous theorem, shows how the inverse scattering transform can be used to solve (1-1).

Theorem 2.3. *Let $q_0\in\mathcal{S}(\mathbb{R})$ and let $\boldsymbol{r}_{q_0}=\boldsymbol{r}_{q_0}(\lambda)$ be the reflection coefficient of L_{q_0}. Define the family*

$$\boldsymbol{r}(\lambda,t)=e^{-i\lambda^2 t}\boldsymbol{r}_{q_0}(\lambda),\quad \lambda\in\mathbb{R},\ t\in\mathbb{R}. \tag{2-9}$$

For each $t \in \mathbb{R}$, let $q = q(\xi, t)$ be the potential in the previous theorem generated by $\boldsymbol{r}(\lambda, t)$. Then, $q = q(\xi, t)$ is the unique classical solution of (1-1) *with the initial datum q_0. Moreover, for every $t \in \mathbb{R}$, the function $\xi \mapsto q(\xi, t)$ lies in $\mathcal{S}(\mathbb{R})$.*

The solutions to the NLSE

$$i\frac{\partial q}{\partial t} = -\frac{\partial^2 q}{\partial \xi^2} + 2|q|^2 q \tag{2-10}$$

behave in an explicit way under some transformations. Specifically, we have:

(a) Dilation: if $q(\xi, t)$ solves (2-10), then $\alpha q(\alpha\xi, \alpha^2 t)$ solves (2-10) for every $\alpha \neq 0$.

(b) Time reversal: if $q(\xi, t)$ solves (2-10), then $\bar{q}(\xi, -t)$ solves (2-10). In particular, if q_0 is real-valued, then $q(\xi, t) = q(\xi, -t)$.

(c) Translation: if $q(\xi, t)$ solves (2-10), then $q(\xi - \ell, t)$ solves (2-10) for every $\ell \in \mathbb{R}$.

(d) Modulation or Galilean symmetry: if $q(\xi, t)$ solves equation (2-10), then $e^{iv\xi - iv^2 t} q(\xi - 2vt, t)$ solves (2-10) for every $v \in \mathbb{R}$.

(e) Rotation: if $q(\xi, t)$ solves (2-10), then $\mu q(\xi, t)$ solves (2-10) for every $\mu \in \mathbb{C}$, $|\mu| = 1$.

These properties can be checked by direct calculation (see, e.g., formula (1.19) in [15] for (d)) and a simple inspection shows that the bound (1-5) is consistent with all these transformations. The statements of Theorem 2.3 and Lemma 2.2 are consistent with these symmetries as well.

Now, we can explain the idea behind the proof of Theorem 1.2.

The idea of the proof for Theorem 1.2. One can proceed as follows. First, we assume that $q_0 \in \mathcal{S}(\mathbb{R})$ and notice that conservation of $|\boldsymbol{r}(\lambda, t)|$, $\lambda \in \mathbb{R}$, guaranteed by (2-9), yields that $\log|a(i, t)|$ is conserved, where $a(z, t)$ is defined for $z \in \mathbb{C}_+$ by (2-8). Separately, for every Dirac operator L_q with $q \in L^2(\mathbb{R})$, we show that $\log|a(i)|$ is equivalent to some explicit quantity $\widetilde{\mathcal{K}}_Q$ that involves q. That quantity was introduced and studied in [1; 2; 3]: it resembles the matrix Muckenhoupt $A_2(\mathbb{R})$ condition and it is equivalent to $H^{-1}(\mathbb{R})$ norm of q provided that $\|q\|_{L^2(\mathbb{R})}$ is under control, e.g., $\|q\|_{L^2(\mathbb{R})} < C$ with some fixed C. Putting things together, we see that Sobolev $H^{-1}(\mathbb{R})$ norm of $q(\,\cdot\,, t)$ does not change much in time provided that the bound $\|q(\,\cdot\,, t)\|_{L^2(\mathbb{R})} < C$ holds. Since $\|q(\,\cdot\,, t)\|_{L^2(\mathbb{R})} = \|q_0\|_{L^2(\mathbb{R})}$ is time-invariant, we arrive at the statement of Theorem 1.2 for $q_0 \in \mathcal{S}(\mathbb{R})$ and $s = -1$. For $s = 0$, the claim of Theorem 1.2 is trivial. The intermediate range of $s \in (-1, 0)$ is handled by interpolation using Galilean invariance of NLSE. The general case when $q_0 \in L^2(\mathbb{R})$ follows by a density argument if one uses the stability of L^2-solutions guaranteed by Theorem 1.1.

There are other methods that use conserved quantities that agree with negative Sobolev norms. The paper [19] uses a representation of $\log|a(i)|$ through a perturbation determinant. Then, the analysis of the perturbation series allows the authors of [19] to obtain estimates similar to (1-4). It is conceivable that this approach can provide results along the same lines as Theorem 1.2.

To focus on the Dirac operator with $q \in L^2(\mathbb{R})$, we first consider this operator on half-line $\mathbb{R}_+$ in connection to Krein systems that were introduced in [22].

2.2. ***Operator L_q and Krein system.*** Let $A : \mathbb{R}_+ \to \mathbb{C}$ be a function on the positive half-line $\mathbb{R}_+ = [0, +\infty)$ such that

$$\int_0^r |A(\xi)|\, d\xi < \infty$$

for every $r \geqslant 0$. Recall that we denote the set of such functions by $L^1_{\mathrm{loc}}(\mathbb{R}_+)$. The Krein system (see (4.52) in [13]) with coefficient A has the form

$$\begin{cases} P'(\xi,\lambda) = i\lambda P(\xi,\lambda) - \overline{A(\xi)}P_*(\xi,\lambda), & P(0,\lambda)=1, \\ P_*'(\xi,\lambda) = -A(\xi)P(\xi,\lambda), & P_*(0,\lambda)=1, \end{cases} \tag{2-11}$$

where the derivative is taken with respect to $\xi \in \mathbb{R}_+$ and $\lambda \in \mathbb{C}$. Let also

$$\begin{cases} \hat{P}'(\xi,\lambda) = i\lambda \hat{P}(\xi,\lambda) + \overline{A(\xi)}\hat{P}_*(\xi,\lambda), & \hat{P}(0,\lambda)=1, \\ \hat{P}_*'(\xi,\lambda) = A(\xi)\hat{P}(\xi,\lambda), & \hat{P}_*(0,\lambda)=1, \end{cases} \tag{2-12}$$

denote the so-called dual Krein system (see Corollary 5.7 in [13]). Set

$$Y(\xi,\lambda) = e^{-i\lambda\xi}\begin{pmatrix} P(2\xi,\lambda) & i\hat{P}(2\xi,\lambda) \\ P_*(2\xi,\lambda) & -i\hat{P}_*(2\xi,\lambda)\end{pmatrix}. \tag{2-13}$$

The matrix-function Z_q, which was defined in (2-6) for $q \in \mathcal{S}(\mathbb{R})$, makes sense if we assume that $q \in L^1_{\mathrm{loc}}(\mathbb{R})$. In the next lemma, we relate Y to Z_q.

Lemma 2.4. *Let $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, $A(2\xi) = -\frac{1}{2}\overline{q(\xi)}$ on $\mathbb{R}_+$, and Y be the corresponding matrix-valued function defined by* (2-13). *Then, $Z_q(\xi, 2\lambda) = \sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$ for $\xi \geqslant 0$ and $\lambda \in \mathbb{C}$.*

Proof. The proof is a computation. We have

$$\begin{aligned} &L_{\bar{q}}Y(\xi,\lambda) \\ &= \lambda\sigma_3 Y(\xi,\lambda) + i\sigma_3 e^{-i\lambda\xi}\frac{d}{d\xi}\begin{pmatrix} P(2\xi,\lambda) & i\hat{P}(2\xi,\lambda) \\ P_*(2\xi,\lambda) & -i\hat{P}_*(2\xi,\lambda)\end{pmatrix} + i(\bar{q}\sigma_- - q\sigma_+)Y(\xi,\lambda), \\ &= 2i\sigma_3 e^{-i\lambda\xi}\begin{pmatrix} i\lambda P(2\xi,\lambda) - \overline{A(2\xi)}P_*(2\xi,\lambda) & -\lambda\hat{P}(2\xi,\lambda) + i\overline{A(2\xi)}\hat{P}_*(2\xi,\lambda) \\ -A(2\xi)P(2\xi,\lambda) & -iA(2\xi)\hat{P}(2\xi,\lambda)\end{pmatrix} \\ &\qquad + i(\bar{q}\sigma_- - q\sigma_+ - i\lambda\sigma_3)Y(\xi,\lambda). \end{aligned}$$

The second summand equals

$$ie^{-i\lambda\xi}\begin{pmatrix}-i\lambda & -q\\ \bar q & i\lambda\end{pmatrix}\begin{pmatrix}P(2\xi,\lambda) & i\hat P(2\xi,\lambda)\\ P_*(2\xi,\lambda) & -i\hat P_*(2\xi,\lambda)\end{pmatrix}$$
$$= ie^{-i\lambda\xi}\begin{pmatrix}-i\lambda P(2\xi,\lambda)-qP_*(2\xi,\lambda) & \lambda\hat P(2\xi,\lambda)+iq\hat P_*(2\xi,\lambda)\\ \bar q P(2\xi,\lambda)+i\lambda P_*(2\xi,\lambda) & i\bar q\hat P(2\xi,\lambda)+\lambda\hat P_*(2\xi,\lambda)\end{pmatrix}.$$

Using relation $2A(2\xi)+\bar q(\xi)=0$, we obtain

$$L_{\bar q}Y(\xi,\lambda)=ie^{-i\lambda\xi}\begin{pmatrix}i\lambda P(2\xi,\lambda) & -\lambda\hat P(2\xi,\lambda)\\ i\lambda P_*(2\xi,\lambda) & \lambda\hat P_*(2\xi,\lambda)\end{pmatrix}=-\lambda Y(\xi,\lambda).$$

Since $\sigma\sigma_3\sigma=-\sigma_3$ and $\sigma\sigma_\pm\sigma=\sigma_\mp$, one has $\sigma L_{\bar q}\sigma=-L_q$. Therefore,

$$L_q(\sigma Y(\xi,\lambda)\sigma)=\lambda(\sigma Y(\xi,\lambda)\sigma).$$

It follows that matrix-valued functions $Z_q(\xi,2\lambda)$ and $\sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$ solve the same Cauchy problem. Thus $Z_q(\xi,2\lambda)=\sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$, as required. □

Lemma 2.5. *Let $q\in L^1_{\mathrm{loc}}(\mathbb{R})$, let $A(2\xi)=\frac12 q(-\xi)$ on $\mathbb{R}_+$, and let Y be the corresponding matrix-valued function defined by* (2-13). *Then, we have that $Z_q(-\xi,2\lambda)=Y(\xi,\lambda)Y^{-1}(0,\lambda)$ for $\xi\geqslant 0$ and $\lambda\in\mathbb{C}$.*

Proof. Recall that matrices σ_3, $\sigma_\pm$, σ are defined in formula (2-2). Using relations $\sigma\sigma_3\sigma=-\sigma_3$ and $\sigma\sigma_\pm\sigma=\sigma_\mp$, we see that $L_{\tilde q}\widetilde Z_q=\frac{\lambda}{2}\widetilde Z_q$, where $\tilde q(\xi)=-\overline{q(-\xi)}$ and $\widetilde Z_q(\xi,\lambda)=\sigma Z_q(-\xi,\lambda)\sigma$. Then, Lemma 2.5 applies to $\tilde q$, $Z_{\tilde q}(\xi,2\lambda)=\widetilde Z_q(\xi,2\lambda)$ and $A(2\xi)=-\frac12\overline{\tilde q(\xi)}=\frac12 q(-\xi)$. It gives $\widetilde Z_q(\xi,2\lambda)=\sigma Y(\xi,\lambda)Y^{-1}(0,\lambda)\sigma$. Now returning to Z_q, we get $Z_q(-\xi,2\lambda)=Y(\xi,\lambda)Y^{-1}(0,\lambda)$. □

Given $q\in L^2(\mathbb{R})$, we define the continuous analogs of Wall polynomials (see [16] and Section 7 in [13]) by

$$\mathfrak{A}^\pm=\tfrac12(P_*^\pm+\hat P_*^\pm),\quad \mathfrak{A}_*^\pm=\tfrac12(P^\pm+\hat P^\pm),$$
$$\mathfrak{B}^\pm=\tfrac12(P_*^\pm-\hat P_*^\pm),\quad \mathfrak{B}_*^\pm=\tfrac12(P^\pm-\hat P^\pm), \tag{2-14}$$

where $P^\pm$, $P_*^\pm$, $\hat P^\pm$, $\hat P_*^\pm$ are the solutions of systems (2-11), (2-12) for the coefficient $A^+(\xi)=-\frac12\overline{q(\xi/2)}$ from Lemma 2.4 and the coefficient $A^-(\xi)=\frac12 q(-\xi/2)$ from Lemma 2.5, correspondingly. Functions $P^\pm$, $P_*^\pm$, $\hat P^\pm$, $\hat P_*^\pm$ are continuous analogs of polynomials orthogonal on the unit circle, they depend on two parameters: $\xi\in\mathbb{R}_+$ and $\lambda\in\mathbb{C}$ and they satisfy identities (see (4.32) in [13]):

$$P_*^\pm(\xi,\lambda)=e^{i\xi\lambda}\overline{P^\pm(\xi,\lambda)},\qquad \hat P_*^\pm(\xi,\lambda)=e^{i\xi\lambda}\overline{\hat P^\pm(\xi,\lambda)} \tag{2-15}$$

for real λ.

We will use the following result (see Lemma 2 in [12]) which contains a stronger statement.

Theorem 2.6. *Let $A \in L^2(\mathbb{R}_+)$ and let P, P_* be the solutions of system* (2-11) *for the coefficient A. Then, the limit*

$$\Pi(\lambda) = \lim_{\xi \to +\infty} P_*(\xi, \lambda) \tag{2-16}$$

exists for every $\lambda \in \mathbb{C}_+$. That function Π is outer in $\mathbb{C}_+$. If $\lambda \in \mathbb{R}$, the convergence in (2-16) *holds in the Lebesgue measure on $\mathbb{R}$ where $\Pi(\lambda)$ now denotes the nontangential boundary value of Π.*

The above theorem allows us to define

$$\mathfrak{a}^{\pm}(\lambda) = \lim_{\xi \to +\infty} \mathfrak{A}^{\pm}(\xi, \lambda), \quad \mathfrak{b}^{\pm}(\lambda) = \lim_{\xi \to +\infty} \mathfrak{B}^{\pm}(\xi, \lambda) \tag{2-17}$$

for every $\lambda \in \mathbb{C}_+$ and for almost every $\lambda \in \mathbb{R}$. Also, Corollary 12.2 of [13] gives

$$|\mathfrak{a}^{\pm}(\lambda)|^2 = 1 + |\mathfrak{b}^{\pm}(\lambda)|^2 \tag{2-18}$$

for a.e. $\lambda \in \mathbb{R}$. For every $\lambda \in \mathbb{C}_+$, we define

$$a(\lambda) = \mathfrak{a}^{+}(\lambda)\, \mathfrak{a}^{-}(\lambda) - \mathfrak{b}^{+}(\lambda)\, \mathfrak{b}^{-}(\lambda).$$

Proposition 2.7. *The function a is outer in $\mathbb{C}_+$.*

Proof. We can write

$$a = \mathfrak{a}^{+}\mathfrak{a}^{-}(1 - s^{+}s^{-}), \quad s^{\pm} := \mathfrak{b}^{\pm}/\mathfrak{a}^{\pm}.$$

It is known that $\mathfrak{a}^{\pm}$ are outer (see (12.9) and (12.29) in [13]) and that $s^{\pm}$ satisfy $|s^{\pm}| < 1$ in $\mathbb{C}_+$. The function $1 - s^{+}s^{-}$ has a positive real part in $\mathbb{C}_+$ and so is an outer function. That shows that a is a product of three outer functions and hence it is outer itself. □

Proposition 2.8. *Let $q \in L^2(\mathbb{R})$ and let Z_q be defined by* (2-6). *Then, the limits in* (2-7) *exist in the Lebesgue measure on $\mathbb{R}$. The matrix $T(\lambda) = T_+^{-1}(0, \lambda)\, T_-(0, \lambda)$ has the form* (2-5) *where*

$$a = \mathfrak{a}^{+}\mathfrak{a}^{-} - \mathfrak{b}^{+}\mathfrak{b}^{-}, \quad b = \mathfrak{a}^{-}\, \overline{\mathfrak{b}^{+}} - \mathfrak{b}^{-}\, \overline{\mathfrak{a}^{+}}, \tag{2-19}$$

and $\mathfrak{a}^{\pm}$, $\mathfrak{b}^{\pm}$ are defined Lebesgue almost everywhere on $\mathbb{R}$ by the convergence in (2-17) *in measure.*

Proof. If $q \in L^2(\mathbb{R})$, the fundamental matrix Z_q and the continuous Wall polynomials (2-14) are related by

$$Z_q(\xi, 2\lambda) = \begin{cases} e^{-i\lambda\xi} \begin{pmatrix} \mathfrak{A}^{+}(2\xi, \lambda) & \mathfrak{B}^{+}(2\xi, \lambda) \\ \mathfrak{B}_*^{+}(2\xi, \lambda) & \mathfrak{A}_*^{+}(2\xi, \lambda) \end{pmatrix}, & \xi \geqslant 0, \\ e^{i\lambda\xi} \begin{pmatrix} \mathfrak{A}_*^{-}(-2\xi, \lambda) & \mathfrak{B}_*^{-}(-2\xi, \lambda) \\ \mathfrak{B}^{-}(-2\xi, \lambda) & \mathfrak{A}^{-}(-2\xi, \lambda) \end{pmatrix}, & \xi < 0. \end{cases} \tag{2-20}$$

Indeed, it is enough to use Lemmas 2.4 and 2.5 and the fact that $Y^{-1}(0,\lambda)=\frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ -i & i\end{smallmatrix}\right)$. Our next step is to prove that the limit

$$T_+^{-1}(0,2\lambda)=\lim_{\xi\to+\infty}E^{-1}(\xi,2\lambda)\,Z_q(\xi,2\lambda) \tag{2-21}$$

exists in Lebesgue measure when $\lambda\in\mathbb{R}$. From (2-15), we obtain

$$\begin{aligned}E^{-1}(\xi,2\lambda)\,Z_q(\xi,2\lambda)&=\begin{pmatrix}1 & 0\\ 0 & e^{-2i\lambda\xi}\end{pmatrix}\begin{pmatrix}\mathfrak{A}^+(2\xi,\lambda) & \mathfrak{B}^+(2\xi,\lambda)\\ \mathfrak{B}_*^+(2\xi,\lambda) & \mathfrak{A}_*^+(2\xi,\lambda)\end{pmatrix}\\ &=\begin{pmatrix}\mathfrak{A}^+(2\xi,\lambda) & \mathfrak{B}^+(2\xi,\lambda)\\ \overline{\mathfrak{B}^+}(2\xi,\lambda) & \overline{\mathfrak{A}^+}(2\xi,\lambda)\end{pmatrix}\end{aligned}$$

for every $\xi\geqslant 0$ and $\lambda\in\mathbb{R}$. Similarly,

$$\begin{aligned}E^{-1}(-\xi,2\lambda)\,Z_q(-\xi,2\lambda)&=\begin{pmatrix}e^{-2i\lambda\xi} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathfrak{A}_*^-(2\xi,\lambda) & \mathfrak{B}_*^-(2\xi,\lambda)\\ \mathfrak{B}^-(2\xi,\lambda) & \mathfrak{A}^-(2\xi,\lambda)\end{pmatrix}\\ &=\begin{pmatrix}\overline{\mathfrak{A}^-}(2\xi,\lambda) & \overline{\mathfrak{B}^-}(2\xi,\lambda)\\ \mathfrak{B}^-(2\xi,\lambda) & \mathfrak{A}^-(2\xi,\lambda)\end{pmatrix}.\end{aligned}$$

Hence, the limits

$$T_\pm^{-1}(0,2\lambda)=\lim_{\xi\to\pm\infty}E^{-1}(\xi,2\lambda)\,Z_q(\xi,2\lambda) \tag{2-22}$$

exist in Lebesgue measure on $\mathbb{R}$ by Theorem 2.6. Moreover,

$$\begin{aligned}T(2\lambda)&=T_+^{-1}(0,2\lambda)\,T_-(0,2\lambda)\\ &=\begin{pmatrix}\mathfrak{a}^+(\lambda) & \mathfrak{b}^+(\lambda)\\ \overline{\mathfrak{b}^+(\lambda)} & \overline{\mathfrak{a}^+(\lambda)}\end{pmatrix}\begin{pmatrix}\overline{\mathfrak{a}^-(\lambda)} & \overline{\mathfrak{b}^-(\lambda)}\\ \mathfrak{b}^-(\lambda) & \mathfrak{a}^-(\lambda)\end{pmatrix}^{-1}\\ &\overset{(2\text{-}18)}{=}\begin{pmatrix}\mathfrak{a}^+(\lambda) & \mathfrak{b}^+(\lambda)\\ \overline{\mathfrak{b}^+(\lambda)} & \overline{\mathfrak{a}^+(\lambda)}\end{pmatrix}\begin{pmatrix}\mathfrak{a}^-(\lambda) & -\overline{\mathfrak{b}^-(\lambda)}\\ -\mathfrak{b}^-(\lambda) & \overline{\mathfrak{a}^-(\lambda)}\end{pmatrix}=\begin{pmatrix}a(\lambda) & \overline{b(\lambda)}\\ b(\lambda) & \overline{a(\lambda)}\end{pmatrix}\end{aligned}$$

and the proposition follows. □

We end this section with a few remarks on reflection coefficients of potentials in $L^2(\mathbb{R})$. We have $|a|^2-|b|^2=1$ almost everywhere on $\mathbb{R}$ due to the fact that $\det T_\pm(0,\lambda)=1$ almost everywhere on $\mathbb{R}$. That can also be established directly using (2-18). Proposition 2.8 then allows to define the reflection coefficient $\boldsymbol{r}_q=b/a$ for every $q\in L^2(\mathbb{R})$. Lemma 2.2 holds for $\boldsymbol{r}_q$ in that case as well. However, not all results about the scattering transform can be generalized from the case $q\in\mathcal{S}(\mathbb{R})$ to $q\in L^2(\mathbb{R})$. For example, the scattering transform is injective on $\mathcal{S}(\mathbb{R})$ by Theorem 2.1, but it is no longer so when extended to $L^2(\mathbb{R})$ (see Example A.8).

3. Another form of Dirac operator, $q \in L^2(\mathbb{R})$, and the entropy function

Suppose $q \in L^2(\mathbb{R})$. The alternative form of writing the Dirac operator L_q on the line is given by

$$\mathcal{D}_Q : X \mapsto JX' + QX, \quad Q := \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{pmatrix}, \tag{3-1}$$

where $\mathcal{D}_Q$ is densely defined self-adjoint operator on the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$ of functions $X : \mathbb{R} \to \mathbb{C}^2$ such that $\|X\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} = \int_{\mathbb{R}} \|X(\xi)\|^2_{\mathbb{C}^2}\, d\xi$ is finite. Operators $\mathcal{D}_Q$ and L_q defined in (2-1) are related by

$$\mathcal{D}_Q = \Sigma L_q \Sigma^{-1}, \quad \Sigma := \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad \Sigma^{-1} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$

One way to study $\mathcal{D}_Q$ is to focus on Dirac operators on half-line $\mathbb{R}_+$ first. Given $q \in L^2(\mathbb{R}_+)$, we define $\mathcal{D}_Q^+$ on $L^2(\mathbb{R}_+, \mathbb{C}^2)$ by

$$\mathcal{D}_Q^+ : X \mapsto JX' + QX, \quad Q := \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{pmatrix} \tag{3-2}$$

on the dense subset of absolutely continuous functions $X \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ such that $\mathcal{D}_Q^+ X \in L^2(\mathbb{R}_+, \mathbb{C}^2)$, $X(0) = \binom{*}{0}$. We will call $\mathcal{D}_Q^+$ the Dirac operator defined on the positive half-line with boundary conditions $X(0) = \binom{*}{0}$ or simply the half-line Dirac operator. Set $A(\xi) = -\frac{1}{2}\overline{q(\xi/2)}$ for $\xi \in \mathbb{R}_+$, and let $P(\xi, \lambda)$, $P_*(\xi, \lambda)$ be the solutions of Krein system (2-11) generated by A. The Krein system with coefficient A and Dirac equation (3-2) are related (see the proof of Lemma A.9 in Appendix) as follows: if N_Q solves the Cauchy problem

$$JN_Q'(\xi, \lambda) + Q(\xi)\, N_Q(\xi, \lambda) = \lambda N_Q(\xi, \lambda), \quad N_Q(0, \lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$N_Q(\xi, \lambda) = \frac{e^{-i\lambda\xi}}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} \mathfrak{A}_*^+(2\xi, \lambda) & \mathfrak{B}_*^+(2\xi, \lambda) \\ \mathfrak{B}^+(2\xi, \lambda) & \mathfrak{A}^+(2\xi, \lambda) \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

where the continuous Wall polynomials $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{A}_*^+, \mathfrak{B}_*^+$ were defined in (2-14). The Weyl function of the operator $\mathcal{D}_Q^+$ coincides (see Lemma A.9) with

$$m_Q(z) := \lim_{\xi \to +\infty} i \frac{\hat{P}_*(\xi, z)}{P_*(\xi, z)}, \quad z \in \mathbb{C}_+. \tag{3-3}$$

It is known (see Theorem 7.3 in [13]) that the limit above exists for every $z \in \mathbb{C}_+$ and defines an analytic function of Herglotz–Nevanlinna class in $\mathbb{C}_+$. The latter means that $m_Q(\mathbb{C}_+) \subset \mathbb{C}_+$. In Theorem 3.1 below, $\operatorname{Im} m_Q(\lambda)$ denotes the nontangential boundary value on $\mathbb{R}$ which exists Lebesgue almost everywhere. It is understood as a nonnegative function $g = \operatorname{Im} m$ on $\mathbb{R}$ and it satisfies $g/(1+\lambda^2) \in L^1(\mathbb{R})$.

Theorem 3.1. *Let $q \in L^2(\mathbb{R}_+)$ and let Q, $\mathcal{D}_Q^+$, m_Q be defined by* (3-2) *and* (3-3). *Denote by N_Q the solution of the Cauchy problem $JN_Q'(\xi) + Q(\xi)\,N_Q(\xi) = 0$, $N_Q(0) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and set $\mathcal{H}_Q = N_Q^* N_Q$. Define also*

$$\mathcal{K}_Q^+ = \log \operatorname{Im} m_Q(i) - \frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_Q(\lambda)\frac{d\lambda}{\lambda^2+1}, \tag{3-4}$$

$$\widetilde{\mathcal{K}}_Q^+ = \sum_{k=0}^{+\infty}\left(\det \int_k^{k+2} \mathcal{H}_Q(\xi)\,d\xi - 4\right). \tag{3-5}$$

Then, we have

$$c_1 \mathcal{K}_Q^+ \leqslant \widetilde{\mathcal{K}}_Q^+ \leqslant c_2\, e^{c_2 \mathcal{K}_Q^+} \tag{3-6}$$

for some positive absolute constants c_1, c_2.

Proof. Lemma A.9 shows that m_Q coincides with the Weyl function for the canonical system with Hamiltonian $\mathcal{H}_Q$. Then, the bounds in (3-6) follow from Theorem 1.2 in [2] (see also Corollary 1.4 in [2]). □

The quantity $\mathcal{K}_Q^+$ will be called *the entropy* of the Dirac operator on $\mathbb{R}_+$. We now turn to (3-1) to define the entropy for the Dirac operator on the whole line. Take $q \in L^2(\mathbb{R})$ and let $A^+(\xi) = -\frac{1}{2}\overline{q(\xi/2)}$ and $A^-(\xi) = \frac{1}{2}q(-\xi/2)$, $\xi \in \mathbb{R}_+$ be the coefficients of Krein systems associated to restrictions of q to the half-lines $\mathbb{R}_+$ and $\mathbb{R}_-$. As in (3-3), the half-line Weyl functions $m_\pm$ are introduced by

$$m_\pm(z) = \lim_{\xi\to+\infty} i\,\frac{\hat{P}_*^\pm(\xi,z)}{P_*^\pm(\xi,z)}, \quad z \in \mathbb{C}_+. \tag{3-7}$$

These Weyl functions $m_\pm$ can be used to construct the spectral representation for the Dirac operator. Let

$$m(z) = -\frac{1}{m_+(z)+m_-(z)}\begin{pmatrix} -2m_+(z)m_-(z) & m_+(z)-m_-(z) \\ m_+(z)-m_-(z) & 2\end{pmatrix}, \quad z \in \mathbb{C}_+. \tag{3-8}$$

Using $\operatorname{Im} m_\pm(z) > 0$, one can show that $\operatorname{Im} m(z)$ is a positive definite matrix for $z \in \mathbb{C}_+$. In other words, m is the matrix-valued Herglotz function. Therefore, there exists a unique matrix-valued measure ρ taking Borel subsets of $\mathbb{R}$ into 2×2 nonnegative matrices such that

$$m(z) = \alpha + \beta z + \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2+1}\right)d\rho(\lambda), \quad z \in \mathbb{C}_+,$$

where α, β are constant 2×2 real matrices, $\beta \geqslant 0$. The importance of ρ becomes clear when we recall the spectral decomposition for $\mathcal{D}_Q$. Specifically, let $N_Q(\xi, z)$

be the solution of the Cauchy problem

$$J\frac{\partial}{\partial\xi}N_Q(\xi,z)+Q(\xi)\,N_Q(\xi,z)=zN_Q(\xi,z),\qquad N_Q(0,z)=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \tag{3-9}$$

where $z\in\mathbb{C}$, $\xi\in\mathbb{R}$. Then, the mapping

$$\mathcal{F}_{\mathcal{D}_Q}: X\mapsto\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}N_Q^*(\xi,\lambda)\,X(\xi)\,d\xi,\qquad \lambda\in\mathbb{R}, \tag{3-10}$$

initially defined on the set of compactly supported smooth functions $X:\mathbb{R}\to\mathbb{C}^2$, extends (see Appendix) to the unitary operator between the Hilbert spaces $L^2(\mathbb{R},\mathbb{C}^2)$ and $L^2(\rho)$ where

$$L^2(\rho)=\left\{Y:\mathbb{R}\to\mathbb{C}^2:\|Y\|^2_{L^2(\rho)}=\int_{\mathbb{R}}Y^*(\lambda)\,d\rho(\lambda)\,Y(\lambda)<\infty\right\}.$$

Moreover, $\mathcal{D}_Q$ is unitarily equivalent to the operator of multiplication by the independent variable in $L^2(\rho)$ and the unitary equivalence is given by the operator $\mathcal{F}_{\mathcal{D}_Q}$. In fact, these properties of ρ will not be used in the paper, we mention them only to motivate the following definition. Let us define the *entropy function* $\mathcal{K}_Q(z)$ by

$$\mathcal{K}_Q(z)=-\frac{1}{\pi}\int_{\mathbb{R}}\log(\det\rho_{ac}(\lambda))\frac{\operatorname{Im} z}{|\lambda-z|^2}d\lambda,\qquad z\in\mathbb{C}_+, \tag{3-11}$$

where ρ_{ac} denotes the absolutely continuous part of the spectral measure ρ and it satisfies $\rho_{ac}(\lambda)=\lim_{\varepsilon\to0,\varepsilon>0}\operatorname{Im} m(\lambda+i\varepsilon)$ for a.e. $\lambda\in\mathbb{R}$. The quantity $\mathcal{K}_Q$ will play a crucial role in our considerations. We first relate it to the coefficient a of the reduced transition matrix T which was introduced in Proposition 2.8.

Lemma 3.2. *We have* $\det\rho_{ac}(\lambda)=|a(\lambda)|^{-2}$ *for almost all* $\lambda\in\mathbb{R}$. *In particular*, $\mathcal{K}_Q(z)=2\log|a(z)|$ *for all* $z\in\mathbb{C}_+$.

Proof. From the definition (or see page 59 in [24]), one has

$$\det\operatorname{Im} m(z)=4\frac{\operatorname{Im} m_+(z)\operatorname{Im} m_-(z)}{|m_+(z)+m_-(z)|^2},\qquad z\in\mathbb{C}_+. \tag{3-12}$$

Substituting expressions for

$$m_\pm(z)=\lim_{\xi\to+\infty}i\frac{\hat{P}^\pm_*(\xi,z)}{P^\pm(\xi,z)}=i\frac{\mathfrak{a}^\pm(z)-\mathfrak{b}^\pm(z)}{\mathfrak{a}^\pm(z)+\mathfrak{b}^\pm(z)},\qquad z\in\mathbb{C}_+$$

into (3-12), we obtain

$$\begin{aligned}\det\rho_{ac}(\lambda)&=\lim_{\varepsilon\to+0}\det\operatorname{Im} m(\lambda+i\varepsilon)\\&=\lim_{\varepsilon\to+0}\frac{(|\mathfrak{a}^+(\lambda+i\varepsilon)|^2-|\mathfrak{b}^+(\lambda+i\varepsilon)|^2)(|\mathfrak{a}^-(\lambda+i\varepsilon)|^2-|\mathfrak{b}^-(\lambda+i\varepsilon)|^2)}{|\mathfrak{a}^+(\lambda+i\varepsilon)\mathfrak{a}^-(\lambda+i\varepsilon)-\mathfrak{b}^+(\lambda+i\varepsilon)\,\mathfrak{b}^-(\lambda+i\varepsilon)|^2}\\&=\frac{1}{|a(\lambda)|^2}\end{aligned}$$

for almost every $\lambda \in \mathbb{R}$ and the first claim of the lemma follows. The second claim is immediate because a is an outer function as we showed in Proposition 2.7. □

Consider again the half-line entropy functions

$$\mathcal{K}_Q^{\pm}(z) = \log \operatorname{Im} m_{\pm}(z) - \frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_{\pm}(\lambda) \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda, \quad z \in \mathbb{C}_+.$$

We see that $\mathcal{K}_Q^{+}(i)$ coincides with the entropy (3-4) for the restriction of Q to $\mathbb{R}_+$ (that explains why we use the same notation for the two objects), and $\mathcal{K}_Q^{-}(z) = \mathcal{K}_{Q_-}^{+}(z)$ for the potential

$$Q_-(\xi) = \begin{pmatrix} -\operatorname{Im} q(-\xi) & \operatorname{Re} q(-\xi) \\ \operatorname{Re} q(-\xi) & \operatorname{Im} q(-\xi) \end{pmatrix}, \quad \xi \in \mathbb{R}_+.$$

Our plan now is to relate $\mathcal{K}_Q^{\pm}(i)$ with $\mathcal{K}_Q(i)$ and then use the fact that the full-line entropy $\mathcal{K}_Q(i)$ is conserved, see Lemma 3.2. That will eventually lead to the proof of Theorem 1.2.

Lemma 3.3. *Let $q \in L^2(\mathbb{R})$ and let $q_\ell(\xi) = q(\xi - \ell)$, where $\ell \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Denote by Q_ℓ the matrix-function in* (3-1) *corresponding to q_ℓ. Then, $\mathcal{K}_{Q_\ell}^{+}(z) \to \mathcal{K}_Q(z)$, $\mathcal{K}_{Q_\ell}^{-}(z) \to 0$ as $\ell \to +\infty$ for every $z \in \mathbb{C}_+$.*

Proof. Take $z \in \mathbb{C}_+$. Expression $\mathcal{K}_{Q_\ell}^{+}(z) + \mathcal{K}_{Q_\ell}^{-}(z)$ equals

$$\log\bigl(\operatorname{Im} m_{\ell,+}(z) \operatorname{Im} m_{\ell,-}(z)\bigr) - \frac{1}{\pi}\int_{\mathbb{R}} \log\bigl(\operatorname{Im} m_{\ell,+}(\lambda) \operatorname{Im} m_{\ell,-}(\lambda)\bigr) \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda$$

for the corresponding Weyl functions $m_{\ell,\pm}$. We also have

$$\log |m_{\ell,+}(z) + m_{\ell,-}(z)|^2 = \frac{1}{\pi}\int_{\mathbb{R}} \log |m_{\ell,+}(\lambda) + m_{\ell,-}(\lambda)|^2 \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda$$

by the mean value theorem for harmonic functions. From (3-12), it follows that

$$\mathcal{K}_{Q_\ell}^{+}(z) + \mathcal{K}_{Q_\ell}^{-}(z) = \log\left(4\frac{\operatorname{Im} m_{\ell,+}(z) \operatorname{Im} m_{\ell,-}(z)}{|m_{\ell,+}(z) + m_{\ell,-}(z)|^2}\right) + \mathcal{K}_{Q_\ell}(z).$$

Notice that $\mathcal{K}_{Q_\ell}(z)$ does not depend on $\ell \in \mathbb{R}$ because the coefficient a in Lemma 3.2 for the potential Q_ℓ does not depend on ℓ. So, we only need to show that

$$\mathcal{K}_{Q_\ell}^{-}(z) \to 0 \quad \text{and} \quad \log\left(4\frac{\operatorname{Im} m_{\ell,+}(z) \operatorname{Im} m_{\ell,-}(z)}{|m_{\ell,+}(z) + m_{\ell,-}(z)|^2}\right) \to 0,$$

when $\ell \to +\infty$ and $z \in \mathbb{C}_+$. The second relation follows from $m_{\ell,+}(z) \to i$, $m_{\ell,-}(z) \to i$, which hold because q_ℓ tends to zero weakly in $L^2(\mathbb{R})$ as $\ell \to +\infty$ and $\|q_\ell\|_{L^2(\mathbb{R})} = \|q\|_{L^2(\mathbb{R})}$ (see Lemma A.10). Moreover, relation $m_{\ell,-}(z) \to i$ implies that $\mathcal{K}_{Q_\ell}^{-}(z) \to 0$ if and only if

$$\frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_{\ell,-}(\lambda) \frac{\operatorname{Im} z}{|\lambda - z|^2}\, d\lambda \to 0. \tag{3-13}$$

In the rest of the proof, we will show that (3-13) holds. Let $\mathfrak{a}_\ell^-$, $\mathfrak{b}_\ell^-$ be the limits of continuous Wall polynomials corresponding to Q_ℓ^-. Consider $s_\ell^- := \mathfrak{b}_\ell^-/\mathfrak{a}_\ell^-$. Formula (12.57) in [13] gives

$$s_\ell^-(z) = \frac{1+im_{\ell,-}(z)}{1-im_{\ell,-}(z)}, \quad m_{\ell,-}(z) = i\,\frac{\mathfrak{a}_\ell^-(z)-\mathfrak{b}_\ell^-(z)}{\mathfrak{a}_\ell^-(z)+\mathfrak{b}_\ell^-(z)}.$$

It implies that $\operatorname{Im} m_{\ell,-}(\lambda) = |\mathfrak{a}_\ell^-(\lambda)+\mathfrak{b}_\ell^-(\lambda)|^{-2}$ when $\lambda \in \mathbb{R}$ and that $s_\ell^-(z) \to 0$ when $\ell \to +\infty$ and $z \in \mathbb{C}_+$. Now, we can write

$$\begin{aligned}\frac{1}{\pi}\int_{\mathbb{R}} \log \operatorname{Im} m_{\ell,-}(\lambda)\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda &= \frac{1}{\pi}\int_{\mathbb{R}} \log\left(\frac{1}{|\mathfrak{a}_\ell^-(\lambda)+\mathfrak{b}_\ell^-(\lambda)|^2}\right)\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda\\ &= \log\frac{1}{|\mathfrak{a}_\ell^-(z)+\mathfrak{b}_\ell^-(z)|^2}\\ &= \log\frac{1}{|\mathfrak{a}_\ell^-(z)|^2} + \log\frac{1}{|1+s_\ell^-(z)|^2}.\end{aligned}$$

Therefore, it remains to show that $|\mathfrak{a}_\ell^-(z)|^2 \to 1$ as $\ell \to +\infty$. That holds because $\|q_{\ell,-}\|_{L^2(\mathbb{R}_+)} \to 0$ as $\ell \to +\infty$ and

$$\begin{aligned}\|q_{\ell,-}\|^2_{L^2(\mathbb{R}_+)} &= \frac{1}{\pi}\int_{\mathbb{R}} \log|\mathfrak{a}_\ell^-(\lambda)|^2\,d\lambda\\ &\geqslant \frac{\operatorname{Im} z}{\pi}\int_{\mathbb{R}} \log|\mathfrak{a}_\ell^-(\lambda)|^2\frac{\operatorname{Im} z}{|\lambda-z|^2}\,d\lambda = \operatorname{Im} z\cdot \log|\mathfrak{a}_\ell^-(z)|^2,\end{aligned}$$

where the first equality follows from $\|q_{\ell,-}\|^2_{L^2(\mathbb{R}_+)} = 2\|A_{\ell,-}\|^2_{L^2(\mathbb{R}_+)}$ and (12.2) in [13]. Since $\log|\mathfrak{a}_\ell^-(z)|^2 \geqslant 0$, equation (3-13) holds and we are done. □

As an immediate corollary of Theorem 3.1 and Lemma 3.3, we get the following estimate.

Lemma 3.4. *Let $q \in L^2(\mathbb{R})$. Denote by N_Q the solution of the Cauchy problem $JN_Q'(\xi) + Q(\xi)\,N_Q(\xi) = 0$, $N_Q(0) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, and set $\mathcal{H}_Q = N_Q^* N_Q$. Consider*

$$\mathcal{K}_Q := \mathcal{K}_Q(i), \quad \widetilde{\mathcal{K}}_Q := \sum_{k\in\mathbb{Z}}\left(\det\int_k^{k+2}\mathcal{H}_Q(\xi)\,d\xi - 4\right). \tag{3-14}$$

Then, we have

$$c_1\mathcal{K}_Q \leqslant \widetilde{\mathcal{K}}_Q \leqslant c_2\,\mathcal{K}_Q\,e^{c_2\mathcal{K}_Q} \tag{3-15}$$

for some positive absolute constants c_1, c_2.

Proof. By Lemma 3.3, we have $\mathcal{K}_Q = \lim_{\ell\to+\infty}\mathcal{K}^+_{Q_\ell}$. It remains to substitute Q_ℓ into the estimate (3-6) and take the limit as $\ell \to +\infty$ for $\ell \in \mathbb{Z}$. □

4. Proof of Theorem 1.2

The following result will play a crucial role in what follows. We postpone its proof to the next section.

Theorem 4.1. *Suppose $q \in L^2(\mathbb{R})$ and let N_Q satisfy $JN_Q' + QN_Q = 0,\ N_Q(0) = I$, where $Q := \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{pmatrix}$. Then,*

$$e^{-C_1 R}\|q\|^2_{H^{-1}(\mathbb{R})} \lesssim \widetilde{\mathcal{K}}_Q \lesssim e^{C_2 R}\|q\|^2_{H^{-1}(\mathbb{R})}, \tag{4-1}$$

where $R := \|q\|_{L^2(\mathbb{R})}$ and C_1, C_2 are two positive absolute constants.

Proof of Theorem 1.2 in the case $s = -1$. First, assume that $q_0 \in \mathcal{S}(\mathbb{R})$ and let $q(\xi, t)$ be the solution of (1-1) with the initial datum q_0. We want to prove that

$$\begin{aligned} C_1(1 + \|q_0\|_{L^2(\mathbb{R})})^{-2}\|q_0\|_{H^{-1}(\mathbb{R})} &\leqslant \|q(\,\cdot\,, t)\|_{H^{-1}(\mathbb{R})} \\ &\leqslant C_2(1 + \|q_0\|_{L^2(\mathbb{R})})^{2}\|q_0\|_{H^{-1}(\mathbb{R})}. \end{aligned} \tag{4-2}$$

We have $\|q(\,\cdot\,, t)\|_{L^2(\mathbb{R})} = \|q_0\|_{L^2(\mathbb{R})}$ for all t, see (4.33) in [14]. Let $a(z, t)$ denote the coefficient in the matrix (2-5) given by $q(\xi, t)$. For each $t \in \mathbb{R}$, define Q by (3-2). Let $\widetilde{\mathcal{K}}_Q(t)$ be as in Lemma 3.4 and $\mathcal{K}_Q(z, t)$ be defined by (3-11). Formulas (2-8) and (2-9) show that $a(z, t)$ is constant in t and Lemma 3.2 says that $\mathcal{K}_Q(z, t)$ is constant in t as well. The bound (3-15) yields

$$c_1\mathcal{K}_Q(i, 0) \leqslant \widetilde{\mathcal{K}}_Q(t) \leqslant c_2\,\mathcal{K}_Q(i, 0)\, e^{c_2\,\mathcal{K}_Q(i,0)}. \tag{4-3}$$

Assume first that $R := \|q_0\|_{L^2(\mathbb{R})} \leqslant 1$. Taking $t = 0$ in (4-3) and applying (4-1) to q_0, we get $\mathcal{K}_Q(i, 0) \lesssim 1$ since $\|q_0\|_{H^{-1}(\mathbb{R})} \leqslant R \leqslant 1$. Hence, in that case (4-3) can be written as $\widetilde{\mathcal{K}}_Q(t) \sim \mathcal{K}_Q(i, 0)$. By Theorem 4.1, $\|q(\,\cdot\,, t)\|^2_{H^{-1}(\mathbb{R})} \sim \widetilde{\mathcal{K}}_Q(t)$, and so $\|q(\,\cdot\,, t)\|^2_{H^{-1}(\mathbb{R})} \sim \|q(\,\cdot\,, 0)\|^2_{H^{-1}(\mathbb{R})}$.

If $R := \|q_0\|_{L^2(\mathbb{R})} > 1$, we use dilation. Consider $q_\alpha(\xi, t) = \alpha q(\alpha\xi, \alpha^2 t)$ which solves the same equation and notice that $\|q_\alpha\|_{L^2(\mathbb{R})} = \alpha^{1/2}R$.

Let $\alpha = \alpha_c = R^{-2} < 1$ making $\|q_{\alpha_c}\|_{L^2(\mathbb{R})} = 1$. Then, for the Sobolev norm:

$$\|q_\alpha(\,\cdot\,, t)\|_{H^{-1}(\mathbb{R})} = \alpha^{1/2}\left(\int_{\mathbb{R}} \frac{1}{1 + \alpha^2\eta^2}|(\mathcal{F}q)(\eta, \alpha^2 t)|^2\, d\eta\right)^{1/2}. \tag{4-4}$$

Since

$$\frac{1}{1 + \eta^2} \leqslant \frac{1}{1 + \alpha_c^2\eta^2} \leqslant \frac{1}{\alpha_c^2(1 + \eta^2)}, \tag{4-5}$$

one has

$$\alpha_c^{1/2}\|q(\,\cdot\,, \alpha_c^2\, t)\|_{H^{-1}(\mathbb{R})} \leqslant \|q_{\alpha_c}(\,\cdot\,, t)\|_{H^{-1}(\mathbb{R})} \leqslant \alpha_c^{-1/2}\|q(\,\cdot\,, \alpha_c^2\, t)\|_{H^{-1}(\mathbb{R})}.$$

In particular, at $t = 0$ we get

$$\alpha_c^{1/2}\|q(\,\cdot\,, 0)\|_{H^{-1}(\mathbb{R})} \leqslant \|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^{-1}(\mathbb{R})} \leqslant \alpha_c^{-1/2}\|q(\,\cdot\,, 0)\|_{H^{-1}(\mathbb{R})}.$$

Since $\|q_{\alpha_c}(\,\cdot\,,0)\|_{L^2(\mathbb{R})}=1$, one can apply the previous bounds to obtain

$$\|q_{\alpha_c}(\,\cdot\,,t)\|_{H^{-1}(\mathbb{R})}\sim\|q_{\alpha_c}(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})}.$$

Then,

$$\alpha_c^{1/2}\|q(\,\cdot\,,\alpha_c^2 t)\|_{H^{-1}(\mathbb{R})}\lesssim\|q_{\alpha_c}(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})}\lesssim\alpha_c^{-1/2}\|q(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})},$$
$$\alpha_c^{-1/2}\|q(\,\cdot\,,\alpha_c^2 t)\|_{H^{-1}(\mathbb{R})}\gtrsim\|q_{\alpha_c}(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})}\gtrsim\alpha_c^{1/2}\|q(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})}.$$

Recalling that $\alpha_c=R^{-2}$, we obtain

$$R^{-2}\|q(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})}\lesssim\|q(\,\cdot\,,t)\|_{H^{-1}(\mathbb{R})}\lesssim R^2\|q(\,\cdot\,,0)\|_{H^{-1}(\mathbb{R})}$$

for all $t\in\mathbb{R}$. Finally, having proved (4-2) for $q_0\in\mathcal{S}(\mathbb{R})$, it is enough to use Theorem 1.1 to extend (4-2) to $q_0\in L^2(\mathbb{R})$. □

Our next goal is to prove the estimate

$$C_1(1+\|q_0\|_{L^2(\mathbb{R})})^{2s}\|q_0\|_{H^s(\mathbb{R})}\leqslant\|q(\,\cdot\,,t)\|_{H^s(\mathbb{R})}\leqslant C_2(1+\|q_0\|_{L^2(\mathbb{R})})^{-2s}\|q_0\|_{H^s(\mathbb{R})},\tag{4-6}$$

where $t\in\mathbb{R}$, $s\in(-1,0]$, and C_1 and C_2 are positive absolute constants. For $s=0$, this bound is trivial. To cover $s\in(-1,0)$, we will need some auxiliary results first. One of the basic properties of NLSE which we discussed in the introduction has to do with modulation: if $q(\xi,t)$ solves (2-10), then $\widetilde{q}_v(\xi,t)=e^{iv\xi-iv^2t}q(\xi-2vt,t)$ solves (2-10) for every $v\in\mathbb{R}$.

Lemma 4.2. *Let $q_0\in L^2(\mathbb{R})$, $t\in\mathbb{R}$. Then,*

$$\|\tilde{q}_v(\,\cdot\,,t)\|^2_{H^{-1}(\mathbb{R})}=\int_{\mathbb{R}}\frac{|(\mathcal{F}q)(\eta,t)|^2}{1+(\eta+v)^2}\,d\eta.$$

Proof. It is clear that $\|e^{-iv^2t}f\|_{H^{-1}(\mathbb{R})}=\|f\|_{H^{-1}(\mathbb{R})}$ for every $f\in H^{-1}(\mathbb{R})$ and $t\in\mathbb{R}$, because e^{-iv^2t} is a unimodular constant. We have

$$\mathcal{F}(e^{iv\xi}q(\xi-2vt,t))(\eta)=(\mathcal{F}q(\xi,t))(\eta-v)\,e^{-2ivt(\eta-v)},\quad \eta\in\mathbb{R}.$$

Since $|e^{-2ivt(\eta-v)}|=1$, it only remains to change the variable of integration in

$$\|\widetilde{q}_v\|_{H^{-1}(\mathbb{R})}=\int_{\mathbb{R}}\frac{|(\mathcal{F}q(\xi,t))(\eta-v)|^2}{1+\eta^2}\,d\eta$$

to get the statement of the lemma. □

The next result is a standard property of convolutions.

Lemma 4.3. *Let $\gamma\in\left(-\frac{1}{2},1\right]$ and set $a_k=\frac{1}{(1+k^2)^\gamma}$ for $k\in\mathbb{Z}$. We have*

$$\sum_{k\in\mathbb{Z}}\frac{a_k}{1+(\eta-k)^2}\sim C_\gamma\frac{1}{(1+\eta^2)^\gamma},\quad \eta\in\mathbb{R}.$$

Proof. After comparing the sum to an integral, it is enough to show that

$$\int_{\mathbb{R}} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \sim C_\gamma \frac{1}{(1+\eta^2)^\gamma}.$$

The function on the left-hand side is even and continuous in η and γ, so we can assume that $\eta > 1$. Then,

$$\int_{|\eta-u|<0.5\eta} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \sim \frac{1}{(1+\eta^2)^\gamma},$$

and

$$\int_{|\eta-u|>0.5\eta} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \lesssim \mathcal{I}_1+\mathcal{I}_2,$$

where

$$\begin{aligned}\mathcal{I}_1 := \int_{u<-\eta/2,\, u>3\eta/2} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} &\lesssim \int_{-\infty}^{-\eta/2} \frac{du}{u^{2+2\gamma}} + \int_{3\eta/2} \frac{du}{u^{2+2\gamma}} \\ &\leqslant C_\gamma \eta^{-1-2\gamma},\end{aligned}$$

$$\mathcal{I}_2 := \int_{|u|<\eta/2} \frac{du}{(1+u^2)^\gamma(1+(\eta-u)^2)} \lesssim \eta^{-2} \int_{|u|<\eta/2} \frac{du}{(1+u^2)^\gamma} \lesssim \eta^{-2\gamma}.$$

Combining these bounds proves the lemma. □

Proof of Theorem 1.2 in the case $s \in (-1, 0)$. We can again assume that $q_0 \in \mathcal{S}(\mathbb{R})$. Recall the estimate (1-5) for $s = -1$:

$$\begin{aligned}C_1(1+\|q_0\|_{L^2(\mathbb{R})})^{-2}\|q_0\|_{H^{-1}(\mathbb{R})} &\leqslant \|q(\,\cdot\,,t)\|_{H^{-1}(\mathbb{R})} \\ &\leqslant C_2(1+\|q_0\|_{L^2(\mathbb{R})})^{2}\|q_0\|_{H^{-1}(\mathbb{R})}.\end{aligned} \tag{4-7}$$

According to Lemma 4.2, we have

$$\|\tilde{q}_v(\,\cdot\,,t)\|^2_{H^{-1}(\mathbb{R})} = \int_{\mathbb{R}} \frac{|(\mathcal{F}q(\,\cdot\,,t))(\eta)|^2}{1+(v+\eta)^2}\,d\eta \tag{4-8}$$

for $\tilde{q}_v(\xi,t) = e^{iv\xi-iv^2t}q(\xi-2vt,t)$. Let a_k, $k\in\mathbb{Z}$, be the coefficients from Lemma 4.3 with $\gamma = -s$. Then, (4-8) and Lemma 4.3 imply

$$\sum_{k\in\mathbb{Z}} a_k\|\tilde{q}_k(\,\cdot\,,t)\|^2_{H^{-1}(\mathbb{R})} \sim C_s\|q(\,\cdot\,,t)\|^2_{H^s(\mathbb{R})}. \tag{4-9}$$

In particular, taking $t = 0$ gives

$$\sum_{k\in\mathbb{Z}} a_k\|\tilde{q}_k(\,\cdot\,,0)\|^2_{H^{-1}(\mathbb{R})} \sim C_s\|q_0\|^2_{H^s(\mathbb{R})}. \tag{4-10}$$

We now apply (4-7) to $\tilde{q}_k$ and use (4-9) and (4-10) to get

$$C_1(s)(1+\|q_0\|_{L^2(\mathbb{R})})^{-2} \leqslant \frac{\|q(\,\cdot\,,t)\|_{H^s(\mathbb{R})}}{\|q_0\|_{H^s(\mathbb{R})}} \leqslant C_2(s)(1+\|q_0\|_{L^2(\mathbb{R})})^2. \tag{4-11}$$

If $R := \|q_0\|_{L^2(\mathbb{R})} \leqslant 1$, we have the statement of our theorem. If $R := \|q_0\|_{L^2(\mathbb{R})} > 1$, we use dilation transformation like in the previous proof for $s = -1$. Consider $q_\alpha(\xi, t) := \alpha q(\alpha\xi, \alpha^2 t)$ which solves the same equation and notice that $\|q_\alpha\|_{L^2(\mathbb{R})} = \alpha^{1/2} R$. Let $\alpha = \alpha_c = R^{-2} < 1$ making $\|q_{\alpha_c}\|_{L^2(\mathbb{R})} = 1$. Then, for the Sobolev norm, we have

$$\|q_\alpha(\,\cdot\,, t)\|_{H^s(\mathbb{R})} = \alpha^{1/2}\left(\int_{\mathbb{R}} \frac{1}{(1+\alpha^2\eta^2)^{|s|}} |(\mathcal{F}q)(\eta, \alpha^2 t)|^2 \, d\eta\right)^{1/2}.$$

From (4-5), we have

$$\frac{1}{(1+\eta^2)^{|s|}} \leqslant \frac{1}{(1+\alpha_c^2\eta^2)^{|s|}} \leqslant \frac{1}{\alpha_c^{2|s|}(1+\eta^2)^{|s|}}.$$

Then, one has

$$\alpha_c^{1/2}\|q(\,\cdot\,, \alpha_c^2 t)\|_{H^s(\mathbb{R})} \leqslant \|q_{\alpha_c}(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \leqslant \alpha_c^{(1/2)-|s|}\|q(\,\cdot\,, \alpha_c^2 t)\|_{H^s(\mathbb{R})}.$$

In particular, taking $t = 0$ gives us

$$\alpha_c^{1/2}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})} \leqslant \|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^s(\mathbb{R})} \leqslant \alpha_c^{(1/2)-|s|}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})}.$$

Now $\|q_{\alpha_c}(\,\cdot\,, 0)\|_{L^2(\mathbb{R})} = 1$ and we can apply the previous bounds to get

$$\|q_{\alpha_c}(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \sim \|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^s(\mathbb{R})}.$$

Then,

$$\alpha_c^{1/2}\|q(\,\cdot\,, \alpha_c^2 t)\|_{H^s(\mathbb{R})} \lesssim \|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^s(\mathbb{R})} \lesssim \alpha_c^{(1/2)-|s|}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})},$$
$$\alpha_c^{(1/2)-|s|}\|q(\,\cdot\,, \alpha_c^2 t)\|_{H^s(\mathbb{R})} \gtrsim \|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^s(\mathbb{R})} \gtrsim \alpha_c^{1/2}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})}.$$

Recalling that $\alpha_c = R^{-2} = \|q_0\|_{L^2(\mathbb{R})}^{-2}$, we obtain

$$\|q_0\|_{L^2(\mathbb{R})}^{-2|s|}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})} \lesssim \|q(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \lesssim \|q_0\|_{L^2(\mathbb{R})}^{2|s|}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})}$$

for all $t \in \mathbb{R}$. □

Our approach also provides the bounds for some positive Sobolev norms. The following proposition slightly improves (1-4) when $s \in \left[0, \frac{1}{2}\right)$, $\|q_0\|_{H^s(\mathbb{R})}$ is large, and $\|q_0\|_{L^2(\mathbb{R})}$ is much smaller than $\|q_0\|_{H^s(\mathbb{R})}$.

Proposition 4.4. *Let $q_0 \in \mathcal{S}(\mathbb{R})$ and let $q = q(\xi, t)$ be the solution of* (1-1) *corresponding to q_0. Then, for each $s \in \left[0, \frac{1}{2}\right)$, we get*

$$\|q(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \sim C_s\|q_0\|_{H^s(\mathbb{R})} \tag{4-12}$$

if $\|q_0\|_{L^2(\mathbb{R})} \leqslant 1$ and

$$\|q(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \lesssim C_s(\|q_0\|_{L^2(\mathbb{R})}^{1+2s} + \|q_0\|_{H^s(\mathbb{R})}) \tag{4-13}$$

if $\|q_0\|_{L^2(\mathbb{R})} > 1$.

Proof. In the case when $\|q\|_{L^2(\mathbb{R})} \leqslant 1$, the proof of proposition repeats the arguments given above to get (4-11) except that the constants in the inequalities depend on s and can blow up when $s \to \frac{1}{2}$. Suppose $\|q\|_{L^2(\mathbb{R})} \geqslant 1$. Then, for the Sobolev norm, we have

$$\|q_\alpha(\,\cdot\,, t)\|_{H^s(\mathbb{R})} = \alpha^{1/2}\left(\int_{\mathbb{R}}(1+\alpha^2\eta^2)^s|(\mathcal{F}q)(\eta, \alpha^2 t)|^2\,d\eta\right)^{1/2}.$$

Take $\alpha = \alpha_c$ and write the following estimate for the integral above:

$$\begin{aligned}\int_{\mathbb{R}}\left(1+\frac{\eta^2}{R^4}\right)^s &|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta \\ &\sim \int_{-R^2}^{R^2}|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta + R^{-4s}\int_{|\eta|>R^2}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta \\ &\lesssim R^2 + R^{-4s}\int_{\mathbb{R}}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta.\end{aligned}$$

We use $\|q_{\alpha_c}(\,\cdot\,, t)\|_{L^2(\mathbb{R})} = 1$ and (4-11) to get $\|q_{\alpha_c}(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \sim C_s\|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^s(\mathbb{R})}$. The previous estimate for $t=0$ yields $\|q_{\alpha_c}(\,\cdot\,, 0)\|_{H^s(\mathbb{R})} \lesssim 1 + R^{-1-2s}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})}$. Hence, $\|q_{\alpha_c}(\,\cdot\,, t)\|_{H^s(\mathbb{R})} \leqslant C_s(1+R^{-1-2s}\|q(\,\cdot\,, 0)\|_{H^s(\mathbb{R})})$. We write a lower bound

$$\begin{aligned}\|q_{\alpha_c}(\,\cdot\,, t)\|^2_{H^s(\mathbb{R})} &= R^{-2}\int_{\mathbb{R}}\left(1+\frac{\eta^2}{R^4}\right)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta \\ &\gtrsim R^{-2-4s}\int_{|\eta|>R^2}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta,\end{aligned}$$

so we have

$$\int_{|\eta|>R^2}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta \leqslant C_s(R^{2+4s} + \|q(\,\cdot\,, 0)\|^2_{H^s(\mathbb{R})}).$$

Write the integral in the left-hand side as a sum of two:

$$\int_{|\eta|>R^2}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta + \int_{|\eta|<R^2}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta.$$

Estimating each of them, we get a bound which holds for all t:

$$\int_{\mathbb{R}}(1+\eta^2)^s|(\mathcal{F}q)(\eta, \alpha_c^2 t)|^2\,d\eta \leqslant C_s(R^{2+4s} + \|q(\,\cdot\,, 0)\|^2_{H^s(\mathbb{R})}),$$

which is the required upper bound (4-13). □

5. Oscillation and Sobolev space $H^{-1}(\mathbb{R})$

In this part of the paper, our goal is to prove Theorem 4.1. Let us recall its statement.

Theorem 4.1. *Suppose $q \in L^2(\mathbb{R})$ and let N_Q satisfy $JN_Q' + QN_Q = 0$, $N_Q(0) = I$, where $Q := \begin{pmatrix} -\operatorname{Im} q & -\operatorname{Re} q \\ -\operatorname{Re} q & \operatorname{Im} q \end{pmatrix}$. Then,*

$$e^{-C_1 R}\|q\|^2_{H^{-1}(\mathbb{R})} \lesssim \widetilde{\mathcal{K}}_Q \lesssim e^{C_2 R}\|q\|^2_{H^{-1}(\mathbb{R})}, \tag{5-1}$$

where $R := \|q\|_{L^2(\mathbb{R})}$ and C_1, C_2 are two positive absolute constants.

Theorem 4.1 is of independent interest in the spectral theory of Dirac operators. For example, Lemma 3.4 shows that $\|q\|_{L^2(\mathbb{R})}$ and $\|q\|_{H^{-1}(\mathbb{R})}$ control the size of $\mathcal{K}_Q$.

The strategy of the proof is the following. In the next subsection, we show that $H^{-1}(\mathbb{R})$-norm of any function can be characterized through BMO-like condition for its "antiderivative". In Section 5.2, we consider the solution to the Cauchy problem $JN' + QN = 0$, $N(0) = I$ on the interval $[0, 1]$ with zero-trace symmetric Q and study the quantity $\det \int_0^1 N^* N\,dx$, which represents a single term in the sum for $\widetilde{\mathcal{K}}_Q$. The results in Section 5.3 show that small value of $\widetilde{\mathcal{K}}_Q$ guarantees that the "local" H^{-1} norm of Q is also small. This rough estimate is used in the proof of Theorem 4.1 which is contained in the proof of Theorem 4.1.

5.1. *One property of Sobolev space $H^{-1}(\mathbb{R})$.* Observe that a function $f \in L^2(\mathbb{R})$ belongs to the Sobolev space $H^{-1}(\mathbb{R})$ if and only if

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)\, \chi_{\mathbb{R}_+}(x-y)\, e^{-(x-y)}\,dy \right|^2 dx < \infty. \tag{5-2}$$

Moreover, the last integral is equal to $\|f\|^2_{H^{-1}(\mathbb{R})}$. Indeed, recall that $\mathcal{F}f$ stands for the Fourier transform of f:

$$(\mathcal{F}f)(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\eta x}\,dx.$$

Then, from Plancherel's identity and formula

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} e^{-x} e^{-ix\eta}\,dx = \frac{1}{\sqrt{2\pi}} \frac{1}{1+i\eta},$$

we obtain

$$\|f\|^2_{H^{-1}(\mathbb{R})} = 2\pi \|(\mathcal{F}f)\mathcal{F}(\chi_{\mathbb{R}_+} e^{-x})\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \frac{|(\mathcal{F}f)(\eta)|^2}{1+\eta^2}\,d\eta$$

by properties of convolutions. We will need the following proposition.

Proposition 5.1. *Suppose that $f \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap H^{-1}(\mathbb{R})$. Let g be an absolutely continuous function on $\mathbb{R}$ such that $g' = f$ almost everywhere on $\mathbb{R}$. Then,*

$$c_1 \|f\|^2_{H^{-1}(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}} \int_{I_k} |g - \langle g\rangle_{I_k}|^2\,dx \leqslant c_2 \|f\|^2_{H^{-1}(\mathbb{R})}, \tag{5-3}$$

where $I_k := [k, k+2]$, $\langle g\rangle_I := \frac{1}{|I|}\int_I g(x)\,dx$, and the positive constants c_1 and c_2 are universal.

Proof. Take a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap H^{-1}(\mathbb{R})$ and let g be an absolutely continuous function on $\mathbb{R}$ such that $g' = f$ almost everywhere on $\mathbb{R}$. Assume first that f has a compact support. The integral under the sum does not change if we add a constant to g, so we can suppose without loss of generality that

$$g(x) = \int_{-\infty}^{x} f(s)\,ds, \qquad x \in \mathbb{R}.$$

Upper bound. Given f, define o_f by

$$o_f(x) = e^{-x}\int_{-\infty}^{x} f(y)\,e^{y}\,dy$$

and recall (see (5-2)) that

$$\|f\|_{H^{-1}(\mathbb{R})} = \|o_f\|_{L^2(\mathbb{R})}. \tag{5-4}$$

Moreover,

$$o_f' + o_f = f. \tag{5-5}$$

For each interval I_k, we use (5-5) for the corresponding term in the sum (5-3):

$$\int_{I_k}\left|\int_k^x f\,dx_1 - \tfrac{1}{2}\int_k^{k+2}\left(\int_k^{x_1} f(x_2)\,dx_2\right)dx_1\right|^2 dx$$
$$= \int_{I_k}\left|\int_k^x o(x_1)\,dx_1 + o(x) - \tfrac{1}{2}\int_k^{k+2}\left(o(x_1) + \int_k^{x_1} o(x_2)\,dx_2\right)dx_1\right|^2 dx \lesssim \int_{I_k}|o|^2\,dx$$

after the Cauchy–Schwarz inequality is applied. Summing these estimates in $k \in \mathbb{Z}$ and using (5-4), we get the upper bound in (5-3) for compactly supported f.

Lower bound. Integration by parts gives

$$\begin{aligned}\int_{-\infty}^{x} f(y)\,e^{-(x-y)}\,dy &= \int_{-\infty}^{x} f(s)\,ds - \int_{-\infty}^{x}\left(\int_{-\infty}^{y} f(s)\,ds\right)e^{-(x-y)}\,dy \\ &= g(x) - \int_{-\infty}^{x} g(y)\,e^{-(x-y)}\,dy \\ &= \int_{-\infty}^{x}(g(x)-g(y))\,e^{-(x-y)}\,dy.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}}\left|\int_{-\infty}^{x} f(y)\,e^{-(x-y)}\,dy\right|^2 dx &\leqslant \sum_{k\in\mathbb{Z}}\int_k^{k+2}\left(\int_{-\infty}^{x}|g(x)-g(y)|^2\,e^{-(x-y)}\,dy\right)dx \\ &\lesssim \sum_{k\in\mathbb{Z}}\sum_{j\leqslant k} e^{-(k-j)}\int_k^{k+2}\int_j^{j+2}|g(x)-g(y)|^2\,dx\,dy.\end{aligned}$$

Using the inequality $(x+y+z)^2 \leqslant 3(x^2+y^2+z^2)$, we continue the estimate:

$$\cdots \lesssim \sum_{k\in\mathbb{Z}} \sum_{j\leqslant k} e^{-(k-j)} \biggl(\int_{I_k} |g - \langle g\rangle_{I_k}|^2 \,dx + |\langle g\rangle_{I_j} - \langle g\rangle_{I_k}|^2 + \int_{I_j} |g - \langle g\rangle_{I_j}|^2 \,dx \biggr).$$

Since

$$\sum_{k\in\mathbb{Z}} \sum_{j\leqslant k} e^{-(k-j)} \biggl(\int_{I_k} |g - \langle g\rangle_{I_k}|^2 \,dx + \int_{I_j} |g - \langle g\rangle_{I_j}|^2 \,dx \biggr) \lesssim \sum_{k\in\mathbb{Z}} \int_{I_k} |g - \langle g\rangle_{I_k}|^2 \,dx,$$

we are left with estimating

$$\sum_{k\in\mathbb{Z}} \sum_{j\leqslant k} e^{-(k-j)} |\langle g\rangle_{I_j} - \langle g\rangle_{I_k}|^2.$$

Applying the Cauchy–Schwarz inequality for the telescoping sum

$$\langle g\rangle_{I_k} - \langle g\rangle_{I_j} = \sum_{s=j+1}^{k} (\langle g\rangle_{I_s} - \langle g\rangle_{I_{s-1}}),$$

we get

$$|\langle g\rangle_{I_j} - \langle g\rangle_{I_k}|^2 \leqslant (k-j) \sum_{j\leqslant s\leqslant k-1} |\langle g\rangle_{I_s} - \langle g\rangle_{I_{s+1}}|^2.$$

Then,

$$\sum_{k\in\mathbb{Z}} \sum_{j\leqslant k} e^{-(k-j)} (k-j) \sum_{j\leqslant s\leqslant k-1} |\langle g\rangle_{I_s} - \langle g\rangle_{I_{s+1}}|^2 = \sum_{s\in\mathbb{Z}} |\langle g\rangle_{I_s} - \langle g\rangle_{I_{s+1}}|^2 \sum_{k,j:\, j\leqslant s\leqslant k-1} (k-j)\, e^{-(k-j)}.$$

We have

$$\begin{aligned} \sum_{k,j: j\leqslant s\leqslant k-1} (k-j)\, e^{-(k-j)} &= \sum_{j\leqslant s} \sum_{m\geqslant 1} (s+m-j)\, e^{-(s+m-j)} \\ &= \sum_{j\leqslant 0} \sum_{m\geqslant 1} (m-j)\, e^{-(m-j)} = \sum_{j\geqslant 0} \sum_{m\geqslant 1} (m+j)\, e^{-(m+j)}. \end{aligned}$$

The last sum is finite and does not depend on index s. Now, the estimate

$$\begin{aligned} |\langle g\rangle_{I_s} - \langle g\rangle_{I_{s+1}}|^2 &= \int_{I_s\cap I_{s+1}} |\langle g\rangle_{I_s} - g + g - \langle g\rangle_{I_{s+1}}|^2 \,dx \\ &\leqslant 2\int_{I_s} |g - \langle g\rangle_{I_s}|^2 \,dx + 2\int_{I_{s+1}} |g - \langle g\rangle_{I_{s+1}}|^2 \,dx \end{aligned}$$

proves that

$$\sum_{k\in\mathbb{Z}} \sum_{j\leqslant k} e^{-(k-j)} |\langle g\rangle_{I_j} - \langle g\rangle_{I_k}|^2 \lesssim \sum_{s\in\mathbb{Z}} \int_{I_s} |g - \langle g\rangle_{I_s}|^2 \,dx.$$

Hence, the lower bound in (5-3) holds for compactly supported f.

Now, take any $f \in L^1_{\text{loc}}(\mathbb{R}) \cap H^{-1}(\mathbb{R})$. The definition (1-2) of $H^{-1}(\mathbb{R})$ implies that $\mathcal{F}f$ can be written as $(1+i\eta)(\mathcal{F}o)$ for some function $o \in L^2(\mathbb{R})$. Moreover, this map $f \mapsto o$ is a bijection between $H^{-1}(\mathbb{R})$ and $L^2(\mathbb{R})$ and $\|f\|_{H^{-1}(\mathbb{R})} = \|o\|_{L^2(\mathbb{R})}$. Taking the inverse Fourier transform of identity $\mathcal{F}f = (1+i\eta)(\mathcal{F}o)$, one gets a formula $f = o + o'$ where o' is understood as a derivative in $\mathcal{S}'(\mathbb{R})$. Since $f \in L^1_{\text{loc}}(\mathbb{R})$ and $o \in L^2(\mathbb{R})$, we have $o' \in L^1_{\text{loc}}(\mathbb{R})$ and, therefore, o is absolutely continuous on $\mathbb{R}$ with the derivative equal to $f - o$. Now, take $o_n(x) = o(x)\mu_n(x)$, where $\mu_n(x)$ is even and

$$\mu_n(x) = \begin{cases} 1, & 0 \leqslant x < n, \\ n+1-x, & x \in [n, n+1), \\ 0, & x \geqslant n+1. \end{cases}$$

Define the corresponding $f_n = o_n + o_n'$. Then, $\{o_n\} \to o$ in $L^2(\mathbb{R})$ and so $\{f_n\} \to f$ in $H^{-1}(\mathbb{R})$ because the mapping $f \mapsto o$ is unitary from $H^{-1}(\mathbb{R})$ onto $L^2(\mathbb{R})$. Also, each f_n is compactly supported and $\{f_n\}$ converges to f uniformly on every finite interval. Define $g_n = \int_0^x f_n\,ds$, $g = \int_0^x f\,ds$, and write (5-3) for f_n. The estimate on the right gives

$$\sum_{|k|\leqslant N} \int_{I_k} |g_n - \langle g_n\rangle_{I_k}|^2\,dx \leqslant c_2 \|f_n\|^2_{H^{-1}(\mathbb{R})}$$

for each $N \in \mathbb{N}$. Sending $n \to \infty$, the bound

$$\sum_{|k|\leqslant N} \int_{I_k} |g - \langle g\rangle_{I_k}|^2\,dx \leqslant c_2 \|f\|^2_{H^{-1}(\mathbb{R})}$$

appears. Taking $N \to \infty$, one has the right estimate in (5-3). In particular, it shows that the sum in (5-3) converges. By construction,

$$\sum_{k\in\mathbb{Z}} \int_{I_k} |g_n - \langle g_n\rangle_{I_k}|^2 dx = \sum_{-n\leqslant k\leqslant n-2} \int_{I_k} |g - \langle g\rangle_{I_k}|^2 dx + \epsilon_n,$$

where ϵ_n is a sum of integrals over $I_{-n-2}, I_{-n-1}, I_{n-1}, I_n$. Since $o \in L^2(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{I_k} |g_n - \langle g_n\rangle_{I_k}|^2 dx = 0, \quad k \in \{-n-2, -n-1, n-1, n\}.$$

Hence, $\lim_{n\to\infty} \epsilon_n = 0$ and, taking $n \to \infty$ in inequality

$$c_1 \|f_n\|^2_{H^{-1}(\mathbb{R})} \leqslant \sum_{k\in\mathbb{Z}} \int_{I_k} |g_n - \langle g_n\rangle_{I_k}|^2 dx,$$

one gets the left bound in (5-3). Since all antiderivatives are different by at most a constant and the integral in (5-3) does not change if we add a constant to g, the proof is finished. □

5.2. ***Auxiliary perturbative results for a single interval.*** Notice that for any real symmetric 2×2 matrix Q with zero trace, we have that $V = JQ$ is also real, symmetric and has zero trace. The converse statement is true as well. Hence, the equation $JN_Q' + QN_Q = 0$ in Theorem 4.1, which is equivalent to $N_Q' = JQN_Q$, can be written as $N_Q' = VN_Q$ with V having the same properties as Q. Let $U_+(x, y)$ denote the solution to

$$\frac{d}{dx} U_+(x, y) = V(x)\, U_+(x, y), \quad U_+(y, y) = I$$

and $U_-(x, y)$ denote the solution to

$$\frac{d}{dx} U_-(x, y) = -V(x)\, U_-(x, y), \quad U_-(y, y) = I.$$

Lemma 5.2. *Suppose $N' = VN$, $N(0) = I$, where V is real-valued, $V \in L^1[0, 1]$, $V = V^*$, and $\operatorname{tr} V = 0$. Then, for $\mathcal{H} := N^* N$, we have*

$$\text{(5-6)} \qquad \det \int_0^1 \mathcal{H}(\xi)\, d\xi = \tfrac{1}{2} \int_0^1 \int_0^1 \operatorname{tr}(U_+^*(x, y)\, U_+(x, y))\, dx\, dy = \tfrac{1}{2} \int_0^1 \int_0^1 \|U_+(x, y)\|_{\mathrm{HS}}^2\, dx\, dy,$$

$$\text{(5-7)} \qquad \det \int_0^1 \mathcal{H}(\xi)\, d\xi - 1 = \tfrac{1}{2} \int_0^1 \int_0^1 \|(U_+(x, y) - U_-(x, y))\, e_1\|^2 dx\, dy.$$

Proof. Notice that $N, U_+, U_- \in \mathrm{SL}(2, \mathbb{R})$ and that every matrix $A \in \mathrm{SL}(2, \mathbb{R})$ satisfies

$$\text{(5-8)} \qquad JA^* = A^{-1} J, \quad AJ = J(A^*)^{-1}.$$

Also, for any real 2×2 matrix B, we have

$$\det B = \langle JBe_1, Be_2 \rangle = -\langle JBe_2, Be_1 \rangle.$$

Hence,

$$\mathcal{I} := \det \int_0^1 \mathcal{H}(\xi)\, d\xi = \int_0^1 \int_0^1 \langle JN^*(x)\, N(x)\, e_1, N^*(y)\, N(y)\, e_2 \rangle\, dx\, dy = -\int_0^1 \int_0^1 \langle JN^*(x)\, N(x)\, e_2, N^*(y)\, N(y)\, e_1 \rangle\, dx\, dy.$$

For the second integrand, we have

$$\langle JN^*(x)\, N(x)\, e_1, N^*(y)\, N(y)\, e_2 \rangle = \langle N^*(y)\, N(y)\, JN^*(x)\, N(x)\, e_1, e_2 \rangle.$$

Then, identities (5-8) imply

$$\begin{aligned} N^*(y)\,N(y)\,JN^*(x)\,N(x) &= N^*(y)\,J(N^*(y))^{-1}N^*(x)\,N(x) \\ &= J(N(y))^{-1}(N^*(y))^{-1}N^*(x)\,N(x) \end{aligned}$$

and, since $Je_1 = e_2$ and $J^* = -J$,

$$\langle JN^*(x)\,N(x)\,e_1, N^*(y)\,N(y)\,e_2\rangle = \langle (N(y))^{-1}(N^*(y))^{-1}N^*(x)\,N(x)\,e_1, e_1\rangle.$$

Similarly,

$$\langle JN^*(x)\,N(x)\,e_2, N^*(y)\,N(y)\,e_1\rangle = -\langle (N(y))^{-1}(N^*(y))^{-1}N^*(x)\,N(x)\,e_2, e_2\rangle.$$

Hence,

$$\begin{aligned} \mathcal{I} &= \tfrac{1}{2}\int_0^1\int_0^1 \sum_{j=1}^{2}\langle (N(y))^{-1}(N^*(y))^{-1}N^*(x)\,N(x)\,e_j, e_j\rangle\,dx\,dy \\ &= \tfrac{1}{2}\int_0^1\int_0^1 \operatorname{tr}\big((N(y))^{-1}(N^*(y))^{-1}N^*(x)\,N(x)\big)\,dx\,dy \\ &= \tfrac{1}{2}\int_0^1\int_0^1 \operatorname{tr}\big((N^*(y))^{-1}N^*(x)\,N(x)(N(y))^{-1}\big)\,dx\,dy. \end{aligned}$$

Now, using the formula $N(x)(N(y))^{-1} = U_+(x, y)$ we rewrite the last expression as

$$\mathcal{I} = \tfrac{1}{2}\int_0^1\int_0^1 \operatorname{tr}(U_+^*(x, y)\,U_+(x, y))\,dx\,dy.$$

Finally, (5-7) follows from $U_+(x, y) \in \mathrm{SL}(2, \mathbb{R})$ by direct inspection after one uses the identities $JU_+(x, y)\,J = -U_-(x, y)$ and $\operatorname{tr}(A^*A) - 2 = \|(A + JAJ)\,e_1\|^2$, which holds for every $A \in \mathrm{SL}(2, \mathbb{R})$. □

Remark. The integrand in (5-6) is symmetric in x, y because $U_+(x, y) = U_+^{-1}(y, x)$ and $U_+(x, y) \in \mathrm{SL}(2, \mathbb{R})$. Notice also that

$$\operatorname{tr}(U_+^*(x, y)\,U_+(x, y)) = \lambda_{x,y}^2 + \lambda_{x,y}^{-2} \geqslant 2,$$

where $\lambda_{x,y}$ is an eigenvalue of $U_+^*(x, y)\,U_+(x, y)$ which explains why the left-hand side in (5-7) is nonnegative.

Lemma 5.3. *Suppose real-valued matrix-function* $V = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}$ *is defined on* $[0, 1]$ *and satisfies* $\|V\|_{L^1[0,1]} < \infty$. *Consider* $\mathcal{H} := N^*N$, *where* $N : N' = VN$, $N(0) = I$. *Then*,

$$\det\int_0^1 \mathcal{H}\,dx - 1 \lesssim \|V\|_{L^1[0,1]}^2 \exp(C\|V\|_{L^1[0,1]}). \tag{5-9}$$

Proof. The integral equation for N is

$$N = I + \int_0^x VN\,ds. \tag{5-10}$$

By Gronwall's inequality,

$$\|N(x)\| \leqslant \exp\left(\int_0^x \|V(s)\|\,ds\right) \leqslant \exp(\|V\|_{L^1[0,1]}). \tag{5-11}$$

Iteration of (5-10) gives

$$N = I + \int_0^x V\,dx_1 + \int_0^x V(x_1)\left(\int_0^{x_1} V(x_2)\,N(x_2)\,dx_2\right)dx_1.$$

Then,

$$\int_0^1 N^* N\,dx = I + 2\int_0^1\left(\int_0^x V(x_1)\,dx_1\right)dx + R,$$
$$\|R\| \lesssim \|V\|^2_{L^1[0,1]} \exp(C\|V\|_{L^1[0,1]}).$$

Since $\operatorname{tr} V = 0$, the identity $\det(I+A) = 1 + \operatorname{tr} A + \det A$, which holds for all 2×2 matrices A, gives

$$\det\int_0^1 \mathcal{H}dx - 1 \lesssim \|V\|^2_{L^1[0,1]} \exp(C\|V\|_{L^1[0,1]}). \qquad \square$$

Lemma 5.4. *Suppose real-valued symmetric matrix-functions V and O are defined on* $[0, 1]$ *and satisfy*

$$V = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix} = O + O', \quad O = O^* = \begin{pmatrix} o_1 & o_2 \\ o_2 & -o_1 \end{pmatrix}, \tag{5-12}$$

$$\delta := \|O\|_{L^2[0,1]} < \infty, \tag{5-13}$$

$$d := \|O'\|_{L^2[0,1]} < \infty. \tag{5-14}$$

Consider $\mathcal{H} := N^*N$ where $N' = VN$, $N(0) = I$. Then, we have

$$\det\int_0^1 \mathcal{H}\,dx - 1 = 4\sum_{j=1}^2 \int_0^1 |g_j - \langle g_j\rangle|^2\,dx + r, \quad |r| \lesssim \delta^{2.5}\exp(C(d+\delta)), \tag{5-15}$$

where

$$g_j := \int_0^x v_j\,dx \tag{5-16}$$

and C is an absolute positive constant. An analogous result holds if O and V are related by $V = O - O'$.

Proof. We will use the formula (5-7) for our analysis. Fix $y \in [0,1]$ and take $U_+(x,y)$ and $U_-(x,y)$ which solve $\frac{d}{dx}U_+(x,y) = V(x)\,U_+(x,y)$, $U_+(y,y) = I$ and $\frac{d}{dx}U_-(x,y) = -V(x)\,U_-(x,y)$, $U_-(y,y) = I$. Iterating the corresponding integral equations, one gets

$$\begin{aligned}
U_+(x,y) = I + \int_y^x V\,dx_1 + \int_y^x V\int_y^{x_1} V\,dx_2\,dx_1 + \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\,dx_3\,dx_2\,dx_1 \\
+ \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} V\,dx_4\,dx_3\,dx_2\,dx_1 \\
+ \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} Vf_+\,dx_4\,dx_3\,dx_2\,dx_1,
\end{aligned}$$

$$f_+(x_4) = \int_y^{x_4} V(s)\,U_+(s,y)\,ds,$$

$$\begin{aligned}
U_-(x,y) = I - \int_y^x V\,dx_1 + \int_y^x V\int_y^{x_1} V\,dx_2\,dx_1 - \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\,dx_3\,dx_2\,dx_1 \\
+ \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} V\,dx_4\,dx_3\,dx_2\,dx_1 \\
- \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} Vf_-dx_4\,dx_3\,dx_2\,dx_1,
\end{aligned}$$

$$f_-(x_4) = \int_y^{x_4} V(s)\,U_-(s,y)\,ds.$$

Taking $U_+(x,y) - U_-(x,y)$ as in (5-7) leaves us with

$$\frac{U_+(x,y) - U_-(x,y)}{2} = \int_y^x V\,dx_1 + \mathcal{I}_1 + \mathcal{I}_2, \tag{5-17}$$

$$\mathcal{I}_1 = \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\,dx_3\,dx_2\,dx_1, \tag{5-18}$$

$$\mathcal{I}_2 = \int_y^x V\int_y^{x_1} V\int_y^{x_2} V\int_y^{x_3} V(f_+ + f_-)\,dx_4\,dx_3\,dx_2\,dx_1. \tag{5-19}$$

Recall that $V = O + O'$ where O satisfies (5-13) and (5-14). These assumptions are to be used in the following proposition. On $\mathbb{R}^2_+$, we define the partial order

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leqslant \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

by requiring that $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$.

Proposition 5.5. *Suppose a matrix-function O is defined on* $[0,1]$ *and denote*

$$\delta = \|O\|_{L^2[0,1]}, \quad d = \|O'\|_{L^2[0,1]}. \tag{5-20}$$

Let an operator $G_{(y)}$ be given by $F \mapsto (G_{(y)}F)(x) = \int_y^x (O + O')F\,ds$ where $y \in [0, 1]$ and a matrix-function F, defined on $[0, 1]$, satisfies $\|F\|_{L^\infty[0,1]} < \infty$ and $\|F'\|_{L^2[0,1]} < \infty$. Then,

$$\begin{bmatrix} \|G_{(y)}F\|_{L^\infty[0,1]} \\ \|(G_{(y)}F)'\|_{L^2[0,1]} \end{bmatrix} \leqslant C\mathcal{M} \begin{bmatrix} \|F\|_{L^\infty[0,1]} \\ \|F'\|_{L^2[0,1]} \end{bmatrix}, \quad \mathcal{M} = \begin{pmatrix} \delta + \sqrt{\delta d} & \delta \\ \delta + d & 0 \end{pmatrix}, \tag{5-21}$$

where C is an absolute positive constant, the norms and derivatives are computed with respect to x.

Proof. Let $b = \|F\|_{L^\infty[0,1]}$, $c = \|F'\|_{L^2[0,1]}$. Write

$$O^*(x)\,O(x) - O^*(y)\,O(y) = \int_y^x ((O^*)'O + O^*O')\,ds. \tag{5-22}$$

Then,

$$\|O(x)\|^2 = \max_{\|\xi\|_{\mathbb{C}^2} \leqslant 1} \langle O^*(x)\,O(x)\,\xi, \xi\rangle \overset{(5\text{-}22)}{\leqslant} \|O(y)\|^2 + 2\int_0^1 \|O'(s)\| \cdot \|O(s)\|\,ds.$$

Applying Cauchy–Schwarz inequality to the integral, integrating in y from 0 to 1 and maximizing in x gives

$$\|O\|_{L^\infty[0,1]} \lesssim \delta + (d\delta)^{1/2}. \tag{5-23}$$

Then,

$$(G_{(y)}F)(x) = \int_y^x OF\,ds + O(x)\,F(x) - O(y)\,F(y) - \int_y^x OF'\,ds$$

and the estimate for the first coordinate in (5-21) follows from Cauchy–Schwarz inequality and (5-23). Since $(G_{(y)}F)' = (O + O')\,F$, we get

$$\|(G_{(y)}F)'\|_{L^2[0,1]} \leqslant (\|O\|_{L^2[0,1]} + \|O'\|_{L^2[0,1]})\|F\|_{L^\infty[0,1]}$$

and the bound for the second coordinate in (5-21) is obtained. □

Continuation of the proof of Lemma 5.4. We apply the proposition to $\mathcal{I}_1$ three times with the initial choice of F: $F = I$. That gives rise to taking the third power of matrix $\mathcal{M}$: $\mathcal{M}^3$, applying it to $(1, 0)^t$, and looking at the first coordinate. As the result, one has $\|\mathcal{I}_1\|_{L^\infty[0,1]} \lesssim \delta^{3/2}(\delta + d)^{3/2}$. Therefore,

$$\|\mathcal{I}_1 e_1\|_{L^\infty([0,1]^2)} \lesssim \delta^{3/2} \exp(\delta + d). \tag{5-24}$$

Similarly, we consider $\mathcal{I}_2$ and use the previous proposition four times making the first choice of F as $F = f_+ + f_-$. Applying the bound (5-11) to U_+ and U_-, we get $\|f_+ + f_-\|_{L^\infty[0,1]} \lesssim (\delta + d)\exp(\delta + d)$, $\|f'_+ + f'_-\|_{L^2[0,1]} \lesssim (\delta + d)\exp(\delta + d)$.

This time, we compute the fourth power of matrix $\mathcal{M}$: $\mathcal{M}^4$, apply it to vector $(\delta+d)\exp(\delta+d)(1,1)^t$, and look at the first coordinate. In the end, one has

$$\|\mathcal{I}_2\, e_1\|_{L^\infty([0,1]^2)} \lesssim \delta^2 \exp(C(d+\delta)). \tag{5-25}$$

The first term in (5-17) can be written as

$$\int_y^x V\,ds = \int_y^x O\,ds + O(x) - O(y)$$

and

$$\left\| \int_y^x O\,ds + O(x) - O(y) \right\|_{L^2([0,1]^2)} \lesssim \delta. \tag{5-26}$$

For any three vectors v_1, v_2 and v_3 in $\mathbb{R}^2$, we have an estimate

$$\big|\|v_1+v_2+v_3\| - \|v_1\|\big| \leqslant \|v_2+v_3\| \leqslant \|v_2\| + \|v_3\|,$$

which follows from the triangle inequality. Multiplying with

$$\|v_1+v_2+v_3\| + \|v_1\| \leqslant 2\|v_1\| + \|v_2\| + \|v_3\|,$$

we get

$$\big|\|v_1+v_2+v_3\|^2 - \|v_1\|^2\big| \leqslant 2\|v_1\|(\|v_2\|+\|v_3\|) + (\|v_2\|+\|v_3\|)^2.$$

Applying it to (5-17) gives

$$\left| \tfrac{1}{4}\|(U_+(x,y)-U_-(x,y))\, e_1\|^2 - \left\| \left(\int_y^x V\,ds\right) e_1 \right\|^2 \right| \\ \lesssim \left\| \left(\int_y^x V\,ds\right) e_1 \right\| \cdot (\|\mathcal{I}_1 e_1\| + \|\mathcal{I}_2 e_1\|) + \|\mathcal{I}_1 e_1\|^2 + \|\mathcal{I}_2 e_1\|^2.$$

Taking the $L^1([0,1]^2)$ norm in the variables x and y of both sides and using (5-24), (5-25), (5-26) and the Cauchy–Schwarz inequality gives

$$\tfrac{1}{4}\int_0^1\int_0^1 \|(U_+(x,y)-U_-(x,y))\, e_1\|^2\,dx\,dy = \int_0^1\int_0^1 \left\| \left(\int_y^x V\,ds\right) e_1 \right\|^2 dx\,dy + r,$$

with $|r| \lesssim \delta^{2.5}\exp(C(d+\delta))$. Recalling the definition (5-16), we get

$$\left\| \left(\int_y^x V\,ds\right) e_1 \right\|^2 = \sum_{j=1}^2 (g_j(x) - g_j(y))^2$$

so

$$\tfrac{1}{2}\int_0^1\int_0^1 \|(U_+(x,y)-U_-(x,y))\, e_1\|^2\,dx\,dy = 4\sum_{j=1}^2 \int_0^1 |g_j - \langle g_j\rangle|^2\,dx + r,$$

where $|r| \lesssim \delta^{2.5}\exp(C(d+\delta))$. Lemma 5.4 is proved. □

Remark. All statements in this subsection can be easily adjusted to any interval but the constants in the inequalities will depend on the size of that interval.

5.3. *Rough bound when $\widetilde{\mathcal{K}}_Q$ is small.*

Lemma 5.6. *Suppose an absolutely continuous function f is defined on* $[0, 1]$ *and satisfies*

$$f \in L^2[0,1], \quad f' = l_1 + l_2, \quad l_1 \in L^1[0,1], \quad l_2 \in L^2[0,1]. \tag{5-27}$$

Then,

$$\|f\|_{L^\infty[0,1]} \leqslant \sqrt{\delta^2 + 2(\delta\tau + \epsilon(\tau + \epsilon + \delta))},$$

where $\delta := \|f\|_{L^2[0,1]}$, $\epsilon := \|l_1\|_{L^1[0,1]}$, $\tau := \|l_2\|_{L^2[0,1]}$.

Proof. There is $\xi \in [0, 1]$ such that $|f(\xi)| \leqslant \delta$ and

$$|f(x) - f(\xi)| \leqslant \left| \int_\xi^x f' \, ds \right| \leqslant \tau + \epsilon.$$

Thus, $\|f\|_{L^\infty[0,1]} \leqslant \tau + \epsilon + \delta$. Then, writing

$$f^2(x) - f^2(y) = 2\int_y^x f f' \, ds,$$

integrating in y and maximizing in x, we get

$$\|f\|^2_{L^\infty[0,1]} \leqslant \delta^2 + 2(\delta\tau + \epsilon(\tau + \epsilon + \delta)). \qquad \square$$

Suppose Q is real-valued, symmetric matrix-function on $\mathbb{R}$ with zero trace and $\|Q\|_{L^2(\mathbb{R})} < \infty$. Define $\mathcal{H}_Q = N^*N$, where $N : N' = JQN,\ N(0) = I$. Notice that we have $\det \int_n^{n+2} S^* \mathcal{H}_Q S \, dx = \det \int_n^{n+2} \mathcal{H}_Q \, dx$ for every constant matrix $S \in \mathrm{SL}(2, \mathbb{R})$. So, we can apply Lemma 5.3 to each interval $[n, n+2]$ by choosing $S = N^{-1}(n)$ and get an estimate which explains how $\|Q\|_{L^2(\mathbb{R})}$ controls $\widetilde{\mathcal{K}}_Q$:

$$\begin{aligned} \widetilde{\mathcal{K}}_Q = \sum_{n\in\mathbb{Z}} \left(\det \int_n^{n+2} \mathcal{H}_Q dx - 4 \right) &\lesssim \sum_{n\in\mathbb{Z}} \|Q\|^2_{L^2[n,n+2]} \exp(C\|Q\|_2) \\ &\lesssim \|Q\|^2_{L^2(\mathbb{R})} \exp(C\|Q\|_2). \end{aligned}$$

The next lemma shows that $\widetilde{\mathcal{K}}_Q$ controls the convolution of Q with the exponential.

Lemma 5.7. *Let Q be real-valued, symmetric 2×2 matrix-function on $\mathbb{R}$ with zero trace and entries in $L^2(\mathbb{R})$. Define $\mathcal{H}_Q = N^*N$ where $N : N' = JQN, N(0) = I$ and assume that $\widetilde{\mathcal{K}}_Q < \infty$. If $O := e^x \int_x^\infty e^{-s} Q \, ds$, then*

$$\|O\|_{L^\infty(\mathbb{R})} \lesssim \exp(C(\|Q\|_{L^2(\mathbb{R})} + \widetilde{\mathcal{K}}_Q))\, \widetilde{\mathcal{K}}_Q^{1/4},$$

where C is a positive absolute constant.

Proof. Let $R = \|Q\|_{L^2(\mathbb{R})}$ and $E = \widetilde{\mathcal{K}}_Q$. We split the proof into several steps.

Step 1 (bound for a single interval [0, 1]**).** The definitions (3-5) and (3-14) imply that $\widetilde{\mathcal{K}}_Q^+ \leqslant E$. From Theorems 1.2 and 3.2 in [2], we know that $\mathcal{H}_Q$ admits the following factorization on $\mathbb{R}_+$: $\mathcal{H}_Q = G^*WG$ where G and W satisfy conditions:

$$G' = J(v_1 + v_2)\, G, \quad \|v_1\|_{L^1(\mathbb{R}_+)} \lesssim E, \quad \|v_2\|_{L^2(\mathbb{R}_+)} \lesssim E^{1/2}, \tag{5-28}$$

$$\det G = 1, \quad v_1 + v_2 = (v_1 + v_2)^*, \tag{5-29}$$

and

$$W \geqslant 0, \quad \det W = 1, \quad \|\operatorname{tr} W - 2\|_{L^1(\mathbb{R}_+)} \lesssim E.$$

Since $\|\operatorname{tr} W - 2\|_{L^1[0,1]} \lesssim E$, we have $\|\lambda + \lambda^{-1} - 2\|_{L^1[0,1]} \lesssim E$, where λ is the largest eigenvalue of W. If one denotes $p = \operatorname{tr} W - 2 = \lambda + \lambda^{-1} - 2$, then

$$\lambda = \frac{2 + p + \sqrt{4p + p^2}}{2}, \quad \lambda^{-1} = \frac{2 + p - \sqrt{4p + p^2}}{2}. \tag{5-30}$$

In particular, that yields

$$\int_0^1 \|W\|\, dx \lesssim 1 + E. \tag{5-31}$$

The given conditions on Q and (5-11) yield

$$\|N(x)\|, \|N^{-1}(x)\| \lesssim \exp(CR), \quad x \in [0, 1],$$

where the second estimate follows from the first since $\det N = 1$. The Hamiltonian $\mathcal{H}_Q = N^*N$ is absolutely continuous on $\mathbb{R}_+$ and

$$0 < \exp(-CR)\, I \lesssim \mathcal{H}_Q(x) \lesssim \exp(CR)\, I \tag{5-32}$$

on [0, 1]. We claim that $\|G(0)\| \lesssim \exp(C(R+E))$ and $\|G^{-1}(0)\| \lesssim \exp(C(R+E))$. Indeed, if X satisfies $X' = J(v_1+v_2)\, X$ and $X(0) = I$, then $G = XG(0)$. Moreover, given conditions on v_1 and v_2 and $\det X = 1$, we have

$$\|X(x)\| \lesssim \exp(CE), \quad \|X^{-1}(x)\| \lesssim \exp(CE) \tag{5-33}$$

uniformly on [0, 1]. Identity $\mathcal{H}_Q = G^*(0)\, X^*WXG(0)$ yields

$$(G^*(0))^{-1}\mathcal{H}_Q(G(0))^{-1} = X^*WX.$$

Taking an arbitrary $\xi \in \mathbb{C}^2$ with $\|\xi\|_{\mathbb{C}^2} = 1$, we get

$$\begin{aligned}\|G^{-1}(0)\,\xi\|^2 &\overset{(5\text{-}32)}{\lesssim} \exp(CR)\int_0^1 \langle \mathcal{H}_Q G^{-1}(0)\,\xi, G^{-1}(0)\,\xi\rangle\, dx \\ &= \exp(CR)\int_0^1 \langle WX\xi, X\xi\rangle\, dx \overset{(5\text{-}31)+(5\text{-}33)}{\lesssim} \exp(C(R+E)),\end{aligned}$$

which implies $\|G^{-1}(0)\| \lesssim \exp(C(R+E))$. We also have $\|G(0)\| \lesssim \exp(C(R+E))$ since $\det G = 1$ and the claim is proved. Finally, we have

$$\|G(x)\| \lesssim \exp(C(R+E)), \quad \|G^{-1}(x)\| \lesssim \exp(C(R+E))$$

for $x \in [0, 1]$ since $G = XG(0)$.

Next, let us study W and W'. Since $W = (G^*)^{-1}N^*NG^{-1}$, one has the inequality $\|W\| \lesssim \exp(C(R+E))$ on $x \in [0, 1]$. Recall that $W \geqslant 0$ and $\det W = 1$, so

$$\exp(-C(R+E))\, I \lesssim W \lesssim \exp(C(R+E))\, I, \quad x \in [0, 1].$$

Since λ is the largest eigenvalue of W and $\lambda \lesssim \exp(C(R+E))$, then (5-30) yields $\|\lambda - 1\|_{L^2[0,1]} \lesssim E^{1/2} \exp(C(R+E))$ and $\|\lambda^{-1} - 1\|_{L^2[0,1]} \lesssim E^{1/2} \exp(C(R+E))$. Introduce $\Upsilon = W - I$. The matrix Υ is unitarily equivalent to $\begin{pmatrix} \lambda-1 & 0 \\ 0 & 1/(\lambda-1) \end{pmatrix}$ and that gives

$$\|\Upsilon\|_{L^2[0,1]} \lesssim E^{1/2} \exp(C(R+E)). \tag{5-34}$$

We need to study Υ', which is equal to W'. To do so, notice that

$$2N^*JQN = \mathcal{H}'_Q = G^*J(v_1+v_2)\,WG + G^*WJ(v_1+v_2)\,G + G^*W'G. \tag{5-35}$$

Hence,

$$\Upsilon' = W' = F_1 + F_2,$$

where

$$F_1 = -J(v_1+v_2)\,W - WJ(v_1+v_2), \quad F_2 = 2(G^*)^{-1}N^*JQNG^{-1}.$$

The previously obtained estimates give us

$$\|F_1\|_{L^1[0,1]} \lesssim E^{1/2} \exp(C(R+E)), \quad \|F_2\|_{L^2[0,1]} \lesssim \exp(C(R+E)). \tag{5-36}$$

Now, we use (5-34) and (5-36) to apply the previous lemma to each component of Υ to obtain

$$\|\Upsilon\|_{L^\infty[0,1]} \lesssim E^{1/4} \exp(C(R+E)). \tag{5-37}$$

The formula (5-35) also gives an expression for Q:

$$Q = -J(H_1 + H_2),$$

where

$$H_1 = 0.5(N^*)^{-1}(G^*J(v_1+v_2)\,WG + G^*WJ(v_1+v_2)\,G)\,N^{-1},$$
$$H_2 = 0.5(N^*)^{-1}(G^*\Upsilon'G)\,N^{-1}.$$

Since $\|H_1\|_{L^1[0,1]} \lesssim E^{1/2} \exp(C(R+E))$, we have

$$\left\| e^x \int_x^1 e^{-s} H_1\, ds \right\|_{L^\infty[0,1]} \lesssim E^{1/2} \exp(C(R+E)).$$

For smooth matrix-functions u_1, u_2, u_3, we have

$$\int_x^1 u_1\, u_2'\, u_3\, ds = u_1u_2u_3|_x^1 - \int_x^1 u_1'\, u_2\, u_3\, ds - \int_x^1 u_1\, u_2\, u_3'\, ds.$$

Then,

$$\begin{aligned}
&2e^x\int_x^1 e^{-s} H_2\, ds\\
&= e^x\big(e^{-1}(N^*(1))^{-1}G^*(1)\Upsilon(1)\, G(1)(N(1))^{-1}\\
&\qquad\qquad - e^{-x}(N^*(x))^{-1}G^*(x)\Upsilon(x)\, G(x)(N(x))^{-1}\big)\\
&\quad - e^x\int_x^1 (e^{-s}(N^*(s))^{-1}G^*)'\Upsilon G N^{-1}\, ds - e^x\int_x^1 e^{-s}(N^*(s))^{-1}G^*\Upsilon(GN^{-1})'\, ds.
\end{aligned}$$

Since $\|(N^{-1})'\|_{L^2[0,1]} \lesssim \exp(C(R+E))$ and $\|G'\|_{L^1[0,1]} \lesssim \exp(C(R+E))$, we have

$$\left\| e^x\int_x^1 e^{-s} H_2\, ds\right\|_{L^\infty[0,1]} \lesssim \|\Upsilon\|_{L^\infty[0,1]} \exp(C(R+E)) \overset{(5\text{-}37)}{\lesssim} E^{1/4}\exp(C(R+E)).$$

Summing up, we get

$$\left\| e^x\int_x^1 e^{-s} Q\, ds\right\|_{L^\infty[0,1]} \lesssim E^{1/4}\exp(C(R+E)). \tag{5-38}$$

Step 2 (handling all intervals $[n, n+1]$ for $n\in\mathbb{Z}$). Take any $n\in\mathbb{Z}$. Our immediate goal is to show the bound

$$\left\| e^x\int_x^{n+1} e^{-s} Q\, ds\right\|_{L^\infty[n,n+1]} \lesssim E^{1/4}\exp(C(R+E)) \tag{5-39}$$

analogous to (5-38) but written for interval $[n, n+1]$. To this end, take the Hamiltonian $\mathcal{H}^{(n)}(x) := \mathcal{H}_Q(x+n)$ defined on $\mathbb{R}_+$. For the corresponding $\widetilde{\mathcal{K}}^+_{(n)}$, we get $\widetilde{\mathcal{K}}^+_{(n)} \leqslant E$ as follows from its definition. Since the $\widetilde{\mathcal{K}}$-characteristics of the Hamiltonians $\mathcal{H}$ and $S^*\mathcal{H}S$ are equal for every constant matrix $S\in \mathrm{SL}(2,\mathbb{R})$, we can instead consider $\hat{\mathcal{H}}^{(n)} = \hat{N}^*\hat{N}$ where $\hat{N}' = JQ(x+n)\hat{N}$, $\hat{N}(0) = I$. Using the arguments in Step 1 for $\hat{\mathcal{H}}^{(n)}$, we get (5-39).

Step 3 (summing up). Denote $O_n(x) = e^x\int_x^\infty e^{-s} Q\cdot \chi_{n<s<n+1}\, ds$. Notice that $O = \sum_{n\in\mathbb{Z}} O_n$. Since $O_n(x) = 0$ for $x > n+1$ and $\|O_n(x)\| \lesssim e^{x-n}\|O_n\|_{L^\infty[n,n+1]}$ for $x < n$, we then get

$$\|O(x)\| \leqslant \sum_{n\in\mathbb{Z}} \|O_n(x)\| \lesssim E^{1/4}\exp(C(R+E))\sum_{n\geqslant 0} e^{-n} \sim E^{1/4}\exp(C(R+E))$$

as follows from (5-39). That finishes the proof of Lemma 5.7. □

Proof of Theorem 4.1. Denote $E = \widetilde{\mathcal{K}}_Q$ and $O = e^x \int_x^\infty e^{-s} Q\, ds$, and recall that $\|O\|_{L^2(\mathbb{R})} \sim \|Q\|_{H^{-1}(\mathbb{R})} \leqslant \|Q\|_{L^2(\mathbb{R})}$.

Step 1 (lower bound). Define $\delta_n = \|O\|_{L^2[n,n+1]}$. By Lemma 5.7, we know that $\sup_n \delta_n \lesssim E^{1/4} \exp(C(R+E))$. Next, we apply Lemma 5.4 to each interval $[n, n+2]$. The remainder r_n in that lemma allows the estimate

$$r_n \lesssim (\delta_n + \delta_{n+1})^{2.5} \exp(C(\delta_n + \delta_{n+1} + R)), \quad n \in \mathbb{Z}.$$

For each $R > 0$ and $\eta > 0$, we find a positive $E_0(R, \eta)$ such that $E \in (0, E_0(R, \eta))$ implies that the remainder r_n is smaller than $\eta(\delta_n^2 + \delta_{n+1}^2)$ uniformly in all n. For example, one can take

$$E_0(R, \eta) \sim e^{-C_\eta R}, \tag{5-40}$$

where C_η is a sufficiently large positive number that depends on η. Therefore, for such E and some positive constant c independent of η, we have

$$\sum_{n\in\mathbb{Z}} (c-\eta)\, \delta_n^2 \lesssim \sum_{n\in\mathbb{Z}} \left(\det \int_n^{n+2} \mathcal{H}_Q dx - 4 \right) \lesssim \sum_{n\in\mathbb{Z}} (c+\eta)\, \delta_n^2,$$

where the Proposition 5.1 has been applied to the terms $\int_n^{n+2} |g_j - \langle g_j \rangle|^2 dx$ in the right-hand side of (5-15), adjusted to the interval $[n, n+2]$, to show that they are comparable to $\delta_n^2 + \delta_{n+1}^2$. Taking $\eta = \frac{c}{2}$, we see that

$$E = \sum_{n\in\mathbb{Z}} \left(\det \int_n^{n+2} \mathcal{H}_Q dx - 4 \right) \sim \sum_{n\in\mathbb{Z}} \delta_n^2 \sim \|O\|_{L^2(\mathbb{R})}^2,$$

for $E \leqslant E_0\big(R, \frac{c}{2}\big)$. If $E > E_0(R, \frac{c}{2})$, one uses inequality $\|O\|_{L^2(\mathbb{R})} \lesssim R$ to get

$$e^{-CR} \|O\|_{L^2(\mathbb{R})}^2 \lesssim \frac{E_0\big(R, \frac{c}{2}\big)}{1+R^2} \|O\|_{L^2(\mathbb{R})}^2 \lesssim E, \tag{5-41}$$

which holds for some positive absolute constant C due to (5-40). That provides the required lower bound.

Step 2 (upper bound). Let $\delta_n = \|O\|_{L^2[n,n+1]}$. For a given value of R, apply Lemma 5.4 and Proposition 5.1 to each interval $[n, n+2]$. That gives

$$E \lesssim \sum_{n\in\mathbb{Z}} \delta_n^2 e^{C(R+\delta_n)},$$

with an absolute constant C. Since $\sum_{n\in\mathbb{Z}} \delta_n^2 \sim \|q\|_{H^{-1}(\mathbb{R})}^2$, $\|q\|_{H^{-1}(\mathbb{R})} \lesssim R$, one has

$$E \lesssim \|q\|_{H^{-1}(\mathbb{R})}^2 \, e^{C(R + \|q\|_{H^{-1}(\mathbb{R})})} \lesssim \|q\|_{H^{-1}(\mathbb{R})}^2 \, e^{C_2 R}. \qquad \square$$

Appendix

Here we collect some auxiliary results used in the main text.

A.1. We start with an example that shows that the scattering transform is not injective when defined on $q \in L^2(\mathbb{R})$. This is an analog of Lemma 17 in [27].

Example A.8. There exist potentials $q_1, q_2 \in L^2(\mathbb{R})$ such that $q_1 \neq q_2$ in $L^2(\mathbb{R})$ but we have $\boldsymbol{r}_{q_1} = \boldsymbol{r}_{q_2}$ a.e. on $\mathbb{R}$ for their reflection coefficients. In other words, the scattering transform $q \mapsto \boldsymbol{r}_q$ is not injective on $L^2(\mathbb{R})$.

Proof. Let us consider

$$\mathfrak{a}_1^+ = 1, \quad \mathfrak{b}_1^+ = 0, \quad \mathfrak{a}_1^- = \mathfrak{a}, \quad \mathfrak{b}_1^- = \mathfrak{b}$$

and

$$\mathfrak{a}_2^+ = \mathfrak{a}, \quad \mathfrak{b}_2^+ = \mathfrak{b}, \quad \mathfrak{a}_2^- = 1, \quad \mathfrak{b}_2^- = 0,$$

where $\mathfrak{a} = 1 + i/x$ and $\mathfrak{b} = i/x$. Note that

$$\int_{\mathbb{R}} \log(1 - |s_k^\pm(x)|^2)\, dx > -\infty, \quad s_k^\pm := \mathfrak{b}^\pm/\mathfrak{a}_k^\pm, \qquad k = 1, 2.$$

Theorem 12.11 in [13] says that for every contractive analytic function s on $\mathbb{C}_+$ whose boundary values on $\mathbb{R}$ satisfy $\log(1 - |s|^2) \in L^1(\mathbb{R})$ there exists a unique coefficient $A \in L^2(\mathbb{R}_+)$ such that $s = \lim_{\xi \to +\infty} \frac{\mathfrak{B}(\xi,\lambda)}{\mathfrak{A}(\xi,\lambda)}$, $\lambda \in \mathbb{C}_+$ for the continuous Wall polynomials generated by A. Moreover, we have

$$2\pi \|A\|^2_{L^2(\mathbb{R}_+)} = \|\log(1 - |s|^2)\|_{L^1(\mathbb{R}_+)}. \tag{A-1}$$

Applying this result, we see that there exist functions $A_1^\pm, A_2^\pm \in L^2(\mathbb{R}_+)$ such that $\mathfrak{a}^\pm_{1,2}$, $\mathfrak{b}^\pm_{1,2}$ are the limits of their continuous Wall polynomials. Now define potentials $q_{1,2} \in L^2(\mathbb{R})$ by relations

$$A^+_{1,2}(\xi) = -\tfrac{1}{2}\overline{q_{1,2}(\xi/2)}, \quad A^-_{1,2}(\xi) = \tfrac{1}{2} q_{1,2}(-\xi/2), \qquad \xi \in \mathbb{R}_+.$$

From Proposition 2.8, we conclude that the coefficients $a_{1,2}, b_{1,2}$ for these potentials satisfy

$$a_1 = \mathfrak{a} = a_2, \quad b_1 = -\mathfrak{b} = \bar{\mathfrak{b}} = b_2$$

on $\mathbb{R}\backslash\{0\}$. Hence, $\boldsymbol{r}_{q_1} = \boldsymbol{r}_{q_2}$ on $\mathbb{R}\backslash\{0\}$. On the other hand, we have $A_1^+ = 0$ and $A_2^- = 0$ by construction. It follows that $\operatorname{supp} q_1 \subset (-\infty, 0]$ and $\operatorname{supp} q_2 \subset [0, +\infty)$. Since q_1 and q_2 are nonzero (they have a nonzero $L^2(\mathbb{R})$-norm as follows from (A-1)), that yields $q_1 \neq q_2$ in $L^2(\mathbb{R})$. $\square$

A.2. Next, we outline how to prove that the spectral representation for the Dirac operator $\mathcal{D}_Q$, defined by relation (3-1), is given by the Weyl–Titchmarsh transform (3-10). To this end, we will use the corresponding result for canonical Hamiltonian systems proved in [24].

At first, we note that if $\mathcal{H}_Q = N_Q^* N_Q$ is the Hamiltonian from Theorem 3.1. Then, $\det \mathcal{H}_Q = 1$ on $\mathbb{R}$ and the operator $V : X \mapsto N_Q^{-1} X$ is unitary from $L^2(\mathbb{R}, \mathbb{C}^2)$ onto the Hilbert space

$$L^2(\mathcal{H}_Q) = \left\{ Y : \mathbb{R} \to \mathbb{C}^2 : \|Y\|^2_{L^2(\mathcal{H}_Q, \mathbb{R})} = \int_{\mathbb{R}} \langle \mathcal{H}_Q(\xi) Y(\xi), Y(\xi) \rangle_{\mathbb{C}^2} \, d\xi < \infty \right\}.$$

Moreover, $V \mathcal{D}_Q V^{-1}$ coincides with the operator $\mathcal{D}_{\mathcal{H}_Q} : Y \mapsto \mathcal{H}^{-1} J Y'$ of the canonical Hamiltonian system generated by the Hamiltonian $\mathcal{H}_Q$. Thus, the operator $\mathcal{D}_Q$ on $L^2(\mathbb{R}, \mathbb{C}^2)$ is unitarily equivalent to the operator $\mathcal{D}_{\mathcal{H}_Q}$ on $L^2(\mathcal{H}_Q)$. Let $\widetilde{M}$ be the solution of Cauchy problem

$$J \widetilde{M}'(\xi, z) = z \mathcal{H}_Q(\xi) \widetilde{M}(\xi, z), \quad \widetilde{M}(0, z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A-2}$$

where $z \in \mathbb{C}$, $\xi \in \mathbb{R}$, and the differentiation is taken with respect to $\xi \in \mathbb{R}$. The Weyl–Titchmarsh transform for $\mathcal{D}_{\mathcal{H}_Q}$ is defined by

$$\mathcal{F}_{\mathcal{D}_{\mathcal{H}_Q}} : Y \mapsto \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \widetilde{M}(\xi, \lambda)^* \mathcal{H}_Q(\xi) Y(\xi) \, d\xi$$

on a dense subset of $L^2(\mathcal{H}_Q)$ of smooth compactly supported functions. This operator is unitary from $L^2(\mathcal{H}_Q)$ onto the space $L^2(\rho)$ defined in the same way as at the beginning of Section 3. Specifically, we let $m_\pm$ be the half-line Weyl functions of $\mathcal{H}_Q$ and define ρ as the representing measure for the matrix-valued Herglotz function m in (3-8). It was proved in Theorem 3.21 in [24] that $\mathcal{F}_{\mathcal{D}_{\mathcal{H}_Q}} \mathcal{D}_{\mathcal{H}_Q} \mathcal{F}^{-1}_{\mathcal{D}_{\mathcal{H}_Q}}$ coincides with the operator of multiplication by the independent variable in $L^2(\rho)$. We also have

$$\mathcal{F}_{\mathcal{D}_{\mathcal{H}_Q}}(V X) = \mathcal{F}_{\mathcal{D}_Q} X, \quad X \in L^2(\mathbb{R}, \mathbb{C}^2).$$

Thus, we only need to check that the Weyl functions $m_\pm$ used in Section 3 coincide with the half-line Weyl functions of the Hamiltonian $\mathcal{H}_Q$. For the $\mathbb{R}_+$-Weyl functions this follows from Lemma A.9 below. Comparing the formulas for A^+, A^- in the beginning of Section 3, we see that the Weyl function m_- for $\mathcal{D}_Q$ corresponds to the Weyl function m_+ for $\mathcal{D}_{\widetilde{Q}}$ where $\widetilde{Q}(\xi) = \sigma_3 Q(-\xi) \sigma_3$. Similarly, in the setting of canonical Hamiltonian systems, the Weyl function m_- for $\mathcal{D}_{\mathcal{H}_Q}$ coincides with the Weyl function m_+ of $\mathcal{D}_{\widetilde{\mathcal{H}}_Q}$ such that $\widetilde{\mathcal{H}}_Q(\xi) = \sigma_3 \mathcal{H}(-\xi) \sigma_3$. Therefore, the statement for A^- follows from Lemma A.9 and from the relation

$$\widetilde{\mathcal{H}}_Q = \sigma_3 \mathcal{H}_Q \sigma_3 = (\sigma_3 N_Q^* \sigma_3)(\sigma_3 N_Q \sigma_3) = \mathcal{H}_{\widetilde{Q}}.$$

Lemma A.9. *Let $q \in L^2(\mathbb{R}_+)$. Define*

$$Q(\xi) = \begin{pmatrix} -\operatorname{Im} q(\xi) & -\operatorname{Re} q(\xi) \\ -\operatorname{Re} q(\xi) & \operatorname{Im} q(\xi) \end{pmatrix}, \qquad A(\xi) = -\tfrac{1}{2}\overline{q(\xi/2)}, \qquad \xi \in \mathbb{R}_+.$$

Let N_Q be defined by

$$JN_Q'(\xi,\lambda) + Q(\xi)\, N_Q(\xi,\lambda) = \lambda N_Q(\xi,\lambda), \quad N_Q(0,\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the Hamiltonian $\mathcal{H}_Q := N_Q^*(\xi,0)\, N_Q(\xi,0)$ on $\mathbb{R}_+$ and let $\widetilde{M} = \left(\begin{smallmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{smallmatrix}\right)$ be defined by

$$J\widetilde{M}'(\xi,z) = z\mathcal{H}_Q(\xi)\,\widetilde{M}(\xi,z), \quad \widetilde{M}(0,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let, finally, P, P_*, $\hat{P}$, $\hat{P}_*$ be the solutions to Krein systems (2-11) *and* (2-12) *for the coefficient A on $\mathbb{R}_+$. Then,*

$$\lim_{\xi\to+\infty} \frac{\widetilde{M}_{22}(\xi,z)}{\widetilde{M}_{21}(\xi,z)} = \lim_{\xi\to+\infty} \frac{(N_Q)_{22}(\xi,z)}{(N_Q)_{21}(\xi,z)} = \lim_{\xi\to+\infty} i\,\frac{\hat{P}_*(\xi,z)}{P_*(\xi,z)}, \quad z \in \mathbb{C}_+. \tag{A-3}$$

In other words, the function m_+ in (3-7) *is the half-line Weyl function for the operators $\mathcal{D}_{\mathcal{H}_Q}$, $\mathcal{D}_Q$.*

Proof. The formula

$$\lim_{\xi\to+\infty} \frac{\widetilde{M}_{22}(\xi,z)}{\widetilde{M}_{21}(\xi,z)} = \lim_{\xi\to+\infty} \frac{(N_Q)_{22}(\xi,z)}{(N_Q)_{21}(\xi,z)}$$

for $\mathcal{D}_Q$ and $\mathcal{D}_{\mathcal{H}_Q}$ is well known and can be derived from the analysis of Weyl circles by using identity $N_Q(\xi,\lambda) = N_Q(\xi,0)\,\widetilde{M}(\xi,\lambda)$ and the invariance of Weyl circles under transforms generated by J-unitary matrices (in our setting, the J-unitary matrix is $N_Q(\xi,0)$: we have $N_Q^*(\xi,0)\,JN_Q(\xi,0) = J$ on $\mathbb{R}$). See, e.g., [4] or Section 8 in [25] for more details on Weyl circles for canonical Hamiltonian systems. Thus, we focus on the second identity in (A-3) and define

$$X(\xi,z) = e^{-i\xi z}\begin{pmatrix} \frac{P(2\xi,z)+P_*(2\xi,z)}{2} & \frac{\hat{P}(2\xi,z)-\hat{P}_*(2\xi,z)}{2i} \\ \frac{P_*(2\xi,z)-P(2\xi,z)}{2i} & \frac{\hat{P}(2\xi,z)+\hat{P}_*(2\xi,z)}{2} \end{pmatrix}, \quad \xi \in \mathbb{R},\ z \in \mathbb{C}.$$

Differentiating, one obtains $JX' + QX = zX$, $X(0,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It follows that $X(\xi,z) = N_Q(\xi,z)$. In particular, we have

$$(N_Q)_{22} = e^{-i\xi z}\,\frac{\hat{P}(2\xi,z) + \hat{P}_*(2\xi,z)}{2},$$
$$(N_Q)_{21} = e^{-i\xi z}\,\frac{P_*(2\xi,z) - P(2\xi,z)}{2i}.$$

Since $P(\xi, z) \to 0$, $P_*(\xi, z) \to \Pi(z) \neq 0$ as $\xi \to +\infty$ (see Theorem 12.1 in [13]), and analogous relations hold for $\hat{P}$ and $\hat{P}_*$, we have

$$\lim_{\xi\to+\infty} \frac{(N_Q)_{22}(\xi, z)}{(N_Q)_{21}(\xi, z)} = \lim_{\xi\to+\infty} i\,\frac{\hat{P}_*(\xi, z)}{P_*(\xi, z)}, \quad z \in \mathbb{C}_+. \qquad \square$$

A.3. Lemma A.9 and some known results for canonical systems can be used to show that weak convergence of potentials of the Dirac operator implies convergence of the corresponding Weyl functions.

Lemma A.10. *Suppose $\{q_\ell\}_{\ell>0}$ is a bounded sequence in $L^2(\mathbb{R}_+)$ which converges to zero weakly. Let Q_ℓ be the associated matrix-functions defined as in* Lemma A.9. *Then, the sequence of corresponding Weyl functions $\{m_{\ell,+}\}$ converges to i locally uniformly in $\mathbb{C}_+$ when $\ell \to +\infty$.*

Proof. For $\ell > 0$, denote by $\mathcal{H}_{Q_\ell}$ the Hamiltonian generated by Q_ℓ as in Lemma A.9. Then, $m_{\ell,+}$ is the Weyl function for the half-line operators $\mathcal{D}_{\mathcal{H}_{Q_\ell}}$ and $\mathcal{D}_{Q_\ell}$. Since $\sup_{\ell>0}\|q_\ell\|_{L^2(\mathbb{R}_+)} < \infty$ and q_ℓ converge to zero weakly in $L^2(\mathbb{R}_+)$ as $\ell \to +\infty$, the Hamiltonians $\mathcal{H}_{Q_\ell}$ tend to the identity matrix $\mathcal{H}_0 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ uniformly on compact subsets on $\mathbb{R}_+$. Then, their Weyl functions $m_{+,\ell}$ tend to the Weyl function $m_+ = i$ of the Hamiltonian $\mathcal{H}_0$ locally uniformly in $\mathbb{C}_+$ by Theorem 5.7(b) in [24]. $\square$

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EXTRINSIC POLYHARMONIC MAPS INTO THE SPHERE

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In the first part we shall prove that the inverse of the stereographic projection $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ ($n \geq 2$) is extrinsically k-harmonic if and only if $n = 2k$. In the second part we shall study minimizing properties and stability of its restriction to the closed ball $B^n(R)$. In this context we shall prove that there exists a small enough positive upper bound R_k^* such that $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ is a *minimizer* provided that $0 < R \leq R_k^*$. By contrast, we shall show that $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ is *not* energy minimizing when $R > 1$. Finally, in some cases we shall obtain stability with respect to rotationally symmetric variations (*equivariant stability*) for values of R which are greater than 1.

1. Introduction and statement of the results

In order to set our work in an appropriate setting, let us first briefly recall some basic facts about some well-known intrinsic energy functionals.

The classical *energy functional*, whose critical points are called *harmonic maps*, is defined by

$$E(u) = \frac{1}{2}\int_M |du|^2 \, dv_g, \tag{1-1}$$

where $u : M \to N$ is a smooth map between two Riemannian manifolds (M, g) and (N, h) of dimension m and n respectively (we refer to [6; 7] for background on harmonic maps). In analytic terms, the condition of harmonicity is equivalent to the fact that the map u is a solution of the Euler–Lagrange equation associated to the energy functional (1-1), i.e.,

$$\tau(u) = -d^* du = \operatorname{trace} \nabla du = 0. \tag{1-2}$$

The left member $\tau(u)$ of (1-2) is a vector field along the map u or, equivalently, a section of the pull-back bundle $u^{-1}(\mathrm{TN})$: it is called *tension field*.

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Let $i:\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ denote the canonical inclusion. In the special case that $N=\mathbb{S}^n$ the harmonicity equation (1-2) takes the following form, where, with a slight abuse of notation, we write u for $i\circ u$:

$$\Delta u+\lambda_1 u=0, \tag{1-3}$$

with

$$\lambda_1=-\langle u,\,\Delta u\rangle=|\nabla u|^2 \tag{1-4}$$

and the sign convention for the Laplacian Δ is such that, for a function $f:M\to\mathbb{R}$,

$$\Delta f=\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}\,g^{ij}\,\frac{\partial f}{\partial x_j}\right). \tag{1-5}$$

A related topic of growing interest deals with the study of the so-called *biharmonic maps*. These maps, which provide a natural generalization of harmonic maps, are the critical points of the *bienergy functional* (as suggested in [6; 8]):

$$E_2(u)=\tfrac{1}{2}\int_M|\tau(u)|^2\,dv_g. \tag{1-6}$$

There have been extensive studies on biharmonic maps (see [5; 15; 21; 22] for an introduction to this topic). For future comparison we point out that, when the target manifold N is the Euclidean sphere $\mathbb{S}^n$, the bienergy functional (1-6) takes the form

$$E_2(u)=\tfrac{1}{2}\int_M|(\Delta u)^T|^2\,dv_g=\tfrac{1}{2}\int_M(|\Delta u|^2-|\nabla u|^4)\,dv_g, \tag{1-7}$$

where, again, we have written u for $i\circ u$, $i:\mathbb{S}^n\hookrightarrow\mathbb{R}^{n+1}$, and $(\,\cdot\,)^T$ denotes the tangential component to $\mathbb{S}^n$.

The inclusion $i:\mathbb{S}^n\hookrightarrow\mathbb{R}^{n+1}$ enables us to consider the Sobolev space

$$W^{2,2}(M^m,\mathbb{S}^n)=\bigl\{u\in W^{2,2}(M^m,\mathbb{R}^{n+1}):u(x)=(u^1(x),\dots,u^{n+1}(x))\in\mathbb{S}^n\text{ a.e.}\bigr\}$$

and so we say that u is a *weak* critical point if it verifies the Euler–Lagrange equation in the sense of distributions.

Since any harmonic map is trivially biharmonic we say that a (weakly) biharmonic map is *proper* if it is *not* (weakly) harmonic.

In general, it is very difficult to apply variational methods and, particularly, direct minimization, to deduce the existence of proper biharmonic maps. The main reason for this is the fact that harmonic maps provide absolute minima for the bienergy. To overcome this difficulty, an interesting variant of (1-7), called *extrinsic bienergy* or *Hessian energy*, has been introduced to study maps into $\mathbb{S}^n$. This new functional (see, for instance, [1; 4; 11; 13; 14; 16; 23]) is defined by

$$E_2^{\text{ext}}(u)=\int_M|\Delta u|^2\,dv_g \tag{1-8}$$

and its Euler–Lagrange equation is

$$\Delta^2 u + \lambda_2 u = 0, \tag{1-9}$$

where

$$\lambda_2 = \Delta(|\nabla u|^2) + |\Delta u|^2 + 2\nabla\Delta u.\nabla u.$$

Here and below,

$$\nabla u = (\nabla u^1, \dots, \nabla u^{n+1}) \quad \text{and} \quad \Delta u = (\Delta u^1, \dots, \Delta u^{n+1}), \tag{1-10}$$

where ∇ is the gradient on (M^m, g) (note that each entry of ∇u is an m-dimensional vector field tangent to M) and we denote by . a scalar product in the sense that

$$\nabla u.\nabla\Delta u = \sum_{j=1}^{n+1} \langle \nabla u^j, \nabla\Delta u^j \rangle_g, \tag{1-11}$$

where $\langle\ ,\ \rangle_g$ is the scalar product associated to the Riemannian metric g.

Next, let us introduce in detail a conformal map which will play a central role in this paper. The inverse of the stereographic projection, denoted by π^{-1}, can be described as

$$\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \left(\frac{2}{1+r^2}\, x, \frac{1-r^2}{1+r^2} \right), \tag{1-12}$$

where $r = |x|$. In some instances, we shall also denote by π^{-1} the restriction of π^{-1} to the n-dimensional ball $B^n(R)$ of radius R.

We point out that, in general, u harmonic does not imply that u is extrinsically biharmonic. For instance, the map π^{-1} is conformal and when $n = 2$ is harmonic, but not extrinsically biharmonic.

When $n = 4$, π^{-1} is not harmonic, but it is a critical point for both (1-7) and (1-8), that is, it is both intrinsically and extrinsically biharmonic. Therefore, it is reasonable to think that, when $n = 6$, π^{-1} could be a critical point of a suitable third-order energy functional.

There are two natural energy functionals of order 3. The first is the intrinsic 3-energy:

$$E_3(u) = \tfrac{1}{2} \int_M |d\tau|^2 \, dv_g. \tag{1-13}$$

Critical points of (1-13) are called *triharmonic* maps. These maps have received plenty of attention in the literature: their study was proposed in [6; 8] and important progresses were made in a series of papers by Maeta (see [17; 18; 19; 20]) who obtained the Euler–Lagrange equations and several general results. Particularly, the study of triharmonic immersions into $\mathbb{S}^n$ provided significant examples. By contrast, as a part of a more general result, we recently proved in [3] that there exists no proper triharmonic rotationally symmetric conformal diffeomorphism from $\mathbb{R}^n$ to $\mathbb{S}^n \backslash \text{Pole}$ $(n \geq 2)$. In particular, the map (1-12) is *not* triharmonic for all $n \geq 3$.

This fact, together with the hope to use more effectively variational methods, suggested to us to turn our attention to a suitable class of extrinsic k-energy functionals, a context in which we expect that conformal maps may play a significant role when $n = 2k$.

More specifically, let (M^m, g) denote a compact Riemannian manifold. Again, we consider the canonical embedding of the unit Euclidean sphere $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ and, if u is a map from M into $\mathbb{S}^n$, we shall write $u = (u^1, \ldots, u^{n+1})$ for $i \circ u$. Assuming that k is a positive integer, we shall work in the Sobolev spaces

$$W^{k,2}(M^m, \mathbb{S}^n) = \left\{ u \in W^{k,2}(M^m, \mathbb{R}^{n+1}) : u(x) = (u^1(x), \ldots, u^{n+1}(x)) \in \mathbb{S}^n \text{ a.e.} \right\}.$$

The *extrinsic k-energy functional* $E_k^{\text{ext}}(u)$ is defined on $W^{k,2}(M^m, \mathbb{S}^n)$ as

$$E_k^{\text{ext}}(u) = \int_M |\Delta^s u|^2 \, dv_g, \qquad \text{when } k = 2s, \tag{1-14}$$

$$E_k^{\text{ext}}(u) = \int_M |\nabla \Delta^s u|^2 \, dv_g, \qquad \text{when } k = 2s+1. \tag{1-15}$$

Of course, if $k = 1$, the extrinsic 1-energy coincides (up to the constant $\frac{1}{2}$) with the classical energy. Therefore, our interest is mainly on the case $k \geq 2$.

We say that $u \in W^{k,2}(M^m, \mathbb{S}^n)$ is an *extrinsic* (*weakly*) *k-harmonic map* if

$$\frac{d}{dt} E_k^{\text{ext}}(u_t) \Big|_{t=0} = 0$$

for all variations

$$u_t = \frac{u + t\phi}{|u + t\phi|},$$

where $\phi \in C_0^\infty(M^m, \mathbb{R}^{n+1})$.

A very important class of critical points are the so-called minimizers. Specifically, a *minimizer*, or *minimizing extrinsic k-harmonic map*, is a map $u \in W^{k,2}(M^m, \mathbb{S}^n)$ such that

$$E_k^{\text{ext}}(u) \leq E_k^{\text{ext}}(v)$$

for all $v \in W^{k,2}(M^m, \mathbb{S}^n)$ such that $u - v \in W_0^{k,2}(M^m, \mathbb{R}^{n+1})$.

If u is an extrinsic k-harmonic map, we say that u is *stable* if

$$\frac{d^2}{dt^2} E_k^{\text{ext}}(u_t) \Big|_{t=0} \geq 0.$$

We point out that if u is an unstable critical point, then it cannot be a minimizer.

The following proposition provides the explicit expression of the Euler–Lagrange equation associated to the extrinsic energy functionals $E_k^{\text{ext}}(u)$.

Proposition 1.1 (see [10]). *Let* (M^m, g) *denote a compact Riemannian manifold. Assume that* $k \geq 2$ *and let* $u \in W^{k,2}(M^m, \mathbb{S}^n)$. *Then* u *is an extrinsic* (*weakly*) k*-harmonic map if and only if*

$$\Delta^k u + \lambda_k u = 0 \tag{1-16}$$

in the sense of distributions. Moreover, *if* (1-16) *holds*, *then*

$$\lambda_k = \Delta^{k-1}(|\nabla u|^2) + \sum_{j=0}^{k-2} \Delta^j(\langle \Delta^{k-1-j} u, \Delta u\rangle) + 2\sum_{j=0}^{k-2} \Delta^j(\nabla u . \nabla \Delta^{k-1-j} u) \tag{1-17}$$

(*note that* $\Delta^0 u = u$ *and* $\langle\, ,\rangle$ *is the scalar product in* $\mathbb{R}^{n+1}$).

Remark 1.2. As we shall detail below, it will be important for us to consider the case that u is defined on a Riemannian manifold M which is *not* compact (for instance, we shall study the inverse of the stereographic projection $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$). Of course, in this context we shall say that u is extrinsically k-harmonic on M if it is such on each bounded domain $\Omega \subset M$ and Proposition 1.1 still applies.

Our first theorem confirms that extrinsic energies are suitable to study analytic and geometric features of π^{-1}. Indeed, we shall prove:

Theorem 1.3. *Assume* $n \geq 2$ *and* $k \geq 1$. *Then the inverse of the stereographic projection* $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ *is an extrinsic* k*-harmonic map if and only if* $n = 2k$.

This result makes it natural to ask whether $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is energy minimizing for the k-energy. It was proved in [9] that, when $k = 2$, the answer is affirmative if and only if $0 < R \leq 1$.

In this context we obtain:

Theorem 1.4. *Let* $0 < R \leq R_3^*$, *where the constant* $R_3^* \approx 0.82$ *is defined in* (3-14). *Then* $\pi^{-1} : B^6(R) \to \mathbb{S}^6$ *is energy minimizing for the extrinsic* 3*-energy.*

As for higher-order energies, we do not have such an explicit upper estimate for R, but we can prove:

Theorem 1.5. *Assume* $n = 2k$ *with* $k \geq 4$. *Then there exists* $0 < R_k^* \leq 1$ *such that* $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ *is energy minimizing for the extrinsic* k*-energy provided that* $0 < R \leq R_k^*$.

Proposition 1.6. *Assume* $n = 2k$ *and* $R > 1$. *Then* $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ *is* ***not*** *energy minimizing for the extrinsic* k*-energy.*

As a consequence of these results, a very natural topic for further investigation is to study when $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is a *stable* critical point for the extrinsic k-energy. The study of this problem is not present in the literature, not even in the case of the bienergy. In general, it seems to be a difficult task to obtain a complete answer depending on R and k. A starting point is to restrict attention to rotationally symmetric variations, i.e., to the so-called *equivariant variations*. In this context, our main result is:

Theorem 1.7. (i) *The extrinsically biharmonic map* $\pi^{-1} : B^4(R) \to \mathbb{S}^4$ *is stable with respect to equivariant variations provided that* $0 < R \leq R_2^{\text{stab}} \approx 1.81$.

(ii) *The extrinsically triharmonic map* $\pi^{-1} : B^6(R) \to \mathbb{S}^6$ *is stable with respect to equivariant variations provided that* $0 < R \leq R_3^{\text{stab}} \approx 1.43$.

In our opinion, an interesting feature of Theorem 1.7 is the stability for values of $R > 1$. In these cases, the image of the map is *not* contained in the closed upper hemisphere.

Our paper is organized as follows. The proof of Theorem 1.3 requires to overcome several technical difficulties and will be carried out in Section 2. In Section 3 we prove Theorems 1.4, 1.5 and Proposition 1.6. The study of the second variation will be carried out in Section 4, where we shall prove Theorem 1.7.

2. Proof of Theorem 1.3

We carry out some preliminary work. Let $r = |x|$ and $u : \mathbb{R}^n \backslash \{O\} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a map of the form

$$x = (x_1, \ldots, x_n) \mapsto (p(r)\, x, q(r)) = (p(r)\, x_1, \ldots, p(r)\, x_n, q(r)), \tag{2-1}$$

where $p(r)$ and $q(r)$ are smooth functions for $r > 0$. We shall need to compute terms involving $\Delta^k u$ and $\nabla \Delta^k u$. To this purpose, it is convenient to define recursively the following functions:

$$\begin{aligned} P_0(r) &= p(r), \\ P_k(r) &= P''_{k-1}(r) + \frac{(n+1)}{r} P'_{k-1}(r), \quad k \geq 1, \\ Q_0(r) &= q(r), \\ Q_k(r) &= Q''_{k-1}(r) + \frac{(n-1)}{r} Q'_{k-1}(r), \quad k \geq 1. \end{aligned} \tag{2-2}$$

We observe that the above functions depend on n. However, since this dependence will always be clear from the context, we have simplified the notation avoiding to write $P_{k,n}(r)$ etc. Next, we have:

Lemma 2.1. *Let* $u : \mathbb{R}^n \backslash \{O\} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *be a map as in* (2-1). *Then, in the notation of* (2-2) *for all* $i, k \geq 0$ *we have*:

(i) $\Delta^k u = (P_k(r)\, x, Q_k(r))$.

(ii) $\langle \Delta^i u, \Delta^k u \rangle = r^2 P_i(r) P_k(r) + Q_i(r)\, Q_k(r)$.

(iii) $$\nabla\Delta^k u = \left(\frac{P_k' x_1}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} P_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \frac{P_k' x_2}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ P_k \\ \vdots \\ 0 \end{bmatrix}, \dots, \frac{P_k' x_n}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ P_k \end{bmatrix}, \frac{Q_k'}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right).$$

(iv) $\nabla\Delta^i u . \nabla\Delta^k u = r^2 P_i' P_k' + n P_i P_k + r P_i' P_k + r P_i P_k' + Q_i' Q_k'$.

(*note that* $\Delta^0 u = u$ *and the scalar product* . *was defined in* (1-11)).

Proof. The proof is a straightforward computation which can be carried out using

$$\nabla p(r) = p'(r)\frac{x}{r}, \quad \Delta p(r) = p''(r) + \frac{(n-1)}{r}\, p'(r),$$
$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g\rangle. \qquad \square$$

Proof of Theorem 1.3. Observe that the smooth map π^{-1} is of the type (2-1) with

(2-3) $$p(r) = \frac{2}{1+r^2} \quad \text{and} \quad q(r) = \frac{1-r^2}{1+r^2}.$$

We need to compute the explicit expression of the functions introduced in (2-2). To this purpose, it is convenient to define the following sets of constants:

(2-4) $$B_k = (-1)^k\, 2^{k+1}\, k! \quad (k \geq 1)$$

and

(2-5) $$a_k[j, n] = \binom{k}{j} \prod_{\ell=j}^{k-1} (n+2\ell+2) \prod_{\ell=0}^{j-1} (n+2\ell-2k) \quad (n \geq 2,\ k \geq 1,\ 0 \leq j \leq k).$$

Note that we make use of the convention

$$\prod_{\ell=q}^{q'} C_\ell = 1, \quad \text{whenever } q' < q.$$

The following lemma is technically difficult, but crucial for our proof.

Lemma 2.2. *Assume that* $n \geq 2$. *In the case of* $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$, *the explicit expression of the functions* $P_k(r)$, $Q_k(r)$ *introduced in* (2-2) *is*

$$P_k(r) = \frac{B_k}{(1+r^2)^{2k+1}} \sum_{j=0}^{k} a_k[j,n]\, r^{2j} \quad (k \geq 0), \tag{2-6}$$

$$\begin{aligned} Q_k(r) &= \frac{B_k}{(1+r^2)^{2k+1}} \sum_{j=0}^{k} a_k[j,n-2]\, r^{2j} \quad (k \geq 1), \\ Q_0(r) &= q(r) \quad (= -1 + P_0(r)). \end{aligned} \tag{2-7}$$

In order to preserve the flow of the exposition, the rather tedious proof of the previous lemma will be given at the end of the proof of the theorem.

Now, using Lemma 2.1(i), we easily see that the Euler–Lagrange equation (1-16) is equivalent to the system

$$\begin{cases} P_k(r) + \lambda_k\, p(r) = 0, \\ Q_k(r) + \lambda_k\, q(r) = 0. \end{cases} \tag{2-8}$$

Next, let us assume that $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is extrinsically k-harmonic. Then, taking into account that $q(r) = -1 + p(r)$, equation (2-8) implies

$$(-1 + p(r))\, P_k(r) - p(r)\, Q_k(r) = 0. \tag{2-9}$$

Using Lemma 2.2, we see immediately that (2-9) has the form

$$\frac{1}{(1+r^2)^{2k+2}} \sum_{j=0}^{k+1} c_k[j,n]\, r^{2j} = 0 \tag{2-10}$$

for some real coefficients $c_k[j,n]$. Next, we observe that

$$\begin{aligned} c_k[0,n] &= B_k(a_k[0,n] - 2a_k[0,n-2]) \\ &= B_k\Bigl(\prod_{\ell=0}^{k-1}(n+2\ell+2) - 2\prod_{\ell=0}^{k-1}(n+2\ell)\Bigr) \\ &= B_k\Bigl((n+2k)\prod_{\ell=1}^{k-1}(n+2\ell) - 2n\prod_{\ell=1}^{k-1}(n+2\ell)\Bigr) \\ &= B_k\Bigl((2k-n)\prod_{\ell=1}^{k-1}(n+2\ell)\Bigr). \end{aligned}$$

Of course, if $c_k[0,n] \neq 0$, then (2-10) can hold only at isolated points. Therefore, a necessary condition for the validity of (2-10) is $c_k[0,n] = 0$, i.e., $n = 2k$. By way of summary, we have proved that $n = 2k$ is necessary for the extrinsic k-harmonicity of $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$.

Conversely, let us now assume that $n = 2k$. If we use this assumption in the definition of $a_k[j, n]$ it is easy to check that

$$a_k[j, 2k] = 0 \quad (j \geq 1), \qquad a_k[0, 2k] = \prod_{\ell=0}^{k-1}(2k + 2\ell + 2).$$

From this, it follows easily that

$$-\frac{P_k(r)}{p(r)} = \frac{A_k}{(1+r^2)^{2k}}, \tag{2-11}$$

where

$$A_k = -\tfrac{1}{2} B_k \prod_{\ell=0}^{k-1}(2k + 2\ell + 2) = (-1)^{k+1}\, 2^{2k}\, (2k)!. \tag{2-12}$$

In a similar fashion we find

$$a_k[j, 2k-2] = 0 \quad (j \geq 2), \qquad a_k[0, 2k-2] = \prod_{\ell=0}^{k-1}(2k + 2\ell),$$
$$a_k[1, 2k-2] = \binom{k}{1} \prod_{\ell=1}^{k-1}(2k + 2\ell) \prod_{\ell=0}^{0}(2\ell - 2) = -\prod_{\ell=0}^{k-1}(2k + 2\ell).$$

From this, using again Lemma 2.2, it is easy to conclude that

$$\begin{aligned} -\frac{Q_k(r)}{q(r)} &= -B_k \prod_{\ell=0}^{k-1}(2k + 2\ell) \frac{1}{(1+r^2)^{2k}} \\ &= -\tfrac{1}{2} B_k \prod_{\ell=0}^{k-1}(2k + 2\ell + 2) \frac{1}{(1+r^2)^{2k}} = \frac{A_k}{(1+r^2)^{2k}}. \end{aligned}$$

The conclusion is that, if $n = 2k$, then $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is extrinsically k-harmonic because it verifies the Euler–Lagrange equation (1-16) with

$$\lambda_k = \frac{A_k}{(1+r^2)^{2k}}, \tag{2-13}$$

where A_k was defined in (2-12). So, it only remains to prove Lemma 2.2.

Proof of Lemma 2.2. First, we prove (2-6). We observe that $n \geq 2$ is a fixed integer and so we proceed by induction on k. It is immediate to check that, independently of n, $P_0(r) = p(r)$. Thus, our proof amounts to check that the functions $P_k(r)$ defined in (2-6) verify the recursive law

$$P_{k+1}(r) = P_k''(r) + \frac{(n+1)}{r} P_k'(r). \tag{2-14}$$

A straightforward direct computation, taking into account the expression (2-6) for $P_k(r)$, shows that the right-hand side of (2-14) is given by

$$\frac{T_k(r)}{(1+r^2)^{3+2k}}, \tag{2-15}$$

where

$$\begin{aligned} T_k(r) = 2B_k \Big\{ &\sum_{j=0}^{k-1}(j+1)(n+2j+2)\, a_k[j+1,n]\, r^{2j} \\ &- (2k+1)(n+2)\sum_{j=0}^{k} a_k[j,n]\, r^{2j} \\ &+ \sum_{j=1}^{k} 2j(-4k+n+2j-2)\, a_k[j,n]\, r^{2j} \\ &+ (2k+1)(4k-n+2)\sum_{j=1}^{k+1} a_k[j-1,n]\, r^{2j} \\ &+ \sum_{j=1}^{k+1}(j-1)(-8k+n+2j-6)\, a_k[j-1,n]\, r^{2j} \Big\} \\ = 2B_k &\sum_{j=0}^{k+1} d_j\, r^{2j} \end{aligned} \tag{2-16}$$

for some real coefficients d_j, $j = 0, \dots, k+1$.

For later use we note that, applying directly the definition of $a_k[j,n]$ given in (2-5), we have the following relations which hold for $1 \le j \le k$:

$$\begin{aligned} a_k[j-1,n] &= \binom{k}{j-1} \prod_{\ell=j-1}^{k-1}(n+2+2\ell) \prod_{\ell=0}^{j-2}(n-2k+2\ell) \\ &= \left[\frac{\binom{k}{j-1}}{\binom{k}{j}}\right]\binom{k}{j}\frac{(n+2j)}{(n-2k+2j-2)}\prod_{\ell=j}^{k-1}(n+2+2\ell)\prod_{\ell=0}^{j-1}(n-2k+2\ell) \\ &= \frac{j}{(k-j+1)}\frac{(n+2j)}{(n-2k+2j-2)}a_k[j,n], \end{aligned} \tag{2-17}$$

and, in a similar way, for $1 \le j \le k-1$:

$$a_k[j+1,n] = \frac{(k-j)}{(j+1)}\frac{(n+2j-2k)}{(n+2j+2)}a_k[j,n]. \tag{2-18}$$

Moreover, we shall also need to use

$$a_{k+1}[j,n] = \frac{(k+1)(n-2k-2)(2k+n+2)}{(-j+k+1)(n+2j-2k-2)}a_k[j,n] \quad (0 \le j \le k). \tag{2-19}$$

This last formula can be proved using again the definition (2-5) and the methods which we employed in (2-17). Indeed,

$$\begin{aligned}
&a_{k+1}[j,n]\\
&=\binom{k+1}{j}\prod_{\ell=j}^{k}(n+2\ell+2)\prod_{\ell=0}^{j-1}(n+2\ell-2k-2)\\
&=\left[\frac{\binom{k+1}{j}}{\binom{k}{j}}\right]\binom{k}{j}(n+2k+2)\left(\prod_{\ell=j}^{k-1}(n+2+2\ell)\right)\frac{(n-2k-2)}{(n-2k+2j-2)}\prod_{\ell=0}^{j-1}(n-2k+2\ell)\\
&=\frac{(k+1)(n-2k-2)(2k+n+2)}{(-j+k+1)(n+2j-2k-2)}a_k[j,n].
\end{aligned}$$

Next, since $B_{k+1}=-2(k+1)\,B_k$ and taking into account the expression (2-6) for $P_{k+1}(r)$, the proof of (2-14) will be completed if we show that the coefficients d_j introduced in (2-16) verify

$$d_j=-(k+1)\,a_{k+1}[j,n]\quad\text{for all }\ 0\le j\le k+1. \tag{2-20}$$

Now, from (2-16), we find

$$\begin{aligned}
d_0&=(n+2)\big[a_k[1,n]-(1+2k)\,a_k[0,n]\big]\\
&=(n+2)\left[k(n-2k)\prod_{\ell=1}^{k-1}(n+2\ell+2)-(1+2k)\,a_k[0,n]\right]\\
&=(n+2)\left[k(n-2k)\frac{a_k[0,n]}{(n+2)}-(1+2k)\,a_k[0,n]\right]\\
&=-(k+1)\,(n+2k+2)\,a_k[0,n]\\
&=-(k+1)\,a_{k+1}[0,n]
\end{aligned}$$

and so (2-20) is proved when $j=0$.

Next, computing and using (2-17)–(2-19), we obtain

$$\begin{aligned}
d_k&=(2+k)(4+2k-n)\,a_k[k-1,n]-(2+8k+4k^2+n)\,a_k[k,n]\\
&=-(k+1)\frac{(k+1)\,n^2-4(k+1)^3}{(n-2)}a_k[k,n]=-(k+1)\,a_{k+1}[k,n],\\
d_{k+1}&=[(2k+1)(4k-n+2)+k(-8k+n+2k-4)]\,a_k[k,n]\\
&=-(k+1)(n-2-2k)\,a_k[k,n]=-(k+1)\,a_{k+1}[k+1,n].
\end{aligned}$$

Thus, we have verified that (2-20) is true also when $j=k$ and $j=k+1$.

As for the other coefficients, again, from (2-16) we find

$$d_j = [(2k+1)(4k-n+2)+(j-1)(-8k+n+2j-6)]\,a_k[j-1,n] + [2j(-4k+n+2j-2)-(2k+1)(n+2)]\,a_k[j,n] + (j+1)(n+2j+2)\,a_k[j+1,n]. \tag{2-21}$$

for $1 \le j \le k-1$.

Next, substituting (2-17) and (2-18) into (2-21), after a routine simplification and using again (2-19), we obtain

$$d_j = -\frac{(k+1)^2(n-2k-2)(2k+n+2)}{(-j+k+1)(n+2j-2k-2)}a_k[j,n] = -(k+1)\,a_{k+1}[j,n]. \tag{2-22}$$

Therefore, the verification of (2-20) is completed and so the proof of (2-6) is ended.

As for (2-7), we observe that the recursive definition (2-2) of $Q_k(r)$ is as that of $P_k(r)$, with the only difference that n is replaced by $n-2$. Moreover, the difference between $q(r)$ and $p(r) = P_0(r)$ is just the additive constant -1, which is irrelevant for $k \ge 1$. Therefore, the explicit expression of $Q_k(r)$ can be obtained replacing n by $n-2$ in the expression of $P_k(r)$ and so the conclusion of the lemma follows immediately. □

In conclusion, the proof of Theorem 1.3 is now ended. □

Remark 2.3. Let $n = 2k$. The extrinsic k-energy of $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is finite. For instance, explicit integration provides the exact value of the extrinsic 3-energy of $\pi^{-1} : \mathbb{R}^6 \to \mathbb{S}^6$:

$$\begin{aligned} E_3^{\text{ext}}(\pi^{-1}) &= \text{Vol}(\mathbb{S}^5)\int_0^{+\infty} [r^2 P_1'^2 + 6P_1^2 + 2r\,P_1' P_1 + Q_1'^2]\,r^5\,dr \\ &= \pi^3 \int_0^{+\infty} \frac{512r^5(7r^4+24r^2+12)}{(r^2+1)^6}\,dr \\ &= -\pi^3 \frac{256(7r^8+26r^6+30r^4+15r^2+3)}{(r^2+1)^5}\bigg|_0^{+\infty} = 768\pi^3. \end{aligned}$$

Note that $E(\pi^{-1}) = 4\pi$ and $E_2^{\text{ext}}(\pi^{-1}) = 64\pi^2$.

3. Proofs of Theorems 1.4, 1.5 and Proposition 1.6

Proof of Theorem 1.4. **Step 1.** Lemma 3.1 below is a version of Lemma 2.3 of [10] in this context. The proof is similar, but we include it for the sake of completeness (to shorten the notation, when the meaning is clear, we shall use "." instead of $\langle\, , \rangle$).

Lemma 3.1. (i) *Let $n = 2k$ and $k = 2s$. If*

$$\int_{B^n(R)} [|\Delta^s \phi|^2 - (\Delta^{2s}\pi^{-1}.\pi^{-1})\,|\phi|^2]\,dx \ge 0 \tag{3-1}$$

for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$, *then* π^{-1} *is energy minimizing for the extrinsic* k*-energy.*

(ii) *Let* $n = 2k$ *and* $k = 2s + 1$. *If*

$$\int_{B^n(R)} [|\nabla \Delta^s \phi|^2 + (\Delta^{2s+1}\pi^{-1}.\pi^{-1})\,|\phi|^2]\,dx \geq 0 \tag{3-2}$$

for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$, *then* π^{-1} *is energy minimizing for the extrinsic* k*-energy.*

Proof. (i) We must show that

$$E_k^{\text{ext}}(\pi^{-1}) \leq E_k^{\text{ext}}(v) \tag{3-3}$$

for all $v \in W^{k,2}(B^n(R), \mathbb{S}^n)$ such that $\pi^{-1} - v \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$.

On the ball $B^n(R)$ the map π^{-1} satisfies

$$\Delta^{2s}\pi^{-1} = (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1} \tag{3-4}$$

strongly. Thus we can multiply both sides of (3-4) by $\phi \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$ and we obtain

$$\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s\phi\,dx = \int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.\phi\,dx.$$

Choosing $\phi = \pi^{-1} - v$ we deduce that

$$\begin{aligned}\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s\pi^{-1}\,dx &- \int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s v\,dx \\ &= \int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,dx - \int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.v\,dx,\end{aligned}$$

which for convenience we rewrite as

$$\begin{aligned}-2\int_{B^n(R)} |\Delta^s\pi^{-1}|^2\,dx &+ 2\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s v\,dx \\ &= -2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,dx + 2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.v\,dx.\end{aligned} \tag{3-5}$$

Next, we apply the hypothesis (3-1) with $\phi = \pi^{-1} - v$. Since π^{-1}, v have values in $\mathbb{S}^n$ we have

$$|\phi|^2 = |\pi^{-1} - v|^2 = 2 - 2\pi^{-1}.v$$

and so we easily find

$$\begin{aligned}\int_{B^n(R)} |\Delta^s v|^2\,dx + \int_{B^n(R)} |\Delta^s\pi^{-1}|^2\,dx - 2\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s v\,dx \\ - 2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,dx + 2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.v\,dx \geq 0.\end{aligned} \tag{3-6}$$

Finally, inserting (3-5) into the second line of (3-6), we obtain

$$\int_{B^n(R)} |\Delta^s v|^2 \, dx - \int_{B^n(R)} |\Delta^s \pi^{-1}|^2 \, dx \geq 0,$$

which is precisely (3-3). The proof of part (ii) is analogous and so we omit the details. □

Step 2. Case $n = 2k,\ k = 2s$. Taking into account (2-12) and (2-13) it is easy to deduce that the inequality (3-1) is equivalent to

$$\int_{B^n(R)} \left[|\Delta^s \phi|^2 - \frac{2^{4s}(4s)!}{(1+r^2)^{4s}} |\phi|^2 \right] dx \geq 0. \tag{3-7}$$

Similarly, when $n = 2k$ and $k = 2s+1$, equation (3-2) can be written as

$$\int_{B^n(R)} \left[|\nabla \Delta^s \phi|^2 - \frac{2^{4s+2}(4s+2)!}{(1+r^2)^{4s+2}} |\phi|^2 \right] dx \geq 0. \tag{3-8}$$

We also point out that in (3-1), (3-2), (3-7) and (3-8) the test function ϕ is a vector function $\phi = (\phi^1, \dots, \phi^{n+1})$. But, since

$$|\Delta^s \phi|^2 = \sum_{i=1}^{n+1} |\Delta^s \phi^i|^2, \quad |\nabla \Delta^s \phi|^2 = \sum_{i=1}^{n+1} |\nabla \Delta^s \phi^i|^2 \quad \text{and} \quad |\phi|^2 = \sum_{i=1}^{n+1} |\phi^i|^2,$$

we easily deduce that it suffices to prove that these inequalities hold for all *scalar* test functions $\phi \in W_0^{k,2}(B^n(R), \mathbb{R})$.

By way of summary, the validity of (3-7) (case $k = 2s$) or (3-8) (case $k = 2s+1$) for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R})$ is sufficient to insure that $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is a minimizer for the extrinsic k-energy.

As a special case, the proof of Theorem 1.4 will be complete if we show that the inequality (3-8), with $s = 1$, holds provided that $0 < R \leq R_3^*$.

To this purpose, we recall that in [12, Theorem 3, p. 2159] the authors proved a general third-order Hardy inequality for bounded domains $\Omega \subset \mathbb{R}^n,\ n \geq 6$. It is convenient for us to state their result in the special case $n = 6$ and $\Omega = B^6(R)$. We set

$$c_1 = 57\Lambda(2), \quad c_2 = 6\Lambda(2)^2 + 4\Lambda(4)^2, \quad c_3 = \Lambda(2)\,\Lambda(4)^2, \tag{3-9}$$

where

$$\Lambda(n) = \inf \left\{ \frac{\int_0^1 |v'(r)|^2 \, r^{n-1} \, dr}{\int_0^1 |v(r)|^2 \, r^{n-1} \, dr} : v \in X \right\},$$

with

$$X = \left\{ v \in C^1([0,1]) : v'(0) = v(1) = 0,\ v \not\equiv 0 \right\}.$$

In other words, $\Lambda(n)$ is the first positive eigenvalue associate to the Dirichlet problem for Δ on B^n, i.e.,

$$\Lambda(n)=\lambda_1=\inf\left\{\frac{\int_{B^n}|\nabla\phi|^2}{\int_{B^n}\phi^2}:\phi\in W_0^{1,2}(B^n),\ \phi\not\equiv 0\right\}.$$

It is also known (see [12]) that

$$\Lambda(n)^2=\inf\left\{\frac{\int_{B^n}(\Delta\phi)^2}{\int_{B^n}\phi^2}:\phi\in W^{2,2}\cap W_0^{1,2}(B^n),\ \phi\not\equiv 0\right\}=\lambda_1^2.$$

The value of λ_1, which depends on n, is related to the first positive zero for a class of Bessel functions.

More precisely (see [2]), let $\nu=\frac{n}{2}-1$. Then $\lambda_1=j_\nu^2$, where j_ν denotes the first positive zero of the Bessel function $J_\nu(r)$. In particular,

$$\Lambda(2)=j_0^2\approx(2.4)^2,\quad \Lambda(4)=j_1^2\approx(3.8)^2. \tag{3-10}$$

Then we have:

Theorem 3.2 [12]. *Let $\phi\in W^{3,2}\cap W_0^{1,2}(B^6(R))$ with $\Delta\phi=0$ on $\partial B^6(R)$, i.e., $\Delta\phi\in W_0^{1,2}(B^6(R))$. Then*

$$\int_{B^6(R)}|\nabla\Delta\phi|^2\,dx\geq\frac{c_1}{R^2}\int_{B^6(R)}\frac{\phi^2}{|x|^4}dx+\frac{c_2}{R^4}\int_{B^6(R)}\frac{\phi^2}{|x|^2}dx+\frac{c_3}{R^6}\int_{B^6(R)}\phi^2\,dx, \tag{3-11}$$

where the constants c_i, $i=1,2,3$, are defined in (3-9).

Now, we are in the position to conclude the proof of Theorem 1.4.

Indeed, since in our context $\phi\in W_0^{3,2}(B^6(R))$, we can apply (3-11) and we deduce ($r=|x|$):

$$\begin{aligned}\int_{B^6(R)}|\nabla\Delta\phi|^2\,dx&\geq\int_{B^6(R)}\left[\frac{c_1}{R^2r^4}+\frac{c_2}{R^4r^2}+\frac{c_3}{R^6}\right]\phi^2\,dx\\&=\int_{B^6(R)}\psi_R(r)\,\phi^2\,dx,\end{aligned} \tag{3-12}$$

where we have set

$$\psi_R(r)=\frac{c_1}{R^2r^4}+\frac{c_2}{R^4r^2}+\frac{c_3}{R^6},\quad 0<r\leq R.$$

Now, taking into account (3-8), we deduce that (3-12) implies that (3-2) with $s=1$ holds provided that

$$\psi_R(r)-\frac{46080}{(1+r^2)^6}\geq 0\quad\text{on }(0,R]. \tag{3-13}$$

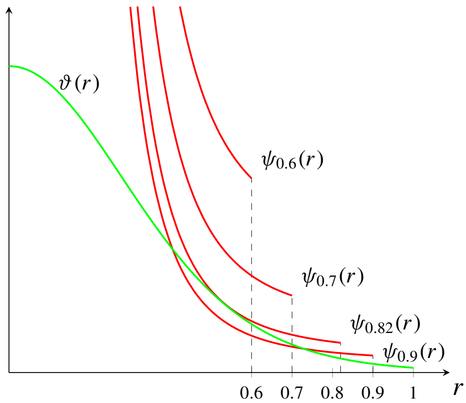

Figure 1. Analysis of condition (3-13). Here $\vartheta(r) = \frac{46080}{(1+r^2)^6}$.

Therefore, it is natural to define

$$R_3^* = \text{Max}\{R > 0 : (3\text{-}13) \text{ holds}\}. \tag{3-14}$$

Now, using (3-9) and (3-10), a routine analysis shows that R_3^* is well defined and (3-13) holds for all $0 < R \le R_3^*$. Finally, a straightforward analysis carried out with Mathematica shows that $R_3^* \approx 0.82$ and so the proof of Theorem 1.4 is completed. We have inserted the output of this study in Figure 1. □

Proof of Theorem 1.5. We follow the method of proof of Theorem 1.4. Then, when $k = 2s \ge 4$, it suffices to prove the existence of $0 < R_k^* \le 1$ such that the inequality (3-7) holds for $0 < R \le R_k^*$. This can be achieved using another result of [12]. Indeed, let us fix $k = 2s$. Then Corollary 2 of [12] enables us to say that there exist positive constants $c_1, \dots, c_k$ such that

$$\int_{B^n(R)} |\Delta^s \phi|^2 \, dx \ge \sum_{\ell=1}^{k} \frac{c_\ell}{R^{2\ell}} \int_{B^n(R)} \frac{\phi^2}{|x|^{2k-2\ell}} \, dx$$

for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R})$. From this we easily deduce that

$$\int_{B^n(R)} |\Delta^s \phi|^2 \, dx \ge \frac{\left(\sum_{\ell=1}^k c_\ell\right)}{R^{4s}} \int_{B^n(R)} \frac{\phi^2}{(1+|x|^2)^{4s}} \, dx. \tag{3-15}$$

Therefore, setting

$$R_k^* = 2 \sqrt[4s]{\frac{\left(\sum_{\ell=1}^k c_\ell\right)}{(2s)!}},$$

it is easy to conclude that (3-7) holds for all $0 < R \le R_k^*$. Moreover, as a consequence of Proposition 1.6, we observe that necessarily $R_k^* \le 1$ and so the proof of the case $k = 2s$ is ended.

Similarly, using again Corollary 2 of [12], we obtain the existence of $0 < R_k^* \leq 1$ such that, when $k = 2s + 1 \geq 5$, (3-8) holds and so the proof is completed. □

Remark 3.3. In order to obtain a numerical value for R_k^*, $k \geq 4$, it would be useful to know the exact optimal values of the positive constants $c_1, \dots, c_k$, but this datum is not available (see [12]). Therefore, in this case, we have preferred to limit technicalities and we do not have introduced a function which could play the role of $\psi_R(r)$ in the proof of Theorem 1.4. Moreover, we point out that a better estimate for R_k^* could be achieved if Conjecture 1 of [12, p. 2164] were true.

Proof of Proposition 1.6. Following an idea of [9], we compare $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ with the map $\tilde{\pi}^{-1} : B^n(R) \to \mathbb{S}^n$ defined as follows: $\tilde{\pi}^{-1}(x) = \pi_N^{-1}(x/R^2)$, where π_N^{-1} is the map obtained from π^{-1} by changing the sign of the last component. In order to clarify the geometric construction, we point out that π represents the stereographic projection from the *south pole*, while π_N is the stereographic projection from the *north pole*. More explicitly, we have

$$\tilde{\pi}^{-1} : B^n(R) \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \left(\frac{2R^2}{R^4 + r^2}\, x, -\frac{R^4 - r^2}{R^4 + r^2} \right), \tag{3-16}$$

where, as usual, $r = |x|$. It is easy to check that π^{-1} and $\tilde{\pi}^{-1}$ coincide on $\partial B^n(R)$. Next, we first observe that

$$(\Delta \tilde{\pi}^{-1})(x) = \frac{1}{R^4} (\Delta \pi_N^{-1}) \left(\frac{x}{R^2} \right)$$

and so

$$(\Delta^s \tilde{\pi}^{-1})(x) = \frac{1}{R^{4s}} (\Delta^s \pi_N^{-1}) \left(\frac{x}{R^2} \right).$$

Now, when $n = 2k$, $k = 2s$, we have:

$$\begin{aligned} \int_{B^n(R)} |(\Delta^s \tilde{\pi}^{-1})(x)|^2 \, dx &= \frac{1}{R^{8s}} \int_{B^n(R)} \left| (\Delta^s \pi_N^{-1}) \left(\frac{x}{R^2} \right) \right|^2 dx \\ \left(\text{using } y = \frac{x}{R^2}\right) \quad &= \frac{1}{R^{8s}} \int_{B^n(1/R)} |(\Delta^s \pi_N^{-1})(y)|^2 R^{2n} \, dy \\ (\text{using } 8s = 2n) \quad &= \int_{B^n(1/R)} |(\Delta^s \pi_N^{-1})(y)|^2 \, dy. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \int_{B^n(R)} |\Delta^s \tilde{\pi}^{-1}|^2 \, dx &= \int_{B^n(1/R)} |\Delta^s \pi_N^{-1}|^2 \, dx \\ &= \int_{B^n(1/R)} |\Delta^s \pi^{-1}|^2 \, dx < \int_{B^n(R)} |\Delta^s \pi^{-1}|^2 \, dx, \end{aligned}$$

which proves Proposition 1.6 in the case $k = 2s$. The case $k = 2s+1$ is analogous. □

Remark 3.4. We point out that the map $\tilde{\pi}^{-1}$ defined in (3-16), although it coincides with π^{-1} on the boundary $\partial B^n(R)$, belongs to a different homotopy class.

4. Second variation and equivariant stability

For convenience we rewrite $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ as

$$\pi^{-1} : B^n(R) \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \Big(\sin\alpha(r)\frac{x}{r}, \cos\alpha(r)\Big), \tag{4-1}$$

where $r = |x|$ and $\alpha(r) = 2\tan^{-1} r$. We consider *equivariant variations*, i.e., rotationally symmetric variations as

$$u_t(x) = \Big(\sin[\alpha(r) + t\,\phi(r)]\frac{x}{r}, \cos[\alpha(r) + t\phi(r)]\Big), \tag{4-2}$$

where $\phi = \phi(r)$ is any smooth real valued function on $[0, R]$ such that u_t is smooth and preserves the boundary data. In particular, we now have $\phi^{(2j)}(0) = 0,\ j \ge 0$, $\phi^{(j)}(R) = 0,\ j \ge 0$.

Therefore, we shall say that $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is *stable with respect to equivariant variations* (shortly, *equivariantly stable*) if

$$\frac{d^2}{dt^2}(E_k^{\text{ext}}(u_t))|_{t=0} \ge 0 \tag{4-3}$$

for all u_t as in (4-2).

Next, we define a map $\pi^{-1}_{\partial/\partial\alpha} : B^n(R) \to \mathbb{S}^n$ which represents the unit vector field $\partial/\partial\alpha$ along π^{-1}:

$$\pi^{-1}_{\partial/\partial\alpha} : B^n(R) \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \Big(\cos\alpha(r)\frac{x}{r}, -\sin\alpha(r)\Big), \tag{4-4}$$

where again $r = |x|$ and $\alpha(r) = 2\tan^{-1} r$. Then a simple computation shows

$$\frac{du_t(x)}{dt}\Big|_{t=0} = \pi^{-1}_{\partial/\partial\alpha}(x)\,\phi \quad \text{and} \quad \frac{d^2u_t(x)}{dt^2}\Big|_{t=0} = -\pi^{-1}(x)\,\phi^2. \tag{4-5}$$

Now, we prove a general, preliminary lemma.

Lemma 4.1. (i) *Let $n = 2k,\ k = 2s$. Then $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ is equivariantly stable for the extrinsic k-energy if and only if*

$$\int_{B^n(R)} [|\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\phi)|^2 - (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\phi^2]dx \ge 0 \quad \textit{for all } \phi \textit{ as in } (4\text{-}2). \tag{4-6}$$

(ii) *Let $n = 2k,\ k = 2s+1$. Then $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ is equivariantly stable for the extrinsic k-energy if and only if*

$$\int_{B^n(R)} [|\nabla\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\phi)|^2 - (\Delta^{2s+1}\pi^{-1}.\pi^{-1})\,\phi^2]dx \ge 0 \quad \textit{for all } \phi \textit{ as in } (4\text{-}2). \tag{4-7}$$

Proof. (i) We consider variations of the type (4-2) and we use (4-5). Then on $B^n(R)$ we have

$$\frac{d}{dt}(\Delta^s u_t)\Big|_{t=0} = \Delta^s\left(\frac{du_t}{dt}\right)\Big|_{t=0} = \Delta^s\left(\left(\frac{du_t}{dt}\right)\Big|_{t=0}\right) = \Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi) \tag{4-8}$$

and

$$\frac{d^2}{dt^2}(\Delta^s u_t)\Big|_{t=0} = \Delta^s\left(\frac{d^2u_t}{dt^2}\right)\Big|_{t=0} = \Delta^s\left(\left(\frac{d^2u_t}{dt^2}\right)\Big|_{t=0}\right) = -\Delta^s(\pi^{-1}\,\phi^2). \tag{4-9}$$

Now,

$$\frac{d^2}{dt^2}(E_k^{\text{ext}}(u_t))\Big|_{t=0} = 2\int_{B^n(R)}\frac{d^2}{dt^2}(\Delta^s(u_t))\Big|_{t=0}.(\Delta^s u_t)|_{t=0}\,dx + 2\int_{B^n(R)}\left|\frac{d}{dt}(\Delta^s(u_t))|_{t=0}\right|^2 dx. \tag{4-10}$$

Substituting (4-8) and (4-9) into (4-10) we get

$$\frac{d^2}{dt^2}(E_k^{\text{ext}}(u_t))\Big|_{t=0} = -2\int_{B^n(R)}\Delta^s(\pi^{-1}\phi^2).\,\Delta^s(\pi^{-1})\,dx + 2\int_{B^n(R)}|\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\phi)|^2\,dx.$$

Then, using the second Green identity, we deduce that the stability condition (4-3) is equivalent to

$$\int_{B^n(R)}[|\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)|^2 - (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\phi^2]\,dx \geq 0$$

for any arbitrary smooth function ϕ as in (4-2), as required.

The proof of the inequality (4-7) of part (ii) is analogous and so we omit it. □

The stability inequalities provided by Lemma 4.1 are rather general, but difficult to deal with because of the terms $\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)$ and $\nabla\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)$.

Therefore, a natural first step is to investigate these inequalities under the assumptions that k is small. In this order of ideas, a straightforward computation using Lemma 2.1 and some integration by parts leads us to the following:

Proposition 4.2. (i) *The map* $\pi^{-1} : B^4(R) \to \mathbb{S}^4$ *is equivariantly stable for the extrinsic bienergy if and only if*

$$\int_0^R\left[r^3\phi''^2(r) + 9r\phi'^2(r) + \left(\frac{9}{r} - \frac{384\,r^3}{(1+r^2)^4}\right)\phi^2(r)\right]dr \geq 0 \tag{4-11}$$

for all $\phi = \phi(r)$ *as in* (4-2).

(ii) *The map* $\pi^{-1} : B^6(R) \to \mathbb{S}^6$ *is equivariantly stable for the extrinsic trienergy if and only if*

$$\int_0^R \left[r^5 \phi'''^2(r) + 30 r^3 \phi''^2(r) + 225 r \phi'^2(r) + \left(\frac{225}{r} - \frac{46080\, r^5}{(1+r^2)^6} \right) \phi^2(r) \right] dr \geq 0 \tag{4-12}$$

for all $\phi = \phi(r)$ *as in* (4-2).

Remark 4.3. (i) As an alternative to (4-2), we could use

$$u_t(x) = \pi^{-1}(x) + t \phi \pi^{-1}_{\partial/\partial\alpha}(x).$$

Then we would have again (4-5) and so we would reobtain the conclusion of Lemma 4.1.

(ii) In the case that $\phi = \phi(r)$ is radial, the inequality (3-8) ($s = 1$) on $B^6(R)$ turns out to be equivalent to

$$\int_0^R \left[r^5 \phi'''^2(r) + 15 r^3 \phi''^2(r) + 45 r \phi'^2(r) - \frac{46080 r^5}{(1+r^2)^6} \phi^2(r) \right] dr \geq 0.$$

We observe that this condition is stronger than (4-12).

Now we can proceed to the proof of our main result in this context.

Proof of Theorem 1.7. (i) According to Proposition 4.2 it suffices to show that (4-11) holds provided that $0 < R \leq R_2^{\mathrm{stab}} \approx 1.81$.

First, a simple computation using integration by parts leads us to

$$\int_0^R [r^3 \phi''^2(r) + 9 r \phi'^2(r)]\, dr = \frac{1}{\mathrm{Vol}(\mathbb{S}^3)} \int_{B^4(R)} |\Delta\phi|^2 dx + \frac{6}{\mathrm{Vol}(\mathbb{S}^3)} \int_{B^4(R)} \frac{|\nabla\phi|^2}{r^2} dx. \tag{4-13}$$

Next, Theorem 4 of [12] gives

$$\int_{B^4(R)} |\Delta\phi|^2 dx \geq \frac{\Lambda((-\Delta)^2, 4)}{R^4} \int_{B^4(R)} \phi^2 dx, \tag{4-14}$$

where, keeping the notation of [12], we have

$$\Lambda((-\Delta)^2, 4) \geq j_1^2 j_2^2 \approx 387.23.$$

Using (8) of [12], we also have

$$\int_{B^4(R)} \frac{|\nabla\phi|^2}{r^2} dx \geq \frac{1}{R^2} \int_{B^4(R)} \left[\frac{1}{r^2} + \frac{\Lambda(2)}{R^2} \right] \phi^2 dx. \tag{4-15}$$

Inserting (4-14) and (4-15) into (4-13) we deduce

$$\int_0^R [r^3\phi''^2(r)+9r\phi'^2(r)]\,dr \geq \int_0^R \left[\frac{(\Lambda(-\Delta)^2,4)}{R^4}+\frac{6}{R^2r^2}+\frac{6\Lambda(2)}{R^4}\right]\phi^2\,dr. \tag{4-16}$$

Next, we set

$$\psi_{R,2}(r)=\frac{(\Lambda(-\Delta)^2,4)+6\Lambda(2)}{R^4}+\frac{6}{R^2r^2}+\frac{9}{r^4},\quad 0<r\leq R.$$

Then it is immediate to conclude that (4-11) holds provided that

$$\psi_{R,2}(r)-\frac{384}{(1+r^2)^4}\geq 0 \quad \text{on } (0,R]. \tag{4-17}$$

Now, an analysis similar to the study of (3-13) (see the proof of Theorem 1.4) shows that (4-17) holds provided that $0<R\leq R_2^{\text{stab}}\approx 1.81$, so ending (i).

(ii) According to Proposition 4.2 it suffices to show that (4-12) holds provided that $0<R\leq R_3^{\text{stab}}\approx 1.43$.

We shall use the following Hardy-type inequalities which again can be deduced from Theorem 4 and inequality (8) of [12], respectively:

$$\int_{B^6(R)}|\Delta\phi|^2\,dx \geq 9\int_{B^6(R)}\frac{\phi^2}{r^4}dx+\frac{6\Lambda(2)}{R^2}\int_{B^6(R)}\frac{\phi^2}{r^2}dx + \frac{(\Lambda(-\Delta)^2,4)}{R^4}\int_{B^6(R)}\phi^2\,dx, \tag{4-18}$$

$$\int_{B^6(R)}|\nabla\phi|^2\,dx \geq 4\int_{B^6(R)}\frac{\phi^2}{r^2}dx+\frac{\Lambda(2)}{R^2}\int_{B^6(R)}\phi^2\,dx. \tag{4-19}$$

Next, a computation similar to (4-13) enables us to write

$$\begin{aligned}\int_0^R|\nabla\Delta\phi|^2\,r^5\,dr&=\int_0^R[r^5\phi'''^2(r)+15r^3\phi''^2(r)+45r\phi'^2(r)]\,dr,\\ \int_0^R|\Delta\phi|^2\,r^5\,dr&=\int_0^R[r^5\phi''^2(r)+5r^3\phi'^2(r)]\,dr,\\ \int_0^R|\nabla\phi|^2\,r^5\,dr&=\int_0^R\phi'^2(r)\,r^5\,dr.\end{aligned} \tag{4-20}$$

Now, using (4-20), we deduce

$$\int_0^R[r^5\phi'''^2+30r^3\phi''^2+225r\phi'^2]\,dr = \int_0^R\left[|\nabla\Delta\phi|^2+15\frac{|\Delta\phi|^2}{r^2}+105\frac{|\nabla\phi|^2}{r^4}\right]r^5\,dr. \tag{4-21}$$

Next, using (3-11), (4-18), (4-19) and (4-21) into (4-12), we obtain that (4-12) holds provided that

$$\psi_{R,3}(r) - \frac{46080}{(1+r^2)^6} \geq 0 \quad \text{on } (0, R], \tag{4-22}$$

where we have defined

$$\psi_{R,3}(r) = \frac{c_1}{R^2 r^4} + \frac{c_2}{R^4 r^2} + \frac{c_3}{R^6} + \frac{15}{R^2}\left(\frac{9}{r^4} + \frac{6\Lambda(2)}{R^2 r^2} + \frac{(\Lambda(-\Delta)^2, 4)}{R^4}\right) + \frac{105}{R^4}\left(\frac{4}{r^2} + \frac{\Lambda(2)}{R^2}\right) + \frac{225}{r^6}.$$

Now, an analysis similar to the study of (3-13) shows that (4-22) holds provided that $0 < R \leq R_3^{\text{stab}} \approx 1.43$, so ending the proof of Theorem 1.7. □

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BALANCED HOMOGENEOUS HARMONIC MAPS BETWEEN CONES

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We study the degrees of homogeneous harmonic maps between simplicial cones. Such maps have been used to model the local behavior of harmonic maps between singular spaces, where the degrees of homogeneous approximations describe the regularity of harmonic maps. In particular the degrees of homogeneous harmonic maps are related to eigenvalues of discrete and edge-based graph Laplacians.

1. Introduction

The study of harmonic maps into singular spaces goes back to the seminal work of Gromov and Schoen [1992], who considered maps into Riemannian simplicial complexes. Of particular interest to our work is their local approximation of such maps by *homogeneous* maps between tangent cones. The study of harmonic maps was extended by Chen [1995] to include simplicial domains, and the theory for simplicial domains was further studied in [Eells and Fuglede 2001; Mese 2004; Daskalopoulos and Mese 2006; 2008], among others. The Hölder continuity of harmonic maps from Riemannian simplicial complexes to metric spaces of nonpositive curvature was established in [Eells and Fuglede 2001]. Further regularity away from the $(n-2)$-skeleton of an n-dimensional simplicial domain was studied in [Mese 2004; Daskalopoulos and Mese 2006; 2008].

The blow-up analysis of [Gromov and Schoen 1992] was emulated by Daskalopoulos and Mese [2006; 2008] to construct tangent maps as homogeneous approximations of harmonic maps with singular domains and targets. The general idea is to fix a point p in the domain of a harmonic map u, and its image $u(p)$. For $\lambda, \mu > 0$ one constructs the map

$$u_{\lambda,\mu}(x) = \mu^{-1} u(\lambda x).$$

A small neighborhood of p in the domain, scaled by the factor λ^{-1}, serves as the domain of $u_{\lambda,\mu}$. The map first rescales the neighborhood to its original size. Then after mapping by the original map u, the image is scaled by a factor of μ^{-1}.

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If λ and μ are small this has the effect of magnifying small neighborhoods of p and $u(p)$. As $\lambda, \mu \to 0$, technical estimates on the relationship between λ and μ ensure the existence of a limit

$$u_* = \lim_{\lambda,\mu\to 0} u_{\lambda,\mu}.$$

The limit map u_* has as domain the tangent cone to the domain of u at the point p, and has as target the tangent cone at $u(p)$ to the target of u. The tangent cone to a smooth Riemannian manifold is just its Euclidean tangent space, while the tangent cone to a simplicial complex is a geometric cone over a small sphere around a point, or its link. The map u is homogeneous in the sense that for each x in the domain cone, $t \mapsto u_*(tx)$ is an (unparametrized) geodesic, and there is some $\alpha > 0$ so that $|u_*(tx)| = t^\alpha |u_*(x)|$. Here $|\cdot|$ denotes the distance from the vertex of the target cone. For more details on the blow-up procedure and homogeneous harmonic maps, see the references cited above, especially [Gromov and Schoen 1992] and [Daskalopoulos and Mese 2006]. A typical application of this blow-up procedure is to say that if at each point p in the domain, the corresponding blow-up map u_* has degree $\alpha \geq 1$, then the original map u is locally Lipschitz continuous.

In related work, the author together with Victòria Gras Andreu [2020; 2021] consider a class of harmonic maps between 2-dimensional simplicial complexes. The maps respect the simplicial structure of the domain and target in the sense that vertices (resp. edges, faces) of the domain are mapped to vertices (resp. edges, faces) of the target. In [Freidin and Gras Andreu 2020] the simplicial complexes are endowed with metrics identifying each face with the ideal hyperbolic triangle. This has the result of putting the vertices of the complexes at infinite distance, effectively puncturing the spaces. In [Freidin and Gras Andreu 2021] the complexes are endowed with simplexwise flat metrics, as well as metrics conformal to flat or ideal hyperbolic metrics. The blow-up procedure described above is repeated in [Freidin and Gras Andreu 2021] to construct tangent maps at edge points and at (nonpunctured) vertices.

In [Daskalopoulos and Mese 2008] the tangent maps of harmonic maps from 2-dimensional simplicial complexes into Riemannian manifolds are related to eigenfunctions of a discrete Laplace operator on the link $\mathrm{Lk}(p)$ of a point p. The link of a point p in a 2-complex X is a graph whose vertices correspond to edges of X containing p, and whose edges correspond to faces of X containing p. The strategy of relating continuous and discrete harmonic maps is explored in detail in [Daskalopoulos and Mese 2008]. The discrete Laplace operator considered in [Daskalopoulos and Mese 2008] assumes the geometry in the link $\mathrm{Lk}(p)$ is very regular, in particular that all faces of the 2-complex are equilateral triangles so that every edge of $\mathrm{Lk}(p)$ has the same length. Our main goals are to extend that analysis to include more of the richness of the geometry that $\mathrm{Lk}(p)$ can have, as well as to the setting of tangent maps between simplicial cones constructed in [Freidin and Gras Andreu 2021].

Before beginning with the main content, we discuss some history of discrete Laplace operators that will become useful later on. For a more detailed discussion, including proofs of quoted results, see, e.g., [Chung 1997] or [Banerjee and Jost 2008]. Future work may be done to generalize the results of this paper to higher dimensional cones, at which point the work of Horak and Jost [2013] on the spectra of simplicial complexes will likely play the role of the discussion below.

Let Γ be a graph with vertices $V = V(\Gamma)$ and edges $E = E(\Gamma)$. One can enumerate the vertices of Γ and thus identify functions $f : V \to \mathbb{R}$ with vectors in $\mathbb{R}^{\#V}$. Then the unweighted graph Laplacian is represented by the matrix Δ with entries

$$\Delta_{ij} = \begin{cases} \deg v_i & \text{if } i = j, \\ -1 & \text{if } v_i \sim v_j, \\ 0 & \text{else.} \end{cases}$$

Here $v_i \sim v_j$ means that the vertices v_i and v_j are adjacent in Γ. This matrix $\Delta = D - A$ is the difference between the diagonal degree matrix D and the adjacency matrix A with entries

$$D_{ij} = \begin{cases} \deg v_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad \text{and} \qquad A_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \not\sim v_j. \end{cases}$$

A normalized Laplace operator, denoted $\mathcal{L}$, is defined by

$$\mathcal{L} = \mathrm{Id} - D^{-1/2} A D^{-1/2} = D^{-1/2} \Delta D^{-1/2}.$$

Both the matrices Δ and $\mathcal{L}$ are symmetric and positive semidefinite. Indeed, they determine quadratic forms

$$\rho \cdot \Delta \rho = \sum_{uv \in E} (\rho(v) - \rho(u))^2 \quad \text{and} \quad \rho \cdot \mathcal{L} \rho = \sum_{uv \in E} \left(\frac{\rho(v)}{\sqrt{\deg v}} - \frac{\rho(u)}{\sqrt{\deg u}} \right)^2.$$

An immediate consequence is that the vector $w = \mathbb{1}$ with $w_i = 1$ for all i is in the kernel of Δ and in fact spans the kernel if Γ is connected. Since $\mathcal{L} = D^{-1/2} \Delta D^{-1/2}$, the vector $D^{1/2}\mathbb{1}$ spans the kernel of $\mathcal{L}$. The operator $\mathcal{L}$ is called normalized because its largest eigenvalue is ≤ 2. The eigenvalue $\lambda = 2$ of $\mathcal{L}$ is achieved if and only if Γ is bipartite, and the first positive eigenvalue of $\mathcal{L}$ is at most 1 unless Γ is a complete graph K_n, in which case the first eigenvalue is $\frac{n}{n-1}$.

If weights are assigned to Γ by a function $w : V \cup E \to \mathbb{R}_{\geq 0}$, then one can modify the adjacency matrix so that $A_{ij} = w(v_i v_j)$ when $v_i \sim v_j$, and the degree matrix so that $D_{ii} = w(v_i)$. In case only edge weights are provided, the vertex weights are determined by $w(v_i) = \sum_{v_j \sim v_i} w(v_i v_j)$. In each of these cases one can still define

$$\Delta = D - A \quad \text{and} \quad \mathcal{L} = \mathrm{Id} - D^{-1/2} A D^{-1/2} = D^{-1/2} \Delta D^{-1/2}.$$

In case $w(e) = 1$ for all $e \in E$ and $w(v) = \deg v$ for $v \in V$ the unweighted operators are recovered. The matrices are still positive semidefinite but their kernels are affected by the weight function.

In both [Chung and Langlands 1996] and [Banerjee and Jost 2008] the asymmetric matrix $L = D^{-1}\Delta$ is studied. This matrix also comes up in our work. Fortunately it is similar to the normalized Laplacian via the similarity

$$L = D^{-1/2}\mathcal{L}D^{1/2}.$$

As a result the spectrum of L is identical to the spectrum of $\mathcal{L}$.

In [Friedman and Tillich 2004] the functions $f : V(\Gamma) \to \mathbb{R}$ are interpreted as continuous, edgewise linear functions $f : \Gamma \to \mathbb{R}$, and the combinatorial Laplacian Δ is interpreted to compute the sum of derivatives of f along the edges meeting at a vertex. That paper studies the problem of finding functions $f : \Gamma \to \mathbb{R}$ that are eigenfunctions of the second derivative operator on each edge and satisfy a boundary condition at each vertex relating the derivatives of f in the incident edges. The problem studied there is equivalent to our problem in the case of maps from a 2-dimensional cone into $\mathbb{R}$, but our approaches differ slightly.

The structure of this paper is as follows. We begin in Section 2 by introducing the spaces and maps to be studied. The spaces include k-pods and simplicial cones, together with the metrics they can carry. The maps of interest are homogeneous maps between cones. In Section 2 we also introduce the notion of harmonic maps as well as the balancing conditions that are essential to the current study.

In Section 3 we define and study balanced homogeneous harmonic maps from simplicial cones into Euclidean spaces. In Section 4 we do the same for maps from Euclidean spaces into k-pods. Section 4 also allows for a domain with a single singular point, as well as p-harmonic maps. In Section 5 we define and study balanced homogeneous harmonic maps between simplicial cones. Finally in Section 6 we generalize Sections 3 and 5 by studying the limits as the angles of each face of the cone tend to 0 and the vertices become punctures.

2. Preliminaries

2.1. ***Spaces and metrics.*** Our main objects of study will be homogeneous maps between cones. The spaces and maps we consider will have a lot more structure, but the basic definitions are quite general.

Definition 2.1 (cone). For any topological space X, the cone $C(X)$ is the quotient

$$C(X) = (X \times \mathbb{R}_{\geq 0})/\sim,$$

where $(x, 0) \sim (y, 0)$ for all $x, y \in X$.

If X is a space of dimension n, then the cone $C(X)$ has dimension $n+1$. Thus 1-dimensional cones can be constructed as the cone over a finite set of points. The common point $(x, 0) \in C(X)$ is called the *vertex* of the cone.

Definition 2.2 (homogeneous map). A map $f : C(X) \to C(Y)$ is homogeneous of degree $\alpha > 0$ if there are functions $y : X \to Y$ and $s : X \to \mathbb{R}_{\geq 0}$ so that

$$f(x, t) = (y(x), s(x)t^{\alpha}).$$

Necessarily the image of the vertex of $C(X)$ under a homogeneous map is the vertex of $C(Y)$.

The remainder of this section is devoted to describing which cones will appear as domains and targets of the maps we consider.

Definition 2.3 (k-pod). For $k \in \mathbb{N}$, consider the space $\{x_1, \dots, x_k\}$ of k discrete points. The k-pod is the cone $C(k) = C(\{x_1, \dots, x_n\})$. The rays $C(x_j) = \{x_j\} \times \mathbb{R}_{\geq 0}$ are called the edges of the k-pod.

Each edge of the k-pod $C(k)$ is homeomorphic to the half-line $\mathbb{R}_{\geq 0}$. Pulling back the standard Euclidean metric from $\mathbb{R}_{\geq 0}$ defines a metric on each edge of $C(k)$. We will also always use these homeomorphisms (now isometries) to give coordinates on each edge.

For a 2-dimensional cone, we must take the cone over a 1-dimensional space. The 1-dimensional spaces we will consider are graphs Γ, with vertices $V(\Gamma)$ and edges $E(\Gamma)$.

Definition 2.4 (2-dimensional cone). For a connected finite simplicial graph Γ, consider the cone $C(\Gamma)$. The rays $C(v) = v \times \mathbb{R}_{\geq 0}$ for vertices $v \in V(\Gamma)$ are edges of the cone, and the sectors $C(e)$ for edges $e \in E(\Gamma)$ are faces of the cone.

Each face of the cone $C(\Gamma)$ is homeomorphic to a sector of the plane $\mathbb{R}^2$. In order to determine the geometry of the cone one needs only specify the angle formed by each face at the vertex of the cone.

Definition 2.5 (metrics). Given a cone $C(\Gamma)$ over a graph Γ, a function $\theta : E(\Gamma) \to (0, \pi)$ determines a metric. For $e \in E(\gamma)$ the face $C(e)$ is isometric to the sector $S_{\theta(e)}$ of the Euclidean plane given in polar coordinates by

$$S_{\theta(e)} = \{(r, \theta) \mid r \geq 0,\ 0 \leq \theta \leq \theta(e)\}.$$

Let $C(\Gamma, \theta)$ denote the space $C(\Gamma)$ endowed with the metric determined by θ.

Remark 2.6. One could also define metrics on $C(\Gamma)$ for functions $\theta : E(\Gamma) \to \mathbb{R}_{>0}$. For $0 < \theta(e) < 2\pi$ one can still model $C(e)$ by the sector $S_{\theta(e)}$. For $\theta \geq 2\pi$ one could use a sector on a cone with cone angle $> 2\pi$. Though most results are unchanged when considering this more general class of metrics, we will not consider them here for two main reasons. First, the simplicial cones considered here are meant to model tangent cones to Euclidean simplicial complexes, all of whose angles are at most π. And second, the analysis for $\theta \geq \pi$ is more subtle and distracts from the main flow of many arguments.

Given a graph Γ and an edge $e = (p, q) \in E(\Gamma)$, one can subdivide e by adding a vertex x in the interior. This produces a graph Γ' with vertex set $V(\Gamma') = V(\Gamma) \cup \{x\}$ and edge set $E(\Gamma') = (E(\Gamma) \setminus \{e\}) \cup \{px, xq\}$.

Given a function $\theta : E(\Gamma) \to (0, \pi)$ and an edge $e \in E(\Gamma)$, when one subdivides the edge e to produce Γ' as above one can define a function $\theta' : E(\Gamma') \to (0, \pi)$. For $e' \neq e$ set $\theta'(e') = \theta(e')$, and set $\theta'(px) = \alpha$ and $\theta'(xq) = \beta$ in any way so that $\alpha + \beta = \theta(e)$. Any pair (Γ^*, θ^*) obtained from (Γ, θ) by a finite sequence of edge subdivisions as in these two paragraphs is called a subdivision of (Γ, θ), and the cones $C(\Gamma^*, \theta^*)$ and $C(\Gamma, \theta)$ are isometric.

In general two cones $C(\Gamma_1, \theta_1)$ and $C(\Gamma_2, \theta_2)$ are isometric if and only if the graphs have a common subdivision (Γ^*, θ^*).

In $C(\Gamma, \theta)$ identify an edge $e \in E(\Gamma)$ with the interval $[0, 1]$. Taking polar coordinates (ρ, φ) in $S_{\theta(e)}$, a concrete isometry $\psi_e : C(e) = C([0, 1]) \to S_{\theta(e)}$ is given by

$$\psi_e(x, t) = (\rho, \varphi) \quad \text{with } \rho = t,\ \varphi = x\theta(e).$$

We will also use these isometries to give coordinates on each face of $C(\Gamma, \theta)$. But note that the map $\psi_e(x, t) = (t, (1 - x)\theta(e))$ also defines an isometry $C(e) \to S_{\theta(e)}$. Our choice of isometry will be determined by which endpoint of e is most relevant at the moment.

Definition 2.7. In a cone $C(\Gamma, \theta)$, fix a face $C(e)$ for some $e \in E(\Gamma)$. If $v \in V(\Gamma)$ is one endpoint of e, then the coordinates on $C(E)$ adapted to v are the coordinates determined by the isometry $\psi_e : C(e) \to S_{\theta(e)}$ that sends the point $(v, 1) \in C(e)$ to $(1, 0) \in S_{\theta(e)}$.

One can also view the Euclidean plane $\mathbb{R}^2$ as a 2-dimensional cone. Namely, for the cycle graph C_n consisting of n vertices and n edges, $\mathbb{R}^2 = C(C_n, \theta)$ for any $\theta : E(C_n) \to (0, \pi)$ with $\sum_{e \in E(C_n)} \theta(e) = 2\pi$. More generally, we will sometimes consider cones over the cycle graph with different metrics.

Definition 2.8. A smooth cone is a geometric cone over the cycle graph. In other words, it is a space $C(C_n, \theta)$ for some $\theta : E(C_n) \to (0, \pi)$.

The curvature at the vertex of a smooth cone is determined by $\sum_{e \in E(C_n)} \theta(e)$. If the sum is less than 2π the vertex has positive curvature, if the sum is equal to 2π the vertex is flat, and if the sum is greater than 2π the vertex has negative curvature.

2.2. ***Harmonic functions on sectors.*** When it comes time to consider harmonic maps between cones, we will need to understand the structure of those harmonic maps. Our domains will always be 2-dimensional cones $C(\Gamma, \theta)$, all of whose faces are isometric to plane sectors S_θ. To begin this section we first aim to understand harmonic functions $f : S_{\theta_0} \to \mathbb{R}$.

Lemma 2.9. *A homogeneous function $u : S = S_{\theta_0} \to \mathbb{R}$ is harmonic if and only if there are constants c_1, c_2 so that*

$$u(r, \theta) = r^\alpha (c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta)).$$

Proof. In polar coordinates, a homogeneous function $u(r, \theta)$ of degree α has the form

$$u(r, \theta) = r^\alpha u(1, \theta) = r^\alpha \rho(\theta).$$

And in polar coordinates the Laplacian of u reads

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^{\alpha-2}(\alpha^2 \rho(\theta) + \rho''(\theta)).$$

To be harmonic, u must satisfy $\Delta u = 0$, and this equation must hold for all (r, θ). Hence ρ must satisfy $\rho''(\theta) + \alpha^2 \rho(\theta) = 0$. The solutions to this ordinary differential equation are precisely

$$\rho(\theta) = c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta).$$ □

In order to turn our study into a discrete problem later, we introduce the following result that says homogeneous harmonic functions are described by discrete data.

Lemma 2.10. *A homogeneous harmonic function $u(r, \theta) = r^\alpha \rho(\theta) : S = S_{\theta_0} \to \mathbb{R}$ is uniquely determined by the two numbers $\rho(0)$ and $\rho(\theta_0)$ unless $\alpha\theta_0 \in \pi\mathbb{Z}$.*

Proof. Given the form of $\rho(\theta)$ from Lemma 2.9, one easily solves the boundary value problem on $S = S_{\theta_0}$:

$$\begin{aligned}
\rho(0) &= c_1 \cos(0) + c_1 \sin(0) \\
&= c_1, \\
\rho(\theta_0) &= c_1 \cos(\alpha\theta_0) + c_2 \sin(\alpha\theta_0) \\
&= \rho(0) \cos(\alpha\theta_0) + c_2 \sin(\alpha\theta_0).
\end{aligned}$$

So as long as $\alpha\theta_0 \notin \pi\mathbb{Z}$, we have $\sin(\alpha\theta_0) \neq 0$ so we can write

$$\rho(\theta) = \rho(0) \cos(\alpha\theta) + \frac{\rho(\theta_0) - \rho(0)\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)} \sin(\alpha\theta).$$

In case $\alpha\theta_0 = n\pi$, c_2 can be any real number but we must have

$$\rho(\theta_0) = \rho(0)\cos(n\pi) = (-1)^n \rho(0).$$

In this case ρ has the form

$$\rho(\theta)=\rho(0)\cos(\alpha\theta)+c_2\sin(\alpha\theta). \qquad \square$$

When we consider maps into k-pods, we recall that each edge of a k-pod is isometric to the half-line, $\mathbb{R}_{\geq 0}$. Thus we must understand which homogeneous harmonic functions have image contained in the half-line.

Proposition 2.11. *A nontrivial homogeneous harmonic map* $u(r,\theta)=r^\alpha\rho(\theta):S=S_{\theta_0}\to\mathbb{R}_{\geq 0}$ *exists if and only if* $0<\alpha\theta_0\leq\pi$. *When* $0<\alpha\theta_0<\pi$, *any pair of numbers* $\rho(0),\rho(\theta_0)\geq 0$, *not both* 0, *uniquely determine such a function.*

Proof. According to Lemma 2.9, a homogeneous harmonic function $u:S_{\theta_0}\to\mathbb{R}$ has the form $u(r,\theta)=r^\alpha\rho(\theta)$ with

$$\rho(r)=c_1\cos(\alpha\theta)+c_2\sin(\alpha\theta).$$

If $\alpha\theta_0>\pi$, then both $\theta=0$ and $\theta=\pi$ are in the domain of ρ. If $c_2\neq 0$ then either $\rho(1,0)$ or $\rho(1,\pi)$ is negative. And if $c_2=0$ but $c_1\neq 0$ then ρ changes signs around $\theta=\pi$. In this case no homogeneous harmonic map $u:S_{\theta_0}\to\mathbb{R}_{\geq 0}$ exists.

If $\alpha\theta_0=\pi$ then $\rho(\pi)=-\rho(0)$, so to be nonnegative $\rho(0)=\rho(\pi)=0$. Now

$$\rho(\theta)=c_2\sin(\alpha\theta).$$

The values of $\sin(\alpha\theta)$ remain positive for $0<\theta<\theta_0$, so for the image of u to lie in $\mathbb{R}_{\geq 0}$ c_2 must be nonnegative.

Finally if $0<\alpha\theta_0<\pi$, Lemma 2.10 tells us that

$$\rho(\theta)=\rho(0)\cos(\alpha\theta)+\frac{\rho(\theta_0)-\rho(0)\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)}\sin(\alpha\theta).$$

Clearly both $\rho(0)$ and $\rho(\theta_0)$ must be nonnegative to ensure $u\geq 0$. Now $\rho(\theta)$ only vanishes when

$$\begin{aligned}0&=\rho(0)\cos(\alpha\theta)+\frac{\rho(\theta_0)-\rho(0)\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)}\sin(\alpha\theta),\\ \rho(0)\cot(\alpha\theta)&=\frac{\rho(0)\cos(\alpha\theta_0)-\rho(\theta_0)}{\sin(\alpha\theta_0)}\\ &=\rho(0)\cot(\alpha\theta_0)-\rho(\theta_0)\csc(\alpha\theta_0).\end{aligned}$$

If $\rho(0)=0$ then $\rho(\theta_0)=0$ also, in which case u is a trivial map. And if $\rho(0)>0$ then $\rho(\theta)$ cannot vanish until $\cot(\alpha\theta)<\cot(\alpha\theta_0)$. But cot is a decreasing function, so this cannot happen for $0<\theta<\theta_0$.

Thus in this last case where $0<\alpha\theta_0<\pi$, any pair of nonnegative numbers $\rho(0)$ and $\rho(\theta_0)$, not both 0, uniquely determine a homogeneous harmonic function $u:S_{\theta_0}\to\mathbb{R}_{\geq 0}$ of degree α. $\square$

Finally, when we discuss harmonic maps between 2-dimensional cones, we will need to understand harmonic maps between sectors. We will only consider those maps that send the boundary of the domain sector to the boundary of the target.

Proposition 2.12. *Fix* $0 < \theta_0, \varphi_0 < \pi$. *Then a nontrivial homogeneous harmonic map* $u : S_{\theta_0} \to S_{\varphi_0}$ *exists if and only if* $0 < \alpha\theta_0 \leq \pi$. *When* $0 < \alpha\theta_0 < \pi$, *any pair of numbers* $|u(1,0)|, |u(1,\theta_0)| \geq 0$, *not both* 0, *uniquely determine such a map.*

Proof. Use polar coordinates (r, θ) in the domain sector S_{θ_0} and rectangular coordinates (x, y) in the target sector S_{φ_0}. Then a homogeneous map $u : S_{\theta_0}$ can be written

$$u(r,\theta) = (x(r,\theta), y(r,\theta)).$$

As $S_{\varphi_0} \subset \mathbb{R}^2$ is flat, the map u is harmonic if and only if the functions x and y are. So Lemma 2.9 says $x(r,\theta) = r^\alpha \chi(\theta)$ and $y(r,\theta) = r^\alpha \eta(\theta)$ with

$$\chi(\theta) = c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta) \quad \text{and} \quad \eta(\theta) = c_3 \cos(\alpha\theta) + c_4 \sin(\alpha\theta).$$

First if $\alpha\theta_0 > \pi$, then just as in Proposition 2.11, $\eta(\theta)$ cannot remain positive for $0 \leq \theta \leq \theta_0$. But every point $(x, y) \in S_{\varphi_0}$ has $y \geq 0$, so in this case no map $u : S_{\theta_0} \to S_{\varphi_0}$ exists.

Now suppose $0 < \alpha\theta_0 \leq \pi$ and let $\rho_0 = |u(1,0)|$ and $\rho_1 = |u(1,\theta_0)|$. In order to map the boundary of S_{θ_0} to the boundary of S_{φ_0}, the maps χ and η must satisfy the conditions

$$\chi(0) = \rho_0, \quad \chi(\theta_0) = \rho_1 \cos(\varphi_0),$$
$$\eta(0) = 0, \quad \eta(\theta_0) = \rho_1 \sin(\varphi_0).$$

In other words,

$$c_1 = \rho_0, \quad c_1 \cos(\alpha\theta_0) + c_2 \sin(\alpha\theta_0) = \rho_1 \cos(\varphi_0),$$
$$c_3 = 0, \quad c_3 \cos(\alpha\theta_0) + c_4 \sin(\alpha\theta_0) = \rho_1 \sin(\varphi_0).$$

In all cases $c_1 = \rho_0$ and $c_3 = 0$ are determined.

If $\alpha\theta_0 = \pi$, then

$$-\rho_0 = \rho_1 \cos(\varphi_0) \quad \text{and} \quad 0 = \rho_1 \sin(\varphi_0).$$

Since $0 < \varphi_0 < \pi$, this means that $\rho_0 = \rho_1 = 0$. Now

$$\chi(\theta) = c_2 \sin(\alpha\theta) \quad \text{and} \quad \eta(\theta) = c_4 \sin(\alpha\theta).$$

As long as $(c_2, c_4) \in S_{\varphi_0}$, the image of the corresponding map u lies in S_{φ_0}.

Finally if $0 < \alpha\theta_0 < \pi$ one can explicitly solve for the constants c_2 and c_4 to find

$$\chi(\theta) = \rho_0 \cos(\alpha\theta) + \frac{\rho_1 \cos(\varphi_0) - \rho_0 \cos(\alpha\theta_0)}{\sin(\alpha\theta_0)} \sin(\alpha\theta),$$
$$\eta(\theta) = \frac{\sin(\varphi_0)}{\sin(\alpha\theta_0)} \rho_1 \sin(\alpha\theta).$$

To verify that the corresponding map u has image in S_{φ_0}, one needs only check that $\eta(\theta) \geq 0$ and $\chi(\theta) \geq \eta(\theta)\cot(\varphi_0)$ for $0 \leq \theta \leq \theta_0$. The first of these is immediate since

$$\frac{\sin(\varphi_0)}{\sin(\alpha\theta_0)}\rho_1 \geq 0 \qquad \text{and} \qquad \sin(\alpha\theta) \geq 0 \quad \text{for } 0 \leq \theta \leq \theta_0.$$

The second condition is also straightforward:

$$\begin{aligned}\chi(\theta) - \eta(\theta)\cot(\varphi_0) \\ &= \rho_0\cos(\alpha\theta) + \frac{\rho_1\cos(\varphi_0) - \rho_0\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)}\sin(\alpha\theta) - \frac{\cos(\varphi_0)}{\sin(\alpha\theta_0)}\rho_1\sin(\alpha\theta) \\ &= \rho_0(\cos(\alpha\theta) - \cot(\alpha\theta_0)\sin(\alpha\theta))\end{aligned}$$

This expression only vanishes when $\cot(\alpha\theta) = \cot(\alpha\theta_0)$, which only happens when $\theta = \theta_0$ since $\cot(\alpha\theta)$ is decreasing on the interval $0 < \theta < \theta_0$. □

2.3. ***Balancing conditions.*** We will mainly be interested in homogeneous maps in three situations:

(1) $f : C(\Gamma, \theta) \to \mathbb{R}^n$ from simplicial cones into Euclidean spaces,

(2) $f : C(C_n, \theta) \to C(k)$ from smooth cones into k-pods, and

(3) $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ between 2-dimensional cones with the same simplicial structure.

In addition to studying maps that are harmonic on the faces of the domain cone, we introduce balancing conditions along the edges of the domain, akin to those in, e.g., [Daskalopoulos and Mese 2006; 2008], that have to do with harmonicity along the edges.

In all cases homogeneous maps send the vertex of one cone to the vertex of the other. In the third situation we will also demand that edges and faces of the cone are sent to themselves. We will then add extra conditions to ensure that our homogeneous maps are as smooth as the singularities of the domain and target allow. In particular, for open sets U, V in top-dimensional faces of the domain and target, respectively, if $f(U) \subset V$ then f can be represented in coordinates as a smooth map between open subsets of Euclidean spaces. Along edges we introduce the so-called balancing conditions.

2.3.1. *Euclidean targets.* A map $f : C(\Gamma, \theta) \to \mathbb{R}^n$ can be written in coordinates as $f = (f_1, \dots, f_n)$ for n functions $f_j : C(\gamma, \theta) \to \mathbb{R}$. So we need only consider the case $n = 1$, i.e., $f : C(\Gamma, \theta) \to \mathbb{R}$.

In this case the balancing conditions relate the normal derivatives in each face along a common edge of the domain.

Definition 2.13. Let $f : C(\Gamma, \theta) \to \mathbb{R}$ be a homogeneous function. Fix a vertex $v \in V(\Gamma)$ and let $\{e_j\}$ enumerate the edges of Γ incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v in order to represent f in these faces by homogeneous functions

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}.$$

The function f is balanced along $C(v)$ if

$$\sum_j \frac{\partial u_j}{\partial \theta}(r, 0) = 0.$$

The function f is balanced if it is balanced along each edge of $C(\Gamma, \theta)$.

For maps $f : C(\Gamma, \theta) \to \mathbb{R}^n$, simply impose the balancing condition for each component of f. We can see the regularity of f along an edge $C(v)$ for a vertex of degree 2 by unfolding the incident faces $C(e_1)$ and $C(e_2)$ to form a larger sector of the plane.

Lemma 2.14. *Let $f : (\Gamma, \theta) \to \mathbb{R}$ be a balanced homogeneous function and let $v \in V(\Gamma)$ be a vertex of degree 2. Then f is C^1 over the edge $C(v) \subset C(\Gamma, \theta)$.*

Proof. Let $e_1, e_2 \in E(\Gamma)$ be the two edges incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v and represent f in these coordinates by homogeneous functions

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}.$$

Now define a function $u : S = \{(r, \theta) \in \mathbb{R}^2 \mid r \geq 0,\ -\theta(e_2) \leq \theta \leq \theta(e_1)\} \to \mathbb{R}$ as

$$u(r, \theta) = \begin{cases} u_1(r, \theta) & \text{if } \theta \geq 0, \\ u_2(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

The maps u_1 and u_2 agree on the x-axis $\{(r, 0) \mid r \geq 0\}$. Thus u is continuous, and moreover $\frac{\partial u_1}{\partial r}(r, 0) = \frac{\partial u_2}{\partial r}(r, 0)$ so $\frac{\partial u}{\partial r}$ is continuous. The $\frac{\partial}{\partial \theta}$ derivatives of u are given by

$$\frac{\partial u}{\partial \theta}(r, \theta) = \begin{cases} \dfrac{\partial u_1}{\partial \theta}(r, \theta) & \text{if } \theta \geq 0, \\ -\dfrac{\partial u}{\partial \theta}(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

At $\theta = 0$ the balancing condition says that these partial derivatives coincide, so $\frac{\partial u}{\partial \theta}$ is continuous too. $\square$

2.3.2. *1-dimensional targets.* In order for a homogeneous map $f = (y(x), s(x)t^\alpha) : C(C_n, \theta) \to C(k)$ into a k-pod to be continuous, the map y must be locally constant where $s \neq 0$. We will also assume that if $s(v) = 0$ at some vertex v with incident edges e_1 and e_2, then $y(e_1) \neq y(e_2)$.

Let $f = (y(x), s(x)t^\alpha) : C(C_n, \theta) \to C(k)$ be a homogeneous map from a smooth cone to a k-pod. If $s(x) = 0$ for some point x in the interior of an edge $e \in E(\Gamma)$, subdivide Γ by introducing a vertex at x. After performing all such subdivisions, we'll abuse notation and call the resulting space $C(C_n,\theta)$ too. Now $s(x)$ vanishes only at the vertices of Γ, and f maps each face $C(e)$ to a single edge of $C(k)$.

Now for a homogeneous map $f : C(C_n,\theta) \to C(k)$ we can define the balancing condition in terms of normal derivatives along each edge of the domain.

Definition 2.15. Let $f : C(C_n,\theta) \to C(k)$ be a homogeneous map. Fix a vertex $v \in V(C_n)$ and let $e_1, e_2 \in E(C_n)$ be the two edges incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v, in order to represent f in these faces by homogeneous maps

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}_{\geq 0}.$$

If $f(v,1)$ is the vertex of $C(k)$ then f is balanced along $C(v)$ if

$$\frac{\partial u_1}{\partial \theta}(r,0) = \frac{\partial u_2}{\partial \theta}(r,0).$$

If $f(v,1)$ lies in the interior of some edge of $C(k)$ then f is balanced along $C(v)$ if

$$\frac{\partial u_1}{\partial \theta}(r,0) + \frac{\partial u_2}{\partial \theta}(r,0) = 0.$$

The map f is balanced if it is balanced along each edge of $C(C_n,\theta)$.

In this case we can again see the regularity of f along each edge $C(v) \subset C(C_n,\theta)$. In addition to unfolding the two faces incident to $C(v)$ into a larger sector, we may have to unfold their image edges in $C(k)$ to form a whole line if $f(v)$ is the vertex of $C(k)$.

Lemma 2.16. *Let $f = (y(x), s(x)t^\alpha) : (C_n,\theta) \to C9k)$ be a balanced homogeneous map and $v \in V(\Gamma)$. Then f is C^1 over the edge $C(v) \subset C(C_n,\theta)$.*

Proof. Let $e_1, e_2 \in E(\Gamma)$ be the two edges incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v.

Consider first the case where $s(v) \neq 0$, so $f(v,1)$ is in the interior of an edge of $C(k)$. The map f is represented in coordinates in the faces $C(e_1)$ and $C(e_2)$ by homogeneous maps into the same half-line $\mathbb{R}_{\geq 0} \subset \mathbb{R}$. Thus Lemma 2.14 already establishes the C^1 regularity of f over the edge $C(v)$.

Now consider $s(v) = 0$, so $f(v,1)$ is the vertex of $C(k)$. Again we can represent the map f in the face $C(e_j)$ by homogeneous maps

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}_{\geq 0}.$$

Since $y(e_1) \neq y(e_2)$, the target half-lines should be thought of as different spaces, and can thus be glued together to form the entire real line. Specifically, define

$u : S = \{(r, \theta) \in \mathbb{R}^2 \mid r \geq 0,\ -\theta(e_2) \leq \theta \leq \theta(e_1)\} \to \mathbb{R}$ as

$$u(r, \theta) = \begin{cases} u_1(r, \theta) & \text{if } \theta \geq 0, \\ -u_2(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

Both u_1 and u_2 map $(r, 0)$ to 0, so u and $\frac{\partial u}{\partial r}$ are continuous over the x-axis. The $\frac{\partial}{\partial \theta}$ derivatives are given by

$$\frac{\partial u}{\partial \theta}(r, \theta) = \begin{cases} \dfrac{\partial u_1}{\partial \theta}(r, \theta) & \text{if } \theta \geq 0, \\ \dfrac{\partial u_2}{\partial \theta}(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

Again the balancing condition says that these derivatives coincide at $(r, 0)$, so $\frac{\partial u}{\partial \theta}$ is continuous. □

2.3.3. *2-dimensional targets.* Let $f = (y(x), s(x)t^\alpha) : C(\Gamma_1, \theta_1) \to C(\Gamma_2, \theta_2)$ be a homogeneous map. If $s(x) \in V(\Gamma_2)$ for some point x in the interior of an edge $e \in E(\Gamma_1)$, subdivide Γ_1 by introducing a vertex at x. Likewise subdivide Γ_2 at any point $s(v)$ for $v \in V(\Gamma_1)$. After performing all such subdivisions, the original map factors as a composition $C(\Gamma_1, \theta_1) \to C(\Gamma_1, \varphi) \to C(\Gamma_2, \theta_2)$ where the first map sends each edge and face of $C(\Gamma_1)$ to itself and the second map sends each face of $C(\Gamma_1, \varphi)$ isometrically to a face of $C(\Gamma_2, \theta_2)$.

As the harmonicity of a map is unaffected by postcomposition with isometries, we may consider only maps $f(x, t) = (y(x), s(x)t^\alpha) : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ such that $y(v) = v$ for all $v \in V(\Gamma)$ and $y|_e$ maps e to e for each $e \in E(\Gamma)$. That is, f maps each edge and face of $C(\Gamma)$ to itself.

Now for a homogeneous map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ we can define the balancing condition in terms of normal derivatives along each edge of the domain. But under the additional assumptions made on our maps, only some components of the normal derivatives are relevant.

Definition 2.17. Let $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ be a homogeneous map. Fix a vertex $v \in V(\Gamma_1)$ and let $\{e_j\}$ enumerate the edges of Γ incident to v. In each face $C(e_j)$ of $C(\Gamma, \theta)$ take coordinates adapted to v, and likewise in the faces of $C(\Gamma, \varphi)$. In this way, represent f in each face by a homogeneous map

$$u_j : S_j = S_{\theta(e_j)} \to S_{\varphi(e_j)} = S'_j.$$

Taking rectangular coordinates on the target S'_j, write $u_j = (x_j, y_j)$. Then the map f is balanced along $C(v)$ if

$$\sum_j \frac{\partial x_j}{\partial \theta}(r, 0) = 0.$$

The map f is balanced if it is balanced along each edge of $C(\Gamma, \theta)$.

From the maps x_j in this definition one can define a map x as follows. Suppose there are N edges e_j incident to v, and fix some index $1 \le j_0 \le N$. Define

$$x(r,\theta) = \begin{cases} x_{j_0}(r,\theta) & \text{if } \theta \ge 0, \\ -x_{j_0}(r,-\theta) + \dfrac{2}{N}\displaystyle\sum_j x_j(r,-\theta) & \text{if } \theta \le 0. \end{cases}$$

The fact that all the functions x_j agree along $\theta = 0$ means that x and $\frac{\partial x}{\partial r}$ are continuous, and the balancing condition now means that $\frac{\partial x}{\partial \theta}$ is continuous.

Remark 2.18. In the case that v has degree 2, the function x constructed just above is the direct analogue of the function u from Lemma 2.14. Unfortunately there is not an analogous smoothness result for a similarly constructed y function. The restriction that f sends $C(v)$ to itself for all $v \in V(\Gamma)$ means that we cannot expect such a smoothness result.

3. Balanced harmonic functions into Euclidean spaces

A function $f : C(\Gamma,\theta) \to \mathbb{R}$ is harmonic if its restriction to each face of $C(\Gamma,\theta)$ is a harmonic function on a plane sector. We investigate the balanced (as in Definition 2.13) homogeneous harmonic functions $f : C(\Gamma,\theta) \to \mathbb{R}$. The analysis will be different if $\alpha\theta(e) \in \pi\mathbb{Z}$ for some edge $e \in E(\Gamma)$; we will refer to such α as *singular degrees*.

We will focus first on the case of nonsingular α. In each case we begin by constructing homogeneous harmonic functions $f : C(\Gamma,\theta) \to \mathbb{R}$, and then investigating which of those functions are balanced.

3.1. *Nonsingular degrees.*

Lemma 3.1. *If α is a nonsingular degree then a homogeneous harmonic function $f : C(\Gamma,\theta) \to \mathbb{R}$ of degree $\alpha > 0$ is uniquely determined by a function $\rho : V(\Gamma) \to \mathbb{R}$.*

Proof. For a fixed edge $e \in E(\Gamma)$ consider the face $C(e) \subset C(\Gamma,\theta)$, which is isometric to $S_{\theta(e)} \subset \mathbb{R}^2$. According to Lemma 2.10 a homogeneous harmonic function $u : S_{\theta(e)} \to \mathbb{R}$ is uniquely determined by the values $u(1,0)$ and $u(1,\theta_0)$ (using polar coordinates (r,θ) in $S_{\theta(e)}$).

Fix a function $\rho : V(\Gamma) \to \mathbb{R}$. For each edge $e = (v_0 v_1) \in E(\Gamma)$, fix coordinates in the face $C(e)$ adapted to v_0, so that $C(e)$ is identified with $S_{\theta(e)}$ and the edge $C(v)$ is identified with the positive x-axis. Define f in $C(e)$ to be represented in these coordinates by the unique homogeneous harmonic function $u : S_{\theta(e)} \to \mathbb{R}$ with $u(1,0) = \rho(v_0)$ and $u(1,\theta(e)) = \rho(v_1)$.

Along each edge $C(v)$, $f(v,t) = \rho(v)t^\alpha$ is defined independently of the choice of incident face. Hence f is continuous over the edges of $C(\Gamma,\theta)$ and harmonic in each face. □

For ease of notation, we will enumerate the vertices of Γ by $\{v_i\}$. Then a function $\rho : V(\Gamma) \to \mathbb{R}$ can be encoded by the vector $(\rho_i = \rho(v_i))$. If $v_i v_j \in E(\Gamma)$ is an edge, we will also define by $\theta_{ij} = \theta(v_i v_j)$.

Theorem 3.2. *The homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of nonsingular degree $\alpha > 0$ determined by $\rho : V(\Gamma) \to \mathbb{R}$ is balanced if and only if, for each vertex v_i,*

$$\sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0.$$

Here $v \sim w$ means that vertices v and w are adjacent, so there is an edge $vw \in E(\Gamma)$.

Proof. In each face $C(v_i v_j)$ choose coordinates adapted to v_i, so that the face is identified with $S_{\theta_{ij}}$ and $C(v_i)$ is identified with the positive x-axis. According to Lemma 2.10, in these coordinates $f|_{C(v_i v_j)}$ is represented by

$$u_j(r, \theta) = r^\alpha \left(\rho_i \cos(\alpha\theta) + \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} \sin(\alpha\theta) \right).$$

And the balancing formula of Definition 2.13 says that f is balanced along $C(v_i)$ if

$$\sum_{v_j \sim v_i} \frac{\partial u_j}{\partial \theta}(r, 0) = 0.$$

Combining these two formulas, we see that f is balanced along $C(v_i)$ if

$$\alpha r^\alpha \sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij}))} = 0.$$

In order to be balanced, f must be balanced along each edge of $C(\Gamma, \theta)$, so every sum of this form must vanish. □

Another way to interpret this result is to say a vector (ρ_j) determines a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of nonsingular degree $\alpha > 0$ if it is in the kernel of the matrix $\Delta_{\alpha\theta}$ with entries

$$(\Delta_{\alpha\theta})_{ij} = \begin{cases} -\sum_{v_k \sim v_i} \cot(\alpha\theta_{ik}) & \text{if } i = j, \\ \csc(\alpha\theta_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{else.} \end{cases}$$

For only certain values of α will $\Delta_{\alpha\theta}$ have a nontrivial kernel. That is, only certain degrees α admit a balanced homogeneous harmonic function.

Remark 3.3. In general one can find a balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to \mathbb{R}^d$ for any d (f may be the 0 map). Both the harmonicity and the balancing condition for f split into the corresponding conditions for the components f_j of $f = (f_1, \dots, f_d)$. By setting the components $\{f_j\}$ to be a basis of the d-dimensional kernel of $\Delta_{\alpha\theta}$ one finds the most geometrically interesting such maps. If one tries to map into a larger-dimensional space, or more generally lets the components of f be dependent, then the image of f lies in a subspace, not taking advantage of the full dimension of the target.

In special cases the collection of degrees α that admit balanced homogeneous harmonic functions is related to other discrete Laplace operators studied extensively in the literature. The following result is a direct generalization of Proposition 13 from [Daskalopoulos and Mese 2008], and also occurs as part of Theorem 3.5 of [Friedman and Tillich 2004].

Proposition 3.4. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$. If $\alpha\theta_0 \notin \pi\mathbb{Z}$ then there exist balanced harmonic functions of degree α if and only if $1 - \cos(\alpha\theta_0)$ is an eigenvalue of the normalized graph Laplacian $\mathcal{L}$ on Γ.*

Remark 3.5. Proposition 3.4 ignores the eigenvalue $\lambda = 0$ that $\mathcal{L}$ necessarily has, and the eigenvalue $\lambda = 2$ that $\mathcal{L}$ may have. These cases correspond to $\alpha\theta_0 = 2n\pi$ and $\alpha\theta_0 = (2n + 1)\pi$, respectively, and will be discussed in Section 3.2.

Proof. Suppose $\theta(e) = \theta_0$ for all e and that (ρ_i) is in the kernel of $\Delta_{\alpha\theta}$. For a fixed vertex $v_i \in V(\Gamma)$ with $\deg v_i$ neighbors,

$$\begin{aligned} 0 &= \sum_{v_j \sim v_i} (\csc(\alpha\theta_0)\rho_j - \cot(\alpha\theta_0)\rho_i) \\ &= \csc(\alpha\theta_0) \sum_{v_j \sim v_i} (\rho_j - \rho_i) + \deg(v_i)(\csc(\alpha\theta_0) - \cot(\alpha\theta_0))\rho_i, \end{aligned}$$

$$(1 - \cos(\alpha\theta_0))\rho_i = \frac{1}{\deg v_i} \sum_{v_j \sim v_i} (\rho_i - \rho_j).$$

Thus $1 - \cos(\alpha\theta_0)$ is an eigenvalue of the vertex-weighted Laplace matrix L with entries

$$L_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\deg v_i} & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \not\sim v_j. \end{cases}$$

Although this matrix L is not symmetric, it is similar to the symmetric normalized Laplace matrix $\mathcal{L}$ (see [Chung and Langlands 1996] or [Banerjee and Jost 2008]) with entries

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \not\sim v_j. \end{cases}$$

Since these matrices are similar they have the same eigenvalues. Their spectra have been studied extensively in the literature, for instance in [Chung 1997] and [Banerjee and Jost 2008]. □

For metrics given by general functions $\theta : E(\Gamma) \to (0, \pi)$, the question of which degrees α admit balanced homogeneous harmonic functions is more subtle. We will study the structure of $\Delta_{\alpha\theta}$ via the quadratic form it determines on $\mathbb{R}^{\#V(\Gamma)}$. This quadratic form will give information about the changing eigenvalues of $\Delta_{\alpha\theta}$. In particular, when an eigenvalue crosses 0 the operator $\Delta_{\alpha\theta}$ will have a nontrivial kernel.

Lemma 3.6. *For each fixed $\rho = (\rho_i) \not\equiv 0$ the quadratic form $\rho \cdot \Delta_{\alpha\theta}\rho$ is strictly increasing in α.*

Proof. First compute the quadratic form as follows:

$$\begin{aligned} \rho \cdot \Delta_{\alpha\theta}\rho &= \sum_i \rho_i \biggl(\sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} \biggr) \\ &= \sum_{v_i v_j \in E(\Gamma)} \biggl(\rho_i \frac{\rho_j - \rho_i \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} + \rho_j \frac{\rho_i - \rho_j \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \biggr) \\ &= \sum_{v_i v_j \in E(\Gamma)} \frac{2\rho_i\rho_j - \cos(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)}{\sin(\alpha\theta_{ij})} \\ &= \sum_{v_i v_j \in E(\Gamma)} (2\csc(\alpha\theta_{ij})\rho_i\rho_j - \cot(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)). \end{aligned}$$

Now the derivative with respect to α is easy to compute:

$$\begin{aligned} \frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) &= \sum_{v_i v_j \in E(\Gamma)} (\theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2) - 2\rho_i\rho_j\theta_{ij}\csc(\alpha\theta_{ij})\cot(\alpha\theta_{ij})) \\ &= \sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2 - 2\rho_i\rho_j\cos(\alpha\theta_{ij})) \\ &\ge \sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i - \rho_j)^2 \ge 0. \end{aligned}$$

Moreover suppose $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) = 0$ at some fixed $\rho = (\rho_i)$ and some fixed α. Then $\rho_i\rho_j\cos(\alpha\theta_{ij}) = \rho_i\rho_j$ for each $v_i v_j \in E(\Gamma)$, and also

$$\sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i - \rho_j)^2 = 0.$$

The expression on the left is clearly nonnegative, so for equality to hold ρ must be constant. And since α is nonsingular, $\cos(\alpha\theta_{ij}) \neq 1$ so that $\rho_i\rho_j = 0$ for all $v_iv_j \in E(\Gamma)$. Since ρ is constant, we have $\rho \equiv 0$. That is, the only way that $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) = 0$ at any α is if $\rho \equiv 0$. □

Theorem 3.7. *In an interval of nonsingular α, the eigenvalues of $\Delta_{\alpha\theta}$ are strictly increasing functions of α.*

Proof. The entries of the matrix $\Delta_{\alpha\theta}$ are analytic in α except where they are not defined, i.e., at singular α. Since we avoid such values, $\Delta_{\alpha\theta}$ is analytic in α. Moreover, $\Delta_{\alpha\theta}$ is symmetric for each real α. A result in the perturbation theory of eigenvalue problems (see [Rellich 1969, Section 1.1] or [Kato 1966, Theorem 6.1]) says that the eigenvalues and eigenvectors of $\Delta_{\alpha\theta}$ are also analytic in α. Namely, there are $\lambda_j(\alpha) \in \mathbb{R}$ and $w_j(\alpha) \in \mathbb{R}^{\#V(\Gamma)}$ depending analytically on α so that

$$\Delta_{\alpha\theta} w_j(\alpha) = \lambda_j(\alpha) w_j(\alpha).$$

Without loss of generality, $w_j(\alpha)$ is a unit vector for each j and each α. The eigenvalues can be recovered from the eigenvectors via the formula

$$\lambda_j(\alpha) = w_j(\alpha) \cdot \Delta_{\alpha\theta} w_j(\alpha).$$

Differentiating this identity with respect to α yields

$$\lambda_j'(\alpha) = 2w_j'(\alpha) \cdot \Delta_{\alpha\theta} w_j(\alpha) + w_j(\alpha) \cdot \frac{\partial \Delta_{\alpha\theta}}{\partial\alpha} w_j(\alpha).$$

As each $w_j(\alpha)$ is a unit vector, it lies in the unit sphere of $\mathbb{R}^{\#V(\Gamma)}$. Hence $w'(\alpha)$ is perpendicular to $w_j(\alpha)$. But $\Delta_{\alpha\theta} w_j(\alpha) = \lambda_j(\alpha) w_j(\alpha)$ is parallel to $w_j(\alpha)$, so the first term in $\lambda'(\alpha)$ vanishes. The second term is strictly positive by Lemma 3.6. Thus

$$\lambda_j'(\alpha) > 0.$$ □

An immediate consequence of Theorem 3.7 is that one can count the number of degrees α in an interval that admit balanced homogeneous harmonic functions in terms of the behavior of $\rho \cdot \Delta_{\alpha\theta}\rho$ at the endpoints of the interval.

Corollary 3.8. *Suppose $[\alpha_0, \alpha_1]$ is an interval of nonsingular degrees. If $\Delta_{\alpha_0\theta}$ has n_0 nonpositive eigenvalues and $\Delta_{\alpha_1\theta}$ has n_1 negative eigenvalues then there are $n_0 - n_1$ degrees $\alpha \in [\alpha_0, \alpha_1]$, counted with multiplicity, for which $\Delta_{\alpha\theta}$ has a nontrivial kernel.*

Remark 3.9. If $\Delta_{\alpha\theta}$ has a nontrivial kernel, the *multiplicity* of α is the dimension of that kernel. This is the number of independent balanced homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α, or equivalently the maximum dimension of the image of a balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to \mathbb{R}^d$.

Proof. Let the eigenvalues of $\Delta_{\alpha\theta}$ be $\{\lambda_j(\alpha)\}$ as in the proof of Theorem 3.7. The operator $\Delta_{\alpha\theta}$ can only have a nontrivial kernel if some $\lambda_j(\alpha) = 0$.

By Theorem 3.7 each $\lambda_j(\alpha)$ is strictly increasing in α. Let $\lambda_1(\alpha_0), \ldots, \lambda_{n_0}(\alpha_0)$ be the nonpositive eigenvalues of $\Delta_{\alpha_0\theta}$. For each of these eigenvalues there is at most one $\alpha \in [\alpha_0, \alpha_1]$ for which $\lambda_j(\alpha) = 0$; before this degree $\lambda_j < 0$ and after it $\lambda_j > 0$. The remaining eigenvalues, $\{\lambda_j(\alpha) \mid j > n_0\}$, all remain positive for $\alpha_0 \le \alpha \le \alpha_1$.

So if there are n_1 negative eigenvalues of $\Delta_{\alpha_1\theta}$ then $n_0 - n_1$ of the original n_0 nonpositive eigenvalues of $\Delta_{\alpha_0\theta}$ were 0 for some $\alpha \in [\alpha_0, \alpha_1]$. □

One can extend this result to open intervals whose endpoints are singular degrees. If $\alpha_0\theta_{ij} = n\pi$ for some edge $v_iv_j \in E_(\Gamma)$ and some $n \in \mathbb{Z}$, the term in the quadratic form $\rho \cdot \Delta_{(\alpha_0+t)\theta}\rho$ corresponding to v_iv_j reads

$$\begin{aligned}\frac{2\rho_i\rho_j - \cos((\alpha_0+t)\theta_{ij})(\rho_i^2+\rho_j^2)}{\sin((\alpha_0+t)\theta_{ij})} &= \frac{2\rho_i\rho_j - ((-1)^n + O(t^2))(\rho_i^2+\rho_j^2)}{(-1)^n\theta_{ij}t + O(t^3)} \\ &= -\frac{(\rho_i - (-1)^n\rho_j)^2 + O(t^2)}{\theta_{ij}t + O(t^3)}.\end{aligned}$$

If $\rho_i - (-1)^n\rho_j \neq 0$ then this term approaches ∞ as $t \to 0$ from the left, and approaches $-\infty$ as $t \to 0$ from the right. But if $\rho_i - (-1)^n\rho_j = 0$ then this term approaches 0 as $t \to 0$.

Suppose $\alpha_0\theta(e) \in \pi\mathbb{Z}$ for more than one edge $e \in E(\Gamma)$. For $\rho \cdot \Delta_{\alpha\theta}\rho$ to remain bounded near α_0, each edge v_iv_j with $\alpha_0\theta_{ij} = n_{ij}\pi$ imposes a relation $\rho_i - (-1)^{n_{ij}}\rho_j = 0$. These relations cut out a subspace $B(\alpha_0) \subset \mathbb{R}^{\#V(\Gamma)}$ on which $\rho \cdot \Delta_{\alpha\theta}\rho$ remains bounded for α near α_0. But, for any $\rho \notin B(\alpha_0)$, $\rho \cdot \Delta_{\alpha\theta}\rho$ satisfies

$$\lim_{\alpha \to \alpha_0^{\pm}} \rho \cdot \Delta_{\alpha\theta}\rho = \mp\infty.$$

Now suppose $\alpha_0 < \alpha_1$ are singular, but each $\alpha \in (\alpha_0, \alpha_1)$ is nonsingular. Let $B_j = B(\alpha_j)$ be the subspaces described above, and let $Q_j(\rho) = \lim_{\alpha\to\alpha_j} \rho \cdot \Delta_{\alpha\theta}\rho$ be a quadratic form defined for $\rho \in B_j$. Then the number of nonpositive eigenvalues of $\Delta_{\alpha\theta}$ for $\alpha = \alpha_0 + \epsilon$, ϵ sufficiently small, is given by the number of negative eigenvalues of Q_0 plus $\dim B_0^{\perp}$. Likewise the number of positive eigenvalues of $\Delta_{\alpha\theta}$ for $\alpha = \alpha_1 - \epsilon$ is given by the number of positive eigenvalues of Q_1 plus $\dim B_1^{\perp}$. The number of degrees $\alpha \in (\alpha_0, \alpha_1)$ admitting a balanced homogeneous harmonic function, counted with multiplicity, is the difference between these two numbers.

The behavior of $\Delta_{\alpha\theta}$ as $\alpha \to 0$ is of particular interest.

Proposition 3.10. *As $\alpha \to 0$ from the right, the limit of the operators $\alpha\Delta_{\alpha\theta}$ converge to an edge weighted Laplace operator on Γ with edge weights $w(e) = \frac{1}{\theta(e)}$. In* *particular $B(0)$ is the span of the constant vector $\mathbb{1}$, with $\mathbb{1}_i = 1$ for all i, and $Q_0 = 0$ on B_0.*

Proof. The limit of the quadratic forms determined by $\alpha\Delta_{\alpha\theta}$ is given by

$$\begin{aligned}\lim_{\alpha\to 0^+} \rho\cdot(\alpha\Delta_{\alpha\theta})\rho &= \lim_{\alpha\to 0^+} \sum_{v_iv_j\in E(\Gamma)} \frac{2\rho_i\rho_j - \cos(\alpha\theta_{ij})(\rho_i^2+\rho_j^2)}{\sin(\alpha\theta_{ij})/\alpha} \\ &= \sum_{v_iv_j\in E(\Gamma)} \frac{2\rho_i\rho_j - \rho_i^2 - \rho_j^2}{\theta_{ij}} \\ &= -\sum_{v_iv_j\in E(\Gamma)} \frac{(\rho_i-\rho_j)^2}{\theta_{ij}} \\ &= \rho\cdot\Delta\rho,\end{aligned}$$

where the operator Δ has entries

$$\Delta_{ij} = \begin{cases} \displaystyle\sum_{v_k\sim v_i} \frac{1}{\theta_{ik}} & \text{if } i=j, \\ \displaystyle\frac{1}{\theta_{ij}} & \text{if } v_i\sim v_j, \\ 0 & \text{else.}\end{cases}$$

This is precisely the edge-weighted Laplace operator described in the statement of the proposition.

Now if $\rho_i = c$ for all i then

$$\lim_{\alpha\to 0^+} \rho\cdot\Delta_{\alpha\theta}\rho = 0.$$

But for any other ρ,

$$\lim_{\alpha\to 0^+} \rho\cdot(\alpha\Delta_{\alpha\theta})\rho = -\sum_{v_iv_j\in E(\Gamma)} \frac{(\rho_i-\rho_j)^2}{\theta_{ij}} < 0,$$

so $\rho\cdot\Delta_{\alpha\theta}\rho \to -\infty$ as $\alpha\to 0$ from the right. □

We are now in a position to give a lower bound on the smallest degree $\alpha > 0$ of any balanced homogeneous harmonic function $f : C(\Gamma,\theta)\to\mathbb{R}$, in terms of the first eigenvalue of the normalized Laplace operator $\mathcal{L}$ with entries

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i=j, \\ \displaystyle -\frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } v_i\sim v_j, \\ 0 & \text{if } v_i\not\sim v_j.\end{cases}$$

Theorem 3.11. *Let* λ_1 *be the first positive eigenvalue of* $\mathcal{L}$, *and let*

$$\theta_{\max} = \max_{e \in E(\Gamma)} \theta(e).$$

If Γ *is not a complete graph and there is a balanced homogeneous harmonic function* $f : C(\Gamma, \theta) \to \mathbb{R}$ *of degree* α, *then*

$$\alpha \geq \frac{1}{\theta_{\max}} \arccos(1 - \lambda_1).$$

If $\Gamma = K_n$ *is the complete graph on n vertices then*

$$\alpha \geq \frac{2}{\theta_{\max}} \arctan \frac{n}{n-1}.$$

Proof. The operator $\mathcal{L}$ is positive semidefinite, which can be seen by the formula

$$w \cdot \mathcal{L} w = \sum_{v_i v_j \in E(\Gamma)} \left(\frac{w_i}{\sqrt{\deg v_i}} - \frac{w_j}{\sqrt{\deg v_j}} \right)^2.$$

Moreover, $\mathcal{L}$ has a kernel spanned by the vector $w = D^{1/2}\mathbb{1}$ with $w_i = \sqrt{\deg v_i}$. If Γ is not complete then $\lambda_1 \leq 1$, and if $\Gamma = K_n$ then $\lambda_1 = \frac{n}{n-1}$ (see [Chung 1997, Lemma 1.7]).

We will proceed by contradiction. In the case where Γ is not complete and $\lambda_1 \leq 1$, we first assume that $\alpha < \arccos(1 - \lambda_1)/\theta_{\max}$. The bound $\lambda_1 \leq 1$ also implies

$$\alpha \theta_{\max} < \frac{\pi}{2}.$$

Since the first singular α happens when $\alpha\theta_{\max} = \pi$, we are within the interval of nonsingular degrees with 0 as an endpoint.

Using the bound $\alpha\theta_{\max} < \pi/2$, bound the quadratic form $\rho \cdot \Delta_{\alpha\theta}\rho$ in terms of $\mathcal{L}$:

$$\begin{aligned}
\rho \cdot \Delta_{\alpha\theta}\rho &= \sum_{v_i v_j \in E(\Gamma)} \frac{2\rho_i \rho_j - \cos(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)}{\sin(\alpha\theta_{ij})} \\
&= \sum_{v_i v_j \in E(\Gamma)} \frac{1 - \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})}(\rho_i^2 + \rho_j^2) - \sum_{v_i v_j \in E(\Gamma)} \frac{(\rho_i - \rho_j)^2}{\sin(\alpha\theta_{ij})} \\
&\leq \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})} \sum_i \deg(v_i)\rho_i^2 - \frac{1}{\sin(\alpha\theta_{\max})} \sum_{v_i v_j \in E(\Gamma)} (\rho_i - \rho_j)^2 \\
&= \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})} \rho \cdot D\rho - \frac{1}{\sin(\alpha\theta_{\max})} \rho \cdot \Delta\rho \\
&= \frac{\rho}{\sin(\alpha\theta_{\max})} \cdot \big((1 - \cos(\alpha\theta_{\max}))D - \Delta\big)\rho \\
&= \frac{D^{1/2}\rho}{\sin(\alpha\theta_{\max})} \cdot (1 - \cos(\alpha\theta_{\max}) - \mathcal{L})D^{1/2}\rho.
\end{aligned}$$

Combining Theorem 3.7 with Proposition 3.10 implies $\Delta_{\alpha\theta}$ has at least one positive eigenvalue for each $0 < \alpha < \pi/\theta_{\max}$. By assumption $1 - \cos(\alpha\theta_{\max}) < \lambda_1$, so the quadratic form above has exactly one positive eigenvalue. The remaining eigenvalues are all strictly negative, and their corresponding eigenvectors span a hyperplane W.

If $\Delta_{\alpha\theta}$ had a second nonnegative eigenvalue then there would be a 2-dimensional space on which $\rho \cdot \Delta_{\alpha\theta}\rho$ is nonnegative. But such a plane necessarily intersects W, on which an upper bound for $\rho \cdot \Delta_{\alpha\theta}\rho$ is negative, a contradiction! Hence $\Delta_{\alpha\theta}$ cannot develop a kernel or a second positive eigenvalue for $0 < \alpha < \arccos(1 - \lambda_1)/\theta_{\max}$.

The proof in the case where $\Gamma = K_n$ and $\lambda_1 = \frac{n}{n-1}$ is similar. Now we assume $\alpha\theta_{\max} < 2\arctan\frac{n}{n-1}$. We are still in the interval before the first singular degree. We can still bound as before

$$\frac{1 - \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \leq \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})}$$

for each edge $v_i v_j$, but now we can only bound $\frac{1}{\sin(\alpha\theta_{ij})} \geq 1$. Thus our bound on the quadratic form now reads

$$\begin{aligned}
\rho \cdot \Delta_{\alpha\theta}\rho &\leq \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})}\rho \cdot D\rho - \rho \cdot \Delta\rho \\
&= \tan\left(\frac{\alpha\theta_{\max}}{2}\right)\rho \cdot D\rho - \rho \cdot \Delta\rho \\
&= D^{1/2}\rho \cdot \left(\tan\left(\frac{\alpha\theta_{\max}}{2}\right) - \mathcal{L}\right)D^{1/2}\rho.
\end{aligned}$$

Just as in the previous case, $\rho \cdot \Delta_{\alpha\theta}\rho$ has at least one positive eigenvalue, and our assumed bound on α implies the quadratic form on the right-hand side has exactly one positive eigenvalue. Thus $\rho \cdot \Delta_{\alpha\theta}\rho$ cannot have developed a kernel or second positive for $0 < \alpha < \frac{2}{\theta_{\max}} \arctan\frac{n}{n-1}$. □

3.2. ***Singular degrees.*** Let us begin with the extension of Proposition 3.4, where $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$. When $\alpha\theta_0 \notin \pi\mathbb{Z}$, that proposition equated $1 - \cos(\alpha\theta_0)$ with the eigenvalues of $\mathcal{L}$. This leaves out the possible eigenvalues of 0 and 2.

The normalized Laplacian $\mathcal{L}$ always has an eigenvalue $\lambda = 0$, as $D^{1/2}\mathbb{1}$ is in the kernel of $\mathcal{L}$. This corresponds to $\alpha\theta_0 = 2n\pi$ for $n \in \mathbb{Z}$; see Theorem 3.13.

The operator $\mathcal{L}$ may also have an eigenvalue $\lambda = 2$, depending on the structure of Γ. In fact, 2 is an eigenvalue of $\mathcal{L}$ precisely when Γ has no odd cycles. The case $\lambda = 2$ corresponds with $\alpha\theta_0 = (2n+1)\pi$; see Theorem 3.14.

Before approaching the balancing conditions in Theorems 3.13 and 3.14 we will first parametrize the space of homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ in the case $\alpha\theta_0 \in \pi\mathbb{Z}$.

Lemma 3.12. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha\theta_0 = n\pi$ for $n \in \mathbb{Z}$, $n > 0$. If n is even, or if n is odd and Γ contains no odd cycles, then a homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α is uniquely determined by a number ρ_0 and a function $c_2 : E(\Gamma) \to \mathbb{R}$. If n is odd and Γ contains an odd cycle then a homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α is uniquely determined by just a function $c_2 : E(\Gamma) \to \mathbb{R}$.*

Proof. Begin by arbitrarily orienting all the edges of Γ. On each face $C(e) \subset C(\Gamma, \theta)$ choose coordinates adapted to the tail of e, identifying $C(e)$ with S_{θ_0} so that the tail corresponds to the positive x-axis and the head corresponds to the line $x = y \cot \theta_0$. In these coordinates Lemma 2.9 says a homogeneous harmonic function is represented by a function

$$u_e(r, \theta) = r^\alpha (c_1(e) \cos(\alpha\theta) + c_2(e) \sin(\alpha\theta)).$$

In the spirit of Lemma 3.1, we can let $\rho : V(\Gamma) \to \mathbb{R}$ be given by $\rho(v) = f(v, 1)$. If edge e has tail v_1 and head v_2, then Lemma 2.10 says that $c_1 = \rho(v_1)$ and $c_1 = \rho(v_1)\cos(\alpha\theta_0)$. If n is even this means that $\rho(v_i) = \rho(v_j)$ for each edge $v_i v_j \in E(\Gamma)$. As Γ is connected, we must have $\rho \equiv \rho_0$ for some constant ρ_0. If n is odd we must have $\rho_i = -\rho_j$ for each edge $v_i v_j \in E(\Gamma)$. If Γ contains no odd cycles then it is bipartite, so we may set $\rho = \rho_0$ on one part of Γ and $\rho = -\rho_0$ on the other part. But an odd cycle in Γ would force $\rho \equiv 0$.

The paragraph above explains the parameter ρ_0 in the cases when n is even or Γ contains no odd cycle, and its absence in the case when n is odd and Γ contains an odd cycle. Once the constants $c_1(e)$ are so specified, the coefficients $c_2(e)$ in front of $\sin(\alpha\theta)$ can be anything and the function f is well-defined and continuous. □

If f is determined by any constant ρ_0 (with $\rho_0 = 0$ in case n is odd and Γ contains an odd cycle) and the function $c_2 \equiv 0$, then f is already balanced because

$$\frac{\partial u_e}{\partial \theta}(r, 0) = \frac{\partial u_e}{\partial \theta}(r, \theta_0) = 0.$$

Considering a more general function $c_2 : E(\Gamma) \to \mathbb{R}$, two linear maps become relevant. In case n is even, we define ∂c_2, and in case n is odd we define dc_2. Separating out a single edge e_1, we will define both ∂ and d on the function

$$c_2(e) = \begin{cases} 1 & \text{if } e = e_1, \\ 0 & \text{if } e \neq e_1. \end{cases}$$

If e_1 has tail v_1 and head v_2, then define

$$\partial c_2(v) = \begin{cases} 1 & \text{if } v = v_2, \\ -1 & \text{if } v = v_1, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \qquad dc_2(v) = \begin{cases} 1 & \text{if } v = v_1 \text{ or } v_2, \\ 0 & \text{else.} \end{cases}$$

Define the maps ∂ and d for arbitrary $c_2 : E(\Gamma) \to \mathbb{R}$ by linearity.

The next two theorems expand on a result of Theorem 3.5 in [Friedman and Tillich 2004].

Theorem 3.13. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha\theta_0 = 2n\pi$ for some $n \in \mathbb{Z}$, $n > 0$. Then the space of balanced homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α has dimension $\#E(\Gamma) - \#V(\Gamma) + 2$.*

Proof. According to Lemma 3.12 a homogeneous harmonic function is determined by $\rho_0 \in \mathbb{R}$ and $c_2 : E(\Gamma) \to \mathbb{R}$. The function f is balanced according to Definition 2.13 if and only if $\partial c_2 = 0$. It thus remains to compute the dimension of the kernel of ∂, or equivalently $\#E(\Gamma) - \dim \operatorname{im} \partial$.

The map ∂ turns out to be the boundary operator from homology, whose structure is well understood. For the sake of completeness we provide a description of its image here as the space of functions $g : V(\Gamma) \to \mathbb{R}$ with $\sum_{v \in V(\Gamma)} g(v) = 0$. From the structure of ∂ it is clear that the image is contained in this space.

Fix a vertex v_1, and a collection of paths p_j from v_1 to v_j for each other vertex v_j. Without loss of generality assume that the orientation along each path points from v_1 towards v_j. Let $\xi_j(e) = 1$ for each edge $e \in p_j$ and $\xi_j(e) = 0$ otherwise, so that

$$\partial \xi_j(v) = \begin{cases} 1 & \text{if } v = v_j, \\ -1 & \text{if } v = v_1, \\ 0 & \text{else.} \end{cases}$$

For any function $g : V(\Gamma) \to \mathbb{R}$ with $\sum_j g(v_j) = 0$, let $\xi = \sum_{j \geq 2} g(v_j)\xi_j$, so that

$$\partial \xi(v_j) = \begin{cases} g(v_j) & \text{if } j \geq 2, \\ -\sum_{i \geq 2} g(v_i) & \text{if } j = 1. \end{cases}$$

Since g satisfies $\sum_j g(v_j) = 0$, we necessarily have $\partial \xi(v_1) = g(v_1)$ also.

The image of ∂ thus has dimension $\#V(\Gamma) - 1$, so its kernel has dimension $\#E(\Gamma) - \#V(\Gamma) + 1$. Combining with the choice of ρ_0, the space of balanced homogeneous harmonic functions of degree α is $\#E(\Gamma) - \#V(\Gamma) + 2$. □

Theorem 3.14. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha\theta_0 = (2n + 1)\pi$ for some $n \in \mathbb{Z}$, $n \geq 0$. Then the space of balanced homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α has dimension $\#E(\Gamma) - \#V(\Gamma) + 2$ if Γ contains no odd cycles, and dimension $\#E(\Gamma) - \#V(\Gamma)$ if Γ does contain an odd cycle.*

Proof. According to Lemma 3.12 a homogeneous harmonic function is determined by $\rho_0 \in \mathbb{R}$ and $c_2 : E(\Gamma) \to \mathbb{R}$. In case Γ contains an odd cycle, though, we must have $\rho_0 = 0$. The function f is balanced according to Definition 2.13 if and only if $dc_2 = 0$. It thus remains to compute the dimension of the kernel of d, or equivalently $\#E(\Gamma) - \dim \operatorname{im} d$.

In case Γ contains an odd cycle we claim the image of the linear map d consists of all functions $g : V(\Gamma) \to \mathbb{R}$. With the presence of a single odd cycle, one can find a cycle of odd length starting and ending at any particular vertex. For a vertex v_j, choose such a path and let $\xi_j(e) = \pm 1$ on the edges of the path in an alternating fashion, so that

$$d\xi_j(v) = \begin{cases} 2 & \text{if } v = v_j, \\ 0 & \text{else.} \end{cases}$$

Defining $\xi = \frac{1}{2}\sum_j g(v_j)\xi_j$, we see that $d\xi = g$. In this case we had no choice of ρ_0, so the dimension of balanced homogeneous harmonic functions is just

$$\dim\ker d = \#E(\Gamma) - \dim\operatorname{im} d = \#E(\Gamma) - \#V(\Gamma).$$

If Γ does not contain an odd cycle then the graph is bipartite. Say the parts of Γ are $A, B \subset V(\Gamma)$, with $A \cup B = V(\Gamma)$, $A \cap B = \varnothing$, and every edge of Γ has one endpoint in A and one endpoint in B. Then we claim the image of d consists of those functions $g : V(\Gamma) \to \mathbb{R}$ with

$$\sum_{v\in A} g(v) = \sum_{v\in B} g(v).$$

Assuming G is connected, one can find odd-length paths p from any vertex in A to any vertex in B. If the path p joins $v_1 \in A$ and $v_2 \in B$, then defining $\xi_p(e) = \pm 1$ along p in an alternating fashion gives

$$d\xi_p(v) = \begin{cases} 1 & \text{if } v = v_1, \\ 1 & \text{if } v = v_2, \\ 0 & \text{else.} \end{cases}$$

Start with $\xi \equiv 0$ and build up as follows. Enumerate the vertices in A by v_j and the vertices in B by w_j, and without loss of generality $\#A = a \geq b = \#B$. First select odd-length paths $p_1, \ldots, p_{a-b}$ from $v_{b+1}, \ldots, v_a$ to w_1 and add $g(v_j)\xi_{p_j}$ to ξ for $b < j \leq a$. Then choose paths from w_1 to v_1, v_1 to w_2, w_2 to v_2, ..., w_b to v_b. Adding appropriate multiples of the corresponding functions ξ_p to ξ will ensure that $d\xi(v) = g(v)$ for all v except perhaps v_a. But the condition that $\sum_{v\in A} g(v) = \sum_{v\in B} g(v)$ will ensure that $d\xi(v_a) = g(v_a)$ as well.

Thus in this case the image of d has dimension $\#V(\Gamma)-1$, so its kernel has dimension $\#E(\Gamma) - \#V(\Gamma) + 1$. Combining with the choice of ρ_0, the space of balanced homogeneous harmonic functions of degree α has dimension $\#E(\Gamma)-\#V(\Gamma)+2$. □

Remark 3.15. Theorem 3.13 can be seen as a corollary of Theorem 3.14 after subdividing edges. In fact, all that is really needed is the statement when $\alpha\theta_0 = \pi$. If $\alpha\theta_0 = k\pi$ simply subdivide each edge into k pieces to reduce to the case $\alpha\theta_0 = \pi$. Lemma 2.14 then implies a balanced function on the subdivided complex is smooth at the introduced vertices and is thus the restriction of a function on the unsubdivided complex.

We now turn to the more complicated situation when $\theta : E(\Gamma) \to (0, \pi)$ is not constant but α is still a singular degree. Let $\Sigma \subset \Gamma$ be the subgraph consisting of all those edges $e \in E(\Gamma)$ with $\alpha\theta(e) \in \pi\mathbb{Z}$, along with their incident vertices. One can see, as in Lemma 3.12, that a homogeneous harmonic function is determined by a function $\rho : V(\Gamma) \to \mathbb{R}$ that lies in the space $B(\alpha)$ described in the previous subsection, along with a function $c_2 : E(\Sigma) \to \mathbb{R}$. Then one could describe which ρ and c_2 describe a balanced function. But it is simpler to use the strategy described in Remark 3.15.

For each edge $e \in E(\Sigma)$, if $\alpha\theta(e) = k\pi$ then subdivide e into k edges, each with $\alpha\theta = \pi$. This turns the original complex $C(\Gamma, \theta)$ into an isometric complex (by abuse of notation also denoted $C(\Gamma, \theta)$) where each edge $e \in E(\Gamma)$ satisfies either $\alpha\theta(e) = \pi$ or $\alpha\theta(e) \notin \pi\mathbb{Z}$. Now we can describe homogeneous harmonic functions.

Lemma 3.16. *A homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of singular degree α is uniquely determined by a function $c_2 : E(\Sigma) \to \mathbb{R}$ together with a function $\rho : V(\Gamma) \to \mathbb{R}$ such that $\rho(v_i) = -\rho(v_j)$ for all $v_i v_j \in E(\Sigma)$.*

Proof. Just as in Lemma 3.1, the function f is uniquely determined on those edges $e \in E(\Gamma) \setminus E(\Sigma)$ by the values $\rho(v)$ on the incident vertices. And just as in Lemma 3.12, the condition $\rho_i = -\rho_j$ together with a choice of $c_{ij} = c_2(v_i v_j)$ on $v_i v_j \in E(\Sigma)$ uniquely determines f in the corresponding face via the representation

$$u(r, \theta) = r^\alpha(\rho_i \cos(\alpha\theta) + c_{ij} \sin(\alpha\theta)). \qquad \square$$

Now the balancing condition is slightly more subtle. On vertices $v_i \in V(\Gamma) \setminus V(\Sigma)$ one must still have

$$\sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0,$$

just as in Theorem 3.2. Unfortunately these conditions alone seem unlikely to be satisfied for $\rho \not\equiv 0$, given the restrictions on ρ coming from Σ. Certainly a necessary condition is that the quadratic form $Q(\rho) = \lim_{\alpha' \to \alpha} \rho \cdot \Delta_{\alpha'\theta}\rho$ on the space $B(\alpha)$ from the previous subsection should have a nontrivial kernel. Since the eigenvalues of $\Delta_{\alpha'\theta}$ are strictly increasing with α' the possibility that one passes through 0 precisely at $\alpha' = \alpha$ seems unlikely.

On those vertices $v_i \in V(\Sigma)$ a combination of the formula from Theorem 3.2 and the linear map d from above is necessary, which we'll define as $\sigma : \mathbb{R}^{\#E(\Sigma)} \to \mathbb{R}^{\#V(\Sigma)}$. The balancing condition says

$$dc_2(v_i) + \sum_{v_i v_j \in E(\Gamma) \setminus E(\Sigma)} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0.$$

Proposition 3.17. *If Σ has a connected component that is not a tree but contains no odd cycles, then there are nontrivial balanced homogeneous harmonic functions of degree α.*

Proof. To produce a nontrivial balanced homogeneous harmonic function of a singular degree α it is enough, according to Lemma 3.16, to specify ρ and $c_2 : E(\Sigma) \to \mathbb{R}$. We will choose $\rho \equiv 0$, and if $\Sigma_0 \subset \Sigma$ is a connected component that is not a tree but contains no odd cycles we set $c_2 = 0$ on $E(\Sigma) \setminus E(\Sigma_0)$.

We will construct a balanced function by specifying $c_2 \not\equiv 0$ on $E(\Sigma_0)$ in the kernel of d. To prove that such a c_2 exists, it suffices to show that the image of d when restricted to functions $c_2 : E(\Sigma_0) \to \mathbb{R}$ has dimension at most $\#E(\Sigma_0) - 1$. But this is precisely the content of Theorem 3.14; in the absence of odd cycles the image of d acting on $\mathbb{R}^{\#E(\Sigma_0)}$ has dimension $\#V(\Sigma_0) - 1$. As Σ_0 is not a tree, $\#E(\Sigma_0) \geq \#V(\Sigma_0)$, so d has a kernel and a balanced homogeneous harmonic function exists. □

3.3. ***Eigenvalue problems on graphs.*** The balanced homogeneous harmonic functions of this section can be reinterpreted in terms of an eigenvalue problem on the graph Γ. Returning to the notation of Definition 2.2, write $f : C(\Gamma, \theta) \to \mathbb{R}$ as

$$f(x, t) = \rho(x) t^{\alpha}.$$

Here x is a point on Γ and $t \in \mathbb{R}_{\geq 0}$. On each edge e determine coordinates by a map $\psi_e : e \to [0, \theta(e)] \subset \mathbb{R}$, and define $\rho_e : [0, \theta(e)] \to \mathbb{R}$ by

$$\rho_e(s) = \rho \circ \psi_e^{-1}(s).$$

Now the original function $f : C(\Gamma, \theta) \to \mathbb{R}$ is harmonic if and only if each ρ_e is an eigenfunction of the second derivative operator, namely $\rho_e'' = -\alpha^2 \rho_e$. And the original function f is balanced if and only if, for each $v \in V(\Gamma)$,

$$\sum_{v \in e} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi_e(v)) = 0.$$

Thus a balanced homogeneous harmonic function f corresponds to a solution of the eigenvalue problem $\rho_e'' = -\alpha^2 \rho_e$ for all $e \in E(\Gamma)$ with the boundary conditions stated above at each $v \in V(\gamma)$.

This eigenvalue problem was studied extensively in [Friedman and Tillich 2004], and existence of solutions was studied variationally by way of the Rayleigh quotient,

$$R[\rho] = \frac{\sum_{e \in E(\Gamma)} \int_0^{\theta(e)} |\rho_e'(s)|^2 \, ds}{\sum_{e \in E(\Gamma)} \int_0^{\theta(e)} |\rho_e(s)|^2 \, ds}.$$

In particular, they show:

Theorem 3.18 [Friedman and Tillich 2004, Theorem 3.2]. *There exist solutions ρ_i of degree α_i to the eigenvalue problem on Γ. The degrees α_i are increasing and tend to ∞, and the ρ_i form a complete orthonormal basis for $L^2(\Gamma)$.*

Friedman and Tillich [2004, Theorem 3.3] proved a Weyl law. In our notation, if N_α counts the number of degrees $t \le \alpha$, with their multiplicities, that occur as degrees of balanced homogeneous harmonic functions, then there is a constant C so that

$$\left| N_\alpha - \frac{\alpha\Theta}{\pi} \right| \le C.$$

Here $\Theta = \sum_{e \in E(\Gamma)} \theta(e)$ is the total length of the graph.

It is interesting to compare this Weyl law with our bounds on the eigenvalues, e.g., in Corollary 3.8. In case $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$, there are precisely $1 + \left\lfloor \frac{\alpha\theta_0}{\pi} \right\rfloor$ singular degrees of the form $\frac{k\pi}{\theta_0}$ between 0 and α. From Theorems 3.13 and 3.14 we know that one eigenvalue of $\Delta_{t\theta}$ approaches 0 as $t \to \frac{k\pi}{\theta_0}$ and the rest approach $\pm\infty$, unless k is odd and Γ is not bipartite in which case all eigenvalues of $\Delta_{t\theta}$ approach $\pm\infty$. Including the balanced homogeneous functions of singular degrees counted in Theorems 3.13 and 3.14, one can say in this case that

$$\left| N_\alpha - \frac{\alpha\Theta}{\pi} \right| \le \#E(\Gamma).$$

In case $\theta(e)$ is not constant over $E(\Gamma)$, the analysis is more subtle. In the most generic case, when $\theta(e) \ne \theta(e')$ whenever $e \ne e'$, $\Delta_{t\theta}$ becomes singular whenever $t\theta_{ij} = k\pi$ for some edge $v_i v_j \in E(\Gamma)$ and $k \in \mathbb{Z}$. Near such a singular degree, $\Delta_{t\theta}\rho$ remains bounded if and only if $\rho_i = (-1)^k \rho_j$. That is, $\Delta_{\alpha\theta}$ remains bounded on a space of codimension 1. It is hard to predict when any particular eigenvalue of $\Delta_{\alpha\theta}$ passes through 0, but since the eigenvalues that remain bounded are continuous across the singular degrees, each time an eigenvalue approaches ∞, it must have passed through 0 at some point. The only exception is the first eigenvalue that approaches ∞ might be the one that was an eigenvalue of $\Delta_{0\theta}$.

For fixed $\alpha > 0$, there are a total of $\sum_{v_i \sim v_j} \left\lfloor \frac{\alpha\theta_{ij}}{\pi} \right\rfloor$ singular degrees between 0 and α, each one contributing at least one balanced eigenfunction. Between the last singular degree and α there is some number $b \le \#V(\Gamma)$ of additional balanced eigenfunctions.

If S_α is the number of balanced eigenfunctions of singular degrees that don't arise as limits of unbalanced eigenfunctions from nearby degrees, then we have

$$N_\alpha = \sum_{v_i \sim v_j} \left\lfloor \frac{\alpha\theta_{ij}}{\pi} \right\rfloor + S_\alpha + b \ge \frac{\alpha\Theta}{\pi} - \#E(\Gamma) + S_\alpha.$$

From the Weyl bound, $N_\alpha \le \frac{\alpha\Theta}{\pi} + C$ for some constant C, and hence

$$S_\alpha \le \#E(\Gamma) + C.$$

In other words, there are only finitely many balanced eigenfunctions that occur spontaneously at singular degrees.

4. Balanced harmonic maps into k-pods

After subdividing the domain if necessary a homogeneous map $f : C(\Gamma, \theta) \to C(k)$ maps each face $C(e) \subset C(\Gamma, \theta)$ to a single edge $C(v_k) \subset C(k)$. The map f is harmonic if its restriction to each face, when viewed in coordinates, is a harmonic function $u : S_{\theta(e)} \to \mathbb{R}_{\geq 0}$. According to Proposition 2.11 such a map exists if and only if $0 < \alpha\theta(e) \leq \pi$ for each edge $e \in E(\Gamma)$.

The balancing condition of Definition 2.15 imposes further restrictions on which degrees α can occur for harmonic maps from smooth cones into k-pods. In fact the degree α and the geometry of the cone are closely related.

Theorem 4.1. *A balanced homogeneous harmonic map $f : C(C_{n'}, \theta') \to C(k)$ of degree α exists if and only if the cone is a subdivision of $C(C_n, \theta)$ where $\theta(e) = \theta_0$ for each edge $e \in E(C_n)$, and*

$$\alpha\theta_0 = \pi.$$

Remark 4.2. When $k = 2$, Theorem 4.1 can be seen to say that the nodal domains of a homogeneous harmonic function $f : \mathbb{R}^2 \to \mathbb{R}$ are equiangular sectors.

Proof. First suppose that there is a balanced homogeneous harmonic map $f : C(C_{n'}, \theta') \to C(k)$ and $f(v_0, 1) = 0$ is the vertex of $C(k)$ for some fixed $v_0 \in V(C_{n'})$. Enumerate the vertices of $C_{n'}$ in order as $\{v_0, v_2, \dots, v_{n'-1}\}$, so $v_i \sim v_{(i+1) \pmod n}$ for each i. In each face $C(v_i v_{i+1}) \subset C(C_{n'}, \theta')$ take coordinates adapted to v_i and represent f in those coordinates by a harmonic function $u_i : S_{\theta(v_i v_{i+1})} \to \mathbb{R}_{\geq 0}$.

Since $f(v_0, 1) = 0$ is the vertex of the k-pod $C(k)$ we must have $\rho(v_0) = 0$. The map u_0 that represents f in $C(v_0 v_1)$ thus has the form

$$u_0(r, \theta) = c_1 r^\alpha \sin(\alpha\theta).$$

If $\alpha\theta(v_0 v_1) = \pi$ then $u_0(r, \theta(v_0 v_1)) = 0$ as well. Otherwise $u_0(r, \theta(v_0 v_1)) \neq 0$ and according to Lemma 2.16 the map u_1 must have the form

$$u_1(r, \theta) = c_1 r^\alpha \sin(\alpha\theta + \alpha\theta(v_0 v_1)).$$

Continuing inductively in this fashion, as long as $u_{i-1}(r, \theta(v_{i-1} v_i)) \neq 0$ Lemma 2.16 says the map u_i representing f in $C(v_i v_{i+1})$ must have the form

$$u_i(r, \theta) = c_1 r^\alpha \sin(\alpha\theta + \alpha\theta(v_0 v_1) + \cdots + \alpha\theta(v_{i-1} v_i)).$$

Since f can only map an edge of $C(C_{n'}, \theta')$ to the vertex of $C(k)$, some u_m must vanish at $(r, \theta(v_m v_{m+1}))$. In this case we have

$$\alpha \sum_{i=1}^{r} \theta(v_i v_{i+1}) = \pi.$$

We can "unsubdivide" the first r edges of $c_{n'}$ to form an edge of a new graph with $\alpha\theta = \pi$ on that edge. Repeating the same argument starting at v_{r+1}, and at each successive vertex of $C_{n'}$ that f maps to the vertex of $C(k)$, one sees that $C(C_{n'}, \theta')$ is a subdivision of some $C(C_n, \theta)$ with $\alpha\theta(e) = \pi$ for each $e \in E(C_n)$.

It remains to see that a balanced homogeneous harmonic map of degree α does exist from $C(C_n, \theta)$ when $\alpha\theta(e) = \pi$ for each $e \in E(C_n)$. Let $\theta_0 = \pi/\alpha$ be the common angle measure in each face of $C(C_n, \theta)$. In each face of $C(C_n, \theta)$ let f be represented in coordinates by the function

$$u(r, \theta) = cr^\alpha \sin(\alpha\theta).$$

Then f will map *each edge* of $C(C_n, \theta)$ into the vertex of $C(k)$. And the balancing condition is satisfied precisely because the constant c in each face is the same. □

Remark 4.3. The process of "unsubdividing" described in this proof may result in a cone $C(C_n, \theta)$ with $\theta(e) \geq \pi$. This is not consistent with Definition 2.5 but it causes only notational difficulty. In particular, if $\theta(e) = \theta_0 \geq \pi$ one can identify the face $C(e)$ with the upper half-plane endowed with a different metric. Indeed, let

$$S = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \quad \text{with metric } ds^2 = dr^2 + \frac{\theta_0^2 r^2}{\pi^2} d\theta^2.$$

This S has an angle of θ_0 at the origin, and the harmonic functions of degree α defined on S have the form

$$u(r, \theta) = r^\alpha (c_1 \cos(\theta) + c_2 \sin(\theta)).$$

In order that u stays positive one must have $c_1 = 0$ and $c_2 > 0$, which is consistent with the functions described in the above proof.

Remark 4.4. According to our assumption from Section 2.3.2, if $v \in V(C_n)$ has incident edges e_1 and e_2, and $f(v, 1)$ is the vertex of $C(k)$, then the images of $f|_{C(e_1)}$ and $f|_{C(e_2)}$ should be distinct edges of $C(k)$. This is a common assumption in applications, and is consistent with the maximum principle for harmonic maps that would prohibit a local minimum value of 0 along an edge, thus ruling out the possibility $k = 1$. With this additional restriction, the number n from the above proof must be at least 3, unless the image of f lies in only two edges of the k-pod (in particular if $k = 2$).

The application we list here was established in [Gromov and Schoen 1992].

Corollary 4.5. *Any homogeneous harmonic map $f : \mathbb{R}^2 \to C(k)$ into a k-pod either has degree* 1 *or degree $\alpha \geq \frac{3}{2}$. If $C(C_n, \theta)$ is a smooth cone whose vertex is positively curved, then any homogeneous harmonic map $f : C(C_n, \theta) \to C(k)$ has degree $\alpha > 1$, and if the total angle of $C(C_n, \theta)$ is at most 3π then any map that visits at least three edges of $C(k)$ has degree $\alpha \geq 1$.*

Proof. We begin by remarking that $\mathbb{R}^2$ is isometric to $C(C_{n'}, \theta')$ whenever

$$\sum_{e \in E(C_{n'})} \theta'(e) = 2\pi.$$

According to Theorem 4.1, the degree α of a homogeneous harmonic map $f : C(C_n, \theta) \to C(k)$ must satisfy

$$\alpha \sum_{e \in E(C_{n'})} \theta'(e) = n\pi,$$

where n is the number of edges of $C(k)$ visited by f counted with multiplicity. By Remark 4.4 we must have $n \geq 2$, or $n \geq 3$ if f visits at least three edges of $C(k)$.

If the vertex of $C(C_{n'}, \theta')$ is flat, this means $\alpha = n/2$. When $n = 2$ we have $\alpha = 1$, and when $n \geq 3$ we have $\alpha \geq \frac{3}{2}$.

If the vertex of $C(C_{n'}, \theta')$ is positively curved then $\sum_e \theta'(e) < 2\pi$ so

$$\alpha = \frac{n\pi}{\sum_{e \in E(C_{n'})} \theta'(e)} > \frac{n}{2}.$$

Again by Remark 4.4 $n \geq 2$ so that $\alpha > 1$.

And in fact if

$$\sum_{e \in E(C_{n'})} \theta'(e) \leq 3\pi \quad \text{and} \quad n \geq 3,$$

then

$$\alpha = \frac{n\pi}{\sum \theta'(e)} \geq 1.$$

□

4.1. ***p-harmonic maps.*** Just as in the previous subsection, we consider p-harmonic maps $f : C(C_n, \theta) \to C(k)$. The basic object of study is analogous to the nonnegative harmonic functions described in Proposition 2.11. The existence of homogeneous p-harmonic functions with Dirichlet boundary conditions in a plane sector were studied in [Tolksdorf 1983] and later in [Porretta and Véron 2009]. Akman, Lewis, and Vogel describe in [Akman et al. 2019] the relationship between the sector S_{θ_0} and the degree α of homogeneity of p-harmonic functions described in [Tolksdorf 1983] and [Porretta and Véron 2009].

Translating the description of [Akman et al. 2019] into the present notation, for each $p > 1$ and each angle $0 < \theta_0 < 2\pi$ there is a unique degree $\alpha > 0$ and a unique (up to scaling) homogeneous p-harmonic function $\phi_\alpha : S_{\theta_0} \to \mathbb{R}_{\geq 0}$ of degree α with $\phi_\alpha(r, 0) = \phi_\alpha(r, \theta_0) = 0$. (There is also a unique degree $\alpha < 0$ with an associated p-harmonic function, but we will not consider that here.) The angle θ_0 and the degree α are related by the equation

$$1 - \frac{\theta_0}{\pi} = \frac{\alpha - 1}{\sqrt{\alpha^2 + \frac{2-p}{p-1}\alpha}}.$$

Lemma 4.6. *The relationship between θ_0 and α is monotone. That is, if $\theta_0 < \theta_1$ support homogeneous p-harmonic functions of degrees α_0 and α_1, respectively, then $\alpha_0 > \alpha_1$.*

Proof. Simply differentiate the equation relating θ_0 and α. This yields

$$-\frac{1}{\pi} = \frac{p\alpha + 2 - p}{2(p-1)\big(\alpha^2 + \frac{2-p}{p-1}\alpha\big)^{3/2}} \cdot \frac{d\alpha}{d\theta_0}.$$

Now, in order that the square root in the equation relating θ and α is real, we must have $\alpha > \frac{p-2}{p-1}$. This rearranges to say that $p\alpha + 2 - p > \alpha$, so the derivative computation above rearranges to

$$\frac{d\alpha}{d\theta_0} < -\frac{2(p-1)\big(\alpha^2 + \frac{2-p}{p-1}\alpha\big)^{3/2}}{\alpha\pi} < 0. \qquad \square$$

Theorem 4.7. *Any homogeneous p-harmonic map $f : \mathbb{R}^2 \to C(k)$ into a k-pod either has degree 1 or degree $\alpha \geq \frac{9}{8}$. If $C(C_n, \theta)$ is a smooth cone whose vertex is positively curved, then any homogeneous p-harmonic map $f : C(C_n, \theta) \to C(k)$ has degree $\alpha > 1$, and if the total angle of $C(C_n, \theta)$ is at most 3π then any map that visits at least three edges of $C(k)$ has degree $\alpha \geq 1$.*

Proof. Just as in Theorem 4.1 we can construct homogeneous p-harmonic maps $f : C(C_n, \theta) \to C(k)$ essentially (i.e., up to subdivision) only when $\theta(e) = \theta_0$ for all $e \in E(C_n)$. In each face $C(e)$, f is represented in coordinates by $c(e)\phi_\alpha$, where α is the degree related to the angle θ_0 in the above equation. Then the balancing condition of Definition 2.15 says that f is balanced if and only if the constants $c(e)$ do not depend on $e \in E(\Gamma)$, that is, $c(e) = c$ for all e.

If the domain is isometric to $\mathbb{R}^2$, i.e., if the vertex of $C(C_n, \theta)$ is flat, that means

$$\sum_{e \in E(C_n)} \theta(e) = n\theta_0 = 2\pi.$$

When $n = 2$ and $\theta_0 = \pi$ the corresponding degree of a p-harmonic function $u : S_\pi \to \mathbb{R}_{\geq 0}$ is $\alpha = 1$. If $n \geq 3$ then $\theta_0 \leq 2\pi/3$. According to Lemma 4.6 smaller angles θ_0 correspond to larger degrees α, so the smallest degree happens when $\theta_0 = 2\pi/3$. Solving the relationship between θ_0 and α in the case $\theta_0 = 2\pi/3$ gives

$$\alpha = \frac{17p - 16 + \sqrt{p^2 + 32p - 32}}{16(p-1)}.$$

This is a decreasing function of p so it is bounded below by its limit as $p \to \infty$, namely $\alpha \geq \frac{9}{8}$.

And if the vertex of $C(C_n, \theta)$ is positively curved then $\sum_e \theta(e) = n\theta_0 < 2\pi$. In this case we have $\theta_0 < \pi$, so by Lemma 4.6 we have $\alpha > 1$.

In fact if $\sum_e \theta(e) = n\theta_0 \le 3\pi$ and the map f visits at least three edges of $C(k)$ then we must have $n \ge 3$ so that $\theta_0 \le \pi$, and again this means $\alpha \ge 1$. □

5. Balanced harmonic maps between cones

Consider a homogeneous map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$, and assume f maps each edge and face of $C(\Gamma)$ to itself. The map f is harmonic if its restriction to each face of $C(\Gamma, \theta)$ is a harmonic map between sectors of the plane. We investigate the balanced (as in Definition 2.17) homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$. According to Proposition 2.12 such a map can only exist if $0 < \alpha\theta(e) \le \pi$ for each $e \in E(\Gamma)$. The analysis will be different if $\alpha\theta(e) = \pi$ for some edge e (necessarily $\theta(e) = \theta_{\max}$); we will call this the singular degree.

We will focus first on the case of nonsingular α. We begin by constructing homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$, and later investigating which of those maps are balanced.

Lemma 5.1. *A homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $0 < \alpha < \pi/\theta_{\max}$ is uniquely determined by a function $\rho : V(\Gamma) \to \mathbb{R}_{\ge 0}$.*

Proof. For a fixed edge $e \in E(\Gamma)$ consider the faces $C(e) \subset C(\Gamma, \theta)$ and $C(e) \subset C(\Gamma, \varphi)$, which are isometric to $S_{\theta(e)}$ and $S_{\varphi(e)}$, respectively. Proposition 2.12 says a homogeneous harmonic map $u : S_{\theta(e)} \to S_{\varphi(e)}$ is uniquely determined by the values $|u(1, 0)|$ and $|u(1, \theta_0)|$ (using polar coordinates (r, θ) in $S_{\theta(e)}$).

Fix a function $\rho : V(\Gamma) \to \mathbb{R}_{\ge 0}$. For each edge $e = (v_0 v_1) \in E(\Gamma)$, fix coordinates in the face $C(e)$ adapted to v_0 for each metric, so that $C(e)$ is identified with $S_{\theta(e)}$ or $S_{\varphi(e)}$ and the edge $C(v)$ is identified with the positive x-axis. Define f in $C(e)$ to be represented in these coordinates by the unique homogeneous harmonic map $u : S_{\theta(e)} \to S_{\varphi(e)}$ with $u(1, 0) = (\rho(v_0), 0)$ and $u(1, \theta(e)) = (\rho(v_1)\cos\varphi(e), \rho(v_1)\sin\varphi(e))$.

Along each edge $C(v)$ of $C(\Gamma)$ the map f is defined independently of the choice of incident edge. Hence f is continuous over the edges of $C(\Gamma, \theta)$ and harmonic in each face. □

For ease of notation, we will enumerate the vertices of Γ by $\{v_i\}$. Then a function $\rho : V(\Gamma) \to \mathbb{R}_{\ge 0}$ can be encoded by the vector $(\rho_i = \rho(v_i))$. If $v_i v_j \in E(\Gamma)$ is an edge, we will also define $\theta_{ij} = \theta(v_i v_j)$ and $\varphi_{ij} = \varphi(v_i v_j)$.

Theorem 5.2. *The homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $0 < \alpha < \pi/\theta_{\max}$ determined by $\rho : V(\Gamma) \to \mathbb{R}_{\ge 0}$ is balanced if and only if, for each vertex v_i,*

$$\sum_{v_j \sim v_i} \frac{\cos(\varphi_{ij})\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0.$$

Here $v \sim w$ means that vertices v and w are adjacent, so there is an edge $vw \in E(\Gamma)$.

Proof. In each face $C(v_i v_j)$ choose coordinates adapted to v_i for each metric, so that the face is identified with $S_{\theta_{ij}}$ or $S_{\varphi_{ij}}$ and $C(v_i)$ is identified with the positive x-axis. By Proposition 2.12, in these coordinates $f|_{C(v_i v_j)}$ is represented by

$$u_j(r,\theta) = r^\alpha \left(\rho_i \cos(\alpha\theta) + \frac{\rho_j \cos(\varphi_{ij}) - \rho_i \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \sin(\alpha\theta), \frac{\sin(\varphi_{ij})}{\sin(\alpha\theta_{ij})} \rho_j \sin(\alpha\theta) \right).$$

And the balancing formula of Definition 2.17 says that f is balanced along $C(v_i)$ if

$$\sum_{v_j \sim v_i} \frac{\partial \chi_j}{\partial \theta}(r, 0) = 0,$$

where $r^\alpha \chi(\theta)$ is the x-coordinate of $u(r,\theta)$.

Combining these two formulas, we see that f is balanced along $C(v_i)$ if

$$\alpha \sum_{v_j \sim v_i} \frac{\cos(\varphi_{ij})\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij}))} = 0.$$

In order to be balanced, f must be balanced along each edge of $C(\Gamma, \theta)$, so every sum of this form must vanish. □

Another way to interpret this result is to say a vector (ρ_j) determines a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $0 < \alpha < \pi/\theta_{\max}$ if it is in the kernel of the matrix $\Delta^\varphi_{\alpha\theta}$ with entries

$$(\Delta^\varphi_{\alpha\theta})_{ij} = \begin{cases} -\sum_{v_k \sim v_i} \cot(\alpha\theta_{ik}) & \text{if } i = j, \\ \cos(\varphi_{ij}) \csc(\alpha\theta_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \not\sim v_j. \end{cases}$$

For only certain values of α will $\Delta^\varphi_{\alpha\theta}$ have a nontrivial kernel. That is, only certain degrees α admit balanced homogeneous harmonic maps.

The situation for harmonic maps between cones is much more rigid than that of harmonic functions. In special cases the degrees α and the corresponding balanced homogeneous harmonic maps are easy to describe.

Proposition 5.3. *Suppose $\theta(e) = \theta_0$ and $\varphi(e) = \varphi_0$ for all $e \in E(\Gamma)$. Then there exist balanced harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $0 < \alpha < \pi/\theta_0$ if and only if $\alpha\theta_0 = \varphi_0$. If $\alpha\theta_0 = \varphi_0 = \pi/2$ then every function $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$ determines such an f, but if $\alpha\theta_0 = \varphi_0 \neq \pi/2$ then only the constant vectors $\rho = c\mathbb{1}$ with $\rho_i = c$ for all i do.*

Proof. First suppose $\varphi_0 = \pi/2$. Then $\cos(\varphi_0) = 0$ so the matrix $\Delta^{\varphi}_{\alpha\theta}$ is diagonal with diagonal entries $\deg(v_i)\cot(\alpha\theta_0)$. A diagonal operator only has a kernel if some of its diagonal entries are 0. In this case the only possibility is if $\cot(\alpha\theta_0) = 0$, i.e., $\alpha\theta_0 = \pi/2$. In this case $\Delta^{\varphi}_{\alpha\theta_0} = 0$, so all vectors (ρ_i) are in its kernel.

Now suppose $\varphi_0 \neq \pi/2$. The matrix $\Delta^{\varphi}_{\alpha\theta}$ can be rewritten

$$\Delta^{\varphi}_{\alpha\theta} = \csc(\alpha\theta_0)(\cos(\varphi_0)A - \cos(\alpha\theta_0)D).$$

A vector ρ is in the kernel of this matrix if and only if

$$\begin{aligned} \cos(\varphi_0)A\rho &= \cos(\alpha\theta_0)D\rho, \\ D^{-1}A\rho &= \frac{\cos(\alpha\theta_0)}{\cos(\varphi_0)}\rho, \\ D^{-1/2}(1 - D^{-1/2}AD^{-1/2})D^{1/2}\rho &= \frac{\cos(\varphi_0) - \cos(\alpha\theta_0)}{\cos(\varphi_0)}\rho, \\ \mathcal{L}D^{1/2}\rho &= \frac{\cos(\varphi_0) - \cos(\alpha\theta_0)}{\cos(\varphi_0)}D^{1/2}\rho. \end{aligned}$$

Thus $D^{1/2}\rho$ is an eigenvector of $\mathcal{L}$ with eigenvalue $\lambda = 1 - \frac{\cos(\alpha\theta_0)}{\cos(\varphi_0)}$.

If $\alpha\theta_0 \neq \varphi_0$ then $\lambda \neq 0$. But the eigenvectors of $\mathcal{L}$ with nonzero eigenvalue are perpendicular to the null-vector $w = D^{1/2}\mathbb{1}$ with $w_i = \sqrt{\deg v_i}$ for each i. Equivalently,

$$0 = D^{1/2}\rho \cdot w = \sum_i \deg(v_i)\rho_i.$$

But in order to determine a balanced homogeneous harmonic map, all coordinates of ρ must be nonnegative! For $\rho \not\equiv 0$ the above equation precludes this possibility, so the only case left to consider is $\alpha\theta_0 = \varphi_0$.

If $\alpha\theta_0 = \varphi_0$ then a vector ρ in the kernel of $\Delta^{\varphi}_{\alpha\theta}$ must satisfy $D^{1/2}\rho = cw = cD^{1/2}\mathbb{1}$, so $\rho = c\mathbb{1}$. □

For metrics given by more general functions $\theta, \varphi : E(\Gamma) \to (0, \pi)$, the question of which degrees α admit balanced homogeneous harmonic functions is more subtle. But we can achieve a lower bound on such α more easily than we did for functions into $\mathbb{R}$.

Theorem 5.4. *If there is a balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ then*

$$\alpha \geq \min\left(\frac{\pi}{2\theta_{\max}}, \min\left\{\frac{\varphi(e)}{\theta(e)} \;\middle|\; e \in E(\Gamma)\right\}\right).$$

Proof. We proceed by the contrapositive. If $\alpha < \pi/2\theta_{\max}$ and $\alpha\theta(e) < \varphi(e)$ for each $e \in E(\Gamma)$, we will show that no balanced homogeneous harmonic map of degree α

can exist. Under these assumptions on α we have, for each edge $v_i v_j$,

$$\cos(\alpha\theta_{ij}) > \cos(\varphi_{ij}) \quad \text{and} \quad \cos(\alpha\theta_{ij}) > 0.$$

Now bound the quadratic form associated to $\Delta^{\varphi}_{\alpha\theta}$ as follows:

$$\begin{aligned}\rho \cdot \Delta^{\varphi}_{\alpha\theta}\rho &= \sum_{v_i v_j \in E(\gamma)} \frac{2\rho_i\rho_j\cos(\varphi_{ij}) - (\rho_i^2+\rho_j^2)\cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \\ &\leq -\sum_{v_i v_j \in E(\Gamma)} \cot(\alpha\theta_{ij})(\rho_i-\rho_j)^2 \leq 0.\end{aligned}$$

This quadratic form is negative definite; if ρ is not the zero vector then $\rho \cdot \Delta^{\varphi}_{\alpha\theta}\rho < 0$. Thus no nontrivial balanced homogeneous harmonic maps of degree α can exist. □

We can further study the quadratic form determined by $\Delta^{\varphi}_{\alpha\theta}$ just as we did in the case of functions to $\mathbb{R}$.

Lemma 5.5. *For each fixed $\rho = (\rho_i) \not\equiv 0$ the quadratic form $\rho \cdot \Delta^{\varphi}_{\alpha\theta}\rho$ is strictly increasing in α.*

Proof. We computed the quadratic form in the previous result:

$$\rho \cdot \Delta^{\varphi}_{\alpha\theta}\rho = \sum_{v_i v_j \in E(\gamma)} (2\rho_i\rho_j\cos(\varphi_{ij})\csc(\alpha\theta_{ij}) - (\rho_i^2+\rho_j^2)\cot(\alpha\theta_{ij})).$$

Its derivative can now be bounded as follows:

$$\begin{aligned}\frac{\partial}{\partial\alpha}(\rho \cdot \Delta^{\varphi}_{\alpha\theta}\rho) &= \sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i^2+\rho_j^2-2\rho_i\rho_j\cos(\varphi_{ij})\cos(\alpha\theta_{ij})) \\ &\geq \sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i-\rho_j)^2 \geq 0.\end{aligned}$$

Moreover suppose $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta^{\varphi}_{\alpha\theta}\rho) = 0$ at some fixed $\rho = (\rho_i)$ and some fixed α. Then $\rho_i\rho_j\cos(\varphi_{ij})\cos(\alpha\theta_{ij}) = \rho_i\rho_j$ for each $v_i v_j \in E(\Gamma)$, and also

$$\sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i-\rho_j)^2 = 0.$$

The expression on the left is clearly nonnegative, so for equality to hold ρ must be constant. And it follows from $0 < \varphi_{ij}, \alpha\theta_{ij} < \pi$, $\cos(\varphi_{ij})\cos(\alpha\theta_{ij}) \neq 1$ that $\rho_i\rho_j = 0$ for all $v_i v_j \in E(\Gamma)$. Since ρ is constant, we have $\rho \equiv 0$. That is, the only way that $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) = 0$ at any α is if $\rho \equiv 0$. □

Theorem 5.6. *In the interval $0 < \alpha < \pi/\theta_{\max}$ the eigenvalues of $\Delta^{\varphi}_{\alpha\theta}$ are strictly increasing functions of α.*

Proof. The entries of the matrix $\Delta^{\varphi}_{\alpha\theta}$ are analytic in α in the interval $0 < \alpha < \pi/\theta_{\max}$. Moreover, $\Delta^{\varphi}_{\alpha\theta}$ is symmetric for each real α. A result in the perturbation theory of eigenvalue problems (see [Rellich 1969, Section 1.1] or [Kato 1966, Theorem 6.1]) says that the eigenvalues and eigenvectors of $\Delta_{\alpha\theta}$ are also analytic in α. Namely, there are $\lambda_j(\alpha) \in \mathbb{R}$ and $w_j(\alpha) \in \mathbb{R}^{\#V(\Gamma)}$ depending analytically on α so that

$$\Delta^{\varphi}_{\alpha\theta} w_j(\alpha) = \lambda_j(\alpha) w_j(\alpha).$$

Without loss of generality, $w_j(\alpha)$ is a unit vector for each j and each α. The eigenvalues can be recovered from the eigenvectors via the formula

$$\lambda_j(\alpha) = w_j(\alpha) \cdot \Delta^{\varphi}_{\alpha\theta} w_j(\alpha).$$

Differentiating this identity with respect to α yields

$$\lambda_j'(\alpha) = 2w_j'(\alpha) \cdot \Delta^{\varphi}_{\alpha\theta} w_j(\alpha) + w_j(\alpha) \cdot \frac{\partial \Delta^{\varphi}_{\alpha\theta}}{\partial \alpha} w_j(\alpha).$$

As each $w_j(\alpha)$ is a unit vector, it lies in the unit sphere of $\mathbb{R}^{\#V(\Gamma)}$. Hence $w'(\alpha)$ is perpendicular to $w_j(\alpha)$. But $\Delta_{\alpha\theta} w_j(\alpha) = \lambda_j(\alpha) w_j(\alpha)$ is parallel to $w_j(\alpha)$, so the first term in $\lambda'(\alpha)$ vanishes. The second term is strictly positive by Lemma 5.5. Thus

$$\lambda_j'(\alpha) > 0. \qquad \square$$

An immediate consequence of Theorem 5.6 is that one can count the number of degrees α admitting balanced homogeneous harmonic functions just as in the case of functions to $\mathbb{R}$. If $\alpha\theta_{ij} = \pi - \epsilon$ for some edge $v_i v_j \in E(\Gamma)$ then the corresponding term in $\rho \cdot \Delta^{\varphi}_{\alpha\theta} \rho$ is

$$\begin{aligned}\frac{2\rho_i\rho_j \cos(\varphi_{ij}) - (\rho_i^2 + \rho_j^2)\cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} &= \frac{2\rho_i\rho_j\cos(\varphi_{ij}) + \rho_i^2 + \rho_j^2 + O(\epsilon^2)}{\epsilon + O(\epsilon^3)} \\ &= \frac{(\rho_i - \rho_j)^2}{\epsilon} + (1 - \cos(\varphi_{ij}))\frac{2\rho_i\rho_j}{\epsilon} + O(\epsilon).\end{aligned}$$

The only way this term does not approach $+\infty$ is if $\rho_i = \rho_j = 0$.

As $\alpha \to \pi/\theta_{\max}$, the quadratic form $\rho \cdot \Delta^{\varphi}_{\alpha\theta} \rho$ stays bounded only if $\rho_i = \rho_j = 0$ for each edge $v_i v_j$ with $\theta_{ij} = \theta_{\max}$. In other words, ρ_i must be 0 at any vertex v_i incident to an edge e with $\theta(e) = \theta_{\max}$. This defines a subspace W on which the quadratic form remains bounded. For all $\rho \notin W$,

$$\lim_{\alpha \to \pi/\theta_{\max}} \rho \cdot \Delta^{\varphi}_{\alpha\theta} \rho = +\infty.$$

We can thus define a quadratic form Q on W via

$$Q(\rho) = \lim_{\alpha \to \pi/\theta_{\max}} \rho \cdot \Delta^{\varphi}_{\alpha\theta} \rho.$$

Corollary 5.7. *The total number of degrees* $0 < \alpha < \pi/\theta_{\max}$ *that admit balanced homogeneous harmonic maps* $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$, *counted with multiplicity, is the sum of the dimension* $\dim W^{\perp}$ *and the number of positive eigenvalues of the quadratic form* Q *defined on* W.

Remark 5.8. If $\Delta^{\varphi}_{\alpha\theta}$ has a nontrivial kernel, the *multiplicity* of α is the dimension of that kernel.

Proof. Following Theorem 5.6, the only time that $\Delta^{\varphi}_{\alpha\theta}$ can have a nontrivial kernel is when one of the eigenvalues $\lambda_j(\alpha)$ crosses 0. According to Theorem 5.4 the matrices $\Delta^{\varphi}_{\alpha\theta}$ have only negative eigenvalues for α small, so we need only count the number of positive eigenvalues as α approaches $\pi/\theta_{\max}$. Any eigenvalue of $\Delta^{\varphi}_{\alpha\theta}$ that approaches $+\infty$ as $\alpha \to \pi/\theta_{\max}$ is certainly positive, and the remaining eigenvalues of $\Delta^{\varphi}_{\alpha\theta}$ approach the eigenvalues of the quadratic form Q on W as described above. □

5.1. ***The singular degree*** $\boldsymbol{\alpha = \pi/\theta_{\max}}$**.** In stark contrast to Section 3.2, balanced homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of the singular degree $\alpha = \pi/\theta_{\max}$ are quite rare. First we must describe homogeneous harmonic maps in this case. Let $\Sigma \subset \Gamma$ be the subgraph consisting of all those edges $e \in E(\Gamma)$ with $\alpha\theta(e) = \pi$, along with their incident vertices.

Lemma 5.9. *A homogeneous harmonic map* $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ *of degree* $\alpha = \pi/\theta_{\max}$ *is uniquely determined by a function* $\rho : V(\Gamma) \setminus V(\Sigma) \to \mathbb{R}_{\geq 0}$, *along with a function*

$$\nu : \{(v, e) \in V(\Sigma) \times E(\Sigma) \mid v \in e\} \to \mathbb{R}$$

subject to the constraints

$$\nu(v, e) \geq \cos(\varphi(e))\nu(w, e) \;\; \textit{and} \;\; \nu(w, e) \geq \cos(\varphi(e))\nu(v, e) \quad \textit{for all } e = vw \in E(\Gamma).$$

Proof. If $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ has degree $\alpha = \pi/\theta_{\max}$, let $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$ be defined by $f(v, 1) = (v, \rho(v))$. Any vertex $v \in V(\Sigma)$ is incident to an edge e with $\alpha\theta(e) = \pi$, so according to Proposition 2.12 $\rho(v) = 0$. Thus we may replace ρ with its restriction to $V(\Gamma) \setminus V(\Sigma)$. Using Proposition 2.12, such a ρ uniquely determines the behavior of f on each face $C(e)$ corresponding to $e \in E(\Gamma) \setminus E(\Sigma)$.

Now we turn to those edges $e \in E(\Sigma)$. According to Proposition 2.12, the harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ can be represented in coordinates in the face $C(e)$ by the map

$$u(r, \theta) = (x, y) = (c_1 r^{\alpha} \sin(\alpha\theta), c_2 r^{\alpha} \sin(\alpha\theta)).$$

This is a map $S_{\theta(e)} \to S_{\varphi(e)}$, and we are using polar coordinates in the domain and rectangular coordinates in the target.

If the positive x-axis corresponds to a vertex $v_1 \in V(\Gamma)$ and the other edge of $S_{\theta(e)}$ (or $S_{\varphi(e))}$) corresponds to $v_2 \in V(\gamma)$, consider $\nu_1 = \nu(v_1, e)$ and $\nu_2 = \nu(v_2, e)$ to be the normal derivatives used in the balancing condition of Definition 2.17. That is,

$$\nu_1 = \frac{\partial x}{\partial \theta}(1, 0) = c_1 \alpha$$

and

$$\nu_2 = -\frac{\partial}{\partial \theta}\bigg|_{r=1,\theta=\theta(e)} (x \cos\varphi(e) + y \sin\varphi(e)) = \alpha(c_1 \cos\varphi(e) + c_2 \sin\varphi(e)).$$

One can easily solve for c_1 and c_2:

$$c_1 = \frac{\nu_1}{\alpha} \quad \text{and} \quad c_2 = \frac{\nu_2 - \nu_1 \cos\varphi(e)}{\alpha \sin\varphi(e)}.$$

Thus the map ν uniquely determines the map f in each face $C(e)$ corresponding to $e \in E(\Sigma)$. The only restriction on ν is that the image of u must lie in S_{φ_0}. The conditions are very similar to those found in Proposition 2.12, namely $c_2 \geq 0$ and $c_1 \geq c_2 \cot\varphi_0$. Using the above equations to express c_1 and c_2 in terms of ν_1 and ν_2 yields precisely

$$\nu_2 \geq \nu_1 \cos\varphi_0 \quad \text{and} \quad \nu_1 \geq \nu_2 \cos\varphi_0. \qquad \square$$

Now we discuss the balancing condition. On vertices $v_i \in V(\Gamma) \setminus V(\Sigma)$ one must still have

$$\sum_{v_j \sim v_i} \frac{\cos(\varphi_{ij})\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0,$$

just as in Theorem 3.2. And on those vertices $v_i \in V(\Sigma)$ one has $\rho_i = 0$ and the balancing condition reads

$$\sum_{v_i v_j \in E(\Sigma)} \nu(v_i, v_i v_j) + \sum_{v_i v_j \in E(\Gamma) \setminus E(\Sigma)} \frac{\cos(\varphi_{ij})}{\sin(\alpha\theta_{ij})}\rho_j = 0.$$

It seems unlikely to find a map balanced at just the vertices in $V(\Gamma) \setminus V(\Sigma)$ without taking $\rho \equiv 0$. Since the eigenvalue of $\Delta^{\varphi}_{\alpha\theta}$ are strictly increasing with α the chance that one passes through 0 precisely at a particular value of α seems low. Despite a similar concern in Section 3.2, we described an abundance of balanced harmonic maps in Proposition 3.17 by choosing $\rho \equiv 0$. Unfortunately such a result does not hold in the current situation.

Theorem 5.10. *The only balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $\alpha = \pi/\theta_{\max}$, determined by $\rho \equiv 0$ and some function ν as in Lemma 5.9, is the trivial map.*

Proof. The map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ determined by the functions $\rho \equiv 0$ and ν must satisfy the conditions listed in Lemma 5.9. For an edge $e = vw \in E(\Sigma)$ this says

$$\nu(v, e) \geq \nu(w, e)\cos(\varphi(e)) \quad \text{and} \quad \nu(w, e) \geq \nu(v, e)\cos(\varphi(e)).$$

Adding these together gives

$$\nu(v, e) + \nu(w, e) \geq (\nu(v, e) + \nu(w, e))\cos(\varphi(e)).$$

Since $\cos(\varphi(e))$ cannot equal 1, we must have

$$\nu(v, e) + \nu(w, e) \geq 0.$$

Summing over all edges of Σ we have

$$0 \leq \sum_{e\in E(\Gamma)} \sum_{v\in e} \nu(v, e) = \sum_{v\in V(\Gamma)} \sum_{e\ni v} \nu(v, e) = 0.$$

The vanishing of the reordered sum follows from the balancing condition at the vertices of Σ. Thus each inequality from Lemma 5.9 must in fact be an equality. So on the edge $e = vw \in E(\Gamma)$,

$$\nu(v, e) = \nu(w, e)\cos(\varphi(e)) = \nu(v, e)\cos^2(\varphi(e)).$$

Since $\cos(\varphi(e))$ never equals 1, $\nu(v, e) = 0$ for each pair (v, e). □

As an immediate consequence we can extend the result of Proposition 5.3, which discussed the case when $\theta(e) = \theta_0$ and $\varphi(e) = \varphi_0$ for all $e \in E(\Gamma)$. That proposition insisted the only balanced harmonic maps had degree $\alpha = \varphi_0/\theta_0$, but did not discuss the possible singular degree $\alpha = \pi/\theta_0$.

Corollary 5.11. *Let $\theta(e) = \theta_0$ and $\varphi(e) = \varphi_0$ for all $e \in E(\Gamma)$. Then there is no nontrivial balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $\alpha = \pi/\theta_0$.*

Proof. If $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha = \pi/\theta_0$ then the subgraph Σ is all of Γ itself. According to Lemma 5.9 a homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ is determined by just a function ν. But now Theorem 5.10 says that in order for f to be balanced ν must vanish. In other words, f must be the trivial map that sends all of $C(\Gamma, \theta)$ to the vertex of $C(\Gamma, \varphi)$. □

If one could balance the vertices in $V(\Gamma) \setminus V(\Sigma)$, then balancing the vertices in $V(\Sigma)$ amounts to choosing a function ν from a $2\#E(\Sigma)$-dimensional space subject to $\#V(\Sigma)$ balancing conditions. Such a ν certainly exists since $2\#E(\Sigma) \geq \#V(\Sigma)$ (in fact the inequality is strict if Σ is not a disjoint collection of edges) and the $\#V(\Sigma)$ balancing conditions are independent (no two of them use the same value of ν). But as was the case in Theorem 5.10 the extra inequalities restricting ν are unlikely to be satisfied.

6. Collapsing cones

We deal with the limits of the cones $C(\Gamma, t\theta)$ as $t \to 0$. These spaces can also be seen as tangent cones to the ideal hyperbolic simplicial complexes of [Freidin and Gras Andreu 2020] at punctured vertices.

The naïve approach is to simply let t tend to 0 in $C(\Gamma, t\theta)$. Each face of $C(\Gamma)$ will collapse to a half-line, its bounding edges merging. If Γ is a connected graph then all the faces and edges of $C(\Gamma)$ will collapse down to a single half-line, $\mathbb{R}_{\geq 0}$.

The results from Section 3 would turn into a search for a homogeneous harmonic map $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, which must be simply a degree-1 map $f(x) = ax$. Such a map also generalizes the results of Section 4, mapping the collapsed domain to a single edge of the k-pod target. If the domain collapses in Section 5 in this way then a similar map can be constructed into a single edge or face of the target. And if only the target collapses then one searches for a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}_{\geq 0}$. Unfortunately no such map exists, as the following result will show.

Proposition 6.1. *If $f : C(\Gamma, \theta) \to \mathbb{R}$ is a balanced homogeneous harmonic map of degree $\alpha > 0$ then the average of f on any ball centered at the vertex of $C(\Gamma, \theta)$ is* 0.

An immediate consequence is that a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}_{\geq 0}$ must be identically 0.

Proof. Just as in Section 3.3, the map $f : C(\Gamma, \theta) \to \mathbb{R}$ can be written as $f(x, t) = \rho(x)t^\alpha$ for some $\rho : \Gamma \to \mathbb{R}$. After choosing coordinate functions $\psi_e : e \to [0, \theta(e)]$ for each $e \in E(\Gamma)$ and defining $\rho_e = \rho \circ \psi_e^{-1}$, ρ is a solution to the problem

$$\begin{aligned} \rho_e'' &= -\alpha^2 \rho_e && \text{for all } e \in E(\gamma), \\ \sum_{v \in e} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi(e)) &= 0 && \text{for all } v \in V(\Gamma). \end{aligned}$$

The average of f over a ball centered at the vertex of $C(\Gamma, \theta)$ can be computed as follows:

$$\begin{aligned} \iint_{B_r(0)} f\,dA &= \sum_{e \in E(\Gamma)} \iint_{C_r(0) \cap C(e)} f\,dA = \sum_{e \in E(\Gamma)} \int_0^r \int_0^{\theta(e)} \rho_e(x) t^\alpha t\,dx\,dt \\ &= \frac{-r^{\alpha+2}}{\alpha^2(\alpha+2)} \sum_{e \in E(\Gamma)} \int_0^{\theta(e)} \rho_e''(x)\,dx = \frac{-r^{\alpha+2}}{\alpha^2(\alpha+2)} \sum_{e \in E(\Gamma)} (\rho_e'(\theta(e)) - \rho_e'(0)) \\ &= \frac{r^{\alpha+2}}{\alpha^2(\alpha+2)} \sum_{v \in V(\Gamma)} \sum_{e \ni v} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi(v)) \\ &= 0. \end{aligned}$$

□

Remark 6.2. This result can also be seen as the L^2-orthogonality between the eigenfunction ρ with eigenvalue α and the eigenfunction ρ_0 with $\rho_0(x) = 1$, whose eigenvalue is 0. Similar integrations by parts imply that all eigenfunctions with distinct eigenvalues are orthogonal in L^2.

Evidently there is not much happening when we allow the metrics on our cones to collapse, so more care must be taken. As we scale the angle measures by $t \to 0$, we will dilate the cones so that the curves originally at distance 1 from the vertex maintain a constant length. In a face S_{θ_0} this can be achieved with the following four step procedure:

(1) In polar coordinates map $(r, \theta) \mapsto (r^t, \theta)$. This does not change S_{θ_0} as a set, but is useful for defining coefficients later on.
(2) In polar coordinates map $(r, \theta) \mapsto (r, t\theta)$. This maps S_{θ_0} to $S_{t\theta_0}$.
(3) In rectangular coordinates map $(x, y) \mapsto (x-1, y)$. This has the effect of moving the curve originally at distance 1 from the vertex so it now passes through the origin.
(4) Dilate the region $S_{t\theta_0}$ by a factor of $1/t$. This moves the vertex to the point $(-1/t, 0)$ in rectangular coordinates, and rescales the curve now passing through the origin to have length θ_0 as it did at the start.

This transformation can be accomplished much more concisely in complex coordinates by the map

$$z \mapsto \frac{z^t - 1}{t}.$$

In the limit as $t \to 0$ this transformation converges to $z \mapsto \log z$, and the image of S_{θ_0} under this transformation is a strip, described in rectangular coordinates by

$$R_{\theta_0} = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le \theta_0\}.$$

Thus we define the space

$$C^*(\Gamma, \theta) = \Gamma \times \mathbb{R}$$

with a product metric. The factor $\mathbb{R}$ is endowed with its usual Euclidean metric, and Γ is endowed with a path metric where each edge $e \in E(\Gamma)$ has length $\theta(e)$. Note that in the definition of $C^*(\Gamma, \theta)$ there is absolutely no difficulty if the function θ takes values larger than π.

6.1. ***Balanced harmonic functions into*** $\mathbb{R}$. A homogeneous map $u : S_{\theta_0} \to \mathbb{R}$ can be described in polar coordinates as $u(r, \theta) = r^\alpha \rho(\theta)$ for some $\alpha > 0$. The complex exponential transforms the rectangular coordinates (x, y) to the polar coordinates

$(r = e^x, \theta = y)$. So precomposing with the complex exponential gives a map $v = u \circ \exp : R_{\theta_0} \to \mathbb{R}$ given in rectangular coordinates by

$$v(x, y) = u(e^x, y) = e^{\alpha x} \rho(y).$$

When we move to functions $f : C^*(\Gamma, \theta) \to \mathbb{R}$ this suggests that we should consider the class of separated functions, a broader class than homogeneous functions.

Definition 6.3. A map $f : C^*(\Gamma, \theta) \to \mathbb{R}$ is separated if there are functions $\rho : \Gamma \to \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ so that

$$f(x, t) = \rho(x)\tau(t).$$

Since the transformation $z \mapsto \log z$ is conformal on the interior of S_{θ_0}, a homogeneous harmonic function $u(r, \theta) = r^\alpha \rho(\theta)$ is transformed to a *harmonic* function $v(x, y) = e^{\alpha x} \rho(y)$. This suggests that the search for balanced harmonic functions $v : R_{\theta_0} \to \mathbb{R}$ will be equivalent to the search in Section 3.

Of course the class of separated harmonic functions is larger than those maps described above, so some care must be taken. Using coordinates (x, t) on $C^*(\Gamma, \theta) = \Gamma \times \mathbb{R}$, a separated function has the form

$$f(x, t) = \rho(x)\tau(t).$$

By the theory of separation of variables, such a function is harmonic if both ρ and τ are eigenfunctions of the second derivative operator with opposite eigenvalues. Choosing isometries $\psi_e : e \to [0, \theta(e)]$ for each $e \in E(\Gamma)$, let $\rho_e = \rho \circ \psi_e^{-1}$. Then for each e we must have $\tau'' = \lambda\tau$ and $\rho_e'' = -\lambda\rho_e$ for some $\lambda \in \mathbb{R}$.

The eigenfunctions $u'' = cu$ are given by

$$u(x) = \begin{cases} A\cosh\big(\sqrt{c}x\big) + B\sinh\big(\sqrt{c}x\big) & \text{if } c > 0, \\ A + Bx & \text{if } c = 0, \\ A\cos\big(\sqrt{-c}x\big) + B\sin\big(\sqrt{-c}x\big) & \text{if } \lambda < 0. \end{cases}$$

Since τ is independent of $e \in E(\gamma)$, the eigenvalue $-\lambda$ for ρ_e must also be independent of e.

Using the isometries $\psi_e : e \to [0, \theta(e)]$ and the representations $\rho_e = \rho \circ \psi_e^{-1}$ the balancing condition of Definition 2.13 can be generalized to separated functions as follows.

Definition 6.4. A separated function $f(x, t) = \rho(x)\tau(t)$ from $C^*(\Gamma, \theta)$ to $\mathbb{R}$ is balanced along the edge $\{v\} \times \mathbb{R} \subset C^*(\Gamma, \theta)$ corresponding to $v \in V(\Gamma)$ if

$$\sum_{e \ni v} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi_e(v)) = 0.$$

The function f is balanced if it is balanced along each edge of $C^*(\Gamma, \theta)$.

Now we are in a position to fully understand balanced separated harmonic functions and their relationship to the balanced homogeneous harmonic functions of Section 3.

Theorem 6.5. *Let $f : C^*(\Gamma, \theta) \to \mathbb{R}$ be a separated harmonic function, $f(x,t) = \rho(x)\tau(t)$. The function f is balanced if and only if one of the following occurs*:

(1) $f(x,t) = A + Bt$, *or*

(2) *there is a number $\alpha > 0$ so that $g(x,t) = \rho(x)t^\alpha : C(\Gamma, \theta) \to \mathbb{R}$ is a balanced homogeneous harmonic function of degree α.*

In the second case $\tau(t) = A\cosh(\alpha t) + B\sinh(\alpha t) = Ce^{\alpha t} + De^{-\alpha t}$.

Remark 6.6. At the beginning of this subsection we saw that the functions coming from balanced homogeneous functions $\tilde{f} : C(\Gamma, \theta) \to \mathbb{R}$ corresponded to $f = \rho\tau$ with $\tau(t) = Ce^{\alpha t}$. We also develop the additional solutions $f(x,t) = A + Bt$, which correspond to $\alpha = 0$. In fact the balanced homogeneous function $\tilde{f}(x,t) = 1$ of degree 0 gives rise to the balanced separated function $f(x,t) = 1$, corresponding to $B = 0$.

See also that the functions $f(x,t) = A$ and $f(x,t) = \rho(x)e^{\alpha t}$ are the only ones with $\lim_{t\to-\infty} f(x,t) = 0$.

Proof. Let $f : C^*(\Gamma, \theta) \to \mathbb{R}$ be a balanced separated harmonic function. With isometries $\psi_e : e \to [0, \theta(e)]$ and representations $\rho_e = f \circ \psi_e^{-1} : [0, \theta(e)] \to \mathbb{R}$ in coordinates, the discussion above implies that the functions ρ_e satisfy

$$\rho_e'' = -\lambda\rho_e \quad \text{for all } e \in E(\Gamma),$$
$$\sum_{e \ni v} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi(v)) = 0 \quad \text{for all } v \in V(\Gamma).$$

This is precisely the eigenvalue problem stated in Section 3.3. According to [Friedman and Tillich 2004], the solutions ρ to this problem are critical points of the Rayleigh quotient

$$R[\rho] = \frac{\sum_e \|\rho_e'\|^2}{\sum_e \|\rho_e\|^2},$$

where $\|-\|$ indicates the L^2 norm on an interval. Moreover the eigenvalue $-\lambda$ of the solution ρ is precisely $-R[\rho]$, so $\lambda \geq 0$. Let $\alpha \geq 0$ be such that $\lambda = \alpha^2$.

If $\alpha = 0$, then $\rho_e(x) = A_e + B_e x$ for each $e \in E(\Gamma)$, i.e., ρ is edgewise linear. Just as in Proposition 3.10, though, ρ is only balanced if it is constant. In this case $\rho \equiv c$ for some constant c. Since $\lambda = 0$ the function $\tau(t)$ solves $\tau'' = 0$, so τ too is linear, say $\tau(t) = a + bt$. In this case

$$f(x,t) = \rho(x)\tau(t) = ac + bct = A + Bt.$$

If $\alpha > 0$ then according to Section 3.3 the function $\rho : \Gamma \to \mathbb{R}$ fits into a balanced homogeneous harmonic function $g : C(\Gamma, \theta) \to \mathbb{R}$ via

$$g(x, t) = \rho(x) t^{\alpha}.$$

In this case $\tau'' = \alpha^2 \tau$, so $\tau(t) = A \cosh(\alpha t) + B \sinh(\alpha t) = C e^{\alpha t} + D e^{-\alpha t}$. □

6.2. ***Balanced harmonic maps between cones.*** Just as in Section 5 we will consider only maps $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ between spaces with the same combinatorial structure, and only those maps that map the edges $v \times \mathbb{R}$ and faces $e \times R$ of $C^*(\Gamma)$ to themselves. If $\psi_e : e \to [0, \theta(e)]$ is an isometry for the metric θ, then we define a map $\Psi_e : e \times \mathbb{R} \to R_{\theta(e)}$ by

$$\Psi_e(x, t) = (t, \psi_e(x)).$$

Taking isometries $\psi_e^\theta : e \to [0, \theta(e)]$ and $\psi_e^\varphi : e \to [0, \varphi(e)]$ for each $e \in E(\Gamma)$, ensure that $\psi_e^\theta(v)/\theta(e) = \psi_e^\varphi(v)/\varphi(e)$ for each $v \in e$, that is, ψ_e^θ and ψ_e^φ induce the same orientation on e. Then the map $f|_{e\times\mathbb{R}}$ can be represented in coordinates by

$$u_e = \Psi_e^\varphi \circ f \circ (\Psi_e^\theta)^{-1} : R_{\theta(e)} \to R_{\varphi(e)}.$$

But only some such maps are limits of homogeneous maps under collapsing metrics.

A homogeneous map $u : S_{\theta_0} \to S_{\varphi_0}$ can be described in polar coordinates as $u(r, \theta) = (r^\alpha \rho(\theta), \phi(\theta))$ for some $\alpha > 0$. In Section 5 we used rectangular coordinates on the target to describe such maps, but the translation to polar coordinates is not difficult. The complex exponential transforms the rectangular coordinates (x, y) to the polar coordinates $(r = e^x, \theta = y)$, so conjugating with the complex exponential gives a map $v = \log \circ u \circ \exp : R_{\theta_0} \to R_{\varphi_0}$ given in rectangular coordinates by

$$v(x, y) = \log u(e^x, y) = \log(r^{\alpha x} \rho(y) e^{i\phi(y)}) = (\alpha x + \log \rho(y), \phi(y)).$$

Though the presence of the complex logarithm above turns a harmonic $u : S_{\theta_0} \to S_{\varphi_0}$ into a map $v : R_{\theta_0} \to R_{\varphi_0}$ that is not necessarily harmonic, some (nonharmonic) choices of homogeneous u still result in harmonic v. As both R_{θ_0} and R_{φ_0} are Euclidean, the map v is harmonic if and only if both $\alpha x + \log \rho(y)$ and $\phi(y)$ are. Unfortunately this only happens when $\rho(y) = e^{\beta y + a}$ and $\phi(y) = \gamma y + b$ for some constants β, γ, in which case f is linear.

When we move to functions $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ we will consider only maps that send each line $\{x\} \times \mathbb{R} \subset C^*(\Gamma, \theta)$ to a corresponding line $\{x'\} \times \mathbb{R} \subset C^*(\Gamma, \varphi)$. Representing the map f in coordinates in the face $e \times \mathbb{R}$ by the map $u_e : R_{\theta(e)} \to R_{\varphi(e)}$, the maps u_e take the form

$$u_e(x, y) = (\rho_e(x, y), z_e(y)).$$

For f to be harmonic each u_e must be harmonic. In particular $z_e'' = 0$, so z_e is linear. And in order to map each edge $v \times \mathbb{R}$ to itself we must in fact have

$$z_e(y) = \frac{\varphi(e)}{\theta(e)} y.$$

And the balancing condition of Definition 2.17 can be generalized as follows.

Definition 6.7. The map $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ is balanced along the edge $v \times \mathbb{R}$ corresponding to $v \in V(\Gamma)$ if

$$\sum_{e \ni v} (-1)^{\psi_e^\theta(v)/\theta(e)} \frac{\partial \rho_e}{\partial y}(\Psi_e^\theta(v, t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

And the map f is balanced if it is balanced along each edge of $C^*(\Gamma, \theta)$.

Unlike in the case of functions to $\mathbb{R}$, the only harmonic maps $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ that came from maps $C(\Gamma, \theta) \to C(\Gamma, \varphi)$ were the piecewise linear ones. And in fact there are very few *balanced* piecewise linear maps.

Proposition 6.8. *If $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ is a balanced harmonic map whose restriction to each face $e \times \mathbb{R}$ is represented by a linear map $u_e : R_{\theta(e)} \to R_{\varphi(e)}$, then each u_e has the form*

$$u_e(x, y) = \left(ax + b, \frac{\varphi(e)}{\theta(e)} y \right)$$

for some constants a, b that are independent of e. In particular, ρ_e is independent of y.

Proof. If each function ρ_e is linear, then write $\rho_e(x, y) = a_e x + b_e y + c_e$. If $f(v, 0) = (v, \rho(v))$ for some function $\rho : V(\Gamma) \to \mathbb{R}$, then

$$\frac{\partial \rho_e}{\partial y} = b_e = \frac{\rho \circ (\psi_e^\theta)^{-1}(\theta(e)) - \rho \circ (\psi_e^\theta)^{-1}(0)}{\theta(e)}.$$

The balancing condition at a vertex $v \in V(\Gamma)$ then reads

$$\begin{aligned} 0 &= \sum_{e \ni v} (-1)^{\psi_e^\theta(v)/\theta(e)} \frac{\rho \circ (\psi_e^\theta)^{-1}(\theta(e)) - \rho \circ (\psi_e^\theta)^{-1}(0)}{\theta(e)} \\ &= \sum_{w \sim v} \frac{\rho(w) - \rho(v)}{\theta(vw)}. \end{aligned}$$

Thus the balancing condition for f says that ρ is in the kernel of the edge-weighted graph Laplacian with edge weights $w(e) = \frac{1}{\theta(e)}$. But the kernel of such an operator is spanned by the vector $\rho = \mathbb{1}$ with $\rho(v) = 1$ for all $v \in V(\Gamma)$.

So if f is balanced then ρ must be constant, i.e., each $b_e = 0$. Then $\rho_e(x, y) = a_e x + c_e$. If $e \cap e' = v$, then we must also have $\rho_e(\Psi_e^\theta(v, t)) = \rho_{e'}(\Psi_{e'}^\theta(v, t))$, i.e.,

$a_e t + c_e = a_{e'} t + c_{e'}$. The connectedness of Γ then implies that a_e and c_e are independent of e, so let $a = a_e$ and $b = c_e$ to find the result claimed. □

Remark 6.9. The above result can then be realized as a companion to case 1 of Theorem 6.5. In fact, in general, any balanced separated harmonic function $g : C^*(\Gamma, \theta) \to \mathbb{R}$ from Theorem 6.5, or any linear combination or convergent series of such separated solutions, gives a balanced harmonic map $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$, for any φ, by the formula

$$f(x, t) = (x, g(x, t)) .$$

However the facewise linear maps f described in Proposition 6.8 are the only ones for which $|\nabla f|^2$ is bounded on $C^*(\Gamma, \theta)$.

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RANK GROWTH OF ELLIPTIC CURVES IN S_4- AND A_4-QUARTIC EXTENSIONS OF THE RATIONALS

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We investigate the rank growth of elliptic curves from $\mathbb{Q}$ to S_4- and A_4-quartic extensions $K/\mathbb{Q}$. In particular, we are interested in the quantity $\mathrm{rk}(E/K) - \mathrm{rk}(E/\mathbb{Q})$ for fixed E and varying K. When $\mathrm{rk}(E/\mathbb{Q}) \le 1$, with E subject to some other conditions, we prove there are infinitely many S_4-quartic extensions $K/\mathbb{Q}$ over which E does not gain rank, i.e., such that $\mathrm{rk}(E/K) - \mathrm{rk}(E/\mathbb{Q}) = 0$. To do so, we show how to control the 2-Selmer rank of E in certain quadratic extensions, which in turn contributes to controlling the rank in families of S_4- and A_4-quartic extensions of $\mathbb{Q}$.

1. Introduction

1A. ***Rank growth.*** For a number field L and an elliptic curve E defined over L, let $E(L)$ be the group of L-rational points of E. The Mordell–Weil theorem says $E(L)$ is a finitely generated abelian group. The rank of $E(L)$, denoted $\mathrm{rk}(E/L)$, has been the subject of much study. Of particular interest here is the behavior of the rank upon base change, i.e., for an extension of number fields K/L, what is $\mathrm{rk}(E/K) - \mathrm{rk}(E/L)$? We call this difference the *rank growth* of E in K/L.

Suppose $L = \mathbb{Q}$ and $K/\mathbb{Q}$ denotes a quadratic extension. Given an elliptic curve $E/\mathbb{Q}$, a conjecture of Goldfeld predicts that 50% of quadratic twists, E^K, of E have analytic rank zero and 50% have analytic rank one. See Section 3B for a definition and discussion of quadratic twists. Recent work of Smith [2017; 2023a; 2023b] studies the distribution of ℓ^∞-Selmer groups and proves a version of Goldfeld's conjecture for 2^∞-Selmer coranks.

We are interested in studying rank growth in higher degree and nonabelian extensions. In this setting, ranks of quadratic twists, E^K, measure the rank growth of E from $\mathbb{Q}$ to K, which will be essential for rank growth in some larger degree extensions. Previously, and in higher degrees, David, Fearnley, and Kisilevsky [David et al. 2007] have given conjectures for how frequently the rank of an elliptic curve grows in cyclic prime degree extensions. Lemke Oliver and Thorne [2021] gave asymptotic lower bounds for the number of S_d-extensions for which an elliptic

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curve E gains rank. Further, Shnidman and Weiss [2023] study rank growth of elliptic curves from a number field L up to an extension $L(\sqrt[2n]{d})$.

1B. ***Results for S_4- and A_4-quartic extensions.*** We investigate the rank growth of elliptic curves in S_4- and A_4-quartic extensions of the rationals. In what follows, unless stated otherwise, we will always assume E is an elliptic curve defined over $\mathbb{Q}$. Further, $K/\mathbb{Q}$ will always be an S_4- or A_4-quartic extension. That is, one for which the normal closure of K over $\mathbb{Q}$ is an S_4- or A_4-Galois extension. For an elliptic curve $E/\mathbb{Q}$ with discriminant Δ_E, we'll consider $\mathrm{rk}(E/K) - \mathrm{rk}(E/\mathbb{Q})$ for many such K. It will often be convenient to use the rank of a Selmer group of an elliptic curve E/K (here we only need the 2-Selmer group) in place of the rank, $\mathrm{rk}(E/K)$. See Definition 3.1 for the definition of the 2-Selmer group, $\mathrm{Sel}_2(E/K)$, of an elliptic curve E/K and Section 3A for a discussion of their utility in our context.

In particular, for an elliptic curve E with Selmer rank zero over the rationals, subject to some mild constraints, we prove there are infinitely many S_4- or A_4-quartic extensions over which E does not gain rank. Further, we give a lower bound on the number of such extensions with bounded discriminant with some fixed cubic resolvent field.

Theorem 1.1. *Let E be an elliptic curve over $\mathbb{Q}$ such that* $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \simeq S_3$, *and* $\mathrm{Sel}_2(E/\mathbb{Q}) = 0$. *Let K_3 be an S_3-cubic (resp., C_3-cubic) extension of $\mathbb{Q}$ such that $\tilde{K}_3$ and $\mathbb{Q}(E[2])$ are linearly disjoint. Suppose also that there is a place v_0 of K_3, unramified in $\tilde{K}_3$, such that either v_0 is real and $\Delta_E < 0$, or $v_0 \nmid 2\infty$, E has multiplicative reduction at v_0 and* $\mathrm{ord}_{v_0}(\Delta_E)$ *is odd. Then there are infinitely many S_4-quartic (resp., A_4-quartic) extensions K over $\mathbb{Q}$ with cubic resolvent K_3 such that*

- *if* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv 0 \pmod 2$, *then* $\mathrm{rk}(E/K) = 0$;
- *if* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv 1 \pmod 2$, *and also assuming the parity condition* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv \mathrm{rk}(E/K_3) \pmod 2$, *then* $\mathrm{rk}(E/K) = 1$.

To prove the theorem above, we utilize two tools. First, we reduce the problem of studying rank growth in our quartic extensions, $K/\mathbb{Q}$, to one of studying rank growth in certain quadratic subextensions of the Galois closure of K, and thus to studying the rank of certain quadratic twists. Second, we use and further develop some Selmer group machinery of Mazur and Rubin. Indeed, in [Mazur and Rubin 2010], they show that under suitable assumptions, an elliptic curve has infinitely many twists of a prescribed Selmer rank. In subsequent work, Klasgbrun, Mazur, and Rubin [Klagsbrun et al. 2014] study the distribution of 2-Selmer ranks of quadratic twists of an elliptic curve. The quadratic twists we study here, see Definition 1.2 below, are thin in the full family of quadratic twists, so we require a different approach. Nonetheless, we use similar ideas to show one can reduce the 2-Selmer ranks of the appropriate quadratic twists either to one or zero. Translating from the language of Selmer groups to ranks yields Theorem 1.1.

To emphasize the extra properties imposed on the quadratic twists we consider here, we make the following definition.

Definition 1.2 (square norm twists). Let E be an elliptic curve defined over a number field L, and let F/L be a quadratic extension. We call a quadratic twist E^F over L a *square norm twist* if $F = L(\sqrt{\alpha})$ where $\alpha \in L^\times/(L^\times)^2$ and $N_{L/\mathbb{Q}}(\alpha)$ is a square.

Such twists will be key to keeping track of an associated S_4-quartic extension when working over a suitable cubic field (see Lemma 2.2).

Likewise, to streamline the discussion and highlight the properties required of the cubic resolvents of our S_4- or A_4-quartic extensions, which are S_3- or C_3-cubic extensions of the rationals, respectively, always denoted K_3, we make the next definition.

Definition 1.3 (admissible cubic resolvent). Let K_3 be a cubic extension of the rationals and fix an elliptic curve $E/\mathbb{Q}$ with discriminant Δ_E. Suppose $E(K_3)[2] = 0$, $\tilde{K}_3$ (the Galois closure of K_3 over $\mathbb{Q}$) and $\mathbb{Q}(E[2])$ are linearly disjoint, and there is a place v_0 of K_3, unramified in $\tilde{K}_3$, such that either v_0 is real and $(\Delta_E)_{v_0} < 0$ or $v_0 \nmid 2\infty$, E has multiplicative reduction at v_0 and $\mathrm{ord}_{v_0}(\Delta_E)$ is odd. For such an extension,

- if K_3 is an S_3-cubic extensions, we call K_3 an *admissible S_3-cubic resolvent for E*;
- if K_3 is a C_3-cubic extension, we call K_3 an *admissible C_3-cubic resolvent for E*.

In the event we do not need to specify one of the two Galois group cases above, we will call a K_3 as in one of the two cases above an *admissible cubic resolvent* for some E.

The restrictions placed on K_3 to make it admissible for some elliptic curve are not overly burdensome. The conditions on the distinguished place should be compared to the assumptions of [Mazur and Rubin 2010, Theorem 1.6].

Theorem 1.1 is a consequence of the following result.

Theorem 1.4. *Let E be an elliptic curve over $\mathbb{Q}$ such that* $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \simeq S_3$, *and* $\mathrm{Sel}_2(E/\mathbb{Q}) = 0$, *and let K_3 be an admissible cubic resolvent for E. Then there are infinitely many square norm twists*, E^F/K_3, *such that*

- *if* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv 0 \pmod 2$, *then* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = 0$;
- *if* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv 1 \pmod 2$, *then* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = 1$.

Remark 1.5. In Theorems 1.1 and 1.4, and in what follows, when we say "infinitely many" we mean the number of such things with the norm of their discriminant bounded above by X is $\gg X^{1/2}/\log X^{\alpha}$ for some $\alpha > 0$. Further, α depends on E and the two-torsion field $K_3(E[2])$. The sizes of these lower bounds, particularly the values of α depending on K_3 and E, are elucidated in Proposition 4.1.

Remark 1.6. In the case that one obtains infinitely many A_4-quartic extensions for which the Mordell or 2-Selmer rank has some prescribed behavior, the "infinitely many" given by the two theorems above differs from the predicted number of A_4-quartic extensions only by some factors of $\log X$. In particular, Malle's conjecture [2004] predicts that the number of A_4-quartic extensions of a number field L with absolute value of the norm of the relative discriminant bounded by X is asymptotic to $c_L X^{1/2} \log X^{b_L}$ for constants b_L and c_L depending on L.

The number of S_4-quartic extensions of L with absolute value of the norm of the relative discriminant bounded by X is asymptotic to $d_L X$ for a constant d_L depending on L [Bhargava et al. 2015]. In this case the "infinitely many" given by the two theorems above differs from S_4-quartic asymptotic by both a logarithmic factor and a factor of $X^{1/2}$.

1C. ***Layout.*** In Section 2 we outline the connection between rank growth in S_4 and A_4-quartic extensions with rank growth in certain quadratic extensions. In Section 3, we recall some facts about Selmer groups and record some related results of Mazur and Rubin [Mazur and Rubin 2010] on quadratic twists. In Sections 4 and 5, we interface the tools of Section 3 with the notion of square norm twists to show we can decrease the 2-Selmer rank of an elliptic curve with a suitable square norm twist and can indeed find many such twists. Finally, in Section 6, we prove the main theorems stated above.

2. Rank growth in S_4-quartics

2A. ***Preliminaries.*** For an extension of number fields $L/\mathbb{Q}$, we write $\tilde{L}$ for the Galois closure of L in some choice of algebraic closure $\bar{\mathbb{Q}}$.

Consider an S_4- or A_4- extension $K/\mathbb{Q}$ with Galois closure $\tilde{K}$. We are principally concerned with the change (or lack of change) in rank in the group of K-rational points vs. the group of $\mathbb{Q}$-rational points of E. We will show this rank change is governed by the rank growth in a quadratic extension of fields between $\mathbb{Q}$ and $\tilde{K}$. K_3 will always denote a cubic resolvent for our quartic extension(s). Of particular interest will be fixing an admissible cubic resolvent K_3 and considering many quartic S_4- or A_4-extensions K with cubic resolvent K_3.

2B. ***S_4- and A_4-quartic extensions.*** Before we turn to the question of rank growth, we record a few facts about S_4- and A_4-quartic extensions with cubic resolvent field K_3.

Lemma 2.1. *Let $K_3/\mathbb{Q}$ be a cubic extension and F be a quadratic extension of K_3. There is always an embedding* $\mathrm{Gal}(\tilde{F}/\mathbb{Q}) \hookrightarrow S_2 \wr S_3$.

Quadratic extensions of S_3-cubics generically have Galois group $S_2 \wr S_3$ over $\mathbb{Q}$. We are interested in the case where the Galois group is instead $S_4 < S_2 \wr S_3$. Likewise for C_3-cubics, we wish to consider the case where quadratic extensions of K_3 have Galois group A_4.

Lemma 2.2. *Fix an S_3- or C_3-cubic extension $K_3/\mathbb{Q}$, and let F denote a quadratic extension of the form $K_3(\sqrt{\alpha})$, where $\alpha \in K_3$ and $\mathbf{N}_{K_3/\mathbb{Q}}(\alpha)$ is a square.*

- *If $K_3/\mathbb{Q}$ is an S_3-cubic, then* $\mathrm{Gal}(\tilde{F}/\mathbb{Q}) \simeq S_4$. *Further, there is a one-to-one correspondence between such quadratic extensions F/K_3 and S_4-quartic extensions of $\mathbb{Q}$ with cubic resolvent K_3.*
- *If $K_3/\mathbb{Q}$ is a C_3-cubic, then* $\mathrm{Gal}(\tilde{F}/\mathbb{Q}) \simeq A_4$. *Further, there is a three-to-one correspondence between such quadratic extensions F/K_3 and A_4-quartic extensions of $\mathbb{Q}$ with cubic resolvent K_3.*

This correspondence is described in detail in, for example, Section 2 of [Cohen and Thorne 2016].

Remark 2.3. Fix a cubic field, K_3. For each $L = K_3(\sqrt{\alpha})$ where $\alpha \in K_3^\times/(K_3^\times)^2$ and $N_{K_3/\mathbb{Q}}(\alpha)$ is a square, there is an S_4- or A_4-quartic extension $K/\mathbb{Q}$ with cubic resolvent K_3. Use Lemma 2.2 and observe that $\tilde{L} = \tilde{K}$. For our purposes, we will fix K_3 and range over quadratic extensions of K_3 as in Lemma 2.2 to range over S_4-quartic extensions of $\mathbb{Q}$ with cubic resolvent K_3. We will then consider a fixed elliptic curve E over these extensions, and consider various differences in rank.

2C. *Measuring rank growth in S_4- and A_4-quartic extensions.* Our aim now is to show that measuring rank growth of an elliptic curve E from $\mathbb{Q}$ to S_4- or A_4-quartic extensions $K/\mathbb{Q}$ with cubic resolvent $K_3/\mathbb{Q}$ is a matter of measuring the rank growth of E from K_3 to a quadratic extension F/K_3, namely the quadratic extension of Lemma 2.2.

Lemma 2.4. *Let E be an elliptic curve defined over $\mathbb{Q}$, K be an S_4- or A_4-quartic extension of $\mathbb{Q}$ with cubic resolvent K_3, and F/K_3 be the quadratic extension of Lemma 2.2. Then*

$$\mathrm{rk}(E/K) - \mathrm{rk}(E/\mathbb{Q}) = \mathrm{rk}(E/F) - \mathrm{rk}(E/K_3). \tag{2-1}$$

The rank relation of Lemma 2.4 is a manifestation of the following more general fact [Dokchitser and Dokchitser 2010, page 572]. Suppose L/k is a Galois extension of number fields with $G = \mathrm{Gal}(L/k)$, and E/k is an elliptic curve. For $H \leq G$, write $\mathbb{1}_H$ for the trivial character on H. If there are subextensions K_i/k and K'_j/k of L, cut out by subgroups H_i and H'_j of G, such that

$$\bigoplus_i \mathrm{Ind}_{H_i}^G \mathbb{1}_{H_i} \simeq \bigoplus_j \mathrm{Ind}_{H'_j}^G \mathbb{1}_{H'_j} \tag{2-2}$$

as complex representations of G, then

$$\sum_i \mathrm{rk}(E/K_j) = \sum_j \mathrm{rk}(E/K'_j). \tag{2-3}$$

The see this relation on the ranks, let χ_L be the character of the representation of the complex representation of the Mordell–Weil group, $E(L)\otimes\mathbb{C}$, and note that

$$\mathrm{rk}(E/K_i) = \langle \mathrm{Ind}_{H_j}^G \mathbb{1}_{H_i}, \chi_L\rangle. \tag{2-4}$$

The same statement can be made for the rank of E over each K'_j as well. Using (2-4) together with (2-2) yields (2-3).

Proof of Lemma 2.4. Consider the following subgroups of $G = \mathrm{Gal}(\tilde{K}/\mathbb{Q}) \simeq S_4$ which fix the subfields K, K_3, and F of $\tilde{K}$. Let $H_K \simeq S_3$ be the subgroup fixing K, $H_{K_3} \simeq D_8$ be the subgroup fixing K_3, and $H_F \simeq V_4$, the Klein four group, be the subgroup fixing F. Then one can verify

$$\mathrm{Ind}_{H_K}^G \mathbb{1}_{H_K} \oplus \mathrm{Ind}_{H_{K_3}}^G \mathbb{1}_{H_{K_3}} \simeq \mathbb{1} \oplus \mathrm{Ind}_{H_F}^G \mathbb{1}_{H_F}. \tag{2-5}$$

The lemma now follows from (2-3). Note that relations like that of (2-5) are an example of those provided in [Bartel and Dokchitser 2015]. □

3. The 2-Selmer groups and quadratic twists

In the previous section, we established that the rank growth from $\mathbb{Q}$ to K is the same as the rank growth from K_3 to a quadratic extension of K_3 determined by K. So we may restrict ourselves to the study of rank growth in quadratic extensions. This is governed by the theory of quadratic twists.

3A. ***The 2-Selmer group.*** We now recall the definition of the 2-Selmer group for an elliptic curve E over a number field L. The multiplication-by-2 map on E gives rise to a short exact sequence of Galois modules:

$$0 \to E[2] \to E(\bar{\mathbb{Q}}) \xrightarrow{\times 2} E(\bar{\mathbb{Q}}) \to 0.$$

This in turn yields a long exact sequence of Galois cohomology groups, which, after quotienting appropriately, gives rise to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(L)/2E(L) & \longrightarrow & H^1(L, E[2]) & \longrightarrow & H^1(L, E)[2] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_v E(L_v)/2E(L_v) & \longrightarrow & \prod_v H^1(L_v, E[2]) & \longrightarrow & \prod_v H^1(L_v, E)[2] & \longrightarrow & 0 \end{array}$$

Now, define subgroups $H^1_f(L_v, E[2])$ of each local cohomology $H^1(L_v, E[2])$ as

$$H^1_f(L_v, E[2]) := \mathrm{Image}\big(E(L_v)/2E(K_v) \to H^1(L_v, E[2])\big).$$

Definition 3.1. The 2-*Selmer group of* E/L, denoted $\mathrm{Sel}_2(E/L)$, is the $\mathbb{F}_2$-vector space defined by the exactness of the sequence

$$0 \to \mathrm{Sel}_2(E/L) \to H^1(L, E[2]) \to \bigoplus_v H^1(L_v, E[2])/H^1_f(L_v, E[2]).$$

We may think of the elements of the 2-Selmer group as being the classes in $H^1(L, E[2])$ which, for every place v of L, land in the image of $E(L_v)/2E(L_v)$; that is, elements of $H^1(L, E[2])$ which everywhere locally satisfy the local conditions determined by $H^1_f(L_v, E[2])$.

Further, the 2-Selmer group fits into a short exact sequence

$$0 \to E(L)/2E(L) \to \mathrm{Sel}_2(E/L) \to Ш(E/L)[2] \to 0,$$

where $Ш(E/L)[2]$ are the elements of the Shafarevich–Tate group of E/L with order dividing 2. We have

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/L) = (\mathrm{rk}(E/L) + \dim_{\mathbb{F}_2} E(L)[2]) + \dim_{\mathbb{F}_2} Ш(E/L)[2] \tag{3-1}$$

and further that $\mathrm{rk}(E/L) \le \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/L)$. It is by this relation that we'll access the ranks of the various elliptic curves and twists discussed later in the paper.

3B. ***Quadratic twists.*** Suppose our elliptic curve E/L is given in short Weierstrass form:

$$E : y^2 = x^3 + Ax + B$$

with $A, B \in L$. A *quadratic twist*, E^F/L, of E/L is an elliptic curve of the form

$$E^F : \delta y^2 = x^3 + Ax + B, \tag{3-2}$$

where $\delta \in L^\times/(L^\times)^2$ and $F = L\big(\sqrt{\delta}\,\big)$. With a change of variables, one can put (3-2) in short Weierstrass form:

$$E^F : y^3 = x^3 + a\delta^2 x + b\delta^3.$$

An elliptic curve E/L and a quadratic twist E^F/L are not, in general, isomorphic as elliptic curves over L but *are* isomorphic as elliptic curves over F. In particular, quadratic twists will be the main tool for measuring growth in quadratic extensions as we have

$$\mathrm{rk}(E^F/L) = \mathrm{rk}(E/F) - \mathrm{rk}(E/L).$$

3C. ***The 2-Selmer of quadratic twists.*** Mazur and Rubin [2010] gave results in which understanding the behavior of an elliptic curve E/L and its 2-Selmer group, $\mathrm{Sel}_2(E/L)$, locally at only a few places of L is sufficient to, under some mild conditions, understand the relation between $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/L)$ and $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/L)$ for some quadratic twists, E^F, of E.

One defines the 2-Selmer group of a twist E^F/L following the definition laid out in Section 3A, just with E^F in place of E. It is important to note that $E^F[2]$ and $E[2]$ are isomorphic as Galois modules, so we may view both Selmer groups inside $H^1(L, E[2])$.

Lemma 2.4 shows that understanding the rank growth in quadratic extensions will be sufficient for understanding rank growth in the quartic extensions of interest. In the reminder of this section we will record some results from [Mazur and Rubin 2010] on how local information about E relates the 2-Selmer rank of E to the 2-Selmer rank of quadratic twists of E.

Lemma 3.2 [Mazur and Rubin 2010, Lemma 2.2]. *With the notation as above*:

- *If* $v \nmid 2\infty$, *then* $\dim_{\mathbb{F}_2} H^1_f(L_v, E[2]) = \dim_{\mathbb{F}_2} E(L_v)[2]$.
- *If* $v \nmid 2\infty$ *and E has good reduction at v, then*

$$H^1_f(L_v, E[2]) \cong E[2]/(\mathrm{Frob}_v - 1)E[2].$$

Definition 3.3. Suppose T is a finite set of places of L. Let loc_T be the sum of the localization maps for each place of T,

$$\mathrm{loc}_T : H^1(L, E[2]) \to \bigoplus_{v \in T} H^1(L_v, E[2]).$$

Also set

$$V_T = \mathrm{loc}_T\,(\mathrm{Sel}_2(E/L)) \subset \bigoplus_{v \in T} H^1_f(L_v, E[2]).$$

We finish the section be recalling two results from [Mazur and Rubin 2010] that we'll later use to control the rank of the 2-Selmer groups in the quadratic extension of Lemma 2.4.

Lemma 3.4 [Mazur and Rubin 2010, Proposition 3.3]. *Let E/L be an elliptic curve, and let F/L be a quadratic extension in which the following places of L split*:

- *all primes where E has additive reduction*;
- *all places v where E has multiplicative reduction such that* $\mathrm{ord}_v(\Delta_E)$ *is even*;
- *all primes above* 2;
- *all real places v with* $(\Delta_E)_v > 0$.

Further, suppose that all v where E has multiplicative reduction and $\mathrm{ord}_v(\Delta_E)$ *is odd are unramified in F/L.*

Let T be the set of finite primes $\mathfrak{p}$ of L such that F/L is ramified at $\mathfrak{p}$ and $E(L_\mathfrak{p})[2] \neq 0$. Then,

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/L) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/L) - \dim_{\mathbb{F}_2}(V_T) + d \tag{3-3}$$

for some d such that

$$0 \le d \le \dim_{\mathbb{F}_2}\Big(\bigoplus_{\mathfrak{p}\in T} H^1_f(L_\mathfrak{p}, E[2])/V_T\Big)$$

and

$$d \equiv \dim_{\mathbb{F}_2}\Big(\bigoplus_{\mathfrak{p}\in T} H^1_f(L_\mathfrak{p}, E[2])/V_T\Big) \pmod 2.$$

An immediate consequence of the above is the following lemma.

Lemma 3.5 [Mazur and Rubin 2010, Corollary 3.4]. *For an elliptic curve E/L and for F/L and T as defined in Lemma 3.4, we have*:

(1) *If* $\dim_{\mathbb{F}_2}\big(\bigoplus_{\mathfrak{p}\in T} H^1_f(L_\mathfrak{p}, E[2])/V_T\big) \le 1$, *then*

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/L) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/L) - 2\dim_{\mathbb{F}_2} V_T + \sum_{\mathfrak{p}\in T} \dim_{\mathbb{F}_2} H^1_f(L_\mathfrak{p}, E[2]).$$

(2) *If T is empty, then* $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/L) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/L)$.

We will use Lemma 3.5, setting L to be some admissible cubic resolvent, say K_3, to understand the 2-Selmer rank of some square norm twists by controlling $\dim_{\mathbb{F}_2} \mathrm{loc}_T(\mathrm{Sel}_2(E/K_3)$ and $\dim_{\mathbb{F}_2} H^1_f((K_3)_\mathfrak{p}, E[2])$ for each $\mathfrak{p} \in T$.

4. Twisting by square norm extensions

We will consider elliptic curves $E/\mathbb{Q}$ together with some K_3 which will always be assumed to be an *admissible cubic resolvent for E* as in Definition 1.3. Recall that among other conditions, we require $E(K_3)[2] = 0$ and K_3 to $\mathbb{Q}(E[2])$ be linearly disjoint.

We are concerned with quadratic twists E^F over K_3 where we impose conditions on F. In Section 1 we introduced Definition 1.2 defining *square norm twists* to keep track of conditions on the twists. Recall that for an elliptic curve defined over a number field L, E/L, these are quadratic twists E^F/L of E/L where $F = L(\sqrt{\alpha})$, $\alpha \in L^\times/(L^\times)^2$, and $N_{L/\mathbb{Q}}(\alpha)$ is a square.

We will be interested in the application of the definition above where $L = K_3$, which, as above, will be the cubic resolvent for some quartic S_4-extensions of $\mathbb{Q}$.

Further, define $N_r^{\square}(E, X)$ as follows to count quadratic extensions F/K_3 with bounded conductor, $\mathfrak{f}(F/K_3)$, that give square norm twists E^F of E with 2-Selmer group of dimension r:

$$N_r^{\square}(E, X) = \#\big\{F = K_3\big(\sqrt{\alpha}\big) \mid \alpha \in K_3^{\times}/(K_3^{\times})^2,\ N_{K_3/\mathbb{Q}}(\alpha) \text{ a square},\\ \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = r,\ N_{K_3/\mathbb{Q}}\mathfrak{f}(F/K_3) < X\big\}.$$

With that in mind, we prove the following.

Proposition 4.1. *Fix an S_3- or C_3-cubic field $K_3/\mathbb{Q}$, an elliptic curve $E/\mathbb{Q}$, and a nonnegative even integer r. Suppose there exists a square norm twist, E^L/K_3, of E/K_3, with $\dim_{\mathbb{F}_2} \mathrm{Sel}(E^L/K_3) = r$. Then we have*:

- *If* $\mathrm{Gal}(K_3/\mathbb{Q}) \simeq S_3$ *and* $\mathrm{Gal}(K_3(E[2])/K_3) \simeq S_3$, *then*
$$N_r^{\square}(E, X) \gg X^{1/2}/\log(X)^{5/6}.$$
- *If* $\mathrm{Gal}(K_3/\mathbb{Q}) \simeq S_3$ *and* $\mathrm{Gal}(K_3(E[2])/K_3) \simeq C_3$, *then*
$$N_r^{\square}(E, X) \gg X^{1/2}/\log(X)^{2/3}.$$
- *If* $\mathrm{Gal}(K_3/\mathbb{Q}) \simeq C_3$ *and* $\mathrm{Gal}(K_3(E[2])/K_3) \simeq S_3$, *then*
$$N_r^{\square}(E, X) \gg X^{1/2}/\log(X)^{8/9}.$$
- *If* $\mathrm{Gal}(K_3/\mathbb{Q}) \simeq C_3$ *and* $\mathrm{Gal}(K_3(E[2])/K_3) \simeq C_3$, *then*
$$N_r^{\square}(E, X) \gg X^{1/2}/\log(X)^{7/9}.$$

Remark 4.2. In Section 5, we will prove the existence of the quadratic extension L/K_3 from the hypotheses of Proposition 4.1. With this, we will use the relationship between the rank growth from $\mathbb{Q}$ to K and the rank growth K_3 to a quadratic extensions F/K_3 to prove Theorem 1.1.

The rest of this section will be devoted to proving Proposition 4.1. Before proceeding, we first need to enumerate ideals in K_3 that allow us to get quadratic extensions for square norm twists. We will then show for each such ideal, there is a square norm twist of E corresponding to that ideal.

Lemma 4.3. *Suppose $E/\mathbb{Q}$ is an elliptic curve where K_3 is an admissible S_3-cubic resolvent for E and* $\mathrm{Gal}(K_3(E[2]/K_3)$ *is C_3 or S_3. Let S be the set of the elements of order* 3 *in* $\mathrm{Gal}(K_3(E[2])/K_3)$, *and N be a ray class field of K_3. Then the number of ideals $\mathfrak{b}$ of K_3 such that*

- $N\mathfrak{b} < X$ *and* $[\mathfrak{b}, N/K_3] = 1$, *and*
- *for every prime ideal $\mathfrak{p}$ dividing $\mathfrak{b}$, $N\mathfrak{p}$ is a square and* $\mathrm{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$

is asymptotic to

$$(C+o(1))\frac{X^{1/2}}{\log(X)^{1-(1/2)\left(|S|/[K_3(E[2]):K_3]\right)}}$$

as $X \to \infty$, *where* C *is some positive constant*, $[-, N/K_3]$ *is the global Artin symbol and* $\boldsymbol{N}$ *is the ideal norm.*

Proof. An unramified, noninert rational prime p can split as a product of primes in two ways in the ring of integers, $\mathcal{O}_{K_3}$, of the cubic field K_3. Either $p\mathcal{O}_{K_3} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ where each factor has degree one, or $p\mathcal{O}_{K_3} = \mathfrak{p}_1\mathfrak{p}_2$ where one factor has degree one and one factor has degree two. Primes of degree two only appear as factors in the latter splitting type.

First we will count rational primes p such that $p\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q}$ where the residue degrees of the prime factors are $f(\mathfrak{p}\,|\,p) = 1$ and $f(\mathfrak{q}\,|\,p) = 2$. For each such $p < X$ we get one prime $\mathfrak{q}$ of K_3 of square norm such that $\boldsymbol{N}\mathfrak{q} < X^2$. Let $\mathcal{S}_{(1,2)}$ be the set of such rational primes, i.e.,

$$\mathcal{S}_{(1,2)} := \{p \in \mathbb{N} \text{ prime} \mid p\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q},\ f(\mathfrak{p}\,|\,p) = 1,\ f(\mathfrak{q}\,|\,p) = 2\} \tag{4-1}$$

Also set

$$\mathcal{P}_{(1)} := \{\mathfrak{p} \subset \mathcal{O}_{K_3} \text{ prime ideal} \mid (\mathfrak{p}\cap\mathbb{Z})\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q},\ f(\mathfrak{p}\,|\,\mathfrak{p}\cap\mathbb{Z}) = 1,\ f(\mathfrak{q}\,|\,\mathfrak{p}\cap\mathbb{Z}) = 2\}$$

and

$$\mathcal{Q}_{(2)} := \{\mathfrak{q} \subset \mathcal{O}_{K_3} \text{ prime ideal} \mid f(\mathfrak{q}\,|\,\mathfrak{q}\cap\mathbb{Z}) = 2\}. \tag{4-2}$$

Likewise, define

- $\mathcal{S}_{(1,2)}(X) := \{p \in \mathcal{S}_{(1,2)} \mid p < X\}$;
- $\mathcal{P}_{(1)}(X) := \{\mathfrak{p} \in \mathcal{P}_{(1)} \mid \boldsymbol{N}\mathfrak{p} < X\}$;
- $\mathcal{Q}_{(2)}(X) := \{\mathfrak{q} \in \mathcal{Q}_{(2)} \mid \boldsymbol{N}\mathfrak{q} < X\}$.

With this notation, the discussion above amounts to

$$\#\mathcal{S}_{(1,2)}(X) = \#\mathcal{P}_{(1)}(X) = \#\mathcal{Q}_{(2)}(X^2). \tag{4-3}$$

A rational prime p belongs to $\mathcal{S}_{(1,2)}$ if and only if $\mathrm{Frob}_p(\tilde{K}_3/\mathbb{Q})$ acts on the three cosets of $\mathrm{Gal}(\tilde{K}_3/\mathbb{Q})/\mathrm{Gal}(\tilde{K}_3/K_3)$ like a transposition. Via the Chebotarev density theorem, this happens with probability $\#\{\text{transpositions in } S_3\}/\#S_3 = \frac{1}{2}$. That is, the density of $\mathcal{S}_{(1,2)}$ in the set of all rational primes is $\frac{1}{2}$.

We can conclude the Dirichlet density, δ_{dir}, of the set of primes $\mathfrak{p}$ in K_3 corresponding to each $p \in \mathcal{S}_{(1,2)}$ is also $\frac{1}{2}$, i.e.,

$$\delta_{\mathrm{dir}}\big(\{\mathfrak{p} \mid f(\mathfrak{p}\,|\,\mathfrak{p}\cap\mathbb{Z}) = 1,\ (\mathfrak{p}\cap\mathbb{Z})\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q},\ f(\mathfrak{q}\,|\,\mathfrak{p}\cap\mathbb{Z}) = 2\}\big) = \delta_{\mathrm{dir}}(\mathcal{P}_{(1)}) = \tfrac{1}{2}. \tag{4-4}$$

Now define some notation. Set $M = K_3(E[2])$ and recall that S is the set of all elements of order 3 in $\mathrm{Gal}(M/K_3)$ and note that S is a union of conjugacy classes when $\mathrm{Gal}(M/K_3) = C_3$ and is a conjugacy class when $\mathrm{Gal}(M/K_3) = S_3$. Now, set

- $\mathcal{P} = \{\mathfrak{p} \in \mathcal{P}_{(1)} \mid \mathfrak{p} \text{ unramified in } NM/K_3,\ \mathrm{Frob}_{\mathfrak{p}}(M/K_3) \subset S\}$;
- $\mathcal{Q} = \{\mathfrak{q} \in \mathcal{Q}_{(2)} \mid \mathfrak{p} \text{ unramified in } NM/K_3,\ \mathrm{Frob}_{\mathfrak{q}}(M/K_3) \subset S\}$;
- $\mathcal{N} = \{\mathfrak{a} \mid \text{squarefree product of ideals from } \mathcal{P}\}$;
- $\mathcal{N}_1 = \{\mathfrak{a} \mid \text{squarefree product of ideals from } \mathcal{P},\ [\mathfrak{a}, N/K_3] = 1\}$;
- $\mathcal{R}_1 = \{\mathfrak{b} \mid \text{squarefree product of ideals from } \mathcal{Q},\ [\mathfrak{b}, N/K_3] = 1\}$.

Our goal is now to access the number of ideals in $\mathcal{N}_1(X)$ via the Dirichlet series $\sum_{\mathfrak{a}\in\mathcal{N}_1} N\mathfrak{a}^{-1}$ and a Tauberian theorem of Wintner. Indeed, we'll see knowing $\mathcal{N}_1(X)$ suffices to understand $\mathcal{R}_1(X^2)$.

To that end, for an irreducible character $\chi : \mathrm{Gal}(N/K_3) \to \mathbb{C}^\times$ where we will write $\chi(\mathfrak{a})$ for $\chi([\mathfrak{a}, N/K_3])$, set

$$f_\chi(s) := \sum_{\mathfrak{a}\in\mathcal{N}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}\in\mathcal{P}} (1 + \chi(\mathfrak{p}) N\mathfrak{p}^{-s}). \tag{4-5}$$

Note that $\mathfrak{p} \in \mathcal{P}$ can't be above a rational prime p which splits completely in $\tilde{K}_3$; if it split completely in $\tilde{K}_3$, then it splits completely in K_3, too. Thus, $\mathrm{Frob}_{\mathfrak{p}}(\tilde{K}_3/K_3)$ isn't trivial.

Let τ be the nontrivial element of $\mathrm{Gal}(\tilde{K}_3/K_3)$ and set

$$S' = \{\tau\} \times S \subset \mathrm{Gal}(\tilde{K}_3 M/K_3) = \mathrm{Gal}(\tilde{K}_3/K_3) \times \mathrm{Gal}(M/K_3) \tag{4-6}$$

and

$$\delta(S, \chi) = \begin{cases} 0 & \text{if } \chi \text{ nontrivial,} \\ \dfrac{1}{2}\dfrac{|S|}{[M : K_3]} & \text{if } \chi \text{ trivial,} \end{cases}$$

noting that, in the χ trivial case,

$$\frac{1}{2}\frac{|S|}{[M : K_3]} = \#S'/\#\mathrm{Gal}(\tilde{K}_3 M/K_3).$$

We write $g_1(s) \sim g_2(s)$ for two complex functions g_1, g_2 on the half plane $\Re s > 1$ if $g_1(s) - g_2(s)$ extends to a holomorphic function on the half plane $\Re s \geq 1$. Now, starting from the logarithm of (4-5) and using the Chebotarev density theorem,

$$\log f_\chi(s) \sim \sum_{\mathfrak{p}\in\mathcal{P}} \chi(\mathfrak{p}) N\mathfrak{p}^{-s} \sim \delta(S, \chi) \sum_{\mathfrak{p} \text{ prime}} \chi(\mathfrak{p}) N\mathfrak{p}^{-s} \sim \delta(S, \chi) \log \frac{1}{s-1}.$$

Using character orthogonality, observe

$$\frac{1}{[N:K_3]}\sum_{\chi} f_\chi(s) = \frac{1}{[N:K_3]}\sum_{\mathfrak{a}\in\mathcal{N}} N\mathfrak{a}^{-s}\sum_{\chi}\chi(\mathfrak{a}) = \sum_{\mathfrak{a}\in\mathcal{N}_1} N\mathfrak{a}^{-s} = (s-1)^{-(1/2)(|S|/[M:K_3])}h(s), \tag{4-7}$$

where the first two sums range over irreducible characters χ of $\mathrm{Gal}(N/K_3)$, and where $h(s)$ is a nonzero, holomorphic function for $\Re s \geq 1$.

Applying a Tauberian theorem of Wintner [1942] to (4-7), we obtain

$$\#\mathcal{N}_1(X) = (C+o(1))\frac{X}{\log(X)^{1-(1/2)(|S|/[M:K_3])}}. \tag{4-8}$$

Now, if $\mathfrak{a}\in\mathcal{N}_1$ we have, for some positive integer m, $\mathfrak{a}=\prod_{i=1}^m \mathfrak{p}_i$, where $\mathfrak{p}_i\in\mathcal{P}$. For each rational prime p_i below $\mathfrak{p}_i$, we have $p_i\mathcal{O}_{K_3}=\mathfrak{p}_i\mathfrak{q}_i$ where $\mathfrak{q}_i\in\mathcal{Q}$. Set $\mathfrak{b}=\prod_{i=1}^m\mathfrak{q}_i$.

First, we'll show $\mathrm{Frob}_{\mathfrak{q}_i}(M/K_3)\subset S$. If $E: y^2=f(T)$, consider the cubic extension $L=K_3(T)/(f(T))$, where $f(T)\in\mathbb{Q}[x]$ is some cubic polynomial, between M and K_3. If $\mathrm{Gal}(M/K_3)=C_3$, then $M=L$. We can look at how $f(T)$ factors modulo $\mathfrak{p}_i$ and $\mathfrak{q}_i$. The only way for $f(T)$ to be irreducible modulo $\mathfrak{q}_i$ (i.e., over $\mathbb{F}_{p_i^2}$) is for $f(T)$ to be irreducible modulo $\mathfrak{p}_i$; this happens precisely when $\mathrm{Frob}_{\mathfrak{p}_i}(M/K_3)\subset S$. If $f(T) \pmod{\mathfrak{q}_i}$ is irreducible, $\mathrm{Frob}_{\mathfrak{q}_i}(M/K_3)$ has order 3. That is, demanding $\mathrm{Frob}_{\mathfrak{p}_i}(M/K_3)\subset S$ forces $\mathrm{Frob}_{\mathfrak{q}_i}(M/K_3)\subset S$.

Second, since each $\mathfrak{p}_i\mathfrak{q}_i$ is principle, knowing $[\mathfrak{a}, N/K_3]=1$ suffices to show $[\mathfrak{b}, N/K_3]=1$, too. Thus, $\mathfrak{b}\in\mathcal{R}_1$.

Finally, since $N\mathfrak{b}=(N\mathfrak{a})^2$, we have established a bijection between $\mathcal{N}_1(X)$ and $\mathcal{R}_1(X^{1/2})$ by mapping $\mathfrak{a}\mapsto\mathfrak{b}$.

This and (4-8) give us

$$\#\mathcal{R}_1(X)\sim(C+o(1))\frac{X^{1/2}}{\log(X)^{1-(1/2)(|S|/[M:K_3])}}$$

for some positive constant C, as needed. □

We now state and prove the analogue of Lemma 4.3 in the case that K_3 is an admissible C_3-cubic resolvent.

Lemma 4.4. *Suppose $E/\mathbb{Q}$ is an elliptic curve where K_3 is an admissible C_3-cubic resolvent for E and* $\mathrm{Gal}(K_3(E[2]/K_3)$ *is C_3 or S_3. Let S be the set of the elements of order* 3 *in* $\mathrm{Gal}(\mathbb{Q}(E[2])/\mathbb{Q})$, *and N be an abelian extension of K_3. Then, the number of ideals $\mathfrak{b}$ of K_3 such that*

- $N\mathfrak{b}<X$, $N\mathfrak{b}$ *is a square, and* $[\mathfrak{b}, N/K_3]=1$; *and*
- *for every prime ideal $\mathfrak{p}$ dividing $\mathfrak{b}$,* $\mathrm{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3)\subset S$

is asymptotic to

$$(D+o(1))\frac{X^{1/2}}{\log(X)^{1-(1/2)\left(|S|/[K_3(E[2]):\mathbb{Q}]\right)}}$$

for some real, positive constant D, *and where* $[-, N/K_3]$ *is the global Artin symbol and* $\mathbf{N}$ *is the ideal norm.*

Proof. Since K_3 is an admissible C_3-cubic resolvent, we have that K_3 and $\mathbb{Q}(E[2])$ are linearly disjoint. Setting $M = K_3(E[2])$, we have $\mathrm{Gal}(M/\mathbb{Q}) = C_3 \times S_3$. First, some notation. Define the following sets:

- $\mathcal{P}_{\mathbb{Q}} = \{p \in \mathbb{N} \text{ prime} \mid \mathrm{Frob}_p(M/\mathbb{Q}) \subset \{1\} \times S,\ p \text{ unramified in } \tilde{N} K_3 \mathbb{Q}(E[2])\}$;
- $\mathcal{A} = \{a \mid a \text{ a squarefree product of } p \in \mathcal{P}_{\mathbb{Q}}\}$;
- $\mathcal{A}_1 = \{a \in \mathcal{A} \mid [(a), \tilde{N}/\mathbb{Q}] = 1\}$ where $[-, \tilde{N}/\mathbb{Q}]$ is the global Artin symbol.

Let $\tilde{N}$ be the normal closure of N over $\mathbb{Q}$. We will use the triviality of the Artin symbol $[-, \tilde{N}/\mathbb{Q}]$ to obtain the triviality of the Artin symbol $[-, N/\mathbb{Q}]$ as in the statement of the lemma. Now, for a character $\psi : \mathrm{Gal}(\tilde{N}/\mathbb{Q}) \to \mathbb{C}^{\times}$, and writing $\psi(a)$ for $\psi([(a), \tilde{N}/\mathbb{Q}])$, let

$$f_{\psi}(s) := \sum_{a \in \mathcal{A}} \psi(a) a^{-s}.$$

For two functions g_1 and g_2 defined on the complex half plane $\Re s > 1$, write $g_1(s) \sim g_2(s)$ to mean $g_1(s)$ and $g_2(s)$ differ by a function which is holomorphic on $\Re s \geq 1$. Taking log of the $f_{\psi}(s)$, substituting the Taylor series for $\log(1-x)$ and truncating the Taylor series after one term, one arrives at

$$\log f_{\psi}(s) = \sum_{p \in \mathcal{P}_{\mathbb{Q}}} \log(1+\psi(p)p^{-s}) \sim \sum_{p \in \mathcal{P}_{\mathbb{Q}}} \psi(p) p^{-s} \sim \delta_{\psi} \log \frac{1}{s-1}, \tag{4-9}$$

where, using the Chebotarev density theorem,

$$\delta_{\psi} = \begin{cases} 0 & \text{if } \psi \text{ is nontrivial,} \\ \dfrac{|S|}{[M:\mathbb{Q}]} & \text{if } \psi \text{ is trivial.} \end{cases}$$

Now, using character orthogonality and summing over irreducible characters ψ of $\mathrm{Gal}(\tilde{N}/\mathbb{Q})$, we have

$$\frac{1}{[\tilde{N}:\mathbb{Q}]} \sum_{\psi} f_{\psi}(s) = \frac{1}{[\tilde{N}:\mathbb{Q}]} \sum_{a \in \mathcal{A}} a^{-s} \sum_{\psi} \psi(a) = \sum_{a \in \mathcal{A}_1} a^{-s}. \tag{4-10}$$

But also, using (4-9), we have

$$\frac{1}{[\tilde{N}:\mathbb{Q}]} \sum_{\psi} f_{\psi}(s) = g(s)(s-1)^{-|S|/[M:\mathbb{Q}]}, \tag{4-11}$$

where $g(s)$ is holomorphic and nonzero on $\Re s \geq 1$. Thus, via (4-10) and (4-11),

$$\sum_{a \in \mathcal{A}_1} a^{-s} = g(s)(s-1)^{-|S|/[M:\mathbb{Q}]}.$$

Applying a Tauberian theorem of Wintner [1942] yields

$$\#\{a \in \mathcal{A}_1 \mid a < X\} = (C + o(1)) \frac{X}{(\log X)^{1-|S|/[M:\mathbb{Q}]}}$$

for some positive, real C.

Now, suppose $a \in \mathcal{A}_1$ and $a = \prod_{i=1}^{r} p_i$, where the p_i are distinct primes and $p_i \in \mathcal{P}_{\mathbb{Q}}$ and each p_i splits completely in $\mathcal{O}_{K_3}$. Set $\mathfrak{a} = a\mathcal{O}_{K_3}$. Then $\mathfrak{a}$ decomposes into prime ideals as $\mathfrak{a} = \prod_{i=1}^{r} \mathfrak{p}_i \mathfrak{p}_i' \mathfrak{p}_i''$ where $\mathfrak{p}_i$, $\mathfrak{p}_i'$, and $\mathfrak{p}_i''$ are the three primes above p_i. For each $\mathfrak{p}_i$ above a prime p_i, pick another prime $\mathfrak{p}_i'$ of K_3 above p_i (there are two choices), and set $\mathfrak{b} = \prod_{i=1}^{r} \mathfrak{p}_i \mathfrak{p}_i'$. Note $\boldsymbol{N}_{K_3/\mathbb{Q}}\mathfrak{b} = a^2$.

In this way, counting $a \in \mathcal{A}_1$ with $a < X$ gives a way of counting ideals $\mathfrak{b}$ in K_3 such that $\boldsymbol{N}_{K_3/\mathbb{Q}}\mathfrak{b} < X^{1/2}$ such that $\boldsymbol{N}_{K_3/\mathbb{Q}}\mathfrak{b}$ is a square and, for each prime $\mathfrak{p}$ dividing $\mathfrak{b}$, $\mathrm{Frob}_{\mathfrak{p}}(M/K_3) \subset S$ and $[\mathfrak{b}, N/K_3] = 1$. The lemma follows. □

For an elliptic curve E and each of the ideals enumerated in Lemma 4.3, there is a twist of E in which the 2-Selmer rank remains the same.

Lemma 4.5. *Keeping the notation of Lemma 4.3, if $\mathfrak{b}$ is an ideal of K_3 such that*

- $\boldsymbol{N}\mathfrak{b} < X$,
- *if a prime ideal $\mathfrak{p}$ divides $\mathfrak{b}$, then $\boldsymbol{N}\mathfrak{p}$ is a square,*
- $\mathrm{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$, *and*
- $[\mathfrak{b}, N/K_3] = 1$,

then there is a quadratic extension F/K_3 of conductor $\mathfrak{b}$ such that

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3).$$

Proof. This is Proposition 4.2 of [Mazur and Rubin 2010], with $N = K_3(8\Delta_E\infty)$, the ray class field of K_3 modulo $8\Delta_E$ and all archimedean places of K_3, applied to the relevant ideals, which are a subset of the ideals discussed in that result. □

For an elliptic curve E and each of the ideals enumerated in Lemma 4.4, there is a twist of E in which the 2-Selmer rank remains the same.

Lemma 4.6. *Keeping the notation of Lemma 4.4, if $\mathfrak{b}$ is an ideal of K_3 such that*

- $\boldsymbol{N}\mathfrak{b} < X$,
- *if a prime ideal $\mathfrak{p}$ divides $\mathfrak{b}$, then $\boldsymbol{N}\mathfrak{p}$ is a square,*
- $\mathrm{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$, *and*
- $[\mathfrak{b}, N/K_3] = 1$,

then there is a quadratic extension F/K_3 of conductor $\mathfrak{b}$ such that

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3).$$

Proof. The proof is the same as that of Lemma 4.5. □

We are now ready to prove the main result of the section. We follow exactly the strategy of the proof of [Mazur and Rubin 2010, Theorem 1.4] with the additional step of keeping track of the square norm condition of the involved quadratic twists.

Proof of Proposition 4.1. As in Lemmas 4.3 and 4.4, let S be the set of order-3 elements in $\mathrm{Gal}(K_3(E[2])/K_3)$. Then if $\mathrm{Gal}(K_3(E[2])/K_3) \simeq S_3$,

$$\frac{|S|}{[K_3(E[2]) : K_3]} = \frac{1}{3},$$

and if $\mathrm{Gal}(K_3(E[2])/K_3) \simeq \mathbb{Z}/3\mathbb{Z}$,

$$\frac{|S|}{[K_3(E[2]) : K_3]} = \frac{2}{3}.$$

We'll consider the case that K_3 is an admissible S_3-cubic resolvent; there are two subcases to consider:

(1) Suppose $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) = r$. By Lemmas 4.3 and 4.5, the number of square norm twists E^F/K_3 such that $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = r$ is

$$\gg \frac{X^{1/2}}{\log X^{1-(1/2)(|S|/[K_3(E[2]):K_3])}}.$$

(2) Suppose $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \neq r$. We have assumed there is a square norm twist E^L/K_3 such that $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^L/K_3) = r$. Note that a square norm twist of a square norm twist results in the square norm twist. That is, a square norm twist $(E^L)^{F'}$ of E^L is itself a square norm twist E^F of E. Now the result follows from Case (1) applied to E^L.

If instead K_3 is an admissible C_3-cubic resolvent for E, the proof is the same as above, but with Lemmas 4.4 and 4.6 in place of Lemmas 4.3 and 4.5, respectively. □

5. Decreasing the 2-Selmer rank

Our strategy will be to use Lemma 3.5(2) to understand the 2-Selmer rank of square norm twists. We'll then use Proposition 4.1 to show there are many square norm twist with prescribed 2-Selmer rank assuming we *have already* a square norm twist of that prescribed 2-Selmer rank. We'll show a square norm twist with that prescribed 2-Selmer rank *must exist* by showing we can take square norm twists that reduce the 2-Selmer rank by two; this is the content of Proposition 5.1.

Proposition 5.1 below can be viewed as analogous to Proposition 5.1(iii) of [Mazur and Rubin 2010]. Except, instead of decreasing the 2-Selmer rank by 1 via a quadratic twist obtained by controlling one local condition, we decrease the 2-Selmer rank by 2 via a square norm twist obtained by controlling two local conditions. Those two local conditions are obtained from two primes in the cubic resolvent above the same rational prime.

Proposition 5.1. *Let E be an elliptic curve defined over $\mathbb{Q}$ and let K_3 be an admissible cubic resolvent for E. Suppose further that*

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/\mathbb{Q}) = 0 \quad \textit{and} \quad \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \geq 2.$$

Then there exists a square norm twist E^F/K_3 such that

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) - 2.$$

Proof. Let Δ_E be the discriminant of (a minimal model of) E. For the admissible cubic resolvent, K_3, of E, set $M = K_3(E[2])$. Note that M/K_3 is a Galois S_3-extension, since K_3 is as in Definition 1.3, and $K_3(\sqrt{\Delta_E})/K_3$ is an intermediate quadratic extension. Let v_0 be the distinguished place of K_3 guaranteed by Definition 1.3.

Let Σ be a finite set that contains the following places of K_3: all infinite places of K_3, places of bad reduction for E, and all primes above 2. Now set $\mathfrak{d} = \prod_{v \in \Sigma \setminus \{v_0\}} v$. Let $K_3(8\mathfrak{d})$ be the ray class field of K_3 with modulus $8\mathfrak{d}$ and let $K_3[8\mathfrak{d}]$ be the maximal extension between $K_3(8\mathfrak{d})$ and K_3 whose degree is a power of 2.

Let $\tilde{K}_3$ be the Galois closure of K_3 over $\mathbb{Q}$ and $\mathfrak{D} = \prod_{V | v, v | \mathfrak{d}} V$ be the product of places of $\tilde{K}_3$ above the places of K_3 dividing $\mathfrak{d}$. Let L be the maximal extension[1] of $\tilde{K}_3$ between $\tilde{K}_3$ and the ray class field of $\tilde{K}_3$ with modulus $8\mathfrak{D}$. Note $L \supseteq K_3[8\mathfrak{d}]$, $[L : K_3]$ is a power of 2, and L is Galois over $\mathbb{Q}$.

By assumption, $\tilde{K}_3$ and $\mathbb{Q}(E[2])$ are linearly disjoint over $\mathbb{Q}$. $K_3[8\mathfrak{d}]$ and M are linearly disjoint as extensions of K_3 since v_0 is ramified in $K_3(\sqrt{\Delta_E})$ but not in $K_3[8\mathfrak{d}]$. By the conditions on v_0 of Definition 1.3, v_0 does not ramify from K_3 to $\tilde{K}_3$, and hence is unramified in L. So likewise, the same consideration of $K_3(\sqrt{\Delta_E})$ shows M and L are linearly disjoint over K_3.

Let σ be an element of the absolute Galois group of K_3 such that $\sigma|_M$ is a transposition in $\mathrm{Gal}(M/K_3) \simeq \mathrm{Aut}(E[2])$ and $\sigma|_L = 1$. The former condition implies $E[2]/(\sigma - 1)E[2] \simeq \mathbb{F}_2$. The latter condition implies $\sigma|_{K_3[8\mathfrak{d}]} = 1$.

For the rest of the proof, fix a nonzero map $\phi : \mathrm{Sel}_2(E/K_3) \to E[2]/(\sigma - 1)E[2]$. By [Mazur and Rubin 2010, Lemma 3.5], there is an element $\gamma \in G_{K_3}$ for which $\gamma = \sigma$ when restricted to $MK_3[8\mathfrak{d}]$ and $c(\gamma) = \phi(c)$ for all $c \in \mathrm{Sel}_2(E/K_3)$.

Let N be a Galois extension of $\mathbb{Q}$ containing M and L for which the restriction of $\mathrm{Sel}_2(E/K_3)$ to N is zero. For instance, take N to be the Galois closure (over $\mathbb{Q}$)

[1] Note that if K_3 is C_3-cubic, then $L = K_3[8\mathfrak{d}]$ since $\tilde{K}_3 = K_3$.

of the compositum of M, L, and the fixed field of the kernel of the restriction to $\mathrm{Hom}(G_M, E[2])$ of every $c \in \mathrm{Sel}_2(E/K_3)$.

Let $\mathcal{P}_\gamma$ be the set of primes $\mathfrak{p}$ of K_3 for which $\mathfrak{p} \notin \Sigma$ and $\mathrm{Frob}_\mathfrak{p}(N/K_3) = \gamma|_N$. By the Cebotarev density theorem, the natural density of $\mathcal{P}_\gamma$ among the primes of K_3 is positive, i.e.,

$$\delta_\gamma := \lim_{X\to\infty} \frac{\#\{\text{primes } \mathfrak{p} \text{ of } K_3 \mid \mathrm{Frob}_\mathfrak{p}(N/K_3) = \gamma|_N,\ \mathfrak{p} \notin \Sigma,\ \boldsymbol{N}\mathfrak{p} < X\}}{\#\{\text{primes } \mathfrak{p} \text{ of } K_3 \mid \boldsymbol{N}\mathfrak{p} < X\}} > 0.$$

Now, let $\mathcal{P}_{\mathrm{sp}}$ be the set of primes of K_3 with inertia degree one over $\mathbb{Q}$, that is, the primes $\mathfrak{p}$ for which the rational prime ideal $\mathfrak{p} \cap \mathbb{Z}$ splits completely in K_3. Recall that $\mathcal{P}_{\mathrm{sp}}$ has natural density one among the primes of K_3. In particular, this means we can pick a prime $\mathfrak{p}_1 \in \mathcal{P}_\gamma \cap \mathcal{P}_{\mathrm{sp}}$. If we could not, then $\mathcal{P}_\gamma$ (which has positive density) would be contained in the complement of $\mathcal{P}_{\mathrm{sp}}$ (which has density zero). Let p be the rational prime below $\mathfrak{p}_1$, and let $\mathfrak{p}_2$ and $\mathfrak{p}_3$ be the other two primes of K_3 above p, in other words, $p\mathcal{O}_{K_3} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$.

Our goal is now to construct a suitable square norm twist from two of $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$. We can understand the 2-Selmer group of this twist using Lemma 3.5(1), which requires us to compute both $H^1_f((K_3)_{\mathfrak{p}_i}, E[2])$ and $\mathrm{loc}_{\mathfrak{p}_i}\mathrm{Sel}_2(E/K_3)$.

First, consider the localization at $\mathfrak{p}_1$. Since $\mathrm{Frob}_{\mathfrak{p}_1} = \gamma$ when restricted to N (and $\sigma = \gamma$ when restricted to $MK_3[8\mathfrak{d}]$) we have both that

$$H^1_f((K_3)_{\mathfrak{p}_1}, E[2]) \simeq E[2]/(\sigma - 1)E[2] \simeq \mathbb{F}_2 \tag{5-1}$$

and $\phi(c) = c(\gamma)$ for all $c \in \mathrm{Sel}_2(E/K_3)$. The localization map

$$\begin{aligned} \mathrm{loc}_{\mathfrak{p}_1} : \mathrm{Sel}_2(E/K_3) \to H^1_f((K_3)_{\mathfrak{p}_1}, E[2]) &\simeq E[2]/(\mathrm{Frob}_{\mathfrak{p}_1} - 1)E[2] \\ &\simeq E[2]/(\sigma - 1)E[2] \simeq \mathbb{F}_2 \end{aligned}$$

is given by evaluation of cocycles at $\mathrm{Frob}_{\mathfrak{p}_1}$, so we can identify

$$\mathrm{loc}_{\mathfrak{p}_1}(\mathrm{Sel}_2(E/K_3)) = \phi(\mathrm{Sel}_2(E/K_3))$$

as subspaces of $\mathbb{F}_2$. Since ϕ is nonzero,

$$\dim_{\mathbb{F}_2} \mathrm{loc}_{\mathfrak{p}_1}(\mathrm{Sel}_2(E/K_3)) = 1. \tag{5-2}$$

It remains to understand the localizations at $\mathfrak{p}_2$ and $\mathfrak{p}_3$. Since p splits completely in K_3, there is an equality of local fields

$$(K_3)_{\mathfrak{p}_1} = (K_3)_{\mathfrak{p}_2} = (K_3)_{\mathfrak{p}_3} = \mathbb{Q}_p$$

and so, together with (5-1), we have

$$\begin{aligned} H^1_f((K_3)_{\mathfrak{p}_1}, E[2]) &= H^1_f((K_3)_{\mathfrak{p}_2}, E[2]) \\ &= H^1_f((K_3)_{\mathfrak{p}_3}, E[2]) = H^1_f(\mathbb{Q}_p, E[2]) \simeq \mathbb{F}_2. \end{aligned} \tag{5-3}$$

Beginning from the following commutative diagram, we will consider the localization of $\mathrm{Sel}_2(E/K_3)$ at primes of K_3 above p and localization at p:

$$\begin{array}{ccc} H^1(K_3, E[2]) & \xrightarrow{\bigoplus_{i=1}^3 \mathrm{loc}_{\mathfrak{p}_i}} & \bigoplus_{i=1}^3 H^1((K_3)_{\mathfrak{p}_i}, E[2]) \\ {\scriptstyle \mathrm{cores}_{K_3/\mathbb{Q}}}\downarrow & & \downarrow{\scriptstyle \bigoplus_{i=1}^3 \mathrm{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p}} \\ H^1(\mathbb{Q}, E[2]) & \xrightarrow{\mathrm{loc}_{\{p\}}} & H^1(\mathbb{Q}_p, E[2]) \end{array}$$

where the vertical map on the left side,

$$\mathrm{cores}_{K_3/\mathbb{Q}} : H^1(K_3, E[2]) \to H^1(\mathbb{Q}, E[2]),$$

is determined by corestriction on Galois cohomology induced by the norm map $N_{K_3/\mathbb{Q}} : K_3 \to \mathbb{Q}$ (see [Milne 2020, Example 1.29] or [Serre 1997]). The vertical map on the right side is the sum of corestriction maps:

$$\bigoplus_{i=1}^{3} \mathrm{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p} : \bigoplus_{i=1}^{3} H^1((K_3)_{\mathfrak{p}_i}, E[2]) \to H^1(\mathbb{Q}_p, E[2]),$$
$$(c_1, c_2, c_3) \mapsto \sum_{i=1}^{3} \mathrm{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p}(c_i).$$

Restricting the left column to the 2-Selmer groups of E over K_3 and $\mathbb{Q}$ and to restricted cohomology on the right column, together with (5-3), we have

$$\begin{array}{ccc} \mathrm{Sel}_2(E/K_3) & \xrightarrow{\mathrm{cores}_{K_3/\mathbb{Q}}} & \mathrm{Sel}_2(E/\mathbb{Q}) = 0 \\ \downarrow{\scriptstyle \bigoplus_{i=1}^3 \mathrm{loc}_{\mathfrak{p}_i}} & & \downarrow{\scriptstyle \mathrm{loc}_p} \\ \bigoplus_{i=1}^3 H^1_f((K_3)_{\mathfrak{p}_i}, E[2]) & \xrightarrow{\bigoplus_{i=1}^3 \mathrm{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p}} & H^1_f(\mathbb{Q}_p, E[2]) \simeq \mathbb{F}_2 \\ \downarrow{\scriptstyle \simeq} & \nearrow{\scriptstyle v_1+v_2+v_3} & \\ H^1_f(\mathbb{Q}_p, E[2])^{\oplus 3} \simeq \mathbb{F}_2^3 & & \end{array}$$

where the diagonal map $\mathbb{F}_2^3 \to \mathbb{F}_2$ is coordinatewise addition of vectors in $\mathbb{F}_2^3$ modulo 2. For $c \in \mathrm{Sel}_2(E/K_3)$ we have $\mathrm{loc}_p\mathrm{cores}(c) = 0$ since $\mathrm{Sel}_2(E/\mathbb{Q}) = 0$. Hence,

$$\mathrm{loc}_{\mathfrak{p}_1}(c) + \mathrm{loc}_{\mathfrak{p}_2}(c) + \mathrm{loc}_{\mathfrak{p}_3}(c) = 0 \quad \text{in } \mathbb{F}_2. \tag{5-4}$$

By (5-2) there is an element $c \in \mathrm{Sel}_2(E/K_3)$ for which $\mathrm{loc}_{\mathfrak{p}_1}(c) = 1$ viewed in $\mathbb{F}_2$. Combining this with (5-4), there is exactly one prime $\mathfrak{p}_i \in \{\mathfrak{p}_2, \mathfrak{p}_3\}$ for which $\mathrm{loc}_{\mathfrak{p}_i}(c) = 1$; suppose, without loss of generality, it is $\mathfrak{p}_2$. Whence,

$$\dim_{\mathbb{F}_2} \mathrm{loc}_{\mathfrak{p}_2}\mathrm{Sel}_2(E/K_3) = 1. \tag{5-5}$$

Finally, we will twist E/K_3 by a quadratic extension F/K_3 ramified only at $\mathfrak{p}_1$ and $\mathfrak{p}_2$ to get our desired result.

Let $\mathfrak{P}$ be a prime of L above $\mathfrak{p}_1$. Since $L/\mathbb{Q}$ is Galois, we have

$$\mathrm{Frob}_{\mathfrak{P}}(L/\mathbb{Q})^{f(\mathfrak{p}_1/p)} = \mathrm{Frob}_{\mathfrak{P}}(L/\mathbb{Q}) = \mathrm{Frob}_{\mathfrak{P}}(L/K_3) = 1,$$

i.e., p splits completely in L, and so p splits completely in $K_3[8\mathfrak{d}]$.

Since our choice of Frobenius class for p is trivial when restricted just to $K_3[8\mathfrak{d}]$, $\mathrm{Frob}_{\mathfrak{p}_1}(K_3[8\mathfrak{d}]/K_3) = \mathrm{Frob}_{\mathfrak{p}_2}(K_3[8\mathfrak{d}]/K_3) = 1$, and since $[K_3(8\mathfrak{d}) : K_3[8\mathfrak{d}]]$ is odd, there will be an odd integer, say h, such that $(\mathfrak{p}_1\mathfrak{p}_2)^h$ is principal with generator α such that $\alpha \equiv 1 \pmod{8\mathfrak{d}}$ and α positive at all real embeddings except possibly v_0.

We are now in a position to construct the quadratic extension F/K_3 by which we will twist E: Define $F = K_3\big(\sqrt{\alpha}\big)$. Only the primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of K_3 ramify in F. Set $p(x) = x^2 - \alpha$ and note $p'(1)^2 = 4$. For any $\mathfrak{q} \in \Sigma \setminus \{v_0\}$, since $\alpha \equiv 1 \pmod{8\mathfrak{d}}$, we also have $p(1) = 1 - \alpha \equiv 0 \pmod{4\mathfrak{q}}$. From Hensel's lemma (see, e.g., [Eisenbud 1995, Theorem 7.3] for an applicable statement), it follows $p(x)$ has a root in $(K_3)_{\mathfrak{q}}$. So $(K_3)_{\mathfrak{q}} \otimes F = (K_3)_{\mathfrak{q}}^2$, that is, $\mathfrak{q}$ splits in F. Thus all primes in $\Sigma \setminus \{v_0\}$ split in F.

Also, $\boldsymbol{N}(\alpha) = \boldsymbol{N}(\mathfrak{p}_1)\boldsymbol{N}(\mathfrak{p}_2) = p^{2h}$, so the quadratic twist E^F/K_3 of E/K_3 is a square norm twist.

Finally, apply Lemma 3.5(1) with $T = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $F = K_3\big(\sqrt{\alpha}\big)$ as above, (5-2) and (5-5), we get

$$\begin{aligned}
&\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) \\
&= \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) - 2\dim_{\mathbb{F}_2} V_T + \sum_{\mathfrak{r}\in T} \dim_{\mathbb{F}_2} H^1_f((K_3)_{\mathfrak{r}}, E[2]) \\
&= \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) - 2\dim_{\mathbb{F}_2} \mathrm{loc}_{\{\mathfrak{p}_1\}}(\mathrm{Sel}_2(E/K_3)) - 2\dim_{\mathbb{F}_2} \mathrm{loc}_{\{\mathfrak{p}_2\}}(\mathrm{Sel}_2(E/K_3)) \\
&\qquad + \dim_{\mathbb{F}_2} H^1_f((K_3)_{\mathfrak{p}_1}, E[2]) + \dim_{\mathbb{F}_2} H^1_f((K_3)_{\mathfrak{p}_2}, E[2]) \\
&= \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) - 2
\end{aligned}$$

Noting again the twist by F above is a square norm twist, we have the desired result. □

6. Proofs of the main theorems

We are now ready to prove Theorem 1.4. We'll then show how it implies Theorem 1.1. Again, the "infinitely many" of both theorems is quantified by Proposition 4.1.

Proof of Theorem 1.4. Let E and K_3 be as in Theorem 1.4. If $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv 0 \pmod 2$, repeated application of Proposition 5.1 gives a square norm twist L such that $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^L/K_3) = 0$. Once we have L, Proposition 4.1 provides infinitely many more square norm twists F with $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = 0$.

Likewise, if $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3) \equiv 1 \pmod 2$, the argument above provides infinitely many more square norm twists F with $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = 1$. □

Now we can prove Theorem 1.1 as a consequence of Theorem 1.4.

Proof of Theorem 1.1. Let E and K_3 be as in the statement of the theorem. Theorem 1.1 is essentially an immediate consequence of Theorem 1.4 coupled with the upper bound the dimension of the 2-Selmer group provides for the rank.

As in Theorem 1.4, there are two cases. In the first case, $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3)$ is even. In this case, Theorem 1.4 provides infinitely many square norm twists E^F/K_3 for which $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) = 0$.

From (3-1), we have $\mathrm{rk}(E/K_3) \leq \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3)$. Thus, our infinitely many square norm twists E^F of 2-Selmer rank zero give us

$$0 = \dim_{\mathbb{F}_2} \mathrm{Sel}_2(E^F/K_3) \geq \mathrm{rk}(E^F/K_3) = \mathrm{rk}(E/F) - \mathrm{rk}(E/K_3).$$

If $K/\mathbb{Q}$ is the S_4-quartic corresponding to quadratic extension F/K_3 corresponding to each square norm twist, then having no rank growth from K_3 to F means we have no rank growth from $\mathbb{Q}$ to K for infinitely many K.

In the second case, $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E/K_3)$ is even. Then, in the same way as above, there are infinitely many square norm twists E^F of 2-Selmer rank one. The result follows if we assume the parity of the rank and 2-Selmer dimension are the same. □

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OBSTRUCTION COMPLEXES IN GRID HOMOLOGY

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Recently, Manolescu–Sarkar constructed a stable homotopy-type for link Floer homology, which uses grid homology and accounts for all domains that do not pass through a specific square. In doing so, they produced an obstruction chain complex of the grid diagram with that square removed. We define the obstruction chain complex of the full grid, without the square removed, and compute its homology. Though this homology is too complicated to immediately extend the Manolescu–Sarkar construction, we give results about the existence of sign assignments in grid homology.

1. Introduction

Link Floer homology, developed by Ozsváth and Szabó [2004a; 2008], and Rasmussen [2003] is an invariant of oriented links in three-manifolds which comes from Heegaard Floer homology, from [Ozsváth and Szabó 2004c; 2004b]. Manolescu et al. [2007; 2009a] and Ozsváth et al. [2015] gave a combinatorial description of the link Floer chain complex for a link in S^3 using grid diagrams, known as grid homology. A toroidal grid diagram is a $n \times n$ grid of squares, with the left and right edges identified and the top and bottom edges identified, together with markings X and O, such that each row and column contains exactly one X and one O. Given a grid diagram $\mathbb{G}$, drawing vertical segments from the X to the O in each column and horizontal segments — going under the vertical segments whenever they cross — from the O to the X in each row gives the diagram of an oriented link L; we say that $\mathbb{G}$ is a grid diagram for L. Figure 1 shows a 5×5 grid diagram for the trefoil. The grid chain complex is generated by unordered n-tuples of intersection points between the horizontal and vertical circles — Figure 1 shows an example of such a generator.

Grid diagrams have been useful in a variety of applications in Heegaard Floer homology. Manolescu et al. [2009b] and Manolescu and Ozsváth [2010] obtain the Heegaard Floer invariants of 3- and 4-manifolds using grid diagrams, which give algorithmically computable descriptions. Sarkar [2011] uses grid homology to give another proof of Milnor's conjecture on the slice genus of torus

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Keywords: link Floer, stable homotopy-type, spectrum, sign assignment.

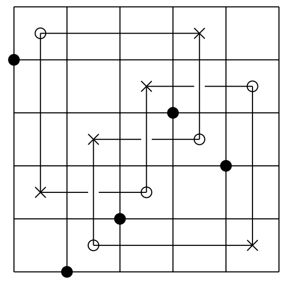


Figure 1. A 5×5 grid diagram for the trefoil, along with the generator [51243] drawn with $\bullet$. Note that the generator is independent of the X and O markings.

knots. Ozsváth et al. [2008], Ng et al. [2008], Chongchitmate and Ng [2013], and Khandhawit and Ng [2010] use a version of grid homology to prove results about Legendrian knots.

Manolescu and Sarkar [2021] constructed a stable homotopy refinement of knot Floer homology from the grid chain complex, using framed flow categories in the sense of [Cohen et al. 1995]. The Manolescu–Sarkar construction uses only those domains that do not pass through a particular square on the grid, and uses obstruction theory. Their obstruction chain complexes CD_* and CDP_*, which we will henceforth denote $\widehat{\mathrm{CD}}_*$ and $\widehat{\mathrm{CDP}}_*$, respectively, have simple enough homology to construct a stable homotopy-type. We will extend them to complexes CD_* and CDP_* which contain all domains in the grid, first with $\mathbb{Z}/2$ coefficients, which can be extended to $\mathbb{Z}$ coefficients using obstruction theory. We take the first step towards extending the Manolescu–Sarkar construction, by computing the homology of CD_* and partially computing the homology of CDP_*.

To state our main results, we fix the following convention throughout the paper. For a ring R, R^{2^n} will denote the chain complex given by $R^{\binom{n}{k}}$ in grading k with no differentials, and $R[U]$ the chain complex given by R in every nonnegative even grading and 0 in every odd grading (which by definition has no differentials). We begin by showing that:

Proposition 1.1. *$H_*(\mathrm{CD}_*; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2[U]$.*

In order to frame the moduli spaces in the Manolescu–Sarkar construction, we will need a sign assignment for the grid diagram. A sign assignment is a particular way of orienting the index 1 domains in Heegaard Floer homology; equivalently, it is a particular assignment of 0 or 1 to each rectangle in the grid. The existence and uniqueness (up to gauge equivalence) of sign assignments for toroidal grid diagrams was constructed in [Manolescu et al. 2007]; see also [Gallais 2008] for an explicit construction. In the course of our later computations, we will provide a different proof of this fact via obstruction theory:

Theorem 1.2. *Sign assignments for* CD_* *exist and are unique up to gauge equivalence* (*equivalently*, *up to* 1*-coboundaries of* CD_* *with* $\mathbb{Z}/2$ *coefficients*).

Given a sign assignment for CD_*, we obtain a definition of CD_* in $\mathbb{Z}$ coefficients. Perhaps unsurprisingly, we then obtain the following analogue of Proposition 1.1.

Proposition 1.3. $H_*(\mathrm{CD}_*; \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}[U]$.

In the end, our eventual goal is to extend the Manolescu–Sarkar construction over the full grid. Since the moduli spaces presented in [Manolescu and Sarkar 2021] exhibit some bubbling, we will compute the first few homology groups of CDP_*. Unfortunately, CDP_* has too much homology to immediately construct a stable homotopy-type. So instead, we will work towards constructing a framed 1-flow category, which is a formulation by Lobb et al. [2020] that still contains all the information needed to define invariants such as the second Steenrod square. This requires only a sign assignment and a frame assignment, whose obstructions lie in the following lower homologies.

Theorem 1.4. *We have that*:

(0) $H_0(\mathrm{CDP}_*; \mathbb{Z}/2)$ *is isomorphic to* $\mathbb{Z}/2$.

(1) $H_1(\mathrm{CDP}_*; \mathbb{Z}/2)$ *is isomorphic to* $(\mathbb{Z}/2)^n$.

(2) $H_2(\mathrm{CDP}_*; \mathbb{Z}/2)$ *is isomorphic to* $(\mathbb{Z}/2)^{\binom{n}{2}+1}$.

(3) $H_3(\mathrm{CDP}_*; \mathbb{Z}/2)$ *is isomorphic to* $(\mathbb{Z}/2)^{\binom{n}{3}+n}$.

In this paper, we'll show existence and uniqueness of sign assignments for CDP_*.

Theorem 1.5. *A sign assignment* s *on* CDP_* *exists*, *and is unique up to gauge transformations and the values of*

$$s_j := s(c_{x^{\mathrm{Id}}}, \vec{e}_j, (1)).$$

(The elements $(c_{x^{\mathrm{Id}}}, \vec{e}_j, (1)) \in \mathrm{CDP}_*$ will be defined later in Section 4.)

Just like for CD_*, we can use Theorem 1.5 to define CDP_* with $\mathbb{Z}$ coefficients. We have the following analogue of Theorem 1.4.

It remains to find a frame assignment for CDP_* using the above homology computation, and to complete the construction of the 1-flow category for the full grid, which we will carry out in a future paper. This present paper may be treated as a prelude thereof.

2. The obstruction complex

Definitions related to grid diagrams are summarized below. For details, see [Manolescu et al. 2007; 2009a; Ozsváth et al. 2015].

- An index n grid diagram $\mathbb{G}$ is a torus together with n α-circles (drawn horizontally) and n β-circles (drawn vertically). The complements of the α (respectively, β) circles are called the horizontal (respectively, vertical) annuli — the complements of the α and β circles are called the square regions.

- Each vertical and horizontal annulus contains exactly one X and O marking, which are arbitrarily labeled $X_1, \dots, X_n$ and $O_1, \dots, O_n$.

- The horizontal (respectively, vertical) annuli can be labeled by which O-marking they pass through — write H_i (respectively, V_i) for the horizontal (respectively, vertical) annulus passing through O_i.

- Given a fixed planar drawing of the grid, we can also label the α circles $\alpha_1, \dots, \alpha_n$ from bottom to top, and the β circles $\beta_1, \dots, \beta_n$ from left to right. The annuli can also be labeled by which sets of α or β circles they lie between — write $H_{(i)}$ (respectively, $V_{(i)}$) for the horizontal annulus between α_i and α_{i+1} (respectively, vertical annulus between β_i and β_{i+1}). Note that $H_{(n)}$ and $V_{(n)}$ lie between α_n and α_1, and β_n and β_1, respectively.

- A generator is an unordered n-tuple of points such that each α and β circle contains exactly one. Generators can equivalently be viewed a $\mathbb{Z}$-linear combination of n points, or alternatively as permutations — for a permutation $\sigma \in S_n$ the generator x^σ is the unique generator with a point at each $\alpha_{\sigma(i)} \cap \beta_i$. In this paper we will use the convention that $[a_1 a_2 \dots a_n]$ denotes the permutation $\sigma \in S_n$ where $\sigma(j) = a_j$ for each j. For instance, Figure 1 shows the generator $x^{[51243]}$, which we will interchangeably denote as [51243].

- A domain is a $\mathbb{Z}$-linear combination of square regions with the property that $\partial(\partial D \cap \alpha) = y - x$ for some generators x, y. We say that D is a domain from x to y, and write $D \in \mathscr{D}(x, y)$. D is said to be positive if none of its coefficients are negative, in which case we would write $D \in \mathscr{D}^+(x, y)$.

- Given $D \in \mathscr{D}(x, y)$, $E \in \mathscr{D}(y, z)$, we get a domain $D * E \in \mathscr{D}(x, z)$ by adding D and E as 2-chains.

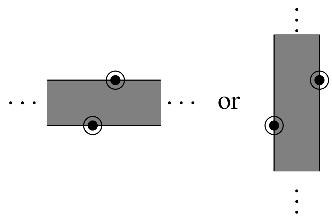
- The constant domain from a generator x to itself is the domain $c_x \in \mathscr{D}(x, x)$ whose coefficients are zero in every square region.

- For every domain D, there is an associated integer $\mu(D)$ called its Maslov index, which satisfies:
  - $\mu(D * E) = \mu(D) + \mu(E)$.
  - For a positive domain D, $\mu(D) \geq 0$, with equality if and only if D is some constant domain.
  - For $D \in \mathscr{D}^+(x, y)$, $\mu(D) = 1$ if and only if D is a rectangle: that is, its bottom left and top right corners are coordinates of x, its bottom right and top left

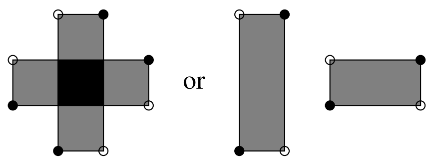
corners are coordinates of y, and the other $n-2$ coordinates of x and y agree and do not lie in D.

- $\mu(D) = k$ if and only if D can be decomposed (not necessarily uniquely) into k rectangles $D = R_1 * \cdots * R_k$.

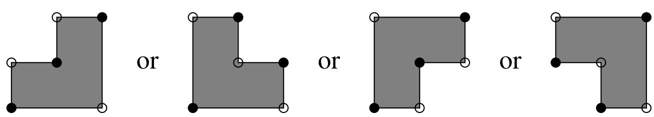
It will be particularly helpful to classify positive index 2 domains, which are exactly those that can be decomposed as two rectangles. Thus every positive index 2 domain $D \in \mathscr{D}^+(x, y)$ is a horizontal or vertical annulus



or two rectangles (overlapping or disjoint)



or a hexagon of the following shape



(Here the generator x is shown with $\bullet$ while y is shown with $\circ$.) Note that while a horizontal or vertical annulus admits exactly one decomposition into rectangles, all the other positive index 2 domains admit exactly two.

Given a grid diagram $\mathbb{G}$, we define the complex of positive domains, on which our desired sign assignment can be constructed as a cochain.

Definition 2.1. The complex of positive domains $\mathrm{CD}_* = \mathrm{CD}_*(\mathbb{G}; \mathbb{Z}/2)$ is freely generated over $\mathbb{Z}/2$ by the positive domains, with the homological grading being the Maslov index

$$\mathrm{CD}_k = \mathbb{Z}/2\langle\{(x, y, D) \mid D \in \mathscr{D}^+(x, y), \mu(D) = k\}\rangle.$$

Sometimes the generators x, y will be omitted. The differential $\partial : \mathrm{CD}_k \to \mathrm{CD}_{k-1}$ of $D \in \mathscr{D}^+(x, y)$ is given by

$$\partial(D) = \sum_{R*E=D} E + \sum_{E*R=D} E,$$

where R is a rectangle, and E is a positive domain.

Note that CD_* is independent of the placement of the X's and O's.

Lemma 2.2. $(\mathrm{CD}_*, \partial)$ *is a chain complex*, *that is*, $\partial^2 = 0$.

Proof. Let R and S denote rectangles, then

$$\partial^2(D) = \sum_{R*S*E=D} E + \sum_{R*E*S=D} E + \sum_{S*E*R=D} E + \sum_{E*S*R=D} E.$$

The second and third terms cancel (modulo 2). If $R * S$ is a hexagon or two rectangles, then it has exactly one other decomposition $R * S = R' * S'$, so $R * S * E$ and $R' * S' * E$ cancel in the first term. Similarly, $E * R * S$ and $E * R' * S'$ cancel in the last term. Finally, if $R * S$ is not a hexagon or two rectangles, it must be a horizontal or vertical annulus, and then the terms $E * R * S$ and $R * S * E$ in the first and last term cancel, and so $\partial^2(D) = 0$. □

We now compute the homology CD_* by constructing filtrations, for which we need the following fact. Given two generators x and y, we say that $x \le y$ if there exists a positive domain from y to x that does not intersect the topmost row $H_{(n)}$ or rightmost column $V_{(n)}$ of the grid. It is clear that the set of generators with $\le$ is a partially ordered set (which actually coincides with the opposite of the Bruhat ordering on the symmetric group S_n, see [Manolescu and Sarkar 2021, Section 3.2]).

Proof of Proposition 1.1. The proof is nearly identical to the proof of [Manolescu and Sarkar 2021, Proposition 3.4], so we present the most relevant parts. To $D \in \mathrm{CD}_*$ associate $A(D) \in \mathbb{N}^n$ by its coefficients in the rightmost vertical annulus. Note that here, unlike in [Manolescu and Sarkar 2021], $A(D)$ is an n-tuple, since there is no assumption that domains do not pass through the top right corner. By definition, the differential only preserves or lowers $A(D)$, so it is a filtration on CD_*. Now let CD_*^a be the associated graded complex in filtration grading $A(D) = a$.

Let $M(D) = \min\{\text{coordinates of } A(D)\}$ — by definition, a positive domain D contains exactly $M(D)$ copies of the rightmost vertical annulus $V_{(n)}$, so write $D = D' * M(D)V_{(n)}$. $A(D')$ contains a 0, so without loss of generality (since the differential of CD_*^a does not change $A(D)$ and thus does not change where the 0 is located) D' does not contain the top right corner. Now let $B(D) \in \mathbb{N}^{n-1}$ be the coordinates of D' in the top row (except the top right corner). Similarly, $B(D)$ is a filtration on the associated graded complex CD_*^a, so let $\mathrm{CD}_*^{a,b}$ be the associated graded complex in grading $A(D) = a$, $B(D) = b$.

Now fix (a, b) and consider the differential ∂ on $\mathrm{CD}_*^{a,b}$. Consider the following new filtration. For any domain $D \in \mathscr{D}^+(x, y)$ with $A(D) = a$, $B(D) = b$, let the generator y be its filtration grading. With respect to the aforementioned partial ordering of the generators, ∂ preserves or decreases y since we only consider removing domains that do not pass through the topmost row and rightmost column. Therefore y is a filtration grading, so let $\mathrm{CD}_*^{a,b,y}$ be the associated graded complex with respect to this filtration. Unless $a=(l,l,\dots,l)$, $b=0$, and $y=x^{\mathrm{Id}}$, the proof in [Manolescu and Sarkar 2021] shows that $\mathrm{CD}_*^{a,b,y}$ is acyclic. When $a=(l,l,\dots,l)$, $b=0$, and $y=x^{\mathrm{Id}}$, the complex $\mathrm{CD}_*^{a,b,y}$ has one generator (since x^{Id} is maximal), which is represented by the domain $lV_{(n)}$, lying in grading $2l$.

Finally, because the associated graded complex has homology only in even gradings, CD_* must have the same homology. □

In order to later remove obstructions in grading 2, we now explicitly find the generator U of $H_2(\mathrm{CD}_*)$. We define the following index 2 domains (see Figure 2):

- $A_1,\dots,A_{n-1}$ where A_i is the vertical annulus in the $(n-i)$-th column from the left from the generator $[n23\dots(n-i)\,1(n-i+1)\dots(n-1)]$ to itself, and A_0 is the rightmost vertical annulus from the identity generator x^{Id} to itself.
- $B_1,\dots,B_{n-1}$ where B_i is the horizontal annulus in the $(n-i)$-th row from the bottom from the generator $[(n-i+1)\,23\dots(n-i)(n-i+2)\dots n1]$ to itself, and B_0 is the topmost horizontal annulus from the identity generator x^{Id} to itself.
- $C_1,\dots,C_{n-2}$ where C_i is a hexagon from the generator

$$[n23\dots(n-i)\,1(n-i+1)\dots(n-1)]$$

to the generator $[12\dots(n-i-1)\,n(n-i)\dots(n-1)]$.

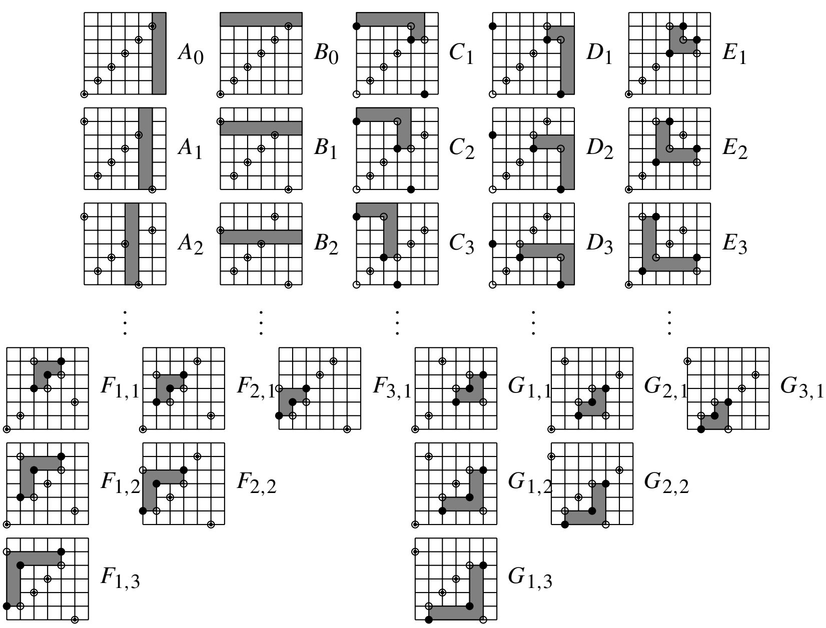


Figure 2. The domains A_i, B_i, C_i, D_i, E_i, $F_{i,j}$, and $G_{i,j}$ in the special case of a 6×6 grid, where each domain is drawn from a generator x (drawn with •) to a generator y (drawn with ∘).

• $D_1, \dots, D_{n-2}$ where D_i is a hexagon from the generator

$$[(n-i+1)\,23\dots(n-i)(n-i+2)\dots n1]$$

to the generator $[12\dots(n-i-1)(n-i+1)\dots n(n-i)]$.

• $E_1, \dots, E_{n-2}$ where E_i is a hexagon from the generator

$$[12\dots(n-i-1)\,n(n-i+1)\dots(n-1)(n-i)]$$

to the generator $[12\dots(n-i-2)\,n(n-i)\dots(n-1)(n-i-1)]$.

• $F_{i,1}, \dots, F_{i,n-i-2}$ for each $i = 1, \dots n-3$, where $F_{i,j}$ is a hexagon from the generator

$$[12\dots(n-i-j-2)(n-i-j)(n-i)(n-i-j+1) \\ \dots(n-i-1)(n-i+1)\dots n(n-i-j-1)]$$

to the generator

$$[12\dots(n-i-j-2)(n-i+1)(n-i-j)(n-i-j+1) \\ \dots(n-i)(n-i+2)\dots n(n-i-j-1)].$$

• $G_{i,1}, \dots, G_{i,n-i-2}$ for each $i = 1, \dots n-3$, where $G_{i,j}$ is a hexagon from the generator

$$[12\dots(n-i-j-2)\,n(n-i-j-1)(n-i-j+1) \\ \dots(n-i-1)(n-i-j)(n-i)\dots(n-1)]$$

to the generator

$$[12\dots(n-i-j-2)\,n(n-i-j)\dots(n-i)(n-i-j-1)(n-i+1)\dots(n-1)].$$

Let

$$U := \sum_{i=0}^{n-1}(A_i + B_i) + \sum_{i=1}^{n-2}(C_i + D_i) + \sum_{i=1}^{n-2} E_i + \sum_{i=i}^{n-3}\sum_{j=1}^{n-i-2}(F_{i,j} + G_{i,j})$$

Proposition 2.3. *U is the generator of* $H_2(\mathrm{CD}_*)$

Proposition 2.3 will follow from the following computational lemmas.

Lemma 2.4. *U is a cycle in* CD_2 (*that is*, $\partial U = 0$).

Proof. We consider the possible rectangles that appear in ∂U, starting with the following rectangles that will be useful to name for the purposes of giving signs later.

• $R_{1,2}, \dots, R_{1,n-1}$ where $R_{1,i}$ is the $1 \times i$ rectangle from

$$[n23\dots(n-i+1)\,1(n-i+2)\dots(n-1)]$$

to $[n23\dots(n-i)\,1(n-i+1)\dots(n-1)]$, and $R_{1,1}$ is the 1×1 rectangle from x^{Id} to $[n23\dots(n-1)1]$.

- $R_{2,2}, \ldots, R_{2,n-1}$ where $R_{2,i}$ is the $1 \times i$ rectangle from

$$[12\ldots(n-i)\,n(n-i+1)\ldots(n-1)]$$

to $[12\ldots(n-i-1)\,n(n-i)\ldots(n-1)]$, and $R_{2,1}$ is the 1×1 rectangle from x^{Id} to $[12\ldots(n-2)\,n(n-1)]$

- $R_{3,2}, \ldots, R_{3,n-1}$ where $R_{3,i}$ is the $i \times 1$ rectangle from

$$[(n-i+1)\,23\ldots(n-i)(n-i+2)\ldots n1]$$

to $[(n-i)\,23\ldots(n-i-1)(n-i+1)\ldots n1]$, and $R_{3,1} = R_{1,1}$.

- $R_{4,2}, \ldots, R_{4,n-1}$ where $R_{4,i}$ is the $i \times 1$ rectangle from

$$[12\ldots(n-i)(n-i+2)\ldots n(n-i+1)]$$

to $[12\ldots(n-i-1)(n-i+1)\ldots n(n-i)]$, and $R_{4,1} = R_{2,1}$.

- $R_{5,1}, \ldots, R_{5,n-2}$, where $R_{5,i}$ is the $1 \times i$ rectangle from

$$[12\ldots(n-i-2)(n-i)\,n(n-i+1)\ldots(n-1)(n-i-1)]$$

to $[12\ldots(n-i-2)\,n(n-i)\ldots(n-1)(n-i-1)]$.

- $R_{6,1}, \ldots, R_{6,n-2}$, where $R_{6,i}$ is the $i \times 1$ rectangle from

$$[12\ldots(n-i-2)\,n(n-i-1)(n-i+1)\ldots(n-1)(n-i)]$$

to $[12\ldots(n-i-2)\,n(n-i)\ldots(n-1)(n-i-1)]$.

We cancel each of these rectangles in the boundary as follows (see Figure 3):

- $R_{1,1}$ occurs in ∂U twice, from ∂A_0 and ∂B_0, so it cancels in ∂U. $R_{1,i}$ occurs in ∂A_{i-1} and ∂C_{i-1} for $i = 2, \ldots n-1$, so they also cancel in ∂U.

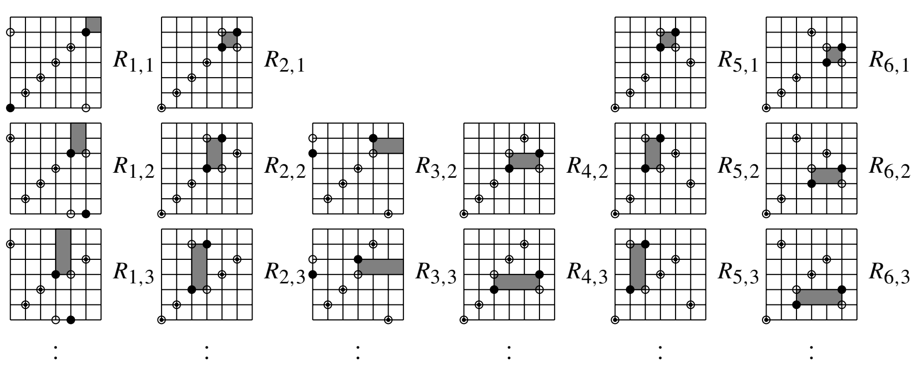


Figure 3. The rectangles $R_{1,i}$, $R_{2,i}$, $R_{3,i}$, $R_{4,i}$, $R_{5,i}$, $R_{6,i}$, where each rectangle is drawn from a generator x (drawn with $\bullet$) to a generator y (drawn with $\circ$).

- $R_{2,1}$ occurs in ∂U twice, from ∂C_1 and ∂D_1, so it cancels in ∂U, $R_{2,n-1}$ occurs in ∂A_{n-1} and ∂E_{n-2}, and for $i = 2, \dots n-2$, $R_{2,i}$ occurs in ∂C_i and ∂E_i, so they also cancel in ∂U.
- For $i = 2, \dots n-1$, $R_{3,i}$ occurs in ∂B_{i-1} and ∂D_{i-1}.
- For $i = 2, \dots n-2$, $R_{4,i}$ occurs in ∂D_i and ∂E_{i-1}.
- $R_{5,1}$ occurs in ∂E_1 and $\partial F_{1,1}$; for $i = 2, \dots n-2$, $R_{5,i}$ occurs in ∂E_i and $\partial F_{1,i-1}$.
- $R_{6,1}$ occurs in ∂E_1 and $\partial G_{1,1}$; for $i = 2, \dots n-2$, $R_{6,i}$ occurs in ∂E_i and $\partial G_{1,i-1}$.

Next, we consider the following rectangles:

- $P_{i,1} \dots P_{i,n-i-1}$ for each $i = 2 \dots n-2$, where $P_{i,j}$ is the $1 \times j$ rectangle from

$$[12\dots(n-i-j-1)(n-i-j+1)(n-i+1)(n-i-j+2)\\ \dots(n-i)(n-i+2)\dots n(n-i-j)]$$

to $[12\dots(n-i-j-1)(n-i+1)(n-i-j+1)\dots(n-i)(n-i+2)\dots n(n-i-j)]$, and $P_{1,j} = R_{5,j}$ for each $j = 1, \dots n-2$.

- $Q_{i,1} \dots Q_{i,n-i-1}$ for each $i = 2 \dots n-2$, where $Q_{i,j}$ is the $j \times 1$ rectangle from

$$[12\dots(n-i-j-1)\,n(n-i-j)(n-i-j+2)\\ \dots(n-i)(n-i-j+1)(n-i+1)\dots(n-1)]$$

to $[12\dots(n-i-j-1)\,n(n-i-j+1)\dots(n-i)(n-i-j)(n-i+1)\dots(n-1)]$, and $Q_{1,j} = R_{6,j}$ for each $j = 1, \dots n-2$.

We cancel each of these rectangles in the boundary as follows (see Figure 4):

- $P_{n-2,1}$ occurs in $\partial F_{n-3,1}$ and ∂B_{n-2}. For $i = 2, \dots n-3$, $P_{i,j}$ occurs in $\partial F_{i-1,j}$ and either $\partial F_{i,j-1}$ if $j \geq 2$ or $\partial F_{i,1}$ if $j = 1$.
- $Q_{n-2,1}$ occurs in $\partial G_{n-3,1}$ and ∂A_{n-2}. For $i = 2, \dots n-3$, $Q_{i,j}$ occurs in $\partial G_{i-1,j}$ and either $\partial G_{i,j-1}$ if $j \geq 2$ or $\partial G_{i,1}$ if $j = 1$.

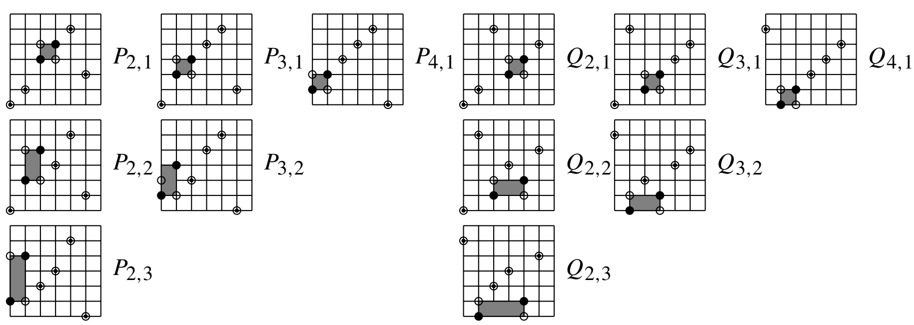


Figure 4. The rectangles $P_{i,j}$ and $Q_{i,j}$ in the special case of a 6×6 grid, where each domain is drawn from a generator x (drawn with $\bullet$) to a generator y (drawn with $\circ$).

Finally, the remaining rectangles have the following form:

- $R'_{1,1} \dots R'_{1,n-1}$, where $R'_{1,i}$ is the $(n-i) \times 1$ rectangle from

$$[n23 \dots (n-i)\, 1(n-i+1) \dots (n-1)]$$

to $[12 \dots (n-i)\, n(n-i+1) \dots (n-1)]$.

- $R'_{2,1} \dots R'_{2,n-1}$, where $R'_{2,i}$ is the $1 \times (n-i)$ rectangle from

$$[(n-i+1)\, 23 \dots (n-i)(n-i+2) \dots n1]$$

to $[12 \dots (n-i)(n-i+2) \dots n(n-i+1)]$.

- $P'_{i,2} \dots P'_{i,n-i-1}$ for $i = 1 \dots n-3$, where $P'_{i,j}$ is the $j \times 1$ rectangle from $[12 \dots (n-i-j-1)(n-i)(n-i-j+1) \dots (n-i-1)(n-i+1) \dots n(n-i-j)]$ to $[12 \dots (n-i-j-1)(n-i+1)(n-i-j+1) \dots (n-i)(n-i+2) \dots n(n-i-j)]$, and $P'_{i,1} = P_{i,1}$.

- $Q'_{i,2} \dots Q'_{i,n-i-1}$ for $i = 1 \dots n-3$, where $Q'_{i,j}$ is the $1 \times j$ rectangle from $[12 \dots (n-i-j-1)\, n(n-i-j+1) \dots (n-i-1)(n-i-j)(n-i) \dots (n-1)]$ to $[12 \dots (n-i-j-1)\, n(n-i-j+1) \dots (n-i)(n-i-j)(n-i+1) \dots (n-1)]$, and $Q'_{i,1} = Q_{i,1}$.

We cancel each of these rectangles in the boundary as follows (see Figure 5):

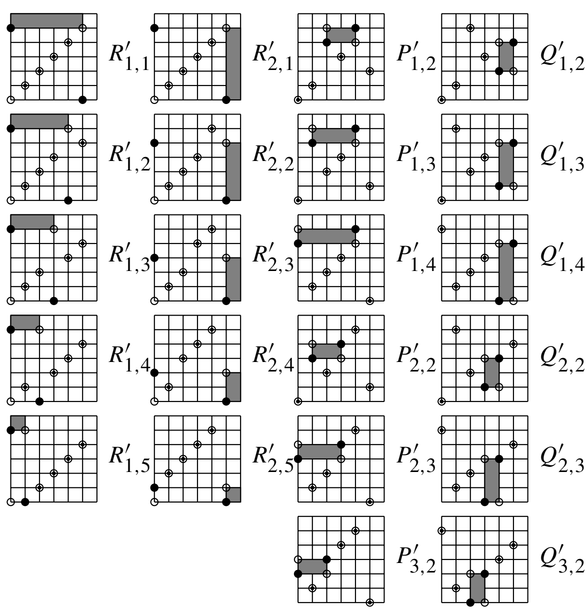


Figure 5. The rectangles $R'_{1,i}$, $R'_{2,i}$, $P'_{i,j}$, $Q'_{i,j}$ in the special case of a 6×6 grid, where each domain is drawn from a generator x (drawn with $\bullet$) to a generator y (drawn with $\circ$).

- $R'_{1,1}$ occurs in ∂B_0 and ∂C_1, $R'_{1,n-1}$ occurs in ∂A_{n-1} and ∂C_{n-2}. $R'_{1,i}$ occurs in ∂C_{i-1} and ∂C_i for $i = 2, \dots, n-2$.
- $R'_{2,1}$ occurs in ∂A_0 and ∂D_1, $R'_{2,n-1}$ occurs in ∂B_{n-1} and ∂D_{n-2}. $R'_{2,i}$ occurs in ∂D_{i-1} and ∂D_i for $i = 2, \dots, n-2$.
- $P'_{i,n-i-1}$ occurs in $\partial F_{i,n-i-2}$ and ∂B_i. For $2 \le j \le n-i-2$, $P'_{i,j}$ occurs in $\partial F_{i,j-1}$ and $\partial F_{i,j}$.
- $Q'_{i,n-i-1}$ occurs in $\partial G_{i,n-i-2}$ and ∂A_i. For $2 \le j \le n-i-2$, $Q'_{i,j}$ occurs in $\partial G_{i,j-1}$ and $\partial G_{i,j}$.

Since these are the only rectangles produced by ∂A_i, ∂B_i, ∂C_i, ∂D_i, ∂E_i, $\partial F_{i,j}$, $\partial G_{i,j}$, we conclude that indeed $\partial U = 0$. □

Lemma 2.5. *U is not homologous to zero in* CD_*.

Proof. Let r be the 2-cochain which is 1 on the rightmost vertical annulus from any generator to itself, and zero on all other domains; we will first show that r is a cocycle, at which point it suffices to show that $r(U) \neq 0$. Let E be an index 3 domain. If E does not contain the rightmost vertical annulus, then clearly $\delta r(E) = 0$. If E does contain the rightmost vertical annulus, then E can be written exactly two ways as the product of the rightmost vertical annulus $V_{(n)}$ with an index 1 domain: $E = D * V_{(n)} = V_{(n)} * D$. So $\delta r(E) = 0$ and therefore r is a cocycle, and $r(U) = 1$ since U contains exactly one copy of the rightmost vertical annulus. □

Proof of Proposition 2.3. This immediately follows from Lemmas 2.4 and 2.5 and Proposition 1.1. □

3. Sign assignments

In order to extend CD_* over $\mathbb{Z}$ coefficients (and to frame some of the 0-dimensional moduli spaces in the Manolescu–Sarkar construction), we need a sign assignment for CD_*, which is a particular $\mathbb{Z}/2$-valued 1-cochain on CD_*. The following conditions for a sign assignment ensures that 1-dimensional moduli spaces are frameable, since their boundaries must have opposite signs; see [Manolescu and Sarkar 2021] for more details, and note also that this agrees with the sign assignments defined in [Manolescu et al. 2007; Gallais 2008], though we are giving a new proof of their existence.

Definition 3.1. A sign assignment for $\mathbb{G}$ is a $\mathbb{Z}/2$-valued 1-cochain s on CD_* such that:

(1) (square rule) If D_1, D_2, D_3, D_4 are distinct rectangles and $D_1 * D_2 = D_3 * D_4 = E$ which is not an annulus, then $s(D_1) + s(D_2) = s(D_3) + s(D_4) + 1$.

(2) (annuli) If D_1, D_2 are rectangles and $D_1 * D_2$ is a vertical annulus, then $s(D_1) = s(D_2) + 1$. If D_1, D_2 are rectangles such that $D_1 * D_2$ is a horizontal annulus, then $s(D_1) = s(D_2)$.

In order to prove that such a sign assignment exists, we will show that the 2-cocycle that we hypothesize to be δs is indeed a 2-coboundary.

Lemma 3.2. *Let T be the 2-cochain with the following values*:

(1) (*square rule*) *For any index 2 domain D that is not an annulus*, $T(D) = 1$.

(2) (*annuli*) $T(V) = 1$ *for all vertical annuli V, and* $T(H) = 0$ *for all horizontal annuli H.*

Then T is a 2-coboundary.

Proof. First, we show that T is a cocycle. Let E be any index 3 domain; we must show that $\langle T, \partial E\rangle = 0$. For every decomposition $E = D * A$, where A is a vertical or horizontal annulus, there is a corresponding decomposition $E = A * D$, so that A occurs an even number of times in ∂E. It now suffices to show that ∂E contains an even number of every other type of index 2 domain.

To every index 3 domain E from a generator x to a generator y, consider a graph with vertices x at level 3, y at level 0, and edges down 1 level corresponding to each way to break off an index 1 domain (see Figure 6 for an example of such a graph). Then each level 2 vertex has an index 2 domain to y, which decomposes into rectangles exactly two ways, so each level 2 vertex has downward degree 2, and each level 1 vertex has an index 2 domain from x, which decomposes into rectangles exactly two ways, so each level 1 vertex has upward degree 2. Therefore there are the same number of level 2 and level 1 vertices, so since each index 2 domain that shows up in ∂E corresponds to a level 2 or 1 vertex, there are an even number of index 2 domains. Since an even number of these are annuli, we must therefore have an even number of hexagons. This shows that $\langle T, \partial E\rangle = 0$, as desired.

By Propositions 1.1 and 2.3 it now suffices to show that $T(U) = 0$ where U is the generator of $H_2(\mathrm{CD}_*)$. By definition, U consists of n annuli A_i, n annuli B_i, $n-2$ hexagons C_i, $n-2$ hexagons D_i, $n-2$ hexagons E_i, $\binom{n-2}{2}$ hexagons $F_{i,j}$, and $\binom{n-2}{2}$ hexagons $G_{i,j}$, so for any T satisfying the conditions of Lemma 3.2,

$$T(U) \equiv n + 3(n-2) + 2\binom{n-2}{2} \equiv 0 \pmod 2$$

so that T is indeed a coboundary. □

Lemma 3.3. *Let T be the 2-coboundary from* Lemma 3.2. *Then $T = \delta s$ if and only if s is a sign assignment.*

Proof. This is clear from the definitions. □

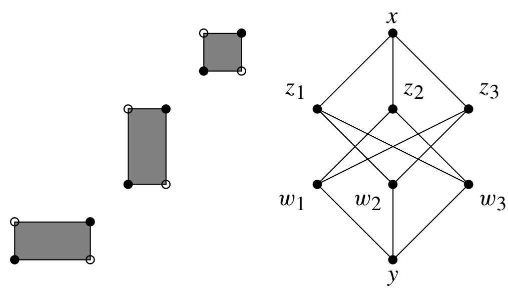


Figure 6. An example of a positive index 3 domain from a generator x (drawn with •) to a generator y (drawn with ∘), along with the graph defined in the proof of Proposition 1.3. The generators z_i are given by ∘ on the i-th rectangle from the left and • on the other two, while the generators w_i are given by • on the i-th rectangle from the left and ∘ on the other two.

Proof of Theorem 1.2. Existence immediately follows from Lemmas 3.2 and 3.3. For uniqueness, suppose $T = \delta s = \delta s'$. Then $\delta(s - s') = 0$, so $s - s'$ is a 1-cocycle, which is cohomologous to zero by Proposition 1.1, so there is a 0-cochain g such that $s = s' + \delta g$. □

Given a sign assignment s, we can use it to redefine CD_* in $\mathbb{Z}$ coefficients:

Definition 3.4. $\mathrm{CD}_*(\mathbb{G}; \mathbb{Z})$ is freely generated over $\mathbb{G}$ by the positive domains, with the homological grading being the Maslov index. The differential $\partial : \mathrm{CD}_k \to \mathrm{CD}_{k-1}$ of $D \in \mathscr{D}^+(x, y)$ is given by

$$\partial(D) = \sum_{R*E=D} (-1)^{s(R)} E + (-1)^k \sum_{E*R=D} (-1)^{s(R)} E,$$

where R is a domain of index 1 from x to some generator z and E is a positive domain from z to y.

We now have analogues of Lemma 2.2 and Proposition 1.1 in $\mathbb{Z}$ coefficients, in the following lemma and Proposition 1.3, respectively.

Lemma 3.5. $(\mathrm{CD}_*, \partial)$ *is a chain complex.*

Proof. The proof is similar to the proof of Lemma 2.2, except we must keep track of signs. □

Proof of Proposition 1.3. The proof is similar to the proof of Proposition 1.1. Specifically, our proof of Proposition 1.1 over $\mathbb{Z}/2$ adapts the proof of [Manolescu and Sarkar 2021, Proposition 3.4]. This proof is over $\mathbb{Z}$, and a similar adaptation will prove Proposition 1.3. □

4. The obstruction complex with partitions

The moduli spaces in the construction of the 1-flow category require more than just positive domains. Since periodic domains (annuli) can bubble, Manolescu and Sarkar [2021] introduces a new complex to keep track of the bubbles — since there are n different types of bubbles (corresponding to bubbling of the j-th horizontal or vertical annulus) which can be at the same or different heights, these correspond to n-tuples of ordered partitions.

It is convenient to use both of the following equivalent definitions of an ordered partition of a positive integer N (and when $N = 0$, a partition of N is the empty set).

- An ordered partition λ is a tuple of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_m)$ such that $N = \sum \lambda_i$ (m is called the *length* of the partition, and is denoted $l(\lambda)$).
- An ordered partition λ is a tuple $\epsilon(\lambda) = (\epsilon_1(\lambda), \dots, \epsilon_{N-1}(\lambda)) \in \{0, 1\}^{N-1}$, where an ϵ_i equaling 1 indicates a split at that point. For instance, the ordered partitions (1, 1, 1), (1, 2), (2, 1), and (3) of 3 are written (1, 1), (1, 0), (0, 1), and (0, 0).

Besides annuli bubbling off (the second type of terms that will be in the differential — the first being terms in the differential of CD_*), there are two other boundary degenerations that occur with existing bubbles. Bubbles of the same type may come to the same height (the third type of term), and bubbles may go to height $\pm\infty$ (the fourth and final type of term). The corresponding changes to the partitions are described below.

Definition 4.1. The following changes to an ordered partition will describe the differential terms (see [Manolescu and Sarkar 2021, Definitions 4.1, 4.2, 4.3] for more details):

- A unit enlargement (at position k) increases N by 1 and adds a 1 to the tuple λ (at position k). The set of unit enlargements of λ is denoted $\mathrm{UE}(\lambda)$.
- An elementary coarsening (at position k) replaces both terms λ_k and λ_{k+1} with one term $\lambda_k + \lambda_{k+1}$. The set of elementary coarsenings of λ is denoted $\mathrm{EC}(\lambda)$.
- An initial reduction removes λ_1 (and decreases N by λ_1), and a final reduction removes λ_m (and decreases N by λ_m). The set of initial reductions (respectively, final reductions) of λ is denoted $\mathrm{IR}(\lambda)$ (respectively, $\mathrm{FR}(\lambda)$), where we consider both sets empty if $N = 0$.

We are now ready to define the complex of domains with partitions, CDP_*.

Definition 4.2. The complex of positive domains with $\mathrm{CDP}_* = \mathrm{CDP}_*(\mathbb{G}; \mathbb{Z}/2)$ is freely generated by triples of the form $D, \vec{N}, \vec{\lambda}$, where:

- $D \in \mathscr{D}^+(x, y)$ is a positive domain.
- $\vec{N} \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, $\vec{N} = (N_1, \dots, N_n)$.

- $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ is an n-tuple of ordered partitions for $\lambda_j = (\lambda_{j,1}, \dots, \lambda_{j,m_j})$, an ordered partition of N_j.

We denote $|\vec{N}| := \sum_{j=1}^{n} N_j$, and define $|l(\vec{\lambda})| := \sum_{j=1}^{n} l(\lambda_j)$ be the total length of $\vec{\lambda}$. The grading of $(D, \vec{N}, \vec{\lambda})$ is given by the Maslov index of D plus $|l(\vec{\lambda})|$. The differential is given by the sum of the following four terms.

- Type I terms, given by taking out a rectangle from D, just like in the differential of CD_*.
- Type II terms, given by taking out a vertical or horizontal annulus passing through O_j from D and performing a unit enlargement to λ_j.
- Type III terms, given by an elementary coarsening of one of the partitions λ_j.
- Type IV terms, given by taking the initial or final reduction of one of the partitions λ_j.

Precisely, we can write $\partial = \partial_1 + \partial_2 + \partial_3 + \partial_4$ where

$$\partial_1(D, \vec{N}, \vec{\lambda}) = \sum_{R*E=D} (E, \vec{N}, \vec{\lambda}) + \sum_{E*R=D} (E, \vec{N}, \vec{\lambda}),$$

$$\partial_2(D, \vec{N}, \vec{\lambda}) = \sum_{j=1}^{n} \sum_{D=E*H_j \text{ or } E*V_j} \sum_{\lambda'_j \in \mathrm{UE}(\lambda_j)} (E, \vec{N} + \vec{e}_j, \vec{\lambda}'),$$

$$\partial_3(D, \vec{N}, \vec{\lambda}) = \sum_{j=1}^{n} \sum_{\lambda'_j \in \mathrm{EC}(\lambda_j)} (D, \vec{N}, \vec{\lambda}'),$$

$$\partial_4(D, \vec{N}, \vec{\lambda}) = \sum_{j=1}^{n} \sum_{\lambda'_j \in \mathrm{IR}(\lambda_j)} (D, \vec{N} - \lambda_{j,1}\vec{e}_j, \vec{\lambda}') + \sum_{j=1}^{n} \sum_{\lambda'_j \in \mathrm{FR}(\lambda_j)} (D, \vec{N} - \lambda_{j,m_j}\vec{e}_j, \vec{\lambda}').$$

As in Definition 2.1, R is a rectangle, and the annuli H_j, V_j are the ones passing through the j-th O marking. We also use $\vec{\lambda}' := (\lambda_1, \dots, \lambda_{j-1}, \lambda'_j, \lambda_{j+1}, \dots, \lambda_n)$, and $\vec{e}_j := (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in the j-th position.

It will help us to classify the lower grading generators, that is, generators of CDP_0, CDP_1, CDP_2, CDP_3 (see [Manolescu and Sarkar 2021, Section 4.2] for details).

(0) CDP_0 is generated by the constant domains with no partitions $(c_x, 0, 0)$ for some generator x.

(1) CDP_1 is generated by rectangles with no partitions $(R, 0, 0)$ as well as triples of the form $(c_x, N\vec{e}_j, (N))$ for a constraint domain c_x.

(2) CDP_2 is generated by Maslov index 2 domains with no partition $(D, 0, 0)$ (for a classification of the kinds of domains D, see above or [Ozsváth et al. 2015]), triples of the form $(R, N\vec{e}_j, (N))$ for a rectangle R, or a constant domain with partitions of total length 2. Specifically, we can have triples of the form $\big(c_x, N\vec{e}_j + M\vec{e}_k, ((N), (M))\big)$ (where $j \neq k$), or $(c_x, (N + M)\,\vec{e}_j, (N, M))$.

(3) Finally, CDP_3 is generated by Maslov index 3 domains with no partition, Maslov index 2 domains with a partition of the form $(D, N\vec{e}_j, (N))$, rectangles with a partition of total length 2, and constant domains with partitions of total length 3, which has the following cases:

- $(c_x, N_j\,\vec{e}_j + N_k\,\vec{e}_k + N_l\,\vec{e}_l, ((N_j), (N_k), (N_l))$ for j, k, l distinct.
- $(c_x, (N_j + M_j)\,\vec{e}_j + N_k\,\vec{e}_k, ((N_j, M_j), (N_k))$ for j, k distinct.
- $(c_x, (N_j + M_j + P_j)\,\vec{e}_j, (N_j, M_j, P_j))$.

Lemma 4.3. $(\mathrm{CDP}_*, \partial)$ *is a chain complex.*

Proof. The proof follows a similar case analysis to [Manolescu and Sarkar 2021, Lemma 4.4]. Write $\partial = \partial_1 + \partial_2 + \partial_3 + \partial_4$, where ∂_k is the type k term in the differential. Since ∂_1 is just the differential from CD_*, we have by Lemma 2.2 that $\partial_1^2 = 0$. Now for ∂_2^2, the terms will correspond to removing two annuli (and doing two unit enlargements). If the annuli pass through two different O_i and O_j, then the corresponding term shows up twice, once in each order. If the annuli pass through the same O_j, then the corresponding term also shows up twice — once for each order in doing the unit enlargements. So, $\partial_2^2 = 0$. We can similarly show that

$$\partial_3^2 = 0, \quad \partial_1\partial_2 + \partial_2\partial_1 = 0, \quad \partial_1\partial_3 + \partial_3\partial_1 = 0, \quad \partial_2\partial_3 + \partial_3\partial_2 = 0, \quad \partial_1\partial_4 + \partial_4\partial_1 = 0$$

by doing the respective operations in two different orders.

Now consider $\partial_2\partial_4 + \partial_4\partial_2$, the terms of which correspond to a unit enlargement and an initial or final reduction, in either order. If one is done to λ_i and another to λ_j where $i \neq j$, then the two commute and cancel just like before. If both are done to λ_i, then all terms follow one of these cases:

- A unit enlargement not at the beginning, followed by an initial reduction. This cancels with the initial reduction followed by doing the enlargement one place earlier.
- A unit enlargement not at the end, followed by a final reduction. This cancels with the final reduction followed by the same enlargement.
- A unit enlargement at the beginning, followed by an initial reduction; or a unit enlargement at the end, followed by a final reduction. These cancel with each other.

Finally, consider the last terms of ∂^2, $\partial_4^2 + \partial_3\partial_4 + \partial_4\partial_3$. Again there are some special types of terms:

- The elementary coarsening of λ_i by combining the first two parts, followed by a initial reduction of λ_i, cancels with two initial reductions of λ_i.
- The elementary coarsening of λ_i by combining the last two parts, followed by a final reduction of λ_i, cancels with two final reductions of λ_i.

where all the other terms cancel by doing the operations in two different orders. □

We would like to compute the homology of CDP_* using successive filtrations, as in the proof of Proposition 1.1.

Proposition 4.4. *There is a filtration on* CDP_* *such that the associated graded has homology* $(\mathbb{Z}/2)^{2^n} \otimes (\mathbb{Z}/2)[U]$.

Proof. We again follow the proof of [Manolescu and Sarkar 2021, Proposition 4.6]. We can filter the complex CDP'_* in several steps. First we filter CDP_* by the quantity $A(D) \in \mathbb{N}^n$ which are the coefficients of D in the rightmost column. As in the proof of Proposition 1.1, we can assume without loss of generality that the minimum of $A(D)$ occurs in the top right corner, and then filter the associated graded CDP^a_* by $B(D) \in \mathbb{N}^{n-1}$ which are the coefficients of D in the topmost row. In the associated graded $\mathrm{CDP}^{a,b}_*$, there are no type II terms in the differential, since such terms must decrease either A or B. Since $|\vec{N}|$ is kept constant by type I and type III terms and decreased by type IV terms, it is a filtration on $\mathrm{CDP}^{a,b}_*$, so filtering by $|\vec{N}|$ and (as in the proof of Proposition 1.1) the end generator y gives a direct sum of complexes $\mathrm{CDP}^{a,b,y,\vec{N}}_*$.

When $a \neq (l, l, \ldots, l)$ or $b \neq 0$ or $y \neq x^{\mathrm{Id}}$, filtering by the total length of $\vec{\lambda}$ removes all type III terms and keeps all type I terms, so $\mathrm{CDP}^{a,b,y,\vec{N}}_*$ is a direct sum of complexes $\mathrm{CD}^{a,b,y}_*$ which were all shown to be acyclic in the proof of [Manolescu and Sarkar 2021, Proposition 3.4]. Additionally, when $a = (l, l, \ldots, l)$, $b = 0$, $y = x^{\mathrm{Id}}$, and at least one $N_j > 1$, every generator of $\mathrm{CDP}^{a,b,y,\vec{N}}_*$ is represented by some $(D, \vec{N}, \vec{\lambda})$ where $D = kV_n$, so we only have type III terms. The partitions of N_j are given by $(\epsilon_1, \ldots, \epsilon_{N_j-1})$, where the elementary coarsenings just change a 1 to a 0. This gives a hypercube complex, which is acyclic. Therefore, we are only left with the associated graded complexes $\mathrm{CDP}^{a,b,y,\vec{N}}$ where $a = (l, l, \ldots, l)$, $b = 0$, $y = x^{\mathrm{Id}}$, and every N_j is 0 or 1. □

Corollary. $H_k(\mathrm{CDP}_*; \mathbb{Z}/2)$ *has rank at most*

$$\sum_{l=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2l}.$$

In the proof of Proposition 2.3, we found a cocycle that detects the generator of $H_2(\mathrm{CD}_*)$. We will use a similar procedure to compute $H_0(\mathrm{CDP}_*)$ through $H_3(\mathrm{CDP}_*)$.

Proof of Theorem 1.4. ($k = 0$). This case is clear.

($k = 1$). The n generators of $H_1(\mathrm{AssGr}(\mathrm{CDP}_*))$ are the triples

$$g_j := (c_{x^{\mathrm{Id}}}, \vec{e}_j, (1)),$$

which are still cycles in CDP_1 (because their initial and final reductions cancel). It will suffice to show that there exist n 1-cocycles r_j such that $r_j(g_k) = 1$ if and only if $j = k$. Let f_j be the 1-cochain in CD_* such that $\delta f_j(D) = 1$ if and only if D is the vertical annulus V_j or the horizontal annulus H_j where f_j exists by Proposition 2.3 because the 2-cocycle which is 1 on V_j and H_j and zero on every other index 2 domain is a coboundary, since it is zero on the generator U of $H_2(\mathrm{CD}_*)$. We can extend f_j to CDP_* by setting it equal to zero on all triples $(c_x, N\vec{e}_j, (N))$. Let N_j be the 1-cocycle that is the value of N_j in the triple $(D, \vec{N}, \vec{\lambda})$, and

$$r_j := N_j + f_j.$$

To show that r_j is a cocycle, we consider all possible triples $(D, \vec{N}, \vec{\lambda})$ in grading 2. If $N_j = 0$ and D is not the annulus V_j or H_j, then by definition $\delta r_j(D, \vec{N}, \vec{\lambda}) = 0$. If $N_j = 0$ and $D = V_j$ or H_j, then $\vec{N} = 0$ and

$$\delta r_j(D, 0, 0) = N_j(c_x, \vec{e}_j, (1)) + (f_j(R_1) + f_j(R_2)) = 1 + 1 = 0 \pmod 2,$$

where $D = R_1 * R_2$ is the decomposition into rectangles. Finally, if $N_j = M > 0$, there are three cases:

- D is a rectangle. In this case $\vec{N} = M\vec{e}_j$ and $\lambda_i = (M)$, so the initial and final reduction of λ_i cancel, and the only other differential terms are removing D. If D is a rectangle from x to y, then

$$\begin{aligned}\delta r_i(D, M\vec{e}_j, (M)) &= N_j(c_x, M\vec{e}_j, (M)) + N_j(c_y, M\vec{e}_j, (M))\\ &= M + M = 0 \pmod 2.\end{aligned}$$

- Some $N_k > 0$, where $k \neq j$. Then D must be a constant domain, and both λ_j and λ_k are length 1 partitions, so their initial and final reductions all cancel.

- D is a constant domain and $\lambda_j = (M_1, M_2)$ is a length 2 partition. In this case we have all of the type III and type IV differentials, which gives

$$\begin{aligned}&\delta r_j(c_x, (M+N)\,\vec{e}_j, (M, N))\\ &\qquad = N_j(c_x, M\vec{e}_j, (M)) + N_j(c_x, N\vec{e}_j, (N)) + N_j(c_x, (M+N)\,\vec{e}_j, (M+N))\\ &\qquad = M + N + (M+N) = 0 \pmod 2.\end{aligned}$$

Therefore r_j is a cocycle for each j, and by definition $r_j(g_j) = 1$ if and only if $j = k$, so the g_j are in fact the generators of $H_1(\mathrm{CDP}_*)$.

$(k=2)$. Now consider 2-cocycles. $(\mathbb{Z}/2)^{\binom{n}{2}}$ of the generators of $H_2(\mathrm{AssGr}(\mathrm{CDP}_*))$ are the triples

$$g_{j,k} := \bigl(c_{x^{\mathrm{Id}}}, \vec{e}_j + \vec{e}_k, ((1), (1))\bigr),$$

which are similarly still cycles in CDP_2. The final generator will be given by a slight modification U' of $(U, 0, 0)$, where U is the generator of $H_2(\mathrm{CD}_*)$. The boundary of U in CDP_* contains only pairs of triples of the form

$$(c_{x^j}, \vec{e}_j, (1)) \quad \text{and} \quad (c_{y^j}, \vec{e}_j, (1))$$

corresponding to type II differentials on the annuli A_j and B_j. Here we have that x^j and y^j are, respectively, the generators $[n23\dots(n-j)\,1(n-i+1)\dots(n-1)]$ and $[(n-j+1)\,23\dots(n-j)(n-j+2)\dots n1]$, as depicted in Figure 2. For each j, the generators x^j and y^j each have a planar domain (that is, a domain that does not intersect the topmost row or rightmost column of the grid), $D_{j,1}$ and $D_{j,2}$ respectively, from the identity generator x^{Id}. From Figure 2, we see that $D_{j,2}$ is the reflection of $D_{j,1}$ about the diagonal from the bottom left to the top right of the grid, so that $(-D_{j,1}) * D_{j,2}$ is an even index planar domain from x^j to y^j. This domain decomposes into an even number of planar rectangles $\pm R_{jk}$ (where each R_{jk} is positive), so that adding each $(R_{jk}, \vec{e}_j, (1))$ to $(U, 0, 0)$ will cancel the rest of its boundary, making a cycle U'.

In the proof of Proposition 2.3, we used the 2-cocycle r which is 1 on the rightmost vertical annulus and zero on every other 2-chain. Extending r to CDP_* by setting it equal to zero on every 2-chain with $|\vec{N}| > 0$ still gives a cocycle, since CDP_* has no new ways to create an annulus in the boundary, and we still have that $r(U') = 1$, while all of the $r(g_{j,k}) = 0$. Now it suffices to find $r_{j,k}$ such that $r_{j,k}(U') = 0$ for all j, k, and $r_{j,k}(g_{l,m}) = 1$ if and only if $\{l, m\} = \{j, k\}$. Let f_j be the 1-cocycles defined in the proof of (1), and let

$$f_j^k(R, \vec{N}, \vec{\lambda}) = N_k f_j(R),$$

where R is a rectangle and $\vec{\lambda}$ has total length 1 (and $f_j^k = 0$ on all other 2-chains). Now let $N_j N_k$ be the 2-cocycle that is the product of the values of N_j and N_k for a triple $(D, \vec{N}, \vec{\lambda})$, and let

$$r_{j,k} := N_j N_k + f_j^k + f_k^j.$$

To show that $r_{j,k}$ is a cocycle, we consider all possible triples $(D, \vec{N}, \vec{\lambda})$ in grading 3. If $N_j = 0$ (respectively, $N_k = 0$) and D does not contain the annulus V_j or H_j (respectively, V_k or H_k), then by definition $\delta r_{j,k}(D, \vec{N}, \vec{\lambda}) = 0$. If $N_j = 0$, $N_k > 0$ (or vice versa), and D contains V_j or H_j, then $D = V_j$ or H_j and all $N_l = 0$ for $l \neq k$, so that $\delta r_{j,k}(D, \vec{N}, \vec{\lambda}) = 0$ similarly to the proof of (1). Finally, if $N_j = M_j > 0$ and $N_k = M_k > 0$, there are three cases:

- D is a rectangle. In this case $\vec{N} = M_j\,\vec{e}_j + M_k\,\vec{e}_k$, $\lambda_j = (M_j)$, and $\lambda_k = (M_k)$, so the initial and final reductions of λ_j and λ_k cancel, and the only other differential terms are removing D. If D is a rectangle from x to y, then

$$\begin{aligned}&\delta r_{j,k}(D, \vec{N}, \vec{\lambda})\\ &= N_j N_k\big(c_x, M_j\,\vec{e}_j + M_k\,\vec{e}_k, ((M_j), (M_k))\big) + N_i N_j\big(c_y, M_j\,\vec{e}_j + M_k\,\vec{e}_k, ((M_j), (M_k))\big)\\ &= M_i M_j + M_i M_j = 0 \pmod 2.\end{aligned}$$

- Some $N_l > 0$, where $l \neq j, k$. Then D must be a constant domain, and all of λ_j, λ_k, and λ_l are length 1 partitions, so their initial and final reductions all cancel.

• D is a constant domain and $\lambda_j = (M_{j,1}, M_{j,2})$ is a length 2 partition (or symmetrically, $\lambda_k = (M_{k,1}, M_{k,2})$). In this case the initial and final reductions of λ_k cancel, but we have all of the type III and type IV differentials of λ_j, which give

$$\begin{aligned}\delta r_{j,k}\big(c_x, (M_{j,1}+M_{j,2})\,\vec{e}_j + M_k\,\vec{e}_k, ((M_{j,1}, M_{j,2}), (M_k))\big)&\\ = N_j N_k\big(c_x, M_{j,1}\vec{e}_j + M_k\,\vec{e}_k, ((M_{j,1}), (M_k))\big)&\\ + N_j N_k\big(c_x, M_{j,2}\vec{e}_j + M_k\,\vec{e}_k, ((M_{j,2}), (M_k))\big)&\\ + N_j N_k\big(c_x, (M_{j,1}+M_{j,2})\,\vec{e}_j + M_k\,\vec{e}_k, ((M_{j,1}+M_{j,2}), (M_k))\big)&\\ = M_k(M_{j,1}+M_{j,2}+(M_{j,1}+M_{j,2})) = 0 \pmod 2.&\end{aligned}$$

Therefore $r_{j,k}$ is a cocycle for all j, k, and by definition $r_{j,k}(U) = 0$. U' only adds an even number of planar rectangles with partitions that can contribute to f_j^k or f_k^j, no two of which can ever form an annulus, so $r_{j,k}(U') = 0$. Finally, by definition $r_{j,k}(g_{l,m}) = 1$ if and only if $\{l, m\} = \{j, k\}$, so these (along with U') are in fact the generators of $H_2(\mathrm{CDP}_*)$.

($k = 3$). $\binom{n}{3}$ of the generators of $H_3(\mathrm{AssGr}(\mathrm{CDP}_*))$ are the triples

$$g_{j,k,l} := \big(c_{x^{\mathrm{Id}}}, \vec{e}_j + \vec{e}_k + \vec{e}_l, ((1), (1), (1))\big),$$

which are similarly still cycles in CDP_3. The other n generators are the triples U'_j obtained from U' by performing unit enlargements on N_j and adding the triples $(R_{jk}, 2\vec{e}_j, (2))$ defined previously (for this fixed j). Let $V_{(n)}$ be the rightmost vertical annulus and define the cochain

$$rr_j(D, \vec{N}, \vec{\lambda}) := \begin{cases} 0, & D \text{ does not contain } V_{(n)},\\ r_j(D * -V_{(n)}, \vec{N}, \vec{\lambda}), & D \text{ contains } V_{(n)},\end{cases}$$

where r_j is the 1-cocycle from the proof of (1). To show that rr_j are cocycles, we consider all $\delta rr_j(D, \vec{N}, \vec{\lambda})$ for triples $(D, \vec{N}, \vec{\lambda}) \in \mathrm{CDP}_4$. If D does not contain $V_{(n)}$, then this quantity is zero by definition. If D contains $V_{(n)}$, then its Maslov index is at least 2, so that we have the following cases:

• D is an index 4 domain. In this case, $\vec{N} = 0$, so let $E = D * (-V_{(n)})$. If E is also an annulus V_k, then

$$\begin{aligned}&\delta rr_j(D, \vec{N}, \vec{\lambda})\\ &= rr_j(A_1 * V_{(n)}, 0, 0) + rr_j(A_2 * V_{(n)}, 0, 0) + rr_j(E * B_1, 0, 0) + rr_j(E * B_2, 0, 0)\\ &\quad + rr_j(V_{(n)}, \vec{e}_k, (1)) + rr_j(E, \vec{e}_n, (1)) \qquad (\text{where } A_1 * A_2 = E,\ B_1 * B_2 = V_{(n)})\\ &= r_j(A_1, 0, 0) + r_j(A_2, 0, 0) + r_j(c_x, \vec{e}_k, (1))\\ &\quad + \big(r_j(B_1, 0, 0) + r_j(B_2, 0, 0) + r_j(c_x, \vec{e}_n, (1))\big) \qquad (\text{added if and only if } k = n)\\ &= 0\end{aligned}$$

by definition of r_j (since this is just $\delta r_j(V_k)$ with possibly $\delta r_j(V_{(n)})$ added if $k=n$). If E is not an annulus, we similarly have

$$\begin{aligned}
&\delta rr_j(D,\vec{N},\vec{\lambda})\\
&=rr_j(A_1*V_{(n)},0,0)+rr_j(A_2*V_{(n)},0,0)+rr_j(E*B_1,0,0)\\
&\quad+rr_j(E*B_2,0,0)+rr_j(E,\vec{e}_n,(1))\qquad(\text{where } A_1*A_2=E,\ B_1*B_2=V_{(n)})\\
&=r_j(A_1,0,0)+r_j(A_2,0,0)+r_j(c_x,\vec{e}_k,(1))=0.
\end{aligned}$$

- $D=R*V_{(n)}$ is an index 3 domain, where R is a rectangle from a generator x to a generator y. In this case, $\vec{N}=N\vec{e}_k$ and $\vec{\lambda}=\lambda_k=(N)$. Suppose that in the domain D, $V_{(n)}=A*B$, and $R=B$ (or symmetrically, $R=A$). In this case,

$$\begin{aligned}
&\delta rr_j(D,\vec{N},\vec{\lambda})\\
&\ =rr_j(R*A,N\vec{e}_k,(N))+rr_j(A*R,N\vec{e}_k,(N))+rr_j\big(R,N\vec{e}_k+e_n,((N),(1))\big)\\
&\ =r_j(c_x,N\vec{e}_k,(N))+r_j(c_y,N\vec{e}_k,(N))=0.
\end{aligned}$$

by definition of r_j. If R is not A or B, we similarly have

$$\begin{aligned}
\delta rr_j(D,\vec{N},\vec{\lambda})&=rr_j(R*A,N\vec{e}_k,(N))+rr_j(R*B,N\vec{e}_k,(N))\\
&\qquad+rr_j(V_{(n)},N\vec{e}_k,(N))\qquad\qquad(\text{at the generator } x)\\
&\qquad+rr_j(V_{(n)},N\vec{e}_k,(N))\qquad\qquad(\text{at } y)\\
&\qquad+rr_j\big(R,N\vec{e}_k+\vec{e}_n,((N),(1))\big)\\
&=r_j(c_x,N\vec{e}_k,(N))+r_j(c_y,N\vec{e}_k,(N))=0.
\end{aligned}$$

- D is an index 2 domain. In this case, we have that $D=V_{(n)}$, $|\vec{\lambda}|=2$, and $\delta rr_j(D,\vec{N},\vec{\lambda})=\delta r_j(c_x,\vec{N},\vec{\lambda})$ since the type I and type II terms cannot possibly have an annulus; note that we previously showed this expression to be zero in showing that r_j is a cocycle.

Therefore rr_j is a cocycle, and by construction $rr_j(U'_k)=1$ if and only if $k=j$, and all of the $rr_j(g_{k,l,m})=0$. It remains to find cocycles $r_{j,k,l}$ such that $r_{j,k,l}(U'_m)=0$ for all j,k,l,m, and $r_{j,k,l}(g_{m,p,q})=1$ if and only if $\{m,p,q\}=\{j,k,l\}$. Let f_j be the 1-cocycles defined in the proof of (1), and let

$$f_j^{k,l}(R,\vec{N},\vec{\lambda})=N_kN_lf_j(R),$$

where R is a rectangle and λ_k,λ_l are both length 1 partitions (and $f_j^{k,l}=0$ on all other 3-chains). Now let $N_jN_kN_l$ be the 3-cocycle that is the product of the values of N_j, N_k, and N_l for a triple $(D,\vec{N},\vec{\lambda})$, and let

$$r_{j,k,l}=N_jN_kN_l+f_j^{k,l}+f_k^{j,l}+f_l^{j,k}.$$

To show that $r_{j,k,l}$ is a cocycle, we consider all possible triples $(D,\vec{N},\vec{\lambda})$ in grading 4. If $N_j=0$ (respectively, $N_k=0$ or $N_l=0$) and D does not contain the annulus V_j

or H_j (respectively, V_k or H_k, or V_l or H_l), then by definition $\delta r_{j,k,l}(D, \vec{N}, \vec{\lambda}) = 0$. If $N_j = 0$, $N_k > 0$, and $N_l > 0$ (or symmetrically, any other case where exactly one is zero), and D contains V_j or H_j, then $D = V_j$ or H_j and all $N_m = 0$ for $m \neq k, l$, so that $\delta r_{j,k,l}(D, \vec{N}, \vec{\lambda}) = 0$ similarly to the proof of (1). Finally, if $N_j = M_j > 0$, $N_k = M_k > 0$, and $N_l = M_l > 0$, there are three cases:

• D is a rectangle. In this case $\vec{N} = M_j\,\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l$, $\lambda_j = (M_j)$, $\lambda_k = (M_k)$, and $\lambda_l = (M_l)$, so the initial and final reductions of λ_j, λ_k and λ_l cancel, and the only other differential terms are removing D. If D is a rectangle from x to y, then

$$\begin{aligned}\delta r_{j,k,l}(D, \vec{N}, \vec{\lambda}) &= N_j N_k N_l\big(c_x, M_j\,\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l, ((M_j), (M_k), (M_l))\big)\\ &\quad + N_j N_k N_l\big(c_y, M_j\,\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l, ((M_j), (M_k), (M_l))\big)\\ &= M_j M_k M_l + M_j M_k M_l = 0 \pmod 2.\end{aligned}$$

• Some $N_m > 0$, where $m \neq j, k, l$. Then D must be a constant domain, and all of λ_j, λ_k, λ_l, and λ_m are length 1 partitions, so their initial and final reductions all cancel.

• D is a constant domain and $\lambda_j = (M_{j,1}, M_{j,2})$ is a length 2 partition (or symmetrically, $\lambda_k = (M_{k,1}, M_{k,2})$ or $\lambda_l = (M_{l,1}, M_{l,2})$). In this case the initial and final reductions of λ_k and λ_l cancel, but we have all of the type III and type IV differentials of λ_j, which give

$$\begin{aligned}&\delta r_{j,k,l}\big(c_x, (M_{j,1} + M_{j,2})\,\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l, ((M_{j,1}, M_{j,2}), (M_k), (M_l))\big)\\ &= N_j N_k N_l\big(c_x, M_{j,1}\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l, ((M_{j,1}), (M_k), (M_l))\big)\\ &\quad + N_j N_k N_l\big(c_x, M_{j,2}\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l, ((M_{j,2}), (M_k), (M_l))\big)\\ &\quad + N_j N_k N_l\big(c_x, (M_{j,1} + M_{j,2})\,\vec{e}_j + M_k\,\vec{e}_k + M_l\,\vec{e}_l, ((M_{j,1} + M_{j,2}), (M_k), (M_l))\big)\\ &= M_k M_l(M_{j,1} + M_{j,2} + (M_{j,1} + M_{j,2})) = 0 \pmod 2\end{aligned}$$

Therefore $r_{j,k,l}$ are cocycles, satisfying $r_{j,k,l}(F''_m) = 0$ for all j, k, l, m, and $r_{j,k,l}\big(c_{x^{\mathrm{Id}}}, \vec{e}_m + \vec{e}_p + \vec{e}_q, ((1), (1))\big) = 1$ if and only if $\{m, p, q\} = \{j, k, l\}$. □

5. Sign assignments for domains with partitions

Similarly to Section 3, we find the criteria that the coboundary of a sign assignment for CDP_* must satisfy:

Definition 5.1. A sign assignment for CDP_* is a 1-cochain s on CDP_* such that:

(1) $\delta s(D, 0, 0) = 1$ for any index 2 domain D that is not an annulus.

(2) $\delta s(V, 0, 0) = 1$ for any vertical annulus V, and $\delta s(H, 0, 0) = 0$ for any horizontal annulus H.

(3) $\delta s(R, (0, N\vec{e}_j, (N)) = 0$ for any rectangle R, any $N > 0$, and any j.

(4) $\delta s\big(c_x, N\vec{e}_j + M\vec{e}_k, ((N), (M))\big) = 0$ for any constant domain c_x, any $N, M > 0$, and any j, k.

(5) $\delta s\big(c_x, (N+M)\,\vec{e}_j, ((N, M))\big) = 0$ for any constant domain c_x, any $N, M > 0$, and any j.

Proof of Theorem 1.5. Let T be the 2-cochain with values given by (1)–(5) of Definition 5.1. To see that T is a cocycle, we evaluate δT on all triples $(D, \vec{N}, \vec{\lambda})$ in grading 3. These are given by:

- $(D, 0, 0)$ where D is an index 3 domain. The proof of Lemma 3.3 shows that the contributions to δT by Type I differential terms all cancel. If D does not contain an annulus, these are all the differential terms. If D does contain an annulus $A = H_j$ or V_j, we can write $D = R * A$ for a rectangle R, so that the type II differential term gives $(R, (0, \vec{e}_j, (1))$, which does not contribute to δT by (3).

- $(D, N\vec{e}_j, (N))$ where D is an index 2 domain. Here the initial and final reduction of the partition both give $(D, 0, 0)$ so their contributions to δT cancel. The decompositions of D into rectangles do not contribute to δT by condition (3), and again if D is not an annulus, then these are the only other boundary terms. If D is an annulus, then either $D = H_j$, $D = V_j$, or D is some other annulus H_k or V_k. In the latter case, the type II differential term gives $\big(c_x, N\vec{e}_j + \vec{e}_k, ((N), (1))\big)$ which does not contribute to δT by (4). In the former case, the type II differential gives two terms, $(c_x, (N+1)\,\vec{e}_j), (1, N))$ and $(c_x, (N+1)\,\vec{e}_j, (N, 1))$, which do not contribute to δT by (5).

- $(R, \vec{N}, \vec{\lambda})$ where $R \in \mathscr{D}^+(x, y)$ is a rectangle and $\vec{\lambda}$ has total length 2. Here the type I differential removes R two ways, leaving either $\big(c_x, M\vec{e}_j + N\vec{e}_k, ((M), (N))\big)$ (and the corresponding term for c_y, which do not contribute to δT by (4)), or $(c_x, (M+N)\,\vec{e}_j), (M, N))$ (and the corresponding term for c_y, which do not contribute by (5)). All type III and type IV terms do not contribute by (3). Since R cannot possibly contain an annulus, there are no further terms so $\delta T(R, \vec{N}, \vec{\lambda}) = 0$.

- $(c_x, \vec{N}, \vec{\lambda})$ where c_x is a constant domain and $\vec{\lambda}$ has total length 3. None of these terms contribute to δT by (4) and (5).

Hence T is a cocycle, so it remains to show T is zero on every generator of $H_2(\mathrm{CDP}_*)$ listed in the proof of Theorem 1.4. By definition, we have that every $T\big(c_x, \vec{e}_j + \vec{e}_k, ((1), (1))\big) = 0$. Also, $T(U, 0, 0) = 0$ by Lemma 3.2, so $T(U') = 0$ by condition (3), since these are the only types of triples added to $(U, 0, 0)$. Therefore T must be the zero cocycle by Theorem 1.4, so $T = \delta s$ for some s. The values s_j uniquely determine the $H^1(\mathrm{CDP}_*)$ class of s by Theorem 1.4, so at that point s is unique up to gauge equivalence (like sign assignments for CD_*). □

There are two types of triples in grading 1 — rectangles with no partitions and constant domains with a length 1 partition. By uniqueness, the sign of a rectangle with no partition in CDP_* agrees with the sign of that rectangle in CD_*, so it remains to compute the signs of constant domains with a length 1 partition.

Proposition 5.2. *For any constant domain c_x and any $N > 0$,*

$$s(c_x, N\vec{e}_j, (N)) = Ns_j \pmod 2.$$

Proof. We first show that the sign is independent of the generator x. Let $R \in \mathscr{D}^+(x, y)$ be a rectangle. By (3) of Definition 5.1, we have

$$0 = \delta s(R, N\vec{e}_j, (N)) = s(c_x, N\vec{e}_j, (N)) + s(R, 0, 0) + s(R, 0, 0) + s(c_y, N\vec{e}_j, (N))$$

so that $s(c_x, N\vec{e}_j, (N)) = s(c_y, N\vec{e}_j, (N))$, and given any domain from x to y, we find a decomposition into rectangles and repeatedly apply this equation. Therefore we can now assume without loss of generality that $x = x^{\mathrm{Id}}$. We will use the uniqueness of s up to the values s_j to proceed by induction on N. The base case is clear, and by (5) of Definition 5.1 we must have that

$$\begin{aligned} 0 &= \delta s(c_x, N\vec{e}_j, (1, N-1)) \\ &= s(c_x, \vec{e}_j, (1)) + s(c_x, (N-1)\,\vec{e}_j), (N-1)) + s(c_x, N\vec{e}_j, (N)) \\ &= s_j + (N-1)\, s_j \pmod 2 \end{aligned}$$

by the inductive hypothesis, so that $s(c_x, N\vec{e}_j, (N)) = Ns_j \pmod 2$, which completes the induction. □

Remark 5.3. It would suffice by uniqueness to define a sign assignment on CDP_* by defining a sign assignment on CD_* and extending it by Proposition 5.2. Doing so would give another proof of Theorem 1.5.

Again, now that we have a sign assignment s, we can extend CDP_* to $\mathbb{Z}$ coefficients. As in CD_*, the sign associated to breaking off a rectangle is the sign of the rectangle $s(R)$ given by the sign assignment. We now describe the sign of the other differential terms.

Definition 5.4. Let s be a sign assignment for CDP_*.

- Given an ordered partition λ and the unit enlargement

$$\lambda' = (\lambda_1, \ldots, \lambda_{k-1}, 1, \lambda_k, \ldots, \lambda_m),$$

the sign of the unit enlargement is

$$s(\lambda, \lambda') = k + 1 \pmod 2.$$

- Given an ordered partition λ and the elementary coarsening

$$\lambda' = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_m),$$

the sign of the elementary coarsening is

$$s(\lambda, \lambda') = k \pmod 2.$$

• Given an ordered partition $\lambda = (\lambda_1, \dots, \lambda_m)$ and its initial reduction λ', the sign of the reduction is given by

$$s(\lambda, \lambda') = \lambda_1 s_j \pmod 2.$$

The sign of its final reduction is given by the same expression, with λ_m replacing λ_1.

Definition 5.5. The complex of positive domains that have the partitions $\mathrm{CDP}_* = \mathrm{CDP}_*(\mathbb{G}; \mathbb{Z})$ is freely generated by triples of the form $D, \vec{N}, \vec{\lambda}$, where:

- $D \in \mathscr{D}^+(x, y)$ is a positive domain.
- $\vec{N} \in \mathbb{N}^n$ is an n-tuple of nonnegative integers, $\vec{N} = (N_1, \dots, N_n)$.
- $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ is an n-tuple of ordered partitions, where $\lambda_j = (\lambda_{j,1}, \dots, \lambda_{j,m_j})$ is an ordered partition of N_j.

The grading of $(D, \vec{N}, \vec{\lambda})$ is given by the Maslov index of D plus the sum of the lengths of the λ_j (which is referred to as the total length of $\vec{\lambda}$). The differential is given by four terms, $\partial = \partial_1 + \partial_2 + \partial_3 + \partial_4$, where

$$\begin{aligned}
\partial_1(D, \vec{N}, \vec{\lambda}) &= \sum_{R * E = D} (-1)^{s(R)} (E, \vec{N}, \vec{\lambda}) + (-1)^{\mu(D)} \sum_{E * R = D} (-1)^{s(R)} (E, \vec{N}, \vec{\lambda}), \\
\partial_2(D, \vec{N}, \vec{\lambda}) &= (-1)^{\mu(D)} \sum_{j=1}^{n} (-1)^{l(\lambda_1) + \cdots + l(\lambda_{j-1})} \sum_{D = E * H_j \text{ (horizontal)}} (-1)^{1 + s(\lambda_j, \lambda'_j)} \\
&\qquad\qquad \sum_{\lambda'_j \in \mathrm{UE}(\lambda_j)} (E, \vec{N} + \vec{e}_j, \vec{\lambda}') \\
&\quad + (-1)^{\mu(D)} \sum_{j=1}^{n} (-1)^{l(\lambda_1) + \cdots + l(\lambda_{j-1})} \sum_{D = E * V_j \text{ (vertical)}} (-1)^{s(\lambda_j, \lambda'_j)} \\
&\qquad\qquad \sum_{\lambda'_j \in \mathrm{UE}(\lambda_j)} (E, \vec{N} + \vec{e}_j, \vec{\lambda}'), \\
\partial_3(D, \vec{N}, \vec{\lambda}) &= (-1)^{\mu(D)} \sum_{j=1}^{n} (-1)^{l(\lambda_1) + \cdots + l(\lambda_{j-1})} \sum_{\lambda'_j \in \mathrm{EC}(\lambda_j)} (-1)^{s(\lambda_j, \lambda'_j)} (D, \vec{N}, \vec{\lambda}'), \\
\partial_4(D, \vec{N}, \vec{\lambda}) &= (-1)^{\mu(D)} \sum_{j=1}^{n} (-1)^{l(\lambda_1) + \cdots + l(\lambda_{j-1})} \\
&\qquad\qquad \sum_{\lambda'_j \in \mathrm{IR}(\lambda_j)} (-1)^{s(\lambda_j, \lambda'_j)} (D, \vec{N} - \lambda_{j,1} \vec{e}_j, \vec{\lambda}') \\
&\quad + (-1)^{\mu(D)} \sum_{j=1}^{n} (-1)^{l(\lambda_1) + \cdots + l(\lambda_j)} \\
&\qquad\qquad \sum_{\lambda'_j \in \mathrm{FR}(\lambda_j)} (-1)^{s(\lambda_j, \lambda'_j)} (D, \vec{N} - \lambda_{j,m_j} \vec{e}_j, \vec{\lambda}').
\end{aligned}$$

Remark 5.6. In the case that all $s_j = 0$, these signs agree with the signs of [Manolescu and Sarkar 2021, Definitions 4.1–4.3], with the exception of the type II differential.

Lemma 5.7. $(\mathrm{CDP}_*, \partial)$ *is a chain complex.*

Proof. The proof is similar to that of [Manolescu and Sarkar 2021, Lemma 4.4], which is the same case analysis of Lemma 4.3, except where we keep track of signs. In the case of

$$(\partial_4)^2 + \partial_3\partial_4 + \partial_4\partial_3 = 0,$$

we still have all but two cases canceling in pairs by reversing the order of the two operations. These two cases are:

- Two initial reductions and an elementary coarsening at the beginning, followed by an initial reduction. The former has sign

$$\lambda_1 s_j + \lambda_2 s_j \pmod 2$$

and the latter has sign

$$1 + (\lambda_1 + \lambda_2)\, s_j \pmod 2,$$

which is the opposite sign.

- Two final reductions and an elementary coarsening at the end, followed by a final reduction. Note that final reductions have an extra sign of $l(\lambda_j)$ compared to initial reductions, so that including this extra sign, the former has sign

$$l(\lambda_j) + (l(\lambda_j) - 1) + \lambda_m s_j + \lambda_{m-1}\, s_j \pmod 2)$$

and the latter has sign

$$(l(\lambda_j) - 1) + (l(\lambda_j) - 1) + (\lambda_{m-1} + \lambda_m)\, s_j \pmod 2,$$

which is the opposite sign.

Finally, although we still have $\partial_2\partial_4 + \partial_4\partial_2 = 0$, the change to the sign of the type II differential gives a new set of cancellations

$$\partial_1^2 + \partial_2\partial_4 + \partial_4\partial_2 = 0.$$

For this case, suppose $D = A * E = E * A$ is the domain where $A = R * S$ is an annulus.

- If A is a vertical annulus V_j, then $s(R)+s(S)=1$, so that removing R then S from the front has sign 1, while the type II differential that produces a unit enlargement at the front of λ_j followed by the initial reduction of λ_j has sign 0. Also, removing S then R from the back has sign 0 (since the Maslov index of the domain decreases once), while the type II differential that produces a unit enlargement at the end of λ_j followed by the final reduction of λ_j has sign $l(\lambda_j) + 1 + l(\lambda_j) = 1 \pmod 2$.
- If A is a horizontal annulus H_j, then $s(R) + s(S) = 0$, so that removing R then S from the front has sign 0, while the type II differential that produces a unit enlargement at the front of λ_j followed by the initial reduction of λ_j has

sign 1. Also, removing S then R from the back has sign 1 (since the Maslov index of the domain decreases once), while the type II differential that produces a unit enlargement at the end of λ_j followed by the final reduction of λ_j has sign $l(\lambda_j) + l(\lambda_j) = 0 \pmod 2$. □

The analogue of Proposition 4.4 also holds over $\mathbb{Z}$.

Proposition 5.8. *There is a filtration on* CDP_* *such that the associated graded has homology* $\mathbb{Z}^{2^n} \otimes \mathbb{Z}[U]$. *In particular*, $H_k(\mathrm{CDP}_*)$ *has rank at most*

$$\sum_{l=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2l}.$$

Proof. The proof is identical to the proof of Proposition 4.4. □

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