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EXTRINSIC POLYHARMONIC MAPS INTO THE SPHERE

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In the first part we shall prove that the inverse of the stereographic projection $\pi^{-1}:\mathbb{R}^n\to\mathbb{S}^n$ ($n\geq 2$) is extrinsically k-harmonic if and only if $n=2k$. In the second part we shall study minimizing properties and stability of its restriction to the closed ball $B^n(R)$. In this context we shall prove that there exists a small enough positive upper bound R_k^* such that $\pi^{-1}:B^n(R)\to\mathbb{S}^n$ is a *minimizer* provided that $0<R\leq R_k^*$. By contrast, we shall show that $\pi^{-1}:B^n(R)\to\mathbb{S}^n$ is *not* energy minimizing when $R>1$. Finally, in some cases we shall obtain stability with respect to rotationally symmetric variations (*equivariant stability*) for values of R which are greater than 1.

1. Introduction and statement of the results

In order to set our work in an appropriate setting, let us first briefly recall some basic facts about some well-known intrinsic energy functionals.

The classical *energy functional*, whose critical points are called *harmonic maps*, is defined by

(1-1) $$E(u)=\frac{1}{2}\int_M |du|^2\,dv_g,$$

where $u:M\to N$ is a smooth map between two Riemannian manifolds (M,g) and (N,h) of dimension m and n respectively (we refer to [6; 7] for background on harmonic maps). In analytic terms, the condition of harmonicity is equivalent to the fact that the map u is a solution of the Euler–Lagrange equation associated to the energy functional (1-1), i.e.,

(1-2) $$\tau(u)=-d^*du=\operatorname{trace}\nabla du=0.$$

The left member $\tau(u)$ of (1-2) is a vector field along the map u or, equivalently, a section of the pull-back bundle $u^{-1}(\mathrm{TN})$: it is called *tension field*.

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Let $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ denote the canonical inclusion. In the special case that $N = \mathbb{S}^n$ the harmonicity equation (1-2) takes the following form, where, with a slight abuse of notation, we write u for $i \circ u$:

$$\Delta u + \lambda_1 u = 0, \tag{1-3}$$

with

$$\lambda_1 = -\langle u, \Delta u \rangle = |\nabla u|^2 \tag{1-4}$$

and the sign convention for the Laplacian Δ is such that, for a function $f : M \to \mathbb{R}$,

$$\Delta f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial f}{\partial x_j} \right). \tag{1-5}$$

A related topic of growing interest deals with the study of the so-called *biharmonic maps*. These maps, which provide a natural generalization of harmonic maps, are the critical points of the *bienergy functional* (as suggested in [6; 8]):

$$E_2(u) = \tfrac{1}{2} \int_M |\tau(u)|^2 \, dv_g. \tag{1-6}$$

There have been extensive studies on biharmonic maps (see [5; 15; 21; 22] for an introduction to this topic). For future comparison we point out that, when the target manifold N is the Euclidean sphere $\mathbb{S}^n$, the bienergy functional (1-6) takes the form

$$E_2(u) = \tfrac{1}{2} \int_M |(\Delta u)^T|^2 \, dv_g = \tfrac{1}{2} \int_M (|\Delta u|^2 - |\nabla u|^4) \, dv_g, \tag{1-7}$$

where, again, we have written u for $i \circ u$, $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$, and $(\,\cdot\,)^T$ denotes the tangential component to $\mathbb{S}^n$.

The inclusion $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ enables us to consider the Sobolev space

$$W^{2,2}(M^m, \mathbb{S}^n) = \left\{ u \in W^{2,2}(M^m, \mathbb{R}^{n+1}) : u(x) = (u^1(x), \ldots, u^{n+1}(x)) \in \mathbb{S}^n \text{ a.e.} \right\}$$

and so we say that u is a *weak* critical point if it verifies the Euler–Lagrange equation in the sense of distributions.

Since any harmonic map is trivially biharmonic we say that a (weakly) biharmonic map is *proper* if it is *not* (weakly) harmonic.

In general, it is very difficult to apply variational methods and, particularly, direct minimization, to deduce the existence of proper biharmonic maps. The main reason for this is the fact that harmonic maps provide absolute minima for the bienergy. To overcome this difficulty, an interesting variant of (1-7), called *extrinsic bienergy* or *Hessian energy*, has been introduced to study maps into $\mathbb{S}^n$. This new functional (see, for instance, [1; 4; 11; 13; 14; 16; 23]) is defined by

$$E_2^{\text{ext}}(u) = \int_M |\Delta u|^2 \, dv_g \tag{1-8}$$

and its Euler–Lagrange equation is

$$\Delta^2 u + \lambda_2 u = 0, \tag{1-9}$$

where

$$\lambda_2 = \Delta(|\nabla u|^2) + |\Delta u|^2 + 2\nabla\Delta u.\nabla u.$$

Here and below,

$$\nabla u = (\nabla u^1, \dots, \nabla u^{n+1}) \quad \text{and} \quad \Delta u = (\Delta u^1, \dots, \Delta u^{n+1}), \tag{1-10}$$

where ∇ is the gradient on (M^m, g) (note that each entry of ∇u is an m-dimensional vector field tangent to M) and we denote by . a scalar product in the sense that

$$\nabla u.\nabla\Delta u = \sum_{j=1}^{n+1} \langle \nabla u^j, \nabla\Delta u^j \rangle_g, \tag{1-11}$$

where $\langle\,,\,\rangle_g$ is the scalar product associated to the Riemannian metric g.

Next, let us introduce in detail a conformal map which will play a central role in this paper. The inverse of the stereographic projection, denoted by π^{-1}, can be described as

$$\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \left(\frac{2}{1+r^2}\, x, \frac{1-r^2}{1+r^2} \right), \tag{1-12}$$

where $r = |x|$. In some instances, we shall also denote by π^{-1} the restriction of π^{-1} to the n-dimensional ball $B^n(R)$ of radius R.

We point out that, in general, u harmonic does not imply that u is extrinsically biharmonic. For instance, the map π^{-1} is conformal and when $n = 2$ is harmonic, but not extrinsically biharmonic.

When $n = 4$, π^{-1} is not harmonic, but it is a critical point for both (1-7) and (1-8), that is, it is both intrinsically and extrinsically biharmonic. Therefore, it is reasonable to think that, when $n = 6$, π^{-1} could be a critical point of a suitable third-order energy functional.

There are two natural energy functionals of order 3. The first is the intrinsic 3-energy:

$$E_3(u) = \tfrac{1}{2} \int_M |d\tau|^2 \, dv_g. \tag{1-13}$$

Critical points of (1-13) are called *triharmonic* maps. These maps have received plenty of attention in the literature: their study was proposed in [6; 8] and important progresses were made in a series of papers by Maeta (see [17; 18; 19; 20]) who obtained the Euler–Lagrange equations and several general results. Particularly, the study of triharmonic immersions into $\mathbb{S}^n$ provided significant examples. By contrast, as a part of a more general result, we recently proved in [3] that there exists no proper triharmonic rotationally symmetric conformal diffeomorphism from $\mathbb{R}^n$ to $\mathbb{S}^n \backslash \mathrm{Pole}$ $(n \geq 2)$. In particular, the map (1-12) is *not* triharmonic for all $n \geq 3$.

This fact, together with the hope to use more effectively variational methods, suggested to us to turn our attention to a suitable class of extrinsic k-energy functionals, a context in which we expect that conformal maps may play a significant role when $n = 2k$.

More specifically, let (M^m, g) denote a compact Riemannian manifold. Again, we consider the canonical embedding of the unit Euclidean sphere $i : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ and, if u is a map from M into $\mathbb{S}^n$, we shall write $u = (u^1, \dots, u^{n+1})$ for $i \circ u$. Assuming that k is a positive integer, we shall work in the Sobolev spaces

$$W^{k,2}(M^m, \mathbb{S}^n) = \left\{ u \in W^{k,2}(M^m, \mathbb{R}^{n+1}) : u(x) = (u^1(x), \dots, u^{n+1}(x)) \in \mathbb{S}^n \text{ a.e.} \right\}.$$

The *extrinsic k-energy functional* $E_k^{\mathrm{ext}}(u)$ is defined on $W^{k,2}(M^m, \mathbb{S}^n)$ as

$$E_k^{\mathrm{ext}}(u) = \int_M |\Delta^s u|^2 \, dv_g, \qquad \text{when } k = 2s, \tag{1-14}$$

$$E_k^{\mathrm{ext}}(u) = \int_M |\nabla \Delta^s u|^2 \, dv_g, \quad \text{when } k = 2s+1. \tag{1-15}$$

Of course, if $k = 1$, the extrinsic 1-energy coincides (up to the constant $\frac{1}{2}$) with the classical energy. Therefore, our interest is mainly on the case $k \geq 2$.

We say that $u \in W^{k,2}(M^m, \mathbb{S}^n)$ is an *extrinsic* (*weakly*) *k-harmonic map* if

$$\frac{d}{dt} E_k^{\mathrm{ext}}(u_t)\Big|_{t=0} = 0$$

for all variations

$$u_t = \frac{u + t\phi}{|u + t\phi|},$$

where $\phi \in C_0^\infty(M^m, \mathbb{R}^{n+1})$.

A very important class of critical points are the so-called minimizers. Specifically, a *minimizer*, or *minimizing extrinsic k-harmonic map*, is a map $u \in W^{k,2}(M^m, \mathbb{S}^n)$ such that

$$E_k^{\mathrm{ext}}(u) \leq E_k^{\mathrm{ext}}(v)$$

for all $v \in W^{k,2}(M^m, \mathbb{S}^n)$ such that $u - v \in W_0^{k,2}(M^m, \mathbb{R}^{n+1})$.

If u is an extrinsic k-harmonic map, we say that u is *stable* if

$$\frac{d^2}{dt^2} E_k^{\mathrm{ext}}(u_t)\Big|_{t=0} \geq 0.$$

We point out that if u is an unstable critical point, then it cannot be a minimizer.

The following proposition provides the explicit expression of the Euler–Lagrange equation associated to the extrinsic energy functionals $E_k^{\mathrm{ext}}(u)$.

Proposition 1.1 (see [10]). *Let (M^m, g) denote a compact Riemannian manifold. Assume that $k \geq 2$ and let $u \in W^{k,2}(M^m, \mathbb{S}^n)$. Then u is an extrinsic (weakly) k-harmonic map if and only if*

$$\Delta^k u + \lambda_k u = 0 \tag{1-16}$$

in the sense of distributions. Moreover, if (1-16) holds, then

$$\lambda_k = \Delta^{k-1}(|\nabla u|^2) + \sum_{j=0}^{k-2} \Delta^j(\langle \Delta^{k-1-j} u, \Delta u\rangle) + 2\sum_{j=0}^{k-2} \Delta^j(\nabla u.\nabla\Delta^{k-1-j} u) \tag{1-17}$$

(*note that $\Delta^0 u = u$ and $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{R}^{n+1}$*).

Remark 1.2. As we shall detail below, it will be important for us to consider the case that u is defined on a Riemannian manifold M which is *not* compact (for instance, we shall study the inverse of the stereographic projection $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$). Of course, in this context we shall say that u is extrinsically k-harmonic on M if it is such on each bounded domain $\Omega \subset M$ and Proposition 1.1 still applies.

Our first theorem confirms that extrinsic energies are suitable to study analytic and geometric features of π^{-1}. Indeed, we shall prove:

Theorem 1.3. *Assume $n \geq 2$ and $k \geq 1$. Then the inverse of the stereographic projection $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is an extrinsic k-harmonic map if and only if $n = 2k$.*

This result makes it natural to ask whether $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is energy minimizing for the k-energy. It was proved in [9] that, when $k = 2$, the answer is affirmative if and only if $0 < R \leq 1$.

In this context we obtain:

Theorem 1.4. *Let $0 < R \leq R_3^*$, where the constant $R_3^* \approx 0.82$ is defined in (3-14). Then $\pi^{-1} : B^6(R) \to \mathbb{S}^6$ is energy minimizing for the extrinsic 3-energy.*

As for higher-order energies, we do not have such an explicit upper estimate for R, but we can prove:

Theorem 1.5. *Assume $n = 2k$ with $k \geq 4$. Then there exists $0 < R_k^* \leq 1$ such that $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ is energy minimizing for the extrinsic k-energy provided that $0 < R \leq R_k^*$.*

Proposition 1.6. *Assume $n = 2k$ and $R > 1$. Then $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ is **not** energy minimizing for the extrinsic k-energy.*

As a consequence of these results, a very natural topic for further investigation is to study when $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is a *stable* critical point for the extrinsic k-energy. The study of this problem is not present in the literature, not even in the case of the bienergy. In general, it seems to be a difficult task to obtain a complete answer depending on R and k. A starting point is to restrict attention to rotationally symmetric variations, i.e., to the so-called *equivariant variations*. In this context, our main result is:

Theorem 1.7. (i) *The extrinsically biharmonic map* $\pi^{-1} : B^4(R) \to \mathbb{S}^4$ *is stable with respect to equivariant variations provided that* $0 < R \leq R_2^{\text{stab}} \approx 1.81$.

(ii) *The extrinsically triharmonic map* $\pi^{-1} : B^6(R) \to \mathbb{S}^6$ *is stable with respect to equivariant variations provided that* $0 < R \leq R_3^{\text{stab}} \approx 1.43$.

In our opinion, an interesting feature of Theorem 1.7 is the stability for values of $R > 1$. In these cases, the image of the map is *not* contained in the closed upper hemisphere.

Our paper is organized as follows. The proof of Theorem 1.3 requires to overcome several technical difficulties and will be carried out in Section 2. In Section 3 we prove Theorems 1.4, 1.5 and Proposition 1.6. The study of the second variation will be carried out in Section 4, where we shall prove Theorem 1.7.

2. Proof of Theorem 1.3

We carry out some preliminary work. Let $r = |x|$ and $u : \mathbb{R}^n \backslash \{O\} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be a map of the form

$$x = (x_1, \dots, x_n) \mapsto (p(r)\, x, q(r)) = (p(r)\, x_1, \dots, p(r)\, x_n, q(r)), \tag{2-1}$$

where $p(r)$ and $q(r)$ are smooth functions for $r > 0$. We shall need to compute terms involving $\Delta^k u$ and $\nabla \Delta^k u$. To this purpose, it is convenient to define recursively the following functions:

$$\begin{aligned} &P_0(r) = p(r),\\ &P_k(r) = P''_{k-1}(r) + \frac{(n+1)}{r} P'_{k-1}(r), \quad k \geq 1,\\ &Q_0(r) = q(r),\\ &Q_k(r) = Q''_{k-1}(r) + \frac{(n-1)}{r} Q'_{k-1}(r), \quad k \geq 1. \end{aligned} \tag{2-2}$$

We observe that the above functions depend on n. However, since this dependence will always be clear from the context, we have simplified the notation avoiding to write $P_{k,n}(r)$ etc. Next, we have:

Lemma 2.1. *Let* $u : \mathbb{R}^n \backslash \{O\} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ *be a map as in* (2-1). *Then, in the notation of* (2-2) *for all* $i, k \geq 0$ *we have*:

(i) $\Delta^k u = (P_k(r)\, x, Q_k(r))$.

(ii) $\langle \Delta^i u, \Delta^k u \rangle = r^2 P_i(r) P_k(r) + Q_i(r)\, Q_k(r)$.

(iii) $$\nabla\Delta^k u = \left(\frac{P_k' x_1}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} P_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \frac{P_k' x_2}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ P_k \\ \vdots \\ 0 \end{bmatrix}, \dots, \frac{P_k' x_n}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ P_k \end{bmatrix}, \frac{Q_k'}{r} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right).$$

(iv) $\nabla\Delta^i u . \nabla\Delta^k u = r^2 P_i' P_k' + n P_i P_k + r P_i' P_k + r P_i P_k' + Q_i' Q_k'$.

(*note that* $\Delta^0 u = u$ *and the scalar product* . *was defined in* (1-11)).

Proof. The proof is a straightforward computation which can be carried out using

$$\nabla p(r) = p'(r)\frac{x}{r}, \quad \Delta p(r) = p''(r) + \frac{(n-1)}{r}\, p'(r),$$
$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle\nabla f, \nabla g\rangle. \qquad \square$$

Proof of Theorem 1.3. Observe that the smooth map π^{-1} is of the type (2-1) with

(2-3) $$p(r) = \frac{2}{1+r^2} \quad \text{and} \quad q(r) = \frac{1-r^2}{1+r^2}.$$

We need to compute the explicit expression of the functions introduced in (2-2). To this purpose, it is convenient to define the following sets of constants:

(2-4) $$B_k = (-1)^k\, 2^{k+1}\, k! \quad (k \geq 1)$$

and

(2-5) $$a_k[j,n] = \binom{k}{j} \prod_{\ell=j}^{k-1} (n+2\ell+2) \prod_{\ell=0}^{j-1} (n+2\ell-2k) \quad (n \geq 2,\ k \geq 1,\ 0 \leq j \leq k).$$

Note that we make use of the convention

$$\prod_{\ell=q}^{q'} C_\ell = 1, \quad \text{whenever } q' < q.$$

The following lemma is technically difficult, but crucial for our proof.

Lemma 2.2. *Assume that* $n \geq 2$. *In the case of* $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$, *the explicit expression of the functions* $P_k(r)$, $Q_k(r)$ *introduced in* (2-2) *is*

$$P_k(r) = \frac{B_k}{(1+r^2)^{2k+1}} \sum_{j=0}^{k} a_k[j,n]\, r^{2j} \quad (k \geq 0), \tag{2-6}$$

$$\begin{aligned} Q_k(r) &= \frac{B_k}{(1+r^2)^{2k+1}} \sum_{j=0}^{k} a_k[j,n-2]\, r^{2j} \quad (k \geq 1), \\ Q_0(r) &= q(r) \quad (= -1 + P_0(r)). \end{aligned} \tag{2-7}$$

In order to preserve the flow of the exposition, the rather tedious proof of the previous lemma will be given at the end of the proof of the theorem.

Now, using Lemma 2.1(i), we easily see that the Euler–Lagrange equation (1-16) is equivalent to the system

$$\begin{cases} P_k(r) + \lambda_k\, p(r) = 0, \\ Q_k(r) + \lambda_k\, q(r) = 0. \end{cases} \tag{2-8}$$

Next, let us assume that $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is extrinsically k-harmonic. Then, taking into account that $q(r) = -1 + p(r)$, equation (2-8) implies

$$(-1 + p(r))\, P_k(r) - p(r)\, Q_k(r) = 0. \tag{2-9}$$

Using Lemma 2.2, we see immediately that (2-9) has the form

$$\frac{1}{(1+r^2)^{2k+2}} \sum_{j=0}^{k+1} c_k[j,n]\, r^{2j} = 0 \tag{2-10}$$

for some real coefficients $c_k[j,n]$. Next, we observe that

$$\begin{aligned} c_k[0,n] &= B_k(a_k[0,n] - 2a_k[0,n-2]) \\ &= B_k\left(\prod_{\ell=0}^{k-1}(n+2\ell+2) - 2\prod_{\ell=0}^{k-1}(n+2\ell)\right) \\ &= B_k\left((n+2k)\prod_{\ell=1}^{k-1}(n+2\ell) - 2n\prod_{\ell=1}^{k-1}(n+2\ell)\right) \\ &= B_k\left((2k-n)\prod_{\ell=1}^{k-1}(n+2\ell)\right). \end{aligned}$$

Of course, if $c_k[0,n] \neq 0$, then (2-10) can hold only at isolated points. Therefore, a necessary condition for the validity of (2-10) is $c_k[0,n] = 0$, i.e., $n = 2k$. By way of summary, we have proved that $n = 2k$ is necessary for the extrinsic k-harmonicity of $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$.

Conversely, let us now assume that $n = 2k$. If we use this assumption in the definition of $a_k[j, n]$ it is easy to check that

$$a_k[j, 2k] = 0 \quad (j \geq 1), \qquad a_k[0, 2k] = \prod_{\ell=0}^{k-1}(2k + 2\ell + 2).$$

From this, it follows easily that

$$-\frac{P_k(r)}{p(r)} = \frac{A_k}{(1+r^2)^{2k}}, \tag{2-11}$$

where

$$A_k = -\tfrac{1}{2} B_k \prod_{\ell=0}^{k-1}(2k + 2\ell + 2) = (-1)^{k+1}\, 2^{2k}\, (2k)!. \tag{2-12}$$

In a similar fashion we find

$$a_k[j, 2k-2] = 0 \quad (j \geq 2), \qquad a_k[0, 2k-2] = \prod_{\ell=0}^{k-1}(2k + 2\ell),$$

$$a_k[1, 2k-2] = \binom{k}{1} \prod_{\ell=1}^{k-1}(2k + 2\ell) \prod_{\ell=0}^{0}(2\ell - 2) = -\prod_{\ell=0}^{k-1}(2k + 2\ell).$$

From this, using again Lemma 2.2, it is easy to conclude that

$$\begin{aligned} -\frac{Q_k(r)}{q(r)} &= -B_k \prod_{\ell=0}^{k-1}(2k+2\ell)\frac{1}{(1+r^2)^{2k}} \\ &= -\tfrac{1}{2} B_k \prod_{\ell=0}^{k-1}(2k+2\ell+2)\frac{1}{(1+r^2)^{2k}} = \frac{A_k}{(1+r^2)^{2k}}. \end{aligned}$$

The conclusion is that, if $n = 2k$, then $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is extrinsically k-harmonic because it verifies the Euler–Lagrange equation (1-16) with

$$\lambda_k = \frac{A_k}{(1+r^2)^{2k}}, \tag{2-13}$$

where A_k was defined in (2-12). So, it only remains to prove Lemma 2.2.

Proof of Lemma 2.2. First, we prove (2-6). We observe that $n \geq 2$ is a fixed integer and so we proceed by induction on k. It is immediate to check that, independently of n, $P_0(r) = p(r)$. Thus, our proof amounts to check that the functions $P_k(r)$ defined in (2-6) verify the recursive law

$$P_{k+1}(r) = P_k''(r) + \frac{(n+1)}{r} P_k'(r). \tag{2-14}$$

A straightforward direct computation, taking into account the expression (2-6) for $P_k(r)$, shows that the right-hand side of (2-14) is given by

$$\frac{T_k(r)}{(1+r^2)^{3+2k}}, \tag{2-15}$$

where

$$\begin{aligned} T_k(r) = 2B_k\Big\{ &\sum_{j=0}^{k-1}(j+1)(n+2j+2)\, a_k[j+1,n]\, r^{2j} \\ &- (2k+1)(n+2)\sum_{j=0}^{k} a_k[j,n]\, r^{2j} \\ &+ \sum_{j=1}^{k} 2j(-4k+n+2j-2)\, a_k[j,n]\, r^{2j} \\ &+ (2k+1)(4k-n+2)\sum_{j=1}^{k+1} a_k[j-1,n]\, r^{2j} \\ &+ \sum_{j=1}^{k+1}(j-1)(-8k+n+2j-6)\, a_k[j-1,n]\, r^{2j}\Big\} \\ = 2B_k &\sum_{j=0}^{k+1} d_j\, r^{2j} \end{aligned} \tag{2-16}$$

for some real coefficients $d_j,\ j=0,\dots,k+1$.

For later use we note that, applying directly the definition of $a_k[j,n]$ given in (2-5), we have the following relations which hold for $1 \le j \le k$:

$$\begin{aligned} a_k[j-1,n] &= \binom{k}{j-1}\prod_{\ell=j-1}^{k-1}(n+2+2\ell)\prod_{\ell=0}^{j-2}(n-2k+2\ell) \\ &= \left[\frac{\binom{k}{j-1}}{\binom{k}{j}}\right]\binom{k}{j}\frac{(n+2j)}{(n-2k+2j-2)}\prod_{\ell=j}^{k-1}(n+2+2\ell)\prod_{\ell=0}^{j-1}(n-2k+2\ell) \\ &= \frac{j}{(k-j+1)}\,\frac{(n+2j)}{(n-2k+2j-2)}\,a_k[j,n], \end{aligned} \tag{2-17}$$

and, in a similar way, for $1 \le j \le k-1$:

$$a_k[j+1,n] = \frac{(k-j)}{(j+1)}\,\frac{(n+2j-2k)}{(n+2j+2)}\,a_k[j,n]. \tag{2-18}$$

Moreover, we shall also need to use

$$a_{k+1}[j,n] = \frac{(k+1)(n-2k-2)(2k+n+2)}{(-j+k+1)(n+2j-2k-2)}\,a_k[j,n] \quad (0 \le j \le k). \tag{2-19}$$

This last formula can be proved using again the definition (2-5) and the methods which we employed in (2-17). Indeed,

$$
\begin{aligned}
&a_{k+1}[j,n] \\
&= \binom{k+1}{j} \prod_{\ell=j}^{k} (n+2\ell+2) \prod_{\ell=0}^{j-1} (n+2\ell-2k-2) \\
&= \left[\frac{\binom{k+1}{j}}{\binom{k}{j}} \right] \binom{k}{j} (n+2k+2) \left(\prod_{\ell=j}^{k-1} (n+2+2\ell) \right) \frac{(n-2k-2)}{(n-2k+2j-2)} \prod_{\ell=0}^{j-1} (n-2k+2\ell) \\
&= \frac{(k+1)(n-2k-2)(2k+n+2)}{(-j+k+1)(n+2j-2k-2)} a_k[j,n].
\end{aligned}
$$

Next, since $B_{k+1} = -2(k+1)\, B_k$ and taking into account the expression (2-6) for $P_{k+1}(r)$, the proof of (2-14) will be completed if we show that the coefficients d_j introduced in (2-16) verify

$$
d_j = -(k+1)\, a_{k+1}[j,n] \quad \text{for all } 0 \le j \le k+1. \tag{2-20}
$$

Now, from (2-16), we find

$$
\begin{aligned}
d_0 &= (n+2)\big[a_k[1,n] - (1+2k)\, a_k[0,n]\big] \\
&= (n+2) \left[k(n-2k) \prod_{\ell=1}^{k-1} (n+2\ell+2) - (1+2k)\, a_k[0,n] \right] \\
&= (n+2) \left[k(n-2k) \frac{a_k[0,n]}{(n+2)} - (1+2k)\, a_k[0,n] \right] \\
&= -(k+1)\, (n+2k+2)\, a_k[0,n] \\
&= -(k+1)\, a_{k+1}[0,n]
\end{aligned}
$$

and so (2-20) is proved when $j=0$.

Next, computing and using (2-17)–(2-19), we obtain

$$
\begin{aligned}
d_k &= (2+k)(4+2k-n)\, a_k[k-1,n] - (2+8k+4k^2+n)\, a_k[k,n] \\
&= -(k+1) \frac{(k+1)\, n^2 - 4(k+1)^3}{(n-2)} a_k[k,n] = -(k+1)\, a_{k+1}[k,n], \\
d_{k+1} &= [(2k+1)(4k-n+2) + k(-8k+n+2k-4)]\, a_k[k,n] \\
&= -(k+1)(n-2-2k)\, a_k[k,n] = -(k+1)\, a_{k+1}[k+1,n].
\end{aligned}
$$

Thus, we have verified that (2-20) is true also when $j=k$ and $j=k+1$.

As for the other coefficients, again, from (2-16) we find

$$\begin{aligned} d_j = {} & [(2k+1)(4k-n+2)+(j-1)(-8k+n+2j-6)]\,a_k[j-1,n] \\ & +[2j(-4k+n+2j-2)-(2k+1)(n+2)]\,a_k[j,n] \\ & +(j+1)(n+2j+2)\,a_k[j+1,n]. \end{aligned} \tag{2-21}$$

for $1 \le j \le k-1$.

Next, substituting (2-17) and (2-18) into (2-21), after a routine simplification and using again (2-19), we obtain

$$d_j = -\frac{(k+1)^2(n-2k-2)(2k+n+2)}{(-j+k+1)(n+2j-2k-2)}a_k[j,n] = -(k+1)\,a_{k+1}[j,n]. \tag{2-22}$$

Therefore, the verification of (2-20) is completed and so the proof of (2-6) is ended.

As for (2-7), we observe that the recursive definition (2-2) of $Q_k(r)$ is as that of $P_k(r)$, with the only difference that n is replaced by $n-2$. Moreover, the difference between $q(r)$ and $p(r) = P_0(r)$ is just the additive constant -1, which is irrelevant for $k \ge 1$. Therefore, the explicit expression of $Q_k(r)$ can be obtained replacing n by $n-2$ in the expression of $P_k(r)$ and so the conclusion of the lemma follows immediately. □

In conclusion, the proof of Theorem 1.3 is now ended. □

Remark 2.3. Let $n = 2k$. The extrinsic k-energy of $\pi^{-1} : \mathbb{R}^n \to \mathbb{S}^n$ is finite. For instance, explicit integration provides the exact value of the extrinsic 3-energy of $\pi^{-1} : \mathbb{R}^6 \to \mathbb{S}^6$:

$$\begin{aligned} E_3^{\text{ext}}(\pi^{-1}) &= \operatorname{Vol}(\mathbb{S}^5)\int_0^{+\infty} [r^2 P_1'^2 + 6P_1^2 + 2r\,P_1' P_1 + Q_1'^2]\,r^5\,dr \\ &= \pi^3 \int_0^{+\infty} \frac{512r^5(7r^4+24r^2+12)}{(r^2+1)^6}dr \\ &= -\pi^3 \frac{256(7r^8+26r^6+30r^4+15r^2+3)}{(r^2+1)^5}\bigg|_0^{+\infty} = 768\pi^3. \end{aligned}$$

Note that $E(\pi^{-1}) = 4\pi$ and $E_2^{\text{ext}}(\pi^{-1}) = 64\pi^2$.

3. Proofs of Theorems 1.4, 1.5 and Proposition 1.6

Proof of Theorem 1.4. **Step 1.** Lemma 3.1 below is a version of Lemma 2.3 of [10] in this context. The proof is similar, but we include it for the sake of completeness (to shorten the notation, when the meaning is clear, we shall use "." instead of $\langle\, ,\rangle$).

Lemma 3.1. (i) *Let $n = 2k$ and $k = 2s$. If*

$$\int_{B^n(R)} [|\Delta^s \phi|^2 - (\Delta^{2s}\pi^{-1}.\pi^{-1})\,|\phi|^2]\,dx \ge 0 \tag{3-1}$$

for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$, *then* π^{-1} *is energy minimizing for the extrinsic* k*-energy.*

(ii) *Let* $n = 2k$ *and* $k = 2s+1$. *If*

$$\int_{B^n(R)} [|\nabla \Delta^s \phi|^2 + (\Delta^{2s+1}\pi^{-1}.\pi^{-1})\,|\phi|^2]\,dx \geq 0 \tag{3-2}$$

for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$, *then* π^{-1} *is energy minimizing for the extrinsic* k*-energy.*

Proof. (i) We must show that

$$E_k^{\text{ext}}(\pi^{-1}) \leq E_k^{\text{ext}}(v) \tag{3-3}$$

for all $v \in W^{k,2}(B^n(R), \mathbb{S}^n)$ such that $\pi^{-1} - v \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$.

On the ball $B^n(R)$ the map π^{-1} satisfies

$$\Delta^{2s}\pi^{-1} = (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1} \tag{3-4}$$

strongly. Thus we can multiply both sides of (3-4) by $\phi \in W_0^{k,2}(B^n(R), \mathbb{R}^{n+1})$ and we obtain

$$\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s\phi\,dx = \int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.\phi\,dx.$$

Choosing $\phi = \pi^{-1} - v$ we deduce that

$$\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s\pi^{-1}\,dx - \int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s v\,dx = \int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,dx - \int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.v\,dx,$$

which for convenience we rewrite as

$$-2\int_{B^n(R)} |\Delta^s\pi^{-1}|^2\,dx + 2\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s v\,dx = -2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,dx + 2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.v\,dx. \tag{3-5}$$

Next, we apply the hypothesis (3-1) with $\phi = \pi^{-1} - v$. Since π^{-1}, v have values in $\mathbb{S}^n$ we have

$$|\phi|^2 = |\pi^{-1} - v|^2 = 2 - 2\pi^{-1}.v$$

and so we easily find

$$\int_{B^n(R)} |\Delta^s v|^2\,dx + \int_{B^n(R)} |\Delta^s\pi^{-1}|^2\,dx - 2\int_{B^n(R)} \Delta^s\pi^{-1}.\Delta^s v\,dx - 2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,dx + 2\int_{B^n(R)} (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\pi^{-1}.v\,dx \geq 0. \tag{3-6}$$

Finally, inserting (3-5) into the second line of (3-6), we obtain

$$\int_{B^n(R)} |\Delta^s v|^2\, dx - \int_{B^n(R)} |\Delta^s \pi^{-1}|^2 dx \ge 0,$$

which is precisely (3-3). The proof of part (ii) is analogous and so we omit the details. □

Step 2. Case $n = 2k,\ k = 2s$. Taking into account (2-12) and (2-13) it is easy to deduce that the inequality (3-1) is equivalent to

$$\int_{B^n(R)} \left[|\Delta^s \phi|^2 - \frac{2^{4s}(4s)!}{(1+r^2)^{4s}} |\phi|^2 \right] dx \ge 0. \tag{3-7}$$

Similarly, when $n = 2k$ and $k = 2s+1$, equation (3-2) can be written as

$$\int_{B^n(R)} \left[|\nabla \Delta^s \phi|^2 - \frac{2^{4s+2}(4s+2)!}{(1+r^2)^{4s+2}} |\phi|^2 \right] dx \ge 0. \tag{3-8}$$

We also point out that in (3-1), (3-2), (3-7) and (3-8) the test function ϕ is a vector function $\phi = (\phi^1, \dots, \phi^{n+1})$. But, since

$$|\Delta^s \phi|^2 = \sum_{i=1}^{n+1} |\Delta^s \phi^i|^2, \quad |\nabla \Delta^s \phi|^2 = \sum_{i=1}^{n+1} |\nabla \Delta^s \phi^i|^2 \quad \text{and} \quad |\phi|^2 = \sum_{i=1}^{n+1} |\phi^i|^2,$$

we easily deduce that it suffices to prove that these inequalities hold for all *scalar* test functions $\phi \in W_0^{k,2}(B^n(R), \mathbb{R})$.

By way of summary, the validity of (3-7) (case $k = 2s$) or (3-8) (case $k = 2s+1$) for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R})$ is sufficient to insure that $\pi^{-1} : B^{2k}(R) \to \mathbb{S}^{2k}$ is a minimizer for the extrinsic k-energy.

As a special case, the proof of Theorem 1.4 will be complete if we show that the inequality (3-8), with $s = 1$, holds provided that $0 < R \le R_3^*$.

To this purpose, we recall that in [12, Theorem 3, p. 2159] the authors proved a general third-order Hardy inequality for bounded domains $\Omega \subset \mathbb{R}^n$, $n \ge 6$. It is convenient for us to state their result in the special case $n = 6$ and $\Omega = B^6(R)$. We set

$$c_1 = 57\Lambda(2), \quad c_2 = 6\Lambda(2)^2 + 4\Lambda(4)^2, \quad c_3 = \Lambda(2)\,\Lambda(4)^2, \tag{3-9}$$

where

$$\Lambda(n) = \inf\left\{ \frac{\int_0^1 |v'(r)|^2\, r^{n-1}\, dr}{\int_0^1 |v(r)|^2\, r^{n-1}\, dr} : v \in X \right\},$$

with

$$X = \left\{ v \in C^1([0,1]) : v'(0) = v(1) = 0,\ v \not\equiv 0 \right\}.$$

In other words, $\Lambda(n)$ is the first positive eigenvalue associate to the Dirichlet problem for Δ on B^n, i.e.,

$$\Lambda(n) = \lambda_1 = \inf\left\{ \frac{\int_{B^n} |\nabla \phi|^2}{\int_{B^n} \phi^2} : \phi \in W_0^{1,2}(B^n),\ \phi \not\equiv 0 \right\}.$$

It is also known (see [12]) that

$$\Lambda(n)^2 = \inf\left\{ \frac{\int_{B^n} (\Delta \phi)^2}{\int_{B^n} \phi^2} : \phi \in W^{2,2} \cap W_0^{1,2}(B^n),\ \phi \not\equiv 0 \right\} = \lambda_1^2.$$

The value of λ_1, which depends on n, is related to the first positive zero for a class of Bessel functions.

More precisely (see [2]), let $\nu = \frac{n}{2} - 1$. Then $\lambda_1 = j_\nu^2$, where j_ν denotes the first positive zero of the Bessel function $J_\nu(r)$. In particular,

$$\Lambda(2) = j_0^2 \approx (2.4)^2, \quad \Lambda(4) = j_1^2 \approx (3.8)^2. \tag{3-10}$$

Then we have:

Theorem 3.2 [12]. *Let $\phi \in W^{3,2} \cap W_0^{1,2}(B^6(R))$ with $\Delta\phi = 0$ on $\partial B^6(R)$, i.e., $\Delta\phi \in W_0^{1,2}(B^6(R))$. Then*

$$\int_{B^6(R)} |\nabla \Delta \phi|^2 \, dx \geq \frac{c_1}{R^2} \int_{B^6(R)} \frac{\phi^2}{|x|^4} dx + \frac{c_2}{R^4} \int_{B^6(R)} \frac{\phi^2}{|x|^2} dx + \frac{c_3}{R^6} \int_{B^6(R)} \phi^2 \, dx, \tag{3-11}$$

where the constants c_i, $i = 1, 2, 3$, are defined in (3-9).

Now, we are in the position to conclude the proof of Theorem 1.4.

Indeed, since in our context $\phi \in W_0^{3,2}(B^6(R))$, we can apply (3-11) and we deduce ($r = |x|$):

$$\begin{aligned} \int_{B^6(R)} |\nabla \Delta \phi|^2 \, dx &\geq \int_{B^6(R)} \left[\frac{c_1}{R^2 r^4} + \frac{c_2}{R^4 r^2} + \frac{c_3}{R^6} \right] \phi^2 \, dx \\ &= \int_{B^6(R)} \psi_R(r) \, \phi^2 \, dx, \end{aligned} \tag{3-12}$$

where we have set

$$\psi_R(r) = \frac{c_1}{R^2 r^4} + \frac{c_2}{R^4 r^2} + \frac{c_3}{R^6}, \quad 0 < r \leq R.$$

Now, taking into account (3-8), we deduce that (3-12) implies that (3-2) with $s = 1$ holds provided that

$$\psi_R(r) - \frac{46080}{(1 + r^2)^6} \geq 0 \quad \text{on } (0, R]. \tag{3-13}$$

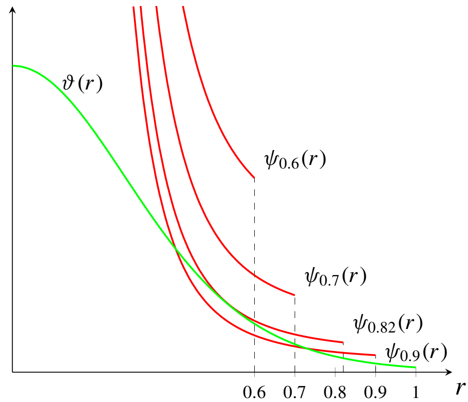

Figure 1. Analysis of condition (3-13). Here $\vartheta(r) = \frac{46080}{(1+r^2)^6}$.

Therefore, it is natural to define

$$R_3^* = \text{Max}\{R > 0 : \text{(3-13) holds}\}. \tag{3-14}$$

Now, using (3-9) and (3-10), a routine analysis shows that R_3^* is well defined and (3-13) holds for all $0 < R \leq R_3^*$. Finally, a straightforward analysis carried out with Mathematica shows that $R_3^* \approx 0.82$ and so the proof of Theorem 1.4 is completed. We have inserted the output of this study in Figure 1. □

Proof of Theorem 1.5. We follow the method of proof of Theorem 1.4. Then, when $k = 2s \geq 4$, it suffices to prove the existence of $0 < R_k^* \leq 1$ such that the inequality (3-7) holds for $0 < R \leq R_k^*$. This can be achieved using another result of [12]. Indeed, let us fix $k = 2s$. Then Corollary 2 of [12] enables us to say that there exist positive constants $c_1, \ldots, c_k$ such that

$$\int_{B^n(R)} |\Delta^s \phi|^2 \, dx \geq \sum_{\ell=1}^{k} \frac{c_\ell}{R^{2\ell}} \int_{B^n(R)} \frac{\phi^2}{|x|^{2k-2\ell}} \, dx$$

for all $\phi \in W_0^{k,2}(B^n(R), \mathbb{R})$. From this we easily deduce that

$$\int_{B^n(R)} |\Delta^s \phi|^2 \, dx \geq \frac{\left(\sum_{\ell=1}^{k} c_\ell\right)}{R^{4s}} \int_{B^n(R)} \frac{\phi^2}{(1+|x|^2)^{4s}} \, dx. \tag{3-15}$$

Therefore, setting

$$R_k^* = 2 \sqrt[4s]{\frac{\left(\sum_{\ell=1}^{k} c_\ell\right)}{(2s)!}},$$

it is easy to conclude that (3-7) holds for all $0 < R \leq R_k^*$. Moreover, as a consequence of Proposition 1.6, we observe that necessarily $R_k^* \leq 1$ and so the proof of the case $k = 2s$ is ended.

Similarly, using again Corollary 2 of [12], we obtain the existence of $0 < R_k^* \leq 1$ such that, when $k = 2s + 1 \geq 5$, (3-8) holds and so the proof is completed. □

Remark 3.3. In order to obtain a numerical value for R_k^*, $k \geq 4$, it would be useful to know the exact optimal values of the positive constants $c_1, \dots, c_k$, but this datum is not available (see [12]). Therefore, in this case, we have preferred to limit technicalities and we do not have introduced a function which could play the role of $\psi_R(r)$ in the proof of Theorem 1.4. Moreover, we point out that a better estimate for R_k^* could be achieved if Conjecture 1 of [12, p. 2164] were true.

Proof of Proposition 1.6. Following an idea of [9], we compare $\pi^{-1} : B^n(R) \to \mathbb{S}^n$ with the map $\tilde{\pi}^{-1} : B^n(R) \to \mathbb{S}^n$ defined as follows: $\tilde{\pi}^{-1}(x) = \pi_N^{-1}(x/R^2)$, where π_N^{-1} is the map obtained from π^{-1} by changing the sign of the last component. In order to clarify the geometric construction, we point out that π represents the stereographic projection from the *south pole*, while π_N is the stereographic projection from the *north pole*. More explicitly, we have

$$\tilde{\pi}^{-1} : B^n(R) \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \left(\frac{2R^2}{R^4 + r^2}\, x, -\frac{R^4 - r^2}{R^4 + r^2} \right), \tag{3-16}$$

where, as usual, $r = |x|$. It is easy to check that π^{-1} and $\tilde{\pi}^{-1}$ coincide on $\partial B^n(R)$. Next, we first observe that

$$(\Delta \tilde{\pi}^{-1})(x) = \frac{1}{R^4} (\Delta \pi_N^{-1}) \left(\frac{x}{R^2} \right)$$

and so

$$(\Delta^s \tilde{\pi}^{-1})(x) = \frac{1}{R^{4s}} (\Delta^s \pi_N^{-1}) \left(\frac{x}{R^2} \right).$$

Now, when $n = 2k$, $k = 2s$, we have:

$$\begin{aligned} \int_{B^n(R)} |(\Delta^s \tilde{\pi}^{-1})(x)|^2 \, dx &= \frac{1}{R^{8s}} \int_{B^n(R)} \left| (\Delta^s \pi_N^{-1}) \left(\frac{x}{R^2} \right) \right|^2 dx \\ \left(\text{using } y = \frac{x}{R^2}\right) \quad &= \frac{1}{R^{8s}} \int_{B^n(1/R)} |(\Delta^s \pi_N^{-1})(y)|^2 R^{2n} \, dy \\ (\text{using } 8s = 2n) \quad &= \int_{B^n(1/R)} |(\Delta^s \pi_N^{-1})(y)|^2 \, dy. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \int_{B^n(R)} |\Delta^s \tilde{\pi}^{-1}|^2 \, dx &= \int_{B^n(1/R)} |\Delta^s \pi_N^{-1}|^2 \, dx \\ &= \int_{B^n(1/R)} |\Delta^s \pi^{-1}|^2 \, dx < \int_{B^n(R)} |\Delta^s \pi^{-1}|^2 \, dx, \end{aligned}$$

which proves Proposition 1.6 in the case $k = 2s$. The case $k = 2s + 1$ is analogous. □

Remark 3.4. We point out that the map $\tilde{\pi}^{-1}$ defined in (3-16), although it coincides with π^{-1} on the boundary $\partial B^n(R)$, belongs to a different homotopy class.

4. Second variation and equivariant stability

For convenience we rewrite $\pi^{-1}: B^n(R) \to \mathbb{S}^n$ as

$$\pi^{-1}: B^n(R) \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \Big(\sin\alpha(r)\frac{x}{r}, \cos\alpha(r)\Big), \tag{4-1}$$

where $r = |x|$ and $\alpha(r) = 2\tan^{-1} r$. We consider *equivariant variations*, i.e., rotationally symmetric variations as

$$u_t(x) = \Big(\sin[\alpha(r) + t\,\phi(r)]\frac{x}{r}, \cos[\alpha(r) + t\phi(r)]\Big), \tag{4-2}$$

where $\phi = \phi(r)$ is any smooth real valued function on $[0, R]$ such that u_t is smooth and preserves the boundary data. In particular, we now have $\phi^{(2j)}(0) = 0,\ j \geq 0$, $\phi^{(j)}(R) = 0,\ j \geq 0$.

Therefore, we shall say that $\pi^{-1}: B^{2k}(R) \to \mathbb{S}^{2k}$ is *stable with respect to equivariant variations* (shortly, *equivariantly stable*) if

$$\frac{d^2}{dt^2}(E_k^{\text{ext}}(u_t))|_{t=0} \geq 0 \tag{4-3}$$

for all u_t as in (4-2).

Next, we define a map $\pi^{-1}_{\partial/\partial\alpha}: B^n(R) \to \mathbb{S}^n$ which represents the unit vector field $\partial/\partial\alpha$ along π^{-1}:

$$\pi^{-1}_{\partial/\partial\alpha}: B^n(R) \to \mathbb{S}^n \subset \mathbb{R}^n \times \mathbb{R}, \quad x \mapsto \Big(\cos\alpha(r)\frac{x}{r}, -\sin\alpha(r)\Big), \tag{4-4}$$

where again $r = |x|$ and $\alpha(r) = 2\tan^{-1} r$. Then a simple computation shows

$$\frac{du_t(x)}{dt}\Big|_{t=0} = \pi^{-1}_{\partial/\partial\alpha}(x)\,\phi \quad \text{and} \quad \frac{d^2u_t(x)}{dt^2}\Big|_{t=0} = -\pi^{-1}(x)\,\phi^2. \tag{4-5}$$

Now, we prove a general, preliminary lemma.

Lemma 4.1. (i) *Let $n = 2k,\ k = 2s$. Then $\pi^{-1}: B^n(R) \to \mathbb{S}^n$ is equivariantly stable for the extrinsic k-energy if and only if*

$$\int_{B^n(R)} [|\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)|^2 - (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\phi^2]\,dx \geq 0 \quad \textit{for all } \phi \textit{ as in (4-2)}. \tag{4-6}$$

(ii) *Let $n = 2k,\ k = 2s + 1$. Then $\pi^{-1}: B^n(R) \to \mathbb{S}^n$ is equivariantly stable for the extrinsic k-energy if and only if*

$$\int_{B^n(R)} [|\nabla\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)|^2 - (\Delta^{2s+1}\pi^{-1}.\pi^{-1})\,\phi^2]\,dx \geq 0 \quad \textit{for all } \phi \textit{ as in (4-2)}. \tag{4-7}$$

Proof. (i) We consider variations of the type (4-2) and we use (4-5). Then on $B^n(R)$ we have

$$\frac{d}{dt}(\Delta^s u_t)\Big|_{t=0} = \Delta^s\left(\frac{du_t}{dt}\right)\Big|_{t=0} = \Delta^s\left(\left(\frac{du_t}{dt}\right)\Big|_{t=0}\right) = \Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi) \tag{4-8}$$

and

$$\frac{d^2}{dt^2}(\Delta^s u_t)\Big|_{t=0} = \Delta^s\left(\frac{d^2u_t}{dt^2}\right)\Big|_{t=0} = \Delta^s\left(\left(\frac{d^2u_t}{dt^2}\right)\Big|_{t=0}\right) = -\Delta^s(\pi^{-1}\,\phi^2). \tag{4-9}$$

Now,

$$\frac{d^2}{dt^2}(E_k^{\text{ext}}(u_t))\Big|_{t=0} = 2\int_{B^n(R)}\frac{d^2}{dt^2}(\Delta^s(u_t))\Big|_{t=0}.(\Delta^s u_t)|_{t=0}\,dx + 2\int_{B^n(R)}\left|\frac{d}{dt}(\Delta^s(u_t))|_{t=0}\right|^2 dx. \tag{4-10}$$

Substituting (4-8) and (4-9) into (4-10) we get

$$\frac{d^2}{dt^2}(E_k^{\text{ext}}(u_t))\Big|_{t=0} = -2\int_{B^n(R)}\Delta^s(\pi^{-1}\phi^2).\,\Delta^s(\pi^{-1})\,dx + 2\int_{B^n(R)}|\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)|^2\,dx.$$

Then, using the second Green identity, we deduce that the stability condition (4-3) is equivalent to

$$\int_{B^n(R)}[|\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)|^2 - (\Delta^{2s}\pi^{-1}.\pi^{-1})\,\phi^2]\,dx \geq 0$$

for any arbitrary smooth function ϕ as in (4-2), as required.

The proof of the inequality (4-7) of part (ii) is analogous and so we omit it. □

The stability inequalities provided by Lemma 4.1 are rather general, but difficult to deal with because of the terms $\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)$ and $\nabla\Delta^s(\pi^{-1}_{\partial/\partial\alpha}\,\phi)$.

Therefore, a natural first step is to investigate these inequalities under the assumptions that k is small. In this order of ideas, a straightforward computation using Lemma 2.1 and some integration by parts leads us to the following:

Proposition 4.2. (i) *The map $\pi^{-1} : B^4(R) \to \mathbb{S}^4$ is equivariantly stable for the extrinsic bienergy if and only if*

$$\int_0^R\left[r^3\phi''^2(r) + 9r\phi'^2(r) + \left(\frac{9}{r} - \frac{384\,r^3}{(1+r^2)^4}\right)\phi^2(r)\right]dr \geq 0 \tag{4-11}$$

for all $\phi = \phi(r)$ as in (4-2).

(ii) *The map* $\pi^{-1} : B^6(R) \to \mathbb{S}^6$ *is equivariantly stable for the extrinsic trienergy if and only if*

$$\int_0^R \left[r^5 \phi'''^2(r) + 30 r^3 \phi''^2(r) + 225 r \phi'^2(r) + \left(\frac{225}{r} - \frac{46080\, r^5}{(1+r^2)^6} \right) \phi^2(r) \right] dr \geq 0 \tag{4-12}$$

for all $\phi = \phi(r)$ *as in* (4-2).

Remark 4.3. (i) As an alternative to (4-2), we could use

$$u_t(x) = \pi^{-1}(x) + t\phi \pi^{-1}_{\partial/\partial\alpha}(x).$$

Then we would have again (4-5) and so we would reobtain the conclusion of Lemma 4.1.

(ii) In the case that $\phi = \phi(r)$ is radial, the inequality (3-8) ($s = 1$) on $B^6(R)$ turns out to be equivalent to

$$\int_0^R \left[r^5 \phi'''^2(r) + 15 r^3 \phi''^2(r) + 45 r \phi'^2(r) - \frac{46080 r^5}{(1+r^2)^6} \phi^2(r) \right] dr \geq 0.$$

We observe that this condition is stronger than (4-12).

Now we can proceed to the proof of our main result in this context.

Proof of Theorem 1.7. (i) According to Proposition 4.2 it suffices to show that (4-11) holds provided that $0 < R \leq R_2^{\text{stab}} \approx 1.81$.

First, a simple computation using integration by parts leads us to

$$\int_0^R [r^3 \phi''^2(r) + 9 r \phi'^2(r)]\, dr = \frac{1}{\operatorname{Vol}(\mathbb{S}^3)} \int_{B^4(R)} |\Delta\phi|^2 dx + \frac{6}{\operatorname{Vol}(\mathbb{S}^3)} \int_{B^4(R)} \frac{|\nabla\phi|^2}{r^2} dx. \tag{4-13}$$

Next, Theorem 4 of [12] gives

$$\int_{B^4(R)} |\Delta\phi|^2 dx \geq \frac{\Lambda((-\Delta)^2, 4)}{R^4} \int_{B^4(R)} \phi^2 dx, \tag{4-14}$$

where, keeping the notation of [12], we have

$$\Lambda((-\Delta)^2, 4) \geq j_1^2 j_2^2 \approx 387.23.$$

Using (8) of [12], we also have

$$\int_{B^4(R)} \frac{|\nabla\phi|^2}{r^2} dx \geq \frac{1}{R^2} \int_{B^4(R)} \left[\frac{1}{r^2} + \frac{\Lambda(2)}{R^2} \right] \phi^2 dx. \tag{4-15}$$

Inserting (4-14) and (4-15) into (4-13) we deduce

$$\text{(4-16)}\quad \int_0^R [r^3\phi''^2(r)+9r\phi'^2(r)]\,dr \ge \int_0^R \left[\frac{(\Lambda(-\Delta)^2,4)}{R^4}+\frac{6}{R^2r^2}+\frac{6\Lambda(2)}{R^4}\right]\phi^2\,dr.$$

Next, we set

$$\psi_{R,2}(r)=\frac{(\Lambda(-\Delta)^2,4)+6\Lambda(2)}{R^4}+\frac{6}{R^2r^2}+\frac{9}{r^4},\quad 0<r\le R.$$

Then it is immediate to conclude that (4-11) holds provided that

$$\text{(4-17)}\qquad \psi_{R,2}(r)-\frac{384}{(1+r^2)^4}\ge 0 \quad \text{on } (0,R].$$

Now, an analysis similar to the study of (3-13) (see the proof of Theorem 1.4) shows that (4-17) holds provided that $0<R\le R_2^{\text{stab}}\approx 1.81$, so ending (i).

(ii) According to Proposition 4.2 it suffices to show that (4-12) holds provided that $0<R\le R_3^{\text{stab}}\approx 1.43$.

We shall use the following Hardy-type inequalities which again can be deduced from Theorem 4 and inequality (8) of [12], respectively:

$$\text{(4-18)}\qquad \int_{B^6(R)}|\Delta\phi|^2\,dx \ge 9\int_{B^6(R)}\frac{\phi^2}{r^4}dx+\frac{6\Lambda(2)}{R^2}\int_{B^6(R)}\frac{\phi^2}{r^2}dx+\frac{(\Lambda(-\Delta)^2,4)}{R^4}\int_{B^6(R)}\phi^2\,dx,$$

$$\text{(4-19)}\qquad \int_{B^6(R)}|\nabla\phi|^2\,dx \ge 4\int_{B^6(R)}\frac{\phi^2}{r^2}dx+\frac{\Lambda(2)}{R^2}\int_{B^6(R)}\phi^2\,dx.$$

Next, a computation similar to (4-13) enables us to write

$$\text{(4-20)}\qquad \begin{aligned}\int_0^R|\nabla\Delta\phi|^2\,r^5\,dr&=\int_0^R[r^5\phi'''^2(r)+15r^3\phi''^2(r)+45r\phi'^2(r)]\,dr,\\ \int_0^R|\Delta\phi|^2\,r^5\,dr&=\int_0^R[r^5\phi''^2(r)+5r^3\phi'^2(r)]\,dr,\\ \int_0^R|\nabla\phi|^2\,r^5\,dr&=\int_0^R\phi'^2(r)\,r^5\,dr.\end{aligned}$$

Now, using (4-20), we deduce

$$\text{(4-21)}\qquad \int_0^R[r^5\phi'''^2+30r^3\phi''^2+225r\phi'^2]\,dr = \int_0^R\left[|\nabla\Delta\phi|^2+15\frac{|\Delta\phi|^2}{r^2}+105\frac{|\nabla\phi|^2}{r^4}\right]r^5\,dr.$$

Next, using (3-11), (4-18), (4-19) and (4-21) into (4-12), we obtain that (4-12) holds provided that

$$\psi_{R,3}(r) - \frac{46080}{(1+r^2)^6} \geq 0 \quad \text{on } (0, R], \tag{4-22}$$

where we have defined

$$\psi_{R,3}(r) = \frac{c_1}{R^2 r^4} + \frac{c_2}{R^4 r^2} + \frac{c_3}{R^6} + \frac{15}{R^2}\left(\frac{9}{r^4} + \frac{6\Lambda(2)}{R^2 r^2} + \frac{(\Lambda(-\Delta)^2, 4)}{R^4}\right) + \frac{105}{R^4}\left(\frac{4}{r^2} + \frac{\Lambda(2)}{R^2}\right) + \frac{225}{r^6}.$$

Now, an analysis similar to the study of (3-13) shows that (4-22) holds provided that $0 < R \leq R_3^{\text{stab}} \approx 1.43$, so ending the proof of Theorem 1.7. □

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