Pacific Journal of Mathematics

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Volume 332 No. 1 September 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 332, No. 1, 2024
https://doi.org/10.2140/pjm.2024.332.69

ON THE TRANSIENT NUMBER OF A KNOT

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The transient number of a knot K, denoted tr(K), is the minimal number of simple arcs that have to be attached to K, in order for K to be homotoped to a trivial knot in a regular neighborhood of the union of K and the arcs. We give a lower bound for tr(K) in terms of the rank of the first homology group of the double branched cover of K. In particular, if tr(K) = 1, then the first homology group of the double branched cover of K is cyclic. Using this, we can calculate the transient number of many knots in the tables and show that there are knots with arbitrarily large transient number.

1. Introduction

Let K be a knot in the 3-sphere and let M be a submanifold of S^3 containing K. We say that K is transient in M if K can be homotoped within M to the trivial knot in S^3; otherwise K is called persistent. For example, K is persistent in a regular neighborhood $\mathcal{N}(K)$ of K, but it is transient in a 3-ball B containing K. Yuya Koda and Makoto Ozawa [2] proved that every knot is transient in a submanifold M if and only if M is unknotted, that is, its complement in S^3 is a union of handlebodies. Then Koda and Ozawa [2] introduced a new invariant of knots, called the transient number of K, which somehow measures, starting with $\mathcal{N}(K)$, how large must be a submanifold in which K is transient.

The transient number is defined as follows: given a knot K in S^3, there is a collection of arcs $\{\tau_1, \tau_2, \dots, \tau_n\}$, disjointly embedded in S^3, each τ_i intersecting K exactly at its endpoints, such that K can be homotoped in a regular neighborhood of K union the arcs, $T = \mathcal{N}(K \cup \tau_1 \cup \cdots \cup \tau_n)$, into the trivial knot. That is, we perform crossing changes and isotopies inside T, until we get the trivial knot K'. Note that any knot K' obtained from K in this way is not trivial in T, i.e., it cannot bound a disk contained in T, but it can be trivial in S^3. The transient number of K, tr(K), is then defined as the minimal number of arcs needed in such a system of arcs. The transient number is related to other knot invariants, namely $\mathrm{tr}(K) \le u(K)$, where $u(K)$ is the unknotting number, and $\mathrm{tr}(K) \le t(K)$, where $t(K)$ is the tunnel number. It is easy to check these inequalities. For the unknotting number, given

MSC2020: primary 57K10; secondary 57M12.
Keywords: knot, transient number, unknotting number, tunnel number, double branched covers.

a sequence of crossing changes that unknot K, consider for each crossing change an arc with endpoints in K that guides the crossing change, such that a regular neighborhood of the arc encapsulates the crossing change. Then clearly K can be made trivial in a neighborhood T of K union the arcs. For the case of the tunnel number, consider a tunnel system and a neighborhood T of the union of K and the arcs, so that the exterior is a handlebody. Isotope T so that it looks like a standard handlebody in S^3. Then K can be projected to the intersection of a plane with T, and guided by this projection to a plane, crossing changes can be performed to K inside T to get the trivial knot.

There is a knot K such that $\mathrm{tr}(K) = 1$ whereas $u(K)$ and $t(k)$ are larger than one. Some examples with this property are given in [2]. However, in that paper no example is given of a knot K with $\mathrm{tr}(K) > 1$. Homology groups of branched covers have been used to bound invariants like $u(K)$ and $t(K)$, which goes back to the work of Wendt [13]. In fact, it is well known that if $\Sigma[K]$ denotes the double branched cover of K, then the rank of the group $H_1(\Sigma[K])$ gives a lower bound for $u(K)$; see [13] or [4]. It is also not difficult to show that the rank of $H_1(\Sigma[K])$ is at most $2t(K)+1$; in particular it is known that if $t(K) = 1$ then $H_1(\Sigma[K])$ is a cyclic group (though not explicitly stated, this follows from the computations of homology of cyclic covers done in [1], or from [8]).

We prove that the rank of the first homology group of a cyclic branched cover of a knot gives lower bounds for the transient number. By using the Montesinos trick, it can be shown that if K is a knot with $u(K) = n$, then $\Sigma[K]$ can be obtained by Dehn surgery on an n-component link in S^3, which implies then the bound for $u(K)$. We do a kind of generalized Montesinos trick. Our main results are the following.

Theorem 1.1. *If K is a knot in S^3 such that* $\mathrm{tr}(K) = n$, *then the first homology group of the double branched cover of K has a presentation with at most* $2n + 1$ *generators.*

Theorem 1.2. *If K is a knot in S^3 such that* $\mathrm{tr}(K) = n$, *then the first homology group of the p-fold cyclic branched cover of K has a presentation with at most $pn + 1$ generators.*

These results imply that $\mathrm{rank}\big(H_1(\Sigma[K])\big) \le 2\,\mathrm{tr}(K) + 1$. If $\Sigma_p[K]$ denotes the p-fold cylic branched cover of K, it follows that $\mathrm{rank}\big(H_1(\Sigma_p[K])\big) \le p\,\mathrm{tr}(K) + 1$.

For the case that $\mathrm{tr}(K) = 1$, we can get a better bound. In fact, by doing a careful calculation of the first homology group of $\Sigma[K]$, we get the following result.

Theorem 1.3. *If K is a knot in S^3 such that* $\mathrm{tr}(K) = 1$, *then the first homology group of the double branched cover of K is cyclic.*

Of course, these results may not be sharp. It would be interesting to find sharp bounds for these inequalities. It would also be interesting to find bounds for the transient number depending on other classical invariants of knots.

Given any knot invariant, it is always interesting to study its behavior under connected sums of knots. We have the following:

Theorem 1.4. *Let K_1, K_2 be knots in S^3. Then* $\operatorname{tr}(K_1 \# K_2) \leq \operatorname{tr}(K_1) + \operatorname{tr}(K_2) + 1$.

The paper is organized as follows. In Section 2 we sketch a proof that the unknotting number and tunnel number are bounded below by the rank of the first homology group of double branched covers. Then we prove the main results. As part of the proofs, we show also that if $t(K) = 1$, then $H_1(\Sigma[K])$ is cyclic; this claim is used to prove Theorem 1.3. In Section 3 we give examples of knots with large transient number and explore the transient number of knots in the tables of KnotInfo [3]. In Section 4 we consider the transient number of a connected sum of knots, prove some facts and propose some problems.

We work in the piecewise linear category. To avoid cumbersome notation we use expressions like the double branched cover of a knot to mean the double cover of S^3 branched along the knot. If Λ is a simple closed curve in the boundary of a 3-manifold M, we say adding a 2-handle along Λ, to mean that we attach a 2-handle $D^2 \times I$ to M, such that $\partial D^2 \times I$ is identified with a regular neighborhood of Λ in ∂M, which is an annulus. Also, if M and T are compact 3-manifolds, with $T \subset M$, then by $M \setminus T$ we mean M minus the interior of T, or the closure in M of $M - T$. If X is a topological space, $|X|$ denotes its number of components.

2. Transient number and double branched covers

This section is inspired by an idea that is used to build the double branched cover of a knot with unknotting number equal to one. Consider a knot K in S^3 with unknotting number equal to one. Let α be an arc embedded in S^3, with endpoints in K, such that a regular neighborhood of it encapsulates the crossing change. So there is a homotopy in $\mathcal{N}(K \cup \alpha)$ between the knot K and the trivial knot, which is denoted by K'. Clearly this homotopy can be taken so that it is constant in $\mathcal{N}(K) \setminus \mathcal{N}(\alpha)$ and that the changes are occurring only in $\mathcal{N}(\alpha)$; so we assume that K' is obtained from K just by taking the two arcs $K \cap \mathcal{N}(\alpha)$ and passing one arc through the other, which would correspond to a crossing change in the corresponding knot diagram. Due to the above we have that $K \cap (S^3 \setminus \mathcal{N}(\alpha)) = K' \cap (S^3 \setminus \mathcal{N}(\alpha))$.

Let $\Sigma(K')$ be the double branched cover of the knot K', with covering function given by $p : \Sigma(K') \to S^3$. Now, since K' is the trivial knot, $\Sigma(K')$ is homeomorphic to S^3. We know that $\mathcal{N}(\alpha)$ is a 3-ball intersecting K' in two arcs, therefore $p^{-1}(\mathcal{N}(\alpha))$ is a solid torus, and $p^{-1}(\partial\mathcal{N}(\alpha))$ is a surface of genus one. Therefore, $S^3 \setminus p^{-1}(\mathcal{N}(\alpha))$ is a double cover of $S^3 \setminus \mathcal{N}(\alpha)$ branched along $K \cap (S^3 \setminus \mathcal{N}(\alpha))$. So to finish building the double branched cover of the knot K, all we have to do is to refill $S^3 \setminus p^{-1}(\mathcal{N}(\alpha))$ appropriately.

Note that there exists a compressing disk for $\partial(\mathcal{N}(\alpha))\setminus K$ contained in $\mathcal{N}(\alpha)\setminus K$; we denote this disk by D. As $K\cap D=\varnothing$, $|K'\cap D|$ is an even number, so the curve ∂D is lifted by p into two curves in $p^{-1}(\partial\mathcal{N}(\alpha))$; we denote these curves by Λ_1 and Λ_2. Let Σ' be the 3-manifold obtained by adding two 2-handles to the 3-manifold $S^3\setminus p^{-1}(\mathcal{N}(\alpha))$, attached along the curves Λ_1 and Λ_2; we denote these 2-handles by $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$, respectively. So $\Sigma'=\left[S^3\setminus p^{-1}(\mathcal{N}(\alpha))\right]\cup\overline{\Lambda}_1\cup\overline{\Lambda}_2$.

We know that $\Lambda_1\cup\Lambda_2$ is a double cover of ∂D with covering function given by $p|_{\Lambda_1\cup\Lambda_2}$. So we can extend the function $p|_{\Lambda_1\cup\Lambda_2}$ to $\overline{\Lambda}_1\cup\overline{\Lambda}_2$, to get that $\overline{\Lambda}_1\cup\overline{\Lambda}_2$ is a double cover of $\mathcal{N}(D)$. Thus, Σ' is a double cover of $\left[S^3\setminus p^{-1}(\mathcal{N}(\alpha))\right]\cup\mathcal{N}(D)$ branched along two arcs of K.

We have that $\partial\big([S^3\setminus\mathcal{N}(\alpha)]\cup\mathcal{N}(D)\big)$ consists of two 2-spheres and $\partial\Sigma'$ also consists of two 2-spheres. The 2-spheres of $\partial\Sigma'$ are a double cover of the two spheres of $\partial\big([S^3\setminus\mathcal{N}(\alpha)]\cup\mathcal{N}(D)\big)$ branched over the points $K\cap\partial\big([S^3\setminus\mathcal{N}(\alpha)]\cup\mathcal{N}(D)\big)$.

Now we can fill the sphere boundary components of Σ' with 3-balls, and extend the function p to these 3-balls in order to get the double covering of S^3 branched along the knot K.

The idea described above is known as the Montesinos trick. Similar to the previous construction, we will build the double branched covers of knots for which we know the tunnel number or the transient number. For the case of the tunnel number, note that if K has tunnel number n, then K is contained in a genus $n+1$ handlebody V, such that its complement is another genus $n+1$ handlebody W. By taking $\Sigma[K]$, V and W lift to genus $2n+1$ handlebodies, that is, give a genus $2n+1$ Heegaard decomposition of $\Sigma[K]$. This shows that $H_1(\Sigma[K])$ is an abelian group of rank at most $2n+1$.

The following lemma is a general result of coverings which we will use often. The proof is a standard argument, so we omit it.

Lemma 2.1. *Let M be a manifold. Let Σ be a double cover of M with covering function $p:\Sigma\to M$; and let $C\subset M$. If M is path connected and $p^{-1}(C)$ is connected then Σ is connected.*

The following theorem is our first important result of this section. We will see that if we are given a transient system of a knot we can construct the double branched cover of this knot and from there calculate its first homology group.

Theorem 2.2. *If K is a knot in S^3 such that* $\operatorname{tr}(K)=n$, *then the first homology group of the double branched cover of K has a presentation with at most* $2n+1$ *generators.*

Proof. Let K be a knot in S^3 such that $\operatorname{tr}(K)=n$, let $\{\tau_1,\tau_2,\dots,\tau_n\}$ be a transient system for K, and let $T=\mathcal{N}(K\cup\tau_1\cup\tau_2\cup\cdots\cup\tau_n)$, this is a genus $n+1$ handlebody. Let $K'\subset T$ be the trivial knot, such that K' is homotopic to K in T.

Let us define a family of compressing disks for ∂T properly embedded in T, say $\{D_1, D_2, \ldots, D_n, D_{n+1}\}$, which satisfy the following properties:

(1) For each $i \in \{1, 2, \ldots, n\}$ the disk D_i is properly embedded in $\mathcal{N}(\tau_i)$.

(2) The disk D_{n+1} is properly embedded in $\mathcal{N}(K)$ and is a compression disk for it.

All of these disks are properly embedded in T, so we can deduce that:

(1) The family $\{D_1, D_2, \ldots, D_n, D_{n+1}\}$ is pairwise disjoint.

(2) For each $i \in \{1, 2, \ldots, n\}$, $|D_i \cup K| = 0$.

(3) $|D_{n+1} \cap K| = 1$.

Let $\Sigma[K']$ be the double branched cover of K' with covering function given by $p : \Sigma[K'] \to S^3$. Note that $\Sigma[K']$ is homeomorphic to S^3.

Claim 2.3. For each $i \in \{1, 2, \ldots, n\}$, $p^{-1}(\partial D_i)$ has exactly two connected components, where each connected component is a simple closed curve in $p^{-1}(\partial T)$; whereas $p^{-1}(\partial D_{n+1})$ is a single simple closed curve in $p^{-1}(\partial T)$. Also, all these curves are disjoint in $p^{-1}(\partial T)$.

Proof. We know that $|D_{n+1} \cap K| = 1$ and $|D_i \cap K| = 0$ for all $i \in \{1, 2, \ldots, n\}$. As K' is homotopic to K in T, $|D_{n+1} \cap K'|$ is an odd integer and $|D_i \cap K'|$ is an even integer for all $i \in \{1, 2, \ldots, n\}$. Therefore, for each $i \in \{1, 2, \ldots, n\}$ we have that $p^{-1}(\partial D_i)$ has exactly two connected components in $p^{-1}(\partial T)$, where each connected component is a simple closed curve; and $p^{-1}(\partial D_{n+1})$ is a simple closed connected curve in $p^{-1}(\partial T)$. Now, since the disks of the family $\{D_1, D_2, \ldots, D_{n+1}\}$ are pairwise disjoint, we have that all the curves are pairwise disjoint. □

Claim 2.4. $p^{-1}(\partial T)$ is a connected, orientable surface with Euler characteristic $-4n$ (and genus $2n + 1$) contained in $\Sigma[K']$.

Proof. Note that ∂T is a genus $n + 1$ surface, then $\chi(\partial T) = -2n$, and therefore $\chi(p^{-1}(\partial T))) = 2\chi(\partial T) = -4n$. Since ∂T is connected, $p^{-1}(\partial T)$ is a double cover of ∂T, $\partial D_{n+1} \subset \partial T$ and $p^{-1}(\partial D_{n+1})$ is a connected curve on $p^{-1}(\partial T)$. Then by Lemma 2.1 we have that $p^{-1}(\partial T)$ is connected. Therefore $p^{-1}(\partial T)$ is a connected orientable surface of Euler characteristic $-4n$ (and of genus $2n + 1$). □

Claim 2.5. $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$ is connected.

Proof. Clearly $\partial T \setminus \bigcup_{j=1}^{n} \partial D_j$ is connected. We have that $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$ is a double cover of $\partial T \setminus \bigcup_{j=1}^{n} \partial D_j$, that $\partial D_{n+1} \subset \partial T \setminus \bigcup_{j=1}^{n} \partial D_j$ and that $p^{-1}(\partial D_{n+1})$ is a connected curve on $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$. Then using Lemma 2.1 we have that $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$ is connected. □

By Claim 2.3 we know that for each $i \in \{1, 2, \ldots, n\}$ the curve ∂D_i lifts, under p, to exactly two simple closed curves in $p^{-1}(\partial T)$. Let us denote by Λ_1^i and Λ_2^i the two liftings of ∂D_i in $p^{-1}(\partial T)$, so $\{\Lambda_1^1, \Lambda_2^1, \Lambda_1^2, \Lambda_2^2, \ldots, \Lambda_1^n, \Lambda_2^n\}$ is a pairwise disjoint collection of simple closed curves in $p^{-1}(\partial T)$. Also, $\Lambda_1^i \cup \Lambda_2^i$ is a double cover of ∂D_i with $p|_{\Lambda_1^i \cup \Lambda_2^i}$ the corresponding covering function. Then the functions $p|_{\Lambda_1^i} : \Lambda_1^i \to \partial D_i$ and $p|_{\Lambda_2^i} : \Lambda_2^i \to \partial D_i$ are homeomorphisms.

By Claim 2.3 we have that $p^{-1}(D_{n+1})$ is a simple closed curve on $p^{-1}(\partial T)$. Let us denote by Λ the curve $p^{-1}(\partial D_{n+1})$. So Λ is a double cover for ∂D_{n+1} with covering function $p|_\Lambda : \Lambda \to \partial D_{n+1}$.

Let us introduce the notation

- $\mathrm{Ext}(T) := S^3 \setminus T$,
- $\Sigma[\mathrm{Ext}(T)] := \Sigma[K'] \setminus p^{-1}(T)$.

Note that $\Sigma[\mathrm{Ext}(T)]$ is a double cover of $\mathrm{Ext}(T)$, and $\partial\Sigma[\mathrm{Ext}(T)] = p^{-1}(\partial T)$.

Let $\Sigma[\mathrm{Ext}(K)]$ be the 3-manifold obtained from $\Sigma[\mathrm{Ext}(T)]$ by adding a 2-handle along each of the members of the family of curves $\{\Lambda_1^1, \Lambda_2^1, \Lambda_1^2, \Lambda_2^2, \dots, \Lambda_1^n, \Lambda_2^n\}$. Since the functions $p|_{\Lambda_r^i}$ are homeomorphisms for each $i \in \{1, 2, \dots, n\}$ and $r \in \{1, 2\}$, we can extend each of these homeomorphisms to a homeomorphism whose domain is a disk whose boundary is the curve Λ_r^i, and which maps to the disk D_i. We then extend these last homeomorphisms to homeomorphisms from the 2-handle added along Λ_r^i to $\mathcal{N}(D_i)$. With this we conclude that $\Sigma[\mathrm{Ext}(K)]$ is a double cover of $\mathrm{Ext}(T) \cup \big(\bigcup_{j=1}^n \mathcal{N}(D_j)\big)$. Recall that the family of disks $\{D_1, D_2, \dots, D_n\}$ was chosen such that $\mathrm{Ext}(T) \cup \big(\bigcup_{j=1}^n \mathcal{N}(Dj)\big)$ is homeomorphic to $\mathrm{Ext}(K)$. Therefore $\Sigma[\mathrm{Ext}(K)]$ is a double cover of $\mathrm{Ext}(K)$.

On the other hand, from Claim 2.4 we know that $p^{-1}(\partial T)$ is an orientable connected surface of genus $2n+1$ and by Claim 2.5 we know that $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^n \partial D_j\big)$ is connected. Since $\{\Lambda_1^1, \Lambda_2^1, \Lambda_1^2, \Lambda_2^2, \dots, \Lambda_1^n, \Lambda_2^n\}$ consist of $2n$ curves and

$$p^{-1}\Big(\partial T \setminus \bigcup_{j=1}^n \partial D\Big) = p^{-1}(\partial T) \setminus \bigcup_{\substack{i \in \{1,2,\dots,n\} \\ r \in \{1,2\}}} \Lambda_r^i,$$

$\partial\Sigma[\mathrm{Ext}(K)]$ is an orientable surface of genus one.

Now, note that $\partial D_{n+1} \subset \partial\mathrm{Ext}(K)$ since $\partial D_{n+1} \subset \partial\mathcal{N}(K)$ and $D_{n+1} \cap D_i = \varnothing$ for all $i \in \{1, 2, \dots, n\}$. Therefore we also have $\Lambda \subset \partial\Sigma[\mathrm{Ext}(K)]$.

Let us define the 3-manifold $\Sigma[K]$ obtained from $\Sigma[\mathrm{Ext}(K)]$ by adding a 2-handle along Λ on $\partial\Sigma[\mathrm{Ext}(K)]$, and then complete with a 3-ball so that $\Sigma[K]$ is a closed 3-manifold. Since $p|_\Lambda$ is a two-to-one covering function then we can extend this function to a function that goes from a disk, whose boundary is Λ, to the disk D_{n+1}, where this extension is two-to-one branched at the point $K \cap D_{n+1}$. This last function is then extended to a function that goes from the 2-handle added along Λ to $\mathcal{N}(D_{n+1})$, where this function is two-to-one branched along the arc $K \cap \mathcal{N}(D_{n+1})$. Finally, this last function is extended to the added 3-ball, thus obtaining a function that goes from $\Sigma[K]$ to S^3 which is two-to-one branched along the knot K. From the above we conclude that $\Sigma[K]$ is the double branched cover of K.

By Claim 2.4, $p^{-1}(\partial T)$ is an orientable connected surface of genus $2n+1$ contained in S^3. Since $\partial\Sigma[\mathrm{Ext}(T)] = p^{-1}(\partial T)$ and $\Sigma[\mathrm{Ext}(T)] \subset \Sigma[K'] = S^3$, $H_1\big(\Sigma[\mathrm{Ext}(T)]\big)$ is a free abelian group of rank $2n+1$. So, let $H_1\big(\Sigma[\mathrm{Ext}(T)]\big) = \langle \theta_1, \theta_2, \ldots, \theta_{2n+1}\rangle$, where θ_i for $i \in \{1, 2, \ldots, 2n+1\}$ are generators.

Thus, $H_1(\Sigma[K]) = \langle \theta_1, \theta_2, \ldots, \theta_{2n+1} \mid \lambda_1^1, \lambda_1^2, \lambda_2^1, \lambda_2^2, \ldots, \lambda_n^1, \lambda_n^2, \lambda\rangle$, where λ and the λ_r^j, for $j \in \{1, 2, \ldots, n\}$ and $r \in \{1, 2\}$, correspond to the homology classes in $H_1\big(\Sigma[\mathrm{Ext}(T)]\big)$ of the respective curves Λ and Λ_r^j. □

In the proof of Theorem 2.2, besides from proving the result, we construct the double cover of S^3 branched along the knot for which we know the transient number. This construction will continue to be repeated throughout this work. Theorem 2.2 can be generalized to p-fold cyclic branched covers, with a similar proof.

Theorem 2.6. *If K is a knot in S^3 such that $\mathrm{tr}(K) = n$, then the first homology group of the p-fold cyclic branched cover of K has a presentation with at most $pn+1$ generators.*

The next lemma is a general result of the algebra of groups, which we will use for the proof of Theorems 2.8 and 2.10.

Lemma 2.7. *Let G_1 and G_2 be abelian groups such that*

$$G_1 = \langle \theta_1, \theta_2, \theta_3 : \lambda_1, \lambda_2, \lambda_3\rangle \quad \textit{and} \quad G_2 = \langle \beta_1, \beta_2 : \delta_1, \delta_2\rangle.$$

Suppose that there exist homomorphisms $\Psi : \langle \theta_1, \theta_2, \theta_3\rangle \to \langle \theta_1, \theta_2, \theta_3\rangle$ and $\Phi : \langle \theta_1, \theta_2, \theta_3\rangle \to \langle \beta_1, \beta_2\rangle$ between free abelian groups such that

$$\begin{aligned}
&\Psi(\theta_1) = \theta_2, \quad &&\Psi(\theta_2) = \theta_1, \quad &&\Psi(\theta_3) = \theta_3,\\
&\Psi(\lambda_1) = \lambda_2, &&\Psi(\lambda_2) = \lambda_1, &&\Psi(\lambda_3) = \lambda_3,\\
&\Phi(\theta_1) = \beta_1, &&\Phi(\theta_2) = \beta_1, &&\Phi(\theta_3) = 2\beta_2,\\
&\Phi(\lambda_1) = \delta_1, &&\Phi(\lambda_2) = \delta_1, &&\Phi(\lambda_3) = 2\delta_2.
\end{aligned}$$

If $\lambda_1 = x\theta_1 + y\theta_2 + z\theta_3$ and G_2 is the trivial group, then G_1 is isomorphic to the finite cyclic group Z_{x-y}.

Proof. Let a_{ij} be integers, with $i, j \in \{1, 2, 3\}$, such that

$$\begin{aligned}
\lambda_1 &= a_{11}\theta_1 + a_{12}\theta_2 + a_{13}\theta_3,\\
\lambda_2 &= a_{21}\theta_1 + a_{22}\theta_2 + a_{23}\theta_3,\\
\lambda_3 &= a_{31}\theta_1 + a_{32}\theta_2 + a_{33}\theta_3.
\end{aligned} \tag{1}$$

Applying the homomorphism Ψ, on both sides of the previous system of equations, we obtain

$$\begin{aligned}
\lambda_2 &= \Psi(\lambda_1) = \Psi(a_{11}\theta_1 + a_{12}\theta_2 + a_{13}\theta_3) = a_{11}\theta_2 + a_{12}\theta_1 + a_{13}\theta_3,\\
\lambda_1 &= \Psi(\lambda_2) = \Psi(a_{21}\theta_1 + a_{22}\theta_2 + a_{23}\theta_3) = a_{21}\theta_2 + a_{22}\theta_1 + a_{23}\theta_3,\\
\lambda_3 &= \Psi(\lambda_3) = \Psi(a_{31}\theta_1 + a_{32}\theta_2 + a_{33}\theta_3) = a_{31}\theta_2 + a_{32}\theta_1 + a_{33}\theta_3.
\end{aligned} \tag{2}$$

From the systems (1) and (2) we get

$$
\begin{aligned}
0 &= (a_{11}-a_{22})\theta_1 + (a_{12}-a_{21})\theta_2 + (a_{13}-a_{23})\theta_3,\\
0 &= (a_{12}-a_{21})\theta_1 + (a_{11}-a_{22})\theta_2 + (a_{13}-a_{23})\theta_3,\\
0 &= (a_{31}-a_{32})\theta_1 + (a_{32}-a_{31})\theta_2.
\end{aligned} \tag{3}
$$

Since $\langle\theta_1, \theta_2, \theta_3\rangle$ is a free abelian group, from the system in (3) we have

$$a_{11} = a_{22}, \quad a_{12} = a_{21}, \quad a_{13} = a_{23}, \quad a_{31} = a_{32}.$$

Then the system (1) can be rewritten as

$$
\begin{aligned}
\lambda_1 &= a_1\theta_1 + a_2\theta_2 + a_3\theta_3,\\
\lambda_2 &= a_2\theta_1 + a_1\theta_2 + a_3\theta_3,\\
\lambda_3 &= a_4\theta_1 + a_4\theta_2 + a_5\theta_3,
\end{aligned} \tag{4}
$$

where $a_1 = a_{11}$, $a_2 = a_{12}$, $a_3 = a_{23}$, $a_4 = a_{31}$ and $a_5 = a_{33}$. Applying the homomorphism Φ to the system (4) we obtain

$$
\begin{aligned}
\delta_1 &= \Phi(\lambda_1) = \Phi(a_1\theta_1 + a_2\theta_2 + a_3\theta_3) = (a_1+a_2)\beta_1 + 2a_3\beta_2,\\
\delta_1 &= \Phi(\lambda_2) = \Phi(a_2\theta_1 + a_1\theta_2 + a_3\theta_3) = (a_2+a_1)\beta_1 + 2a_3\beta_2,\\
2\delta_2 &= \Phi(\lambda_3) = \Phi(a_4\theta_1 + a_4\theta_2 + a_5\theta_3) = 2a_4\beta_1 + 2a_5\beta_2.
\end{aligned} \tag{5}
$$

By properties of free abelian groups, we obtain from the last equation of the system (5) that

$$\delta_2 = a_4\beta_1 + a_5\beta_2.$$

So the system in (5) can be rewritten as

$$
\begin{aligned}
\delta_1 &= (a_1+a_2)\beta_1 + 2a_3\beta_2,\\
\delta_2 &= a_4\beta_1 + a_5\beta_2.
\end{aligned} \tag{6}
$$

From the system (6) we see that the matrix A, given by

$$A = \begin{pmatrix} a_1+a_2 & 2a_3 \\ a_4 & a_5 \end{pmatrix}$$

is the representation matrix of the group $G_2 = \langle\beta_1, \beta_2 : \delta_1, \delta_2\rangle$. From the system in (4), doing an operation on rows, we see that the matrix $\tilde{A}$, given by

$$\tilde{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1+a_2 & a_1+a_2 & 2a_3 \\ a_4 & a_4 & a_5 \end{pmatrix}$$

is a representation matrix of the group G_1.

By the Smith normal form theorem, there exists matrices S_1 and S_2 of order 2×2, invertible and with integer entries such that the matrix S_1AS_2 is a diagonal matrix with integer entries. From the Smith normal form theorem it is also known that the inverse matrices of S_1 and S_2 have integer entries, therefore $\det S_1 = \pm 1$ and $\det S_2 = \pm 1$. Now, since G_2 is the trivial group, $\det A = \pm 1$. So the matrix S_1AS_2 is of the form

$$S_1AS_2 = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{7}$$

From (7) we can ensure that there is a matrix S of order 2×2, invertible and with integer entries that satisfies

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Let us define the matrix

$$\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{S} \end{pmatrix}.$$

Clearly the matrix $\tilde{S}$ has integer entries and using the result in (8) we have

$$\tilde{S}\tilde{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Using elementary operations, from the matrix in (9) we obtain

$$\begin{pmatrix} a_1-a_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the above matrix we conclude that the group $\langle \theta_1, \theta_2, \theta_3 : \lambda_1, \lambda_2, \lambda_3 \rangle$ is isomorphic to $Z_{a_1-a_2}$, therefore the group G_1 is isomorphic to $Z_{a_1-a_2}$. □

The following result is well known to experts. We include a proof for completeness and because it will help us as a lemma in the proof of Theorem 2.10.

Theorem 2.8. *If K is a knot in S^3 such that $t(K) = 1$, then the first homology group of the double branched cover of K is cyclic.*

Proof. Let K be a knot in S^3 such that $t(K) = 1$, and let τ be an unknotting tunnel for K. Let $T = \mathcal{N}(K \cup \tau)$ and $\mathrm{Ext}(T) = S^3 \setminus T$, so $\mathrm{Ext}(T)$ is a genus two handlebody. Since $\mathrm{Ext}(T)$ is a handlebody, we can ensure that there exists a knot $K' \subset T$ such that K' is a trivial knot in S^3 and it is homotopic to the knot K in T. Let $\Sigma(K')$ be the double branched cover of the knot K' and let $p : \Sigma(K') \to S^3$ be the associated covering function. It is easy to notice, for the way it is defined T, that there are meridian disks D_1 and D_2 in T such that $|D_1 \cap K| = 0$ and $|D_2 \cap K| = 1$. Since K' is homotopic to K in T, $|D_1 \cap K'|$ is an even integer and $|D_2 \cap K'|$ is an odd integer. Therefore ∂D_1 lifts, under p, to two simple closed curves; while ∂D_2 lifts to exactly a single simple closed curve. Let us denote by Λ_1 and Λ_2 the liftings of ∂D_1 and by Λ_3 the lifting of ∂D_2. For each $i \in \{1, 2, 3\}$ we attach a 2-handle to $p^{-1}(\text{Ext}(T))$ along $\Lambda_i \subset \partial\big(p^{-1}(\text{Ext}(T))\big)$; let us denote the 2-handle attached along Λ_i by $\overline{\Lambda}_i$. Let Σ be the 3-manifold obtained by attaching to $p^{-1}(\text{Ext}(T))$ the 2-handles $\overline{\Lambda}_i$, that is: $\Sigma := p^{-1}(\text{Ext}(T)) \cup \big(\bigcup_{i=1}^{3} \overline{\Lambda}_i\big)$.

Let us note the following observations:

(1) $\partial p^{-1}(\text{Ext}(T))$ is a genus three connected surface.

(2) $p^{-1}(\text{Ext}(T))$ is a double covering of $\text{Ext}(T)$.

(3) The function p can be extended to Σ, such that $\overline{\Lambda}_1 \cup \overline{\Lambda}_2$ is a double covering of $\mathcal{N}(D_1)$ and Λ_3 is a double covering of $\mathcal{N}(D_2)$ branched along $K \cap \mathcal{N}(D_2)$.

(4) $\partial\Sigma$ is a 2-sphere.

Let $\Sigma(K)$ be the 3-manifold obtained by attaching a 3-ball to Σ along its boundary. So, we can extend the covering function $p|_{p^{-1}(\text{Ext}(T))} : p^{-1}(\text{Ext}(T)) \to \text{Ext}(T)$ to a covering function $p' : \Sigma(K) \to S^3$ which branches along the knot K. Therefore $\Sigma(K)$ is the double covering of S^3 branched along K with covering function given by p'.

We know that $\text{Ext}(T)$ is a genus two handlebody, therefore $H_1(\text{Ext}(T))$ is a free abelian group in two generators. Note that $p^{-1}(\text{Ext}(T))$ is a genus three handlebody, therefore $H_1(p^{-1}(\text{Ext}(T)))$ is a free abelian group in three generators.

Claim 2.9. There are two connected simple closed curves in $\text{Ext}(T)$, denoted by B_1 and B_2, such that B_1 lifts, by p, to two closed and connected simple curves, denoted by Θ_1 and Θ_2; while B_2 lifts, by p, to exactly one simple curve closed, denoted by Θ_3. If β_j is the homology class of B_j in $H_1(\text{Ext}(T))$ and θ_i is the homology class of Θ_i in $H_1\big(p^{-1}(\text{Ext}(T))\big)$ for all $j \in \{1, 2\}$ and $i \in \{1, 2, 3\}$, then $H_1(\text{Ext}(T)) = \langle \beta_1, \beta_2 \rangle$, $H_1\big(p^{-1}(\text{Ext}(T))\big) = \langle \theta_1, \theta_2, \theta_3 \rangle$.

Proof. Note that $\text{Ext}(T)$ is a genus two handlebody, call it V. Let D be a disk in V which splits it in two solid tori V_1 and V_2. Note that $p^{-1}(V_i)$ double covers V_i; thus, it is either a set of two solid tori or a solid torus that coves V_i two-to-one. There are two possibilities:

(1) V_1 is covered by two solid tori, say V_1^1 and V_1^2, and V_2 is covered two-to-one by a solid torus V_2'. See Figure 1.

(2) V_1 and V_2 are covered both two-to-one by solid tori V_1' and V_2'. See Figure 2.

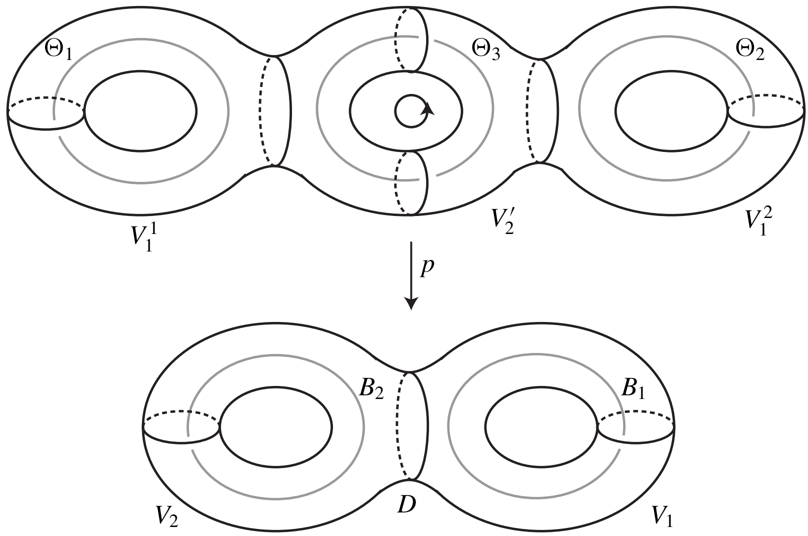

Figure 1. V_1 is covered by two solid tori, say V_1^1 and V_1^2, and V_2 is covered two-to-one by a solid torus V_2'.

In case (1), take as B_i, $i = 1, 2$, a core of the solid tori V_i. Clearly B_1 lifts to two simple closed curves Θ_1 and Θ_2, which are a core of the solid tori V_1^1 and V_1^2, and B_2 lifts to a simple closed curve Θ_3 which is a core of the solid tori V_2', and which cover two-to-one the curve B_2. In this case it is clear that the homology classes of the curves satisfy the required properties. See Figure 1.

In case (2), take as B_1 a curve that goes once around each of the cores of V_1 and V_2 and intersects D in two points. In this case B_1 lifts to two simple closed curves Θ_1 and Θ_2, each of which goes once around V_1^1 and V_1^2. Take as B_2 a core of V_1, then clearly it lifts to a curve Θ_3 which covers B_2 two-to-one. It is clear that the homology classes of the curves satisfy the required properties. See Figure 2. □

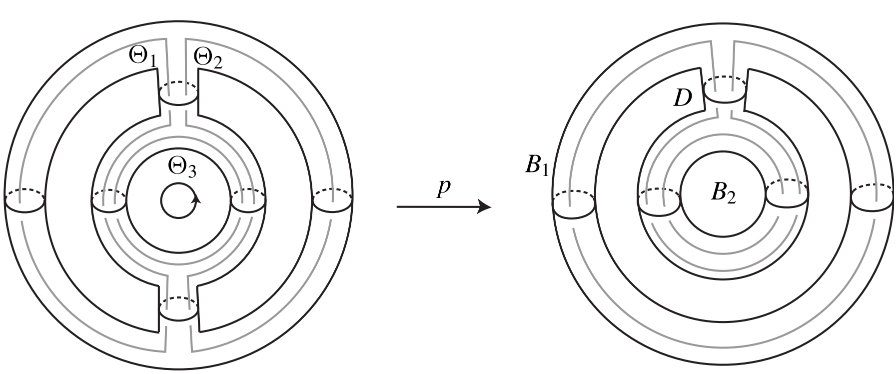

Figure 2. V_1 and V_2 are covered both two-to-one by solid tori V_1' and V_2'.

We know that $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$, with covering function given by the restriction of p. Let $p_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1(\mathrm{Ext}(T))$ be the homomorphism associated with the restriction of p. For each $i \in \{1, 2, 3\}$, let us denote by λ_i the homology class in $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ associated to the curve Λ_i. Note that $H_1(\Sigma[K]) = \langle \theta_1, \theta_2, \theta_3 : \lambda_1, \lambda_2, \lambda_3 \rangle$. For each $j \in \{1, 2\}$, let us denote by δ_j the homology class in $H_1(\mathrm{Ext}(T))$ associated to the curve ∂D_j. We have that $H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 : \delta_1, \delta_2 \rangle$. By choosing orientations conveniently, assume that

$$p_*(\lambda_1) = \delta_1, \quad p_*(\lambda_2) = \delta_1, \quad p_*(\lambda_3) = 2\delta_2. \tag{10}$$

According to Claim 2.9, we have that

$$p_*(\theta_1) = \beta_1, \quad p_*(\theta_2) = \beta_1, \quad p_*(\theta_3) = 2\beta_2. \tag{11}$$

Let $q : p^{-1}(\mathrm{Ext}(T)) \to p^{-1}(\mathrm{Ext}(T))$ be the nontrivial covering transformation associated to the covering function $p|_{p^{-1}(\mathrm{Ext}(T))}$. Let $q_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ be the homomorphism induced by the covering transformation q. By Claim 2.9 we have that

$$\begin{aligned} q_*(\theta_1) &= \theta_2, \quad q_*(\theta_2) = \theta_1, \quad q_*(\theta_3) = \theta_3, \\ q_*(\lambda_1) &= \lambda_2, \quad q_*(\lambda_2) = \lambda_1, \quad q_*(\lambda_3) = \lambda_3. \end{aligned} \tag{12}$$

Then, applying Lemma 2.7 directly we have that $H_1(\Sigma_2(K)) = \mathbb{Z}_{x-y}$, where $\lambda_1 = x\theta_1 + y\theta_2 + z\theta_3$. □

Now we prove our main result.

Theorem 2.10. *If K is a knot in S^3 such that* $\mathrm{tr}(K) = 1$ *then the first homology group of the double branched cover of K is cyclic.*

Proof. Let K be a knot in S^3 such that $\mathrm{tr}(K) = 1$, and let $\{\tau\}$ be a transient system for the knot K. Let $T = \mathcal{N}(K \cup \tau)$ and let $K' \subset T$ be a trivial knot in S^3 such that K' is homotopic to K in T. Define also the 3-manifold $\mathrm{Ext}(T)$ as $\mathrm{Ext}(T) := S^3 \setminus \mathrm{Int}(T)$.

As ∂T is a genus two surface in the exterior of the knot K', which is trivial, it follows that ∂T is compressible in $\mathrm{Ext}(K')$, that is, there is a compression disk E_1 for ∂T disjoint from K'.

There are two possibilities for the disk E_1:

(1) The disk E_1 is a compression disk for ∂T lying in the interior of T.

(2) The disk E_1 is a compression disk for ∂T lying in the exterior of T.

Suppose first that we have case (1), that is, E_1 lies in the interior of T. If E_1 separates T, then by cutting along E_1 we get two solid tori, one of them contains K', and then there is a compression disk in the other solid tori which is nonseparating in T. So, we can assume that there is a compression disk E_1 for ∂T, lying in T, and which does not separate T.

Claim 2.11. There exist a knot K'' and a disk E_2 in T such that:

(1) E_2 is a compression disk for ∂T which is properly embedded in T.

(2) K'' is a trivial knot in S^3 and it is homotopic to K in T.

(3) $|E_2 \cap K''| = 1$.

Proof. By cutting T along E_1, we get a solid torus V. The knot K' lies in V, and as K' represents a primitive element in $\pi_1(T)$, it must be homotopic to the core of V. If V is knotted, then ∂V is incompressible in $\mathrm{Ext}(K')$, which is not possible, for K' is the trivial knot. Then V must be a standard solid torus in S^3. Then K' can be further homotoped to the core of V, which is a trivial knot in the 3-sphere. Then there is a disk E_2 in V such that $|E_2 \cap K''| = 1$. □

Let $\Sigma[K'']$ be the double cover of S^3 branched along K'' with covering function given by $p : \Sigma[K''] \to S^3$. The disks E_1 and E_2 form a meridian disk system for T, and as K'' is disjoint from E_1 and intersects E_2 in one point, it follows that $p^{-1}(T)$ is a genus three handlebody, $p^{-1}(E_1)$ consists of two disks and $p^{-1}(E_2)$ consists of a single disk which covers two-to-one the disk E_2. Note that these disks form a meridian system for $p^{-1}(T)$. Let $B_i = \partial E_i$, $i = 1, 2$. Denote by Θ_1 and Θ_2 the two components of $p^{-1}(B_1)$, and let $\Theta_3 = p^{-1}(B_2)$. As $\Sigma[K'']$ is the 3-sphere, and $p^{-1}(T)$ is a genus three handlebody, it follows that the homology classes of the curves Θ_i, $i = 1, 2, 3$, generate $H_1(p^{-1}(\mathrm{Ext}(T))$.

Let D_1 and D_2 be compression disks in the interior of T such that D_1 is properly embedded in $\mathcal{N}(\tau)$ and D_2 is properly embedded in $\mathcal{N}(K)$, such that $|D_1 \cap K| = 0$ and $|D_2 \cap K| = 1$. Note that the disks D_1 and D_2 do not separate T. As K'' is homotopic to K in T, $|D_1 \cap K''|$ is an even number and $|D_2 \cap K''|$ is an odd number. Therefore ∂D_1 lifts, under p, to two simple closed curves, while ∂D_2 lifts exactly to a single simple closed curve. Denote by Λ_1 and Λ_2 the liftings of ∂D_1 and by Λ_3 the lifting of ∂D_2. Attach 2-handles to the 3-manifold $p^{-1}(\mathrm{Ext}(T))$ along the curves Λ_i, note that these curves lie in $\partial(p^{-1}(\mathrm{Ext}(T)))$, and denote the 2-handle attached along Λ_i by $\overline{\Lambda}_i$. Let Σ be the 3-manifold obtained by attaching to $p^{-1}(\mathrm{Ext}(T))$ the 2-handles $\overline{\Lambda}_i$.

Note that $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$, with covering function p' given by $p' = p|_{p^{-1}(\mathrm{Ext}(T))}$. The function p' can be extended to a function $p' : \Sigma \to \mathrm{Ext}(T) \cup N(D_1) \cup N(D_2)$, such that $\overline{\Lambda}_1 \cup \overline{\Lambda}_2$ is a double covering of $\mathcal{N}(D_1)$ and $\overline{\Lambda}_3$ is a double covering of $\mathcal{N}(D_2)$ branched along $K \cap \mathcal{N}(D_2)$.

Note that $\partial\Sigma$ is a 2-sphere. Let $\Sigma(K)$ be the 3-manifold obtained by attaching a 3-ball to Σ along its boundary. We can extend the covering function p' to a covering function $\hat{p} : \Sigma(K) \to S^3$, which branches along K. Therefore $\Sigma(K)$ is the double cover of S^3 branched along K with covering function given by $\hat{p}$.

As $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$, with covering function given by the restriction of p, let $p_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1(\mathrm{Ext}(T))$ be the homomorphism induced by p. For each $i \in \{1, 2, 3\}$ denote by λ_i the homology class in $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ associated to the curve Λ_i. For each $j \in \{1, 2\}$ denote by δ_j the homology class in $H_1(\mathrm{Ext}(T))$ associated to the curve ∂D_j. Then

$$p_*(\lambda_1) = \delta_1, \quad p_*(\lambda_2) = \delta_1, \quad p_*(\lambda_3) = 2\delta_2. \tag{13}$$

Note that $H_1(\mathrm{Ext}(T))$ is a free abelian group in two generators, generated by the homology classes of the curves B_1 and B_2, which we denote by β_1 and β_2. As we said before, $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ is a free abelian group in three generators, generated by the homology classes of the curves Θ_i, which we denote by θ_i, $i = 1, 2, 3$. We have that

$$H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 \rangle, \quad H_1(p^{-1}(\mathrm{Ext}(T))) = \langle \theta_1, \theta_2, \theta_3 \rangle \tag{14}$$

We also obtain that

$$H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 : \delta_1, \delta_2 \rangle, \quad H_1(\Sigma[K]) = \langle \theta_1, \theta_2, \theta_3; \lambda_1, \lambda_2, \lambda_3 \rangle, \tag{15}$$

$$p_*(\theta_1) = \beta_1, \quad p_*(\theta_2) = \beta_1, \quad p(\theta_3) = 2\beta_2. \tag{16}$$

Let $q : p^{-1}(\mathrm{Ext}(T)) \to p^{-1}(\mathrm{Ext}(T))$ be the nontrivial covering transformation, associated to the covering function p. Let $q_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1\big(p{-}1(\mathrm{Ext}(T))\big)$ be the homomorphism associated to the covering transformation q. By the way that θ_i and the λ_i were defined we have that

$$\begin{aligned} &q_*(\theta_1) = \theta_2, \quad q_*(\theta_2) = \theta_1, \quad q_*(\theta_3) = \theta_3,\\ &q_*(\lambda_1) = \lambda_2, \quad q_*(\lambda_2) = \lambda_1, \quad q_*(\lambda_3) = \lambda_3. \end{aligned} \tag{17}$$

Applying Lemma 2.7 we have that $H_1(\Sigma(K)) = \mathbb{Z}_{x-y}$, where $\lambda_1 = x\theta_1 + y\theta_2 + z\theta_3$. So, we have proved that if the compression disk E_1 is contained in T, then the homology group of the double branched cover of K is cyclic.

Now suppose that we have case (2), that is, the compression disk E_1 is contained in $\mathrm{Ext}(T)$. In this situation we can suppose that $\mathrm{Ext}(T)$ is not a handlebody, for otherwise we have that $t(K) = 1$ and by Theorem 2.8 we get the desired result. Suppose first that the disk E_1 does not separate $\mathrm{Ext}(T)$. Define $\Gamma = T \cup \mathcal{N}(E_1)$. As E_1 does not divide ∂T then $\partial\Gamma$ is a connected genus one surface, and it must bound a solid torus. Then Γ is a solid torus, for otherwise $\mathrm{Ext}(T)$ will be a genus two handlebody. So, Γ is a knotted solid torus and K' lies on it. As K' is a trivial knot, it must lie in a 3-ball contained in Γ, for otherwise there will be an incompressible torus in $\mathrm{Ext}(K)$. In particular, K' has winding number zero in Γ. Then K is also of winding number zero in Γ, as it is homotopic to K' in $T \subset \Gamma$. Embed Γ in S^3 so that it is an standard solid torus V, and that a preferred longitude of Γ goes to a preferred longitude of V. Let $\overline{K}$ be the image of K in V. Then K is a satellite knot with pattern given by $\overline{K}$. As $\overline{K}$ has winding number zero in V, it follows that $H_1(\Sigma[\overline{K}])$ is isomorphic to $H_1(\Sigma[K])$, by [12]. Let $\overline{T}$ be the image of T in V, clearly $\overline{T}$ is the neighborhood of $\overline{K}$ union a transient arc, and the exterior of $\overline{T}$ is the exterior of V, which is a solid torus, union a 1-handle given by the image of the disk E_1. This shows $\overline{K}$ is a tunnel number one knot and then $H_1(\Sigma[\overline{K}])$ is a cyclic group, which implies then that $H_1(\Sigma[K])$ is also cyclic.

Suppose now that the disk E_1 separates $\text{Ext}(T)$ and that there is no nonseparating compression disk in $\text{Ext}(T)$. Let $\Gamma = T \cup \mathcal{N}(E_1)$. As E_1 is separating, $\partial\Gamma$ consist of two tori, say S_1 and S_2. Then S_1 bounds a solid torus V_1 which contains Γ, and also contains S_2. Then V_1 is a knotted solid torus, and as K' is contained in V_1, it must lie inside a 3-ball, and then as in the previous case, K has winding number zero in V_1. Embed V_1 in S^3 so that it is an standard solid torus V_2, and such that a preferred longitude of V_1 goes to a preferred longitude of V_2. Let $\overline{K}$ be the image of K in V_2. Then K is a satellite knot with pattern given by $\overline{K}$. As $\overline{K}$ has winding number zero in V, it follows that $H_1(\Sigma[\overline{K}])$ is isomorphic to $H_1(\Sigma[K])$, by [12]. Let $\overline{T}$ be the image of T in V, clearly $\overline{T}$ is the neighborhood of $\overline{K}$ union a transient arc, and the exterior of $\overline{T}$ is the exterior of V, which is a solid torus union a manifold bounded by the image of S_2 plus 1-handle given by the image of the disk E_1. It follows that $\overline{K}$ is a transient number one knot such that the exterior of the knot union a transient arc is compressible, and it has a nonseparating compression disk. By the previous case, $H_1(\Sigma[\overline{K}])$ is a cyclic group, which implies then that $H_1(\Sigma[K])$ is also cyclic. □

3. Knots with large transient number

By the results of Section 2, we can now estimate the transient number of some knots.

Theorem 3.1. *Let K be a knot such that its double branched cover is not a homology sphere, that is, $H_1(\Sigma[K])$ is not trivial. Then*

(1) $\text{tr}(K \# K) \geq 2$;

(2) $\text{tr}(K_n) \geq (n-1)/2$, *where $K_n = K \# K \# \cdots \# K$, is the connected sum of n copies of K.*

Proof. It is known that $\Sigma[K_n] = \Sigma[K] \# \Sigma[K] \# \cdots \# \Sigma[K]$, the connected sum of n copies of $\Sigma[K]$. As $H_1(\Sigma[K])$ is not trivial, $H_1(\Sigma[K_n])$ has rank at least n. By Theorem 2.2, $\text{tr}(K_n) \geq (n-1)/2$, this shows (2). In particular $H_1(\Sigma[K_2]) = H_1(\Sigma[K]) + H_1(\Sigma[K])$, which is not cyclic, and this implies (1). □

This shows that there are knots with arbitrarily large transient number, which answers a question of Koda and Ozawa [2].

Now we concentrate on the tables of knots up to crossing number 10.

10_{99}	{2, {9, 9}}, {6, {2, 2, 6, 6, 0, 0, 0, 0}}
10_{123}	{2, {11, 11}}, {5, {2, 2, 2, 2, 2, 2, 2, 2}}
$12a_{427}$	{2, {15, 15}}, {4, {3, 3, 3, 3, 15, 15}}, {6, {4, 4, 20, 20, 0, 0, 0, 0}}
$12a_{435}$	{2, {3, 75}}, {6, {2, 2, 8, 200, 0, 0, 0, 0}}
$12a_{465}$	{6, {2, 2, 2, 2, 2, 2, 38, 9158}}
$12a_{466}$	{6, {2, 2, 2, 2, 2, 2, 26, 5434}}
$12a_{475}$	{6, {2, 2, 2, 10, 20, 340, 0, 0}}
$12a_{647}$	{2, {3, 51}}, {6, {2, 2, 2, 34, 0, 0, 0, 0}}
$12a_{868}$	{5, {2, 2, 2, 2, 8, 8, 88, 88}}
$12a_{975}$	{2, {5, 45}}, {4, {5, 5, 5, 5, 5, 45}}
$12a_{990}$	{2, {3, 75}} {6, {2, 2, 8, 200, 0, 0, 0, 0}}
$12a_{1019}$	{2, {19, 19}}, {5, {6, 6, 6, 6, 6, 6, 6, 6}}
$12a_{1102}$	{6, {2, 2, 2, 2, 2, 2, 112, 34160}}
$12a_{1105}$	{2, {17, 17}}, {6, {2, 2, 2, 2, 10, 10, 170, 170}}
$12a_{1167}$	{5, {2, 2, 2, 2, 2, 2, 82, 82}}
$12a_{1229}$	{5, {2, 2, 2, 2, 8, 8, 8, 8}}
$12a_{1288}$	{2, {3, 39}}, {6, {2, 2, 2, 26, 0, 0, 0, 0}}
$12n_{518}$	{2, {3, 21}}, {6, {2, 2, 4, 28, 0, 0, 0, 0}}
$12n_{533}$	{6, {2, 2, 2, 2, 2, 42, 0, 0}}
$12n_{604}$	{2, {3, 27}}, {6, {2, 2, 2, 18, 0, 0, 0, 0}}
$12n_{605}$	{2, {3, 3}}, {6, {2, 2, 2, 2, 0, 0, 0, 0}}
$12n_{706}$	{2, {7, 7}}, {5, {3, 3, 3, 3, 3, 3, 3, 3}}, {6, {2, 2, 2, 2, 2, 2, 14, 14}}
$12n_{840}$	{6, {2, 2, 2, 2, 2, 2, 10, 1190}}
$12n_{879}$	{5, {2, 2, 2, 2, 4, 4, 4, 4}}
$12n_{888}$	{2, {3, 15}}, {6, {2, 2, 2, 10, 0, 0, 0, 0}}

Table 1. List of the knots with the corresponding homology group needed for the proof of Theorem 3.3.

Theorem 3.2. (1) *These knots have transient number* 2: 8_{18}, 9_{35}, 9_{37}, 9_{40}, 9_{41}, 9_{46}, 9_{47}, 9_{48}, 9_{49}, 10_{74}, 10_{75}, 10_{98}, 10_{99}, 10_{103}, 10_{123}, 10_{155}, 10_{157}.

(2) *These knots have transient number at most* 2: 8_{16}, 9_{29}, 9_{32}, 9_{38}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{67}, 10_{68}, 10_{69}, 10_{79}, 10_{80}, 10_{81}, 10_{83}, 10_{85}, 10_{86}, 10_{87}, 10_{89}, 10_{90}, 10_{92}, 10_{93}, 10_{94}, 10_{96}, 10_{97}, 10_{100}, 10_{101}, 10_{105}, 10_{106}, 10_{108}, 10_{109}, 10_{110}, 10_{111}, 10_{112}, 10_{115}, 10_{116}, 10_{117}, 10_{120}, 10_{121}, 10_{122}, 10_{140}, 10_{142}, 10_{144}, 10_{148}, 10_{149}, 10_{150}, 10_{151}, 10_{152}, 10_{153}, 10_{154}, 10_{158}, 10_{160}, 10_{162}, 10_{163}, 10_{165}.

(3) *Any other knot of crossing number at most* 10 *has transient number one.*

Proof. According to the information given in KnotInfo [3], the knots in (1) and (2) are precisely the knots with crossing number up to 10, whose unknotting number and tunnel number are both larger that 1. So, any other knot has unknotting number or tunnel number equal to 1, and then has transient number 1. The knots in (1) are precisely the knots whose double branched cover has noncyclic first homology group, and furthermore these knots have tunnel number 2. Therefore its transient number must be two. The knots in (2) have tunnel number two but their double branched cover has cyclic first homology group; hence we cannot detect the transient number yet. □

A similar result can be done for the knots of crossing number 11 or 12.

The following knots are interesting, for we use the homology of p-branched covers of a knot to determine the transient number.

Theorem 3.3. *These knots have transient number* 2: 10_{99}, 10_{123}, $12a_{427}$, $12a_{435}$, $12a_{465}$, $12a_{466}$, $12a_{475}$, $12a_{647}$, $12a_{742}$, $12a_{801}$, $12a_{868}$, $12a_{975}$, $12a_{990}$, $12a_{1019}$, $12a_{1102}$, $12a_{1105}$, $12a_{1167}$, $12a_{1206}$, $12a_{1229}$, $12a_{1288}$, $12n_{518}$, $12n_{533}$, $12n_{604}$, $12n_{605}$, $12n_{642}$, $12n_{706}$, $12n_{840}$, $12n_{879}$, $12n_{888}$.

Proof. According to Theorem 2.6, if K has $\mathrm{tr}(K) = 1$, then $\mathrm{rank}\big(H_1(\Sigma_p[K])\big) \leq p+1$. Using this and the information given in KnotInfo [3], we show that these knots cannot have transient number one. As they have tunnel number two, they must also have transient number two. In Table 1, there is a list of the knots with the corresponding homology group needed for the proof. For some of them, it is enough to use the homology of $\Sigma[K]$, but not for all. A symbol {6, {2, 2, 2, 10, 20, 340, 0, 0}} means that $H_1(\Sigma_6[K]) = \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_{10} + \mathbb{Z}_{20} + \mathbb{Z}_{340} + \mathbb{Z} + \mathbb{Z}$. □

4. Transient number and connected sums

It is natural to consider the behavior of a knot invariant with respect to connected sums. It is easy to see that

$$u(K_1 \# K_2) \leq u(K_1) + u(K_2),$$

and equality is conjectured. It is also not difficult to see that

$$t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1.$$

There are known examples of knots with $t(K_1 \# K_2) = t(K_1) + t(K_2) + 1$ [6], examples with $t(K_1 \# K_2) = t(K_1) + t(K_2)$, and examples with $t(K_1 \# K_2) < t(K_1) + t(K_2)$ [4]. So, we can expect a similar inequality for the transient number.

Theorem 4.1. *Let K_1, K_2 be knots in S^3. Then*

$$\mathrm{tr}(K_1 \# K_2) \leq \mathrm{tr}(K_1) + \mathrm{tr}(K_2) + 1.$$

Proof. Let K_1 be a knot with transient number $\mathrm{tr}(K) = n$, and let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a system of transient arcs for K_1. Let $T_1 = \mathcal{N}(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$. Then T_1 is a genus $n+1$ handlebody with the property that K_1 can be homotoped in the interior of T_1 to the trivial knot in S^3. We can assume that the homotopy that transform K_1 into the trivial knot can be realized by a sequence of ambient isotopies of K_1 and crossing changes. So, suppose that after making isotopies, all crossing changes are performed simultaneously. Suppose r crossing changes are performed, numbered $1, 2, \ldots, r$, and for each crossing change let α_i be an arc with endpoints in K_1 which remembers the crossing change, that is, if B_i is a regular neighborhood of α_i, in fact a 3-ball that intersects K_1 in two unknotted arcs, then a crossing change can be performed inside each B_i to get the trivial knot. Make an isotopy to move K_1 to its original position, and then $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ is a collection of disjoint arcs with endpoints in K_1 contained in T_1. Let δ_1 be an arc in T_1 with an endpoint in K_1 and the other in $\partial N(K_1)$, such that δ_1 is disjoint from the arcs α_i.

If K_2 is knot with $\operatorname{tr}(K_2) = m$, then as above there is a genus $m+1$ handlebody that is the neighborhood of K union a system of transient arcs $\{\gamma'_1, \gamma'_2, \ldots, \gamma'_m\}$, and there is a collection of arcs $\{\beta_1, \ldots, \beta_s\}$ that determines crossing changes that unknot K_2. Let δ_2 be an arc in T_2 with an endpoint in K_2 and the other in $\partial N(K_2)$, such that δ_2 is disjoint from the arcs β_i.

Suppose that T_1 and T_2 lie in disjoint 3-balls C_1 and C_2 contained in S^3. Suppose that $\partial T_i \cap \partial C_i$ consists of a disk D_i, such that the endpoint of δ_i lying in ∂T_i, it lies in D_i, for $i = 1, 2$. Do a disk sum of T_1 and T_2, identifying D_1 and D_2, such that the endpoints of δ_1 and δ_2 coincide. Let $\delta = \delta_1 \cup \delta_2$, this is an arc with an endpoints in K_1 and K_2. Following δ, do a band sum of K_1 and K_2. As K_1 and K_2 lie in disjoint 3-balls, this band sum is in fact a connected sum $K_1 \# K_2$. Let $T = T_1 \cup T_2$, this is a genus $n+m+2$ handlebody, and $K_1 \# K_2$ can be homotoped to the trivial knot inside it, to see that just do crossing changes following the arcs α_i and β_j. Now note that T is the regular neighborhood of $K_1 \# K_2$ and a system of $n+m+1$ arcs, that is, the n arcs for a system of K_1, the m arcs for a system of K_2, plus one more arc which is dual to the band used to perform the connected sum of K_1 and K_2; see Figure 3. This shows that the transient number of $K_1 \# K_2$ is at most $n+m+1$. □

In many cases we can ensure that $\operatorname{tr}(K_1 \# K_2)$ is at most $\operatorname{tr}(K_1) + \operatorname{tr}(K_2)$. For example, if the arc systems that unknot K_1 and K_2 are disjoint from a meridian disk E_1 for $N(K_1)$ and a meridian disk E_2 for $N(K_2)$, then at most $\operatorname{tr}(K_1) + \operatorname{tr}(K_2)$ arcs are needed to unknot $K_1 \# K_2$. To see this consider handlebodies T_1 and T_2 as above, and disjoint 3-balls C_1 and C_2 that contain them, such that T_i and C_i intersect in a disk D_i. We can suppose that the boundary of the disk E_i intersects D_i in a single arc. Instead of doing a band sum of T_1 and T_2, cut T_i along E_i, and identify the two copies of E_1 with the corresponding copies for E_2, this is like doing a connected sum $T_2 \# T_2$ between T_1 and T_2. We get a genus $n+m+1$ handlebody T'. Note that $K_1 \# K_2$ is contained in the handlebody T'. Now consider the arcs α_i and β_j as in the above proof. These arcs are disjoint from the meridian

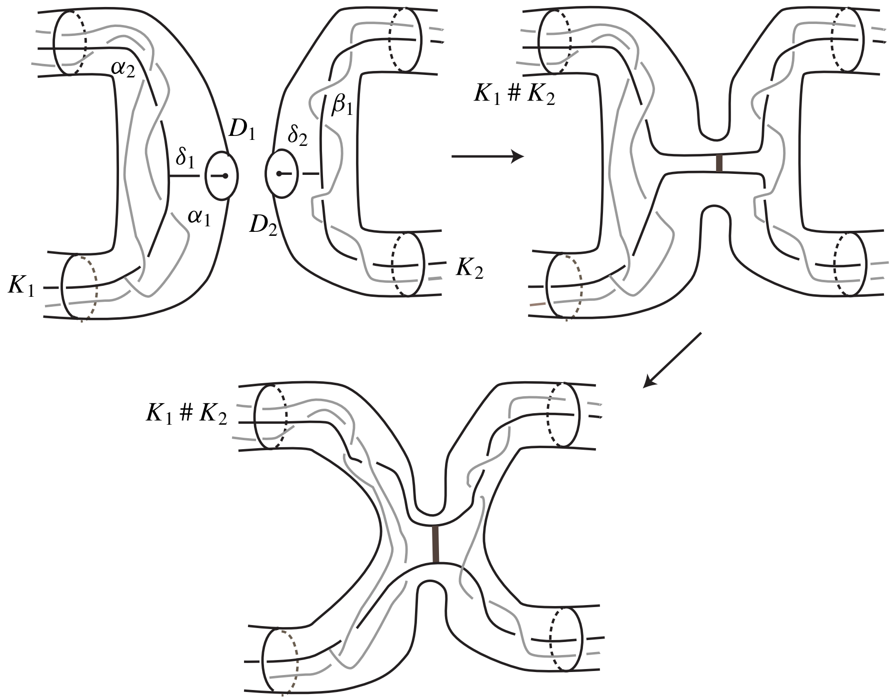

Figure 3. The arcs and sum $K_1 \# K_2$ described in the proof of Theorem 4.1.

disks E_i, and then they are contained in T'. Then these arcs can be used to unknot $K_1 \# K_2$, which then have transient number at most $n+m = \mathrm{tr}(K_1)+\mathrm{tr}(K_2)$.

There are examples of knots K_1 and K_2, such that $t(K_1) = 1 = t(K_2)$, but $t(K_1 \# K_2) = 3$ [6]. For these examples, it is clear that

$$\mathrm{tr}(K_1) = 1 = \mathrm{tr}(K_2),$$

but it is not clear what is $\mathrm{tr}(K_1 \# K_2)$.

There are also examples of knots K_1 and K_2, such that $t(K_1) = 2$, $t(K_2) = 1$, but $t(K_1 \# K_2) = 2$ [5]. In this case

$$\mathrm{tr}(K_2) = 1 \quad \text{and} \quad \mathrm{tr}(K_1 \# K_2) \leq 2,$$

but it is not clear whether $\mathrm{tr}(K_1) = 1$ or 2.

It is well known that knots with unknotting number one or tunnel number one are prime, but the proofs are not so easy. The first proof that knots K with $u(K) = 1$ are prime [10] uses heavy combinatorial arguments, a second proof uses sutured manifold theory [11], and a third proof depends on double branched covers and deep results on Dehn surgery on knots [14]. There are also two proofs that tunnel number one knots are prime, one uses combinatorial group theory [7], and the other uses combinatorial arguments [9]. A proof that transient number one knots are prime would imply both, that unknotting number one and tunnel number one knots are prime, so it may not be easy to prove that. However it seems reasonable to conjecture the following.

Conjecture 4.2. If K is a knot with $\mathrm{tr}(K) = 1$ then K is prime.

Theorem 3.1(1) gives some evidence for this conjecture.

Acknowledgments

This research was supported by a grant from the National Autonomous University of Mexico, UNAM-PAPIIT IN116720.

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Received July 26, 2023. Revised August 4, 2024.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 332 No. 1 September 2024
