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LIOUVILLE EQUATIONS ON COMPLETE SURFACES WITH NONNEGATIVE GAUSS CURVATURE

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We study finite total curvature solutions of the Liouville equation $\Delta u + e^{2u} = 0$ on a complete surface (M, g) with nonnegative Gauss curvature. It turns out that the asymptotic behavior of the solution separates into two extremal cases: on the one end, if the solution decays not too fast, then (M, g) must be isometric to the standard Euclidean plane; on the other end, if (M, g) is isometric to the flat cylinder $S^1 \times \mathbb{R}$, then solutions must decay linearly and can be completely classified.

1. Introduction

In their seminal work [1991], Chen and Li obtained the radial symmetry of the solution of

$$\Delta u + e^{2u} = 0 \tag{1-1}$$

on $\mathbb{R}^2$, provided that $\int_{\mathbb{R}^2} e^{2u}\,dx < \infty$. Putting the center of symmetry at the origin and up to a rescaling, we have

$$u(x) = \ln \frac{2}{1+|x|^2}.$$

The geometric meaning of above equation is that the conformal metric $g = e^{2u} g_0$ has constant Gauss curvature 1. It is tempting to think that g is isometric to the standard round sphere. It is indeed true as the solution is the pull back of the round metric via stereographic projection. Nevertheless this line of reasoning is valid only if one establishes the precise asymptotic behavior of u at ∞, so that the metric extends to a smooth metric on the sphere from $\mathbb{R}^2$. The readers are referred to [Li and Tang 2020] for this line of reasoning; see also [Gui and Li 2021] regarding metric completion of solutions to more general equations.

The assumption $\int_{\mathbb{R}^2} e^{2u}\,dx < \infty$ is natural since there are infinitely many solutions to (1-1) with $\int_{\mathbb{R}^2} e^{2u}\,dx = \infty$. One way to obtain such a solution is to pull back the

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spherical metric via a univalent holomorphic map from $\mathbb{C}$ to $\overline{\mathbb{C}}$. Recently, there appeared some interesting studies on (1-1) subject to $\int_{\mathbb{R}^2} e^{2u}\,dx = \infty$. Eremenko, Gui, Li and Xu [Eremenko et al. 2022] give a complete classification of solutions of (1-1) which are bounded from above. We also refer to [Gui and Li 2023; Bergweiler et al. 2023; Lytchak 2023] for some studies on (1-1) from a geometric point of view.

The story in higher dimensions was accomplished even earlier. For $n \geq 3$, let u be a positive solution of

$$\Delta u + u^{(n+2)/(n-2)} = 0. \tag{1-2}$$

We refer to it as the scalar curvature equation as the conformal metric $g = u^{4/(n-2)} g_0$ has positive constant scalar curvature. Gidas, Ni and Nirenberg [Gidas et al. 1981] first proved the radial symmetry of the solutions under the assumption $u(x) \sim O(|x|^{2-n})$ as $|x| \to \infty$. This can be viewed as an analytical proof of a famous result of Obata on the classification of constant scalar curvature metrics which are conformal to an Einstein metric. In a remarkable paper [Caffarelli et al. 1989], Caffarelli, Gidas and Spruck established the radial symmetry of the solution without any assumption on the asymptotic behavior of u.

The scalar curvature equation for conformal metrics has critical Sobolev power. In the subcritical case,

$$\Delta u + u^p = 0, \quad 1 < p < \frac{n+2}{n-2}. \tag{1-3}$$

Gidas and Spruck [1981] showed that any nonnegative solution must be trivial. Recently, Catino and Monticelli [2024] carried out a systematic study of (1-1)–(1-3) on complete manifolds with nonnegative Ricci curvature. Among many results, one particular case is a full extension of Caffarelli, Gidas and Spruck's result in dimension three to complete manifolds with nonnegative Ricci curvature.

Inspired by Catino and Monticelli's work, we study the Liouville equation (1-1) on complete surfaces with nonnegative Gauss curvature; in particular, we are able to connect the asymptotic behavior of the solution with the underlying manifold.

To be more precise, let (M, g) be a complete surface with nonnegative Gauss curvature. We study the Liouville equation

$$\Delta_g u + e^{2u} = 0 \tag{1-4}$$

on M. A solution is said to have *finite total curvature* if $\int_M e^{2u}\,dg < \infty$.

In view of the Cohn-Vossen splitting theorem, a complete surface (M, g) with nonnegative Gauss curvature is

- either isometric to the flat cylinder $S^1 \times \mathbb{R}$ (orientable case) or the flat Möbius band (nonorientable case),
- or diffeomorphic to $(\mathbb{R}^2, g_0)$.

In the latter case, by Huber's theorem [1957], (M, g) is conformal to $(\mathbb{R}^2, g_0)$.

Without loss of generality, we assume from now on that M is orientable. In the former case, we have the following classification of solutions to (1-4).

Theorem 1. *Let u be a solution of* (1-4) *with finite total curvature on the flat cylinder $(S^1 \times \mathbb{R}, g_{\text{prod}})$. Then there exists $\mu \in [0, \infty)$ and $\beta \in (-1, \infty)$, such that either β is an integer or $\mu = 0$, and up to a rescaling, we have*

$$e^{2u(z)} = \frac{(2\beta+2)^2|z|^{2\beta+2}}{(|1+\mu z^{\beta+1}|^2+|z|^{2\beta+2})^2} \quad \textit{on} \ \left(\mathbb{C}-\{0\}, \frac{1}{|z|^2}g_0\right).$$

The classification result is not new. Since the Gauss curvature for the flat cylinder is identically zero, (1-4) has a geometric meaning that the conformal metric $e^{2u}g_{\text{prod}}$ has Gauss curvature 1. Note the flat cylinder is conformal to $(\mathbb{R}^2 \setminus \{0\}, g_0)$; thus (1-4) can be translated to the Liouville equation on $\mathbb{R}^2 \setminus \{0\}$. Then the theorem follows from a combination of results of Chou and Wan [1994, Theorem 5], Chen and Li [1995, Theorem 3.1] and Troyanov [1989, Theorem II].

Our main theorem is the following rigidity result.

Theorem 2. *Let u be a solution of* (1-4) *with finite total curvature on a complete surface (M, g) with nonnegative Gauss curvature. Let $r(x)$ be the distance function on M with respect to a fixed point. If $u(x) \geq -2\ln r(x) + o(\ln r(x))$, for $r(x)$ large, then (M, g) must be isometric to $(\mathbb{R}^2, g_0)$. Moreover, -2 is optimal in the sense that there exists nonflat (M, g) which admits solutions satisfying $u(x) \sim \gamma \ln r(x)$ for any $\gamma < -2$.*

A similar result has been proved in [Catino and Monticelli 2024, Theorem 1.10]. Our contribution here has two-folds. On the one hand, our assumption on u is weaker than that in [Catino and Monticelli 2024] and our treatment emphasizes the analysis of asymptotic behavior of the solution which helps to identify the threshold where the rigidity occurs. On the other hand, by setting the stage on the complete surfaces with nonnegative Gauss curvature, we unite two works of Chen and Li [1991; 1995].

The strategy of our proof is study the asymptotic behavior of the solution. If (M, g) is conformal to $(\mathbb{R}^2, g_0)$, we write $g = e^{2f}g_0$. Then (1-4) becomes

$$\Delta u + e^{2f}e^{2u} = 0 \quad \text{on } \mathbb{R}^2. \tag{1-5}$$

This is the so-called prescribing Gauss curvature equation on $\mathbb{R}^2$, which has been investigated intensively over the past few decades. Under a suitable decay assumption of e^{2f} near infinity, Cheng and Lin [1997, Theorem 1.1] showed that the solution u of (1-5) has the asymptotic behavior

$$\lim_{x\to\infty} \frac{u(x)}{\ln|x|} = -\frac{1}{2\pi}\left(\int_{\mathbb{R}^2} e^{2f}e^{2u}\,dx\right)$$

if and only if $\int_{\mathbb{R}^2} e^{2f} e^{2u}\, dx < \infty$. However, a priori, there is not any decay control for e^{2f}. In fact, f satisfies the similar equation

$$\Delta f + K_g e^{2f} = 0,$$

where K_g is the Gauss curvature of g. The only information here is that $K_g \geq 0$. Nevertheless, using Arsove and Huber's result [1973], there exists an $m \in [0, 1]$ and an exceptional set E which is thin at infinity such that

$$\lim_{\substack{x\to\infty \\ x\notin E}} \frac{f(x)}{\ln|x|} = \liminf_{x\to\infty} \frac{f(x)}{\ln|x|} = -m. \tag{1-6}$$

Here the thinness of a set at infinity is a concept concerning the logarithmic capacity. For a complete conformal metric $e^{2f} g_0$ on $\mathbb{R}^n$ $(n \geq 3)$ with nonnegative Ricci curvature, Ma and Qing [2021] obtained a similar asymptotic behavior for the conformal factor f.

While Cheng and Lin's and Arsove and Huber's works are the main analytical inspirations for us, we also benefit from two interesting geometric ingredients: the first is Li and Tam's work [1991] on a comparison between the intrinsic distance and the Euclidean distance on $(\mathbb{R}^2, e^{2f} g_0)$ (see Lemma 2.2) and the second is an isoperimetric inequality on complete surfaces with nonnegative Gauss curvature established recently by Brendle [2023] (see Lemma 2.3).

We shall present proofs of Theorems 1 and 2 in the next section.

2. Proof of the main theorem

Proof of Theorem 1. The flat cylinder $\mathbb{S}^1 \times \mathbb{R}$ is conformal to $(\mathbb{R}^2 \setminus \{0\}, g_0)$ since

$$dt^2 + d\theta^2 = \frac{1}{r^2} dr^2 + d\theta^2 = \frac{1}{r^2} g_0,$$

by setting $t = \ln r$. Let $e^{2w(x)} = (1/|x|^2) e^{2u(x)}$. Then $\Delta_g u + e^{2u} = 0$ is equivalent to

$$\begin{cases} \Delta w + e^{2w} = 0 & \text{on } \mathbb{R}^2 \setminus \{0\}, \\ \displaystyle\int_{\mathbb{R}^2} e^{2w(x)}\, dx < \infty. \end{cases} \tag{2-1}$$

Chou and Wan's complex analysis argument [1994, Theorem 5] shows that

$$w(x) = \beta_1 \ln|x| + O(1) \quad \text{as } x \to 0, \text{ for some } \beta_1 > -1.$$

Let $\widetilde{w}(x) = w(x/|x|^2) - 2\ln|x|$, it is easy to see that $\widetilde{w}$ satisfies

$$\begin{cases} \Delta \widetilde{w} + e^{2\widetilde{w}} = 0 & \text{on } \mathbb{R}^2 \setminus \{0\}, \\ \displaystyle\int_{\mathbb{R}^2} e^{2\widetilde{w}(x)}\, dx < \infty. \end{cases}$$

Applying Chou and Wan's asymptotic result [1994, Theorem 5] to $\widetilde{w}$ and tracing back to w, we get

$$w(x) = \beta_2 \ln|x| + O(1) \quad \text{as } x \to \infty, \text{ for some } \beta_2 < -1.$$

Therefore, $w(x)$ is a solution of (2-1) with conical singularities at $x = 0$ and $x = \infty$. Hence the classification result of Troyanov [1989, Theorem II] yields that there exists $\mu \in [0, \infty)$ and $\beta \in (-1, \infty)$ such that either β is an integer or $\mu = 0$, and up to a rescaling, we have

$$e^{2w(z)} = \frac{(2\beta+2)^2|z|^{2\beta}}{(|1+\mu z^{\beta+1}|^2 + |z|^{2\beta+2})^2} \quad \text{on } \mathbb{C} - \{0\}.$$

Then the desired result follows since $e^{2u(z)} = |z|^2 e^{2w(z)}$. Note if both cone angles are less than 2π $(\beta \in (-1, 0))$, Chen and Li [1995, Theorem 3.1] also obtained such a classification. □

Next, we give the complete proof of Theorem 2.

First we exclude the case of a flat cylinder in Theorem 2: Suppose u is a finite total curvature solution of (1-4) on the flat cylinder. Then Theorem 1 implies

$$u(x) \sim -(\beta+1)r(x) \quad \text{for } r(x) \text{ large,}$$

where $\beta > -1$ is a constant. This is a contradiction with the assumption that $u(x) \geq -2\ln r(x) + o(\ln r(x))$ for $r(x)$ large. In conclusion, (M, g) cannot be the flat cylinder and thus is conformal to $(\mathbb{R}^2, g_0)$ by the Cohn-Vossen splitting theorem and Huber's theorem.

We write $g = e^{2f} g_0$. Then the finite total curvature solution u of (1-4) becomes

$$\begin{cases} \Delta u + e^{2f} e^{2u} = 0 & \text{on } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} e^{2f+2u}\, dx < \infty. \end{cases} \tag{2-2}$$

To fix the notation, we consider the quantity

$$\alpha := -\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{2f+2u}\, dx. \tag{2-3}$$

The strategy of our proof is as follows: using the asymptotic lower bound assumption of the solution u, we establish a lower bound of α by analyzing carefully the asymptotic upper bound of the solution to (2-2). On the other hand, with the help of Brendle's isoperimetric inequality, we prove that the reversed inequality still holds. Hence the equality is obtained and the rigidity part of the isoperimetric inequality brings the rigidity of the underlying manifold.

First, we aim at getting the lower bound of α. It is tempting to obtain a pointwise upper bound of the solution u to (2-2) in terms of α so that the lower bound assumption on u could imply immediately the lower bound of α. However, due to the lack of a uniform asymptotic behavior of the conformal factor f, it's impossible to derive such a pointwise bound for u. Instead, we shall give an upper bound of the integral average of u on small balls. The argument is based on that of [Cheng and Lin 1997].

Lemma 2.1. *Let $(M, g) = (\mathbb{R}^2, e^{2f} g_0)$ be a complete surface with nonnegative Gauss curvature. Assume $u \in C^2(\mathbb{R}^2)$ is a solution to* (2-2). *Then for any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$, there holds*

$$\frac{1}{\pi\rho^2}\int_{B_\rho(x)} u(y)\,dy \leq (\alpha+\epsilon)\ln|x| + C,$$

where α is given by (2-3) *and C is a constant depending on ϵ, σ, R.*

Proof. Construct an auxiliary function

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(y)\ln\frac{|x-y|}{|y|}\,dy,$$

where $\psi(y) = e^{2f(y)+2u(y)}$.

The proof consists of three claims:

(1) $v(x) \leq -\alpha\ln|x| + C$ for $|x| \geq 2$.

(2) $u + v$ is a constant.

(3) For any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$,

$$u(x) \leq (\alpha+\epsilon)\ln|x| + \frac{1}{2\pi}\int_{B_\rho(x)} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + C. \tag{2-4}$$

Proof of claim (1): For fixed x with $|x| \geq 2$,

$$\begin{aligned}2\pi\, v(x) &= \int_{T_1}\psi(y)\ln\frac{|x-y|}{|y|}\,dy + \int_{T_2}\psi(y)\ln\frac{|x-y|}{|y|}\,dy + \int_{T_3}\psi(y)\ln\frac{|x-y|}{|y|}\,dy\\ &\overset{\text{def}}{=} I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}T_1 &= \{y : |y| \leq 2\},\\ T_2 &= \left\{y : |y-x| \leq \frac{|x|}{2},\ |y| \geq 2\right\},\\ T_3 &= \left\{y : |y-x| \geq \frac{|x|}{2},\ |y| \geq 2\right\}.\end{aligned}$$

For $|x| \geq 2$ and $y \in T_1$, we have $\ln|x-y| \leq \ln(|x|+2) \leq \ln|x| + \ln 2$. Thus

$$\begin{aligned} I_1 &= \int_{T_1} \psi(y) \ln|x-y|\, dy - \int_{T_1} \psi(y) \ln|y|\, dy \\ &\leq \int_{T_1} \psi(y)(\ln|x| + \ln 2)\, dy - \int_{T_1} \psi(y) \ln|y|\, dy \\ &= (\ln|x|) \int_{T_1} \psi(y)\, dy + C. \end{aligned}$$

Now for $y \in T_2$, we have $|x-y| \leq |x|/2 \leq |y|$. Thus

$$I_2 \leq 0.$$

For $y \in T_3$ and $|x| \geq 2$, there holds $|x-y| \leq |x| + |y| \leq |x|\,|y|$. Therefore

$$I_3 \leq (\ln|x|) \int_{T_3} \psi(y)\, dy.$$

We conclude that

$$2\pi v(x) = I_1 + I_2 + I_3 \leq -2\pi\alpha \ln|x| + C.$$

The proof of claim (1) is finished.

Proof of claim (2): It is easy to see that $\Delta v = e^{2f+2u}$ and $u+v$ is a harmonic function on $\mathbb{R}^2$. Hence there exists an entire function $f(z)$ such that $\operatorname{Re} f = 2(u+v)$. Let $F(z) = e^{f(z)}$. Clearly, by claim (1) we get

$$|F(z)| = e^{2u+2v} \leq C|z|^{-2\alpha} e^{2u},$$

for $|z| \geq 2$. Using the lower bound (1-6) for the conformal factor f ($e^{2f} \geq |z|^{-2m}$), we get that for some R_0 large enough,

$$\int_{|z|\geq R_0} |F(z)|\,|z|^{2\alpha}|z|^{-2m}\, dx \leq C \int_{|z|\geq R_0} e^{2u} e^{2f}\, dx < \infty.$$

Let $M(\rho) = \max_{|z|=\rho} |F(z)|$. We shall show that $M(\rho) \leq C\rho^{2m-2\alpha}$ for $\rho \geq R_0+1$. In fact, assume $|z_0| = \rho$ and $M(\rho) = |F(z_0)|$. The mean value property implies

$$|F(z_0)| \leq \frac{1}{\pi} \int_{B_1(z_0)} |F(z)|\, dx \leq \frac{1}{\pi} \int_{\rho-1\leq|z|\leq\rho+1} |F(z)|\, dx.$$

Hence we get

$$\begin{aligned} M(\rho)\rho^{2\alpha-2m} &\leq \frac{1}{\pi} \int_{\rho-1\leq|z|\leq\rho+1} |F(z)|\rho^{2\alpha-2m}\, dx \\ &\leq \frac{1}{\pi} \int_{\rho-1\leq|z|\leq\rho+1} |F(z)|\,|z|^{2\alpha-2m} \left(\frac{\rho}{\rho+1}\right)^{2\alpha-2m} dx \\ &\leq \frac{2^{2m-2\alpha}}{\pi} \int_{|z|\geq\rho-1} |F(z)|\,|z|^{2\alpha-2m}\, dx < \infty. \end{aligned}$$

Therefore, the order of the entire function $F(z)$ is

$$\lambda := \limsup_{\rho\to\infty} \frac{\ln\ln M(\rho)}{\ln\rho} = 0.$$

By a theorem of Hadamard (see Theorem 8 on p. 209 in [Ahlfors 1978]), we conclude that the genus of $F(z)$ is zero and $F(z)$ is a constant since F has no zeros. The proof of claim (2) is completed.

Proof of claim (3): For any $\epsilon > 0$, $\sigma > 0$, choose $R > 0$ large enough such that

$$(\sigma+1)\int_{|y|\ge R} \psi(y)\,dy \le \pi\epsilon,$$

where $\psi(y) = e^{2f(y)+2u(y)}$. By claim (2), we have

$$\begin{aligned}2\pi u(x) &= C + \int_{\mathbb{R}^2} \psi(y)\ln\frac{|y|}{|x-y|}\,dy\\ &= C + \int_{\widetilde{T}_1} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{\widetilde{T}_2} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{\widetilde{T}_3} \psi(y)\ln\frac{|y|}{|x-y|}\,dy\\ &\overset{\text{def}}{=} \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3,\end{aligned}$$

where

$$\begin{aligned}\widetilde{T}_1 &= \{y : |y| \le R\},\\ \widetilde{T}_2 &= \left\{y : |y-x| \le \frac{|x|}{2},\ |y| \ge R\right\},\\ \widetilde{T}_3 &= \left\{y : |y-x| \ge \frac{|x|}{2},\ |y| \ge R\right\}.\end{aligned}$$

Now for $|x| \ge R^2/(R-1)$ and $y \in \widetilde{T}_1$, we have $\ln|x-y| \ge \ln(|x|-R) \ge \ln|x| - \ln R$. Thus

$$\begin{aligned}\tilde{I}_1 &= \int_{\widetilde{T}_1} \psi(y)\ln|y|\,dy - \int_{\widetilde{T}_1} \psi(y)\ln|x-y|\,dy\\ &\le C - (\ln|x|)\int_{\widetilde{T}_1} \psi(y)\,dy + (\ln R)\int_{\widetilde{T}_1} \psi(y)\,dy\\ &\le -(\ln|x|)\int_{\widetilde{T}_1} \psi(y)\,dy + C.\end{aligned}$$

To estimate $\tilde{I}_2$, let $\widetilde{T}^\sigma = \{y : |y-x| \le |x|^{-\sigma},\ |y| \ge R\}$. Then we have

$$\begin{aligned}\tilde{I}_2 &= \int_{\widetilde{T}^\sigma} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{|x|^{-\sigma}\le|y-x|\le|x|/2,|y|\ge R} \psi(y)\ln\frac{|y|}{|x-y|}\,dy\\ &\le \int_{\widetilde{T}^\sigma} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{|y|\ge R} \psi(y)\ln\frac{\frac{3}{2}|x|}{|x|^{-\sigma}}\,dy\\ &\le \int_{|y-x|\le|x|^{-\sigma}} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + (\sigma+1)\int_{|y|\ge R} \psi(y)\,dy + C.\end{aligned}$$

Now for $y \in \widetilde{T}_3$, one easily gets $|y| \le 4|x-y|$. Therefore

$$\tilde{I}_3 = \int_{\widetilde{T}_3} \psi(y) \ln \frac{|y|}{|x-y|}\, dy \le (\ln 4) \int_{\widetilde{T}_3} \psi(y)\, dy \le C.$$

In conclusion, for $|x| \ge R^2/(R-1)$, there holds

$$\begin{aligned} 2\pi u(x) &= \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 \\ &\le C - (\ln|x|) \int_{|y|\le R} \psi(y)\, dy + \int_{|y-x|\le |x|^{-\sigma}} \psi(y) \ln \frac{|y|}{|x-y|}\, dy \\ &\qquad\qquad + (\sigma+1) \int_{|y|\ge R} \psi(y)\, dy \\ &\le C + 2\pi(\alpha+\epsilon) \ln|x| + \int_{|y-x|\le |x|^{-\sigma}} \psi(y) \ln \frac{|y|}{|x-y|}\, dy. \end{aligned}$$

The proof of claim (3) is completed.

Finally, we give the upper bound of the integral average of u. By Green's formula,

$$u(x) = \frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy + \frac{1}{2\pi} \int_{B_\rho(x)} \psi(y) \ln \frac{\rho}{|x-y|}\, dy,$$

for every $x \in \mathbb{R}^2$ and $\rho > 0$. Combined with (2-4), we have for any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \ge R$ and $\rho = |x|^{-\sigma}$,

$$\frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy \le (\alpha+\epsilon) \ln|x| + \frac{1}{2\pi} \int_{B_\rho(x)} \psi(y) \ln \frac{|y|}{\rho}\, dy + C. \tag{2-5}$$

Since $|y|/\rho \le (|x|+\rho)/\rho = |x|^{\sigma+1} + 1 \le |x|^{\sigma+2}$ for $|x|$ large enough, the second term in the right-hand side of (2-5) could be estimated as

$$\frac{1}{2\pi} \int_{B_\rho(x)} \psi(y) \ln \frac{|y|}{\rho}\, dy \le \frac{\sigma+2}{2\pi} (\ln|x|) \int_{|y|\ge R/2} \psi(y)\, dy \le \epsilon \ln|x|,$$

for $|x| \ge R$ provided R is large enough. Inserting this into (2-5), the proof of the lemma is completed. □

To derive the lower bound of α, we need a lower bound of u in terms of the Euclidean distance $\ln|x|$ rather than the intrinsic distance $\ln r(x)$ that appeared in the hypotheses of Theorem 2. Fortunately, the comparison of these two distances is established by Li and Tam [1991, Corollary 3.3]. Hartman [1964, Theorem 7.1] revealed the connection between this limit and the asymptotic volume ratio of the manifold. Their results are combined as follows.

Lemma 2.2 (Hartman, Li–Tam). *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete manifold with nonnegative Gauss curvature* K. *Then*

$$\lim_{x\to\infty} \frac{\ln r(x)}{\ln |x|} = 1 - \frac{1}{2\pi} \int_{\mathbb{R}^2} K\, dg = \beta,$$

where

$$\beta := \lim_{t\to\infty} \frac{\text{Area}(B(p,t))}{\pi t^2} \in [0, 1]$$

is the asymptotic volume ratio of the manifold $(\mathbb{R}^2, e^{2f} g_0)$.

Given this asymptotic behavior of $r(x)$, the a priori assumption on u could be applied to obtain the lower bound of α in terms of the asymptotic volume ratio.

Proposition 2.1. *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative Gauss curvature. Let u be a solution of* (2-2). *Assume*

$$u(x) \geq -2 \ln r(x) + o(\ln r(x)),$$

for $r(x)$ *large. Then*

$$\alpha \geq -2\beta,$$

where α *is given by* (2-3) *and* β *is the asymptotic volume ratio of* $(\mathbb{R}^2, e^{2f} g_0)$.

Proof. By our assumption on u and Lemma 2.2, we get for any $\epsilon > 0$, there exists $R > 0$ such that for $r(x) \geq R$,

$$u(x) \geq -2 \ln r(x) + o(\ln r(x)) \geq (-2\beta - 2\epsilon) \ln |x| + o(\ln |x|).$$

Lemma 2.1 yields that for any $\epsilon > 0, \sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$,

$$\frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy \leq (\alpha + \epsilon) \ln |x| + C,$$

where C is a constant depending on ϵ, σ, R.

We conclude that for any $\epsilon > 0, \sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$,

$$\begin{aligned}(\alpha + \epsilon) \ln |x| + C &\geq \frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy \\ &\geq (-2\beta - 2\epsilon) \frac{1}{\pi\rho^2} \int_{B_\rho(x)} \ln |y|\, dy + o(\ln |x|) \\ &\geq (-2\beta - 2\epsilon)(\ln |x| - \epsilon) + o(\ln |x|).\end{aligned}$$

Letting $x \to \infty$, we get $\alpha + \epsilon \geq -2\beta - 2\epsilon$. Since ϵ could be arbitrarily small, we get

$$\alpha \geq -2\beta. \qquad \square$$

We shall see that the reversed inequality also holds, and thus the equality is obtained. For this, we need the isoperimetric inequality on nonnegatively curved surfaces established by Brendle [2023, Corollary 1.3], and it also helps to get the rigidity of the underlying manifold in our setting.

Lemma 2.3 (Brendle). *Let (M^2, g) be a complete noncompact manifold with nonnegative Gauss curvature. Let D be a compact domain in M with boundary ∂D. Then*

$$L(\partial D)^2 \geq 4\pi\beta A(D),$$

where $L(\partial D)$ and $A(D)$ represent the length of ∂D and the area of D, respectively, and β is the asymptotic volume ratio of (M, g). The equality holds if and only if (M, g) is isometric to Euclidean space and D is a ball.

Now with the help of Lemma 2.3, one could mimic the argument in [Chen and Li 1991] to give the upper bound of α.

Proposition 2.2. *Let $(\mathbb{R}^2, e^{2f} g_0)$ be a complete surface with nonnegative Gauss curvature. Let u be a solution of* (2-2). *Then*

$$\alpha \leq -2\beta,$$

where α is given by (2-3) *and β is the asymptotic volume ratio of* $(\mathbb{R}^2, e^{2f} g_0)$.

Proof. Consider $F(t) := \int_{\Omega_t} e^{2u}\, dg$, where $\Omega_t = \{x : u(x) > t\}$ is the upper level set of u.

The finite total curvature assumption $\int_M e^{2u}\, dg < \infty$ implies $A(\Omega_t) < \infty$, where $A(\Omega_t)$ represents the area of Ω_t in $(\mathbb{R}^2, g = e^{2f} g_0)$.

It follows from (1-4) and the divergence theorem that

$$F(t) = \int_{\Omega_t} e^{2u}\, dg = -\int_{\Omega_t} \Delta u\, dg = -\int_{\partial\Omega_t} \langle \nabla u, \eta \rangle\, dS_g = \int_{\partial\Omega_t} |\nabla u|\, dS_g.$$

By the coarea formula,

$$F'(t) = -\int_{\partial\Omega_t} \frac{e^{2u}}{|\nabla u|}\, dS_g = -e^{2t} \int_{\partial\Omega_t} \frac{1}{|\nabla u|}\, dS_g.$$

Then the Hölder inequality and the isoperimetric inequality (Lemma 2.3) imply

$$\begin{aligned} (F^2(t))' &= -2e^{2t} \int_{\partial\Omega_t} |\nabla u|\, dS_g \int_{\partial\Omega_t} \frac{1}{|\nabla u|}\, dS_g \\ &\leq -2e^{2t} L(\partial\Omega_t)^2 \\ &\leq -8\pi\beta\, e^{2t} A(\Omega_t). \end{aligned} \tag{2-6}$$

Note that the isoperimetric inequality still holds for noncompact regions whose area are finite, since the length of its boundary must be infinite by the completeness of $(\mathbb{R}^2, e^{2f} g_0)$.

Finally integrating (2-6) from $-\infty$ to ∞ yields

$$\begin{aligned} -\left(\int_M e^{2u}\,dg\right)^2 &\le -8\pi\beta \int_{-\infty}^{\infty} e^{2t} A(\{x : e^{2u(x)} > e^{2t}\})\,dt \\ &= -4\pi\beta \int_0^{\infty} A(\{x : e^{2u(x)} > \lambda\})\,d\lambda \\ &= -4\pi\beta \int_M e^{2u}\,dg. \end{aligned}$$

Thus the desired inequality holds. □

Proof of Theorem 2. By Propositions 2.1 and 2.2, we get

$$\alpha = -2\beta.$$

Inspecting the proof of Proposition 2.2 shows that $L(\partial\Omega_t)^2 = 4\pi\beta A(\Omega_t)$ for every $t \in \mathbb{R}$. Hence Lemma 2.3 tells us $(\mathbb{R}^2, e^{2f} g_0)$ must be isometric to the Euclidean space $(\mathbb{R}^2, g_0)$.

To see the sharpness of the -2 in the assumption $u(x) \ge -2\ln r(x) + o(\ln r(x))$, consider the following examples.

Let $e^{2f(x)} = \gamma/(1+|x|^2)^{2-2\gamma}$. Then for $\gamma \in \left[\frac{1}{2}, 1\right]$, $(\mathbb{R}^2, g = e^{2f} g_0)$ is a complete surface with nonnegative Gauss curvature $K_g = 4(1-\gamma)/\big(\gamma(1+|x|^2)^{2\gamma}\big)$.

Taking $e^{2u(x)} = 4/(1+|x|^2)^{2\gamma}$, it is easy to see that $\Delta u + e^{2f} e^{2u} = 0$, that is,

$$\Delta_g u + e^{2u} = 0.$$

Moreover, $\int_{\mathbb{R}^2} e^{2u}\,dg = \int_{\mathbb{R}^2} 4\gamma/(1+|x|^2)^2\,dx = 4\pi\gamma < \infty$.

Direct computation shows

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln r(x)}{\ln|x|} &= 2\gamma - 1 \quad \text{for } \gamma \in \left(\frac{1}{2}, 1\right], \\ \lim_{x\to\infty} \frac{r(x)}{\ln|x|} &= 1 \qquad\qquad \text{for } \gamma = \frac{1}{2}. \end{aligned}$$

Thus for $\gamma \in \left(\frac{1}{2}, 1\right)$, we have

$$u(x) \sim -2\gamma \ln|x| \sim -\frac{2\gamma}{2\gamma - 1} \ln r(x),$$

where $-2\gamma/(2\gamma-1) \in (-\infty, -2)$.

In conclusion, for any $k < -2$, there exists a complete surface $(\mathbb{R}^2, e^{2f} g_0)$ with nonnegative Gauss curvature which admits a finite total curvature solution u of (1-4) with $u(x) \sim k \ln r(x)$. □

We conclude this paper with the following remark. Theorem 2 states the rigidity of the underlying manifold under the assumption $u(x) \geq -2\ln r(x) + o(\ln r(x))$. However, on the other end, we cannot expect such rigidity as illustrated by examples above. More precisely, when $\gamma = \frac{1}{2}$, it readily follows that the solution u decays linearly with respect to the distance induced by the metric. Hence one cannot distinguish the flat cylinder by imposing linear decay conditions on the solution.

Nevertheless, when the solution decays sufficiently fast, we can get the volume growth control of the underlying manifold. We record this as a result of independent interest.

Proposition 2.3. *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative Gauss curvature. Let u be a solution of* (2-2) *satisfying*

$$\liminf_{x\to\infty} \frac{u(x)}{\ln r(x)} = -\infty.$$

Then the asymptotic volume ratio of $(\mathbb{R}^2, e^{2f} g_0)$ *is zero.*

Proof. Suppose the asymptotic volume ratio β is positive. According to the claims (1) and (2) in Lemma 2.1, there holds

$$u(x) \geq \alpha \ln|x| + C \quad \text{for } |x| \geq 2,$$

where $\alpha = -(1/(2\pi)) \int_{\mathbb{R}^2} e^{2f} e^{2u}\, dx$. Combined with Lemma 2.2 one gets

$$\liminf_{x\to\infty} \frac{u(x)}{\ln r(x)} = \liminf_{x\to\infty} \frac{u(x)}{\ln|x|} \frac{\ln|x|}{\ln r(x)} \geq \frac{\alpha}{\beta} > -\infty.$$

This contradicts the hypothesis. Hence we get $\beta = 0$. □

We also have a partial converse to Proposition 2.3.

Proposition 2.4. *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative and bounded Gauss curvature. Suppose the asymptotic volume ratio* β *equals* 0. *Then there exists a solution of* (2-2) *satisfying*

$$\lim_{x\to\infty} \frac{u(x)}{\ln r(x)} = -\infty.$$

Proof. Recall that f satisfies

$$\Delta f + K e^{2f} = 0,$$

where $0 \leq K \leq C$ by assumption. Based on a work of Taliaferro [1999], Bonini, Ma and Qing [Bonini et al. 2018, Lemma 4.2] showed that

$$e^{2f} \sim |x|^{-2(1-\beta)} = |x|^{-2} \quad \text{as } |x| \to \infty.$$

In view of the existence theorem of McOwen [1985, Theorem 1], for any $\alpha \in (-2, 0)$, there exists a solution u of (2-2) satisfying

$$u(x) \sim \alpha \ln|x| + O(1) \quad \text{at } \infty.$$

Since Lemma 2.2 still holds for $\beta = 0$, the conclusion readily follows. □

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