BALANCED HOMOGENEOUS
HARMONIC MAPS BETWEEN CONES

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**We study the degrees of homogeneous harmonic maps between simplicial
cones. Such maps have been used to model the local behavior of harmonic
maps between singular spaces, where the degrees of homogeneous approxi-
mations describe the regularity of harmonic maps. In particular the degrees
of homogeneous harmonic maps are related to eigenvalues of discrete and
edge-based graph Laplacians.**

1. Introduction

The study of harmonic maps into singular spaces goes back to the seminal work
of Gromov and Schoen [1992], who considered maps into Riemannian simplicial
complexes. Of particular interest to our work is their local approximation of such
maps by *homogeneous* maps between tangent cones. The study of harmonic maps
was extended by Chen [1995] to include simplicial domains, and the theory for
simplicial domains was further studied in [Eells and Fuglede 2001; Mese 2004;
Daskalopoulos and Mese 2006; 2008], among others. The Hölder continuity
of harmonic maps from Riemannian simplicial complexes to metric spaces of
nonpositive curvature was established in [Eells and Fuglede 2001]. Further regularity
away from the $(n-2)$-skeleton of an n-dimensional simplicial domain was studied
in [Mese 2004; Daskalopoulos and Mese 2006; 2008].

The blow-up analysis of [Gromov and Schoen 1992] was emulated by Daskalopou-
los and Mese [2006; 2008] to construct tangent maps as homogeneous approxima-
tions of harmonic maps with singular domains and targets. The general idea is to
fix a point p in the domain of a harmonic map u, and its image $u(p)$. For $\lambda, \mu > 0$
one constructs the map

$$u_{\lambda,\mu}(x) = \mu^{-1} u(\lambda x).$$

A small neighborhood of p in the domain, scaled by the factor λ^{-1}, serves as
the domain of $u_{\lambda,\mu}$. The map first rescales the neighborhood to its original size.
Then after mapping by the original map u, the image is scaled by a factor of μ^{-1}.

If λ and μ are small this has the effect of magnifying small neighborhoods of p and $u(p)$. As $\lambda, \mu \to 0$, technical estimates on the relationship between λ and μ ensure the existence of a limit

$$u_* = \lim_{\lambda,\mu \to 0} u_{\lambda,\mu}.$$

The limit map u_* has as domain the tangent cone to the domain of u at the point p, and has as target the tangent cone at $u(p)$ to the target of u. The tangent cone to a smooth Riemannian manifold is just its Euclidean tangent space, while the tangent cone to a simplicial complex is a geometric cone over a small sphere around a point, or its link. The map u is homogeneous in the sense that for each x in the domain cone, $t \mapsto u_*(tx)$ is an (unparametrized) geodesic, and there is some $\alpha > 0$ so that $|u_*(tx)| = t^\alpha |u_*(x)|$. Here $|\cdot|$ denotes the distance from the vertex of the target cone. For more details on the blow-up procedure and homogeneous harmonic maps, see the references cited above, especially [Gromov and Schoen 1992] and [Daskalopoulos and Mese 2006]. A typical application of this blow-up procedure is to say that if at each point p in the domain, the corresponding blow-up map u_* has degree $\alpha \geq 1$, then the original map u is locally Lipschitz continuous.

In related work, the author together with Victòria Gras Andreu [2020; 2021] consider a class of harmonic maps between 2-dimensional simplicial complexes. The maps respect the simplicial structure of the domain and target in the sense that vertices (resp. edges, faces) of the domain are mapped to vertices (resp. edges, faces) of the target. In [Freidin and Gras Andreu 2020] the simplicial complexes are endowed with metrics identifying each face with the ideal hyperbolic triangle. This has the result of putting the vertices of the complexes at infinite distance, effectively puncturing the spaces. In [Freidin and Gras Andreu 2021] the complexes are endowed with simplexwise flat metrics, as well as metrics conformal to flat or ideal hyperbolic metrics. The blow-up procedure described above is repeated in [Freidin and Gras Andreu 2021] to construct tangent maps at edge points and at (nonpunctured) vertices.

In [Daskalopoulos and Mese 2008] the tangent maps of harmonic maps from 2-dimensional simplicial complexes into Riemannian manifolds are related to eigenfunctions of a discrete Laplace operator on the link Lk(p) of a point p. The link of a point p in a 2-complex X is a graph whose vertices correspond to edges of X containing p, and whose edges correspond to faces of X containing p. The strategy of relating continuous and discrete harmonic maps is explored in detail in [Daskalopoulos and Mese 2008]. The discrete Laplace operator considered in [Daskalopoulos and Mese 2008] assumes the geometry in the link Lk(p) is very regular, in particular that all faces of the 2-complex are equilateral triangles so that every edge of Lk(p) has the same length. Our main goals are to extend that analysis to include more of the richness of the geometry that Lk(p) can have, as well as to the setting of tangent maps between simplicial cones constructed in [Freidin and Gras Andreu 2021].

Before beginning with the main content, we discuss some history of discrete Laplace operators that will become useful later on. For a more detailed discussion, including proofs of quoted results, see, e.g., [Chung 1997] or [Banerjee and Jost 2008]. Future work may be done to generalize the results of this paper to higher dimensional cones, at which point the work of Horak and Jost [2013] on the spectra of simplicial complexes will likely play the role of the discussion below.

Let Γ be a graph with vertices $V = V(\Gamma)$ and edges $E = E(\Gamma)$. One can enumerate the vertices of Γ and thus identify functions $f : V \to \mathbb{R}$ with vectors in $\mathbb{R}^{\#V}$. Then the unweighted graph Laplacian is represented by the matrix Δ with entries

$$\Delta_{ij} = \begin{cases} \deg v_i & \text{if } i = j, \\ -1 & \text{if } v_i \sim v_j, \\ 0 & \text{else.} \end{cases}$$

Here $v_i \sim v_j$ means that the vertices v_i and v_j are adjacent in Γ. This matrix $\Delta = D - A$ is the difference between the diagonal degree matrix D and the adjacency matrix A with entries

$$D_{ij} = \begin{cases} \deg v_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \text{and} \quad A_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \nsim v_j. \end{cases}$$

A normalized Laplace operator, denoted $\mathcal{L}$, is defined by

$$\mathcal{L} = \text{Id} - D^{-1/2} A D^{-1/2} = D^{-1/2} \Delta D^{-1/2}.$$

Both the matrices Δ and $\mathcal{L}$ are symmetric and positive semidefinite. Indeed, they determine quadratic forms

$$\rho \cdot \Delta \rho = \sum_{uv \in E} (\rho(v) - \rho(u))^2 \quad \text{and} \quad \rho \cdot \mathcal{L} \rho = \sum_{uv \in E} \left(\frac{\rho(v)}{\sqrt{\deg v}} - \frac{\rho(u)}{\sqrt{\deg u}} \right)^2.$$

An immediate consequence is that the vector $w = \mathbb{1}$ with $w_i = 1$ for all i is in the kernel of Δ and in fact spans the kernel if Γ is connected. Since $\mathcal{L} = D^{-1/2} \Delta D^{-1/2}$, the vector $D^{1/2} \mathbb{1}$ spans the kernel of $\mathcal{L}$. The operator $\mathcal{L}$ is called normalized because its largest eigenvalue is ≤ 2. The eigenvalue $\lambda = 2$ of $\mathcal{L}$ is achieved if and only if Γ is bipartite, and the first positive eigenvalue of $\mathcal{L}$ is at most 1 unless Γ is a complete graph K_n, in which case the first eigenvalue is $\frac{n}{n-1}$.

If weights are assigned to Γ by a function $w : V \cup E \to \mathbb{R}_{\geq 0}$, then one can modify the adjacency matrix so that $A_{ij} = w(v_i v_j)$ when $v_i \sim v_j$, and the degree matrix so that $D_{ii} = w(v_i)$. In case only edge weights are provided, the vertex weights are determined by $w(v_i) = \sum_{v_j \sim v_i} w(v_i v_j)$. In each of these cases one can still define

$$\Delta = D - A \quad \text{and} \quad \mathcal{L} = \text{Id} - D^{-1/2} A D^{-1/2} = D^{-1/2} \Delta D^{-1/2}.$$

In case $w(e) = 1$ for all $e \in E$ and $w(v) = \deg v$ for $v \in V$ the unweighted operators are recovered. The matrices are still positive semidefinite but their kernels are affected by the weight function.

In both [Chung and Langlands 1996] and [Banerjee and Jost 2008] the asymmetric matrix $L = D^{-1}\Delta$ is studied. This matrix also comes up in our work. Fortunately it is similar to the normalized Laplacian via the similarity

$$L = D^{-1/2}\mathcal{L}D^{1/2}.$$

As a result the spectrum of L is identical to the spectrum of $\mathcal{L}$.

In [Friedman and Tillich 2004] the functions $f : V(\Gamma) \to \mathbb{R}$ are interpreted as continuous, edgewise linear functions $f : \Gamma \to \mathbb{R}$, and the combinatorial Laplacian Δ is interpreted to compute the sum of derivatives of f along the edges meeting at a vertex. That paper studies the problem of finding functions $f : \Gamma \to \mathbb{R}$ that are eigenfunctions of the second derivative operator on each edge and satisfy a boundary condition at each vertex relating the derivatives of f in the incident edges. The problem studied there is equivalent to our problem in the case of maps from a 2-dimensional cone into $\mathbb{R}$, but our approaches differ slightly.

The structure of this paper is as follows. We begin in Section 2 by introducing the spaces and maps to be studied. The spaces include k-pods and simplicial cones, together with the metrics they can carry. The maps of interest are homogeneous maps between cones. In Section 2 we also introduce the notion of harmonic maps as well as the balancing conditions that are essential to the current study.

In Section 3 we define and study balanced homogeneous harmonic maps from simplicial cones into Euclidean spaces. In Section 4 we do the same for maps from Euclidean spaces into k-pods. Section 4 also allows for a domain with a single singular point, as well as p-harmonic maps. In Section 5 we define and study balanced homogeneous harmonic maps between simplicial cones. Finally in Section 6 we generalize Sections 3 and 5 by studying the limits as the angles of each face of the cone tend to 0 and the vertices become punctures.

2. Preliminaries

2.1. *Spaces and metrics.* Our main objects of study will be homogeneous maps between cones. The spaces and maps we consider will have a lot more structure, but the basic definitions are quite general.

Definition 2.1 (cone). For any topological space X, the cone $C(X)$ is the quotient

$$C(X) = (X \times \mathbb{R}_{\geq 0})/\sim,$$

where $(x, 0) \sim (y, 0)$ for all $x, y \in X$.

If X is a space of dimension n, then the cone $C(X)$ has dimension $n+1$. Thus 1-dimensional cones can be constructed as the cone over a finite set of points. The common point $(x, 0) \in C(X)$ is called the *vertex* of the cone.

Definition 2.2 (homogeneous map). A map $f : C(X) \to C(Y)$ is homogeneous of degree $\alpha > 0$ if there are functions $y : X \to Y$ and $s : X \to \mathbb{R}_{\geq 0}$ so that

$$f(x, t) = (y(x), s(x)t^{\alpha}).$$

Necessarily the image of the vertex of $C(X)$ under a homogeneous map is the vertex of $C(Y)$.

The remainder of this section is devoted to describing which cones will appear as domains and targets of the maps we consider.

Definition 2.3 (k-pod). For $k \in \mathbb{N}$, consider the space $\{x_1, \ldots, x_k\}$ of k discrete points. The k-pod is the cone $C(k) = C(\{x_1, \ldots, x_n\})$. The rays $C(x_j) = \{x_j\} \times \mathbb{R}_{\geq 0}$ are called the edges of the k-pod.

Each edge of the k-pod $C(k)$ is homeomorphic to the half-line $\mathbb{R}_{\geq 0}$. Pulling back the standard Euclidean metric from $\mathbb{R}_{\geq 0}$ defines a metric on each edge of $C(k)$. We will also always use these homeomorphisms (now isometries) to give coordinates on each edge.

For a 2-dimensional cone, we must take the cone over a 1-dimensional space. The 1-dimensional spaces we will consider are graphs Γ, with vertices $V(\Gamma)$ and edges $E(\Gamma)$.

Definition 2.4 (2-dimensional cone). For a connected finite simplicial graph Γ, consider the cone $C(\Gamma)$. The rays $C(v) = v \times \mathbb{R}_{\geq 0}$ for vertices $v \in V(\Gamma)$ are edges of the cone, and the sectors $C(e)$ for edges $e \in E(\Gamma)$ are faces of the cone.

Each face of the cone $C(\Gamma)$ is homeomorphic to a sector of the plane $\mathbb{R}^2$. In order to determine the geometry of the cone one needs only specify the angle formed by each face at the vertex of the cone.

Definition 2.5 (metrics). Given a cone $C(\Gamma)$ over a graph Γ, a function $\theta : E(\Gamma) \to (0, \pi)$ determines a metric. For $e \in E(\gamma)$ the face $C(e)$ is isometric to the sector $S_{\theta(e)}$ of the Euclidean plane given in polar coordinates by

$$S_{\theta(e)} = \{(r, \theta) \mid r \geq 0, \ 0 \leq \theta \leq \theta(e)\}.$$

Let $C(\Gamma, \theta)$ denote the space $C(\Gamma)$ endowed with the metric determined by θ.

Remark 2.6. One could also define metrics on $C(\Gamma)$ for functions $\theta : E(\Gamma) \to \mathbb{R}_{>0}$. For $0 < \theta(e) < 2\pi$ one can still model $C(e)$ by the sector $S_{\theta(e)}$. For $\theta \geq 2\pi$ one could use a sector on a cone with cone angle $> 2\pi$. Though most results are unchanged when considering this more general class of metrics, we will not consider them here for two main reasons. First, the simplicial cones considered here are meant to

model tangent cones to Euclidean simplicial complexes, all of whose angles are at most π. And second, the analysis for $\theta \geq \pi$ is more subtle and distracts from the main flow of many arguments.

Given a graph Γ and an edge $e = (p, q) \in E(\Gamma)$, one can subdivide e by adding a vertex x in the interior. This produces a graph Γ' with vertex set $V(\Gamma') = V(\Gamma) \cup \{x\}$ and edge set $E(\Gamma') = (E(\Gamma) \setminus \{e\}) \cup \{px, xq\}$.

Given a function $\theta : E(\Gamma) \to (0, \pi)$ and an edge $e \in E(\Gamma)$, when one subdivides the edge e to produce Γ' as above one can define a function $\theta' : E(\Gamma') \to (0, \pi)$. For $e' \neq e$ set $\theta'(e') = \theta(e')$, and set $\theta'(px) = \alpha$ and $\theta'(xq) = \beta$ in any way so that $\alpha + \beta = \theta(e)$. Any pair (Γ^*, θ^*) obtained from (Γ, θ) by a finite sequence of edge subdivisions as in these two paragraphs is called a subdivision of (Γ, θ), and the cones $C(\Gamma^*, \theta^*)$ and $C(\Gamma, \theta)$ are isometric.

In general two cones $C(\Gamma_1, \theta_1)$ and $C(\Gamma_2, \theta_2)$ are isometric if and only if the graphs have a common subdivision (Γ^*, θ^*).

In $C(\Gamma, \theta)$ identify an edge $e \in E(\Gamma)$ with the interval $[0, 1]$. Taking polar coordinates (ρ, φ) in $S_{\theta(e)}$, a concrete isometry $\psi_e : C(e) = C([0, 1]) \to S_{\theta(e)}$ is given by

$$\psi_e(x, t) = (\rho, \varphi) \quad \text{with } \rho = t, \ \varphi = x\theta(e).$$

We will also use these isometries to give coordinates on each face of $C(\Gamma, \theta)$. But note that the map $\psi_e(x, t) = (t, (1 - x)\theta(e))$ also defines an isometry $C(e) \to S_{\theta(e)}$. Our choice of isometry will be determined by which endpoint of e is most relevant at the moment.

Definition 2.7. In a cone $C(\Gamma, \theta)$, fix a face $C(e)$ for some $e \in E(\Gamma)$. If $v \in V(\Gamma)$ is one endpoint of e, then the coordinates on $C(E)$ adapted to v are the coordinates determined by the isometry $\psi_e : C(e) \to S_{\theta(e)}$ that sends the point $(v, 1) \in C(e)$ to $(1, 0) \in S_{\theta(e)}$.

One can also view the Euclidean plane $\mathbb{R}^2$ as a 2-dimensional cone. Namely, for the cycle graph C_n consisting of n vertices and n edges, $\mathbb{R}^2 = C(C_n, \theta)$ for any $\theta : E(C_n) \to (0, \pi)$ with $\sum_{e \in E(C_n)} \theta(e) = 2\pi$. More generally, we will sometimes consider cones over the cycle graph with different metrics.

Definition 2.8. A smooth cone is a geometric cone over the cycle graph. In other words, it is a space $C(C_n, \theta)$ for some $\theta : E(C_n) \to (0, \pi)$.

The curvature at the vertex of a smooth cone is determined by $\sum_{e \in E(C_n)} \theta(e)$. If the sum is less than 2π the vertex has positive curvature, if the sum is equal to 2π the vertex is flat, and if the sum is greater than 2π the vertex has negative curvature.

2.2. *Harmonic functions on sectors.*

When it comes time to consider harmonic maps between cones, we will need to understand the structure of those harmonic

maps. Our domains will always be 2-dimensional cones $C(\Gamma, \theta)$, all of whose faces are isometric to plane sectors S_θ. To begin this section we first aim to understand harmonic functions $f : S_{\theta_0} \to \mathbb{R}$.

Lemma 2.9. *A homogeneous function* $u : S = S_{\theta_0} \to \mathbb{R}$ *is harmonic if and only if there are constants* c_1, c_2 *so that*

$$u(r, \theta) = r^\alpha(c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta)).$$

Proof. In polar coordinates, a homogeneous function $u(r, \theta)$ of degree α has the form

$$u(r, \theta) = r^\alpha u(1, \theta) = r^\alpha \rho(\theta).$$

And in polar coordinates the Laplacian of u reads

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = r^{\alpha-2}(\alpha^2 \rho(\theta) + \rho''(\theta)).$$

To be harmonic, u must satisfy $\Delta u = 0$, and this equation must hold for all (r, θ). Hence ρ must satisfy $\rho''(\theta) + \alpha^2 \rho(\theta) = 0$. The solutions to this ordinary differential equation are precisely

$$\rho(\theta) = c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta). \qquad \square$$

In order to turn our study into a discrete problem later, we introduce the following result that says homogeneous harmonic functions are described by discrete data.

Lemma 2.10. *A homogeneous harmonic function* $u(r, \theta) = r^\alpha \rho(\theta) : S = S_{\theta_0} \to \mathbb{R}$ *is uniquely determined by the two numbers* $\rho(0)$ *and* $\rho(\theta_0)$ *unless* $\alpha\theta_0 \in \pi\mathbb{Z}$.

Proof. Given the form of $\rho(\theta)$ from [Lemma 2.9], one easily solves the boundary value problem on $S = S_{\theta_0}$:

$$\rho(0) = c_1 \cos(0) + c_1 \sin(0)$$

$$= c_1,$$

$$\rho(\theta_0) = c_1 \cos(\alpha\theta_0) + c_2 \sin(\alpha\theta_0)$$

$$= \rho(0) \cos(\alpha\theta_0) + c_2 \sin(\alpha\theta_0).$$

So as long as $\alpha\theta_0 \notin \pi\mathbb{Z}$, we have $\sin(\alpha\theta_0) \neq 0$ so we can write

$$\rho(\theta) = \rho(0) \cos(\alpha\theta) + \frac{\rho(\theta_0) - \rho(0) \cos(\alpha\theta_0)}{\sin(\alpha\theta_0)} \sin(\alpha\theta).$$

In case $\alpha\theta_0 = n\pi$, c_2 can be any real number but we must have

$$\rho(\theta_0) = \rho(0) \cos(n\pi) = (-1)^n \rho(0).$$

In this case ρ has the form

$$\rho(\theta) = \rho(0)\cos(\alpha\theta) + c_2\sin(\alpha\theta). \qquad \square$$

When we consider maps into k-pods, we recall that each edge of a k-pod is isometric to the half-line, $\mathbb{R}_{\geq 0}$. Thus we must understand which homogeneous harmonic functions have image contained in the half-line.

Proposition 2.11. *A nontrivial homogeneous harmonic map* $u(r, \theta) = r^\alpha \rho(\theta)$:
$S = S_{\theta_0} \to \mathbb{R}_{\geq 0}$ *exists if and only if* $0 < \alpha\theta_0 \leq \pi$. *When* $0 < \alpha\theta_0 < \pi$, *any pair of numbers* $\rho(0)$, $\rho(\theta_0) \geq 0$, *not both* 0, *uniquely determine such a function.*

Proof. According to Lemma 2.9, a homogeneous harmonic function $u : S_{\theta_0} \to \mathbb{R}$ has the form $u(r, \theta) = r^\alpha \rho(\theta)$ with

$$\rho(r) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta).$$

If $\alpha\theta_0 > \pi$, then both $\theta = 0$ and $\theta = \pi$ are in the domain of ρ. If $c_2 \neq 0$ then either $\rho(1, 0)$ or $\rho(1, \pi)$ is negative. And if $c_2 = 0$ but $c_1 \neq 0$ then ρ changes signs around $\theta = \pi$. In this case no homogeneous harmonic map $u : S_{\theta_0} \to \mathbb{R}_{\geq 0}$ exists.

If $\alpha\theta_0 = \pi$ then $\rho(\pi) = -\rho(0)$, so to be nonnegative $\rho(0) = \rho(\pi) = 0$. Now

$$\rho(\theta) = c_2\sin(\alpha\theta).$$

The values of $\sin(\alpha\theta)$ remain positive for $0 < \theta < \theta_0$, so for the image of u to lie in $\mathbb{R}_{\geq 0}$ c_2 must be nonnegative.

Finally if $0 < \alpha\theta_0 < \pi$, Lemma 2.10 tells us that

$$\rho(\theta) = \rho(0)\cos(\alpha\theta) + \frac{\rho(\theta_0) - \rho(0)\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)}\sin(\alpha\theta).$$

Clearly both $\rho(0)$ and $\rho(\theta_0)$ must be nonnegative to ensure $u \geq 0$. Now $\rho(\theta)$ only vanishes when

$$0 = \rho(0)\cos(\alpha\theta) + \frac{\rho(\theta_0) - \rho(0)\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)}\sin(\alpha\theta),$$

$$\rho(0)\cot(\alpha\theta) = \frac{\rho(0)\cos(\alpha\theta_0) - \rho(\theta_0)}{\sin(\alpha\theta_0)}$$

$$= \rho(0)\cot(\alpha\theta_0) - \rho(\theta_0)\csc(\alpha\theta_0).$$

If $\rho(0) = 0$ then $\rho(\theta_0) = 0$ also, in which case u is a trivial map. And if $\rho(0) > 0$ then $\rho(\theta)$ cannot vanish until $\cot(\alpha\theta) < \cot(\alpha\theta_0)$. But cot is a decreasing function, so this cannot happen for $0 < \theta < \theta_0$.

Thus in this last case where $0 < \alpha\theta_0 < \pi$, any pair of nonnegative numbers $\rho(0)$ and $\rho(\theta_0)$, not both 0, uniquely determine a homogeneous harmonic function $u : S_{\theta_0} \to \mathbb{R}_{\geq 0}$ of degree α. $\qquad \square$

Finally, when we discuss harmonic maps between 2-dimensional cones, we will need to understand harmonic maps between sectors. We will only consider those maps that send the boundary of the domain sector to the boundary of the target.

Proposition 2.12. *Fix $0 < \theta_0, \varphi_0 < \pi$. Then a nontrivial homogeneous harmonic map $u : S_{\theta_0} \to S_{\varphi_0}$ exists if and only if $0 < \alpha\theta_0 \leq \pi$. When $0 < \alpha\theta_0 < \pi$, any pair of numbers $|u(1, 0)|, |u(1, \theta_0)| \geq 0$, not both 0, uniquely determine such a map.*

Proof. Use polar coordinates (r, θ) in the domain sector S_{θ_0} and rectangular coordinates (x, y) in the target sector S_{φ_0}. Then a homogeneous map $u : S_{\theta_0}$ can be written

$$u(r, \theta) = (x(r, \theta), y(r, \theta)).$$

As $S_{\varphi_0} \subset \mathbb{R}^2$ is flat, the map u is harmonic if and only if the functions x and y are. So Lemma 2.9 says $x(r, \theta) = r^\alpha \chi(\theta)$ and $y(r, \theta) = r^\alpha \eta(\theta)$ with

$$\chi(\theta) = c_1 \cos(\alpha\theta) + c_2 \sin(\alpha\theta) \quad \text{and} \quad \eta(\theta) = c_3 \cos(\alpha\theta) + c_4 \sin(\alpha\theta).$$

First if $\alpha\theta_0 > \pi$, then just as in Proposition 2.11, $\eta(\theta)$ cannot remain positive for $0 \leq \theta \leq \theta_0$. But every point $(x, y) \in S_{\varphi_0}$ has $y \geq 0$, so in this case no map $u : S_{\theta_0} \to S_{\varphi_0}$ exists.

Now suppose $0 < \alpha\theta_0 \leq \pi$ and let $\rho_0 = |u(1, 0)|$ and $\rho_1 = |u(1, \theta_0)|$. In order to map the boundary of S_{θ_0} to the boundary of S_{φ_0}, the maps χ and η must satisfy the conditions

$$\chi(0) = \rho_0, \quad \chi(\theta_0) = \rho_1 \cos(\varphi_0),$$
$$\eta(0) = 0, \quad \eta(\theta_0) = \rho_1 \sin(\varphi_0).$$

In other words,

$$c_1 = \rho_0, \quad c_1 \cos(\alpha\theta_0) + c_2 \sin(\alpha\theta_0) = \rho_1 \cos(\varphi_0),$$
$$c_3 = 0, \quad c_3 \cos(\alpha\theta_0) + c_4 \sin(\alpha\theta_0) = \rho_1 \sin(\varphi_0).$$

In all cases $c_1 = \rho_0$ and $c_3 = 0$ are determined.

If $\alpha\theta_0 = \pi$, then

$$-\rho_0 = \rho_1 \cos(\varphi_0) \quad \text{and} \quad 0 = \rho_1 \sin(\varphi_0).$$

Since $0 < \varphi_0 < \pi$, this means that $\rho_0 = \rho_1 = 0$. Now

$$\chi(\theta) = c_2 \sin(\alpha\theta) \quad \text{and} \quad \eta(\theta) = c_4 \sin(\alpha\theta).$$

As long as $(c_2, c_4) \in S_{\varphi_0}$, the image of the corresponding map u lies in S_{φ_0}.

Finally if $0 < \alpha\theta_0 < \pi$ one can explicitly solve for the constants c_2 and c_4 to find

$$\chi(\theta) = \rho_0 \cos(\alpha\theta) + \frac{\rho_1 \cos(\varphi_0) - \rho_0 \cos(\alpha\theta_0)}{\sin(\alpha\theta_0)} \sin(\alpha\theta),$$
$$\eta(\theta) = \frac{\sin(\varphi_0)}{\sin(\alpha\theta_0)} \rho_1 \sin(\alpha\theta).$$

To verify that the corresponding map u has image in S_{φ_0}, one needs only check that $\eta(\theta) \geq 0$ and $\chi(\theta) \geq \eta(\theta)\cot(\varphi_0)$ for $0 \leq \theta \leq \theta_0$. The first of these is immediate since

$$\frac{\sin(\varphi_0)}{\sin(\alpha\theta_0)}\rho_1 \geq 0 \qquad \text{and} \qquad \sin(\alpha\theta) \geq 0 \quad \text{for } 0 \leq \theta \leq \theta_0.$$

The second condition is also straightforward:

$$\chi(\theta) - \eta(\theta)\cot(\varphi_0)$$
$$= \rho_0 \cos(\alpha\theta) + \frac{\rho_1\cos(\varphi_0) - \rho_0\cos(\alpha\theta_0)}{\sin(\alpha\theta_0)}\sin(\alpha\theta) - \frac{\cos(\varphi_0)}{\sin(\alpha\theta_0)}\rho_1\sin(\alpha\theta)$$
$$= \rho_0(\cos(\alpha\theta) - \cot(\alpha\theta_0)\sin(\alpha\theta))$$

This expression only vanishes when $\cot(\alpha\theta) = \cot(\alpha\theta_0)$, which only happens when $\theta = \theta_0$ since $\cot(\alpha\theta)$ is decreasing on the interval $0 < \theta < \theta_0$. $\qquad\square$

2.3. *Balancing conditions.* We will mainly be interested in homogeneous maps in three situations:

(1) $f : C(\Gamma, \theta) \to \mathbb{R}^n$ from simplicial cones into Euclidean spaces,

(2) $f : C(C_n, \theta) \to C(k)$ from smooth cones into k-pods, and

(3) $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ between 2-dimensional cones with the same simplicial structure.

In addition to studying maps that are harmonic on the faces of the domain cone, we introduce balancing conditions along the edges of the domain, akin to those in, e.g., [Daskalopoulos and Mese 2006; 2008], that have to do with harmonicity along the edges.

In all cases homogeneous maps send the vertex of one cone to the vertex of the other. In the third situation we will also demand that edges and faces of the cone are sent to themselves. We will then add extra conditions to ensure that our homogeneous maps are as smooth as the singularities of the domain and target allow. In particular, for open sets U, V in top-dimensional faces of the domain and target, respectively, if $f(U) \subset V$ then f can be represented in coordinates as a smooth map between open subsets of Euclidean spaces. Along edges we introduce the so-called balancing conditions.

2.3.1. *Euclidean targets.* A map $f : C(\Gamma, \theta) \to \mathbb{R}^n$ can be written in coordinates as $f = (f_1, \ldots, f_n)$ for n functions $f_j : C(\gamma, \theta) \to \mathbb{R}$. So we need only consider the case $n = 1$, i.e., $f : C(\Gamma, \theta) \to \mathbb{R}$.

In this case the balancing conditions relate the normal derivatives in each face along a common edge of the domain.

Definition 2.13. Let $f : C(\Gamma, \theta) \to \mathbb{R}$ be a homogeneous function. Fix a vertex $v \in V(\Gamma)$ and let $\{e_j\}$ enumerate the edges of Γ incident to v. In the corresponding

faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v in order to represent f in these faces by homogeneous functions

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}.$$

The function f is balanced along $C(v)$ if

$$\sum_j \frac{\partial u_j}{\partial \theta}(r, 0) = 0.$$

The function f is balanced if it is balanced along each edge of $C(\Gamma, \theta)$.

For maps $f : C(\Gamma, \theta) \to \mathbb{R}^n$, simply impose the balancing condition for each component of f. We can see the regularity of f along an edge $C(v)$ for a vertex of degree 2 by unfolding the incident faces $C(e_1)$ and $C(e_2)$ to form a larger sector of the plane.

Lemma 2.14. *Let $f : (\Gamma, \theta) \to \mathbb{R}$ be a balanced homogeneous function and let $v \in V(\Gamma)$ be a vertex of degree 2. Then f is C^1 over the edge $C(v) \subset C(\Gamma, \theta)$.*

Proof. Let $e_1, e_2 \in E(\Gamma)$ be the two edges incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v and represent f in these coordinates by homogeneous functions

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}.$$

Now define a function $u : S = \{(r, \theta) \in \mathbb{R}^2 \mid r \geq 0, \ -\theta(e_2) \leq \theta \leq \theta(e_1)\} \to \mathbb{R}$ as

$$u(r, \theta) = \begin{cases} u_1(r, \theta) & \text{if } \theta \geq 0, \\ u_2(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

The maps u_1 and u_2 agree on the x-axis $\{(r, 0) \mid r \geq 0\}$. Thus u is continuous, and moreover $\frac{\partial u_1}{\partial r}(r, 0) = \frac{\partial u_2}{\partial r}(r, 0)$ so $\frac{\partial u}{\partial r}$ is continuous. The $\frac{\partial}{\partial \theta}$ derivatives of u are given by

$$\frac{\partial u}{\partial \theta}(r, \theta) = \begin{cases} \dfrac{\partial u_1}{\partial \theta}(r, \theta) & \text{if } \theta \geq 0, \\ -\dfrac{\partial u}{\partial \theta}(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

At $\theta = 0$ the balancing condition says that these partial derivatives coincide, so $\frac{\partial u}{\partial \theta}$ is continuous too. $\qquad \square$

2.3.2. *1-dimensional targets.*

In order for a homogeneous map $f = (y(x), s(x)t^\alpha) : C(C_n, \theta) \to C(k)$ into a k-pod to be continuous, the map y must be locally constant where $s \neq 0$. We will also assume that if $s(v) = 0$ at some vertex v with incident edges e_1 and e_2, then $y(e_1) \neq y(e_2)$.

Let $f = (y(x), s(x)t^\alpha) : C(C_n, \theta) \to C(k)$ be a homogeneous map from a smooth cone to a k-pod. If $s(x) = 0$ for some point x in the interior of an edge $e \in E(\Gamma)$,

subdivide Γ by introducing a vertex at x. After performing all such subdivisions, we'll abuse notation and call the resulting space $C(C_n, \theta)$ too. Now $s(x)$ vanishes only at the vertices of Γ, and f maps each face $C(e)$ to a single edge of $C(k)$.

Now for a homogeneous map $f : C(C_n, \theta) \to C(k)$ we can define the balancing condition in terms of normal derivatives along each edge of the domain.

Definition 2.15. Let $f : C(C_n, \theta) \to C(k)$ be a homogeneous map. Fix a vertex $v \in V(C_n)$ and let $e_1, e_2 \in E(C_n)$ be the two edges incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v, in order to represent f in these faces by homogeneous maps

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}_{\geq 0}.$$

If $f(v, 1)$ is the vertex of $C(k)$ then f is balanced along $C(v)$ if

$$\frac{\partial u_1}{\partial \theta}(r, 0) = \frac{\partial u_2}{\partial \theta}(r, 0).$$

If $f(v, 1)$ lies in the interior of some edge of $C(k)$ then f is balanced along $C(v)$ if

$$\frac{\partial u_1}{\partial \theta}(r, 0) + \frac{\partial u_2}{\partial \theta}(r, 0) = 0.$$

The map f is balanced if it is balanced along each edge of $C(C_n, \theta)$.

In this case we can again see the regularity of f along each edge $C(v) \subset C(C_n, \theta)$. In addition to unfolding the two faces incident to $C(v)$ into a larger sector, we may have to unfold their image edges in $C(k)$ to form a whole line if $f(v)$ is the vertex of $C(k)$.

Lemma 2.16. *Let $f = (y(x), s(x)t^\alpha) : (C_n, \theta) \to C9k)$ be a balanced homogeneous map and $v \in V(\Gamma)$. Then f is C^1 over the edge $C(v) \subset C(C_n, \theta)$.*

Proof. Let $e_1, e_2 \in E(\Gamma)$ be the two edges incident to v. In the corresponding faces take coordinates $\psi_j : C(e_j) \to S_{\theta(e_j)}$ adapted to v.

Consider first the case where $s(v) \neq 0$, so $f(v, 1)$ is in the interior of an edge of $C(k)$. The map f is represented in coordinates in the faces $C(e_1)$ and $C(e_2)$ by homogeneous maps into the same half-line $\mathbb{R}_{\geq 0} \subset \mathbb{R}$. Thus Lemma 2.14 already establishes the C^1 regularity of f over the edge $C(v)$.

Now consider $s(v) = 0$, so $f(v, 1)$ is the vertex of $C(k)$. Again we can represent the map f in the face $C(e_j)$ by homogeneous maps

$$u_j = f \circ \psi_j^{-1} : S_{\theta(e_j)} \to \mathbb{R}_{\geq 0}.$$

Since $y(e_1) \neq y(e_2)$, the target half-lines should be thought of as different spaces, and can thus be glued together to form the entire real line. Specifically, define

$u : S = \{(r, \theta) \in \mathbb{R}^2 \mid r \geq 0, \; -\theta(e_2) \leq \theta \leq \theta(e_1)\} \to \mathbb{R}$ as

$$u(r, \theta) = \begin{cases} u_1(r, \theta) & \text{if } \theta \geq 0, \\ -u_2(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

Both u_1 and u_2 map $(r, 0)$ to 0, so u and $\frac{\partial u}{\partial r}$ are continuous over the x-axis. The $\frac{\partial}{\partial \theta}$ derivatives are given by

$$\frac{\partial u}{\partial \theta}(r, \theta) = \begin{cases} \dfrac{\partial u_1}{\partial \theta}(r, \theta) & \text{if } \theta \geq 0, \\[2mm] \dfrac{\partial u_2}{\partial \theta}(r, -\theta) & \text{if } \theta \leq 0. \end{cases}$$

Again the balancing condition says that these derivatives coincide at $(r, 0)$, so $\frac{\partial u}{\partial \theta}$ is continuous. $\qquad\square$

2.3.3. *2-dimensional targets.* Let $f = (y(x), s(x)t^\alpha) : C(\Gamma_1, \theta_1) \to C(\Gamma_2, \theta_2)$ be a homogeneous map. If $s(x) \in V(\Gamma_2)$ for some point x in the interior of an edge $e \in E(\Gamma_1)$, subdivide Γ_1 by introducing a vertex at x. Likewise subdivide Γ_2 at any point $s(v)$ for $v \in V(\Gamma_1)$. After performing all such subdivisions, the original map factors as a composition $C(\Gamma_1, \theta_1) \to C(\Gamma_1, \varphi) \to C(\Gamma_2, \theta_2)$ where the first map sends each edge and face of $C(\Gamma_1)$ to itself and the second map sends each face of $C(\Gamma_1, \varphi)$ isometrically to a face of $C(\Gamma_2, \theta_2)$.

As the harmonicity of a map is unaffected by postcomposition with isometries, we may consider only maps $f(x, t) = (y(x), s(x)t^\alpha) : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ such that $y(v) = v$ for all $v \in V(\Gamma)$ and $y|_e$ maps e to e for each $e \in E(\Gamma)$. That is, f maps each edge and face of $C(\Gamma)$ to itself.

Now for a homogeneous map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ we can define the balancing condition in terms of normal derivatives along each edge of the domain. But under the additional assumptions made on our maps, only some components of the normal derivatives are relevant.

Definition 2.17. Let $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ be a homogeneous map. Fix a vertex $v \in V(\Gamma_1)$ and let $\{e_j\}$ enumerate the edges of Γ incident to v. In each face $C(e_j)$ of $C(\Gamma, \theta)$ take coordinates adapted to v, and likewise in the faces of $C(\Gamma, \varphi)$. In this way, represent f in each face by a homogeneous map

$$u_j : S_j = S_{\theta(e_j)} \to S_{\varphi(e_j)} = S_j'.$$

Taking rectangular coordinates on the target S_j', write $u_j = (x_j, y_j)$. Then the map f is balanced along $C(v)$ if

$$\sum_j \frac{\partial x_j}{\partial \theta}(r, 0) = 0.$$

The map f is balanced if it is balanced along each edge of $C(\Gamma, \theta)$.

From the maps x_j in this definition one can define a map x as follows. Suppose there are N edges e_j incident to v, and fix some index $1 \le j_0 \le N$. Define

$$x(r, \theta) = \begin{cases} x_{j_0}(r, \theta) & \text{if } \theta \ge 0, \\ -x_{j_0}(r, -\theta) + \dfrac{2}{N} \displaystyle\sum_j x_j(r, -\theta) & \text{if } \theta \le 0. \end{cases}$$

The fact that all the functions x_j agree along $\theta = 0$ means that x and $\frac{\partial x}{\partial r}$ are continuous, and the balancing condition now means that $\frac{\partial x}{\partial \theta}$ is continuous.

Remark 2.18. In the case that v has degree 2, the function x constructed just above is the direct analogue of the function u from Lemma 2.14. Unfortunately there is not an analogous smoothness result for a similarly constructed y function. The restriction that f sends $C(v)$ to itself for all $v \in V(\Gamma)$ means that we cannot expect such a smoothness result.

3. Balanced harmonic functions into Euclidean spaces

A function $f : C(\Gamma, \theta) \to \mathbb{R}$ is harmonic if its restriction to each face of $C(\Gamma, \theta)$ is a harmonic function on a plane sector. We investigate the balanced (as in Definition 2.13) homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$. The analysis will be different if $\alpha\theta(e) \in \pi\mathbb{Z}$ for some edge $e \in E(\Gamma)$; we will refer to such α as *singular degrees*.

We will focus first on the case of nonsingular α. In each case we begin by constructing homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$, and then investigating which of those functions are balanced.

3.1. *Nonsingular degrees.*

Lemma 3.1. *If α is a nonsingular degree then a homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree $\alpha > 0$ is uniquely determined by a function $\rho : V(\Gamma) \to \mathbb{R}$.*

Proof. For a fixed edge $e \in E(\Gamma)$ consider the face $C(e) \subset C(\Gamma, \theta)$, which is isometric to $S_{\theta(e)} \subset \mathbb{R}^2$. According to Lemma 2.10 a homogeneous harmonic function $u : S_{\theta(e)} \to \mathbb{R}$ is uniquely determined by the values $u(1, 0)$ and $u(1, \theta_0)$ (using polar coordinates (r, θ) in $S_{\theta(e)}$).

Fix a function $\rho : V(\Gamma) \to \mathbb{R}$. For each edge $e = (v_0 v_1) \in E(\Gamma)$, fix coordinates in the face $C(e)$ adapted to v_0, so that $C(e)$ is identified with $S_{\theta(e)}$ and the edge $C(v)$ is identified with the positive x-axis. Define f in $C(e)$ to be represented in these coordinates by the unique homogeneous harmonic function $u : S_{\theta(e)} \to \mathbb{R}$ with $u(1, 0) = \rho(v_0)$ and $u(1, \theta(e)) = \rho(v_1)$.

Along each edge $C(v)$, $f(v, t) = \rho(v)t^\alpha$ is defined independently of the choice of incident face. Hence f is continuous over the edges of $C(\Gamma, \theta)$ and harmonic in each face. $\qquad\square$

For ease of notation, we will enumerate the vertices of Γ by $\{v_i\}$. Then a function $\rho : V(\Gamma) \to \mathbb{R}$ can be encoded by the vector $(\rho_i = \rho(v_i))$. If $v_i v_j \in E(\Gamma)$ is an edge, we will also define by $\theta_{ij} = \theta(v_i v_j)$.

Theorem 3.2. *The homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of nonsingular degree $\alpha > 0$ determined by $\rho : V(\Gamma) \to \mathbb{R}$ is balanced if and only if, for each vertex v_i,*

$$\sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha \theta_{ij}) \rho_i}{\sin(\alpha \theta_{ij})} = 0.$$

Here $v \sim w$ means that vertices v and w are adjacent, so there is an edge $vw \in E(\Gamma)$.

Proof. In each face $C(v_i v_j)$ choose coordinates adapted to v_i, so that the face is identified with $S_{\theta_{ij}}$ and $C(v_i)$ is identified with the positive x-axis. According to Lemma 2.10, in these coordinates $f|_{C(v_i v_j)}$ is represented by

$$u_j(r, \theta) = r^\alpha \left(\rho_i \cos(\alpha\theta) + \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} \sin(\alpha\theta) \right).$$

And the balancing formula of Definition 2.13 says that f is balanced along $C(v_i)$ if

$$\sum_{v_j \sim v_i} \frac{\partial u_j}{\partial \theta}(r, 0) = 0.$$

Combining these two formulas, we see that f is balanced along $C(v_i)$ if

$$\alpha r^\alpha \sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij}))} = 0.$$

In order to be balanced, f must be balanced along each edge of $C(\Gamma, \theta)$, so every sum of this form must vanish. $\qquad\square$

Another way to interpret this result is to say a vector (ρ_j) determines a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of nonsingular degree $\alpha > 0$ if it is in the kernel of the matrix $\Delta_{\alpha\theta}$ with entries

$$(\Delta_{\alpha\theta})_{ij} = \begin{cases} -\sum_{v_k \sim v_i} \cot(\alpha\theta_{ik}) & \text{if } i = j, \\ \csc(\alpha\theta_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{else.} \end{cases}$$

For only certain values of α will $\Delta_{\alpha\theta}$ have a nontrivial kernel. That is, only certain degrees α admit a balanced homogeneous harmonic function.

Remark 3.3. In general one can find a balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to \mathbb{R}^d$ for any d (f may be the 0 map). Both the harmonicity and the balancing condition for f split into the corresponding conditions for the components f_j of $f = (f_1, \ldots, f_d)$. By setting the components $\{f_j\}$ to be a basis of the d-dimensional kernel of $\Delta_{\alpha\theta}$ one finds the most geometrically interesting such maps. If one tries to map into a larger-dimensional space, or more generally lets the components of f be dependent, then the image of f lies in a subspace, not taking advantage of the full dimension of the target.

In special cases the collection of degrees α that admit balanced homogeneous harmonic functions is related to other discrete Laplace operators studied extensively in the literature. The following result is a direct generalization of Proposition 13 from [Daskalopoulos and Mese 2008], and also occurs as part of Theorem 3.5 of [Friedman and Tillich 2004].

Proposition 3.4. *Suppose* $\theta(e) = \theta_0$ *for all* $e \in E(\Gamma)$. *If* $\alpha\theta_0 \notin \pi\mathbb{Z}$ *then there exist balanced harmonic functions of degree* α *if and only if* $1 - \cos(\alpha\theta_0)$ *is an eigenvalue of the normalized graph Laplacian* $\mathcal{L}$ *on* Γ.

Remark 3.5. Proposition 3.4 ignores the eigenvalue $\lambda = 0$ that $\mathcal{L}$ necessarily has, and the eigenvalue $\lambda = 2$ that $\mathcal{L}$ may have. These cases correspond to $\alpha\theta_0 = 2n\pi$ and $\alpha\theta_0 = (2n + 1)\pi$, respectively, and will be discussed in Section 3.2.

Proof. Suppose $\theta(e) = \theta_0$ for all e and that (ρ_i) is in the kernel of $\Delta_{\alpha\theta}$. For a fixed vertex $v_i \in V(\Gamma)$ with $\deg v_i$ neighbors,

$$0 = \sum_{v_j \sim v_i} (\csc(\alpha\theta_0)\rho_j - \cot(\alpha\theta_0)\rho_i)$$

$$= \csc(\alpha\theta_0) \sum_{v_j \sim v_i} (\rho_j - \rho_i) + \deg(v_i)(\csc(\alpha\theta_0) - \cot(\alpha\theta_0))\rho_i,$$

$$(1 - \cos(\alpha\theta_0))\rho_i = \frac{1}{\deg v_i} \sum_{v_j \sim v_i} (\rho_i - \rho_j).$$

Thus $1 - \cos(\alpha\theta_0)$ is an eigenvalue of the vertex-weighted Laplace matrix L with entries

$$L_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\deg v_i} & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \nsim v_j. \end{cases}$$

Although this matrix L is not symmetric, it is similar to the symmetric normalized Laplace matrix $\mathcal{L}$ (see [Chung and Langlands 1996] or [Banerjee and Jost 2008])

with entries

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \not\sim v_j. \end{cases}$$

Since these matrices are similar they have the same eigenvalues. Their spectra have been studied extensively in the literature, for instance in [Chung 1997] and [Banerjee and Jost 2008]. □

For metrics given by general functions $\theta : E(\Gamma) \to (0, \pi)$, the question of which degrees α admit balanced homogeneous harmonic functions is more subtle. We will study the structure of $\Delta_{\alpha\theta}$ via the quadratic form it determines on $\mathbb{R}^{\#V(\Gamma)}$. This quadratic form will give information about the changing eigenvalues of $\Delta_{\alpha\theta}$. In particular, when an eigenvalue crosses 0 the operator $\Delta_{\alpha\theta}$ will have a nontrivial kernel.

Lemma 3.6. *For each fixed $\rho = (\rho_i) \not\equiv 0$ the quadratic form $\rho \cdot \Delta_{\alpha\theta}\rho$ is strictly increasing in α.*

Proof. First compute the quadratic form as follows:

$$\rho \cdot \Delta_{\alpha\theta}\rho = \sum_i \rho_i \left(\sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} \right)$$

$$= \sum_{v_i v_j \in E(\Gamma)} \left(\rho_i \frac{\rho_j - \rho_i \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} + \rho_j \frac{\rho_i - \rho_j \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \right)$$

$$= \sum_{v_i v_j \in E(\Gamma)} \frac{2\rho_i \rho_j - \cos(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)}{\sin(\alpha\theta_{ij})}$$

$$= \sum_{v_i v_j \in E(\Gamma)} (2\csc(\alpha\theta_{ij})\rho_i \rho_j - \cot(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)).$$

Now the derivative with respect to α is easy to compute:

$$\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) = \sum_{v_i v_j \in E(\Gamma)} (\theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2) - 2\rho_i \rho_j \theta_{ij}\csc(\alpha\theta_{ij})\cot(\alpha\theta_{ij}))$$

$$= \sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2 - 2\rho_i \rho_j \cos(\alpha\theta_{ij}))$$

$$\geq \sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i - \rho_j)^2 \geq 0.$$

Moreover suppose $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) = 0$ at some fixed $\rho = (\rho_i)$ and some fixed α. Then $\rho_i \rho_j \cos(\alpha\theta_{ij}) = \rho_i \rho_j$ for each $v_i v_j \in E(\Gamma)$, and also

$$\sum_{v_i v_j \in E(\Gamma)} \theta_{ij}\csc^2(\alpha\theta_{ij})(\rho_i - \rho_j)^2 = 0.$$

The expression on the left is clearly nonnegative, so for equality to hold ρ must be constant. And since α is nonsingular, $\cos(\alpha\theta_{ij}) \neq 1$ so that $\rho_i\rho_j = 0$ for all $v_iv_j \in E(\Gamma)$. Since ρ is constant, we have $\rho \equiv 0$. That is, the only way that $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta}\rho) = 0$ at any α is if $\rho \equiv 0$. $\hfill\square$

Theorem 3.7. *In an interval of nonsingular α, the eigenvalues of $\Delta_{\alpha\theta}$ are strictly increasing functions of α.*

Proof. The entries of the matrix $\Delta_{\alpha\theta}$ are analytic in α except where they are not defined, i.e., at singular α. Since we avoid such values, $\Delta_{\alpha\theta}$ is analytic in α. Moreover, $\Delta_{\alpha\theta}$ is symmetric for each real α. A result in the perturbation theory of eigenvalue problems (see [Rellich 1969, Section 1.1] or [Kato 1966, Theorem 6.1]) says that the eigenvalues and eigenvectors of $\Delta_{\alpha\theta}$ are also analytic in α. Namely, there are $\lambda_j(\alpha) \in \mathbb{R}$ and $w_j(\alpha) \in \mathbb{R}^{\#V(\Gamma)}$ depending analytically on α so that

$$\Delta_{\alpha\theta}w_j(\alpha) = \lambda_j(\alpha)w_j(\alpha).$$

Without loss of generality, $w_j(\alpha)$ is a unit vector for each j and each α. The eigenvalues can be recovered from the eigenvectors via the formula

$$\lambda_j(\alpha) = w_j(\alpha) \cdot \Delta_{\alpha\theta}w_j(\alpha).$$

Differentiating this identity with respect to α yields

$$\lambda'_j(\alpha) = 2w'_j(\alpha) \cdot \Delta_{\alpha\theta}w_j(\alpha) + w_j(\alpha) \cdot \frac{\partial\Delta_{\alpha\theta}}{\partial\alpha}w_j(\alpha).$$

As each $w_j(\alpha)$ is a unit vector, it lies in the unit sphere of $\mathbb{R}^{\#V(\Gamma)}$. Hence $w'(\alpha)$ is perpendicular to $w_j(\alpha)$. But $\Delta_{\alpha\theta}w_j(\alpha) = \lambda_j(\alpha)w_j(\alpha)$ is parallel to $w_j(\alpha)$, so the first term in $\lambda'(\alpha)$ vanishes. The second term is strictly positive by Lemma 3.6. Thus

$$\lambda'_j(\alpha) > 0. \hfill\square$$

An immediate consequence of Theorem 3.7 is that one can count the number of degrees α in an interval that admit balanced homogeneous harmonic functions in terms of the behavior of $\rho \cdot \Delta_{\alpha\theta}\rho$ at the endpoints of the interval.

Corollary 3.8. *Suppose $[\alpha_0, \alpha_1]$ is an interval of nonsingular degrees. If $\Delta_{\alpha_0\theta}$ has n_0 nonpositive eigenvalues and $\Delta_{\alpha_1\theta}$ has n_1 negative eigenvalues then there are $n_0 - n_1$ degrees $\alpha \in [\alpha_0, \alpha_1]$, counted with multiplicity, for which $\Delta_{\alpha\theta}$ has a nontrivial kernel.*

Remark 3.9. If $\Delta_{\alpha\theta}$ has a nontrivial kernel, the *multiplicity* of α is the dimension of that kernel. This is the number of independent balanced homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α, or equivalently the maximum dimension of the image of a balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to \mathbb{R}^d$.

Proof. Let the eigenvalues of $\Delta_{\alpha\theta}$ be $\{\lambda_j(\alpha)\}$ as in the proof of Theorem 3.7. The operator $\Delta_{\alpha\theta}$ can only have a nontrivial kernel if some $\lambda_j(\alpha) = 0$.

By Theorem 3.7 each $\lambda_j(\alpha)$ is strictly increasing in α. Let $\lambda_1(\alpha_0), \ldots, \lambda_{n_0}(\alpha_0)$ be the nonpositive eigenvalues of $\Delta_{\alpha_0\theta}$. For each of these eigenvalues there is at most one $\alpha \in [\alpha_0, \alpha_1]$ for which $\lambda_j(\alpha) = 0$; before this degree $\lambda_j < 0$ and after it $\lambda_j > 0$. The remaining eigenvalues, $\{\lambda_j(\alpha) \mid j > n_0\}$, all remain positive for $\alpha_0 \leq \alpha \leq \alpha_1$.

So if there are n_1 negative eigenvalues of $\Delta_{\alpha_1\theta}$ then $n_0 - n_1$ of the original n_0 nonpositive eigenvalues of $\Delta_{\alpha_0\theta}$ were 0 for some $\alpha \in [\alpha_0, \alpha_1]$. $\qquad\square$

One can extend this result to open intervals whose endpoints are singular degrees. If $\alpha_0\theta_{ij} = n\pi$ for some edge $v_iv_j \in E(\Gamma)$ and some $n \in \mathbb{Z}$, the term in the quadratic form $\rho \cdot \Delta_{(\alpha_0+t)\theta}\rho$ corresponding to v_iv_j reads

$$\frac{2\rho_i\rho_j - \cos((\alpha_0+t)\theta_{ij})(\rho_i^2 + \rho_j^2)}{\sin((\alpha_0+t)\theta_{ij})} = \frac{2\rho_i\rho_j - ((-1)^n + O(t^2))(\rho_i^2 + \rho_j^2)}{(-1)^n\theta_{ij}t + O(t^3)}$$
$$= -\frac{(\rho_i - (-1)^n\rho_j)^2 + O(t^2)}{\theta_{ij}t + O(t^3)}.$$

If $\rho_i - (-1)^n\rho_j \neq 0$ then this term approaches ∞ as $t \to 0$ from the left, and approaches $-\infty$ as $t \to 0$ from the right. But if $\rho_i - (-1)^n\rho_j = 0$ then this term approaches 0 as $t \to 0$.

Suppose $\alpha_0\theta(e) \in \pi\mathbb{Z}$ for more than one edge $e \in E(\Gamma)$. For $\rho \cdot \Delta_{\alpha\theta}\rho$ to remain bounded near α_0, each edge v_iv_j with $\alpha_0\theta_{ij} = n_{ij}\pi$ imposes a relation $\rho_i - (-1)^{n_{ij}}\rho_j = 0$. These relations cut out a subspace $B(\alpha_0) \subset \mathbb{R}^{\#V(\Gamma)}$ on which $\rho \cdot \Delta_{\alpha\theta}\rho$ remains bounded for α near α_0. But, for any $\rho \notin B(\alpha_0)$, $\rho \cdot \Delta_{\alpha\theta}\rho$ satisfies

$$\lim_{\alpha\to\alpha_0^\pm} \rho \cdot \Delta_{\alpha\theta}\rho = \mp\infty.$$

Now suppose $\alpha_0 < \alpha_1$ are singular, but each $\alpha \in (\alpha_0, \alpha_1)$ is nonsingular. Let $B_j = B(\alpha_j)$ be the subspaces described above, and let $Q_j(\rho) = \lim_{\alpha\to\alpha_j} \rho \cdot \Delta_{\alpha\theta}\rho$ be a quadratic form defined for $\rho \in B_j$. Then the number of nonpositive eigenvalues of $\Delta_{\alpha\theta}$ for $\alpha = \alpha_0 + \epsilon$, ϵ sufficiently small, is given by the number of negative eigenvalues of Q_0 plus $\dim B_0^\perp$. Likewise the number of positive eigenvalues of $\Delta_{\alpha\theta}$ for $\alpha = \alpha_1 - \epsilon$ is given by the number of positive eigenvalues of Q_1 plus $\dim B_1^\perp$. The number of degrees $\alpha \in (\alpha_0, \alpha_1)$ admitting a balanced homogeneous harmonic function, counted with multiplicity, is the difference between these two numbers.

The behavior of $\Delta_{\alpha\theta}$ as $\alpha \to 0$ is of particular interest.

Proposition 3.10. *As $\alpha \to 0$ from the right, the limit of the operators $\alpha\Delta_{\alpha\theta}$ converge to an edge weighted Laplace operator on Γ with edge weights $w(e) = \frac{1}{\theta(e)}$. In*

particular $B(0)$ is the span of the constant vector $\mathbb{1}$, with $\mathbb{1}_i = 1$ for all i, and $Q_0 = 0$ on B_0.

Proof. The limit of the quadratic forms determined by $\alpha \Delta_{\alpha\theta}$ is given by

$$\lim_{\alpha \to 0^+} \rho \cdot (\alpha \Delta_{\alpha\theta}) \rho = \lim_{\alpha \to 0^+} \sum_{v_i v_j \in E(\Gamma)} \frac{2\rho_i \rho_j - \cos(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)}{\sin(\alpha\theta_{ij})/\alpha}$$

$$= \sum_{v_i v_j \in E(\Gamma)} \frac{2\rho_i \rho_j - \rho_i^2 - \rho_j^2}{\theta_{ij}}$$

$$= - \sum_{v_i v_j \in E(\Gamma)} \frac{(\rho_i - \rho_j)^2}{\theta_{ij}}$$

$$= \rho \cdot \Delta \rho,$$

where the operator Δ has entries

$$\Delta_{ij} = \begin{cases} \displaystyle\sum_{v_k \sim v_i} \frac{1}{\theta_{ik}} & \text{if } i = j, \\[2mm] \dfrac{1}{\theta_{ij}} & \text{if } v_i \sim v_j, \\[2mm] 0 & \text{else.} \end{cases}$$

This is precisely the edge-weighted Laplace operator described in the statement of the proposition.

Now if $\rho_i = c$ for all i then

$$\lim_{\alpha \to 0^+} \rho \cdot \Delta_{\alpha\theta} \rho = 0.$$

But for any other ρ,

$$\lim_{\alpha \to 0^+} \rho \cdot (\alpha \Delta_{\alpha\theta}) \rho = - \sum_{v_i v_j \in E(\Gamma)} \frac{(\rho_i - \rho_j)^2}{\theta_{ij}} < 0,$$

so $\rho \cdot \Delta_{\alpha\theta} \rho \to -\infty$ as $\alpha \to 0$ from the right. $\qquad \square$

We are now in a position to give a lower bound on the smallest degree $\alpha > 0$ of any balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$, in terms of the first eigenvalue of the normalized Laplace operator $\mathcal{L}$ with entries

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j, \\[2mm] -\dfrac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & \text{if } v_i \sim v_j, \\[2mm] 0 & \text{if } v_i \nsim v_j. \end{cases}$$

Theorem 3.11. *Let λ_1 be the first positive eigenvalue of $\mathcal{L}$, and let*

$$\theta_{\max} = \max_{e \in E(\Gamma)} \theta(e).$$

If Γ is not a complete graph and there is a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α, then

$$\alpha \geq \frac{1}{\theta_{\max}} \arccos(1 - \lambda_1).$$

If $\Gamma = K_n$ is the complete graph on n vertices then

$$\alpha \geq \frac{2}{\theta_{\max}} \arctan \frac{n}{n-1}.$$

Proof. The operator $\mathcal{L}$ is positive semidefinite, which can be seen by the formula

$$w \cdot \mathcal{L} w = \sum_{v_i v_j \in E(\Gamma)} \left(\frac{w_i}{\sqrt{\deg v_i}} - \frac{w_j}{\sqrt{\deg v_j}} \right)^2.$$

Moreover, $\mathcal{L}$ has a kernel spanned by the vector $w = D^{1/2}\mathbb{1}$ with $w_i = \sqrt{\deg v_i}$. If Γ is not complete then $\lambda_1 \leq 1$, and if $\Gamma = K_n$ then $\lambda_1 = \frac{n}{n-1}$ (see [Chung 1997, Lemma 1.7]).

We will proceed by contradiction. In the case where Γ is not complete and $\lambda_1 \leq 1$, we first assume that $\alpha < \arccos(1 - \lambda_1)/\theta_{\max}$. The bound $\lambda_1 \leq 1$ also implies

$$\alpha\theta_{\max} < \frac{\pi}{2}.$$

Since the first singular α happens when $\alpha\theta_{\max} = \pi$, we are within the interval of nonsingular degrees with 0 as an endpoint.

Using the bound $\alpha\theta_{\max} < \pi/2$, bound the quadratic form $\rho \cdot \Delta_{\alpha\theta}\rho$ in terms of $\mathcal{L}$:

$$\begin{aligned}
\rho \cdot \Delta_{\alpha\theta}\rho &= \sum_{v_i v_j \in E(\Gamma)} \frac{2\rho_i\rho_j - \cos(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2)}{\sin(\alpha\theta_{ij})} \\
&= \sum_{v_i v_j \in E(\Gamma)} \frac{1 - \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})}(\rho_i^2 + \rho_j^2) - \sum_{v_i v_j \in E(\Gamma)} \frac{(\rho_i - \rho_j)^2}{\sin(\alpha\theta_{ij})} \\
&\leq \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})} \sum_i \deg(v_i)\rho_i^2 - \frac{1}{\sin(\alpha\theta_{\max})} \sum_{v_i v_j \in E(\Gamma)} (\rho_i - \rho_j)^2 \\
&= \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})} \rho \cdot D\rho - \frac{1}{\sin(\alpha\theta_{\max})} \rho \cdot \Delta\rho \\
&= \frac{\rho}{\sin(\alpha\theta_{\max})} \cdot \big((1 - \cos(\alpha\theta_{\max}))D - \Delta\big)\rho \\
&= \frac{D^{1/2}\rho}{\sin(\alpha\theta_{\max})} \cdot (1 - \cos(\alpha\theta_{\max}) - \mathcal{L})D^{1/2}\rho.
\end{aligned}$$

Combining [Theorem 3.7](#) with [Proposition 3.10](#) implies $\Delta_{\alpha\theta}$ has at least one positive eigenvalue for each $0 < \alpha < \pi/\theta_{\max}$. By assumption $1 - \cos(\alpha\theta_{\max}) < \lambda_1$, so the quadratic form above has exactly one positive eigenvalue. The remaining eigenvalues are all strictly negative, and their corresponding eigenvectors span a hyperplane W.

If $\Delta_{\alpha\theta}$ had a second nonnegative eigenvalue then there would be a 2-dimensional space on which $\rho \cdot \Delta_{\alpha\theta}\rho$ is nonnegative. But such a plane necessarily intersects W, on which an upper bound for $\rho \cdot \Delta_{\alpha\theta}\rho$ is negative, a contradiction! Hence $\Delta_{\alpha\theta}$ cannot develop a kernel or a second positive eigenvalue for $0 < \alpha < \arccos(1 - \lambda_1)/\theta_{\max}$.

The proof in the case where $\Gamma = K_n$ and $\lambda_1 = \frac{n}{n-1}$ is similar. Now we assume $\alpha\theta_{\max} < 2\arctan\frac{n}{n-1}$. We are still in the interval before the first singular degree. We can still bound as before

$$\frac{1 - \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \leq \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})}$$

for each edge $v_i v_j$, but now we can only bound $\frac{1}{\sin(\alpha\theta_{ij})} \geq 1$. Thus our bound on the quadratic form now reads

$$\rho \cdot \Delta_{\alpha\theta}\rho \leq \frac{1 - \cos(\alpha\theta_{\max})}{\sin(\alpha\theta_{\max})}\rho \cdot D\rho - \rho \cdot \Delta\rho$$

$$= \tan\left(\frac{\alpha\theta_{\max}}{2}\right)\rho \cdot D\rho - \rho \cdot \Delta\rho$$

$$= D^{1/2}\rho \cdot \left(\tan\left(\frac{\alpha\theta_{\max}}{2}\right) - \mathcal{L}\right)D^{1/2}\rho.$$

Just as in the previous case, $\rho \cdot \Delta_{\alpha\theta}\rho$ has at least one positive eigenvalue, and our assumed bound on α implies the quadratic form on the right-hand side has exactly one positive eigenvalue. Thus $\rho \cdot \Delta_{\alpha\theta}\rho$ cannot have developed a kernel or second positive for $0 < \alpha < \frac{2}{\theta_{\max}}\arctan\frac{n}{n-1}$. $\square$

3.2. Singular degrees.

Let us begin with the extension of [Proposition 3.4](#), where $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$. When $\alpha\theta_0 \notin \pi\mathbb{Z}$, that proposition equated $1 - \cos(\alpha\theta_0)$ with the eigenvalues of $\mathcal{L}$. This leaves out the possible eigenvalues of 0 and 2.

The normalized Laplacian $\mathcal{L}$ always has an eigenvalue $\lambda = 0$, as $D^{1/2}\mathbb{1}$ is in the kernel of $\mathcal{L}$. This corresponds to $\alpha\theta_0 = 2n\pi$ for $n \in \mathbb{Z}$; see [Theorem 3.13](#).

The operator $\mathcal{L}$ may also have an eigenvalue $\lambda = 2$, depending on the structure of Γ. In fact, 2 is an eigenvalue of $\mathcal{L}$ precisely when Γ has no odd cycles. The case $\lambda = 2$ corresponds with $\alpha\theta_0 = (2n+1)\pi$; see [Theorem 3.14](#).

Before approaching the balancing conditions in Theorems [3.13](#) and [3.14](#) we will first parametrize the space of homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ in the case $\alpha\theta_0 \in \pi\mathbb{Z}$.

Lemma 3.12. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha\theta_0 = n\pi$ for $n \in \mathbb{Z}$, $n > 0$. If n is even, or if n is odd and Γ contains no odd cycles, then a homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α is uniquely determined by a number ρ_0 and a function $c_2 : E(\Gamma) \to \mathbb{R}$. If n is odd and Γ contains an odd cycle then a homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α is uniquely determined by just a function $c_2 : E(\Gamma) \to \mathbb{R}$.*

Proof. Begin by arbitrarily orienting all the edges of Γ. On each face $C(e) \subset C(\Gamma, \theta)$ choose coordinates adapted to the tail of e, identifying $C(e)$ with S_{θ_0} so that the tail corresponds to the positive x-axis and the head corresponds to the line $x = y \cot\theta_0$. In these coordinates Lemma 2.9 says a homogeneous harmonic function is represented by a function

$$u_e(r, \theta) = r^\alpha(c_1(e)\cos(\alpha\theta) + c_2(e)\sin(\alpha\theta)).$$

In the spirit of Lemma 3.1, we can let $\rho : V(\Gamma) \to \mathbb{R}$ be given by $\rho(v) = f(v, 1)$. If edge e has tail v_1 and head v_2, then Lemma 2.10 says that $c_1 = \rho(v_1)$ and $c_1 = \rho(v_1)\cos(\alpha\theta_0)$. If n is even this means that $\rho(v_i) = \rho(v_j)$ for each edge $v_i v_j \in E(\Gamma)$. As Γ is connected, we must have $\rho \equiv \rho_0$ for some constant ρ_0. If n is odd we must have $\rho_i = -\rho_j$ for each edge $v_i v_j \in E(\Gamma)$. If Γ contains no odd cycles then it is bipartite, so we may set $\rho = \rho_0$ on one part of Γ and $\rho = -\rho_0$ on the other part. But an odd cycle in Γ would force $\rho \equiv 0$.

The paragraph above explains the parameter ρ_0 in the cases when n is even or Γ contains no odd cycle, and its absence in the case when n is odd and Γ contains an odd cycle. Once the constants $c_1(e)$ are so specified, the coefficients $c_2(e)$ in front of $\sin(\alpha\theta)$ can be anything and the function f is well-defined and continuous. $\square$

If f is determined by any constant ρ_0 (with $\rho_0 = 0$ in case n is odd and Γ contains an odd cycle) and the function $c_2 \equiv 0$, then f is already balanced because

$$\frac{\partial u_e}{\partial \theta}(r, 0) = \frac{\partial u_e}{\partial \theta}(r, \theta_0) = 0.$$

Considering a more general function $c_2 : E(\Gamma) \to \mathbb{R}$, two linear maps become relevant. In case n is even, we define ∂c_2, and in case n is odd we define dc_2. Separating out a single edge e_1, we will define both ∂ and d on the function

$$c_2(e) = \begin{cases} 1 & \text{if } e = e_1, \\ 0 & \text{if } e \neq e_1. \end{cases}$$

If e_1 has tail v_1 and head v_2, then define

$$\partial c_2(v) = \begin{cases} 1 & \text{if } v = v_2, \\ -1 & \text{if } v = v_1, \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad dc_2(v) = \begin{cases} 1 & \text{if } v = v_1 \text{ or } v_2, \\ 0 & \text{else.} \end{cases}$$

Define the maps ∂ and d for arbitrary $c_2 : E(\Gamma) \to \mathbb{R}$ by linearity.

The next two theorems expand on a result of Theorem 3.5 in [Friedman and Tillich 2004].

Theorem 3.13. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha\theta_0 = 2n\pi$ for some $n \in \mathbb{Z}$, $n > 0$. Then the space of balanced homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α has dimension $\#E(\Gamma) - \#V(\Gamma) + 2$.*

Proof. According to Lemma 3.12 a homogeneous harmonic function is determined by $\rho_0 \in \mathbb{R}$ and $c_2 : E(\Gamma) \to \mathbb{R}$. The function f is balanced according to Definition 2.13 if and only if $\partial c_2 = 0$. It thus remains to compute the dimension of the kernel of ∂, or equivalently $\#E(\Gamma) - \dim \operatorname{im} \partial$.

The map ∂ turns out to be the boundary operator from homology, whose structure is well understood. For the sake of completeness we provide a description of its image here as the space of functions $g : V(\Gamma) \to \mathbb{R}$ with $\sum_{v \in V(\Gamma)} g(v) = 0$. From the structure of ∂ it is clear that the image is contained in this space.

Fix a vertex v_1, and a collection of paths p_j from v_1 to v_j for each other vertex v_j. Without loss of generality assume that the orientation along each path points from v_1 towards v_j. Let $\xi_j(e) = 1$ for each edge $e \in p_j$ and $\xi_j(e) = 0$ otherwise, so that

$$\partial\xi_j(v) = \begin{cases} 1 & \text{if } v = v_j, \\ -1 & \text{if } v = v_1, \\ 0 & \text{else.} \end{cases}$$

For any function $g : V(\Gamma) \to \mathbb{R}$ with $\sum_j g(v_j) = 0$, let $\xi = \sum_{j \geq 2} g(v_j)\xi_j$, so that

$$\partial\xi(v_j) = \begin{cases} g(v_j) & \text{if } j \geq 2, \\ -\sum_{i \geq 2} g(v_i) & \text{if } j = 1. \end{cases}$$

Since g satisfies $\sum_j g(v_j) = 0$, we necessarily have $\partial\xi(v_1) = g(v_1)$ also.

The image of ∂ thus has dimension $\#V(\Gamma) - 1$, so its kernel has dimension $\#E(\Gamma) - \#V(\Gamma) + 1$. Combining with the choice of ρ_0, the space of balanced homogeneous harmonic functions of degree α is $\#E(\Gamma) - \#V(\Gamma) + 2$. $\qquad\square$

Theorem 3.14. *Suppose $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha\theta_0 = (2n + 1)\pi$ for some $n \in \mathbb{Z}$, $n \geq 0$. Then the space of balanced homogeneous harmonic functions $f : C(\Gamma, \theta) \to \mathbb{R}$ of degree α has dimension $\#E(\Gamma) - \#V(\Gamma) + 2$ if Γ contains no odd cycles, and dimension $\#E(\Gamma) - \#V(\Gamma)$ if Γ does contain an odd cycle.*

Proof. According to Lemma 3.12 a homogeneous harmonic function is determined by $\rho_0 \in \mathbb{R}$ and $c_2 : E(\Gamma) \to \mathbb{R}$. In case Γ contains an odd cycle, though, we must have $\rho_0 = 0$. The function f is balanced according to Definition 2.13 if and only if $dc_2 = 0$. It thus remains to compute the dimension of the kernel of d, or equivalently $\#E(\Gamma) - \dim \operatorname{im} d$.

In case Γ contains an odd cycle we claim the image of the linear map d consists of all functions $g : V(\Gamma) \to \mathbb{R}$. With the presence of a single odd cycle, one can find a cycle of odd length starting and ending at any particular vertex. For a vertex v_j, choose such a path and let $\xi_j(e) = \pm 1$ on the edges of the path in an alternating fashion, so that

$$d\xi_j(v) = \begin{cases} 2 & \text{if } v = v_j, \\ 0 & \text{else.} \end{cases}$$

Defining $\xi = \frac{1}{2} \sum_j g(v_j)\xi_j$, we see that $d\xi = g$. In this case we had no choice of ρ_0, so the dimension of balanced homogeneous harmonic functions is just

$$\dim \ker d = \#E(\Gamma) - \dim \operatorname{im} d = \#E(\Gamma) - \#V(\Gamma).$$

If Γ does not contain an odd cycle then the graph is bipartite. Say the parts of Γ are $A, B \subset V(\Gamma)$, with $A \cup B = V(\Gamma)$, $A \cap B = \varnothing$, and every edge of Γ has one endpoint in A and one endpoint in B. Then we claim the image of d consists of those functions $g : V(\Gamma) \to \mathbb{R}$ with

$$\sum_{v \in A} g(v) = \sum_{v \in B} g(v).$$

Assuming G is connected, one can find odd-length paths p from any vertex in A to any vertex in B. If the path p joins $v_1 \in A$ and $v_2 \in B$, then defining $\xi_p(e) = \pm 1$ along p in an alternating fashion gives

$$d\xi_p(v) = \begin{cases} 1 & \text{if } v = v_1, \\ 1 & \text{if } v = v_2, \\ 0 & \text{else.} \end{cases}$$

Start with $\xi \equiv 0$ and build up as follows. Enumerate the vertices in A by v_j and the vertices in B by w_j, and without loss of generality $\#A = a \geq b = \#B$. First select odd-length paths $p_1, \ldots, p_{a-b}$ from $v_{b+1}, \ldots, v_a$ to w_1 and add $g(v_j)\xi_{p_j}$ to ξ for $b < j \leq a$. Then choose paths from w_1 to v_1, v_1 to w_2, w_2 to v_2, $\ldots$, w_b to v_b. Adding appropriate multiples of the corresponding functions ξ_p to ξ will ensure that $d\xi(v) = g(v)$ for all v except perhaps v_a. But the condition that $\sum_{v \in A} g(v) = \sum_{v \in B} g(v)$ will ensure that $d\xi(v_a) = g(v_a)$ as well.

Thus in this case the image of d has dimension $\#V(\Gamma) - 1$, so its kernel has dimension $\#E(\Gamma) - \#V(\Gamma) + 1$. Combining with the choice of ρ_0, the space of balanced homogeneous harmonic functions of degree α has dimension $\#E(\Gamma) - \#V(\Gamma) + 2$. $\qquad\square$

Remark 3.15. Theorem 3.13 can be seen as a corollary of Theorem 3.14 after subdividing edges. In fact, all that is really needed is the statement when $\alpha\theta_0 = \pi$. If $\alpha\theta_0 = k\pi$ simply subdivide each edge into k pieces to reduce to the case $\alpha\theta_0 = \pi$. Lemma 2.14 then implies a balanced function on the subdivided complex is smooth

at the introduced vertices and is thus the restriction of a function on the unsubdivided complex.

We now turn to the more complicated situation when $\theta : E(\Gamma) \to (0, \pi)$ is not constant but α is still a singular degree. Let $\Sigma \subset \Gamma$ be the subgraph consisting of all those edges $e \in E(\Gamma)$ with $\alpha\theta(e) \in \pi\mathbb{Z}$, along with their incident vertices. One can see, as in Lemma 3.12, that a homogeneous harmonic function is determined by a function $\rho : V(\Gamma) \to \mathbb{R}$ that lies in the space $B(\alpha)$ described in the previous subsection, along with a function $c_2 : E(\Sigma) \to \mathbb{R}$. Then one could describe which ρ and c_2 describe a balanced function. But it is simpler to use the strategy described in Remark 3.15.

For each edge $e \in E(\Sigma)$, if $\alpha\theta(e) = k\pi$ then subdivide e into k edges, each with $\alpha\theta = \pi$. This turns the original complex $C(\Gamma, \theta)$ into an isometric complex (by abuse of notation also denoted $C(\Gamma, \theta)$) where each edge $e \in E(\Gamma)$ satisfies either $\alpha\theta(e) = \pi$ or $\alpha\theta(e) \notin \pi\mathbb{Z}$. Now we can describe homogeneous harmonic functions.

Lemma 3.16. *A homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}$ of singular degree α is uniquely determined by a function $c_2 : E(\Sigma) \to \mathbb{R}$ together with a function $\rho : V(\Gamma) \to \mathbb{R}$ such that $\rho(v_i) = -\rho(v_j)$ for all $v_i v_j \in E(\Sigma)$.*

Proof. Just as in Lemma 3.1, the function f is uniquely determined on those edges $e \in E(\Gamma) \setminus E(\Sigma)$ by the values $\rho(v)$ on the incident vertices. And just as in Lemma 3.12, the condition $\rho_i = -\rho_j$ together with a choice of $c_{ij} = c_2(v_i v_j)$ on $v_i v_j \in E(\Sigma)$ uniquely determines f in the corresponding face via the representation

$$u(r, \theta) = r^\alpha (\rho_i \cos(\alpha\theta) + c_{ij} \sin(\alpha\theta)). \qquad \square$$

Now the balancing condition is slightly more subtle. On vertices $v_i \in V(\Gamma) \setminus V(\Sigma)$ one must still have

$$\sum_{v_j \sim v_i} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0,$$

just as in Theorem 3.2. Unfortunately these conditions alone seem unlikely to be satisfied for $\rho \not\equiv 0$, given the restrictions on ρ coming from Σ. Certainly a necessary condition is that the quadratic form $Q(\rho) = \lim_{\alpha' \to \alpha} \rho \cdot \Delta_{\alpha'\theta}\rho$ on the space $B(\alpha)$ from the previous subsection should have a nontrivial kernel. Since the eigenvalues of $\Delta_{\alpha'\theta}$ are strictly increasing with α' the possibility that one passes through 0 precisely at $\alpha' = \alpha$ seems unlikely.

On those vertices $v_i \in V(\Sigma)$ a combination of the formula from Theorem 3.2 and the linear map d from above is necessary, which we'll define as $\sigma : \mathbb{R}^{\#E(\Sigma)} \to \mathbb{R}^{\#V(\Sigma)}$. The balancing condition says

$$dc_2(v_i) + \sum_{v_i v_j \in E(\Gamma) \setminus E(\Sigma)} \frac{\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0.$$

Proposition 3.17. *If Σ has a connected component that is not a tree but contains no odd cycles, then there are nontrivial balanced homogeneous harmonic functions of degree α.*

Proof. To produce a nontrivial balanced homogeneous harmonic function of a singular degree α it is enough, according to Lemma 3.16, to specify ρ and $c_2 :$ $E(\Sigma) \to \mathbb{R}$. We will choose $\rho \equiv 0$, and if $\Sigma_0 \subset \Sigma$ is a connected component that is not a tree but contains no odd cycles we set $c_2 = 0$ on $E(\Sigma) \setminus E(\Sigma_0)$.

We will construct a balanced function by specifying $c_2 \not\equiv 0$ on $E(\Sigma_0)$ in the kernel of d. To prove that such a c_2 exists, it suffices to show that the image of d when restricted to functions $c_2 : E(\Sigma_0) \to \mathbb{R}$ has dimension at most $\#E(\Sigma_0) - 1$. But this is precisely the content of Theorem 3.14; in the absence of odd cycles the image of d acting on $\mathbb{R}^{\#E(\Sigma_0)}$ has dimension $\#V(\Sigma_0) - 1$. As Σ_0 is not a tree, $\#E(\Sigma_0) \geq \#V(\Sigma_0)$, so d has a kernel and a balanced homogeneous harmonic function exists. $\square$

3.3. *Eigenvalue problems on graphs.* The balanced homogeneous harmonic functions of this section can be reinterpreted in terms of an eigenvalue problem on the graph Γ. Returning to the notation of Definition 2.2, write $f : C(\Gamma, \theta) \to \mathbb{R}$ as

$$f(x, t) = \rho(x)t^\alpha.$$

Here x is a point on Γ and $t \in \mathbb{R}_{\geq 0}$. On each edge e determine coordinates by a map $\psi_e : e \to [0, \theta(e)] \subset \mathbb{R}$, and define $\rho_e : [0, \theta(e)] \to \mathbb{R}$ by

$$\rho_e(s) = \rho \circ \psi_e^{-1}(s).$$

Now the original function $f : C(\Gamma, \theta) \to \mathbb{R}$ is harmonic if and only if each ρ_e is an eigenfunction of the second derivative operator, namely $\rho_e'' = -\alpha^2 \rho_e$. And the original function f is balanced if and only if, for each $v \in V(\Gamma)$,

$$\sum_{v \in e} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi_e(v)) = 0.$$

Thus a balanced homogeneous harmonic function f corresponds to a solution of the eigenvalue problem $\rho_e'' = -\alpha^2 \rho_e$ for all $e \in E(\Gamma)$ with the boundary conditions stated above at each $v \in V(\gamma)$.

This eigenvalue problem was studied extensively in [Friedman and Tillich 2004], and existence of solutions was studied variationally by way of the Rayleigh quotient,

$$R[\rho] = \frac{\sum_{e \in E(\Gamma)} \int_0^{\theta(e)} |\rho_e'(s)|^2 \, ds}{\sum_{e \in E(\Gamma)} \int_0^{\theta(e)} |\rho_e(s)|^2 \, ds}.$$

In particular, they show:

Theorem 3.18 [Friedman and Tillich 2004, Theorem 3.2]. *There exist solutions ρ_i of degree α_i to the eigenvalue problem on Γ. The degrees α_i are increasing and tend to ∞, and the ρ_i form a complete orthonormal basis for $L^2(\Gamma)$.*

Friedman and Tillich [2004, Theorem 3.3] proved a Weyl law. In our notation, if N_α counts the number of degrees $t \leq \alpha$, with their multiplicities, that occur as degrees of balanced homogeneous harmonic functions, then there is a constant C so that

$$\left| N_\alpha - \frac{\alpha\Theta}{\pi} \right| \leq C.$$

Here $\Theta = \sum_{e \in E(\Gamma)} \theta(e)$ is the total length of the graph.

It is interesting to compare this Weyl law with our bounds on the eigenvalues, e.g., in Corollary 3.8. In case $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$, there are precisely $1 + \lfloor \frac{\alpha\theta_0}{\pi} \rfloor$ singular degrees of the form $\frac{k\pi}{\theta_0}$ between 0 and α. From Theorems 3.13 and 3.14 we know that one eigenvalue of $\Delta_{t\theta}$ approaches 0 as $t \to \frac{k\pi}{\theta_0}$ and the rest approach $\pm\infty$, unless k is odd and Γ is not bipartite in which case all eigenvalues of $\Delta_{t\theta}$ approach $\pm\infty$. Including the balanced homogeneous functions of singular degrees counted in Theorems 3.13 and 3.14, one can say in this case that

$$\left| N_\alpha - \frac{\alpha\Theta}{\pi} \right| \leq \#E(\Gamma).$$

In case $\theta(e)$ is not constant over $E(\Gamma)$, the analysis is more subtle. In the most generic case, when $\theta(e) \neq \theta(e')$ whenever $e \neq e'$, $\Delta_{t\theta}$ becomes singular whenever $t\theta_{ij} = k\pi$ for some edge $v_i v_j \in E(\Gamma)$ and $k \in \mathbb{Z}$. Near such a singular degree, $\Delta_{t\theta}\rho$ remains bounded if and only if $\rho_i = (-1)^k \rho_j$. That is, $\Delta_{\alpha\theta}$ remains bounded on a space of codimension 1. It is hard to predict when any particular eigenvalue of $\Delta_{\alpha\theta}$ passes through 0, but since the eigenvalues that remain bounded are continuous across the singular degrees, each time an eigenvalue approaches ∞, it must have passed through 0 at some point. The only exception is the first eigenvalue that approaches ∞ might be the one that was an eigenvalue of $\Delta_{0\theta}$.

For fixed $\alpha > 0$, there are a total of $\sum_{v_i \sim v_j} \lfloor \frac{\alpha\theta_{ij}}{\pi} \rfloor$ singular degrees between 0 and α, each one contributing at least one balanced eigenfunction. Between the last singular degree and α there is some number $b \leq \#V(\Gamma)$ of additional balanced eigenfunctions.

If S_α is the number of balanced eigenfunctions of singular degrees that don't arise as limits of unbalanced eigenfunctions from nearby degrees, then we have

$$N_\alpha = \sum_{v_i \sim v_j} \left\lfloor \frac{\alpha\theta_{ij}}{\pi} \right\rfloor + S_\alpha + b \geq \frac{\alpha\Theta}{\pi} - \#E(\Gamma) + S_\alpha.$$

From the Weyl bound, $N_\alpha \leq \frac{\alpha\Theta}{\pi} + C$ for some constant C, and hence

$$S_\alpha \leq \#E(\Gamma) + C.$$

In other words, there are only finitely many balanced eigenfunctions that occur spontaneously at singular degrees.

4. Balanced harmonic maps into k-pods

After subdividing the domain if necessary a homogeneous map $f : C(\Gamma, \theta) \to C(k)$ maps each face $C(e) \subset C(\Gamma, \theta)$ to a single edge $C(v_k) \subset C(k)$. The map f is harmonic if its restriction to each face, when viewed in coordinates, is a harmonic function $u : S_{\theta(e)} \to \mathbb{R}_{\geq 0}$. According to Proposition 2.11 such a map exists if and only if $0 < \alpha\theta(e) \leq \pi$ for each edge $e \in E(\Gamma)$.

The balancing condition of Definition 2.15 imposes further restrictions on which degrees α can occur for harmonic maps from smooth cones into k-pods. In fact the degree α and the geometry of the cone are closely related.

Theorem 4.1. *A balanced homogeneous harmonic map $f : C(C_{n'}, \theta') \to C(k)$ of degree α exists if and only if the cone is a subdivision of $C(C_n, \theta)$ where $\theta(e) = \theta_0$ for each edge $e \in E(C_n)$, and*

$$\alpha\theta_0 = \pi.$$

Remark 4.2. When $k = 2$, Theorem 4.1 can be seen to say that the nodal domains of a homogeneous harmonic function $f : \mathbb{R}^2 \to \mathbb{R}$ are equiangular sectors.

Proof. First suppose that there is a balanced homogeneous harmonic map $f : C(C_{n'}, \theta') \to C(k)$ and $f(v_0, 1) = 0$ is the vertex of $C(k)$ for some fixed $v_0 \in V(C_{n'})$. Enumerate the vertices of $C_{n'}$ in order as $\{v_0, v_2, \ldots, v_{n'-1}\}$, so $v_i \sim v_{(i+1) \pmod{n}}$ for each i. In each face $C(v_i v_{i+1}) \subset C(C_{n'}, \theta')$ take coordinates adapted to v_i and represent f in those coordinates by a harmonic function $u_i : S_{\theta(v_i v_{i+1})} \to \mathbb{R}_{\geq 0}$.

Since $f(v_0, 1) = 0$ is the vertex of the k-pod $C(k)$ we must have $\rho(v_0) = 0$. The map u_0 that represents f in $C(v_0 v_1)$ thus has the form

$$u_0(r, \theta) = c_1 r^\alpha \sin(\alpha\theta).$$

If $\alpha\theta(v_0 v_1) = \pi$ then $u_0(r, \theta(v_0 v_1)) = 0$ as well. Otherwise $u_0(r, \theta(v_0 v_1)) \neq 0$ and according to Lemma 2.16 the map u_1 must have the form

$$u_1(r, \theta) = c_1 r^\alpha \sin(\alpha\theta + \alpha\theta(v_0 v_1)).$$

Continuing inductively in this fashion, as long as $u_{i-1}(r, \theta(v_{i-1} v_i)) \neq 0$ Lemma 2.16 says the map u_i representing f in $C(v_i v_{i+1})$ must have the form

$$u_i(r, \theta) = c_1 r^\alpha \sin(\alpha\theta + \alpha\theta(v_0 v_1) + \cdots + \alpha\theta(v_{i-1} v_i)).$$

Since f can only map an edge of $C(C_{n'}, \theta')$ to the vertex of $C(k)$, some u_m must vanish at $(r, \theta(v_m v_{m+1}))$. In this case we have

$$\alpha \sum_{i=1}^{r} \theta(v_i v_{i+1}) = \pi.$$

We can "unsubdivide" the first r edges of $c_{n'}$ to form an edge of a new graph with $\alpha\theta = \pi$ on that edge. Repeating the same argument starting at v_{r+1}, and at each successive vertex of C_n that f maps to the vertex of $C(k)$, one sees that $C(C_{n'}, \theta')$ is a subdivision of some $C(C_n, \theta)$ with $\alpha\theta(e) = \pi$ for each $e \in E(C_n)$.

It remains to see that a balanced homogeneous harmonic map of degree α does exist from $C(C_n, \theta)$ when $\alpha\theta(e) = \pi$ for each $e \in E(C_n)$. Let $\theta_0 = \pi/\alpha$ be the common angle measure in each face of $C(C_n, \theta)$. In each face of $C(C_n, \theta)$ let f be represented in coordinates by the function

$$u(r, \theta) = cr^\alpha \sin(\alpha\theta).$$

Then f will map *each edge* of $C(C_n, \theta)$ into the vertex of $C(k)$. And the balancing condition is satisfied precisely because the constant c in each face is the same. $\square$

Remark 4.3. The process of "unsubdividing" described in this proof may result in a cone $C(C_n, \theta)$ with $\theta(e) \geq \pi$. This is not consistent with Definition 2.5 but it causes only notational difficulty. In particular, if $\theta(e) = \theta_0 \geq \pi$ one can identify the face $C(e)$ with the upper half-plane endowed with a different metric. Indeed, let

$$S = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \quad \text{with metric } ds^2 = dr^2 + \frac{\theta_0^2 r^2}{\pi^2} d\theta^2.$$

This S has an angle of θ_0 at the origin, and the harmonic functions of degree α defined on S have the form

$$u(r, \theta) = r^\alpha (c_1 \cos(\theta) + c_2 \sin(\theta)).$$

In order that u stays positive one must have $c_1 = 0$ and $c_2 > 0$, which is consistent with the functions described in the above proof.

Remark 4.4. According to our assumption from Section 2.3.2, if $v \in V(C_n)$ has incident edges e_1 and e_2, and $f(v, 1)$ is the vertex of $C(k)$, then the images of $f|_{C(e_1)}$ and $f|_{C(e_2)}$ should be distinct edges of $C(k)$. This is a common assumption in applications, and is consistent with the maximum principle for harmonic maps that would prohibit a local minimum value of 0 along an edge, thus ruling out the possibility $k = 1$. With this additional restriction, the number n from the above proof must be at least 3, unless the image of f lies in only two edges of the k-pod (in particular if $k = 2$).

The application we list here was established in [Gromov and Schoen 1992].

Corollary 4.5. *Any homogeneous harmonic map $f : \mathbb{R}^2 \to C(k)$ into a k-pod either has degree 1 or degree $\alpha \geq \frac{3}{2}$. If $C(C_n, \theta)$ is a smooth cone whose vertex is positively curved, then any homogeneous harmonic map $f : C(C_n, \theta) \to C(k)$ has degree $\alpha > 1$, and if the total angle of $C(C_n, \theta)$ is at most 3π then any map that visits at least three edges of $C(k)$ has degree $\alpha \geq 1$.*

Proof. We begin by remarking that $\mathbb{R}^2$ is isometric to $C(C_{n'}, \theta')$ whenever

$$\sum_{e \in E(C_{n'})} \theta'(e) = 2\pi.$$

According to Theorem 4.1, the degree α of a homogeneous harmonic map $f : C(C_n, \theta) \to C(k)$ must satisfy

$$\alpha \sum_{e \in E(C_{n'})} \theta'(e) = n\pi,$$

where n is the number of edges of $C(k)$ visited by f counted with multiplicity. By Remark 4.4 we must have $n \geq 2$, or $n \geq 3$ if f visits at least three edges of $C(k)$.

If the vertex of $C(C_{n'}, \theta')$ is flat, this means $\alpha = n/2$. When $n = 2$ we have $\alpha = 1$, and when $n \geq 3$ we have $\alpha \geq \frac{3}{2}$.

If the vertex of $C(C_{n'}, \theta')$ is positively curved then $\sum_e \theta'(e) < 2\pi$ so

$$\alpha = \frac{n\pi}{\sum_{e \in E(C_{n'})} \theta'(e)} > \frac{n}{2}.$$

Again by Remark 4.4 $n \geq 2$ so that $\alpha > 1$.

And in fact if

$$\sum_{e \in E(C_{n'})} \theta'(e) \leq 3\pi \quad \text{and} \quad n \geq 3,$$

then

$$\alpha = \frac{n\pi}{\sum \theta'(e)} \geq 1. \qquad \square$$

4.1. *p-harmonic maps.* Just as in the previous subsection, we consider p-harmonic maps $f : C(C_n, \theta) \to C(k)$. The basic object of study is analogous to the nonnegative harmonic functions described in Proposition 2.11. The existence of homogeneous p-harmonic functions with Dirichlet boundary conditions in a plane sector were studied in [Tolksdorf 1983] and later in [Porretta and Véron 2009]. Akman, Lewis, and Vogel describe in [Akman et al. 2019] the relationship between the sector S_{θ_0} and the degree α of homogeneity of p-harmonic functions described in [Tolksdorf 1983] and [Porretta and Véron 2009].

Translating the description of [Akman et al. 2019] into the present notation, for each $p > 1$ and each angle $0 < \theta_0 < 2\pi$ there is a unique degree $\alpha > 0$ and a unique (up to scaling) homogeneous p-harmonic function $\phi_\alpha : S_{\theta_0} \to \mathbb{R}_{\geq 0}$ of degree α with $\phi_\alpha(r, 0) = \phi_\alpha(r, \theta_0) = 0$. (There is also a unique degree $\alpha < 0$ with an associated p-harmonic function, but we will not consider that here.) The angle θ_0 and the degree α are related by the equation

$$1 - \frac{\theta_0}{\pi} = \frac{\alpha - 1}{\sqrt{\alpha^2 + \frac{2-p}{p-1}\alpha}}.$$

Lemma 4.6. *The relationship between θ_0 and α is monotone. That is, if $\theta_0 < \theta_1$ support homogeneous p-harmonic functions of degrees α_0 and α_1, respectively, then $\alpha_0 > \alpha_1$.*

Proof. Simply differentiate the equation relating θ_0 and α. This yields

$$-\frac{1}{\pi} = \frac{p\alpha + 2 - p}{2(p-1)\left(\alpha^2 + \frac{2-p}{p-1}\alpha\right)^{3/2}} \cdot \frac{d\alpha}{d\theta_0}.$$

Now, in order that the square root in the equation relating θ and α is real, we must have $\alpha > \frac{p-2}{p-1}$. This rearranges to say that $p\alpha + 2 - p > \alpha$, so the derivative computation above rearranges to

$$\frac{d\alpha}{d\theta_0} < -\frac{2(p-1)\left(\alpha^2 + \frac{2-p}{p-1}\alpha\right)^{3/2}}{\alpha\pi} < 0. \qquad \square$$

Theorem 4.7. *Any homogeneous p-harmonic map $f : \mathbb{R}^2 \to C(k)$ into a k-pod either has degree 1 or degree $\alpha \geq \frac{9}{8}$. If $C(C_n, \theta)$ is a smooth cone whose vertex is positively curved, then any homogeneous p-harmonic map $f : C(C_n, \theta) \to C(k)$ has degree $\alpha > 1$, and if the total angle of $C(C_n, \theta)$ is at most 3π then any map that visits at least three edges of $C(k)$ has degree $\alpha \geq 1$.*

Proof. Just as in Theorem 4.1 we can construct homogeneous p-harmonic maps $f : C(C_n, \theta) \to C(k)$ essentially (i.e., up to subdivision) only when $\theta(e) = \theta_0$ for all $e \in E(C_n)$. In each face $C(e)$, f is represented in coordinates by $c(e)\phi_\alpha$, where α is the degree related to the angle θ_0 in the above equation. Then the balancing condition of Definition 2.15 says that f is balanced if and only if the constants $c(e)$ do not depend on $e \in E(\Gamma)$, that is, $c(e) = c$ for all e.

If the domain is isometric to $\mathbb{R}^2$, i.e., if the vertex of $C(C_n, \theta)$ is flat, that means

$$\sum_{e \in E(C_n)} \theta(e) = n\theta_0 = 2\pi.$$

When $n = 2$ and $\theta_0 = \pi$ the corresponding degree of a p-harmonic function $u : S_\pi \to \mathbb{R}_{\geq 0}$ is $\alpha = 1$. If $n \geq 3$ then $\theta_0 \leq 2\pi/3$. According to Lemma 4.6 smaller angles θ_0 correspond to larger degrees α, so the smallest degree happens when $\theta_0 = 2\pi/3$. Solving the relationship between θ_0 and α in the case $\theta_0 = 2\pi/3$ gives

$$\alpha = \frac{17p - 16 + \sqrt{p^2 + 32p - 32}}{16(p-1)}.$$

This is a decreasing function of p so it is bounded below by its limit as $p \to \infty$, namely $\alpha \geq \frac{9}{8}$.

And if the vertex of $C(C_n, \theta)$ is positively curved then $\sum_e \theta(e) = n\theta_0 < 2\pi$. In this case we have $\theta_0 < \pi$, so by Lemma 4.6 we have $\alpha > 1$.

In fact if $\sum_e \theta(e) = n\theta_0 \leq 3\pi$ and the map f visits at least three edges of $C(k)$ then we must have $n \geq 3$ so that $\theta_0 \leq \pi$, and again this means $\alpha \geq 1$. ☐

5. Balanced harmonic maps between cones

Consider a homogeneous map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$, and assume f maps each edge and face of $C(\Gamma)$ to itself. The map f is harmonic if its restriction to each face of $C(\Gamma, \theta)$ is a harmonic map between sectors of the plane. We investigate the balanced (as in Definition 2.17) homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$. According to Proposition 2.12 such a map can only exist if $0 < \alpha\theta(e) \leq \pi$ for each $e \in E(\Gamma)$. The analysis will be different if $\alpha\theta(e) = \pi$ for some edge e (necessarily $\theta(e) = \theta_{\max}$); we will call this the singular degree.

We will focus first on the case of nonsingular α. We begin by constructing homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$, and later investigating which of those maps are balanced.

Lemma 5.1. *A homogeneous harmonic map* $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ *of degree* $0 < \alpha < \pi/\theta_{\max}$ *is uniquely determined by a function* $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$.

Proof. For a fixed edge $e \in E(\Gamma)$ consider the faces $C(e) \subset C(\Gamma, \theta)$ and $C(e) \subset C(\Gamma, \varphi)$, which are isometric to $S_{\theta(e)}$ and $S_{\varphi(e)}$, respectively. Proposition 2.12 says a homogeneous harmonic map $u : S_{\theta(e)} \to S_{\varphi(e)}$ is uniquely determined by the values $|u(1, 0)|$ and $|u(1, \theta_0)|$ (using polar coordinates (r, θ) in $S_{\theta(e)}$).

Fix a function $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$. For each edge $e = (v_0 v_1) \in E(\Gamma)$, fix coordinates in the face $C(e)$ adapted to v_0 for each metric, so that $C(e)$ is identified with $S_{\theta(e)}$ or $S_{\varphi(e)}$ and the edge $C(v)$ is identified with the positive x-axis. Define f in $C(e)$ to be represented in these coordinates by the unique homogeneous harmonic map $u : S_{\theta(e)} \to S_{\varphi(e)}$ with $u(1, 0) = (\rho(v_0), 0)$ and $u(1, \theta(e)) = (\rho(v_1) \cos \varphi(e), \rho(v_1) \sin \varphi(e))$.

Along each edge $C(v)$ of $C(\Gamma)$ the map f is defined independently of the choice of incident edge. Hence f is continuous over the edges of $C(\Gamma, \theta)$ and harmonic in each face. ☐

For ease of notation, we will enumerate the vertices of Γ by $\{v_i\}$. Then a function $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$ can be encoded by the vector $(\rho_i = \rho(v_i))$. If $v_i v_j \in E(\Gamma)$ is an edge, we will also define $\theta_{ij} = \theta(v_i v_j)$ and $\varphi_{ij} = \varphi(v_i v_j)$.

Theorem 5.2. *The homogeneous harmonic map* $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ *of degree* $0 < \alpha < \pi/\theta_{\max}$ *determined by* $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$ *is balanced if and only if, for each vertex* v_i,

$$\sum_{v_j \sim v_i} \frac{\cos(\varphi_{ij})\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij})} = 0.$$

Here $v \sim w$ means that vertices v and w are adjacent, so there is an edge $vw \in E(\Gamma)$.

Proof. In each face $C(v_i v_j)$ choose coordinates adapted to v_i for each metric, so that the face is identified with $S_{\theta_{ij}}$ or $S_{\varphi_{ij}}$ and $C(v_i)$ is identified with the positive x-axis. By Proposition 2.12, in these coordinates $f|_{C(v_i v_j)}$ is represented by

$$u_j(r, \theta) = r^\alpha \left(\rho_i \cos(\alpha\theta) + \frac{\rho_j \cos(\varphi_{ij}) - \rho_i \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} \sin(\alpha\theta), \frac{\sin(\varphi_{ij})}{\sin(\alpha\theta_{ij})} \rho_j \sin(\alpha\theta) \right).$$

And the balancing formula of Definition 2.17 says that f is balanced along $C(v_i)$ if

$$\sum_{v_j \sim v_i} \frac{\partial \chi_j}{\partial \theta}(r, 0) = 0,$$

where $r^\alpha \chi(\theta)$ is the x-coordinate of $u(r, \theta)$.

Combining these two formulas, we see that f is balanced along $C(v_i)$ if

$$\alpha \sum_{v_j \sim v_i} \frac{\cos(\varphi_{ij})\rho_j - \cos(\alpha\theta_{ij})\rho_i}{\sin(\alpha\theta_{ij}))} = 0.$$

In order to be balanced, f must be balanced along each edge of $C(\Gamma, \theta)$, so every sum of this form must vanish. $\square$

Another way to interpret this result is to say a vector (ρ_j) determines a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $0 < \alpha < \pi/\theta_{max}$ if it is in the kernel of the matrix $\Delta_{\alpha\theta}^\varphi$ with entries

$$(\Delta_{\alpha\theta}^\varphi)_{ij} = \begin{cases} -\sum_{v_k \sim v_i} \cot(\alpha\theta_{ik}) & \text{if } i = j, \\ \cos(\varphi_{ij}) \csc(\alpha\theta_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{if } v_i \not\sim v_j. \end{cases}$$

For only certain values of α will $\Delta_{\alpha\theta}^\varphi$ have a nontrivial kernel. That is, only certain degrees α admit balanced homogeneous harmonic maps.

The situation for harmonic maps between cones is much more rigid than that of harmonic functions. In special cases the degrees α and the corresponding balanced homogeneous harmonic maps are easy to describe.

Proposition 5.3. *Suppose $\theta(e) = \theta_0$ and $\varphi(e) = \varphi_0$ for all $e \in E(\Gamma)$. Then there exist balanced harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $0 < \alpha < \pi/\theta_0$ if and only if $\alpha\theta_0 = \varphi_0$. If $\alpha\theta_0 = \varphi_0 = \pi/2$ then every function $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$ determines such an f, but if $\alpha\theta_0 = \varphi_0 \neq \pi/2$ then only the constant vectors $\rho = c\mathbb{1}$ with $\rho_i = c$ for all i do.*

Proof. First suppose $\varphi_0 = \pi/2$. Then $\cos(\varphi_0) = 0$ so the matrix $\Delta_{\alpha\theta}^{\varphi}$ is diagonal with diagonal entries $\deg(v_i)\cot(\alpha\theta_0)$. A diagonal operator only has a kernel if some of its diagonal entries are 0. In this case the only possibility is if $\cot(\alpha\theta_0) = 0$, i.e., $\alpha\theta_0 = \pi/2$. In this case $\Delta_{\alpha\theta_0}^{\varphi} = 0$, so all vectors (ρ_i) are in its kernel.

Now suppose $\varphi_0 \neq \pi/2$. The matrix $\Delta_{\alpha\theta}^{\varphi}$ can be rewritten

$$\Delta_{\alpha\theta}^{\varphi} = \csc(\alpha\theta_0)(\cos(\varphi_0)A - \cos(\alpha\theta_0)D).$$

A vector ρ is in the kernel of this matrix if and only if

$$\cos(\varphi_0)A\rho = \cos(\alpha\theta_0)D\rho,$$

$$D^{-1}A\rho = \frac{\cos(\alpha\theta_0)}{\cos(\varphi_0)}\rho,$$

$$D^{-1/2}(1 - D^{-1/2}AD^{-1/2})D^{1/2}\rho = \frac{\cos(\varphi_0) - \cos(\alpha\theta_0)}{\cos(\varphi_0)}\rho,$$

$$\mathcal{L}D^{1/2}\rho = \frac{\cos(\varphi_0) - \cos(\alpha\theta_0)}{\cos(\varphi_0)}D^{1/2}\rho.$$

Thus $D^{1/2}\rho$ is an eigenvector of $\mathcal{L}$ with eigenvalue $\lambda = 1 - \frac{\cos(\alpha\theta_0)}{\cos(\varphi_0)}$.

If $\alpha\theta_0 \neq \varphi_0$ then $\lambda \neq 0$. But the eigenvectors of $\mathcal{L}$ with nonzero eigenvalue are perpendicular to the null-vector $w = D^{1/2}\mathbb{1}$ with $w_i = \sqrt{\deg v_i}$ for each i. Equivalently,

$$0 = D^{1/2}\rho \cdot w = \sum_i \deg(v_i)\rho_i.$$

But in order to determine a balanced homogeneous harmonic map, all coordinates of ρ must be nonnegative! For $\rho \not\equiv 0$ the above equation precludes this possibility, so the only case left to consider is $\alpha\theta_0 = \varphi_0$.

If $\alpha\theta_0 = \varphi_0$ then a vector ρ in the kernel of $\Delta_{\alpha\theta}^{\varphi}$ must satisfy $D^{1/2}\rho = cw = cD^{1/2}\mathbb{1}$, so $\rho = c\mathbb{1}$. $\qquad\square$

For metrics given by more general functions $\theta, \varphi : E(\Gamma) \to (0, \pi)$, the question of which degrees α admit balanced homogeneous harmonic functions is more subtle. But we can achieve a lower bound on such α more easily than we did for functions into $\mathbb{R}$.

Theorem 5.4. *If there is a balanced homogeneous harmonic map* $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ *then*

$$\alpha \geq \min\left(\frac{\pi}{2\theta_{\max}}, \min\left\{\frac{\varphi(e)}{\theta(e)} \,\Big|\, e \in E(\Gamma)\right\}\right).$$

Proof. We proceed by the contrapositive. If $\alpha < \pi/2\theta_{\max}$ and $\alpha\theta(e) < \varphi(e)$ for each $e \in E(\Gamma)$, we will show that no balanced homogeneous harmonic map of degree α

can exist. Under these assumptions on α we have, for each edge $v_i v_j$,

$$\cos(\alpha\theta_{ij}) > \cos(\varphi_{ij}) \quad \text{and} \quad \cos(\alpha\theta_{ij}) > 0.$$

Now bound the quadratic form associated to $\Delta^\varphi_{\alpha\theta}$ as follows:

$$\rho \cdot \Delta^\varphi_{\alpha\theta} \rho = \sum_{v_i v_j \in E(\gamma)} \frac{2\rho_i \rho_j \cos(\varphi_{ij}) - (\rho_i^2 + \rho_j^2) \cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})}$$

$$\leq - \sum_{v_i v_j \in E(\Gamma)} \cot(\alpha\theta_{ij})(\rho_i - \rho_j)^2 \leq 0.$$

This quadratic form is negative definite; if ρ is not the zero vector then $\rho \cdot \Delta^\varphi_{\alpha\theta} \rho < 0$. Thus no nontrivial balanced homogeneous harmonic maps of degree α can exist. $\square$

We can further study the quadratic form determined by $\Delta^\varphi_{\alpha\theta}$ just as we did in the case of functions to $\mathbb{R}$.

Lemma 5.5. *For each fixed $\rho = (\rho_i) \not\equiv 0$ the quadratic form $\rho \cdot \Delta^\varphi_{\alpha\theta} \rho$ is strictly increasing in α.*

Proof. We computed the quadratic form in the previous result:

$$\rho \cdot \Delta^\varphi_{\alpha\theta} \rho = \sum_{v_i v_j \in E(\gamma)} (2\rho_i \rho_j \cos(\varphi_{ij}) \csc(\alpha\theta_{ij}) - (\rho_i^2 + \rho_j^2) \cot(\alpha\theta_{ij})).$$

Its derivative can now be bounded as follows:

$$\frac{\partial}{\partial\alpha}(\rho \cdot \Delta^\varphi_{\alpha\theta} \rho) = \sum_{v_i v_j \in E(\Gamma)} \theta_{ij} \csc^2(\alpha\theta_{ij})(\rho_i^2 + \rho_j^2 - 2\rho_i \rho_j \cos(\varphi_{ij}) \cos(\alpha\theta_{ij}))$$

$$\geq \sum_{v_i v_j \in E(\Gamma)} \theta_{ij} \csc^2(\alpha\theta_{ij})(\rho_i - \rho_j)^2 \geq 0.$$

Moreover suppose $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta^\varphi_{\alpha\theta} \rho) = 0$ at some fixed $\rho = (\rho_i)$ and some fixed α. Then $\rho_i \rho_j \cos(\varphi_{ij}) \cos(\alpha\theta_{ij}) = \rho_i \rho_j$ for each $v_i v_j \in E(\Gamma)$, and also

$$\sum_{v_i v_j \in E(\Gamma)} \theta_{ij} \csc^2(\alpha\theta_{ij})(\rho_i - \rho_j)^2 = 0.$$

The expression on the left is clearly nonnegative, so for equality to hold ρ must be constant. And it follows from $0 < \varphi_{ij}, \alpha\theta_{ij} < \pi$, $\cos(\varphi_{ij}) \cos(\alpha\theta_{ij}) \neq 1$ that $\rho_i \rho_j = 0$ for all $v_i v_j \in E(\Gamma)$. Since ρ is constant, we have $\rho \equiv 0$. That is, the only way that $\frac{\partial}{\partial\alpha}(\rho \cdot \Delta_{\alpha\theta} \rho) = 0$ at any α is if $\rho \equiv 0$. $\square$

Theorem 5.6. *In the interval $0 < \alpha < \pi/\theta_{\max}$ the eigenvalues of $\Delta^\varphi_{\alpha\theta}$ are strictly increasing functions of α.*

Proof. The entries of the matrix $\Delta_{\alpha\theta}^{\varphi}$ are analytic in α in the interval $0 < \alpha < \pi/\theta_{\max}$. Moreover, $\Delta_{\alpha\theta}^{\varphi}$ is symmetric for each real α. A result in the perturbation theory of eigenvalue problems (see [Rellich 1969, Section 1.1] or [Kato 1966, Theorem 6.1]) says that the eigenvalues and eigenvectors of $\Delta_{\alpha\theta}$ are also analytic in α. Namely, there are $\lambda_j(\alpha) \in \mathbb{R}$ and $w_j(\alpha) \in \mathbb{R}^{\#V(\Gamma)}$ depending analytically on α so that

$$\Delta_{\alpha\theta}^{\varphi} w_j(\alpha) = \lambda_j(\alpha) w_j(\alpha).$$

Without loss of generality, $w_j(\alpha)$ is a unit vector for each j and each α. The eigenvalues can be recovered from the eigenvectors via the formula

$$\lambda_j(\alpha) = w_j(\alpha) \cdot \Delta_{\alpha\theta}^{\varphi} w_j(\alpha).$$

Differentiating this identity with respect to α yields

$$\lambda_j'(\alpha) = 2w_j'(\alpha) \cdot \Delta_{\alpha\theta}^{\varphi} w_j(\alpha) + w_j(\alpha) \cdot \frac{\partial \Delta_{\alpha\theta}^{\varphi}}{\partial \alpha} w_j(\alpha).$$

As each $w_j(\alpha)$ is a unit vector, it lies in the unit sphere of $\mathbb{R}^{\#V(\Gamma)}$. Hence $w'(\alpha)$ is perpendicular to $w_j(\alpha)$. But $\Delta_{\alpha\theta} w_j(\alpha) = \lambda_j(\alpha) w_j(\alpha)$ is parallel to $w_j(\alpha)$, so the first term in $\lambda'(\alpha)$ vanishes. The second term is strictly positive by Lemma 5.5. Thus

$$\lambda_j'(\alpha) > 0. \qquad \square$$

An immediate consequence of Theorem 5.6 is that one can count the number of degrees α admitting balanced homogeneous harmonic functions just as in the case of functions to $\mathbb{R}$. If $\alpha\theta_{ij} = \pi - \epsilon$ for some edge $v_i v_j \in E(\Gamma)$ then the corresponding term in $\rho \cdot \Delta_{\alpha\theta}^{\varphi} \rho$ is

$$\frac{2\rho_i\rho_j \cos(\varphi_{ij}) - (\rho_i^2 + \rho_j^2)\cos(\alpha\theta_{ij})}{\sin(\alpha\theta_{ij})} = \frac{2\rho_i\rho_j \cos(\varphi_{ij}) + \rho_i^2 + \rho_j^2 + O(\epsilon^2)}{\epsilon + O(\epsilon^3)}$$

$$= \frac{(\rho_i - \rho_j)^2}{\epsilon} + (1 - \cos(\varphi_{ij}))\frac{2\rho_i\rho_j}{\epsilon} + O(\epsilon).$$

The only way this term does not approach $+\infty$ is if $\rho_i = \rho_j = 0$.

As $\alpha \to \pi/\theta_{\max}$, the quadratic form $\rho \cdot \Delta_{\alpha\theta}^{\varphi} \rho$ stays bounded only if $\rho_i = \rho_j = 0$ for each edge $v_i v_j$ with $\theta_{ij} = \theta_{\max}$. In other words, ρ_i must be 0 at any vertex v_i incident to an edge e with $\theta(e) = \theta_{\max}$. This defines a subspace W on which the quadratic form remains bounded. For all $\rho \notin W$,

$$\lim_{\alpha \to \pi/\theta_{\max}} \rho \cdot \Delta_{\alpha\theta}^{\varphi} \rho = +\infty.$$

We can thus define a quadratic form Q on W via

$$Q(\rho) = \lim_{\alpha \to \pi/\theta_{\max}} \rho \cdot \Delta_{\alpha\theta}^{\varphi} \rho.$$

Corollary 5.7. *The total number of degrees $0 < \alpha < \pi/\theta_{\max}$ that admit balanced homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$, counted with multiplicity, is the sum of the dimension $\dim W^{\perp}$ and the number of positive eigenvalues of the quadratic form Q defined on W.*

Remark 5.8. If $\Delta_{\alpha\theta}^{\varphi}$ has a nontrivial kernel, the *multiplicity* of α is the dimension of that kernel.

Proof. Following [Theorem 5.6](), the only time that $\Delta_{\alpha\theta}^{\varphi}$ can have a nontrivial kernel is when one of the eigenvalues $\lambda_j(\alpha)$ crosses 0. According to [Theorem 5.4]() the matrices $\Delta_{\alpha\theta}^{\varphi}$ have only negative eigenvalues for α small, so we need only count the number of positive eigenvalues as α approaches $\pi/\theta_{\max}$. Any eigenvalue of $\Delta_{\alpha\theta}^{\varphi}$ that approaches $+\infty$ as $\alpha \to \pi/\theta_{\max}$ is certainly positive, and the remaining eigenvalues of $\Delta_{\alpha\theta}^{\varphi}$ approach the eigenvalues of the quadratic form Q on W as described above. $\square$

5.1. *The singular degree $\alpha = \pi/\theta_{\max}$.*

In stark contrast to [Section 3.2](), balanced homogeneous harmonic maps $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of the singular degree $\alpha = \pi/\theta_{\max}$ are quite rare. First we must describe homogeneous harmonic maps in this case. Let $\Sigma \subset \Gamma$ be the subgraph consisting of all those edges $e \in E(\Gamma)$ with $\alpha\theta(e) = \pi$, along with their incident vertices.

Lemma 5.9. *A homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $\alpha = \pi/\theta_{\max}$ is uniquely determined by a function $\rho : V(\Gamma) \setminus V(\Sigma) \to \mathbb{R}_{\geq 0}$, along with a function*

$$\nu : \{(v, e) \in V(\Sigma) \times E(\Sigma) \mid v \in e\} \to \mathbb{R}$$

subject to the constraints

$$\nu(v, e) \geq \cos(\varphi(e))\nu(w, e) \quad \text{and} \quad \nu(w, e) \geq \cos(\varphi(e))\nu(v, e) \quad \text{for all } e = vw \in E(\Gamma).$$

Proof. If $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ has degree $\alpha = \pi/\theta_{\max}$, let $\rho : V(\Gamma) \to \mathbb{R}_{\geq 0}$ be defined by $f(v, 1) = (v, \rho(v))$. Any vertex $v \in V(\Sigma)$ is incident to an edge e with $\alpha\theta(e) = \pi$, so according to [Proposition 2.12]() $\rho(v) = 0$. Thus we may replace ρ with its restriction to $V(\Gamma) \setminus V(\Sigma)$. Using [Proposition 2.12](), such a ρ uniquely determines the behavior of f on each face $C(e)$ corresponding to $e \in E(\Gamma) \setminus E(\Sigma)$.

Now we turn to those edges $e \in E(\Sigma)$. According to [Proposition 2.12](), the harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ can be represented in coordinates in the face $C(e)$ by the map

$$u(r, \theta) = (x, y) = (c_1 r^{\alpha} \sin(\alpha\theta), c_2 r^{\alpha} \sin(\alpha\theta)).$$

This is a map $S_{\theta(e)} \to S_{\varphi(e)}$, and we are using polar coordinates in the domain and rectangular coordinates in the target.

If the positive x-axis corresponds to a vertex $v_1 \in V(\Gamma)$ and the other edge of $S_{\theta(e)}$ (or $S_{\varphi(e)}$) corresponds to $v_2 \in V(\gamma)$, consider $v_1 = v(v_1, e)$ and $v_2 = v(v_2, e)$ to be the normal derivatives used in the balancing condition of Definition 2.17. That is,

$$v_1 = \frac{\partial x}{\partial \theta}(1, 0) = c_1 \alpha$$

and

$$v_2 = -\frac{\partial}{\partial \theta}\Big|_{r=1, \theta=\theta(e)}(x \cos \varphi(e) + y \sin \varphi(e)) = \alpha(c_1 \cos \varphi(e) + c_2 \sin \varphi(e)).$$

One can easily solve for c_1 and c_2:

$$c_1 = \frac{v_1}{\alpha} \quad \text{and} \quad c_2 = \frac{v_2 - v_1 \cos \varphi(e)}{\alpha \sin \varphi(e)}.$$

Thus the map v uniquely determines the map f in each face $C(e)$ corresponding to $e \in E(\Sigma)$. The only restriction on v is that the image of u must lie in S_{φ_0}. The conditions are very similar to those found in Proposition 2.12, namely $c_2 \geq 0$ and $c_1 \geq c_2 \cot \varphi_0$. Using the above equations to express c_1 and c_2 in terms of v_1 and v_2 yields precisely

$$v_2 \geq v_1 \cos \varphi_0 \quad \text{and} \quad v_1 \geq v_2 \cos \varphi_0. \qquad \square$$

Now we discuss the balancing condition. On vertices $v_i \in V(\Gamma) \setminus V(\Sigma)$ one must still have

$$\sum_{v_j \sim v_i} \frac{\cos(\varphi_{ij})\rho_j - \cos(\alpha \theta_{ij})\rho_i}{\sin(\alpha \theta_{ij})} = 0,$$

just as in Theorem 3.2. And on those vertices $v_i \in V(\Sigma)$ one has $\rho_i = 0$ and the balancing condition reads

$$\sum_{v_i v_j \in E(\Sigma)} v(v_i, v_i v_j) + \sum_{v_i v_j \in E(\Gamma) \setminus E(\Sigma)} \frac{\cos(\varphi_{ij})}{\sin(\alpha \theta_{ij})} \rho_j = 0.$$

It seems unlikely to find a map balanced at just the vertices in $V(\Gamma) \setminus V(\Sigma)$ without taking $\rho \equiv 0$. Since the eigenvalue of $\Delta_{\alpha\theta}^{\varphi}$ are strictly increasing with α the chance that one passes through 0 precisely at a particular value of α seems low. Despite a similar concern in Section 3.2, we described an abundance of balanced harmonic maps in Proposition 3.17 by choosing $\rho \equiv 0$. Unfortunately such a result does not hold in the current situation.

Theorem 5.10. *The only balanced homogeneous harmonic map* $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ *of degree* $\alpha = \pi / \theta_{\max}$, *determined by* $\rho \equiv 0$ *and some function* v *as in Lemma 5.9, is the trivial map.*

Proof. The map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ determined by the functions $\rho \equiv 0$ and ν must satisfy the conditions listed in Lemma 5.9. For an edge $e = vw \in E(\Sigma)$ this says

$$\nu(v, e) \geq \nu(w, e) \cos(\varphi(e)) \quad \text{and} \quad \nu(w, e) \geq \nu(v, e) \cos(\varphi(e)).$$

Adding these together gives

$$\nu(v, e) + \nu(w, e) \geq (\nu(v, e) + \nu(w, e)) \cos(\varphi(e)).$$

Since $\cos(\varphi(e))$ cannot equal 1, we must have

$$\nu(v, e) + \nu(w, e) \geq 0.$$

Summing over all edges of Σ we have

$$0 \leq \sum_{e \in E(\Gamma)} \sum_{v \in e} \nu(v, e) = \sum_{v \in V(\Gamma)} \sum_{e \ni v} \nu(v, e) = 0.$$

The vanishing of the reordered sum follows from the balancing condition at the vertices of Σ. Thus each inequality from Lemma 5.9 must in fact be an equality. So on the edge $e = vw \in E(\Gamma)$,

$$\nu(v, e) = \nu(w, e) \cos(\varphi(e)) = \nu(v, e) \cos^2(\varphi(e)).$$

Since $\cos(\varphi(e))$ never equals 1, $\nu(v, e) = 0$ for each pair (v, e). $\square$

As an immediate consequence we can extend the result of Proposition 5.3, which discussed the case when $\theta(e) = \theta_0$ and $\varphi(e) = \varphi_0$ for all $e \in E(\Gamma)$. That proposition insisted the only balanced harmonic maps had degree $\alpha = \varphi_0/\theta_0$, but did not discuss the possible singular degree $\alpha = \pi/\theta_0$.

Corollary 5.11. *Let $\theta(e) = \theta_0$ and $\varphi(e) = \varphi_0$ for all $e \in E(\Gamma)$. Then there is no nontrivial balanced homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ of degree $\alpha = \pi/\theta_0$.*

Proof. If $\theta(e) = \theta_0$ for all $e \in E(\Gamma)$ and $\alpha = \pi/\theta_0$ then the subgraph Σ is all of Γ itself. According to Lemma 5.9 a homogeneous harmonic map $f : C(\Gamma, \theta) \to C(\Gamma, \varphi)$ is determined by just a function ν. But now Theorem 5.10 says that in order for f to be balanced ν must vanish. In other words, f must be the trivial map that sends all of $C(\Gamma, \theta)$ to the vertex of $C(\Gamma, \varphi)$. $\square$

If one could balance the vertices in $V(\Gamma) \setminus V(\Sigma)$, then balancing the vertices in $V(\Sigma)$ amounts to choosing a function ν from a $2\#E(\Sigma)$-dimensional space subject to $\#V(\Sigma)$ balancing conditions. Such a ν certainly exists since $2\#E(\Sigma) \geq \#V(\Sigma)$ (in fact the inequality is strict if Σ is not a disjoint collection of edges) and the $\#V(\Sigma)$ balancing conditions are independent (no two of them use the same value of ν). But as was the case in Theorem 5.10 the extra inequalities restricting ν are unlikely to be satisfied.

6. Collapsing cones

We deal with the limits of the cones $C(\Gamma, t\theta)$ as $t \to 0$. These spaces can also be seen as tangent cones to the ideal hyperbolic simplicial complexes of [Freidin and Gras Andreu 2020] at punctured vertices.

The naïve approach is to simply let t tend to 0 in $C(\Gamma, t\theta)$. Each face of $C(\Gamma)$ will collapse to a half-line, its bounding edges merging. If Γ is a connected graph then all the faces and edges of $C(\Gamma)$ will collapse down to a single half-line, $\mathbb{R}_{\geq 0}$.

The results from Section 3 would turn into a search for a homogeneous harmonic map $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$, which must be simply a degree-1 map $f(x) = ax$. Such a map also generalizes the results of Section 4, mapping the collapsed domain to a single edge of the k-pod target. If the domain collapses in Section 5 in this way then a similar map can be constructed into a single edge or face of the target. And if only the target collapses then one searches for a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}_{\geq 0}$. Unfortunately no such map exists, as the following result will show.

Proposition 6.1. *If $f : C(\Gamma, \theta) \to \mathbb{R}$ is a balanced homogeneous harmonic map of degree $\alpha > 0$ then the average of f on any ball centered at the vertex of $C(\Gamma, \theta)$ is 0.*

An immediate consequence is that a balanced homogeneous harmonic function $f : C(\Gamma, \theta) \to \mathbb{R}_{\geq 0}$ must be identically 0.

Proof. Just as in Section 3.3, the map $f : C(\Gamma, \theta) \to \mathbb{R}$ can be written as $f(x, t) = \rho(x)t^\alpha$ for some $\rho : \Gamma \to \mathbb{R}$. After choosing coordinate functions $\psi_e : e \to [0, \theta(e)]$ for each $e \in E(\Gamma)$ and defining $\rho_e = \rho \circ \psi_e^{-1}$, ρ is a solution to the problem

$$\rho_e'' = -\alpha^2 \rho_e \quad \text{for all } e \in E(\gamma),$$

$$\sum_{v \in e} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi(e)) = 0 \qquad \text{for all } v \in V(\Gamma).$$

The average of f over a ball centered at the vertex of $C(\Gamma, \theta)$ can be computed as follows:

$$\iint_{B_r(0)} f \, dA = \sum_{e \in E(\Gamma)} \iint_{C_r(0) \cap C(e)} f \, dA = \sum_{e \in E(\Gamma)} \int_0^r \int_0^{\theta(e)} \rho_e(x) t^\alpha t \, dx \, dt$$

$$= \frac{-r^{\alpha+2}}{\alpha^2(\alpha+2)} \sum_{e \in E(\Gamma)} \int_0^{\theta(e)} \rho_e''(x) \, dx = \frac{-r^{\alpha+2}}{\alpha^2(\alpha+2)} \sum_{e \in E(\Gamma)} (\rho_e'(\theta(e)) - \rho_e'(0))$$

$$= \frac{r^{\alpha+2}}{\alpha^2(\alpha+2)} \sum_{v \in V(\Gamma)} \sum_{e \ni v} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi(v))$$

$$= 0. \qquad \qquad \square$$

Remark 6.2. This result can also be seen as the L^2-orthogonality between the eigenfunction ρ with eigenvalue α and the eigenfunction ρ_0 with $\rho_0(x) = 1$, whose eigenvalue is 0. Similar integrations by parts imply that all eigenfunctions with distinct eigenvalues are orthogonal in L^2.

Evidently there is not much happening when we allow the metrics on our cones to collapse, so more care must be taken. As we scale the angle measures by $t \to 0$, we will dilate the cones so that the curves originally at distance 1 from the vertex maintain a constant length. In a face S_{θ_0} this can be achieved with the following four step procedure:

(1) In polar coordinates map $(r, \theta) \mapsto (r^t, \theta)$. This does not change S_{θ_0} as a set, but is useful for defining coefficients later on.

(2) In polar coordinates map $(r, \theta) \mapsto (r, t\theta)$. This maps S_{θ_0} to $S_{t\theta_0}$.

(3) In rectangular coordinates map $(x, y) \mapsto (x - 1, y)$. This has the effect of moving the curve originally at distance 1 from the vertex so it now passes through the origin.

(4) Dilate the region $S_{t\theta_0}$ by a factor of $1/t$. This moves the vertex to the point $(-1/t, 0)$ in rectangular coordinates, and rescales the curve now passing through the origin to have length θ_0 as it did at the start.

This transformation can be accomplished much more concisely in complex coordinates by the map

$$z \mapsto \frac{z^t - 1}{t}.$$

In the limit as $t \to 0$ this transformation converges to $z \mapsto \log z$, and the image of S_{θ_0} under this transformation is a strip, described in rectangular coordinates by

$$R_{\theta_0} = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le \theta_0\}.$$

Thus we define the space

$$C^*(\Gamma, \theta) = \Gamma \times \mathbb{R}$$

with a product metric. The factor $\mathbb{R}$ is endowed with its usual Euclidean metric, and Γ is endowed with a path metric where each edge $e \in E(\Gamma)$ has length $\theta(e)$. Note that in the definition of $C^*(\Gamma, \theta)$ there is absolutely no difficulty if the function θ takes values larger than π.

6.1. *Balanced harmonic functions into* $\mathbb{R}$.

A homogeneous map $u : S_{\theta_0} \to \mathbb{R}$ can be described in polar coordinates as $u(r, \theta) = r^\alpha \rho(\theta)$ for some $\alpha > 0$. The complex exponential transforms the rectangular coordinates (x, y) to the polar coordinates

$(r = e^x, \theta = y)$. So precomposing with the complex exponential gives a map $v = u \circ \exp : R_{\theta_0} \to \mathbb{R}$ given in rectangular coordinates by

$$v(x, y) = u(e^x, y) = e^{\alpha x} \rho(y).$$

When we move to functions $f : C^*(\Gamma, \theta) \to \mathbb{R}$ this suggests that we should consider the class of separated functions, a broader class than homogeneous functions.

Definition 6.3. A map $f : C^*(\Gamma, \theta) \to \mathbb{R}$ is separated if there are functions $\rho : \Gamma \to \mathbb{R}$ and $\tau : \mathbb{R} \to \mathbb{R}$ so that

$$f(x, t) = \rho(x)\tau(t).$$

Since the transformation $z \mapsto \log z$ is conformal on the interior of S_{θ_0}, a homogeneous harmonic function $u(r, \theta) = r^\alpha \rho(\theta)$ is transformed to a *harmonic* function $v(x, y) = e^{\alpha x} \rho(y)$. This suggests that the search for balanced harmonic functions $v : R_{\theta_0} \to \mathbb{R}$ will be equivalent to the search in Section 3.

Of course the class of separated harmonic functions is larger than those maps described above, so some care must be taken. Using coordinates (x, t) on $C^*(\Gamma, \theta) = \Gamma \times \mathbb{R}$, a separated function has the form

$$f(x, t) = \rho(x)\tau(t).$$

By the theory of separation of variables, such a function is harmonic if both ρ and τ are eigenfunctions of the second derivative operator with opposite eigenvalues. Choosing isometries $\psi_e : e \to [0, \theta(e)]$ for each $e \in E(\Gamma)$, let $\rho_e = \rho \circ \psi_e^{-1}$. Then for each e we must have $\tau'' = \lambda\tau$ and $\rho_e'' = -\lambda\rho_e$ for some $\lambda \in \mathbb{R}$.

The eigenfunctions $u'' = cu$ are given by

$$u(x) = \begin{cases} A\cosh(\sqrt{c}x) + B\sinh(\sqrt{c}x) & \text{if } c > 0, \\ A + Bx & \text{if } c = 0, \\ A\cos(\sqrt{-c}x) + B\sin(\sqrt{-c}x) & \text{if } \lambda < 0. \end{cases}$$

Since τ is independent of $e \in E(\gamma)$, the eigenvalue $-\lambda$ for ρ_e must also be independent of e.

Using the isometries $\psi_e : e \to [0, \theta(e)]$ and the representations $\rho_e = \rho \circ \psi_e^{-1}$ the balancing condition of Definition 2.13 can be generalized to separated functions as follows.

Definition 6.4. A separated function $f(x, t) = \rho(x)\tau(t)$ from $C^*(\Gamma, \theta)$ to $\mathbb{R}$ is balanced along the edge $\{v\} \times \mathbb{R} \subset C^*(\Gamma, \theta)$ corresponding to $v \in V(\Gamma)$ if

$$\sum_{e \ni v} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi_e(v)) = 0.$$

The function f is balanced if it is balanced along each edge of $C^*(\Gamma, \theta)$.

Now we are in a position to fully understand balanced separated harmonic functions and their relationship to the balanced homogeneous harmonic functions of Section 3.

Theorem 6.5. *Let* $f : C^*(\Gamma, \theta) \to \mathbb{R}$ *be a separated harmonic function,* $f(x, t) = \rho(x)\tau(t)$. *The function* f *is balanced if and only if one of the following occurs*:

(1) $f(x, t) = A + Bt$, *or*

(2) *there is a number* $\alpha > 0$ *so that* $g(x, t) = \rho(x)t^\alpha : C(\Gamma, \theta) \to \mathbb{R}$ *is a balanced homogeneous harmonic function of degree* α.

In the second case $\tau(t) = A \cosh(\alpha t) + B \sinh(\alpha t) = C e^{\alpha t} + D e^{-\alpha t}$.

Remark 6.6. At the beginning of this subsection we saw that the functions coming from balanced homogeneous functions $\tilde{f} : C(\Gamma, \theta) \to \mathbb{R}$ corresponded to $f = \rho\tau$ with $\tau(t) = C e^{\alpha t}$. We also develop the additional solutions $f(x, t) = A + Bt$, which correspond to $\alpha = 0$. In fact the balanced homogeneous function $\tilde{f}(x, t) = 1$ of degree 0 gives rise to the balanced separated function $f(x, t) = 1$, corresponding to $B = 0$.

See also that the functions $f(x, t) = A$ and $f(x, t) = \rho(x)e^{\alpha t}$ are the only ones with $\lim_{t \to -\infty} f(x, t) = 0$.

Proof. Let $f : C^*(\Gamma, \theta) \to \mathbb{R}$ be a balanced separated harmonic function. With isometries $\psi_e : e \to [0, \theta(e)]$ and representations $\rho_e = f \circ \psi_e^{-1} : [0, \theta(e)] \to \mathbb{R}$ in coordinates, the discussion above implies that the functions ρ_e satisfy

$$\rho_e'' = -\lambda \rho_e \quad \text{for all } e \in E(\Gamma),$$

$$\sum_{e \ni v} (-1)^{\psi_e(v)/\theta(e)} \rho_e'(\psi(v)) = 0 \qquad \text{for all } v \in V(\Gamma).$$

This is precisely the eigenvalue problem stated in Section 3.3. According to [Friedman and Tillich 2004], the solutions ρ to this problem are critical points of the Rayleigh quotient

$$R[\rho] = \frac{\sum_e \|\rho_e'\|^2}{\sum_e \|\rho_e\|^2},$$

where $\|-\|$ indicates the L^2 norm on an interval. Moreover the eigenvalue $-\lambda$ of the solution ρ is precisely $-R[\rho]$, so $\lambda \geq 0$. Let $\alpha \geq 0$ be such that $\lambda = \alpha^2$.

If $\alpha = 0$, then $\rho_e(x) = A_e + B_e x$ for each $e \in E(\Gamma)$, i.e., ρ is edgewise linear. Just as in Proposition 3.10, though, ρ is only balanced if it is constant. In this case $\rho \equiv c$ for some constant c. Since $\lambda = 0$ the function $\tau(t)$ solves $\tau'' = 0$, so τ too is linear, say $\tau(t) = a + bt$. In this case

$$f(x, t) = \rho(x)\tau(t) = ac + bct = A + Bt.$$

If $\alpha > 0$ then according to Section 3.3 the function $\rho : \Gamma \to \mathbb{R}$ fits into a balanced homogeneous harmonic function $g : C(\Gamma, \theta) \to \mathbb{R}$ via

$$g(x, t) = \rho(x)t^\alpha.$$

In this case $\tau'' = \alpha^2 \tau$, so $\tau(t) = A \cosh(\alpha t) + B \sinh(\alpha t) = Ce^{\alpha t} + De^{-\alpha t}$.   □

6.2. Balanced harmonic maps between cones. Just as in Section 5 we will consider only maps $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ between spaces with the same combinatorial structure, and only those maps that map the edges $v \times \mathbb{R}$ and faces $e \times R$ of $C^*(\Gamma)$ to themselves. If $\psi_e : e \to [0, \theta(e)]$ is an isometry for the metric θ, then we define a map $\Psi_e : e \times \mathbb{R} \to R_{\theta(e)}$ by

$$\Psi_e(x, t) = (t, \psi_e(x)).$$

Taking isometries $\psi_e^\theta : e \to [0, \theta(e)]$ and $\psi_e^\varphi : e \to [0, \varphi(e)]$ for each $e \in E(\Gamma)$, ensure that $\psi_e^\theta(v)/\theta(e) = \psi_e^\varphi(v)/\varphi(e)$ for each $v \in e$, that is, ψ_e^θ and ψ_e^φ induce the same orientation on e. Then the map $f|_{e \times \mathbb{R}}$ can be represented in coordinates by

$$u_e = \Psi_e^\varphi \circ f \circ (\Psi_e^\theta)^{-1} : R_{\theta(e)} \to R_{\varphi(e)}.$$

But only some such maps are limits of homogeneous maps under collapsing metrics.

A homogeneous map $u : S_{\theta_0} \to S_{\varphi_0}$ can be described in polar coordinates as $u(r, \theta) = (r^\alpha \rho(\theta), \phi(\theta))$ for some $\alpha > 0$. In Section 5 we used rectangular coordinates on the target to describe such maps, but the translation to polar coordinates is not difficult. The complex exponential transforms the rectangular coordinates (x, y) to the polar coordinates $(r = e^x, \theta = y)$, so conjugating with the complex exponential gives a map $v = \log \circ u \circ \exp : R_{\theta_0} \to R_{\varphi_0}$ given in rectangular coordinates by

$$v(x, y) = \log u(e^x, y) = \log(r^{\alpha x} \rho(y)e^{i\phi(y)}) = (\alpha x + \log \rho(y), \phi(y)).$$

Though the presence of the complex logarithm above turns a harmonic $u : S_{\theta_0} \to S_{\varphi_0}$ into a map $v : R_{\theta_0} \to R_{\varphi_0}$ that is not necessarily harmonic, some (nonharmonic) choices of homogeneous u still result in harmonic v. As both R_{θ_0} and R_{φ_0} are Euclidean, the map v is harmonic if and only if both $\alpha x + \log \rho(y)$ and $\phi(y)$ are. Unfortunately this only happens when $\rho(y) = e^{\beta y + a}$ and $\phi(y) = \gamma y + b$ for some constants β, γ, in which case f is linear.

When we move to functions $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ we will consider only maps that send each line $\{x\} \times \mathbb{R} \subset C^*(\Gamma, \theta)$ to a corresponding line $\{x'\} \times \mathbb{R} \subset C^*(\Gamma, \varphi)$. Representing the map f in coordinates in the face $e \times \mathbb{R}$ by the map $u_e : R_{\theta(e)} \to R_{\varphi(e)}$, the maps u_e take the form

$$u_e(x, y) = (\rho_e(x, y), z_e(y)).$$

For f to be harmonic each u_e must be harmonic. In particular $z_e'' = 0$, so z_e is linear. And in order to map each edge $v \times \mathbb{R}$ to itself we must in fact have

$$z_e(y) = \frac{\varphi(e)}{\theta(e)} y.$$

And the balancing condition of Definition 2.17 can be generalized as follows.

Definition 6.7. The map $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ is balanced along the edge $v \times \mathbb{R}$ corresponding to $v \in V(\Gamma)$ if

$$\sum_{e \ni v} (-1)^{\psi_e^\theta(v)/\theta(e)} \frac{\partial \rho_e}{\partial y} (\Psi_e^\theta(v, t)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

And the map f is balanced if it is balanced along each edge of $C^*(\Gamma, \theta)$.

Unlike in the case of functions to $\mathbb{R}$, the only harmonic maps $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ that came from maps $C(\Gamma, \theta) \to C(\Gamma, \varphi)$ were the piecewise linear ones. And in fact there are very few *balanced* piecewise linear maps.

Proposition 6.8. *If $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$ is a balanced harmonic map whose restriction to each face $e \times \mathbb{R}$ is represented by a linear map $u_e : R_{\theta(e)} \to R_{\varphi(e)}$, then each u_e has the form*

$$u_e(x, y) = \left(ax + b, \frac{\varphi(e)}{\theta(e)} y \right)$$

for some constants a, b that are independent of e. In particular, ρ_e is independent of y.

Proof. If each function ρ_e is linear, then write $\rho_e(x, y) = a_e x + b_e y + c_e$. If $f(v, 0) = (v, \rho(v))$ for some function $\rho : V(\Gamma) \to \mathbb{R}$, then

$$\frac{\partial \rho_e}{\partial y} = b_e = \frac{\rho \circ (\psi_e^\theta)^{-1}(\theta(e)) - \rho \circ (\psi_e^\theta)^{-1}(0)}{\theta(e)}.$$

The balancing condition at a vertex $v \in V(\Gamma)$ then reads

$$0 = \sum_{e \ni v} (-1)^{\psi_e^\theta(v)/\theta(e)} \frac{\rho \circ (\psi_e^\theta)^{-1}(\theta(e)) - \rho \circ (\psi_e^\theta)^{-1}(0)}{\theta(e)}$$

$$= \sum_{w \sim v} \frac{\rho(w) - \rho(v)}{\theta(vw)}.$$

Thus the balancing condition for f says that ρ is in the kernel of the edge-weighted graph Laplacian with edge weights $w(e) = \frac{1}{\theta(e)}$. But the kernel of such an operator is spanned by the vector $\rho = \mathbb{1}$ with $\rho(v) = 1$ for all $v \in V(\Gamma)$.

So if f is balanced then ρ must be constant, i.e., each $b_e = 0$. Then $\rho_e(x, y) = a_e x + c_e$. If $e \cap e' = v$, then we must also have $\rho_e(\Psi_e^\theta(v, t)) = \rho_{e'}(\Psi_{e'}^\theta(v, t))$, i.e.,

$a_e t + c_e = a_{e'} t + c_{e'}$. The connectedness of Γ then implies that a_e and c_e are independent of e, so let $a = a_e$ and $b = c_e$ to find the result claimed. $\square$

Remark 6.9. The above result can then be realized as a companion to case 1 of Theorem 6.5. In fact, in general, any balanced separated harmonic function $g : C^*(\Gamma, \theta) \to \mathbb{R}$ from Theorem 6.5, or any linear combination or convergent series of such separated solutions, gives a balanced harmonic map $f : C^*(\Gamma, \theta) \to C^*(\Gamma, \varphi)$, for any φ, by the formula

$$f(x, t) = (x, g(x, t)).$$

However the facewise linear maps f described in Proposition 6.8 are the only ones for which $|\nabla f|^2$ is bounded on $C^*(\Gamma, \theta)$.

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