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RANK GROWTH OF ELLIPTIC CURVES IN S_4 - AND A_4 -QUARTIC EXTENSIONS OF THE RATIONALS

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We investigate the rank growth of elliptic curves from \mathbb{Q} to S_4 - and A_4 -quartic extensions K/\mathbb{Q} . In particular, we are interested in the quantity $\text{rk}(E/K) - \text{rk}(E/\mathbb{Q})$ for fixed E and varying K . When $\text{rk}(E/\mathbb{Q}) \leq 1$, with E subject to some other conditions, we prove there are infinitely many S_4 -quartic extensions K/\mathbb{Q} over which E does not gain rank, i.e., such that $\text{rk}(E/K) - \text{rk}(E/\mathbb{Q}) = 0$. To do so, we show how to control the 2-Selmer rank of E in certain quadratic extensions, which in turn contributes to controlling the rank in families of S_4 - and A_4 -quartic extensions of \mathbb{Q} .

1. Introduction

1A. Rank growth. For a number field L and an elliptic curve E defined over L , let $E(L)$ be the group of L -rational points of E . The Mordell–Weil theorem says $E(L)$ is a finitely generated abelian group. The rank of $E(L)$, denoted $\text{rk}(E/L)$, has been the subject of much study. Of particular interest here is the behavior of the rank upon base change, i.e., for an extension of number fields K/L , what is $\text{rk}(E/K) - \text{rk}(E/L)$? We call this difference the *rank growth* of E in K/L .

Suppose $L = \mathbb{Q}$ and K/\mathbb{Q} denotes a quadratic extension. Given an elliptic curve E/\mathbb{Q} , a conjecture of Goldfeld predicts that 50% of quadratic twists, E^K , of E have analytic rank zero and 50% have analytic rank one. See Section 3B for a definition and discussion of quadratic twists. Recent work of Smith [2017; 2023a; 2023b] studies the distribution of ℓ^∞ -Selmer groups and proves a version of Goldfeld’s conjecture for 2^∞ -Selmer coranks.

We are interested in studying rank growth in higher degree and nonabelian extensions. In this setting, ranks of quadratic twists, E^K , measure the rank growth of E from \mathbb{Q} to K , which will be essential for rank growth in some larger degree extensions. Previously, and in higher degrees, David, Fearnley, and Kisilevsky [David et al. 2007] have given conjectures for how frequently the rank of an elliptic curve grows in cyclic prime degree extensions. Lemke Oliver and Thorne [2021] gave asymptotic lower bounds for the number of S_d -extensions for which an elliptic

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curve E gains rank. Further, Shnidman and Weiss [2023] study rank growth of elliptic curves from a number field L up to an extension $L(\sqrt[2n]{d})$.

1B. Results for S_4 - and A_4 -quartic extensions. We investigate the rank growth of elliptic curves in S_4 - and A_4 -quartic extensions of the rationals. In what follows, unless stated otherwise, we will always assume E is an elliptic curve defined over \mathbb{Q} . Further, K/\mathbb{Q} will always be an S_4 - or A_4 -quartic extension. That is, one for which the normal closure of K over \mathbb{Q} is an S_4 - or A_4 -Galois extension. For an elliptic curve E/\mathbb{Q} with discriminant Δ_E , we'll consider $\text{rk}(E/K) - \text{rk}(E/\mathbb{Q})$ for many such K . It will often be convenient to use the rank of a Selmer group of an elliptic curve E/K (here we only need the 2-Selmer group) in place of the rank, $\text{rk}(E/K)$. See Definition 3.1 for the definition of the 2-Selmer group, $\text{Sel}_2(E/K)$, of an elliptic curve E/K and Section 3A for a discussion of their utility in our context.

In particular, for an elliptic curve E with Selmer rank zero over the rationals, subject to some mild constraints, we prove there are infinitely many S_4 - or A_4 -quartic extensions over which E does not gain rank. Further, we give a lower bound on the number of such extensions with bounded discriminant with some fixed cubic resolvent field.

Theorem 1.1. *Let E be an elliptic curve over \mathbb{Q} such that $\text{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \simeq S_3$, and $\text{Sel}_2(E/\mathbb{Q}) = 0$. Let K_3 be an S_3 -cubic (resp., C_3 -cubic) extension of \mathbb{Q} such that \tilde{K}_3 and $\mathbb{Q}(E[2])$ are linearly disjoint. Suppose also that there is a place v_0 of K_3 , unramified in \tilde{K}_3 , such that either v_0 is real and $\Delta_E < 0$, or $v_0 \nmid 2\infty$, E has multiplicative reduction at v_0 and $\text{ord}_{v_0}(\Delta_E)$ is odd. Then there are infinitely many S_4 -quartic (resp., A_4 -quartic) extensions K over \mathbb{Q} with cubic resolvent K_3 such that*

- if $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv 0 \pmod{2}$, then $\text{rk}(E/K) = 0$;
- if $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv 1 \pmod{2}$, and also assuming the parity condition $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv \text{rk}(E/K_3) \pmod{2}$, then $\text{rk}(E/K) = 1$.

To prove the theorem above, we utilize two tools. First, we reduce the problem of studying rank growth in our quartic extensions, K/\mathbb{Q} , to one of studying rank growth in certain quadratic subextensions of the Galois closure of K , and thus to studying the rank of certain quadratic twists. Second, we use and further develop some Selmer group machinery of Mazur and Rubin. Indeed, in [Mazur and Rubin 2010], they show that under suitable assumptions, an elliptic curve has infinitely many twists of a prescribed Selmer rank. In subsequent work, Klagsbrun, Mazur, and Rubin [Klagsbrun et al. 2014] study the distribution of 2-Selmer ranks of quadratic twists of an elliptic curve. The quadratic twists we study here, see Definition 1.2 below, are thin in the full family of quadratic twists, so we require a different approach. Nonetheless, we use similar ideas to show one can reduce the 2-Selmer

ranks of the appropriate quadratic twists either to one or zero. Translating from the language of Selmer groups to ranks yields Theorem 1.1.

To emphasize the extra properties imposed on the quadratic twists we consider here, we make the following definition.

Definition 1.2 (square norm twists). Let E be an elliptic curve defined over a number field L , and let F/L be a quadratic extension. We call a quadratic twist E^F over L a *square norm twist* if $F = L(\sqrt{\alpha})$ where $\alpha \in L^\times / (L^\times)^2$ and $N_{L/\mathbb{Q}}(\alpha)$ is a square.

Such twists will be key to keeping track of an associated S_4 -quartic extension when working over a suitable cubic field (see Lemma 2.2).

Likewise, to streamline the discussion and highlight the properties required of the cubic resolvents of our S_4 - or A_4 -quartic extensions, which are S_3 - or C_3 -cubic extensions of the rationals, respectively, always denoted K_3 , we make the next definition.

Definition 1.3 (admissible cubic resolvent). Let K_3 be a cubic extension of the rationals and fix an elliptic curve E/\mathbb{Q} with discriminant Δ_E . Suppose $E(K_3)[2] = 0$, \tilde{K}_3 (the Galois closure of K_3 over \mathbb{Q}) and $\mathbb{Q}(E[2])$ are linearly disjoint, and there is a place v_0 of K_3 , unramified in \tilde{K}_3 , such that either v_0 is real and $(\Delta_E)_{v_0} < 0$ or $v_0 \nmid 2\infty$, E has multiplicative reduction at v_0 and $\text{ord}_{v_0}(\Delta_E)$ is odd. For such an extension,

- if K_3 is an S_3 -cubic extensions, we call K_3 an *admissible S_3 -cubic resolvent for E* ;
- if K_3 is a C_3 -cubic extension, we call K_3 an *admissible C_3 -cubic resolvent for E* .

In the event we do not need to specify one of the two Galois group cases above, we will call a K_3 as in one of the two cases above an *admissible cubic resolvent* for some E .

The restrictions placed on K_3 to make it admissible for some elliptic curve are not overly burdensome. The conditions on the distinguished place should be compared to the assumptions of [Mazur and Rubin 2010, Theorem 1.6].

Theorem 1.1 is a consequence of the following result.

Theorem 1.4. *Let E be an elliptic curve over \mathbb{Q} such that $\text{Gal}(\mathbb{Q}(E[2])/\mathbb{Q}) \simeq S_3$, and $\text{Sel}_2(E/\mathbb{Q}) = 0$, and let K_3 be an admissible cubic resolvent for E . Then there are infinitely many square norm twists, E^F/K_3 , such that*

- if $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv 0 \pmod{2}$, then $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = 0$;
- if $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv 1 \pmod{2}$, then $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = 1$.

Remark 1.5. In Theorems 1.1 and 1.4, and in what follows, when we say “infinitely many” we mean the number of such things with the norm of their discriminant

bounded above by X is $\gg X^{1/2}/\log X^\alpha$ for some $\alpha > 0$. Further, α depends on E and the two-torsion field $K_3(E[2])$. The sizes of these lower bounds, particularly the values of α depending on K_3 and E , are elucidated in Proposition 4.1.

Remark 1.6. In the case that one obtains infinitely many A_4 -quartic extensions for which the Mordell or 2-Selmer rank has some prescribed behavior, the “infinitely many” given by the two theorems above differs from the predicted number of A_4 -quartic extensions only by some factors of $\log X$. In particular, Malle’s conjecture [2004] predicts that the number of A_4 -quartic extensions of a number field L with absolute value of the norm of the relative discriminant bounded by X is asymptotic to $c_L X^{1/2} \log X^{b_L}$ for constants b_L and c_L depending on L .

The number of S_4 -quartic extensions of L with absolute value of the norm of the relative discriminant bounded by X is asymptotic to $d_L X$ for a constant d_L depending on L [Bhargava et al. 2015]. In this case the “infinitely many” given by the two theorems above differs from S_4 -quartic asymptotic by both a logarithmic factor and a factor of $X^{1/2}$.

1C. Layout. In Section 2 we outline the connection between rank growth in S_4 and A_4 -quartic extensions with rank growth in certain quadratic extensions. In Section 3, we recall some facts about Selmer groups and record some related results of Mazur and Rubin [Mazur and Rubin 2010] on quadratic twists. In Sections 4 and 5, we interface the tools of Section 3 with the notion of square norm twists to show we can decrease the 2-Selmer rank of an elliptic curve with a suitable square norm twist and can indeed find many such twists. Finally, in Section 6, we prove the main theorems stated above.

2. Rank growth in S_4 -quartics

2A. Preliminaries. For an extension of number fields L/\mathbb{Q} , we write \tilde{L} for the Galois closure of L in some choice of algebraic closure $\bar{\mathbb{Q}}$.

Consider an S_4 - or A_4 -extension K/\mathbb{Q} with Galois closure \tilde{K} . We are principally concerned with the change (or lack of change) in rank in the group of K -rational points vs. the group of \mathbb{Q} -rational points of E . We will show this rank change is governed by the rank growth in a quadratic extension of fields between \mathbb{Q} and \tilde{K} . K_3 will always denote a cubic resolvent for our quartic extension(s). Of particular interest will be fixing an admissible cubic resolvent K_3 and considering many quartic S_4 - or A_4 -extensions K with cubic resolvent K_3 .

2B. S_4 - and A_4 -quartic extensions. Before we turn to the question of rank growth, we record a few facts about S_4 - and A_4 -quartic extensions with cubic resolvent field K_3 .

Lemma 2.1. *Let K_3/\mathbb{Q} be a cubic extension and F be a quadratic extension of K_3 . There is always an embedding $\text{Gal}(\tilde{F}/\mathbb{Q}) \hookrightarrow S_2 \wr S_3$.*

Quadratic extensions of S_3 -cubics generically have Galois group $S_2 \wr S_3$ over \mathbb{Q} . We are interested in the case where the Galois group is instead $S_4 < S_2 \wr S_3$. Likewise for C_3 -cubics, we wish to consider the case where quadratic extensions of K_3 have Galois group A_4 .

Lemma 2.2. *Fix an S_3 - or C_3 -cubic extension K_3/\mathbb{Q} , and let F denote a quadratic extension of the form $K_3(\sqrt{\alpha})$, where $\alpha \in K_3$ and $N_{K_3/\mathbb{Q}}(\alpha)$ is a square.*

- *If K_3/\mathbb{Q} is an S_3 -cubic, then $\text{Gal}(\tilde{F}/\mathbb{Q}) \simeq S_4$. Further, there is a one-to-one correspondence between such quadratic extensions F/K_3 and S_4 -quartic extensions of \mathbb{Q} with cubic resolvent K_3 .*
- *If K_3/\mathbb{Q} is a C_3 -cubic, then $\text{Gal}(\tilde{F}/\mathbb{Q}) \simeq A_4$. Further, there is a three-to-one correspondence between such quadratic extensions F/K_3 and A_4 -quartic extensions of \mathbb{Q} with cubic resolvent K_3 .*

This correspondence is described in detail in, for example, Section 2 of [Cohen and Thorne 2016].

Remark 2.3. Fix a cubic field, K_3 . For each $L = K_3(\sqrt{\alpha})$ where $\alpha \in K_3^\times / (K_3^\times)^2$ and $N_{K_3/\mathbb{Q}}(\alpha)$ is a square, there is an S_4 - or A_4 -quartic extension K/\mathbb{Q} with cubic resolvent K_3 . Use Lemma 2.2 and observe that $\tilde{L} = \tilde{K}$. For our purposes, we will fix K_3 and range over quadratic extensions of K_3 as in Lemma 2.2 to range over S_4 -quartic extensions of \mathbb{Q} with cubic resolvent K_3 . We will then consider a fixed elliptic curve E over these extensions, and consider various differences in rank.

2C. Measuring rank growth in S_4 - and A_4 -quartic extensions. Our aim now is to show that measuring rank growth of an elliptic curve E from \mathbb{Q} to S_4 - or A_4 -quartic extensions K/\mathbb{Q} with cubic resolvent K_3/\mathbb{Q} is a matter of measuring the rank growth of E from K_3 to a quadratic extension F/K_3 , namely the quadratic extension of Lemma 2.2.

Lemma 2.4. *Let E be an elliptic curve defined over \mathbb{Q} , K be an S_4 - or A_4 -quartic extension of \mathbb{Q} with cubic resolvent K_3 , and F/K_3 be the quadratic extension of Lemma 2.2. Then*

$$(2-1) \quad \text{rk}(E/K) - \text{rk}(E/\mathbb{Q}) = \text{rk}(E/F) - \text{rk}(E/K_3).$$

The rank relation of Lemma 2.4 is a manifestation of the following more general fact [Dokchitser and Dokchitser 2010, page 572]. Suppose L/k is a Galois extension of number fields with $G = \text{Gal}(L/k)$, and E/k is an elliptic curve. For $H \leq G$,

write $\mathbb{1}_H$ for the trivial character on H . If there are subextensions K_i/k and K'_j/k of L , cut out by subgroups H_i and H'_j of G , such that

$$(2-2) \quad \bigoplus_i \text{Ind}_{H_i}^G \mathbb{1}_{H_i} \simeq \bigoplus_j \text{Ind}_{H'_j}^G \mathbb{1}_{H'_j}$$

as complex representations of G , then

$$(2-3) \quad \sum_i \text{rk}(E/K_i) = \sum_j \text{rk}(E/K'_j).$$

The see this relation on the ranks, let χ_L be the character of the representation of the complex representation of the Mordell–Weil group, $E(L) \otimes \mathbb{C}$, and note that

$$(2-4) \quad \text{rk}(E/K_i) = \langle \text{Ind}_{H_i}^G \mathbb{1}_{H_i}, \chi_L \rangle.$$

The same statement can be made for the rank of E over each K'_j as well. Using (2-4) together with (2-2) yields (2-3).

Proof of Lemma 2.4. Consider the following subgroups of $G = \text{Gal}(\tilde{K}/\mathbb{Q}) \simeq S_4$ which fix the subfields K , K_3 , and F of \tilde{K} . Let $H_K \simeq S_3$ be the subgroup fixing K , $H_{K_3} \simeq D_8$ be the subgroup fixing K_3 , and $H_F \simeq V_4$, the Klein four group, be the subgroup fixing F . Then one can verify

$$(2-5) \quad \text{Ind}_{H_K}^G \mathbb{1}_{H_K} \oplus \text{Ind}_{H_{K_3}}^G \mathbb{1}_{H_{K_3}} \simeq \mathbb{1} \oplus \text{Ind}_{H_F}^G \mathbb{1}_{H_F}.$$

The lemma now follows from (2-3). Note that relations like that of (2-5) are an example of those provided in [Bartel and Dokchitser 2015]. □

3. The 2-Selmer groups and quadratic twists

In the previous section, we established that the rank growth from \mathbb{Q} to K is the same as the rank growth from K_3 to a quadratic extension of K_3 determined by K . So we may restrict ourselves to the study of rank growth in quadratic extensions. This is governed by the theory of quadratic twists.

3A. The 2-Selmer group. We now recall the definition of the 2-Selmer group for an elliptic curve E over a number field L . The multiplication-by-2 map on E gives rise to a short exact sequence of Galois modules:

$$0 \rightarrow E[2] \rightarrow E(\bar{\mathbb{Q}}) \xrightarrow{\times 2} E(\bar{\mathbb{Q}}) \rightarrow 0.$$

This in turn yields a long exact sequence of Galois cohomology groups, which, after quotienting appropriately, gives rise to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E(L)/2E(L) & \longrightarrow & H^1(L, E[2]) & \longrightarrow & H^1(L, E)[2] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_v E(L_v)/2E(L_v) & \longrightarrow & \prod_v H^1(L_v, E[2]) & \longrightarrow & \prod_v H^1(L_v, E)[2] \longrightarrow 0 \end{array}$$

Now, define subgroups $H_f^1(L_v, E[2])$ of each local cohomology $H^1(L_v, E[2])$ as

$$H_f^1(L_v, E[2]) := \text{Image}(E(L_v)/2E(K_v) \rightarrow H^1(L_v, E[2])).$$

Definition 3.1. The 2-Selmer group of E/L , denoted $\text{Sel}_2(E/L)$, is the \mathbb{F}_2 -vector space defined by the exactness of the sequence

$$0 \rightarrow \text{Sel}_2(E/L) \rightarrow H^1(L, E[2]) \rightarrow \bigoplus_v H^1(L_v, E[2])/H_f^1(L_v, E[2]).$$

We may think of the elements of the 2-Selmer group as being the classes in $H^1(L, E[2])$ which, for every place v of L , land in the image of $E(L_v)/2E(L_v)$; that is, elements of $H^1(L, E[2])$ which everywhere locally satisfy the local conditions determined by $H_f^1(L_v, E[2])$.

Further, the 2-Selmer group fits into a short exact sequence

$$0 \rightarrow E(L)/2E(L) \rightarrow \text{Sel}_2(E/L) \rightarrow \text{III}(E/L)[2] \rightarrow 0,$$

where $\text{III}(E/L)[2]$ are the elements of the Shafarevich–Tate group of E/L with order dividing 2. We have

$$(3-1) \quad \dim_{\mathbb{F}_2} \text{Sel}_2(E/L) = (\text{rk}(E/L) + \dim_{\mathbb{F}_2} E(L)[2]) + \dim_{\mathbb{F}_2} \text{III}(E/L)[2]$$

and further that $\text{rk}(E/L) \leq \dim_{\mathbb{F}_2} \text{Sel}_2(E/L)$. It is by this relation that we'll access the ranks of the various elliptic curves and twists discussed later in the paper.

3B. Quadratic twists. Suppose our elliptic curve E/L is given in short Weierstrass form:

$$E : y^2 = x^3 + Ax + B$$

with $A, B \in L$. A quadratic twist, E^F/L , of E/L is an elliptic curve of the form

$$(3-2) \quad E^F : \delta y^2 = x^3 + Ax + B,$$

where $\delta \in L^\times/(L^\times)^2$ and $F = L(\sqrt{\delta})$. With a change of variables, one can put (3-2) in short Weierstrass form:

$$E^F : y^3 = x^3 + a\delta^2x + b\delta^3.$$

An elliptic curve E/L and a quadratic twist E^F/L are not, in general, isomorphic as elliptic curves over L but *are* isomorphic as elliptic curves over F . In particular, quadratic twists will be the main tool for measuring growth in quadratic extensions as we have

$$\text{rk}(E^F/L) = \text{rk}(E/F) - \text{rk}(E/L).$$

3C. The 2-Selmer of quadratic twists. Mazur and Rubin [2010] gave results in which understanding the behavior of an elliptic curve E/L and its 2-Selmer group, $\text{Sel}_2(E/L)$, locally at only a few places of L is sufficient to, under some mild conditions, understand the relation between $\dim_{\mathbb{F}_2} \text{Sel}_2(E/L)$ and $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/L)$ for some quadratic twists, E^F , of E .

One defines the 2-Selmer group of a twist E^F/L following the definition laid out in Section 3A, just with E^F in place of E . It is important to note that $E^F[2]$ and $E[2]$ are isomorphic as Galois modules, so we may view both Selmer groups inside $H^1(L, E[2])$.

Lemma 2.4 shows that understanding the rank growth in quadratic extensions will be sufficient for understanding rank growth in the quartic extensions of interest. In the remainder of this section we will record some results from [Mazur and Rubin 2010] on how local information about E relates the 2-Selmer rank of E to the 2-Selmer rank of quadratic twists of E .

Lemma 3.2 [Mazur and Rubin 2010, Lemma 2.2]. *With the notation as above:*

- If $v \nmid 2\infty$, then $\dim_{\mathbb{F}_2} H_f^1(L_v, E[2]) = \dim_{\mathbb{F}_2} E(L_v)[2]$.
- If $v \nmid 2\infty$ and E has good reduction at v , then

$$H_f^1(L_v, E[2]) \cong E[2]/(\text{Frob}_v - 1)E[2].$$

Definition 3.3. Suppose T is a finite set of places of L . Let loc_T be the sum of the localization maps for each place of T ,

$$\text{loc}_T : H^1(L, E[2]) \rightarrow \bigoplus_{v \in T} H^1(L_v, E[2]).$$

Also set

$$V_T = \text{loc}_T(\text{Sel}_2(E/L)) \subset \bigoplus_{v \in T} H_f^1(L_v, E[2]).$$

We finish the section by recalling two results from [Mazur and Rubin 2010] that we'll later use to control the rank of the 2-Selmer groups in the quadratic extension of Lemma 2.4.

Lemma 3.4 [Mazur and Rubin 2010, Proposition 3.3]. *Let E/L be an elliptic curve, and let F/L be a quadratic extension in which the following places of L split:*

- all primes where E has additive reduction;
- all places v where E has multiplicative reduction such that $\text{ord}_v(\Delta_E)$ is even;
- all primes above 2;
- all real places v with $(\Delta_E)_v > 0$.

Further, suppose that all v where E has multiplicative reduction and $\text{ord}_v(\Delta_E)$ is odd are unramified in F/L .

Let T be the set of finite primes \mathfrak{p} of L such that F/L is ramified at \mathfrak{p} and $E(L_{\mathfrak{p}})[2] \neq 0$. Then,

$$(3-3) \quad \dim_{\mathbb{F}_2} \text{Sel}_2(E^F/L) = \dim_{\mathbb{F}_2} \text{Sel}_2(E/L) - \dim_{\mathbb{F}_2}(V_T) + d$$

for some d such that

$$0 \leq d \leq \dim_{\mathbb{F}_2} \left(\bigoplus_{\mathfrak{p} \in T} H_f^1(L_{\mathfrak{p}}, E[2])/V_T \right)$$

and

$$d \equiv \dim_{\mathbb{F}_2} \left(\bigoplus_{\mathfrak{p} \in T} H_f^1(L_{\mathfrak{p}}, E[2])/V_T \right) \pmod{2}.$$

An immediate consequence of the above is the following lemma.

Lemma 3.5 [Mazur and Rubin 2010, Corollary 3.4]. *For an elliptic curve E/L and for F/L and T as defined in Lemma 3.4, we have:*

(1) *If $\dim_{\mathbb{F}_2}(\bigoplus_{\mathfrak{p} \in T} H_f^1(L_{\mathfrak{p}}, E[2])/V_T) \leq 1$, then*

$$\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/L) = \dim_{\mathbb{F}_2} \text{Sel}_2(E/L) - 2 \dim_{\mathbb{F}_2} V_T + \sum_{\mathfrak{p} \in T} \dim_{\mathbb{F}_2} H_f^1(L_{\mathfrak{p}}, E[2]).$$

(2) *If T is empty, then $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/L) = \dim_{\mathbb{F}_2} \text{Sel}_2(E/L)$.*

We will use Lemma 3.5, setting L to be some admissible cubic resolvent, say K_3 , to understand the 2-Selmer rank of some square norm twists by controlling $\dim_{\mathbb{F}_2} \text{loc}_T(\text{Sel}_2(E/K_3))$ and $\dim_{\mathbb{F}_2} H_f^1((K_3)_{\mathfrak{p}}, E[2])$ for each $\mathfrak{p} \in T$.

4. Twisting by square norm extensions

We will consider elliptic curves E/\mathbb{Q} together with some K_3 which will always be assumed to be an *admissible cubic resolvent* for E as in Definition 1.3. Recall that among other conditions, we require $E(K_3)[2] = 0$ and K_3 to $\mathbb{Q}(E[2])$ be linearly disjoint.

We are concerned with quadratic twists E^F over K_3 where we impose conditions on F . In Section 1 we introduced Definition 1.2 defining *square norm twists* to keep track of conditions on the twists. Recall that for an elliptic curve defined over a number field L , E/L , these are quadratic twists E^F/L of E/L where $F = L(\sqrt{\alpha})$, $\alpha \in L^\times/(L^\times)^2$, and $N_{L/\mathbb{Q}}(\alpha)$ is a square.

We will be interested in the application of the definition above where $L = K_3$, which, as above, will be the cubic resolvent for some quartic S_4 -extensions of \mathbb{Q} .

Further, define $N_r^\square(E, X)$ as follows to count quadratic extensions F/K_3 with bounded conductor, $\mathfrak{f}(F/K_3)$, that give square norm twists E^F of E with 2-Selmer group of dimension r :

$$N_r^\square(E, X) = \#\{F = K_3(\sqrt{\alpha}) \mid \alpha \in K_3^\times / (K_3^\times)^2, N_{K_3/\mathbb{Q}}(\alpha) \text{ a square,} \\ \dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = r, N_{K_3/\mathbb{Q}}\mathfrak{f}(F/K_3) < X\}.$$

With that in mind, we prove the following.

Proposition 4.1. *Fix an S_3 - or C_3 -cubic field K_3/\mathbb{Q} , an elliptic curve E/\mathbb{Q} , and a nonnegative even integer r . Suppose there exists a square norm twist, E^L/K_3 , of E/K_3 , with $\dim_{\mathbb{F}_2} \text{Sel}(E^L/K_3) = r$. Then we have:*

- If $\text{Gal}(K_3/\mathbb{Q}) \simeq S_3$ and $\text{Gal}(K_3(E[2])/K_3) \simeq S_3$, then

$$N_r^\square(E, X) \gg X^{1/2} / \log(X)^{5/6}.$$

- If $\text{Gal}(K_3/\mathbb{Q}) \simeq S_3$ and $\text{Gal}(K_3(E[2])/K_3) \simeq C_3$, then

$$N_r^\square(E, X) \gg X^{1/2} / \log(X)^{2/3}.$$

- If $\text{Gal}(K_3/\mathbb{Q}) \simeq C_3$ and $\text{Gal}(K_3(E[2])/K_3) \simeq S_3$, then

$$N_r^\square(E, X) \gg X^{1/2} / \log(X)^{8/9}.$$

- If $\text{Gal}(K_3/\mathbb{Q}) \simeq C_3$ and $\text{Gal}(K_3(E[2])/K_3) \simeq C_3$, then

$$N_r^\square(E, X) \gg X^{1/2} / \log(X)^{7/9}.$$

Remark 4.2. In Section 5, we will prove the existence of the quadratic extension L/K_3 from the hypotheses of Proposition 4.1. With this, we will use the relationship between the rank growth from \mathbb{Q} to K and the rank growth K_3 to a quadratic extensions F/K_3 to prove Theorem 1.1.

The rest of this section will be devoted to proving Proposition 4.1. Before proceeding, we first need to enumerate ideals in K_3 that allow us to get quadratic extensions for square norm twists. We will then show for each such ideal, there is a square norm twist of E corresponding to that ideal.

Lemma 4.3. *Suppose E/\mathbb{Q} is an elliptic curve where K_3 is an admissible S_3 -cubic resolvent for E and $\text{Gal}(K_3(E[2])/K_3)$ is C_3 or S_3 . Let S be the set of the elements of order 3 in $\text{Gal}(K_3(E[2])/K_3)$, and N be a ray class field of K_3 . Then the number of ideals \mathfrak{b} of K_3 such that*

- $N\mathfrak{b} < X$ and $[\mathfrak{b}, N/K_3] = 1$, and
- for every prime ideal \mathfrak{p} dividing \mathfrak{b} , $N\mathfrak{p}$ is a square and $\text{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$

is asymptotic to

$$(C + o(1)) \frac{X^{1/2}}{\log(X)^{1-(1/2)(|S|/[K_3(E[2]:K_3])}}$$

as $X \rightarrow \infty$, where C is some positive constant, $[-, N/K_3]$ is the global Artin symbol and N is the ideal norm.

Proof. An unramified, noninert rational prime p can split as a product of primes in two ways in the ring of integers, \mathcal{O}_{K_3} , of the cubic field K_3 . Either $p\mathcal{O}_{K_3} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ where each factor has degree one, or $p\mathcal{O}_{K_3} = \mathfrak{p}_1\mathfrak{p}_2$ where one factor has degree one and one factor has degree two. Primes of degree two only appear as factors in the latter splitting type.

First we will count rational primes p such that $p\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q}$ where the residue degrees of the prime factors are $f(\mathfrak{p}|p) = 1$ and $f(\mathfrak{q}|p) = 2$. For each such $p < X$ we get one prime \mathfrak{q} of K_3 of square norm such that $N\mathfrak{q} < X^2$. Let $\mathcal{S}_{(1,2)}$ be the set of such rational primes, i.e.,

$$(4-1) \quad \mathcal{S}_{(1,2)} := \{p \in \mathbb{N} \text{ prime} \mid p\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q}, f(\mathfrak{p}|p) = 1, f(\mathfrak{q}|p) = 2\}$$

Also set

$$\mathcal{P}_{(1)} := \{\mathfrak{p} \subset \mathcal{O}_{K_3} \text{ prime ideal} \mid (\mathfrak{p} \cap \mathbb{Z})\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q}, f(\mathfrak{p}|\mathfrak{p} \cap \mathbb{Z}) = 1, f(\mathfrak{q}|\mathfrak{p} \cap \mathbb{Z}) = 2\}$$

and

$$(4-2) \quad \mathcal{Q}_{(2)} := \{\mathfrak{q} \subset \mathcal{O}_{K_3} \text{ prime ideal} \mid f(\mathfrak{q}|\mathfrak{q} \cap \mathbb{Z}) = 2\}.$$

Likewise, define

- $\mathcal{S}_{(1,2)}(X) := \{p \in \mathcal{S}_{(1,2)} \mid p < X\}$;
- $\mathcal{P}_{(1)}(X) := \{\mathfrak{p} \in \mathcal{P}_{(1)} \mid N\mathfrak{p} < X\}$;
- $\mathcal{Q}_{(2)}(X) := \{\mathfrak{q} \in \mathcal{Q}_{(2)} \mid N\mathfrak{q} < X\}$.

With this notation, the discussion above amounts to

$$(4-3) \quad \#\mathcal{S}_{(1,2)}(X) = \#\mathcal{P}_{(1)}(X) = \#\mathcal{Q}_{(2)}(X^2).$$

A rational prime p belongs to $\mathcal{S}_{(1,2)}$ if and only if $\text{Frob}_p(\tilde{K}_3/\mathbb{Q})$ acts on the three cosets of $\text{Gal}(\tilde{K}_3/\mathbb{Q})/\text{Gal}(\tilde{K}_3/K_3)$ like a transposition. Via the Chebotarev density theorem, this happens with probability $\#\{\text{transpositions in } S_3\}/\#S_3 = \frac{1}{2}$. That is, the density of $\mathcal{S}_{(1,2)}$ in the set of all rational primes is $\frac{1}{2}$.

We can conclude the Dirichlet density, δ_{dir} , of the set of primes \mathfrak{p} in K_3 corresponding to each $p \in \mathcal{S}_{(1,2)}$ is also $\frac{1}{2}$, i.e.,

$$(4-4) \quad \delta_{\text{dir}}(\{\mathfrak{p} \mid f(\mathfrak{p}|\mathfrak{p} \cap \mathbb{Z}) = 1, (\mathfrak{p} \cap \mathbb{Z})\mathcal{O}_{K_3} = \mathfrak{p}\mathfrak{q}, f(\mathfrak{q}|\mathfrak{p} \cap \mathbb{Z}) = 2\}) = \delta_{\text{dir}}(\mathcal{P}_{(1)}) = \frac{1}{2}.$$

Now define some notation. Set $M = K_3(E[2])$ and recall that S is the set of all elements of order 3 in $\text{Gal}(M/K_3)$ and note that S is a union of conjugacy classes when $\text{Gal}(M/K_3) = C_3$ and is a conjugacy class when $\text{Gal}(M/K_3) = S_3$. Now, set

- $\mathcal{P} = \{\mathfrak{p} \in \mathcal{P}_{(1)} \mid \mathfrak{p} \text{ unramified in } NM/K_3, \text{Frob}_{\mathfrak{p}}(M/K_3) \subset S\}$;
- $\mathcal{Q} = \{\mathfrak{q} \in \mathcal{Q}_{(2)} \mid \mathfrak{p} \text{ unramified in } NM/K_3, \text{Frob}_{\mathfrak{q}}(M/K_3) \subset S\}$;
- $\mathcal{N} = \{\mathfrak{a} \mid \text{squarefree product of ideals from } \mathcal{P}\}$;
- $\mathcal{N}_1 = \{\mathfrak{a} \mid \text{squarefree product of ideals from } \mathcal{P}, [\mathfrak{a}, N/K_3] = 1\}$;
- $\mathcal{R}_1 = \{\mathfrak{b} \mid \text{squarefree product of ideals from } \mathcal{Q}, [\mathfrak{b}, N/K_3] = 1\}$.

Our goal is now to access the number of ideals in $\mathcal{N}_1(X)$ via the Dirichlet series $\sum_{\mathfrak{a} \in \mathcal{N}_1} N\mathfrak{a}^{-1}$ and a Tauberian theorem of Wintner. Indeed, we'll see knowing $\mathcal{N}_1(X)$ suffices to understand $\mathcal{R}_1(X^2)$.

To that end, for an irreducible character $\chi : \text{Gal}(N/K_3) \rightarrow \mathbb{C}^\times$ where we will write $\chi(\mathfrak{a})$ for $\chi([\mathfrak{a}, N/K_3])$, set

$$(4-5) \quad f_\chi(s) := \sum_{\mathfrak{a} \in \mathcal{N}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \prod_{\mathfrak{p} \in \mathcal{P}} (1 + \chi(\mathfrak{p}) N\mathfrak{p}^{-s}).$$

Note that $\mathfrak{p} \in \mathcal{P}$ can't be above a rational prime p which splits completely in \tilde{K}_3 ; if it split completely in \tilde{K}_3 , then it splits completely in K_3 , too. Thus, $\text{Frob}_{\mathfrak{p}}(\tilde{K}_3/K_3)$ isn't trivial.

Let τ be the nontrivial element of $\text{Gal}(\tilde{K}_3/K_3)$ and set

$$(4-6) \quad S' = \{\tau\} \times S \subset \text{Gal}(\tilde{K}_3 M/K_3) = \text{Gal}(\tilde{K}_3/K_3) \times \text{Gal}(M/K_3)$$

and

$$\delta(S, \chi) = \begin{cases} 0 & \text{if } \chi \text{ nontrivial,} \\ \frac{1}{2} \frac{|S|}{[M : K_3]} & \text{if } \chi \text{ trivial,} \end{cases}$$

noting that, in the χ trivial case,

$$\frac{1}{2} \frac{|S|}{[M : K_3]} = \#S' / \#\text{Gal}(\tilde{K}_3 M/K_3).$$

We write $g_1(s) \sim g_2(s)$ for two complex functions g_1, g_2 on the half plane $\Re s > 1$ if $g_1(s) - g_2(s)$ extends to a holomorphic function on the half plane $\Re s \geq 1$. Now, starting from the logarithm of (4-5) and using the Chebotarev density theorem,

$$\log f_\chi(s) \sim \sum_{\mathfrak{p} \in \mathcal{P}} \chi(\mathfrak{p}) N\mathfrak{p}^{-s} \sim \delta(S, \chi) \sum_{\mathfrak{p} \text{ prime}} \chi(\mathfrak{p}) N\mathfrak{p}^{-s} \sim \delta(S, \chi) \log \frac{1}{s-1}.$$

Using character orthogonality, observe

$$(4-7) \quad \frac{1}{[N : K_3]} \sum_{\chi} f_{\chi}(s) = \frac{1}{[N : K_3]} \sum_{\mathfrak{a} \in \mathcal{N}} N \mathfrak{a}^{-s} \sum_{\chi} \chi(\mathfrak{a}) \\ = \sum_{\mathfrak{a} \in \mathcal{N}_1} N \mathfrak{a}^{-s} = (s - 1)^{-(1/2)(|S|/[M:K_3])} h(s),$$

where the first two sums range over irreducible characters χ of $\text{Gal}(N/K_3)$, and where $h(s)$ is a nonzero, holomorphic function for $\Re s \geq 1$.

Applying a Tauberian theorem of Wintner [1942] to (4-7), we obtain

$$(4-8) \quad \#\mathcal{N}_1(X) = (C + o(1)) \frac{X}{\log(X)^{1-(1/2)(|S|/[M:K_3])}}.$$

Now, if $\mathfrak{a} \in \mathcal{N}_1$ we have, for some positive integer m , $\mathfrak{a} = \prod_{i=1}^m \mathfrak{p}_i$, where $\mathfrak{p}_i \in \mathcal{P}$. For each rational prime p_i below \mathfrak{p}_i , we have $p_i \mathcal{O}_{K_3} = \mathfrak{p}_i \mathfrak{q}_i$ where $\mathfrak{q}_i \in \mathcal{Q}$. Set $\mathfrak{b} = \prod_{i=1}^m \mathfrak{q}_i$.

First, we'll show $\text{Frob}_{\mathfrak{q}_i}(M/K_3) \subset S$. If $E : y^2 = f(T)$, consider the cubic extension $L = K_3(T)/(f(T))$, where $f(T) \in \mathbb{Q}[x]$ is some cubic polynomial, between M and K_3 . If $\text{Gal}(M/K_3) = C_3$, then $M = L$. We can look at how $f(T)$ factors modulo \mathfrak{p}_i and \mathfrak{q}_i . The only way for $f(T)$ to be irreducible modulo \mathfrak{q}_i (i.e., over $\mathbb{F}_{p_i^2}$) is for $f(T)$ to be irreducible modulo \mathfrak{p}_i ; this happens precisely when $\text{Frob}_{\mathfrak{p}_i}(M/K_3) \subset S$. If $f(T) \pmod{\mathfrak{q}_i}$ is irreducible, $\text{Frob}_{\mathfrak{q}_i}(M/K_3)$ has order 3. That is, demanding $\text{Frob}_{\mathfrak{p}_i}(M/K_3) \subset S$ forces $\text{Frob}_{\mathfrak{q}_i}(M/K_3) \subset S$.

Second, since each $\mathfrak{p}_i \mathfrak{q}_i$ is principle, knowing $[\mathfrak{a}, N/K_3] = 1$ suffices to show $[\mathfrak{b}, N/K_3] = 1$, too. Thus, $\mathfrak{b} \in \mathcal{R}_1$.

Finally, since $N\mathfrak{b} = (N\mathfrak{a})^2$, we have established a bijection between $\mathcal{N}_1(X)$ and $\mathcal{R}_1(X^{1/2})$ by mapping $\mathfrak{a} \mapsto \mathfrak{b}$.

This and (4-8) give us

$$\#\mathcal{R}_1(X) \sim (C + o(1)) \frac{X^{1/2}}{\log(X)^{1-(1/2)(|S|/[M:K_3])}}$$

for some positive constant C , as needed. □

We now state and prove the analogue of Lemma 4.3 in the case that K_3 is an admissible C_3 -cubic resolvent.

Lemma 4.4. *Suppose E/\mathbb{Q} is an elliptic curve where K_3 is an admissible C_3 -cubic resolvent for E and $\text{Gal}(K_3(E[2])/K_3)$ is C_3 or S_3 . Let S be the set of the elements of order 3 in $\text{Gal}(\mathbb{Q}(E[2])/\mathbb{Q})$, and N be an abelian extension of K_3 . Then, the number of ideals \mathfrak{b} of K_3 such that*

- $N\mathfrak{b} < X$, $N\mathfrak{b}$ is a square, and $[\mathfrak{b}, N/K_3] = 1$; and
- for every prime ideal \mathfrak{p} dividing \mathfrak{b} , $\text{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$

is asymptotic to

$$(D + o(1)) \frac{X^{1/2}}{\log(X)^{1-(1/2)(|S|/[K_3(E[2]):\mathbb{Q}])}}$$

for some real, positive constant D , and where $[-, N/K_3]$ is the global Artin symbol and N is the ideal norm.

Proof. Since K_3 is an admissible C_3 -cubic resolvent, we have that K_3 and $\mathbb{Q}(E[2])$ are linearly disjoint. Setting $M = K_3(E[2])$, we have $\text{Gal}(M/\mathbb{Q}) = C_3 \times S_3$. First, some notation. Define the following sets:

- $\mathcal{P}_{\mathbb{Q}} = \{p \in \mathbb{N} \text{ prime} \mid \text{Frob}_p(M/\mathbb{Q}) \subset \{1\} \times S, p \text{ unramified in } \tilde{N}K_3\mathbb{Q}(E[2])\}$;
- $\mathcal{A} = \{a \mid a \text{ a squarefree product of } p \in \mathcal{P}_{\mathbb{Q}}\}$;
- $\mathcal{A}_1 = \{a \in \mathcal{A} \mid [(a), \tilde{N}/\mathbb{Q}] = 1\}$ where $[-, \tilde{N}/\mathbb{Q}]$ is the global Artin symbol.

Let \tilde{N} be the normal closure of N over \mathbb{Q} . We will use the triviality of the Artin symbol $[-, \tilde{N}/\mathbb{Q}]$ to obtain the triviality of the Artin symbol $[-, N/\mathbb{Q}]$ as in the statement of the lemma. Now, for a character $\psi : \text{Gal}(\tilde{N}/\mathbb{Q}) \rightarrow \mathbb{C}^\times$, and writing $\psi(a)$ for $\psi([(a), \tilde{N}/\mathbb{Q}])$, let

$$f_\psi(s) := \sum_{a \in \mathcal{A}} \psi(a) a^{-s}.$$

For two functions g_1 and g_2 defined on the complex half plane $\Re s > 1$, write $g_1(s) \sim g_2(s)$ to mean $g_1(s)$ and $g_2(s)$ differ by a function which is holomorphic on $\Re s \geq 1$. Taking log of the $f_\psi(s)$, substituting the Taylor series for $\log(1 - x)$ and truncating the Taylor series after one term, one arrives at

$$(4-9) \quad \log f_\psi(s) = \sum_{p \in \mathcal{P}_{\mathbb{Q}}} \log(1 + \psi(p)p^{-s}) \sim \sum_{p \in \mathcal{P}_{\mathbb{Q}}} \psi(p)p^{-s} \sim \delta_\psi \log \frac{1}{s-1},$$

where, using the Chebotarev density theorem,

$$\delta_\psi = \begin{cases} 0 & \text{if } \psi \text{ is nontrivial,} \\ \frac{|S|}{[M:\mathbb{Q}]} & \text{if } \psi \text{ is trivial.} \end{cases}$$

Now, using character orthogonality and summing over irreducible characters ψ of $\text{Gal}(\tilde{N}/\mathbb{Q})$, we have

$$(4-10) \quad \frac{1}{[\tilde{N}:\mathbb{Q}]} \sum_{\psi} f_\psi(s) = \frac{1}{[\tilde{N}:\mathbb{Q}]} \sum_{a \in \mathcal{A}} a^{-s} \sum_{\psi} \psi(a) = \sum_{a \in \mathcal{A}_1} a^{-s}.$$

But also, using (4-9), we have

$$(4-11) \quad \frac{1}{[\tilde{N}:\mathbb{Q}]} \sum_{\psi} f_\psi(s) = g(s)(s-1)^{-|S|/[M:\mathbb{Q}]},$$

where $g(s)$ is holomorphic and nonzero on $\Re s \geq 1$. Thus, via (4-10) and (4-11),

$$\sum_{a \in \mathcal{A}_1} a^{-s} = g(s)(s - 1)^{-|S|/[M:\mathbb{Q}]}$$

Applying a Tauberian theorem of Wintner [1942] yields

$$\#\{a \in \mathcal{A}_1 \mid a < X\} = (C + o(1)) \frac{X}{(\log X)^{1-|S|/[M:\mathbb{Q}]}}$$

for some positive, real C .

Now, suppose $a \in \mathcal{A}_1$ and $a = \prod_{i=1}^r p_i$, where the p_i are distinct primes and $p_i \in \mathcal{P}_{\mathbb{Q}}$ and each p_i splits completely in \mathcal{O}_{K_3} . Set $\mathfrak{a} = a\mathcal{O}_{K_3}$. Then \mathfrak{a} decomposes into prime ideals as $\mathfrak{a} = \prod_{i=1}^r \mathfrak{p}_i \mathfrak{p}'_i \mathfrak{p}''_i$ where \mathfrak{p}_i , \mathfrak{p}'_i , and \mathfrak{p}''_i are the three primes above p_i . For each \mathfrak{p}_i above a prime p_i , pick another prime \mathfrak{p}'_i of K_3 above p_i (there are two choices), and set $\mathfrak{b} = \prod_{i=1}^r \mathfrak{p}_i \mathfrak{p}'_i$. Note $N_{K_3/\mathbb{Q}} \mathfrak{b} = a^2$.

In this way, counting $a \in \mathcal{A}_1$ with $a < X$ gives a way of counting ideals \mathfrak{b} in K_3 such that $N_{K_3/\mathbb{Q}} \mathfrak{b} < X^{1/2}$ such that $N_{K_3/\mathbb{Q}} \mathfrak{b}$ is a square and, for each prime \mathfrak{p} dividing \mathfrak{b} , $\text{Frob}_{\mathfrak{p}}(M/K_3) \subset S$ and $[\mathfrak{b}, N/K_3] = 1$. The lemma follows. \square

For an elliptic curve E and each of the ideals enumerated in Lemma 4.3, there is a twist of E in which the 2-Selmer rank remains the same.

Lemma 4.5. *Keeping the notation of Lemma 4.3, if \mathfrak{b} is an ideal of K_3 such that*

- $N\mathfrak{b} < X$,
- if a prime ideal \mathfrak{p} divides \mathfrak{b} , then $N\mathfrak{p}$ is a square,
- $\text{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$, and
- $[\mathfrak{b}, N/K_3] = 1$,

then there is a quadratic extension F/K_3 of conductor \mathfrak{b} such that

$$\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3).$$

Proof. This is Proposition 4.2 of [Mazur and Rubin 2010], with $N = K_3(8\Delta_E\infty)$, the ray class field of K_3 modulo $8\Delta_E$ and all archimedean places of K_3 , applied to the relevant ideals, which are a subset of the ideals discussed in that result. \square

For an elliptic curve E and each of the ideals enumerated in Lemma 4.4, there is a twist of E in which the 2-Selmer rank remains the same.

Lemma 4.6. *Keeping the notation of Lemma 4.4, if \mathfrak{b} is an ideal of K_3 such that*

- $N\mathfrak{b} < X$,
- if a prime ideal \mathfrak{p} divides \mathfrak{b} , then $N\mathfrak{p}$ is a square,
- $\text{Frob}_{\mathfrak{p}}(K_3(E[2])/K_3) \subset S$, and
- $[\mathfrak{b}, N/K_3] = 1$,

then there is a quadratic extension F/K_3 of conductor \mathfrak{b} such that

$$\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3).$$

Proof. The proof is the same as that of Lemma 4.5. □

We are now ready to prove the main result of the section. We follow exactly the strategy of the proof of [Mazur and Rubin 2010, Theorem 1.4] with the additional step of keeping track of the square norm condition of the involved quadratic twists.

Proof of Proposition 4.1. As in Lemmas 4.3 and 4.4, let S be the set of order-3 elements in $\text{Gal}(K_3(E[2])/K_3)$. Then if $\text{Gal}(K_3(E[2])/K_3) \simeq S_3$,

$$\frac{|S|}{[K_3(E[2]) : K_3]} = \frac{1}{3},$$

and if $\text{Gal}(K_3(E[2])/K_3) \simeq \mathbb{Z}/3\mathbb{Z}$,

$$\frac{|S|}{[K_3(E[2]) : K_3]} = \frac{2}{3}.$$

We'll consider the case that K_3 is an admissible S_3 -cubic resolvent; there are two subcases to consider:

(1) Suppose $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) = r$. By Lemmas 4.3 and 4.5, the number of square norm twists E^F/K_3 such that $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = r$ is

$$\gg \frac{X^{1/2}}{\log X^{1-(1/2)(|S|/[K_3(E[2]):K_3])}}.$$

(2) Suppose $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \neq r$. We have assumed there is a square norm twist E^L/K_3 such that $\dim_{\mathbb{F}_2} \text{Sel}_2(E^L/K_3) = r$. Note that a square norm twist of a square norm twist results in the square norm twist. That is, a square norm twist $(E^L)^{F'}$ of E^L is itself a square norm twist E^F of E . Now the result follows from Case (1) applied to E^L .

If instead K_3 is an admissible C_3 -cubic resolvent for E , the proof is the same as above, but with Lemmas 4.4 and 4.6 in place of Lemmas 4.3 and 4.5, respectively. □

5. Decreasing the 2-Selmer rank

Our strategy will be to use Lemma 3.5(2) to understand the 2-Selmer rank of square norm twists. We'll then use Proposition 4.1 to show there are many square norm twist with prescribed 2-Selmer rank assuming we *have already* a square norm twist of that prescribed 2-Selmer rank. We'll show a square norm twist with that prescribed 2-Selmer rank *must exist* by showing we can take square norm twists that reduce the 2-Selmer rank by two; this is the content of Proposition 5.1.

Proposition 5.1 below can be viewed as analogous to Proposition 5.1(iii) of [Mazur and Rubin 2010]. Except, instead of decreasing the 2-Selmer rank by 1 via a quadratic twist obtained by controlling one local condition, we decrease the 2-Selmer rank by 2 via a square norm twist obtained by controlling two local conditions. Those two local conditions are obtained from two primes in the cubic resolvent above the same rational prime.

Proposition 5.1. *Let E be an elliptic curve defined over \mathbb{Q} and let K_3 be an admissible cubic resolvent for E . Suppose further that*

$$\dim_{\mathbb{F}_2} \text{Sel}_2(E/\mathbb{Q}) = 0 \quad \text{and} \quad \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \geq 2.$$

Then there exists a square norm twist E^F/K_3 such that

$$\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) - 2.$$

Proof. Let Δ_E be the discriminant of (a minimal model of) E . For the admissible cubic resolvent, K_3 , of E , set $M = K_3(E[2])$. Note that M/K_3 is a Galois S_3 -extension, since K_3 is as in Definition 1.3, and $K_3(\sqrt{\Delta_E})/K_3$ is an intermediate quadratic extension. Let v_0 be the distinguished place of K_3 guaranteed by Definition 1.3.

Let Σ be a finite set that contains the following places of K_3 : all infinite places of K_3 , places of bad reduction for E , and all primes above 2. Now set $\mathfrak{d} = \prod_{v \in \Sigma \setminus \{v_0\}} v$. Let $K_3(8\mathfrak{d})$ be the ray class field of K_3 with modulus $8\mathfrak{d}$ and let $K_3[8\mathfrak{d}]$ be the maximal extension between $K_3(8\mathfrak{d})$ and K_3 whose degree is a power of 2.

Let \tilde{K}_3 be the Galois closure of K_3 over \mathbb{Q} and $\mathfrak{D} = \prod_{V|v, v|\mathfrak{d}} V$ be the product of places of \tilde{K}_3 above the places of K_3 dividing \mathfrak{d} . Let L be the maximal extension¹ of \tilde{K}_3 between \tilde{K}_3 and the ray class field of \tilde{K}_3 with modulus $8\mathfrak{D}$. Note $L \supseteq K_3[8\mathfrak{d}]$, $[L : K_3]$ is a power of 2, and L is Galois over \mathbb{Q} .

By assumption, \tilde{K}_3 and $\mathbb{Q}(E[2])$ are linearly disjoint over \mathbb{Q} . $K_3[8\mathfrak{d}]$ and M are linearly disjoint as extensions of K_3 since v_0 is ramified in $K_3(\sqrt{\Delta_E})$ but not in $K_3[8\mathfrak{d}]$. By the conditions on v_0 of Definition 1.3, v_0 does not ramify from K_3 to \tilde{K}_3 , and hence is unramified in L . So likewise, the same consideration of $K_3(\sqrt{\Delta_E})$ shows M and L are linearly disjoint over K_3 .

Let σ be an element of the absolute Galois group of K_3 such that $\sigma|_M$ is a transposition in $\text{Gal}(M/K_3) \simeq \text{Aut}(E[2])$ and $\sigma|_L = 1$. The former condition implies $E[2]/(\sigma - 1)E[2] \simeq \mathbb{F}_2$. The latter condition implies $\sigma|_{K_3[8\mathfrak{d}]} = 1$.

For the rest of the proof, fix a nonzero map $\phi : \text{Sel}_2(E/K_3) \rightarrow E[2]/(\sigma - 1)E[2]$. By [Mazur and Rubin 2010, Lemma 3.5], there is an element $\gamma \in G_{K_3}$ for which $\gamma = \sigma$ when restricted to $MK_3[8\mathfrak{d}]$ and $c(\gamma) = \phi(c)$ for all $c \in \text{Sel}_2(E/K_3)$.

Let N be a Galois extension of \mathbb{Q} containing M and L for which the restriction of $\text{Sel}_2(E/K_3)$ to N is zero. For instance, take N to be the Galois closure (over \mathbb{Q})

¹Note that if K_3 is C_3 -cubic, then $L = K_3[8\mathfrak{d}]$ since $\tilde{K}_3 = K_3$.

of the compositum of M , L , and the fixed field of the kernel of the restriction to $\text{Hom}(G_M, E[2])$ of every $c \in \text{Sel}_2(E/K_3)$.

Let \mathcal{P}_γ be the set of primes \mathfrak{p} of K_3 for which $\mathfrak{p} \notin \Sigma$ and $\text{Frob}_\mathfrak{p}(N/K_3) = \gamma|_N$. By the Chebotarev density theorem, the natural density of \mathcal{P}_γ among the primes of K_3 is positive, i.e.,

$$\delta_\gamma := \lim_{X \rightarrow \infty} \frac{\#\{\text{primes } \mathfrak{p} \text{ of } K_3 \mid \text{Frob}_\mathfrak{p}(N/K_3) = \gamma|_N, \mathfrak{p} \notin \Sigma, N\mathfrak{p} < X\}}{\#\{\text{primes } \mathfrak{p} \text{ of } K_3 \mid N\mathfrak{p} < X\}} > 0.$$

Now, let \mathcal{P}_{sp} be the set of primes of K_3 with inertia degree one over \mathbb{Q} , that is, the primes \mathfrak{p} for which the rational prime ideal $\mathfrak{p} \cap \mathbb{Z}$ splits completely in K_3 . Recall that \mathcal{P}_{sp} has natural density one among the primes of K_3 . In particular, this means we can pick a prime $\mathfrak{p}_1 \in \mathcal{P}_\gamma \cap \mathcal{P}_{\text{sp}}$. If we could not, then \mathcal{P}_γ (which has positive density) would be contained in the complement of \mathcal{P}_{sp} (which has density zero). Let p be the rational prime below \mathfrak{p}_1 , and let \mathfrak{p}_2 and \mathfrak{p}_3 be the other two primes of K_3 above p , in other words, $p\mathcal{O}_{K_3} = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$.

Our goal is now to construct a suitable square norm twist from two of $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$. We can understand the 2-Selmer group of this twist using Lemma 3.5(1), which requires us to compute both $H_f^1((K_3)_{\mathfrak{p}_i}, E[2])$ and $\text{loc}_{\mathfrak{p}_i} \text{Sel}_2(E/K_3)$.

First, consider the localization at \mathfrak{p}_1 . Since $\text{Frob}_{\mathfrak{p}_1} = \gamma$ when restricted to N (and $\sigma = \gamma$ when restricted to MK_3 [8d]) we have both that

$$(5-1) \quad H_f^1((K_3)_{\mathfrak{p}_1}, E[2]) \simeq E[2]/(\sigma - 1)E[2] \simeq \mathbb{F}_2$$

and $\phi(c) = c(\gamma)$ for all $c \in \text{Sel}_2(E/K_3)$. The localization map

$$\begin{aligned} \text{loc}_{\mathfrak{p}_1} : \text{Sel}_2(E/K_3) &\rightarrow H_f^1((K_3)_{\mathfrak{p}_1}, E[2]) \simeq E[2]/(\text{Frob}_{\mathfrak{p}_1} - 1)E[2] \\ &\simeq E[2]/(\sigma - 1)E[2] \simeq \mathbb{F}_2 \end{aligned}$$

is given by evaluation of cocycles at $\text{Frob}_{\mathfrak{p}_1}$, so we can identify

$$\text{loc}_{\mathfrak{p}_1}(\text{Sel}_2(E/K_3)) = \phi(\text{Sel}_2(E/K_3))$$

as subspaces of \mathbb{F}_2 . Since ϕ is nonzero,

$$(5-2) \quad \dim_{\mathbb{F}_2} \text{loc}_{\mathfrak{p}_1}(\text{Sel}_2(E/K_3)) = 1.$$

It remains to understand the localizations at \mathfrak{p}_2 and \mathfrak{p}_3 . Since p splits completely in K_3 , there is an equality of local fields

$$(K_3)_{\mathfrak{p}_1} = (K_3)_{\mathfrak{p}_2} = (K_3)_{\mathfrak{p}_3} = \mathbb{Q}_p$$

and so, together with (5-1), we have

$$(5-3) \quad \begin{aligned} H_f^1((K_3)_{\mathfrak{p}_1}, E[2]) &= H_f^1((K_3)_{\mathfrak{p}_2}, E[2]) \\ &= H_f^1((K_3)_{\mathfrak{p}_3}, E[2]) = H_f^1(\mathbb{Q}_p, E[2]) \simeq \mathbb{F}_2. \end{aligned}$$

Beginning from the following commutative diagram, we will consider the localization of $\text{Sel}_2(E/K_3)$ at primes of K_3 above p and localization at p :

$$\begin{array}{ccc} H^1(K_3, E[2]) & \xrightarrow{\bigoplus_{i=1}^3 \text{loc}_{\mathfrak{p}_i}} & \bigoplus_{i=1}^3 H^1((K_3)_{\mathfrak{p}_i}, E[2]) \\ \text{cores}_{K_3/\mathbb{Q}} \downarrow & & \bigoplus_{i=1}^3 \text{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p} \downarrow \\ H^1(\mathbb{Q}, E[2]) & \xrightarrow{\text{loc}_{(p)}} & H^1(\mathbb{Q}_p, E[2]) \end{array}$$

where the vertical map on the left side,

$$\text{cores}_{K_3/\mathbb{Q}} : H^1(K_3, E[2]) \rightarrow H^1(\mathbb{Q}, E[2]),$$

is determined by corestriction on Galois cohomology induced by the norm map $N_{K_3/\mathbb{Q}} : K_3 \rightarrow \mathbb{Q}$ (see [Milne 2020, Example 1.29] or [Serre 1997]). The vertical map on the right side is the sum of corestriction maps:

$$\begin{aligned} \bigoplus_{i=1}^3 \text{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p} : \bigoplus_{i=1}^3 H^1((K_3)_{\mathfrak{p}_i}, E[2]) &\rightarrow H^1(\mathbb{Q}_p, E[2]), \\ (c_1, c_2, c_3) &\mapsto \sum_{i=1}^3 \text{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p}(c_i). \end{aligned}$$

Restricting the left column to the 2-Selmer groups of E over K_3 and \mathbb{Q} and to restricted cohomology on the right column, together with (5-3), we have

$$\begin{array}{ccc} \text{Sel}_2(E/K_3) & \xrightarrow{\text{cores}_{K_3/\mathbb{Q}}} & \text{Sel}_2(E/\mathbb{Q}) = 0 \\ \downarrow \bigoplus_{i=1}^3 \text{loc}_{\mathfrak{p}_i} & & \downarrow \text{loc}_p \\ \bigoplus_{i=1}^3 H_f^1((K_3)_{\mathfrak{p}_i}, E[2]) & \xrightarrow{\bigoplus_{i=1}^3 \text{cores}_{(K_3)_{\mathfrak{p}_i}/\mathbb{Q}_p}} & H_f^1(\mathbb{Q}_p, E[2]) \simeq \mathbb{F}_2 \\ \downarrow \simeq & \nearrow v_1+v_2+v_3 & \\ H_f^1(\mathbb{Q}_p, E[2])^{\oplus 3} \simeq \mathbb{F}_2^3 & & \end{array}$$

where the diagonal map $\mathbb{F}_2^3 \rightarrow \mathbb{F}_2$ is coordinatewise addition of vectors in \mathbb{F}_2^3 modulo 2. For $c \in \text{Sel}_2(E/K_3)$ we have $\text{loc}_p \text{cores}(c) = 0$ since $\text{Sel}_2(E/\mathbb{Q}) = 0$. Hence,

$$(5-4) \quad \text{loc}_{\mathfrak{p}_1}(c) + \text{loc}_{\mathfrak{p}_2}(c) + \text{loc}_{\mathfrak{p}_3}(c) = 0 \quad \text{in } \mathbb{F}_2.$$

By (5-2) there is an element $c \in \text{Sel}_2(E/K_3)$ for which $\text{loc}_{\mathfrak{p}_1}(c) = 1$ viewed in \mathbb{F}_2 . Combining this with (5-4), there is exactly one prime $\mathfrak{p}_i \in \{\mathfrak{p}_2, \mathfrak{p}_3\}$ for which $\text{loc}_{\mathfrak{p}_i}(c) = 1$; suppose, without loss of generality, it is \mathfrak{p}_2 . Whence,

$$(5-5) \quad \dim_{\mathbb{F}_2} \text{loc}_{\mathfrak{p}_2} \text{Sel}_2(E/K_3) = 1.$$

Finally, we will twist E/K_3 by a quadratic extension F/K_3 ramified only at \mathfrak{p}_1 and \mathfrak{p}_2 to get our desired result.

Let \mathfrak{P} be a prime of L above \mathfrak{p}_1 . Since L/\mathbb{Q} is Galois, we have

$$\text{Frob}_{\mathfrak{P}}(L/\mathbb{Q})^{f(\mathfrak{p}_1/p)} = \text{Frob}_{\mathfrak{P}}(L/\mathbb{Q}) = \text{Frob}_{\mathfrak{P}}(L/K_3) = 1,$$

i.e., p splits completely in L , and so p splits completely in $K_3[8\delta]$.

Since our choice of Frobenius class for p is trivial when restricted just to $K_3[8\delta]$, $\text{Frob}_{\mathfrak{p}_1}(K_3[8\delta]/K_3) = \text{Frob}_{\mathfrak{p}_2}(K_3[8\delta]/K_3) = 1$, and since $[K_3(8\delta) : K_3[8\delta]]$ is odd, there will be an odd integer, say h , such that $(\mathfrak{p}_1\mathfrak{p}_2)^h$ is principal with generator α such that $\alpha \equiv 1 \pmod{8\delta}$ and α positive at all real embeddings except possibly v_0 .

We are now in a position to construct the quadratic extension F/K_3 by which we will twist E : Define $F = K_3(\sqrt{\alpha})$. Only the primes \mathfrak{p}_1 and \mathfrak{p}_2 of K_3 ramify in F . Set $p(x) = x^2 - \alpha$ and note $p'(1)^2 = 4$. For any $\mathfrak{q} \in \Sigma \setminus \{v_0\}$, since $\alpha \equiv 1 \pmod{8\delta}$, we also have $p(1) = 1 - \alpha \equiv 0 \pmod{4\mathfrak{q}}$. From Hensel's lemma (see, e.g., [Eisenbud 1995, Theorem 7.3] for an applicable statement), it follows $p(x)$ has a root in $(K_3)_{\mathfrak{q}}$. So $(K_3)_{\mathfrak{q}} \otimes F = (K_3)_{\mathfrak{q}}^2$, that is, \mathfrak{q} splits in F . Thus all primes in $\Sigma \setminus \{v_0\}$ split in F .

Also, $N(\alpha) = N(\mathfrak{p}_1)N(\mathfrak{p}_2) = p^{2h}$, so the quadratic twist E^F/K_3 of E/K_3 is a square norm twist.

Finally, apply Lemma 3.5(1) with $T = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $F = K_3(\sqrt{\alpha})$ as above, (5-2) and (5-5), we get

$$\begin{aligned} & \dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) \\ &= \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) - 2 \dim_{\mathbb{F}_2} V_T + \sum_{\mathfrak{t} \in T} \dim_{\mathbb{F}_2} H_f^1((K_3)_{\mathfrak{t}}, E[2]) \\ &= \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) - 2 \dim_{\mathbb{F}_2} \text{loc}_{\{\mathfrak{p}_1\}}(\text{Sel}_2(E/K_3)) - 2 \dim_{\mathbb{F}_2} \text{loc}_{\{\mathfrak{p}_2\}}(\text{Sel}_2(E/K_3)) \\ & \quad + \dim_{\mathbb{F}_2} H_f^1((K_3)_{\mathfrak{p}_1}, E[2]) + \dim_{\mathbb{F}_2} H_f^1((K_3)_{\mathfrak{p}_2}, E[2]) \\ &= \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) - 2 \end{aligned}$$

Noting again the twist by F above is a square norm twist, we have the desired result. \square

6. Proofs of the main theorems

We are now ready to prove Theorem 1.4. We'll then show how it implies Theorem 1.1. Again, the "infinitely many" of both theorems is quantified by Proposition 4.1.

Proof of Theorem 1.4. Let E and K_3 be as in Theorem 1.4. If $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv 0 \pmod{2}$, repeated application of Proposition 5.1 gives a square norm twist L such that $\dim_{\mathbb{F}_2} \text{Sel}_2(E^L/K_3) = 0$. Once we have L , Proposition 4.1 provides infinitely many more square norm twists F with $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = 0$.

Likewise, if $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3) \equiv 1 \pmod{2}$, the argument above provides infinitely many more square norm twists F with $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = 1$. \square

Now we can prove Theorem 1.1 as a consequence of Theorem 1.4.

Proof of Theorem 1.1. Let E and K_3 be as in the statement of the theorem. Theorem 1.1 is essentially an immediate consequence of Theorem 1.4 coupled with the upper bound the dimension of the 2-Selmer group provides for the rank.

As in Theorem 1.4, there are two cases. In the first case, $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3)$ is even. In this case, Theorem 1.4 provides infinitely many square norm twists E^F/K_3 for which $\dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) = 0$.

From (3-1), we have $\text{rk}(E/K_3) \leq \dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3)$. Thus, our infinitely many square norm twists E^F of 2-Selmer rank zero give us

$$0 = \dim_{\mathbb{F}_2} \text{Sel}_2(E^F/K_3) \geq \text{rk}(E^F/K_3) = \text{rk}(E/F) - \text{rk}(E/K_3).$$

If K/\mathbb{Q} is the S_4 -quartic corresponding to quadratic extension F/K_3 corresponding to each square norm twist, then having no rank growth from K_3 to F means we have no rank growth from \mathbb{Q} to K for infinitely many K .

In the second case, $\dim_{\mathbb{F}_2} \text{Sel}_2(E/K_3)$ is even. Then, in the same way as above, there are infinitely many square norm twists E^F of 2-Selmer rank one. The result follows if we assume the parity of the rank and 2-Selmer dimension are the same. \square

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
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