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JAN ČERMÁK, LUCIE FEDORKOVÁ AND JIŘÍ JÁNSKÝ

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Root properties of a general complex trinomial have been explored in numerous papers. Two questions have attracted a significant attention: the relationships between the moduli of these roots and the trinomial's entries, and the location of the roots in the complex plane. We consider several particular problems connected with these topics, and provide new insights into them. As two main results, we describe the set of all trinomials having a root with a given modulus, and derive explicit formula for calculations of the arguments of such roots. In this fashion, we obtain a comprehensive characterization of these roots. In addition, we develop a procedure enabling us to compute moduli and arguments of all roots of a general complex trinomial with arbitrary precision. This procedure is based on the derivation of a family of real transcendental equations for the roots' moduli, and it is supported by the formula for their arguments. All our findings are compared with the existing results.

1. Introduction

We consider a trinomial of the form

$$p(z) = z^k + az^\ell + b,$$

where z, a, b are complex numbers, and $k > \ell$ are positive integers. Because of a lack of formula expressing the roots of p in terms of its entries, many theoretical works analyzed the relationship between the k roots of p, and the values a, b, k, ℓ. More precisely, dependence of moduli and arguments of these roots on a, b, k, ℓ was investigated, and, vice versa, for a given configuration of roots of p, the corresponding parameter space of coefficients was studied.

The list of particular problems discussed in this connection is pretty long. Among others, it includes the following fundamental questions on moduli of the roots of p (we still assume here that a, b are complex numbers, and $k > \ell$ are positive integers):

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(A) What is, for given a, b, k, ℓ and a positive real ϱ, the number of roots of p with moduli lower than ϱ? Alternatively, the inverse of this problem is, for a given ϱ and an integer n, $0 \le n \le k$, to describe the set of all a, b, k, ℓ such that p has just n roots with modulus lower than ϱ.

(B) What is, for given b, k, ℓ, and a positive real ϱ, the geometric structure of the set of all complex numbers a such that p has a root with modulus ϱ?

(C) What is, for given b, k, ℓ, the geometric structure of the set of all complex numbers a such that p has two (or more) roots with the same modulus? Alternatively, problems (B) and (C) can be considered for a fixed a instead of b.

(D) What is, for given a, b, k, ℓ, the geometric description of the location of k roots of p in the complex plane?

Problem (A) has an interesting history. It was completely answered in [Bohl 1908]. However, perhaps due to language reasons ([Bohl 1908] was written in German), this result remained nearly unnoticed by the mathematical community. Some of its particular cases were later rediscovered, among other studied things, for example, in [Brilleslyper and Schaubroeck 2014; 2018; Dilcher et al. 1992; Howell and Kyle 2018].

The inverse of problem (A) is of a great importance in the stability and asymptotic theory of difference equations. Although it is closely related to problem (A) itself (in fact, its solution can be deduced directly from Bohl's result), its various particular cases were investigated in dozens of works (see, for example, [Dannan 2004; Kipnis and Nigmatullin 2016; Kuruklis 1994; Matsunaga and Hajiri 2010; Papanicolaou 1996; Čermák and Jánský 2015]) — again independently of the existence of Bohl's result. Only recently, [Bohl 1908] has attracted attention corresponding to its relevance (see, for example, [Barrera et al. 2022; 2023a; Theobald and de Wolff 2016; Čermák and Fedorková 2023]).

Problems (B) and (C) were formulated and answered in [Theobald and de Wolff 2016], along with a comprehensive historical survey. Using the amoeba theory, these answers revealed a nice geometric and topological structure of the parameter space of trinomials p with respect to the moduli of their roots.

Problem (D) on locating and describing the geometry of roots of p in the complex plane is a classical matter. Starting with [Nekrassoff 1887], a series of papers on sectors in the complex plane, each containing a root of p, appeared (see, for example, [Egerváry 1930]). The strongest results in this sense, namely disjoint annular sectors smaller than those described in previous works, were obtained in [Melman 2012]. In these investigations, Rouché's theorem and other tools of complex analysis turned out to be very useful. For other relevant results on roots of complex trinomials, we refer to [Barrera et al. 2023b; Botta and da Silva 2019; Fell 1980; Szabó 2010; Čermák et al. 2022].

Our main goal is twofold. First, we wish to present new insights into problems (A)–(D), and offer alternate answers to some of them. Second, we aim to learn more about arguments of roots of p as well. Keeping in mind these outlines, the paper is organized as follows.

In Section 2, we recall Bohl's result answering problem (A). Using this result, we discuss an inverse version of problem (A), namely characterization of all couples (a, b) such that p has a prescribed number of roots whose modulus is lower than a prescribed real number. Section 3 deals with problem (B), and formulates explicit necessary and sufficient conditions guaranteeing that p has a root with a given modulus. A formula for calculation of the arguments of such roots is derived as well. Considerations performed in Section 4 are motivated by problem (D), and result in theoretical justification of an algorithm that enables us to localize all k roots of p with arbitrary precision. This algorithm is based on the derivation of k (real) transcendental equations for moduli of these roots, supported by a formula for calculation of their arguments. In Section 5, we illustrate our results and compare them with the existing ones. Doing so, we consider assertions and examples from earlier papers, and clarify contributions of our results to the existing theory on complex trinomials. The final section summarizes the key parts of the paper, outlines possible applications, and poses some open problems.

The main results of this paper are contained in Sections 3 and 4. Our intention was to derive them without any support of advanced theoretical tools, using only some basic facts from linear algebra, mathematical analysis and number theory.

Throughout the text, the following simplifications and notation are utilized. Without loss of generality, we assume that the integers k, ℓ are coprime (the opposite case can be easily reduced to this one), and the complex numbers a, b are nonzero (the opposite case is trivial). Further, we assume that the arguments of complex numbers are taken from the interval $(-\pi, \pi]$, and introduce the notation $\theta_a = \arg(a)$, $\theta_b = \arg(b)$,

$$\alpha_\varrho = \arccos \frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2}{2|ab|\varrho^\ell}, \quad \beta_\varrho = \arccos \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}, \tag{1-1}$$

$$\theta = k\theta_a - (k-\ell)\theta_b + (k-\ell)\pi, \tag{1-2}$$

and

$$\tau_\varrho^\pm = \frac{\theta}{2\pi} \pm \frac{k\alpha_\varrho + \ell\beta_\varrho}{2\pi}. \tag{1-3}$$

Also, we utilize the notation

$$\varphi \equiv \psi \pmod{2\pi}$$

for the arguments $\varphi \in (-\pi, \pi]$ of appropriate complex numbers, meaning that the difference between φ and a real number ψ is an integer multiple of 2π.

Finally, we call the roots of p with modulus lower than ϱ (or equal to ϱ) ϱ-interior (or ϱ-modular, respectively). If $\varrho = 1$, then the ϱ-modular roots of p are called unimodular.

2. A number of ϱ-interior roots of p

Let n_ϱ be the number of ϱ-interior roots of p. As we have already mentioned, the problem of finding an explicit formula for n_ϱ with respect to given a, b, k, ℓ and ϱ was solved in [Bohl 1908]. Since the original formulation of this result has a rather geometric character, we use here its equivalent analytical reformulation (see also [Čermák and Fedorková 2023]).

Theorem 2.1. *Let a, b be nonzero complex numbers, $k > \ell$ be coprime positive integers, ϱ be a positive real number, and let θ be given by (1-2).*

(i) *If $|b| \geq \varrho^k + |a|\varrho^\ell$ then $n_\varrho = 0$.*

(ii) *If $\varrho^k > |a|\varrho^\ell + |b|$ then $n_\varrho = k$.*

(iii) *If $|a|\varrho^\ell > \varrho^k + |b|$ then $n_\varrho = \ell$.*

(iv) *If $|a|\varrho^\ell = \varrho^k + |b| < \frac{k}{\ell}\varrho^k$ and $\frac{\theta}{\pi} + \ell$ is an even integer, then $n_\varrho = \ell$.*

(v) *If*

$$|b| < \varrho^k + |a|\varrho^\ell, \quad \varrho^k \leq |a|\varrho^\ell + |b|, \quad |a|\varrho^\ell \leq \varrho^k + |b|, \tag{2-1}$$

and at least one of the assumptions of (iv) *does not hold, then*

$$n_\varrho = \lceil \tau_\varrho^+ \rceil - \lfloor \tau_\varrho^- \rfloor - 1, \tag{2-2}$$

where $\tau_\varrho^\pm$ is given by (1-3), and the symbols $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ mean the upper and lower integer part.

Remark 2.2. The conditions (i)–(v) of Theorem 2.1 cover all (nontrivial) possibilities for the complex coefficients a, b and exponents k, ℓ of an arbitrary trinomial p. One can observe some interesting geometric connections hidden behind the inequalities forming these conditions. In fact, the conditions (i)–(iv) reflect a dominance of monomials z^k, az^ℓ, b in the sense that one of them exceeds or equals to (in modulus) the sum of the remaining ones. The condition (v) is related to the opposite situation when there exists a triangle with edges of lengths ϱ^k, $|a|\varrho^\ell$ and $|b|$. In this geometric interpretation, the values α_ϱ and β_ϱ are nothing more than the angles between the edges of lengths $|a|\varrho^\ell$, $|b|$ and ϱ^k, $|b|$, respectively.

Also note that the above stated dominance of monomials z^k, az^ℓ, b is closely related to the essential concepts of tropical geometry. In particular, using some basic tools of tropical geometry, the problem of finding the roots of a tropical polynomial (composed of the monomials z^k, az^ℓ, b such that one of them is dominant) inside the circle of radius ϱ simultaneously provides the number of ϱ-interior roots of p (see, for example, [Brugallé et al. 2015; Viro 2011]).

Now we consider an opposite problem: for a given real $\varrho > 0$ and a given integer $n \geq 0$, we search for the set of all a, b, k, ℓ such that $n = n_\varrho$, i.e., p has just n ϱ-interior roots.

Since the relationship between n_ϱ and a, b, k, ℓ is elementary in the properties (i)–(iv) of Theorem 2.1, it is enough to analyze the formula (2-2) forming the core of the property (v). On this account, we introduce the function $\omega = \omega(x)$ to be a 2π-periodic extension of $\omega^*(x) = |x|$, $x \in [-\pi, \pi]$. Then the following holds:

Corollary 2.3. *Let $\varrho > 0$ be a real number, n be a nonnegative integer, let a, b be nonzero complex numbers, and $k > \ell$ be coprime positive integers. Further, assume that (2-1) holds, whereas at least one of the assumptions of the property (iv) of Theorem 2.1 does not hold. Then p has just n ϱ-interior roots if and only if either*

$$n \text{ is even} \quad \text{and} \quad n\pi - \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq n\pi + \omega(\theta), \tag{2-3}$$

or

$$n \text{ is odd} \quad \text{and} \quad (n-1)\pi + \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq (n+1)\pi - \omega(\theta), \tag{2-4}$$

where α_ϱ, β_ϱ and θ are given by (1-1) and (1-2), respectively.

Proof. First, we assume that

$$2m_1\pi \leq |\theta| < (2m_1 + 1)\pi$$

for a nonnegative integer m_1. Then, using

$$\theta = (2m_1\pi + \omega(\theta)) \operatorname{sgn}(\theta),$$

(2-2) implies that $n = n_\varrho$ just when

$$n = \left\lceil \frac{\omega(\theta)\operatorname{sgn}(\theta) + k\alpha_\varrho + \ell\beta_\varrho}{2\pi} \right\rceil - \left\lfloor \frac{\omega(\theta)\operatorname{sgn}(\theta) - (k\alpha_\varrho + \ell\beta_\varrho)}{2\pi} \right\rfloor - 1. \tag{2-5}$$

Now we distinguish two cases leading to (2-3) and (2-4). If

$$2m_2\pi - \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq 2m_2\pi + \omega(\theta), \quad \omega(\theta) \neq 0, \tag{2-6}$$

for an integer m_2, then (2-5) becomes

$$n = m_2 + 1 + m_2 - 1 = 2m_2 \qquad (\text{if } \theta > 0),$$
$$n = m_2 - (-m_2 - 1) - 1 = 2m_2 \quad (\text{if } \theta < 0).$$

Then, it is enough to substitute $2m_2 = n$ in (2-6) to get (2-3).

Similarly, if

$$2m_2\pi + \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq (2m_2 + 2)\pi - \omega(\theta), \quad \omega(\theta) \neq \pi, \tag{2-7}$$

for an integer m_2, then

$$n = m_2 + 1 + m_2 + 1 - 1 = 2m_2 + 1.$$

Consequently, $2m_2 = n - 1$, and (2-7) yields (2-4).

Now, we assume that

$$(2m_1 - 1)\pi \le |\theta| < 2m_1\pi$$

for a positive integer m_1. Then

$$\theta = (2m_1\pi - \omega(\theta))\,\mathrm{sgn}(\theta),$$

and $n = n_\varrho$ is equivalent to

$$n = \left\lceil \frac{-\omega(\theta)\,\mathrm{sgn}(\theta) + k\alpha_\varrho + \ell\beta_\varrho}{2\pi} \right\rceil - \left\lfloor \frac{-\omega(\theta)\,\mathrm{sgn}(\theta) - (k\alpha_\varrho + \ell\beta_\varrho)}{2\pi} \right\rfloor - 1$$

due to (2-2). Thus, using the same line of arguments as given above, we arrive at (2-3) and (2-4). □

3. Existence of ϱ-modular roots of p

We formulate easily applicable conditions verifying whether p has a root with a prescribed modulus. In the affirmative case, we find explicit formulae for arguments of such roots. Thus, we find an effective answer to a more general version of problem (B).

We start with recapitulation of some useful facts from elementary number theory. Let $k > \ell$ be coprime positive integers. For a given integer τ, we consider the linear Diophantine equation in two integer variables u, v

(3-1) $$ku + (k - \ell)v = \tau.$$

If we put

(3-2) $$u = \tau u_0, \quad v = \tau v_0,$$

then (3-1) can be reduced to

(3-3) $$ku_0 + (k - \ell)v_0 = 1,$$

whose integer solutions (u_0, v_0) are so-called *Bézout coefficients* for a couple $(k, k-\ell)$. It is well known that (3-3) admits infinitely many integer solutions; indeed, if (u_0^*, v_0^*) are Bézout coefficients for $(k, k - \ell)$, then all integer solutions (u_0, v_0) of (3-3) can be written as

(3-4) $$u_0 = u_0^* + (k - \ell)m, \quad v_0 = v_0^* - km, \qquad m \in \mathbb{Z}.$$

There are several algorithms to determine a couple (u_0, v_0) satisfying (3-3); the most often used is the extended Euclidean algorithm applied to $(k, k-\ell)$ (see, for example, [Fuhrmann 2012]).

Now we come back to problem (B). The next assertion presents a simple condition verifying that p admits a ϱ-modular root. Moreover, for all a, b, k, ℓ meeting this condition, we give an explicit evaluation of arguments of ϱ-modular roots. Thus, we are able to provide their complete identification.

Theorem 3.1. *Let a, b be nonzero complex numbers, let $k > \ell$ be coprime positive integers, and let $\tau_\varrho^\pm$ be given by (1-3). Then p has a ϱ-modular root $z = \varrho \exp(\mathrm{i}\varphi)$, $\varphi \in (-\pi, \pi]$, if and only if*

$$|b| \le \varrho^k + |a|\varrho^\ell, \quad \varrho^k \le |a|\varrho^\ell + |b|, \quad |a|\varrho^\ell \le \varrho^k + |b|, \tag{3-5}$$

and at least one of the values $\tau_\varrho^\pm$ is an integer.

An explicit dependence of φ on ϱ can be expressed by the formula

$$\varphi = \varphi^\pm \equiv \begin{cases} \dfrac{(2v_0\tau_\varrho^+ - 1)\pi + \beta_\varrho + \theta_b}{k} \pmod{2\pi} & \text{if } \tau_\varrho^+ \text{ is an integer,} \\ \dfrac{(2v_0\tau_\varrho^- - 1)\pi - \beta_\varrho + \theta_b}{k} \pmod{2\pi} & \text{if } \tau_\varrho^- \text{ is an integer,} \end{cases} \tag{3-6}$$

where v_0 is the second component of a couple of Bézout coefficients (u_0, v_0) satisfying (3-3), and β_ϱ is given by (1-1).

Remark 3.2. (a) If just one of the values $\tau_\varrho^\pm$ is an integer, then there exists a unique ϱ-modular root of p (whose argument is $\varphi = \varphi^+$, or $\varphi = \varphi^-$ if τ_ϱ^+ is an integer, or τ_ϱ^- is an integer, respectively). If $\tau_\varrho^\pm$ are two distinct integers, then they generate, along with two arguments $\varphi = \varphi^+$ and $\varphi = \varphi^-$, two distinct ϱ-modular roots of p. Note that this situation occurs just when

$$|b| \ne \varrho^k + |a|\varrho^\ell, \quad k\theta_a - (k-\ell)\theta_b = j_1\pi, \quad k\alpha_\varrho + \ell\beta_\varrho = j_2\pi \tag{3-7}$$

for a couple of integers (j_1, j_2) satisfying $(-1)^{j_1+k} = (-1)^{j_2+\ell}$.

(b) All ϱ-modular roots of p described in Theorem 3.1 are simple except for those generated by the conditions

$$|a|\varrho^\ell = \varrho^k + |b|, \quad |a|\,|b|^{(\ell-k)/k} = \frac{k}{k-\ell}\left(\frac{k-\ell}{\ell}\right)^{\ell/k}, \quad \frac{\theta}{2\pi} + \frac{\ell}{2} \in \mathbb{Z} \tag{3-8}$$

with θ given by (1-2). In this case, p has a double ϱ-modular root z_d with the argument $\varphi = \varphi^+ = \varphi^-$ (in fact, the values φ^+ and φ^- coincide in such a case). Note that this property was already known before since it is a special instance of classical $\mathcal{A}$-discriminant theory. Indeed, applying the formula (1.38), p. 406, of [Gelfand et al. 1994], it is easy to verify that the discriminant of p vanishes just when (3-8) holds. The formula (1.28), p. 404, of [Gelfand et al. 1994] yields an algorithm for direct detection of the double root z_d of p (more precisely, k and ℓ powers of z_d can be expressed as linear rational functions of the coefficients of p). In general, $\mathcal{A}$-discriminants are useful to explain why many trinomial properties can be described explicitly (contrary to polynomials with more than three terms).

(c) The formula for argument φ can be equivalently expressed in the form

$$\varphi=\varphi^{\pm}\equiv\begin{cases}\dfrac{(2u_0\tau_\varrho^{+}+2v_0\tau_\varrho^{+}-1)\pi-\alpha_\varrho-\theta_a+\theta_b}{\ell} \pmod{2\pi} & \text{if } \tau_\varrho^{+}\in\mathbb{Z},\\ \dfrac{(2u_0\tau_\varrho^{-}+2v_0\tau_\varrho^{-}-1)\pi+\alpha_\varrho-\theta_a+\theta_b}{\ell} \pmod{2\pi} & \text{if } \tau_\varrho^{-}\in\mathbb{Z},\end{cases} \tag{3-9}$$

where u_0 is the first component of a couple of Bézout coefficients (u_0, v_0), and α_ϱ is given by (1-1). Because of (3-4), both the formulae (3-6) and (3-9) are independent of a concrete choice of Bézout coefficients.

Proof of Theorem 3.1 and Remark 3.2. Let $z=\varrho\exp(\mathrm{i}\varphi)$, $\varphi\in(-\pi,\pi]$, be a ϱ-modular root of p. Then

$$\varrho^k\exp(\mathrm{i}k\varphi)+|a|\varrho^\ell\exp(\mathrm{i}(\ell\varphi+\theta_a))+|b|\exp(\mathrm{i}\theta_b)=0,$$

i.e.,

$$\begin{aligned}|a|\varrho^\ell\cos(\ell\varphi+\theta_a)+|b|\cos(\theta_b)&=-\varrho^k\cos(k\varphi),\\ |a|\varrho^\ell\sin(\ell\varphi+\theta_a)+|b|\sin(\theta_b)&=-\varrho^k\sin(k\varphi).\end{aligned} \tag{3-10}$$

We solve (3-10) with respect to (positive real) unknowns $|a|$, $|b|$.

First, let the system matrix be singular, that is, $\sin(\ell\varphi+\theta_a-\theta_b)=0$. Then (3-10) has a solution $|a|$, $|b|$ if and only if

$$k\varphi-\theta_b=j_1\pi,\quad (k-\ell)\varphi-\theta_a=j_2\pi$$

for suitable integers j_1, j_2. Equivalently,

$$\varphi=\frac{j_1\pi+\theta_b}{k}=\frac{(j_1-j_2)\pi-\theta_a+\theta_b}{\ell}, \tag{3-11}$$

which implies

$$\frac{\theta}{\pi}=-j_2k+(j_1+1)(k-\ell) \tag{3-12}$$

due to (1-2). Substituting (3-11) into (3-10) one gets

$$\begin{aligned}|a|\varrho^\ell\cos(\theta_b+(j_1-j_2)\pi)+|b|\cos(\theta_b)&=-\varrho^k\cos(\theta_b+j_1\pi),\\ |a|\varrho^\ell\sin(\theta_b+(j_1-j_2)\pi)+|b|\sin(\theta_b)&=-\varrho^k\sin(\theta_b+j_1\pi),\end{aligned}$$

i.e.,

$$(-1)^{j_1-j_2}|a|\varrho^\ell\cos(\theta_b)+|b|\cos(\theta_b)=(-1)^{j_1+1}\varrho^k\cos(\theta_b),$$
$$(-1)^{j_1-j_2}|a|\varrho^\ell\sin(\theta_b)+|b|\sin(\theta_b)=(-1)^{j_1+1}\varrho^k\sin(\theta_b).$$

This yields

$$(-1)^{j_1}\varrho^k+(-1)^{j_1-j_2}|a|\varrho^\ell+|b|=0. \tag{3-13}$$

The case when both j_1 and j_2 are even cannot occur in (3-13). We explore the remaining parity variants.

If j_1 is odd, j_2 even, then (3-13) becomes $|b|=\varrho^k+|a|\varrho^\ell$, i.e., $\alpha_\varrho=\beta_\varrho=0$. Moreover, the right-hand side of (3-12) is even, which implies the first couple of conditions for the existence of a ϱ-modular root of p in the form

$$|b|=\varrho^k+|a|\varrho^\ell,\quad \tau_\varrho^\pm=\frac{\theta}{2\pi}\text{ is an integer.} \tag{3-14}$$

In addition, (3-12) is the linear Diophantine equation (3-1) with

$$u=-\frac{j_2}{2},\quad v=\frac{j_1+1}{2},\quad\text{and}\quad \tau=\tau_\varrho^\pm=\frac{\theta}{2\pi}.$$

Then (3-2) and (3-4) imply

$$j_1=2(v_0-km)\tau_\varrho^\pm-1,\quad j_2=-2(u_0+(k-\ell)m)\tau_\varrho^\pm \tag{3-15}$$

for an integer m. Now it is enough to substitute $(3\text{-}15)_1$ into $(3\text{-}11)_1$ to obtain

$$\varphi\equiv\frac{(2v_0\tau_\varrho^\pm-1)\pi+\theta_b}{k}\pmod{2\pi}. \tag{3-16}$$

If both j_1 and j_2 are odd, then (3-13) yields $\varrho^k=|a|\varrho^\ell+|b|$, that is, $\alpha_\varrho=\pi$, $\beta_\varrho=0$. Also, (3-12) can be written as

$$\frac{\theta}{\pi}+k=(1-j_2)k+(j_1+1)(k-\ell). \tag{3-17}$$

Since the right-hand side of (3-17) is even, we get another couple of conditions

$$\varrho^k=|a|\varrho^\ell+|b|,\quad \tau_\varrho^+=\frac{\theta}{2\pi}+\frac{k}{2}\text{ is an integer.} \tag{3-18}$$

Furthermore, (3-17) is (3-1) with

$$u=\frac{1-j_2}{2},\quad v=\frac{j_1+1}{2},\quad\text{and}\quad \tau=\tau_\varrho^+=\frac{\theta}{2\pi}+\frac{k}{2}.$$

Then (3-2) and (3-4) yield

$$j_1=2\tau_\varrho^+(v_0-km)-1,\quad j_2=1-2\tau_\varrho^+(u_0+(k-\ell)m)$$

for an integer m; hence (3-11)$_1$ becomes

$$\varphi \equiv \frac{(2v_0\tau_\varrho^+ - 1)\pi + \theta_b}{k} \pmod{2\pi}. \tag{3-19}$$

If j_1 is even, j_2 odd, then (3-13) and (3-12) imply $|a|\varrho^\ell = \varrho^k + |b|$, that is, $\alpha_\varrho = 0$, $\beta_\varrho = \pi$, and

$$\frac{\theta}{\pi} + \ell = (1-j_2)k + j_1(k-\ell), \tag{3-20}$$

respectively. Since the right-hand side of (3-20) is even, we get the third couple of conditions for the existence of a ϱ-modular root of p, namely

$$|a|\varrho^\ell = \varrho^k + |b|, \quad \tau_\varrho^+ = \frac{\theta}{2\pi} + \frac{\ell}{2} \text{ is an integer.} \tag{3-21}$$

Obviously, (3-20) is (3-1) with

$$u = \frac{1-j_2}{2}, \quad v = \frac{j_1}{2}, \quad \text{and} \quad \tau = \tau_\varrho^+ = \frac{\theta}{2\pi} + \frac{\ell}{2},$$

which along with (3-2) and (3-4) yields

$$j_1 = 2\tau_\varrho^+(v_0 - km), \quad j_2 = 1 - 2\tau_\varrho^+(u_0 + (k-\ell)m) \tag{3-22}$$

for an integer m. Then we substitute (3-22)$_1$ into (3-11)$_1$ to get

$$\varphi \equiv \frac{2v_0\tau_\varrho^+\pi + \theta_b}{k} \pmod{2\pi}. \tag{3-23}$$

Now we assume the system matrix of (3-10) to be regular, in other words, $\sin(\ell\varphi + \theta_a - \theta_b) \neq 0$. In this case, the solution of (3-10) is given by

$$|a| = -\varrho^{k-\ell}\frac{\sin(k\varphi - \theta_b)}{\sin(\ell\varphi + \theta_a - \theta_b)}, \quad |b| = \varrho^k \frac{\sin((k-\ell)\varphi - \theta_a)}{\sin(\ell\varphi + \theta_a - \theta_b)}. \tag{3-24}$$

At the same time, we solve (3-10) with respect to φ. To do this, we square and sum (3-10) to obtain

$$\cos(\ell\varphi + \theta_a - \theta_b) = \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} - |b|^2}{2|ab|\varrho^\ell}. \tag{3-25}$$

Alternatively, we can write (3-10) as

$$\begin{aligned}\varrho^k\cos(k\varphi) + |b|\cos(\theta_b) &= -|a|\varrho^\ell\cos(\ell\varphi + \theta_a),\\ \varrho^k\sin(k\varphi) + |b|\sin(\theta_b) &= -|a|\varrho^\ell\sin(\ell\varphi + \theta_a),\end{aligned}$$

where repeated squaring and summation yield

$$\cos(k\varphi - \theta_b) = -\frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}. \tag{3-26}$$

Now we analyze (3-25) and (3-26). Since the values of their right-hand sides have to lie between -1 and 1, straightforward calculations imply

$$|b| < \varrho^k + |a|\varrho^\ell, \quad \varrho^k < |a|\varrho^\ell + |b|, \quad |a|\varrho^\ell < \varrho^k + |b| \tag{3-27}$$

(the strict inequalities occur here due to $\sin(\ell\varphi + \theta_a - \theta_b) \neq 0$, $\sin(k\varphi - \theta_b) \neq 0$). Further, to express φ from (3-25) and (3-26), we discuss the arguments of appropriate goniometric functions appearing here.

Let $\sin(\ell\varphi + \theta_a - \theta_b) > 0$. Then $|a|$, $|b|$ given by (3-24) are positive if and only if

$$\sin(k\varphi - \theta_b) < 0, \quad \sin((k-\ell)\varphi - \theta_a) > 0.$$

Equivalently, there exist some integers u, v such that

$$\begin{gathered} (2v-1)\pi < k\varphi - \theta_b < 2v\pi, \\ -2u\pi < (k-\ell)\varphi - \theta_a < (-2u+1)\pi, \\ (2u+2v-2)\pi < \ell\varphi + \theta_a - \theta_b < (2u+2v-1)\pi. \end{gathered} \tag{3-28}$$

Then, using (3-28), we can rewrite (3-25) and (3-26) into

$$\ell\varphi + \theta_a - \theta_b - (2u+2v-2)\pi = \arccos \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} - |b|^2}{2|ab|\varrho^\ell}$$

and

$$2v\pi - (k\varphi - \theta_b) = \arccos \frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} - |b|^2}{2|b|\varrho^k},$$

respectively. From here,

$$\varphi = \frac{(2v-1)\pi + \beta_\varrho + \theta_b}{k} = \frac{(2u+2v-1)\pi - \alpha_\varrho - \theta_a + \theta_b}{\ell}. \tag{3-29}$$

The equality of two ratios in (3-29) is equivalent to

$$ku + (k-\ell)v = \frac{\theta}{2\pi} + \frac{k\alpha_\varrho + \ell\beta_\varrho}{2\pi}. \tag{3-30}$$

This particularly implies that

$$\tau_\varrho^+ = \frac{\theta}{2\pi} + \frac{k\alpha_\varrho + \ell\beta_\varrho}{2\pi} \text{ is an integer.} \tag{3-31}$$

Moreover, (3-30) is the Diophantine equation (3-1) with $\tau = \tau_\varrho^+$, which along with (3-2), (3-4) and $(3\text{-}29)_1$ yields

$$\varphi = \varphi^+ \equiv \frac{(2v_0\tau_\varrho^+ - 1)\pi + \beta_\varrho + \theta_b}{k} \pmod{2\pi}. \tag{3-32}$$

Now let $\sin(\ell\varphi+\theta_a-\theta_b)<0$. An analogous argumentation yields

$$\begin{aligned}(2v-2)\pi &< k\varphi-\theta_b < (2v-1)\pi,\\ (-2u-1)\pi &< (k-\ell)\varphi-\theta_a < -2u\pi,\\ (2u+2v-1)\pi &< \ell\varphi+\theta_a-\theta_b < (2u+2v)\pi\end{aligned}$$

for some integers u, v, and

$$\varphi=\frac{(2v-1)\pi-\beta_\varrho+\theta_b}{k}=\frac{(2u+2v-1)\pi+\alpha_\varrho-\theta_a+\theta_b}{\ell}. \tag{3-33}$$

This implies

$$ku+(k-\ell)v=\frac{\theta}{2\pi}-\frac{k\alpha_\varrho+\ell\beta_\varrho}{2\pi}. \tag{3-34}$$

Therefore,

$$\tau_\varrho^-=\frac{\theta}{2\pi}-\frac{k\alpha_\varrho+\ell\beta_\varrho}{2\pi}\ \text{is an integer,} \tag{3-35}$$

and (3-29)$_1$ supported by (3-34), (3-2), (3-4) results in

$$\varphi=\varphi^-\equiv\frac{(2v_0\tau_\varrho^- -1)\pi-\beta_\varrho+\theta_b}{k}\pmod{2\pi}. \tag{3-36}$$

If we summarize the conditions (3-14), (3-18), (3-21), (3-27), (3-31), and (3-35), then we get the first assertion of Theorem 3.1. The formula (3-6) for arguments of appropriate ϱ-modular roots then follows from (3-16), (3-19), (3-23), (3-32), and (3-36).

Finally, we verify the observations mentioned in Remark 3.2. The part (a) follows directly from the above proof procedures. We confirm the conditions for the appearance of a ϱ-modular double root of p stated in the part (b). Let $z=\varrho\exp(i\varphi)$ be such a root. Then substitution into $p(z)=p'(z)=0$ (supported by some straightforward calculations) yields

$$|a|\varrho^\ell=\varrho^k+|b|=\frac{k}{\ell}\varrho^k. \tag{3-37}$$

This implies $\alpha_\varrho=0$, $\beta_\varrho=\pi$ and $\theta/(2\pi)+\ell/2$ is an integer. It is easy to verify that (3-37) is equivalent with (3-8)$_1$ and (3-8)$_2$. Also,

$$\varphi^+\equiv\frac{2v_0\tau_\varrho^+\pi+\theta_b}{k}\pmod{2\pi},\qquad \varphi^-\equiv\frac{(2v_0\tau_\varrho^- -2)\pi+\theta_b}{k}\pmod{2\pi}$$

due to (3-6). Since

$$\tau_\varrho^+=\frac{\theta}{2\pi}+\frac{\ell}{2},\qquad \tau_\varrho^-=\frac{\theta}{2\pi}-\frac{\ell}{2},$$

we get $\varphi^+ = \varphi^-$ by use of (3-3). Regarding the observation (c), the formula (3-9) follows from alternate expressions of φ in (3-11), (3-29) and (3-33). More precisely, while the use of first expressions in (3-11), (3-29) and (3-33) results into (3-6), the use of the latter ones implies (3-9). □

Based on Theorem 3.1 and Remark 3.2, it is possible to deduce some other basic root properties of complex trinomials. For example, assume that p has a couple of complex conjugate imaginary roots $z = \varrho \exp(\pm \mathrm{i}\varphi) = \varrho \exp(\mathrm{i}\varphi^\pm)$, i.e., $\varphi^+ = -\varphi^-$. Then, in addition to (3-7), the condition

$$\frac{(2v_0\tau_\varrho^+ - 1)\pi + \beta_\varrho + \theta_b}{k} + \frac{(2v_0\tau_\varrho^- - 1)\pi - \beta_\varrho + \theta_b}{k} = 2j\pi,$$

or equivalently,

$$(k\theta_a - (k-\ell)\theta_b)v_0 + ((k-\ell)v_0 - 1)\pi + \theta_b = kj\pi, \tag{3-38}$$

has to be met for an integer j. Obviously, (3-7) and (3-38) imply $\theta_a, \theta_b \in \{0, \pi\}$. Thus we get:

Corollary 3.3. *Suppose p admits a pair of complex conjugate imaginary roots. Then its coefficients a, b have to be real numbers.*

To summarize our previous investigations: Theorem 3.1 is effective in the sense that, for any $\varrho > 0$, it enables us to describe the set of all a, b, k, ℓ such that p admits a ϱ-modular root. Moreover, we are able to specify arguments of such roots. Then a crucial question arises, namely whether (and possibly how) this conclusion can contribute to discussions on location of trinomial roots in the complex plane (see problem (D)). In the next section, we are going to discuss this matter in more detail.

4. Calculating moduli and arguments of roots of *p*

We consider the trinomial p with arbitrary but fixed nonzero complex numbers a, b and coprime positive integers $k > \ell$. Let $z = \varrho \exp(\mathrm{i}\varphi)$, $\varphi \in (-\pi, \pi]$, be a root of p. Our task is to find conditions on ϱ and φ expressed in terms of entry parameters a, b, k, ℓ.

By Theorem 3.1, z is a root of p if and only if (3-5) is true and

$$\theta + k\alpha_\varrho + \ell\beta_\varrho = 2s\pi \quad \text{or} \quad \theta - (k\alpha_\varrho + \ell\beta_\varrho) = 2s\pi \qquad \text{for an integer } s, \tag{4-1}$$

α_ϱ, β_ϱ and θ being given by (1-1) and (1-2), respectively. To analyze (4-1), we introduce the function

$$F(\varrho) = k\alpha_\varrho + \ell\beta_\varrho = k \arccos \frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2}{2|ab|\varrho^\ell} + \ell \arccos \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}$$

whose domain is described just by the triplet of inequalities (3-5). Then (4-1) becomes

$$F(\varrho) = |\theta - 2s\pi|, \quad s \text{ is an integer.} \tag{4-2}$$

In the sequel, we describe some basic properties of F, namely its domain $D(F)$, image $H(F)$ and monotony.

To clarify the domain of F with respect to (3-5), we need to perform a proper sign analysis of real trinomials

$$Q_1(\varrho) = \varrho^k + |a|\varrho^\ell - |b|, \quad Q_2(\varrho) = \varrho^k - |a|\varrho^\ell - |b|, \quad Q_3(\varrho) = \varrho^k - |a|\varrho^\ell + |b|$$

considered for $\varrho \geq 0$. On this account, we put

$$\sigma(k, \ell) = \frac{k}{k-\ell}\left(\frac{k-\ell}{\ell}\right)^{\ell/k}. \tag{4-3}$$

Then, based on elementary calculations, the following observations hold.

Proposition 4.1. (i) *There is a unique positive root ξ_L of Q_1 such that $Q_1(\varrho) < 0$ for all $0 \leq \varrho < \xi_L$, and $Q_1(\varrho) > 0$ for all $\varrho > \xi_L$.*

(ii) *There is a unique positive root ξ_R of Q_2 such that $Q_2(\varrho) < 0$ for all $0 \leq \varrho < \xi_R$, and $Q_2(\varrho) > 0$ for all $\varrho > \xi_R$.*

(iii) *If $|a|\,|b|^{(\ell-k)/k} < \sigma(k, \ell)$, then Q_3 is positive for all $\varrho \geq 0$.*

(iv) *If $|a|\,|b|^{(\ell-k)/k} = \sigma(k, \ell)$, then a unique positive double root ξ_M of Q_3 appears (and Q_3 is positive otherwise).*

(v) *If $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$, then Q_3 has a couple of positive roots $\xi_{M_1} < \xi_{M_2}$ such that $Q_3(\varrho) > 0$ if either $0 \leq \varrho < \xi_{M_1}$, or $\varrho > \xi_{M_2}$, and $Q_3(\varrho) < 0$ whenever $\xi_{M_1} < \varrho < \xi_{M_2}$.*

(vi) *If the assumption of the property (iii), or (iv), or (v) holds, then*

$$\xi_L < \xi_R, \quad \text{or} \quad \xi_L < \xi_M < \xi_R, \quad \text{or} \quad \xi_L < \xi_{M_1} < \xi_{M_2} < \xi_R,$$

respectively.

Remark 4.2. In a slightly different context, the properties (i)–(vi) were described for a triplet of related trinomials Φ, χ, Ψ in [Melman 2012].

Thus, keeping in mind (3-5), we get the following description of the domain of F.

Lemma 4.3. *Let ξ_L, ξ_R and ξ_{M_1}, ξ_{M_2} be positive roots of Q_1, Q_2 and Q_3, respectively, whose existence is guaranteed by the conditions of Proposition 4.1.*

(i) *If $|a|\,|b|^{(\ell-k)/k} \leq \sigma(k, \ell)$, then $D(F) = [\xi_L, \xi_R]$.*

(ii) *If $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$, then $D(F) = [\xi_L, \xi_{M_1}] \cup [\xi_{M_2}, \xi_R]$.*

The next assertion reveals other important properties of F (we still assume that ξ_L, ξ_R and ξ_{M_1}, ξ_{M_2} are positive roots of Q_1, Q_2 and Q_3, respectively).

Lemma 4.4. *The function F defines a strictly increasing mapping of $D(F)$ onto $H(F) = [0, k\pi]$. This mapping is continuous on $[\xi_L, \xi_R]$ provided $|a|\,|b|^{(\ell-k)/k} \le \sigma(k, \ell)$, and it is continuous on $[\xi_L, \xi_{M_1}]$ and $[\xi_{M_2}, \xi_R]$ provided $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. In this case, $F(\xi_{M_1}) = F(\xi_{M_2}) = \ell\pi$.*

Proof. Obviously, the values of F are nonnegative. More precisely, $F(\xi_L) = 0$, and $F(\varrho) > 0$ for all $\varrho \in D(F)$, $\varrho > \xi_L$. Direct calculations confirm that $F(\xi_R) = k\pi$ and $F(\xi_{M_1}) = F(\xi_{M_2}) = \ell\pi$. It remains to show that F is strictly increasing on its domain.

After some straightforward calculations and using the relations

$$(2\,|ab|\,\varrho^\ell)^2 - (-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2)^2 = -Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\,\varrho^\ell + |b|),$$
$$(2\,|b|\,\varrho^k)^2 - (\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2)^2 = -Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\,\varrho^\ell + |b|),$$

one gets the derivative of F with respect to ϱ in the form

$$F'(\varrho) = \frac{-G(\varrho)}{\sqrt{-Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\,\varrho^\ell + |b|)}},$$

where

$$G(\varrho) = 2k\,|ab|\,\varrho^\ell\left(\frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2}{2\,|ab|\,\varrho^\ell}\right)' + 2\ell\,|b|\,\varrho^k\left(\frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2\,|b|\,\varrho^k}\right)'.$$

Then

$$F'(\varrho) = \frac{2(k^2\varrho^{2k} - k\ell\varrho^{2k} - k\ell\,|a|^2\varrho^{2\ell} + \ell^2|a|^2\varrho^{2\ell} + k\ell\,|b|^2)}{\varrho\sqrt{-Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\,\varrho^\ell + |b|)}},$$

i.e.,

$$F'(\varrho) = \frac{2(-k\varrho^k + \ell\,|a|\,\varrho^\ell)^2 - 2k\ell\,Q_2(\varrho)Q_3(\varrho)}{\varrho\sqrt{-Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\,\varrho^\ell + |b|)}}, \qquad Q_1(\varrho)Q_2(\varrho)Q_3(\varrho) \neq 0.$$

Since $Q_2(\varrho)Q_3(\varrho) \le 0$ on $D(F)$, F' is positive on (ξ_L, ξ_R) provided $|a|\,|b|^{(\ell-k)/k} \le \sigma(k, \ell)$, and it is positive on $(\xi_L, \xi_{M_1}) \cup (\xi_{M_2}, \xi_R)$ provided $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. Since $F(\xi_{M_1}) = F(\xi_{M_2})$, we can conclude that F is strictly increasing on $D(F)$. ☐

Now we come back to the analysis of (4-2). Based on Lemma 4.4, its geometric interpretation is the following: we search for intersections ϱ of the function F (whose values are monotonically varying on $H(F) = [0, k\pi]$), and the modulus of the value θ moved via an integer multiple s of 2π.

To ensure the existence of such an intersection (and thus also solvability of (4-2)), we have to require

$$-k\pi \le \theta - 2s\pi \le k\pi.$$

This condition generates k integer values of s up to the case when $\theta/\pi + k$ is an even integer. In this case, there are $k+1$ integer values of s meeting the previous inequality; in particular, it is satisfied in the form of equality for two values of s, namely for $s = \theta/(2\pi) - k/2$ and $s = \theta/(2\pi) + k/2$. Substitution of both these values into (4-2) yields the same equation $F(\varrho) = k\pi$ having the unique root ξ_R. Also, (3-6) yields the same value of the argument φ for both the values. Altogether, these two values of s generate the same (simple) root of p; hence we can restrict to

$$-k\pi < \theta - 2s\pi \le k\pi,$$

i.e.,

$$s = \left\lceil \frac{\theta}{2\pi} - \frac{k}{2} \right\rceil, \left\lceil \frac{\theta}{2\pi} - \frac{k}{2} \right\rceil + 1, \dots, \left\lceil \frac{\theta}{2\pi} + \frac{k}{2} \right\rceil - 1. \tag{4-4}$$

Now let $(s_1, \dots, s_k)$ be a permutation of the k-tuple of integers from (4-4) such that

$$|\theta - 2s_j\pi| \le |\theta - 2s_{j+1}\pi| \quad \text{for all } j = 1, \dots, k-1. \tag{4-5}$$

For the sake of uniqueness, if the equality sign occurs here (it happens just when θ is an integer multiple of π), then we assume $\theta - 2s_j\pi > 0$ and $\theta - 2s_{j+1}\pi < 0$. To get a more explicit prescription for s_j, it is enough to rewrite (4-5) as

$$\left| \frac{\theta}{2\pi} - s_j \right| \le \left| \frac{\theta}{2\pi} - s_{j+1} \right| \quad \text{for all } j = 1, \dots, k-1,$$

i.e., values s_j are ordered with respect to their distance from $\theta/(2\pi)$. Using this geometric interpretation, it is easy to check that

$$s_1 = \operatorname{round}\left(\frac{\theta}{2\pi}\right), \qquad s_j = s_1 + \kappa \left\lfloor \frac{j}{2} \right\rfloor, \quad j = 2, \dots, k, \tag{4-6}$$

where round$(\,\cdot\,)$ means the nearest integer value (if θ/π is an odd integer, then we put $s_1 = \theta/(2\pi) - \frac{1}{2}$), and $\kappa = (-1)^j$ if $s_1 < \theta/(2\pi)$, or $\kappa = (-1)^{j+1}$ if $s_1 \ge \theta/(2\pi)$.

Now we are ready to formulate an algorithm for computations of moduli and arguments of roots of a given trinomial.

Theorem 4.5. *Let* $z_j = \varrho_j \exp(\mathrm{i}\varphi_j)$, $\varphi_j \in (-\pi, \pi]$, $j = 1, \dots, k$, *be roots of* p, *where* a, b *are nonzero complex numbers and* $k > \ell$ *are coprime positive integers. Further, let* s_j, $j = 1, \dots, k$, *be given by* (4-6). *Then* ϱ_j *are* (*unique*) *roots of*

$$F(\varrho) = |\theta - 2s_j\pi|, \quad j = 1, \dots, k, \tag{4-7}$$

and

$$\varphi_j \equiv \begin{cases} \dfrac{(2v_0 s_j - 1)\pi + \beta_{\varrho_j} + \theta_b}{k} \pmod{2\pi} & \text{if } \theta - 2s_j\pi \le 0, \\[2ex] \dfrac{(2v_0 s_j - 1)\pi - \beta_{\varrho_j} + \theta_b}{k} \pmod{2\pi} & \text{if } \theta - 2s_j\pi > 0. \end{cases} \tag{4-8}$$

Here, v_0 is the second component of a couple of Bézout coefficients (u_0, v_0) satisfying (3-3), and β_{ϱ_j} are given by (1-1) with $\varrho = \varrho_j$.

Remark 4.6. Theorem 4.5 offers a computational procedure for finding moduli ϱ_j and arguments φ_j of all roots of p. Based on this procedure, we are able to find all k moduli ϱ_j as roots of k (real) transcendental equations (4-7) with appropriate integer values s_j, $j = 1, \dots, k$, given by (4-6). In particular, we can a priori determine s_j such that (4-7) generates the maximal modulus. Then, a deeper analysis of such an equation may result in strong bounds of the maximal modulus. Note that this matter is crucial (and still insufficiently analyzed) in the frame of asymptotic theory for autonomous difference equations.

Despite a lot of literature on solving sparse polynomials (see, for example, [Tonelli-Cueto and Tsigaridas 2023]), we believe that Theorem 4.5 can provide a new insight into the distribution problem of trinomial roots. By (4-7), the moduli ϱ_j are the intersections of the transcendental function F (depending on moduli of a, b), and a constant function (depending on arguments of a, b) that is moved (in modulus) via an integer multiple of 2π. Furthermore, (4-8) yields the exact formula for the dependence of arguments φ_j on moduli ϱ_j. Besides its direct meaning, this formula might be useful in discussions on argument discrepancies and related equidistribution properties of roots of complex trinomials (see, for example, [D'Andrea et al. 2014] and [Erdős and Turán 1950]).

Some other comments on the application of Theorem 4.5 are presented in Example 5.3.

Proof of Theorem 4.5. The part describing calculations of the moduli ϱ_j follows from observations preceding this assertion. The formula (4-8) for the values of arguments φ_j is a direct consequence of (3-6). ☐

Now we present two consequences of Theorem 4.5. The first assertion answers problem (C), that is, formulates conditions under which p has two roots with the same modulus.

Corollary 4.7. *Let ϱ_j, $j = 1, \dots, k$, be moduli of roots of p labeled with respect to (4-7). Further, let ξ_L, ξ_R and ξ_M, ξ_{M_1}, ξ_{M_2} be positive roots of Q_1, Q_2 and Q_3, respectively (their existence is guaranteed by Proposition 4.1). Finally, let θ and $\sigma(k, \ell)$ be given by (1-2) and (4-3), respectively. We distinguish two cases:*

(i) *If θ/π is not an integer, then the ϱ_j satisfy the strict inequality*

$$\xi_L < \varrho_1 < \varrho_2 < \cdots < \varrho_k < \xi_R.$$

(ii) *If θ/π is an integer, then the ordering of ϱ_j is summarized in Table 1. Here, we use the symbols E or O if the appropriate integer values are even or odd, respectively, and put $\Sigma = \Sigma(a, b, k, \ell) = |a|\,|b|^{(\ell-k)/k}/\sigma(k, \ell)$.*

θ/π	k	ℓ	Σ	moduli ϱ_j $(j=1,\dots,k)$: their ordering and specific values
E	O	O		$\xi_L = \varrho_1 < \varrho_2 = \varrho_3 < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
O	O	E		$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_{k-2} = \varrho_{k-1} < \varrho_k = \xi_R$
O	E	O	<1	$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_\ell = \varrho_{\ell+1} < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
			$=1$	$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_\ell = \xi_M = \varrho_{\ell+1} < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
			>1	$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_\ell = \xi_{M_1} < \xi_{M_2} = \varrho_{\ell+1} < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
E	E	O		$\xi_L = \varrho_1 < \varrho_2 = \varrho_3 < \cdots < \varrho_{k-2} = \varrho_{k-1} < \varrho_k = \xi_R$
E	O	E	<1	$\xi_L = \varrho_1 < \varrho_2 = \varrho_3 < \cdots < \varrho_\ell = \varrho_{\ell+1} < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
			$=1$	$\xi_L = \varrho_1 < \varrho_2 = \varrho_3 < \cdots < \varrho_\ell = \xi_M = \varrho_{\ell+1} < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
			>1	$\xi_L = \varrho_1 < \varrho_2 = \varrho_3 < \cdots < \varrho_\ell = \xi_{M_1} < \xi_{M_2} = \varrho_{\ell+1} < \cdots < \varrho_{k-1} = \varrho_k < \xi_R$
O	O	O	<1	$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_\ell = \varrho_{\ell+1} < \cdots < \varrho_{k-2} = \varrho_{k-1} < \varrho_k = \xi_R$
			$=1$	$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_\ell = \xi_M = \varrho_{\ell+1} < \cdots < \varrho_{k-2} = \varrho_{k-1} < \varrho_k = \xi_R$
			>1	$\xi_L < \varrho_1 = \varrho_2 < \cdots < \varrho_\ell = \xi_{M_1} < \xi_{M_2} = \varrho_{\ell+1} < \cdots < \varrho_{k-2} = \varrho_{k-1} < \varrho_k = \xi_R$

Table 1. Ordering of moduli ϱ_j $(j=1,\dots,k)$ provided θ/π is an integer (for several specific values of k and ℓ, some parts of the presented inequalities lose a formal sense; in such cases, these inequalities need to be simplified appropriately).

Proof. If θ/π is not an integer, then the strict ordering of moduli ϱ_j follows from the strict monotony of the sequence $(|\theta - 2s_j\pi|)_{j=1}^k$. If θ/π is an integer, then $|\theta - 2s_j\pi| = |\theta - 2s_{j+1}\pi|$; hence $\varrho_j = \varrho_{j+1}$ for some $j = 1,\dots,k-1$. Furthermore, θ/π is an integer if and only if at least one of the numbers

$$\frac{\theta}{\pi}, \quad \frac{\theta}{\pi} + k, \quad \frac{\theta}{\pi} + \ell$$

is even (note that all three values cannot be simultaneously even). This implies six parity variants concerning θ/π, k and ℓ that produce slightly different conclusions on the ordering of the ϱ_j presented in Table 1 (this ordering reflects a type of monotony of the sequence $(|\theta - 2s_j\pi|)_{j=1}^{k}$, and its derivation is quite straightforward for any of the six variants). □

Remark 4.8. If θ/π is an integer, then Table 1 immediately implies the following location of moduli ϱ_j, $j = 1, \ldots, k$ (we still assume that the ϱ_j are labeled with respect to (4-7)): Let $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. Then

(4-9) $\xi_L \le \varrho_j \le \xi_{M_1}$, $j = 1, \ldots, \ell$, and $\xi_{M_2} \le \varrho_j \le \xi_R$, $j = \ell+1, \ldots, k$.

In fact, it is easy to check that (4-9) holds for noninteger values of θ/π as well. Indeed, by Proposition 4.1 (the case (vi)) and Lemma 4.4, it is enough to search for a number of solutions of (4-2), where $\xi_L \le \varrho \le \xi_{M_1}$, i.e., $0 \le F(\varrho) \le \ell\pi$. Thus, we need to find all s_j from (4-6) such that $\ell\pi \ge |\theta - 2s_j\pi|$. Equivalently,

$$\frac{\theta}{2\pi} - \frac{\ell}{2} < s_j < \frac{\theta}{2\pi} + \frac{\ell}{2}.$$

Exactly ℓ values of s_j from (4-6) (namely $s_1, \ldots, s_\ell$) satisfy these inequalities.

As the second consequence of Theorem 4.5, we state some interesting connections between moduli and arguments of roots z_ℓ, $z_{\ell+1}$ of p. In particular, we describe the situation when p has two roots with the same argument.

Corollary 4.9. *Let ϱ_j and φ_j, $j = \ell, \ell+1$, be moduli and arguments of roots z_ℓ, $z_{\ell+1}$ of p labeled with respect to (4-7) and (4-8), respectively. Let θ and $\sigma(k, \ell)$ be given by (1-2) and (4-3), respectively, and let θ/π and ℓ have the same parity. Then three qualitatively different relations between moduli and arguments of z_ℓ, $z_{\ell+1}$ can occur:*

(i) *If $|a|\,|b|^{(\ell-k)/k} < \sigma(k, \ell)$, then $\varrho_\ell = \varrho_{\ell+1}$ and $\varphi_\ell \ne \varphi_{\ell+1}$ (i.e., we have two distinct simple roots $z_\ell \ne z_{\ell+1}$ with the same moduli and different arguments).*

(ii) *If $|a|\,|b|^{(\ell-k)/k} = \sigma(k, \ell)$, then $\varrho_\ell = \varrho_{\ell+1}$ and $\varphi_\ell = \varphi_{\ell+1}$ (i.e., we have a double root $z_\ell = z_{\ell+1}$).*

(iii) *If $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$, then $\varrho_\ell < \varrho_{\ell+1}$ and $\varphi_\ell = \varphi_{\ell+1}$ (i.e., we have two distinct simple roots $z_\ell \ne z_{\ell+1}$ with the same arguments and different moduli).*

Proof. Since θ/π and ℓ have the same parity, the properties (i) and (ii) follow from Table 1 (with respect to the conditions of Remark 3.2 (b)).

Let $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. Then $\varrho_\ell = \xi_{M_1} < \xi_{M_2} = \varrho_{\ell+1}$. Because of (3-6) and the fact that $\alpha_{\varrho_\ell} = \alpha_{\varrho_{\ell+1}} = 0$, $\beta_{\varrho_\ell} = \beta_{\varrho_{\ell+1}} = \pi$, the arguments φ_ℓ and $\varphi_{\ell+1}$ are generated by the formula

$$\varphi^{\pm} \equiv \frac{(\theta \pm \ell\pi)v_0 - \pi \pm \pi + \theta_b}{k} \pmod{2\pi}.$$

Since

$$\frac{(\theta - \ell\pi)v_0 - 2\pi + \theta_b}{k} = \frac{(\theta + \ell\pi)v_0 + \theta_b}{k} - \frac{\ell v_0 + 1}{k} 2\pi,$$

both the values $\varphi^{\pm}$ coincide, that is, $\varphi_\ell = \varphi_{\ell+1}$. □

5. Some comparisons with existing results

We compare conclusions of our main results (Theorems 3.1 and 4.5) with previous answers to problems (B)–(D).

5.1. ***Problem (B) — Theorem 3.1 and some earlier results.*** We start with comparisons between our Theorem 3.1 and Theorem 4.1 of [Theobald and de Wolff 2016] solving problem (B). Its formulation uses a roulette curve called a hypotrochoid which depends on three general positive real constants r, R, d with $r < R$. In the Gauss a-plane, this curve can be described by the parametric equation

$$\Re(a) + \mathrm{i}\Im(a) = (R-r)\exp(\mathrm{i}t) + d\exp\left(\mathrm{i}\frac{r-R}{r}t\right),$$

where t is a real parameter. If R/r is a rational number, then the hypotrochoid is a closed curve (for more interesting properties of this curve see [Lockwood 2007]). The assertion of Theorem 4.1 of [Theobald and de Wolff 2016] now can be formulated as follows:

> *Let b be a nonzero complex number, $k > \ell$ be coprime positive integers, and let ϱ be a positive real number. Then p has a ϱ-modular root z if and only if its complex coefficient a is located on a hypotrochoid up to a rotation with parameters*
>
> $$R = \frac{k\varrho^{k-\ell}}{\ell}, \quad r = \frac{(k-\ell)\varrho^{k-\ell}}{\ell}, \quad d = |b|\,\varrho^{-\ell}.$$

Equivalently, this condition says that p has a ϱ-modular root z if and only if its complex coefficient a satisfies

$$\begin{aligned} \Re(a) &= -(R-r)\cos\left(t + \frac{r}{R}\theta_b\right) - d\cos\left(\frac{r-R}{r}t + \frac{r}{R}\theta_b\right), \\ \Im(a) &= -(R-r)\sin\left(t + \frac{r}{R}\theta_b\right) - d\sin\left(\frac{r-R}{r}t + \frac{r}{R}\theta_b\right) \end{aligned} \tag{5-1}$$

for a suitable $t \in (-(k-\ell)\pi, (k-\ell)\pi]$.

To compare this result with Theorem 3.1, we rearrange the conclusions of Theorem 3.1 in the following way: if b is considered to be fixed, then we need to find all complex values a such that

$$\left|\varrho^{k-\ell} - |b|\,\varrho^{-\ell}\right| \le |a| \le \varrho^{k-\ell} + |b|\,\varrho^{-\ell}, \tag{5-2}$$

and at least one of the values $\tau_\varrho^\pm$ is an integer. Equivalently,

$$\theta_a = \pm\left(\alpha_\varrho + \frac{\ell}{k}\beta_\varrho\right) + \frac{(k-\ell)(\theta_b - \pi)}{k} + \frac{2m\pi}{k}, \tag{5-3}$$

where m is a positive integer such that $-\pi < \theta_a \le \pi$ (it is easy to check that there exist at most $2k$ integer values of m with this property). Thus, (5-3) offers a polar representation of all coefficients a such that p has a ϱ-modular root z. Notice that this representation provides an explicit dependence of the argument θ_a on modulus $|a|$ (we recall that $|a|$ is involved in α_ϱ and β_ϱ introduced by (1-1)). In the Gauss a-plane, (5-3) defines a closed curve whose parts are generated by the above specified integer values of m.

To demonstrate the usefulness of this polar representation, we utilize Example 4.2 of [Theobald and de Wolff 2016] serving as an illustration of Theorem 4.1 of the same article.

Example 5.1. We describe the set of all complex numbers a such that the trinomial

$$f(z) = z^5 + az + 1$$

has a unimodular root. By Theorem 4.1 of [Theobald and de Wolff 2016], this occurs if and only if the complex coefficient a is located on the trajectory of the hypotrochoid with parameters $R = 5$, $r = 4$, $d = 1$, or equivalently (according to (5-1)), a satisfies the equations

$$\Re(a) = -\cos(t) - \cos\left(\frac{t}{4}\right), \quad \Im(a) = -\sin(t) + \sin\left(\frac{t}{4}\right),$$

where $t \in (-4\pi, 4\pi]$. Now we apply conclusions of our previous considerations following from Theorem 3.1. If we put $b = 1$, $k = 5$, $\ell = 1$ and $\varrho = 1$, then

$$\alpha_1 = \arccos\frac{|a|}{2}, \quad \beta_1 = \arccos\frac{2-|a|^2}{2} = \pi - 2\arccos\frac{|a|}{2},$$

and (5-2), (5-3) become

$$0 \le |a| \le 2, \quad \theta_a = \pm\frac{1}{5}\left(\pi + 3\arccos\frac{|a|}{2}\right) + \frac{2m\pi - 4\pi}{5}. \tag{5-4}$$

In the case of the plus variant, (5-4) can be rewritten (using straightforward calculations) as

$$|a| = 2\cos\left(\frac{5}{3}\theta_a + s^+\pi\right), \quad \theta_a \in \left[\frac{-6s^+\pi}{10}, \frac{3\pi - 6s^+\pi}{10}\right], \tag{5-5}$$

where $s^+ = \frac{5}{3}, 1, \frac{1}{3}, -\frac{1}{3}, -1$. (To be consistent with the assumption $-\pi < \theta_a \le \pi$, we formally remove the left endpoint from this interval if $s^+ = \frac{5}{3}$.) Similarly, if the minus sign is considered in (5-4), then we get the remaining set of conditions, namely

$$|a| = 2\cos\left(\frac{5}{3}\theta_a + s^-\pi\right), \quad \theta_a \in \left[\frac{-3\pi - 6s^-\pi}{10}, \frac{-6s^-\pi}{10}\right], \tag{5-6}$$

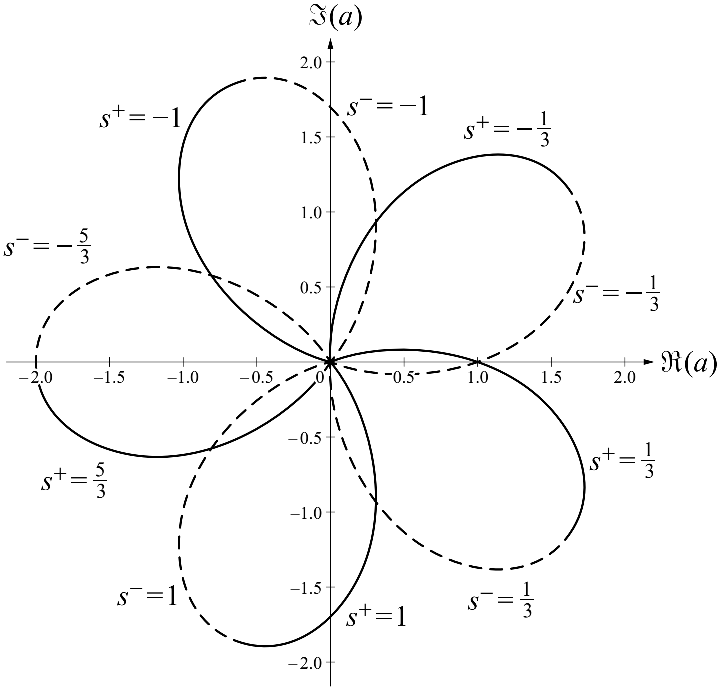

Figure 1. Parts of hypotrochoid described by (5-5) (curves labeled with the corresponding values of s^+) and (5-6) (dashed curves labeled with the corresponding values of s^-).

where $s^- = 1, \frac{1}{3}, -\frac{1}{3}, -1, -\frac{5}{3}$. Thus, (5-5) and (5-6) yield polar descriptions of all coefficients a such that f has a unimodular root. The corresponding curves are depicted in the a-plane in Figure 1.

Analogously, we can proceed in a more general case when the powers $k = 5$ and $\ell = 1$ in f are replaced by general coprime integers $k > \ell$. In this case, the trinomial

$$g(z) = z^k + az^\ell + 1$$

has a unimodular root if and only if

$$\text{(5-7)}\quad 0 \le |a| \le 2, \quad \theta_a = \pm\frac{1}{k}\left(\ell\pi + (k-2\ell)\arccos\frac{|a|}{2}\right) - \frac{(k-\ell)\pi}{k} + \frac{2m\pi}{k},$$

where m is a positive integer such that $-\pi < \theta_a \le \pi$. Moreover, (5-7) can possibly be converted into polar forms analogous to (5-5) and (5-6).

Such polar representations can offer a better insight into the structure of all complex numbers a such that p has a root with a given modulus. Among others, they enable us to decide immediately whether a given complex number a has this property. If the answer is affirmative, then Theorem 3.1 provides an additional benefit, namely calculation of the argument of such a root.

Example 5.2. We consider the trinomial f with $a = 1 + \sqrt{2}/2 + (\sqrt{2}/2)\mathrm{i}$, and discuss the existence of its unimodular root. First, we apply directly Theorem 3.1. After checking (3-5), one gets $\tau_1^+ = 3$, $\tau_1^- = 1.625$, which implies that f (with the above specified coefficient a) actually has a unique unimodular root. The Bézout coefficients for the couple $(5, 4)$ are $u_0 = 1$, $v_0 = -1$ due to (3-3). Then, using (3-6) with $\beta_1 = 3\pi/4$ and $\theta_b = 0$, one can find the exact form of this unimodular root, namely $z = \exp(-5\pi\mathrm{i}/4)$. Of course, verification of the existence of this unimodular root of f can be equivalently done by (5-5), (5-6).

On the other hand, application of Theorem 4.1 of [Theobald and de Wolff 2016] to this problem is more complicated. In general, numerical solution of a nonlinear equation with unknown parameter t is required.

5.2. ***Problem (D) — Theorem 4.5 and some earlier results.*** Now we turn our attention to Theorem 4.5 and its relevance with respect to previous results dealing with problem (D). In [Avendaño et al. 2018], the explicit metric bounds, circumscribing the annuli where log-moduli of roots of p cluster, are derived by use of a concept of Archimedean tropical variety. However, currently strongest bounds on moduli and arguments of roots of p were presented in [Melman 2012]; hence we shortly comment on the main results of this paper. Here, k disjoint annular sectors, each containing just one root of p were derived. More precisely, [Melman 2012] analyzes location of roots of p with $b = -1$ (which can be done without loss of generality). On this account, we involve this formal simplification in our next considerations as well. Forms of these sectors slightly differ with respect to the cases $|a| > \sigma(k, \ell)$ and $|a| < \sigma(k, \ell)$. In the sequel, we comment on the first case (discussions of the latter one are analogous).

If $|a| > \sigma(k, \ell)$, then Theorem 4.1 of [Melman 2012] describes several bounds on moduli and arguments of roots $z_j = \varrho_j \exp(\mathrm{i}\varphi_j)$ of p labeled so that $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, k-1$. The strongest bounds on the ϱ_j following from Theorem 4.1 and Remarks 4.2 of [Melman 2012] are given by (4-9) (thus, our observations made in Remark 4.8 confirm these bounds). Similarly, the arguments φ_j are located in k disjoint intervals whose lengths again depend on whether j belongs to the set $\{1, \dots, \ell\}$, or $\{\ell + 1, \dots, k\}$ (for more details, including bounds not utilizing ξ_L, ξ_R, ξ_{M_1}, ξ_{M_2}, see [Melman 2012]).

Now we clarify the position of Theorem 4.5 with respect to problem (D) and results from [Melman 2012]. Theorem 4.5 enables us to calculate moduli ϱ_j of the k roots of p as a numerical solution of the k transcendental equations (4-7) which differ from each other only by an additive constant appearing on their right-hand side. Also, the interval $[\xi_L, \xi_R]$ was introduced here as a localization interval containing just one root for each of these equations. Notice that this interval can be slightly precised due to (4-9). Then, using an appropriate procedure (such as the bisection method) applied to (4-7), lengths of these localization intervals can be made arbitrarily small, and moduli ϱ_j of all roots of p can be computed with any prescribed precision. Finally, having at our disposal moduli of roots, their arguments are given directly by (4-8).

We demonstrate this algorithm via Example 4.4 of [Melman 2012] that illustrated bounds derived in Theorem 4.1 of that same article.

Example 5.3. We consider the trinomial

$$h(z) = z^{10} - (1.6+\mathrm{i})z^7 - 1.$$

In this case, $a = -1.6 - \mathrm{i}$, $b = -1$, $k = 10$ and $\ell = 7$; hence $|a| = \sqrt{89}/5 = 1.8868$ and $\sigma(10, 7) = 1.8420$. Let $z_j = \varrho_j \exp(\mathrm{i}\varphi_j)$, $j = 1, \dots, 10$, be roots of h labeled in a way so that $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, 9$. Using tropical methods, one can obtain that the z_j cluster around the circles with radii $r_1 = 0.9133$ and $r_2 = 1.2357$ for $j = 1, \dots, 7$ and $j = 8, 9, 10$, respectively. The explicit bounds given by Theorem 1.5 of [Avendaño et al. 2018] are nevertheless too wide, namely

$$0.4566 \le \varrho_j \le 2.4714, \quad j = 1, \dots, 10.$$

Example 4.4 of [Melman 2012] presents several tighter bounds for ϱ_j; the best of them, based on (4-9), yield

$$\begin{aligned} 0.8746 &\le \varrho_j \le 1.0389, \quad j = 1, \dots, 7,\\ 1.1438 &\le \varrho_j \le 1.2744, \quad j = 8, 9, 10. \end{aligned} \tag{5-8}$$

We employ Theorem 4.5 to specify values of these moduli. Doing this, we follow the algorithm summarized in Remark 4.6. First, (4-6) with $\theta = -25.8299$ yields

$$\begin{aligned} &s_1 = -4, \quad s_2 = -5, \quad s_3 = -3, \quad s_4 = -6, \quad s_5 = -2,\\ &s_6 = -7, \quad s_7 = -1, \quad s_8 = -8, \quad s_9 = 0, \quad s_{10} = -9. \end{aligned}$$

Then, by (4-7), moduli ϱ_j of roots z_j are solutions of

$$\begin{aligned} 10\arccos(&-0.2650\varrho^{13} + 0.9434\varrho^7 + 0.2650\varrho^{-7})\\ &+ 7\arccos(0.5\varrho^{10} - 1.78\varrho^4 + 0.5\varrho^{-10})\\ &= |25.8299 + 2s_j\pi|, \quad j = 1, \dots, 10. \end{aligned} \tag{5-9}$$

A clear geometric interpretation of these equations, depicted in Figure 2, offers a better understanding of the location of all the moduli (obviously, $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, 9$). As localization intervals for the use of appropriate numerical calculations, the bounds (5-8) can be used (of course, lengths of these intervals can be made

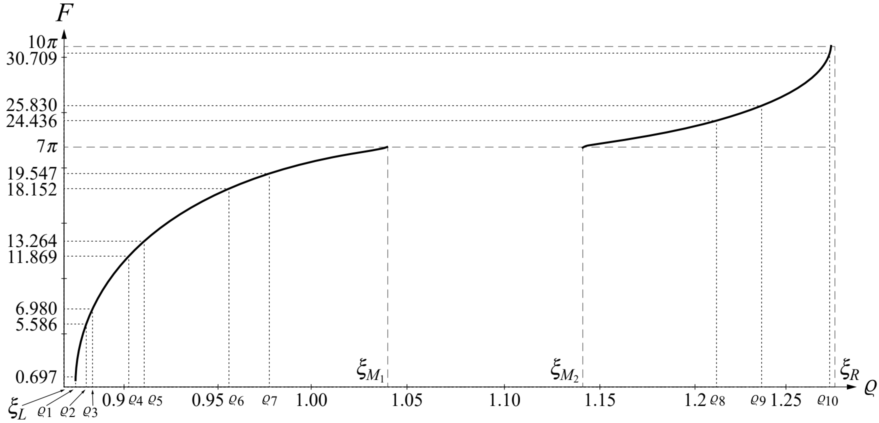

Figure 2. Geometric illustration of (5-9), and specification of some significant values.

arbitrarily small due to (5-9)). MATLAB's `fzero` routine[1] provides (in its default setting) the roots of (5-9) as

$$\varrho_1 = 0.8746, \quad \varrho_2 = 0.8803, \quad \varrho_3 = 0.8836, \quad \varrho_4 = 0.9027, \quad \varrho_5 = 0.9108,$$
$$\varrho_6 = 0.9557, \quad \varrho_7 = 0.9771, \quad \varrho_8 = 1.2140, \quad \varrho_9 = 1.2379, \quad \varrho_{10} = 1.2739.$$

Knowing moduli of all 10 roots of h now enables us, along with the formula (4-8), to directly compute their arguments as

$$\varphi_1 = 1.2632, \quad \varphi_2 = 3.0882, \quad \varphi_3 = -0.5613, \quad \varphi_4 = -1.3737, \quad \varphi_5 = -2.3808,$$
$$\varphi_6 = 0.4322, \quad \varphi_7 = 2.1059, \quad \varphi_8 = 2.3238, \quad \varphi_9 = 0.1467, \quad \varphi_{10} = -1.9025.$$

5.3. ***Problem (C) — Corollaries 4.7, 4.9, and some earlier results.*** Finally, we consider some consequences of Theorem 4.5 (discussed in Corollaries 4.7, 4.9, and summarized in Table 1) confirming and extending a series of assertions from [Theobald and de Wolff 2016] that are related to problem (C). In particular, our previous discussions confirmed that at most two roots of p have the same modulus (see also Proposition 4.3 of [Theobald and de Wolff 2016]), and formulated conditions under which two roots of p share the same modulus (see also Theorems 4.4 and 4.9 of [Theobald and de Wolff 2016]). Also, we clarified whether equality or strict inequality occurs between moduli ϱ_ℓ and $\varrho_{\ell+1}$ of roots z_ℓ and $z_{\ell+1}$ of a given trinomial p (see also Corollary 4.13 of [Theobald and de Wolff 2016]).

[1]The `fzero` algorithm uses a combination of bisection, secant, and inverse quadratic interpolation methods — the so-called Brent's method. It is known that the order of convergence is superlinear for well-behaved functions.

θ_a	$\lvert a\rvert$		$j = 1, \dots, 5$
$\pi/2$	6	ϱ_j:	$0.5426 < 0.5498 < 0.5587 < 2.4397 < 2.4593$
		φ_j:	$0.5312,\ -1.5875,\ 2.6274,\ 2.3521,\ -0.7815$
0	6	ϱ_j:	$0.5416 < 0.5546 = 0.5546 < 2.4498 = 2.4498$
		φ_j:	$\pi,\ 1.0318 \neq -1.0318,\ -1.5765 \neq 1.5765$
π	1.95	ϱ_j:	$0.7534 = 0.7534 < 1.0842 = 1.0842 < 1.4990$
		φ_j:	$2.0297 \neq -2.0297,\ -0.0415 \neq 0.0415,\ \pi$
	$\sigma(5, 3)$	ϱ_j:	$0.7524 = 0.7524 < 1.0845 = 1.0845 < 1.5018$
		φ_j:	$2.0301 \neq -2.0301,\ 0 = 0,\ \pi$
	1.97	ϱ_j:	$0.7516 = 0.7516 < 1.0413 < 1.1301 < 1.5045$
		φ_j:	$2.0304 \neq -2.0304,\ 0 = 0,\ \pi$

Table 2. Moduli ϱ_j and arguments φ_j of roots z_j of q with given a, $j = 1, \dots, 5$.

We illustrate these observations by an extended version of a part of Example 4.10 of [Theobald and de Wolff 2016] that supported theoretical results obtained in that same article.

Example 5.4. We consider the trinomial

$$q(z) = z^5 + az^3 + 1.$$

Again, let $z_j = \varrho_j \exp(i\varphi_j)$ be roots of q labeled so that $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, 4$. Note that q with $a = 6$ was considered in Example 4.10 of [Theobald and de Wolff 2016] where the resulting relations between moduli ϱ_j appeared in the form

$$\varrho_1 < \varrho_2 = \varrho_3 < \varrho_4 = \varrho_5.$$

Based on Corollaries 4.7, 4.9 and Table 1, we perform the same discussion on ϱ_j with respect to the variable complex number a considered in q. Also, we state relations between some values of the arguments φ_j. Doing this, we first notice that considering the trinomial q we have $k = 5$, $\ell = 3$, $b = 1$, $\theta = 5\theta_a + 2\pi$ and $\sigma(5, 3) = 1.9601$. Then observations made in Corollaries 4.7, 4.9 and summarized in Table 1 imply (see Table 2):

- Let $\theta_a \neq j\pi/5$ for any integer j. Then

$$\varrho_1 < \varrho_2 < \varrho_3 < \varrho_4 < \varrho_5.$$

- Let $\theta_a = j\pi/5$ for some $j = 0, \pm 2, \pm 4$ (we note that this case includes the above choice $a = 6$). Then

$$\varrho_1 < \varrho_2 = \varrho_3 < \varrho_4 = \varrho_5, \quad \varphi_2 \neq \varphi_3,\ \varphi_4 \neq \varphi_5.$$

- Let $\theta_a = j\pi/5$ for some $j = \pm 1, \pm 3, 5$, and let $|a| < \sigma(5, 3)$. Then
$$\varrho_1 = \varrho_2 < \varrho_3 = \varrho_4 < \varrho_5, \quad \varphi_1 \neq \varphi_2,\ \varphi_3 \neq \varphi_4.$$
- Let $\theta_a = j\pi/5$ for some $j = \pm 1, \pm 3, 5$, and let $|a| = \sigma(5, 3)$. Then
$$\varrho_1 = \varrho_2 < \varrho_3 = \varrho_4 < \varrho_5, \quad \varphi_1 \neq \varphi_2,\ \varphi_3 = \varphi_4.$$
- Let $\theta_a = j\pi/5$ for some $j = \pm 1, \pm 3, 5$, and let $|a| > \sigma(5, 3)$. Then
$$\varrho_1 = \varrho_2 < \varrho_3 < \varrho_4 < \varrho_5, \quad \varphi_1 \neq \varphi_2,\ \varphi_3 = \varphi_4.$$

6. Concluding remarks

We focused on several basic questions concerning moduli and arguments of roots of complex trinomials. Keeping in mind that similar problems were topics of many earlier investigations, we aimed to offer new views and new answers to these questions.

Our two main results analyzed problems (B) and (D) stated in the introduction. The assertion of Theorem 3.1 enabled us to describe the set of all entry parameters of a general trinomial p such that p has a root with a prescribed modulus. We were able to calculate the arguments of such roots, and thus obtain their complete identification. Theorem 4.5 described a procedure how to localize and compute moduli and arguments of roots of complex trinomials with arbitrary precision.

We believe that these results and their consequences can contribute not only to trinomial theory itself, but also to other areas connected with questions we discussed. In particular, conclusions of Corollary 2.3 and Theorem 3.1 have a considerable application potential towards qualitative theory of autonomous difference equations (stability of their equilibria, existence of periodic solutions, asymptotic bounds of solutions). Also, the comparisons performed in Section 5 indicate alternate possibilities of numerical evaluation of roots of complex trinomials, and new insights into analytic descriptions of some roulette curves (hypotrochoids, epitrochoids). Of course, investigations of problems (A)–(D) in the context of polynomials with more than three terms remain the main (and probably very difficult) challenge.

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JAN ČERMÁK
INSTITUTE OF MATHEMATICS
BRNO UNIVERSITY OF TECHNOLOGY
BRNO
CZECH REPUBLIC
cermak.j@fme.vutbr.cz

LUCIE FEDORKOVÁ
INSTITUTE OF MATHEMATICS
BRNO UNIVERSITY OF TECHNOLOGY
BRNO
CZECH REPUBLIC
lucie.fedorkova@vutbr.cz

JIŘÍ JÁNSKÝ
DEPARTMENT OF MATHEMATICS AND PHYSICS
UNIVERSITY OF DEFENCE
BRNO
CZECH REPUBLIC
jiri.jansky@unob.cz

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