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EFFICIENT CYCLES OF HYPERBOLIC MANIFOLDS

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Let N be a complete finite-volume hyperbolic n-manifold. An efficient cycle for N is the limit (in an appropriate measure space) of a sequence of fundamental cycles whose ℓ^1-norm converges to the simplicial volume of N. Gromov and Thurston's smearing construction exhibits an explicit efficient cycle, and Jungreis and Kuessner proved that, in dimension $n \geq 3$, such a cycle actually is the unique efficient cycle for a huge class of finite-volume hyperbolic manifolds, including all the closed ones. We prove that, for $n \geq 3$, the class of finite-volume hyperbolic manifolds for which the uniqueness of the efficient cycle does not hold is exactly the commensurability class of the figure-8 knot complement (or, equivalently, of the Gieseking manifold).

Introduction

The simplicial volume is a homotopy invariant of manifolds introduced by Gromov in his pioneering paper [1982]. If N is a compact connected oriented n-manifold (possibly with boundary) the simplicial volume $\|N\|$ of N is defined by

$$\|N\| = \inf\left\{\sum_{i=1}^{k} |a_i| : \left[\sum_{i=1}^{k} a_i\sigma_i\right] = [N] \in H_n(N, \partial N)\right\},$$

where $[N]$ denotes the real fundamental class of N, and $H_n(N, \partial N)$ denotes the relative singular homology module of the pair $(N, \partial N)$ with real coefficients.

Computing the simplicial volume is usually a very difficult task. Many vanishing theorems are available by now, but positive exact values of the simplicial volume are known only for a few classes of manifolds, such as complete finite-volume hyperbolic manifolds [Gromov 1982; Thurston 1979], closed manifolds isometrically covered by the product of two copies of the hyperbolic plane [Bucher-Karlsson 2008], some 3-manifolds with higher genus boundary [Bucher et al. 2015] and special families of 4-manifolds [Heuer and Löh 2021]. Even when the simplicial volume of a manifold N is known, characterizing (or, at least, exhibiting) *almost*

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minimal fundamental cycles (i.e., fundamental cycles whose norm is close to $\|N\|$) may be surprisingly difficult. For example, it is known that the simplicial volume of any closed simply connected manifold N vanishes, but there is no recipe, in general, which describes fundamental cycles of N of arbitrarily small norm; in a similar spirit, even if the value of the simplicial volume of the product $\Sigma \times \Sigma'$ of two hyperbolic surfaces has been computed in [Bucher-Karlsson 2008], exhibiting a sequence of fundamental cycles whose norm approximates $\|\Sigma \times \Sigma'\|$ seems very challenging [Marasco 2023].

The situation is better understood for hyperbolic manifolds: the computation by Gromov and Thurston of the simplicial volume of such manifolds explicitly constructs almost minimal cycles via an averaging operator called *smearing* [Thurston 1979]. A natural question is to which extent this construction is unique, that is, whether there exist sequences of almost minimal fundamental cycles which do not come from smearing: this problem has been partially addressed by Jungreis [1997] and Kuessner [2003].

We improve their results by showing that, in dimension $n \geq 3$, the unique hyperbolic manifolds admitting "exotic" almost minimal fundamental cycles are those which are commensurable with the Gieseking manifold (it is known that hyperbolic surfaces admit many almost minimal efficient cycles which do not come from smearing; see, e.g., [Jungreis 1997, Remark at page 647]).

In order to state more precisely our results, let us introduce some notation. Let N be a complete finite-volume hyperbolic n-manifold. If N is closed, we denote by $\|N\|$ its simplicial volume. If N is noncompact, it is the internal part of a compact manifold with boundary $\overline{N}$, and for the sake of simplicity we still denote by $\|N\|$ the simplicial volume of $(\overline{N}, \partial\overline{N})$. In fact, by replacing finite chains with locally finite ones, the definition of simplicial volume may be extended to open manifolds, and for finite-volume hyperbolic manifolds this notion of simplicial volume coincides with the simplicial volume of the compactification (see, for example, [Kim and Kuessner 2015]). In order to better compare our results with Kuessner's we prefer to work with the relative simplicial volume of compact manifolds with boundary rather than the simplicial volume of open manifolds, even if our proofs can be easily adapted to the latter framework.

Let c_i, $i \in \mathbb{N}$, be a sequence of (relative) fundamental cycles such that

$$\lim_{i\to+\infty} \|c_i\| = \|N\|.$$

Any possible limit μ of such a sequence naturally sits in the space $\mathcal{M}(\overline{S}_n^*(N))$ of signed measures on the space of (nondegenerate and possibly ideal) geodesic simplices in N, and will be called an *efficient cycle* for N: thus, an efficient cycle is a measure rather than a classical chain (we refer the reader to Section 1 for more details). In fact, it is not difficult to prove that an efficient cycle μ is supported on the subspace $\mathrm{Reg}(N)$ of *regular ideal* simplices, which may be identified with $\Gamma\backslash\mathrm{Isom}(\mathbb{H}^n)$, where Γ is the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ such that $N = \Gamma\backslash\mathbb{H}^n$ (see Lemma 2.3). The Haar measure on $\mathrm{Isom}(\mathbb{H}^n)$ may then be exploited to define a uniformly distributed measure μ_{eq} on $\mathrm{Reg}(N)$, and Gromov and Thurston's smearing procedure constructs sequences of fundamental cycles converging exactly to a suitable multiple of μ_{eq}.

Jungreis and Kuessner provided a complete characterization of efficient cycles for all finite-volume hyperbolic n-manifolds, $n \geq 3$, except for the so-called *Gieseking-like* manifolds.

Definition 1. Let $N = \Gamma\backslash\mathbb{H}^3$ be a cusped hyperbolic 3-manifold. Let us fix an identification between $\partial\mathbb{H}^3$ and the space $\mathbb{C}\cup\{\infty\}$, and let $\mathcal{P} = \mathbb{Q}(e^{i\pi/3})\cup\{\infty\} \subseteq \partial\mathbb{H}^3$. Then N is *Gieseking-like* if there exists a conjugate Γ' of Γ in $\mathrm{Isom}(\mathbb{H}^3)$ such that $\mathcal{P}$ is contained in the set of parabolic fixed points of Γ'.

The well-known Gieseking manifold is indeed Gieseking-like. Being Gieseking-like is invariant with respect to commensurability; hence all hyperbolic 3-manifolds which are commensurable with the Gieseking manifold (like, for example, the figure-8 knot complement) are Gieseking-like. It is still unknown whether the class of Gieseking-like manifolds coincides with the commensurability class of the Gieseking manifold, or it is strictly larger (see [Long and Reid 2002]).

Let v_n be the volume of a regular ideal simplex in hyperbolic space $\mathbb{H}^n$. We are now ready to state Jungreis' and Kuessner's results:

Theorem 2 [Jungreis 1997]. *Let N be a closed orientable n-hyperbolic manifold with $n \geq 3$. Then N admits a unique efficient cycle, which is given by the measure*

$$\frac{1}{2v_n} \cdot \mu_{\mathrm{eq}}.$$

Theorem 3 [Kuessner 2003]. *Let N be a complete finite-volume n-hyperbolic manifold, $n \geq 3$, and suppose that N is not Gieseking-like (this condition is automatically satisfied if $n \geq 4$). Then every efficient cycle of N is a multiple of μ_{eq}.*

In fact, Kuessner [2003, Theorem 4.5] proved that any efficient cycle is a nonvanishing multiple of μ_{eq}, without explicitly computing the proportionality coefficient $1/(2v_n)$ appearing in Jungreis' theorem. When N is noncompact, the space of straight simplices in N is noncompact, which introduces some issues when dealing with the weak-* convergence of measures (namely, by passing to the limit there could be some loss of mass).

Let us say that a measure in $\mathcal{M}(\mathrm{Reg}(N))$ is *equidistributed* if it is a multiple of μ_{eq}. Our main results strengthen and clarify Kuessner's result in three directions:

(1) We prove that the total variation of any efficient cycle of a cusped hyperbolic manifold is equal to its simplicial volume, thus showing that, in the non-Gieseking like case, also for cusped manifolds the proportionality coefficient between any efficient cycle and μ_{eq} is equal to $1/(2v_n)$, as in Jungreis' theorem.

(2) We show that if a cusped 3-manifold N admits nonequidistributed efficient cycles, then it is commensurable with the Gieseking manifold (a condition which is potentially stronger than being Gieseking-like).

(3) For any such manifold we exhibit nonequidistributed efficient cycles, thus obtaining a complete characterization of hyperbolic manifolds with nonunique efficient cycles.

Let us state more precisely our results:

Theorem 4 (no loss of mass). *Let N be a complete finite-volume hyperbolic n-manifold, $n \geq 3$, and let μ be an efficient cycle for N. Then $\|\mu\| = \|N\|$.*

Theorem 5. *Let N be a complete finite-volume hyperbolic manifold. Then N admits nonequidistributed efficient cycles if and only if it is commensurable with the Gieseking manifold.*

Putting together Theorems 4 and 5 we can then deduce the following:

Theorem 6. *Let N be a complete finite-volume hyperbolic n-manifold with $n \geq 3$.*

(1) *If N is not commensurable with the Gieseking manifold and c_i, $i \in \mathbb{N}$, is any minimizing sequence for N, then*

$$\lim_{i \to +\infty} c_i = \frac{1}{2v_n} \mu_{\mathrm{eq}}.$$

(2) *If N is commensurable with the Gieseking manifold, then N admits nonequidistributed efficient cycles.*

We can be more precise. If N is commensurable with the Gieseking manifold, then a finite cover M of N admits a decomposition T into regular ideal tetrahedra. The triangulation T induces a measure cycle $\mu_T \in \mathcal{M}(\mathrm{Reg}(M))$ which is a finite sum of atomic measures supported on the regular ideal tetrahedra appearing in T (see Section 4.4 for the precise definition of μ_T). We then have the following:

Theorem 7. *Let M be a complete finite-volume 3-manifold admitting a decomposition T into regular ideal tetrahedra. Then μ_T is an efficient cycle for M.*

From the nonequidistributed efficient cycle μ_T for M one can then easily construct a nonequidistributed efficient cycle for N. The proof of Theorem 7 exploits a construction described in [Francaviglia et al. 2012, Section 5.4], which allows one to replace an ideal triangulation T of a cusped manifold with a classical triangulation of its compactification in a very controlled way. By applying this procedure to a suitably chosen tower of coverings of M and pushing-forward the resulting classical triangulations to M we obtain a minimizing sequence whose limit is equal to μ_T.

Plan of the paper. In Section 1 we recall the definition of simplicial volume, of minimizing sequence, of efficient cycle and of equidistributed efficient cycle. To this aim we also introduce the measure spaces we will exploit throughout the paper. Section 2 is devoted to the proof of some fundamental properties of efficient cycles, including Theorem 4. In Section 3 we prove that, if a complete finite-volume hyperbolic n-manifold, $n \geq 3$, admits a nonequidistributed efficient cycle, then it is necessarily commensurable with the Gieseking manifold, while Section 4 is devoted to the construction of nonequidistributed efficient cycles for manifolds which are commensurable with the Gieseking manifold.

1. Preliminaries

1.1. ***Simplicial volume.*** Let X be a topological space. For every $k \in \mathbb{N}$, we denote by $\widehat{S}_k(X)$ the set of singular k-simplices with values in X, and by $C_k(X)$ the chain module of singular k-chains with *real* coefficients, i.e., the real vector space with free basis $\widehat{S}_k(X)$. If $Y \subseteq X$, we denote by $C_*(X, Y)$ the chain complex of relative singular cochains with real coefficients, and by $H_*(X, Y)$ the corresponding homology module. We endow $C_*(X)$ with the ℓ^1-norm $\|\cdot\|$ defined by

$$\left\| \sum_{i=1} a_i \sigma_i \right\| = \sum_{i=1}^{k} |a_i|.$$

This norm descends to a norm on $C_*(X, Y)$ and, by taking the infimum over representatives, to a seminorm on $H_*(X, Y)$, still denoted by $\|\cdot\|$.

If N is a compact oriented n-dimensional manifold (possibly with boundary), then the singular homology module with integral coefficients $H_n(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the *integral fundamental class* $[N]_{\mathbb{Z}} \in H_n(N, \partial N, \mathbb{Z})$. Under the change of coefficient homomorphism induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, the class $[N]_{\mathbb{Z}}$ is sent to the *real fundamental class* $[N] \in H_n(N, \partial N; \mathbb{R})$.

Definition 1.1 [Gromov 1982]. The simplicial volume of N is

$$\|N\| = \|[N]\|.$$

Henceforth, all the homology modules will be understood with real coefficients, and the coefficients will be omitted from our notation.

1.2. ***Straight chains on hyperbolic manifolds.*** Let $N = \Gamma\backslash\mathbb{H}^n$ be a cusped oriented hyperbolic n-manifold, where Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. For every $k \in \mathbb{N}$, if $\sigma : \Delta^k \to \mathbb{H}^n$ is a singular simplex, we denote by $\widetilde{\mathrm{str}}_k(\sigma)$ the *straightening* of σ, that is, the singular simplex obtained by suitably parametrizing the convex hull of the vertices of σ (see, for instance, [Frigerio 2017, Section 8.7] or [Martelli 2022, Chapter III.13]). We denote by $S_k(\mathbb{H}^n) \subseteq \widehat{S}_k(\mathbb{H}^n)$ the image of $\widetilde{\mathrm{str}}_k$, i.e., the subset of *straight* hyperbolic k-simplices, and observe that there is a natural identification

$$S_k(\mathbb{H}^n) = (\mathbb{H}^n)^{k+1}$$

sending a straight simplex to the (ordered) set of its vertices, which will be understood henceforth. With a slight abuse, we still denote by $\widetilde{\mathrm{str}}_k : C_k(\mathbb{H}^n) \to C_k(\mathbb{H}^n)$ the $\mathbb{R}$-linear extension of $\widetilde{\mathrm{str}}_k$ to the space of singular chains, and recall that $\widetilde{\mathrm{str}}_* : C_*(\mathbb{H}^n) \to C_*(\mathbb{H}^n)$ is in fact a chain map (see, for instance, [Frigerio 2017, Proposition 8.11]).

Being $\mathrm{Isom}(\mathbb{H}^n)$-equivariant (and equivariantly homotopic to the identity), the map $\widetilde{\mathrm{str}}_k$ is in particular Γ-equivariant, and induces a well-defined chain map $\mathrm{str}_* : C_*(N) \to C_*(N)$, which is chain homotopic to the identity. We denote by $S_k(N)$ the image of $\widehat{S}_k(N)$ via str_k, i.e., the set of straight simplices in N, and we observe that there is a natural identification

$$S_k(N) = \Gamma\backslash S_k(\mathbb{H}^n) = \Gamma\backslash(\mathbb{H}^n)^{k+1}.$$

A chain is called *straight* if it is supported on straight simplices or, equivalently, if it lies in the image of the chain map str_* (or $\widetilde{\mathrm{str}}_*$).

We denote by $\mathrm{SC}_*(\mathbb{H}^n) \subseteq C_*(\mathbb{H}^n)$ (resp. $\mathrm{SC}_*(N) \subseteq C_*(N)$) the complex of straight chains in $\mathbb{H}^n$ (resp. in N). By construction, under the above identification between the set of straight k-simplices and $(\mathbb{H}^n)^{k+1}$ (resp. $\Gamma\backslash(\mathbb{H}^n)^{k+1}$), the complex $\mathrm{SC}_*(\mathbb{H}^n)$ (resp. $\mathrm{SC}_*(N)$) is identified with the free vector space with basis $(\mathbb{H}^n)^{*+1}$ (resp. $\Gamma\backslash(\mathbb{H}^n)^{*+1}$), with boundary operators which linearly extend the maps

$$\partial_k(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i (v_0, \ldots, \widehat{v}_i, \ldots, v_k)$$

$\big($resp. $\partial_k[(v_0, \ldots, v_k)] = \sum_{i=0}^{k} (-1)^i [(v_0, \ldots, \widehat{v}_i, \ldots, v_k)]\big)$.

If $\sigma = (v_0, \dots, v_k) \in (\mathbb{H}^n)^{k+1}$ is a straight simplex, the alternating chain associated to σ is defined by

$$\mathrm{alt}_k(\sigma) = \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_{k+1}} \varepsilon(\tau) \cdot (v_{\tau(0)}, \dots, v_{\tau(k)}),$$

where $\mathfrak{S}_{k+1}$ is the group of the permutations of the set $\{0, \dots, k\}$, and $\varepsilon(\tau) = \pm 1$ is the sign of τ for every $\tau \in \mathfrak{S}_{k+1}$. The maps alt_k linearly extend to a chain map $\mathrm{alt}_* : \mathrm{SC}_*(\mathbb{H}^n) \to \mathrm{SC}_*(\mathbb{H}^n)$ which is Γ-equivariant, and Γ-equivariantly homotopic to the identity (see, for example, [Fujiwara and Manning 2011, Appendix A]). In particular, alt_* induces a well-defined chain map $\mathrm{alt}_* : \mathrm{SC}_*(N) \to \mathrm{SC}_*(N)$, which is homotopic to the identity. A chain in $\mathrm{SC}_*(\mathbb{H}^n)$ (or in $\mathrm{SC}_*(N)$) is *alternating* if it lies in the image of alt_*.

1.3. ***Thick-thin decomposition of hyperbolic manifolds.*** For every $\varepsilon > 0$ we denote by N_ε the ε-thick part of N, that is, the set of points of N whose injectivity radius is not smaller than ε. We will always choose $\varepsilon > 0$ small enough so that N_ε is a compact submanifold with boundary of N, obtained from N by removing open neighborhoods of its cusps. We denote by $\overline{N}$ the natural compactification of N, which is diffeomorphic to N_ε. The inclusion $(N_\varepsilon, \partial N_\varepsilon) \to (N, N \setminus \mathrm{int}(N_\varepsilon))$ and the obvious deformation retraction $r : (N, N \setminus \mathrm{int}(N_\varepsilon)) \to (N_\varepsilon, \partial N_\varepsilon)$ are the homotopy inverses of each other, and they induce norm nonincreasing maps in homology. Therefore, in order to compute the simplicial volume of $\overline{N}$ we may consider relative fundamental cycles in $C_n(N, N \setminus \mathrm{int}(N_\varepsilon))$. The complement in $\mathbb{H}^n$ of the preimage of N_ε under the covering projection is an equivariant family of disjoint open horoballs. Since horoballs are convex in $\mathbb{H}^n$, the straightening operator induces a well-defined chain map on the relative chain complex $C_*(N, N \setminus \mathrm{int}(N_\varepsilon))$. Finally, since both the straightening and the alternating operators are obviously norm nonincreasing (and they induce the identity also on relative homology), in order to compute the simplicial volume of N it is not restrictive to consider only straight and alternating relative cycles in $C_*(N, N \setminus \mathrm{int}(N_\varepsilon))$.

1.4. ***Minimizing sequences and efficient cycles.*** We say that a sequence $c_i \in C_n(N)$ of chains is a *minimizing sequence* if the following conditions hold:

(1) Each c_i is straight and alternating.

(2) For all sufficiently large $i \in \mathbb{N}$, c_i is a relative cycle in $C_n(N_{2^{-i}}, N \setminus \mathrm{int}(N_{2^{-i}}))$.

(3) Under the identification $H_n(N_{2^{-i}}, N \setminus \mathrm{int}(N_{2^{-i}})) \cong H_n(\overline{N}, \partial \overline{N})$ described above, the relative cycle c_i represents the fundamental class of $(\overline{N}, \partial \overline{N})$.

(4) $\|c_i\| \le \|N\| + 2^{-i}$ for all sufficiently large $i \in \mathbb{N}$.

Of course, in the definition of minimizing sequence the values 2^{-i} may be replaced by any infinitesimal sequence η_i; we decided to choose this specific sequence just to simplify the notation.

We now introduce the measure spaces we are interested in. Recall that $S_n(N) = \Gamma\backslash(\mathbb{H}^n)^{n+1}$ is the space of straight simplices with values in N. Of course, this space does not contain any ideal simplex; hence we need to enlarge it in order to construct a measure space which could support possible limits of minimizing sequences (recall from the introduction that efficient cycles are supported on regular ideal simplices). The natural space to look at is then $\overline{S}_n(N) = \Gamma\backslash(\overline{\mathbb{H}^n})^{n+1}$ but, unfortunately, the action of Γ on $\overline{S}_n(\mathbb{H}^n) = (\overline{\mathbb{H}^n})^{n+1}$ has not closed orbits, so that the quotient space is not Hausdorff. In order to avoid this inconvenience, and for other later purposes, we introduce the following:

Definition 1.2. A straight simplex in $\overline{S}_n(\mathbb{H}^n)$ is *degenerate* if its vertices (hence, its image) lie on (the closure at infinity of) a hyperplane of $\mathbb{H}^n$ or, equivalently, if its image has volume equal to 0. A straight simplex in $\overline{S}_n(N) = \Gamma\backslash\overline{S}_n(\mathbb{H}^n)$ is degenerate if it is the image of a degenerate simplex in $\overline{S}_n(\mathbb{H}^n)$.

We denote by $\overline{S}^*_n(\mathbb{H}^n)$ (resp. $\overline{S}^*_n(N)$) the set of nondegenerate straight simplices in $\overline{S}_n(\mathbb{H}^n)$ (resp. $\overline{S}_n(N)$).

It is not difficult to show that, when endowed with the quotient topology, the space $\overline{S}^*_n(N)$ is metrizable and locally compact (see, e.g., [Kuessner 2003, Lemma 2.6] for a similar result and the proof of Lemma 2.7 here below). We denote by $\mathcal{M}(\overline{S}^*_n(N))$ the space of signed regular measures on $\overline{S}^*_n(N)$. If $\sigma : \Delta_n \to N$ is a straight simplex, then we denote by $\delta_\sigma \in \mathcal{M}(\overline{S}^*_n(N))$ the atomic measure concentrated on σ. The map $\sigma \mapsto \delta_\sigma$ linearly extends to a map

$$\Theta : \mathrm{SC}_n(N) \to \mathcal{M}(\overline{S}^*_n(N)). \tag{1}$$

We are now ready to define the notion of efficient cycle for complete finite-volume hyperbolic manifolds:

Definition 1.3. A measure $\mu \in \mathcal{M}(\overline{S}^*_n(N))$ is an *efficient cycle* for N if there exists a minimizing sequence c_i, $i \in \mathbb{N}$, such that

$$\mu = \lim_{i\to+\infty} \Theta(c_i),$$

where the limit is taken with respect to the weak-* topology on $\mathcal{M}(\overline{S}^*_n(N))$.

1.5. ***Equidistributed measure cycles.*** As explained in the introduction, we are going to prove that, if N is *not* commensurable with the Gieseking manifold, then there exists a unique efficient cycle for N, which is concentrated on (classes of) regular ideal straight simplices, and is equidistributed on such simplices. Let us formally describe what we mean by equidistributed measure on (classes of) regular ideal straight simplices.

We define

$$\mathrm{Reg}(\mathbb{H}^n) = \{(v_0, \dots, v_n) \in (\partial\mathbb{H}^n)^{n+1} \mid v_0, \dots, v_n \text{ span a regular ideal straight simplex}\}$$

and we denote by $\mathrm{Reg}^+(\mathbb{H}^n)$ (resp. $\mathrm{Reg}^-(\partial\mathbb{H}^n)$) the subset of $\mathrm{Reg}(\mathbb{H}^n)$ corresponding to positively oriented (resp. negatively oriented) simplices. We then set

$$\mathrm{Reg}^\pm(N) = \Gamma\backslash\mathrm{Reg}^\pm(\mathbb{H}^n) \subseteq \overline{S}^*_n(N).$$

Since N is oriented, elements of Γ are orientation-preserving; hence the sets $\mathrm{Reg}^+(N)$ and $\mathrm{Reg}^-(N)$ are disjoint.

Let $\Delta_0 = (v_0, \dots, v_n) \in (\partial\mathbb{H}^n)^{n+1}$ be the (ordered) $(n+1)$-tuple of vertices of a fixed positively oriented regular ideal hyperbolic simplex. We then have bijections

$$\mathrm{Isom}^\pm(\mathbb{H}^n) \to \mathrm{Reg}^\pm(\mathbb{H}^n), \quad g \mapsto g \cdot \Delta_0 = (g(v_0), \dots, g(v_n)),$$

which induce bijections

$$\Gamma\backslash\mathrm{Isom}^\pm(\mathbb{H}^n) \to \mathrm{Reg}^\pm(N).$$

We denote by the symbol Haar the Haar measure on $\mathrm{Isom}(\mathbb{H}^n)$, normalized in such a way that, for every measurable subset $\Omega \subseteq \mathbb{H}^n$ and any $x_0 \in \mathbb{H}^n$,

$$\mathrm{Haar}\{g \in \mathrm{Isom}(\mathbb{H}^n) \mid g(x_0) \in \Omega\} = \mathrm{Vol}(\Omega).$$

Being bi-invariant, the Haar measure induces well-defined finite measures $\mathrm{Haar}_\pm$ on $\Gamma/\mathrm{Isom}^\pm(\mathbb{H}^n)$, hence on $\mathrm{Reg}^\pm(N)$ via the above identifications. We finally set

$$\mu_{\mathrm{eq}} = \mathrm{Haar}_+ - \mathrm{Haar}_- \in \mathcal{M}(\mathrm{Reg}(N)) \subseteq \mathcal{M}(\overline{S}^*_n(N)),$$

where the subscript "eq" stands for "equidistributed". Using again the bi-invariance of Haar one can easily check that the definition of $\mathrm{Haar}_\pm$ (hence of μ_{eq}) on $\mathrm{Reg}(\mathbb{H}^n)$ does not depend on the choice of Δ_0.

2. Some properties of efficient cycles

For every $\varepsilon > 0$ we denote by $\omega_\varepsilon : C_n(N, N \setminus \mathrm{int}(N_\varepsilon)) \to \mathbb{R}$ the restriction of the volume cochain to N_ε, i.e., the cochain such that

$$\omega_\varepsilon(c) = \int_c d\,\mathrm{Vol}_\varepsilon,$$

where $d\,\mathrm{Vol}_\varepsilon$ is the (discontinuous) n-form that coincides with the hyperbolic volume form on N_ε and is equal to 0 on $N \setminus N_\varepsilon$ (for our purposes, it is sufficient to define ω_ε on straight chains, which of course are C^1, so the integral above makes sense). If $c \in C_n(N)$ is a straight relative fundamental cycle for $(N, N \setminus \mathrm{int}(N_\varepsilon))$, then we have

$$\omega_\varepsilon(c) = \mathrm{Vol}(N_\varepsilon).$$

If σ is a straight simplex with values in N, then it is immediate to check that $\omega_\varepsilon(\sigma) = \pm\,\mathrm{Vol}(\widetilde{\sigma} \cap \widetilde{N}_\varepsilon)$, where $\widetilde{\sigma}$ is a lift of σ to $\mathbb{H}^n$, the space $\widetilde{N}_\varepsilon$ is the preimage of N_ε in $\mathbb{H}^n$, and the sign is positive (resp. negative) if σ is positively oriented (resp. negatively oriented).

The following lemma shows that, in a minimizing sequence, the orientation of simplices has to be coherent with the sign of their coefficients, at least asymptotically.

Lemma 2.1. *Let c_i, $i \in \mathbb{N}$, be a minimizing sequence, and for every $i \in \mathbb{N}$ let*

$$c_i = \sum_{k=1}^{n_i} a_{i,k}\sigma_{i,k}$$

be the reduced form of c_i (that is, $\sigma_{i,k} \neq \sigma_{i,k'}$ whenever $k \neq k'$, where the $\sigma_{i,k}$ are straight singular simplices and the $a_{i,k}$ are real coefficients). For every i, k, set $b_{i,k} = a_{i,k}$ if $a_{i,k} > 0$ and $\sigma_{i,k}$ is not positively oriented or $a_{i,k} < 0$ and $\sigma_{i,k}$ is not negatively oriented, and $b_{i,k} = 0$ otherwise. If $c'_i = \sum_{k=1}^{n_i} b_{i,k}\sigma_{i,k}$, then

$$\lim_{i\to+\infty} \|c'_i\| = 0.$$

Proof. By definition of minimizing sequence we have

$$\lim_{i\to+\infty} \omega_{2^{-i}}(c_i) = \lim_{i\to+\infty} \mathrm{Vol}(N_{2^{-i}}) = \mathrm{Vol}(N);$$

hence

$$\lim_{i\to+\infty} \frac{\omega_{2^{-i}}(c_i)}{v_n} = \frac{\mathrm{Vol}(N)}{v_n} = \|N\| = \lim_{i\to+\infty} \|c_i\|. \tag{2}$$

Since the hyperbolic volume of any straight simplex is at most v_n, we have

$$\omega_{2^{-i}}(c_i - c'_i) \le \|c_i - c'_i\| \cdot v_n,$$

while our definition of c'_i readily implies that $\omega_{2^{-i}}(c'_i) \le 0$. Therefore,

$$\frac{\omega_{2^{-i}}(c_i)}{v_n} = \frac{\omega_{2^{-i}}(c_i - c'_i) + \omega_{2^{-i}}(c'_i)}{v_n} \le \|c_i - c'_i\| = \|c_i\| - \|c'_i\|,$$

where the last equality follows from the fact that, by construction, the set of simplices appearing in c'_i is disjoint from the set of simplices appearing in $c_i - c'_i$, so that

$\|c_i\| = \|(c_i - c_i') + c_i'\| = \|c_i - c_i'\| + \|c_i'\|$. The conclusion follows from this inequality and (2). □

A very similar argument shows that the volume of "most" simplices appearing in minimizing sequences converges to v_n. We properly state and prove this result, since we will need it later.

Lemma 2.2. *Let c_i, $i \in \mathbb{N}$, be a minimizing sequence, and for every $i \in \mathbb{N}$ let*

$$c_i = \sum_{k=1}^{n_i} a_{i,k}\sigma_{i,k}$$

be the reduced form of c_i, as in the previous lemma. Let $\varepsilon > 0$ be fixed, and, for every i, k, set $b_{i,k} = a_{i,k}$ if the hyperbolic volume of a lift of $\sigma_{i,k}$ to $\mathbb{H}^n$ is smaller that $v_n - \varepsilon$, and $b_{i,k} = 0$ otherwise. If $c_i' = \sum_{k=1}^{n_i} b_{i,k}\sigma_{i,k}$, then

$$\lim_{i\to+\infty} \|c_i'\| = 0.$$

Proof. Just as in the proof of the previous lemma we have $\|c_i - c_i'\| = \|c_i\| - \|c_i'\|$. Using this fact, since the hyperbolic volume of any straight simplex is at most v_n, our definition of c_i' implies that

$$|\omega_{2^{-i}}(c_i)| \leq |\omega_{2^{-i}}(c_i - c_i')| + |\omega_{2^{-i}}(c_i')| \leq \|c_i - c_i'\| \cdot v_n + \|c_i'\|(v_n - \varepsilon) = \|c_i\| v_n - \|c_i'\|\varepsilon,$$

whence

$$\|N\| = \lim_{i\to+\infty} \frac{|\omega_{2^{-i}}(c_i)|}{v_n} \leq \lim_{i\to+\infty} \|c_i\| - \frac{\varepsilon}{v_n} \limsup_{i\to+\infty} \|c_i'\| = \|N\| - \frac{\varepsilon}{v_n} \limsup_{i\to+\infty} \|c_i'\|.$$

The conclusion follows. □

The previous lemma may be exploited to prove that efficient cycles are supported on regular ideal straight simplices:

Lemma 2.3 [Kuessner 2003, Lemma 3.5]. *Let μ be an efficient cycle for N. Then μ is supported on* $\mathrm{Reg}(N) \subseteq \overline{S}_n^*(N)$.

Therefore, we will consider μ both as an element of $\mathcal{M}(\overline{S}_n^*(N))$ and as an element of $\mathcal{M}(\mathrm{Reg}(N))$.

We are now going to prove that the total variation of an efficient cycle is equal to the simplicial volume of N (recall that the total variation is only lower semicontinuous with respect to weak-* convergence; hence the total variation of an efficient cycle could be strictly smaller than $\|N\|$ a priori).

To this aim we need the definition of incenter and inradius of a straight hyperbolic simplex. Consider a nondegenerate straight n-simplex $\Delta \in \overline{S}^*(\mathbb{H}^n)$ (recall that a straight simplex is nondegenerate if its image is not contained in a hyperplane). For every point $p \in \Delta \cap \mathbb{H}^n$ we denote by $r_\Delta(p)$ the radius of the maximal hyperbolic ball centered in p and contained in Δ. Since the volume of any straight n-simplex is smaller than v_n and the volume of hyperbolic balls diverges as the radius diverges, there exists a constant $r_n > 0$ such that $r_\Delta(p) \leqslant r_n$ for every $\Delta \in \overline{S}_n^*(\mathbb{H}^n)$ and $p \in \Delta$.

Definition 2.4. Take a nondegenerate straight simplex $\Delta \in \overline{S}_n^*(\mathbb{H}^n)$. The *inradius* $r(\Delta)$ of Δ is

$$r(\Delta) = \sup_{p \in \Delta \cap \mathbb{H}^n} r_\Delta(p) \in (0, r_n]$$

(observe that $r(\Delta) > 0$ since Δ is nondegenerate). The *incenter* $\mathrm{inc}(\Delta)$ is the unique point $p \in \Delta \cap \mathbb{H}^n$ such that $r_\Delta(p) = r(\Delta)$.

It is shown in [Francaviglia et al. 2012, Lemma 3.12] that the incenter is well-defined, and that the functions

$$\mathrm{inc} : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{H}^n, \quad r : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{R}$$

are continuous.

If Δ is a straight simplex in $\overline{S}_n^*(N)$, we define its inradius $r(\Delta)$ as the inradius of any lift of Δ to $\mathbb{H}^n$, and its incenter $\mathrm{inc}(\Delta)$ as the projection in N of the incenter of any lift of Δ to $\mathbb{H}^n$ (the fact that these notions are well-defined is easily checked).

Lemma 2.5. *Let $\bar{\delta}$ be the inradius of the n-dimensional regular ideal straight simplex, and let $\Delta_i \in \overline{S}_n^*(\mathbb{H}^n)$, $i \in \mathbb{N}$, be a sequence such that $\lim_{i\to+\infty} \mathrm{Vol}(\Delta_i) = v_n$. Then $\lim_{i\to+\infty} r(\Delta_i) = \bar{\delta}$.*

Proof. By [Francaviglia et al. 2012, Proposition 3.14], for every $i \in \mathbb{N}$ there exists an element $g_i \in \mathrm{Isom}(\mathbb{H}^n)$ such that $\lim_{i\to+\infty} g_i(\Delta_i) = \overline{\Delta}$, where $\overline{\Delta}$ is a regular ideal straight simplex. Since the map $r : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{R}$ is continuous, we thus get

$$\lim_{i\to+\infty} r(\Delta_i) = \lim_{i\to+\infty} r(g_i(\Delta_i)) = r(\overline{\Delta}) = \bar{\delta}. \qquad \square$$

Lemma 2.6. *Let $K \subseteq N$ be compact, and let $\delta_0 > 0$. Then the set*

$$\Lambda = \{\Delta \in \overline{S}_n^*(N) \mid \mathrm{inc}(\Delta) \in K,\ r(\Delta) \geq \delta_0\}$$

is compact.

Proof. Let $\widetilde{K} \subseteq \mathbb{H}^n$ be a compact subset such that $\pi(\widetilde{K}) = K$ (for example, if $\pi : \mathbb{H}^n \to N$ is the universal covering, then $\widetilde{K}$ may be chosen as the intersection between $\pi^{-1}(K)$ and a Dirichlet domain for the action of Γ on $\mathbb{H}^3$), and let

$$\widetilde{\Lambda} = \{\widetilde{\Delta} \in \overline{S}_n^*(\mathbb{H}^n) \mid \mathrm{inc}(\widetilde{\Delta}) \in \widetilde{K},\ r(\widetilde{\Delta}) \geq \delta_0\}.$$

Under the projection $\overline{S}_n^*(\mathbb{H}^n) \to \overline{S}_n^*(N)$, the set $\widetilde{\Lambda}$ is sent to Λ; hence in order to conclude it suffices to show that $\widetilde{\Lambda}$ is compact or, equivalently, sequentially compact (being a subset of $(\overline{\mathbb{H}^n})^{n+1}$, the space $\widetilde{\Lambda}$ is metrizable).

Let $\widetilde{\Delta}_i = (v_0^i, v_1^i, \dots, v_n^i) \in (\overline{\mathbb{H}}^n)^{n+1}$, $i \in \mathbb{N}$, be a sequence of elements in $\widetilde{\Lambda}$. Since $(\overline{\mathbb{H}}^n)^{n+1}$ is compact, up to passing to a subsequence we may assume that $\widetilde{\Delta}_i$ tends to $\widetilde{\Delta}_\infty \in (\overline{\mathbb{H}}^n)^{n+1}$. Since the maps $r : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{R}$ and $\mathrm{inc} : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{H}^n$ are continuous and $\widetilde{K}$ is closed, we have $r(\widetilde{\Delta}_\infty) \geq \delta_0$ and $\mathrm{inc}(\widetilde{\Delta}_\infty) \in \widetilde{K}$. Thus in order to conclude it is sufficient to observe that $\widetilde{\Delta}_\infty$ is nondegenerate, since it contains the hyperbolic ball of radius $r(\widetilde{\Delta}_\infty) > 0$ centered at $\mathrm{inc}(\widetilde{\Delta}_\infty)$; hence it cannot be contained in a hyperplane. □

Lemma 2.7. *Let $n = \dim N \geq 3$, and let Λ be a compact subset of $\overline{S}_n^*(N)$. Then, there exists a compactly supported continuous function $g : \overline{S}_n^*(N) \to [-1, 1]$ such that $g(\Delta) = 1$ for every positively oriented simplex in Λ and $g(\Delta) = -1$ for every negatively oriented simplex in Λ.*

Proof. Let us first prove that $\overline{S}_n^*(N)$ is metrizable. Let $SS_n(\mathbb{H}^n)$ be the space of (possibly ideal) straight simplices with pairwise distinct vertices, i.e.,

$$SS_n(\mathbb{H}^n) = \{(v_0, \dots, v_n) \in (\overline{\mathbb{H}}^n)^{n+1} \mid v_i \neq v_j \text{ for } i \neq j\}.$$

It is proved in [Kuessner 2003, Lemma 2.6] that the action of Γ on $SS_n(\mathbb{H}^n)$ is free and properly discontinuous, and that the quotient space $SS_n(N) = \Gamma \backslash SS_n(\mathbb{H}^n)$ is metrizable. But $\overline{S}_n^*(N)$ is clearly a subspace of $SS_n(N)$, and its topology as a quotient of $\overline{S}_n^*(\overline{\mathbb{H}}^n)$ coincides with the topology it inherits as a subspace of $SS_n(N)$. Therefore, $\overline{S}_n^*(N)$ is metrizable. Indeed, since the action of Γ on $\overline{S}_n^*(\mathbb{H}^n)$ is free and properly discontinuous, and the space $\overline{S}_n^*(\mathbb{H}^n)$ is a topological manifold with boundary (being an open subset of the topological manifold $(\overline{\mathbb{H}}^n)^{n+1}$)), also the space $\overline{S}_n^*(N)$ is a topological manifold. In particular, it is locally compact.

Let now $h : \overline{S}_n^*(N) \to [-1, 1]$ be such that $h(\Delta) = 1$ if Δ is positively oriented, and $h(\Delta) = -1$ if Δ is negatively oriented. Since the subspace of positively oriented (resp. negatively oriented) simplices in $\overline{S}_n^*(N)$ is clopen in $\overline{S}_n^*(N)$, the map h is continuous. Since $\overline{S}_n^*(N)$ is locally compact, we may choose a relatively compact open neighborhood U of Λ. By the Urysohn lemma, there exists a continuous function $\psi : \overline{S}_n^*(N) \to [0, 1]$ such that $\psi(\Delta) = 1$ for every $\Delta \in \Lambda$ and $\psi(\Delta) = 0$ for every $\Delta \notin U$. By construction, the function $g = f \cdot \psi$ is continuous and compactly supported, takes values in $[-1, 1]$ and is such that $g(\Delta) = 1$ for every positively oriented simplex in Λ and $g(\Delta) = -1$ for every negatively oriented simplex in Λ. □

We are now ready to prove Theorem 4 from the introduction, which we recall here for the convenience of the reader:

Theorem 4. *Let N be a complete finite-volume hyperbolic n-manifold, $n \geq 3$, and let μ be an efficient cycle for N. Then $\|\mu\| = \|N\|$.*

Proof. Let r_n be a universal upper bound for the inradius of any nondegenerate n-dimensional straight simplex, as above. For every $\varepsilon > 0$ we set

$$\mathrm{th}_\varepsilon = \{\Delta \in \overline{S}_n^*(N) \mid \mathrm{inc}(\Delta) \in B(N_\varepsilon, r_n)\},$$

where $B(N_\varepsilon, r_n)$ denotes the closed r_n-neighborhood of N_ε in N; in other words, th_ε denotes the set of nondegenerate straight simplices of N whose incenter lies in the closed r_n-neighborhood of the ε-thick part of N.

Let $\bar{\delta}$ be the inradius of the regular ideal straight n-simplex, and fix some constant $0 < \delta_0 < \bar{\delta}$. Also denote by V_0 the hyperbolic volume of a hyperbolic n-ball of radius δ_0, and set $\Omega_{\delta_0} = \{\Delta \in \overline{S}_n^*(N) \mid r(\Delta) \geq \delta_0\}$.

Let now c_i, $i \in \mathbb{N}$, be a minimizing sequence, and let us fix $\varepsilon > 0$. We choose $i_0 \in \mathbb{N}$ such that $\mathrm{Vol}(N \setminus N_{2^{-i_0}}) \leq \varepsilon v_n$. Let $i \geq i_0$, and consider the following partition of the space of nondegenerate straight simplices in N:

$$\Lambda_1 = \overline{S}_n^*(N) \setminus \Omega_{\delta_0}, \quad \Lambda_2 = \Omega_{\delta_0} \cap \mathrm{th}_{2^{-i_0}}, \quad \Lambda_3 = \Omega_{\delta_0} \setminus \mathrm{th}_{2^{-i_0}}.$$

We denote by $c_i = c_i^1 + c_i^2 + c_i^3$ the corresponding decomposition of c_i, i.e., we assume that the simplices appearing in c_i^j belong to Λ_j for $j = 1, 2, 3$. By Lemma 2.5, since δ_0 is smaller than the inradius of the regular ideal tetrahedron, the volume of the lifts to $\mathbb{H}^n$ of the simplices in Λ_1 is bounded above by a constant strictly smaller than v_n. By Lemma 2.2, we then have

$$\lim_{i \to +\infty} \|c_i^1\| = 0. \tag{3}$$

Let now $\Delta \in \Lambda_3$, i.e., suppose that $r(\Delta) \geq \delta_0$ and $\mathrm{inc}(\Delta) \notin B(N_{2^{-i_0}}, r_n)$. Since $\delta_0 \leq r_n$, the ball $B(\mathrm{inc}(\Delta), \delta_0) \subseteq \Delta$ does not intersect $N_{2^{-i_0}}$; hence $|\omega_{2^{-i_0}}(\Delta)| \leq v_n - V_0$. Thus

$$|\omega_{2^{-i_0}}(c_i^3)| \leq \|c_i^3\| \cdot (v_n - V_0).$$

Observe now that c_i, being a relative fundamental cycle for $N_{2^{-i}}$, is a fortiori a relative fundamental cycle for $N_{2^{-i_0}}$. Hence

$$\begin{aligned}\mathrm{Vol}(N_{2^{-i_0}}) = |\omega_{2^{-i_0}}(c_i)| &\leq |\omega_{2^{-i_0}}(c_i^1)| + |\omega_{2^{-i_0}}(c_i^2)| + |\omega_{2^{-i_0}}(c_i^3)| \\ &\leq \|c_i^1\| \cdot v_n + \|c_i^2\| \cdot v_n + \|c_i^3\| \cdot (v_n - V_0) = \|c_i\| \cdot v_n - \|c_i^3\| \cdot V_0.\end{aligned}$$

After dividing by v_n we obtain

$$\frac{\mathrm{Vol}(N_{2^{-i_0}})}{v_n} \leq \|c_i\| - \|c_i^3\| \cdot \frac{V_0}{v_n},$$

whence

$$\|N\|-\varepsilon=\frac{\mathrm{Vol}(N)}{v_n}-\varepsilon\le\frac{\mathrm{Vol}(N_{2^{-i_0}})}{v_n}\le\|c_i\|-\|c_i^3\|\cdot\frac{V_0}{v_n}\le\|N\|+2^{-i}-\|c_i^3\|\cdot\frac{V_0}{v_n}$$

and

$$\|c_i^3\|\le\frac{v_n}{V_0}(\varepsilon+2^{-i}).$$

In particular, we have

$$\limsup_{i\to+\infty}\|c_i^3\|\le\frac{v_n}{V_0}\varepsilon. \tag{4}$$

Since $\|c_i\|\ge\|N\|$ we also have

$$\|c_i^2\|=\|c_i\|-\|c_i^1\|-\|c_i^3\|\ge\|N\|-\|c_i^1\|-\frac{v_n}{V_0}(\varepsilon+2^{-i});$$

hence (recalling that $\|c_i^1\|\to 0$ as $i\to+\infty$)

$$\liminf_{i\to+\infty}\|c_i^2\|\ge\|N\|-\varepsilon\frac{v_n}{V_0}. \tag{5}$$

Observe that, thanks to Lemma 2.6, the set Λ_2 is compact. Therefore, by Lemma 2.7 one may construct a compactly supported continuous function $g:\overline{S}^*(N)\to[-1,1]$ such that $g(\Delta)=1$ for every positively oriented $\Delta\in\Lambda_2$ and $g(\Delta)=-1$ for every negatively oriented $\Delta\in\Lambda_2$. Just as in Lemma 2.1, decompose c_i^2 (resp. c_i) as $c_i^2=(c_i^2-(c_i^2)')$ (resp. $c_i=(c_i-c_i')+c_i'$), where $(c_i^2)'$ (resp. c_i') is a linear combination of positively oriented simplices with negative coefficients and of negatively oriented simplices with positive coefficients. We know from Lemma 2.1 that $\lim_{i\to+\infty}\|c_i'\|=0$; hence, a fortiori,

$$\lim_{i\to+\infty}\|(c_i^2)'\|=0 \tag{6}$$

(hence, also $\liminf_{i\to+\infty}\|c_i^2-(c_i^2)'\|=\liminf_{i\to+\infty}\|c_i^2\|$).

Since the supports of $\Theta(c_1^i)$ and of $\Theta(c_3^i)$ are disjoint from Λ_2, we have

$$\int_{\Lambda_2}g\,d\Theta(c_i)=\int_{\Lambda_2}g\,d\Theta(c_i^2)=\int_{\Lambda_2}g\,d\Theta(c_i^2-(c_i^2)')+\int_{\Lambda_2}g\,d\Theta((c_i^2)'). \tag{7}$$

By definition, every simplex appearing in the chain c_i-c_i' has a positive coefficient if it is positively oriented, and a negative coefficient otherwise. Therefore, by the very definition of the function g, we have

$$\int_{\Lambda_2}g\,d\Theta(c_i-(c_i^2)')=\|c_i^2-(c_i^2)'\|. \tag{8}$$

Finally, since $\|g\|_\infty \le 1$, we have

$$\left|\int_{\Lambda_2} g\, d\Theta((c_i^2)')\right| \le \|(c_i^2)'\|. \tag{9}$$

Putting together the (in)equalities (6)–(9), we then have

$$\liminf_{i\to+\infty} \int_{\Lambda_2} g\, d\Theta(c_i) = \liminf_{i\to+\infty} \|c_i^2\|. \tag{10}$$

By definition of weak-* convergence, if $\mu = \lim_{i\to+\infty} \Theta(c_i)$, then from (3)–(5) and (10) (and the fact that $\|g\|_\infty \le 1$) we obtain

$$\begin{aligned}
\left|\int_{\overline{S}^*(N)} g\, d\mu\right| &= \left|\lim_{i\to+\infty}\int_{\Lambda_1} g\, d\Theta(c_i) + \lim_{i\to+\infty}\int_{\Lambda_2} g\, d\Theta(c_i) + \lim_{i\to+\infty}\int_{\Lambda_3} g\, d\Theta(c_i)\right| \\
&\ge -\left|\lim_{i\to+\infty}\int_{\Lambda_1} g\, d\Theta(c_i)\right| + \left|\lim_{i\to+\infty}\int_{\Lambda_2} g\, d\Theta(c_i)\right| - \left|\lim_{i\to+\infty}\int_{\Lambda_3} g\, d\Theta(c_i)\right| \\
&\ge -\limsup_{i\to+\infty} \|c_i^1\| + \liminf_{i\to+\infty} \|c_i^2\| - \limsup_{i\to+\infty} \|c_i^3\| \\
&\ge \|N\| - \frac{2\varepsilon v_n}{V_0}.
\end{aligned}$$

Since $\|g\|_\infty \le 1$, this inequality implies that the total variation of μ is not smaller than $\|N\| - 2\varepsilon v_n/V_0$. Due to the arbitrariness of ε, we may conclude that $\|\mu\| \ge \|N\|$. On the other hand, it is well known that the total variation is lower semicontinuous with respect to the weak-* convergence; hence $\|\mu\| \le \lim_{i\to+\infty} \|\Theta(c_i)\| = \|N\|$. This concludes the proof. □

Our normalization of the Haar measure implies that $\|\mu_{\text{eq}}\| = 2\,\text{Vol}(N)$. Therefore, Theorem 4 readily implies the following:

Corollary 2.8. *Let $k \in \mathbb{R}$ and suppose that the measure $\mu = k\mu_{\text{eq}}$ is an efficient cycle for N. Then $k = 1/(2v_n)$.*

3. Manifolds admitting a unique efficient cycle

Jungreis and Kuessner proved that, if N is a non-Gieseking like hyperbolic manifold, then every efficient cycle of N is equidistributed. We strengthen this result by showing that the same conclusion holds under the supposedly less restrictive requirement that N be noncommensurable with the Gieseking manifold. We may concentrate our attention on the three-dimensional case, the higher-dimensional case being covered by the results proved in [Kuessner 2003]. Therefore, throughout this section we denote by N a complete finite-volume hyperbolic 3-manifold.

Henceforth we fix a regular ideal straight simplex (with ordered vertices) $\Delta_0 \in \mathrm{Reg}(\mathbb{H}^3)$, which we exploit to fix an identification $\mathrm{Reg}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3)$, as explained at the end of Section 1. For $i = 0, \ldots, 3$, let $r_i \in \mathrm{Isom}^-(\mathbb{H}^3)$ be the hyperbolic reflection with respect to the plane containing the i-th face of Δ_0. Under the identification $\mathrm{Reg}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3)$, the *right* multiplication

$$\mathrm{Isom}(\mathbb{H}^3) \to \mathrm{Isom}(\mathbb{H}^3), \quad g \mapsto g \cdot r_i,$$

corresponds to the map $\rho_i : \mathrm{Reg}(\mathbb{H}^3) \to \mathrm{Reg}(\mathbb{H}^3)$ sending any simplex $\Delta \in \mathrm{Reg}(\mathbb{H}^3)$ to the simplex obtained by reflecting Δ with respect to the plane containing its i-th face. We denote by R the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ generated by the r_i, $i = 0, \ldots, 3$, and we set $R^\pm = R \cap \mathrm{Isom}^\pm(\mathbb{H}^3)$. Observe that, since the (left) action of Γ and the (right) action of R on $\mathrm{Isom}(\mathbb{H}^3)$ commute, the groups R, R^+ also act on $\mathrm{Reg}(N)$.

Recall that any efficient cycle for N is supported on $\mathrm{Reg}(N)$, so that we can consider efficient cycles as elements of $\mathcal{M}(\mathrm{Reg}(N))$. The following result is proved in [Kuessner 2003, Lemmas 3.9 and 3.10] (see also [Jungreis 1997, Lemma 2.2]):

Lemma 3.1. *Let $\mu \in \mathcal{M}(\mathrm{Reg}(N))$ be any efficient cycle for N. Then μ is invariant with respect to the right action of R^+ on* $\mathrm{Reg}(N)$. *For every $r \in R^-$ we have $r_*(\mu) = -\mu$.*

If $\Delta = (v_0, v_1, v_2, v_3) \in \mathrm{Reg}(\mathbb{H}^3)$ is an arbitrary regular ideal straight simplex, we denote by $T_\Delta \subseteq \mathrm{Reg}(\mathbb{H}^3)$ the set defined as follows: $\Delta' = (v'_0, v'_1, v'_2, v'_3) \in \mathrm{Reg}(\mathbb{H}^3)$ belongs to T_Δ if and only if its vertices v'_0, v'_1, v'_2, v'_3 span a simplex of the unique tiling of $\mathbb{H}^3$ by regular ideal straight tetrahedra containing the simplex spanned by v_0, v_1, v_2, v_3. We also denote by $\mathrm{Aut}(T_\Delta) < \mathrm{Isom}(\mathbb{H}^3)$ the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ leaving T_Δ invariant. It is easy to check that $\mathrm{Aut}(T_\Delta)$ is discrete.

Recall that two subgroups Γ_1, Γ_2 of $\mathrm{Isom}(\mathbb{H}^3)$ are *commensurable* if there exists $g \in \mathrm{Isom}(\mathbb{H}^3)$ such that $(g\Gamma_1 g^{-1}) \cap \Gamma_2$ has finite index both in $g\Gamma_1 g^{-1}$ and in Γ_2. If Γ_1 and Γ_2 are discrete and torsion-free, this is equivalent to requiring that the hyperbolic manifolds $\Gamma_1 \backslash \mathbb{H}^3$ and $\Gamma_2 \backslash \mathbb{H}^3$ admit a common finite-sheeted Riemannian covering.

Lemma 3.2. *For every $\Delta \in \mathrm{Reg}(\mathbb{H}^3)$, the group* $\mathrm{Aut}(T_\Delta)$ *is commensurable with R. Both these groups are commensurable with the fundamental group of the Gieseking manifold.*

Proof. If $g \in \mathrm{Isom}(\mathbb{H}^3)$ is such that $g(\Delta_0) = \Delta$, then $g \cdot \mathrm{Aut}(T_{\Delta_0}) \cdot g^{-1} = \mathrm{Aut}(T_\Delta)$. Moreover, $R < \mathrm{Aut}(T_{\Delta_0})$ and the index of R is finite, since $\mathrm{Aut}(T_{\Delta_0})$ is discrete and R has finite covolume. This implies that $\mathrm{Aut}(T_\Delta)$ is commensurable with R.

Up to conjugacy, we may suppose that Δ_0 is a fundamental domain for the action of the fundamental group G of the Gieseking manifold on $\mathbb{H}^3$. Then $G < \mathrm{Aut}(T_{\Delta_0})$ and, as above, the index of G in $\mathrm{Aut}(T_{\Delta_0})$ is finite because G has finite covolume (in fact, this index is equal to $4! = 24$). This concludes the proof. □

Theorem 3.3. *Suppose there is a nonequidistributed efficient cycle* $\mu \in \mathcal{M}(\mathrm{Reg}(N))$. *Then* N *is commensurable with the Gieseking manifold.*

Proof. Let $N = \Gamma\backslash\mathbb{H}^3$. As proved in [Kuessner 2003, Section 4], the efficient cycle μ decomposes into the sum of a multiple of μ_{eq} and a measure $\mu' \in \mathrm{Reg}(N)$ which is supported on tetrahedra whose lifts in $\mathbb{H}^3$ have all their vertices in parabolic fixed points of Γ. Since μ is nonequidistributed, we may assume that $\mu' \neq 0$.

Since parabolic fixed points of Γ are in countable number, the support of μ' is also countable, and this implies in turn that μ' is purely atomic. Moreover, since $\mu' = \mu - k\mu_{\mathrm{eq}}$ for some $k \in \mathbb{R}$, the measure μ' also satisfies $r_*(\mu') = \mu'$ for every $r \in R^+$ and $r_*(\mu') = -\mu'$ for every $r \in R^-$. Let us set

$$\Omega = \big\{[\Delta] \in \mathrm{Reg}(N) \mid \mu'\big(\{[\Delta]\}\big) \neq 0\big\} \neq \varnothing.$$

Due to the R-equivariance of μ', the countable set Ω is R-invariant. Let us fix a nonempty R-orbit $\overline{\Omega} \subseteq \Omega$. The absolute value of the measure μ' is constant on elements of $\overline{\Omega}$. Since μ' has finite total variation, this implies that

$$\overline{\Omega} = \{[\Delta_1], \ldots, [\Delta_k]\}$$

is finite. For every $i = 1, \ldots, k$, let Δ_i be a lift of $[\Delta_n] \in \mathrm{Reg}(N)$ in $\mathrm{Reg}(\mathbb{H}^3)$. By looking at the definition of the actions of R and of Γ on $\mathrm{Reg}(\mathbb{H}^3)$, we deduce that the R-orbit of Δ_1 in $\mathrm{Reg}(\mathbb{H}^3)$ is contained in

$$\Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_k.$$

Observe now that the R-orbit of Δ_1 in $\mathrm{Reg}(\mathbb{H}^3)$ realizes a tiling of $\mathbb{H}^3$ by regular ideal tetrahedra. Therefore, up to adding to the Δ_j, $j = 1, \ldots, k$, all the straight simplices obtained by permuting their vertices (which are still in finite number), we may assume that

$$T_{\Delta_1} \subseteq \Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_k.$$

Moreover, up to reordering the Δ_j, we may assume that $T_{\Delta_1} \cap (\Gamma \cdot \Delta_j) \neq \varnothing$ for every $j = 1, \ldots, k'$ and $T_{\Delta_1} \cap (\Gamma \cdot \Delta_j) = \varnothing$ for every $j = k' + 1, \ldots, k$, for some $k' \leq k$. By construction, we still have

$$T_{\Delta_1} \subseteq \Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_{k'}. \tag{11}$$

We are now going to show that the group $\Gamma\cap\mathrm{Aut}(T_{\Delta_1})$ has finite index in $\mathrm{Aut}(T_{\Delta_1})$. To this aim we will just exploit (11). For $j = 1, \ldots, k'$, up to replacing Δ_j with another simplex in its Γ-orbit, we suppose that $\Delta_j \in T_{\Delta_1}$. Observe now that T_{Δ_1} is the orbit of Δ_1 under the action of $\mathrm{Aut}(T_{\Delta_1})$; hence, thanks to (11), for every $j = 1, \ldots, k'$ there exists $g_j \in \mathrm{Aut}(T_{\Delta_1})$ such that $g_j \cdot \Delta_j = \Delta_1$.

Let us fix $g \in \mathrm{Aut}(T_{\Delta_1})$. Since $g \cdot \Delta_1 \in T_{\Delta_1} \subseteq \Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_{k'}$, there exist $\gamma \in \Gamma$, $j \in \{1, \ldots, k'\}$ such that $g \cdot \Delta_1 = \gamma \cdot \Delta_j$, whence $(\gamma^{-1}g) \cdot \Delta_1 = \Delta_j$ and $(g_j\gamma^{-1}g) \cdot \Delta_1 = g_j \cdot \Delta_j = \Delta_1$. However, since the unique hyperbolic isometry which fixes the vertices of a regular ideal tetrahedron is the identity, the stabilizer of Δ_1 in $\mathrm{Aut}(T_{\Delta_1})$ is trivial; hence $g_j\gamma^{-1}g = 1$, i.e., $g = \gamma g_j^{-1}$ (and, in particular, $\gamma \in \Gamma \cap \mathrm{Aut}(T_{\Delta_1})$). We have thus shown that the set $\{g_1, \ldots, g_{k'}\}$ contains a set of representatives for the set of right lateral classes of $\Gamma \cap \mathrm{Aut}(T_{\Delta_1})$ in $\mathrm{Aut}(T_{\Delta_1})$.

Since Γ is discrete and $\Gamma \cap \mathrm{Aut}(T_{\Delta_1})$ has finite covolume (being a finite index subgroup of $\mathrm{Aut}(T_{\Delta_1})$), the group $\Gamma \cap \mathrm{Aut}(T_{\Delta_1})$ has finite index also in Γ. Thus Γ is commensurable with $\mathrm{Aut}(T_{\Delta_1})$; hence N is commensurable with the Gieseking manifold by Lemma 3.2. □

Putting together Theorem 3.3 and Corollary 2.8 we obtain the following:

Corollary 3.4. *Let N be a complete finite-volume n-hyperbolic manifold*, $n \geq 3$, *and suppose that N is not commensurable with the Gieseking manifold. Then N admits a unique efficient cycle*, *which is given by the measure*

$$\frac{1}{2v_n} \cdot \mu_{\mathrm{eq}}.$$

4. Manifolds admitting nonequidistributed efficient cycles

We prove that manifolds that are commensurable with the Gieseking manifold admit nonequidistributed efficient cycles. We will first prove that this phenomenon occurs for manifolds admitting an ideal triangulation by regular ideal tetrahedra, and we will then deduce the general case from the fact that any manifold which is commensurable with the Gieseking manifold admits a finite covering with such a triangulation.

4.1. ***Triangulations and ideal triangulations.*** Let $\overline{N}$ be a compact 3-manifold with nonempty boundary made of tori. We recall the well-known notions of triangulation and ideal triangulation, widely used in 3-dimensional topology.

A *triangulation* of $\overline{N}$ is a realization of $\overline{N}$ via a simplicial face-pairing of finitely many tetrahedra. A triangulation of $\overline{N}$ naturally induces a triangulation of its boundary.

An *ideal triangulation* of $\overline{N}$ (or of N) is a realization of $N = \mathrm{int}(\overline{N})$ as a simplicial face-pairing of finitely many tetrahedra, with all their vertices removed. The removed vertices are called *ideal* and they correspond to the boundary components of $\overline{N}$; the link of every ideal vertex is a triangulation of the corresponding boundary component of $\overline{N}$.

We say as usual that $\overline{N}$ is *hyperbolic* if its interior has a finite-volume complete hyperbolic metric. If $\overline{N}$ is hyperbolic, every geometric decomposition of N into hyperbolic ideal tetrahedra is an example of ideal triangulation, that we call a *geometric ideal triangulation* of $\overline{N}$ (or of N). We still do not know whether every hyperbolic 3-manifold has a geometric ideal triangulation, but we know it does so virtually [Luo et al. 2008].

We are interested here in transforming a geometric ideal triangulation into a triangulation in an efficient way. One method called *inflation* was introduced by Jaco and Rubinstein [2014]. Here we introduce a similar method where we employ the dual viewpoint of simple spines, as Matveev [1990; 2003], in a similar fashion as in [Francaviglia et al. 2012, Section 5.4].

Consider a geometric ideal triangulation T of N. We lift it to a geometric ideal triangulation $\widetilde{T}$ of the universal cover $\mathbb{H}^3$. We choose some disjoint cusp sections in N; their preimage consists of infinitely many disjoint horoballs in $\mathbb{H}^3$, centered at the vertices of $\widetilde{T}$.

For $\varepsilon > 0$ sufficiently small, the ε-thick part N_ε of N is obtained by removing from N sufficiently deep cusp sections, and it is homeomorphic to $\overline{N}$. The ideal triangulation T of N restricts to a decomposition of N_ε into truncated tetrahedra. To obtain a triangulation for N_ε would now suffice to take its barycentric subdivision; however, this operation is not useful for us because it produces too many tetrahedra: we are looking for a triangulation for N_ε which contains the same number of tetrahedra as T, plus only a few more.

We explain our request more precisely. We say that a triangulation T' of N_ε is *adapted* to the geometric ideal triangulation T if there is an injective map i from the set of ideal tetrahedra of T to the set of tetrahedra of T' such that for every tetrahedron Δ of T, every lift of $i(\Delta)$ is a tetrahedron in $\mathbb{H}^3$ whose vertices lie in the boundary of the 4 removed horoballs whose centers are the vertices of a lift of Δ. (We do not require the lift of $i(\Delta)$ to be a straight tetrahedron, only a topological one.) In some sense we require Δ and $i(\Delta)$ to be close. Every tetrahedron of T' that is not in the image of i is called *residual*.

We will need the following lemma, which says that for any hyperbolic manifold N with a geometric ideal triangulation T it is possible to construct a tower of finite

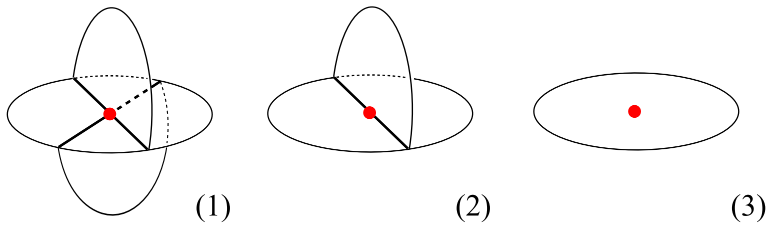

Figure 1. Neighborhoods of points in a simple polyhedron.

coverings, each equipped with an adapted triangulation T_i' whose residual tetrahedra grow sublinearly with respect to the degree of the cover.

Proposition 4.1. *Let N be a hyperbolic manifold equipped with a geometric ideal triangulation T. There is a tower of finite coverings $W_i \to \overline{N}$ of degree d_i such that the following holds: every W_i admits a triangulation T_i' adapted to the geometric ideal triangulation T_i obtained by lifting T, with r_i residual tetrahedra, such that*

$$\lim_{i\to\infty} \frac{r_i}{d_i} \to 0.$$

Subsections 4.2 and 4.3 are devoted to a proof of this proposition.

4.2. ***Construction of an adapted triangulation.*** We introduce an efficient method to transform a geometric ideal triangulation T of a hyperbolic manifold N into a triangulation T' that is adapted to T.

A compact 2-dimensional polyhedron X is *simple* if every point x of X has a star neighborhood PL-homeomorphic to one of the three models shown in Figure 1. Points of type (1) are called *vertices*. Points of type (2) and (3) form respectively some manifolds of dimension 1 and 2: their connected components are called respectively *edges* and *regions*. A simple polyhedron X is *special* if every edge is an open segment and every region is an open disc, so in particular it has a natural CW structure.

Let $\overline{N}$ be a compact 3-manifold with (possibly empty) boundary. A compact 2-dimensional subpolyhedron $X \subset N = \mathrm{int}(\overline{N})$ is a *spine* of $\overline{N}$ if $\overline{N} \setminus X$ consists of an open collar of $\partial\overline{N}$ and some (possibly none) open balls (the presence of some open balls is necessary when $\partial\overline{N} = \varnothing$).

Let $\overline{N}$ be a compact manifold with boundary made of tori. Suppose that N is hyperbolic and equipped with a geometric ideal triangulation T. We now describe a method to construct a triangulation T' for $\overline{N} \cong N_\varepsilon$ adapted to T.

First, we dualize the ideal triangulation T to get a special spine X of $\overline{N}$ with one vertex at the barycenter of each ideal tetrahedron as shown in Figure 2.

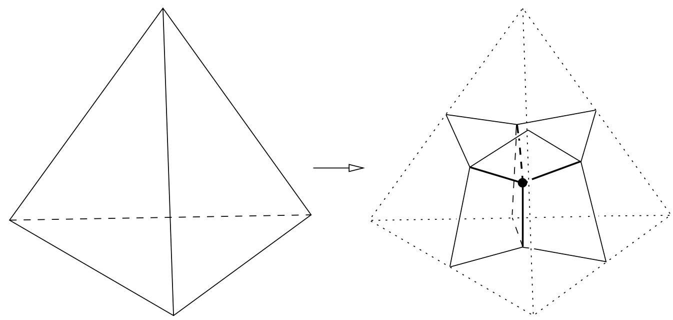

Figure 2. By dualizing an ideal triangulation we get a simple spine.

Second, we add some cells to X to obtain a new special polyhedron X', so that by dualizing X' back we will get our desired triangulation T'. We construct X' as follows. By construction $\overline{N} \setminus X$ consists of an open collar of $\partial \overline{N}$, that is a finite union of products $S \times (0, 1]$ where S is a torus and $S \times \{1\}$ is a boundary component of $\overline{N}$. Choose a θ-shaped graph $Y \subset S$ that is itself a spine of S, i.e., $S \setminus Y$ consists of an open disc. Add to X the polyhedron

$$Y \times (0, 1] \cup S \times \{1\}.$$

If we do this at each product $S \times (0, 1]$ in $\overline{N} \setminus X$, we obtain a 2-dimensional polyhedron $X' \subset \overline{N}$ that contains $\partial \overline{N}$. If Y is chosen generically, the polyhedron X' is special. The complement $\overline{N} \setminus X'$ consists of open balls, one for each boundary component of $\overline{N}$.

As we mentioned above, the triangulation T' for $\overline{N}$ is constructed by dualizing X' in the appropriate way. Every boundary torus S of $\overline{N}$ inherits from X' a cellularization with two vertices, three edges, and one disc (the cellularization depends on the chosen θ-shaped spine Y); this cellularization is dualized to a one-vertex triangulation for S. This triangulation extends from $\partial \overline{N}$ to $\overline{N}$ as follows: every disc, edge, and vertex of X' that is not adjacent to $\partial \overline{N}$ dualizes to an edge, a triangle, and a tetrahedron for T'.

The resulting triangulation T' has the smallest possible number of vertices: one for each boundary component. The tetrahedra of T are in natural 1-1 correspondence with the vertices of X. The tetrahedra of T' are in natural 1-1 correspondence with the vertices of X' that are not contained in $\partial \overline{N}$. Since every vertex of X is also a vertex of X' of this kind, we get a natural injection i from the set of tetrahedra of T into the set of tetrahedra of T'.

Lemma 4.2. *If T is a geometric ideal triangulation for $\overline{N}$, the triangulation T' is adapted to T.*

Proof. We fix some disjoint horocusp sections and truncate N along these, to obtain a smaller copy N_ε of $\overline{N}$. Their preimage in $\mathbb{H}^3$ consists of horospheres. When passing from X to X' we add the cusp sections ∂N_ε and some products $Y \times (0, 1]$. In $\mathbb{H}^3$ this corresponds to adding the horospheres and some products $\widetilde{Y} \times (0, 1]$. The resulting dual triangulation T' has all its vertices in the cusp sections, which lift to vertices in the horospheres. By construction for every ideal tetrahedron Δ in T the corresponding $i(\Delta)$ has its vertices in the same horospheres that are crossed by the edges of Δ. □

The residual tetrahedra correspond to the vertices of X' contained in the interior of $\overline{N}$ that were not themselves vertices of X, and that were created by attaching the products $Y \times (0, 1]$ along some generic map $Y \to X$. We now need to construct some tower of coverings where this kind of vertices grow sublinearly in number.

4.3. ***Characteristic coverings.*** We now build the tower of coverings for $\overline{N}$ needed in Proposition 4.1. We will use some results of Hamilton [Hamilton 2001] on coverings determined by characteristic subgroups. A similar construction was made in [Francaviglia et al. 2012, Section 5.3].

Recall that a *characteristic subgroup* of a group G is a subgroup $H < G$ which is invariant by any automorphism of G. For a natural number $x \in \mathbb{N}$, the *x-characteristic* subgroup of $\mathbb{Z} \times \mathbb{Z}$ is the subgroup $x(\mathbb{Z} \times \mathbb{Z})$ generated by $(x, 0)$ and $(0, x)$. It has index x^2 if $x > 0$ and ∞ if $x = 0$. The characteristic subgroups of $\mathbb{Z} \times \mathbb{Z}$ are precisely the x-characteristic subgroups with $x \in \mathbb{N}$. It is easy to prove that a subgroup of $\mathbb{Z} \times \mathbb{Z}$ of index x contains the x-characteristic subgroup.

A covering map $p : \widetilde{T} \to T$ of tori is *x-characteristic* if $p_*(\pi_1(\widetilde{T}))$ is the x-characteristic subgroup of $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$. A covering map $p : \widetilde{\overline{N}} \to \overline{N}$ of 3-manifolds bounded by tori is *x-characteristic* if the restriction of p to each boundary component of $\widetilde{\overline{N}}$ is x-characteristic.

Lemma 5 from [Hamilton 2001] implies the following.

Lemma 4.3 (E. Hamilton). *Let $\overline{N}$ be a hyperbolic compact, orientable 3-manifold with boundary tori. For every integer $i > 0$ there exist an integer $k > 0$ and a finite-index normal subgroup $K_i \lhd \pi_1(\overline{N})$ such that $K_i \cap \pi_1(T_j)$ is the characteristic subgroup of index $(ik)^2$ in $\pi_1(T_j)$, for each component T_j of $\partial\overline{N}$. Hence the covering $W_i \to \overline{N}$ corresponding to K_i is (ik)-characteristic.*

We can now prove Proposition 4.1. We restate it for the sake of clarity.

Proposition 4.1. *Let N be a hyperbolic manifold equipped with a geometric ideal triangulation T. There is a tower of finite coverings $W_i \to \overline{N}$ of degree d_i such that*

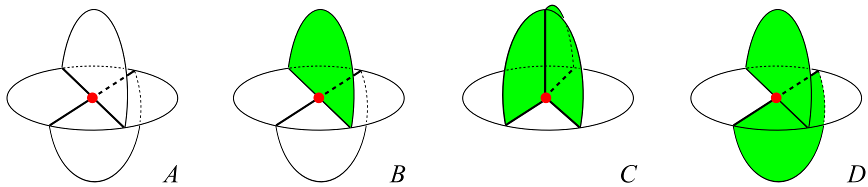

Figure 3. We color in green the regions of the inserted portions $Y \times (0, 1) \cup S \times \{1\}$. There are four types of vertices A, B, C, and D in the spine Q, according to the colors of the incident regions.

the following holds: every W_i admits a triangulation T_i' adapted to the geometric ideal triangulation T_i obtained by lifting T, with r_i residual tetrahedra, such that

$$\lim_{i\to\infty} \frac{r_i}{d_i} \to 0.$$

Proof. Let X be the spine dual to T. Following Section 4.2 we enlarge X to a special polyhedron X' by adding one piece

$$Y \times (0, 1] \cup S \times \{1\}$$

for each boundary torus S of $\overline{N}$, inside the corresponding collar $S \times (0, 1]$ in $\overline{N} \setminus X$. This operation depends on the choice of a generic θ-shaped spine $Y \subset S$.

The polyhedron X' has all the vertices of X, plus some additional ones that we now investigate carefully. The following discussion is similar to [Francaviglia et al. 2012, proof of Lemma 5.9]. Color in white the regions of X and in green the regions in the products $Y \times (0, 1] \cup S \times \{1\}$ that are attached to X. There are five types A, B, C, D, E of vertices in X' according to the colors of the incident regions: the vertices of type A, B, C, D are shown in Figure 3, while those of type E are those that lie in $\partial\overline{N}$ and that are incident to green regions only. The vertices of type A are precisely those of X. The vertices of type B, C, D are dual to the residual tetrahedra of T', and we want to control their number. Those of type E are not interesting here.

For every boundary torus S, the collar map $S \to X$ is a (possibly noninjective) immersion, and the cellularization of X pulls back to a cellularization of S, which is in fact dual to the triangulation link of the corresponding ideal vertex of T. The θ-shaped spine Y is generic, transverse to this cellularization as in Figure 4 (left). The four types of vertices A, B, C, D that may arise are shown in Figure 4 (right).

Let v_A, v_B, v_C, and v_D be the number of vertices of type A, B, C, and D in X'. The number of residual tetrahedra in T' is $v_B + v_C + v_D$.

We build the tower of coverings. By Lemma 4.3, for every integer $i \geq 1$, there are a $k_i > 0$ and an (ik_i)-characteristic covering $W_i \to \overline{N}$.

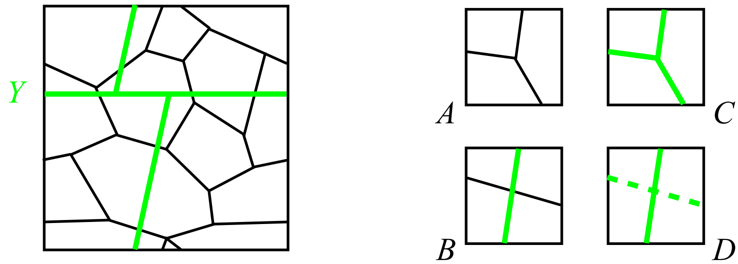

Figure 4. The cellularization of a boundary torus S induced by the collar map $S \to X$, and the θ-shaped spine Y of S colored in green (left). The four types of vertices A, B, C, D (right).

We now construct the triangulation T_i' adapted to the lifted geometric ideal triangulation T_i of W_i. The preimage of X is a spine X_i of W_i dual to T_i. To construct the adapted triangulation T_i', we choose an appropriate θ-shaped spine inside every boundary torus of W_i. We explain now how to make this choice.

Since the covering $W_i \to \overline{N}$ is (ik_i)-characteristic, every boundary torus $\widetilde{S}$ of W_i covers a torus S of $\overline{N}$ as an (ik_i)-characteristic covering. The case $ik_i = 3$ is shown in Figure 5. We have chosen in the previous paragraphs a spine Y for S; see Figure 4. As shown in Figure 5 (left), the preimage $\widetilde{Y}$ of Y in $\widetilde{S}$ is a spine of $\widetilde{S}$, whose complement in S consists of $(ik_i)^2$ discs. Figure 5 (right) shows that we can eliminate most vertices and edges of $\widetilde{Y}$ and obtain a simpler spine $\widetilde{Y}' \subset \widetilde{Y}$ of $\widetilde{S}$, whose complement in $\widetilde{S}$ consists of only one disc. This is the θ-shaped spine that we use on each boundary component $\widetilde{S}$ of W_i.

It remains to estimate the number r_i of residual tetrahedra in T_i'. Recall that

$$r_i = v_B^i + v_C^i + v_D^i,$$

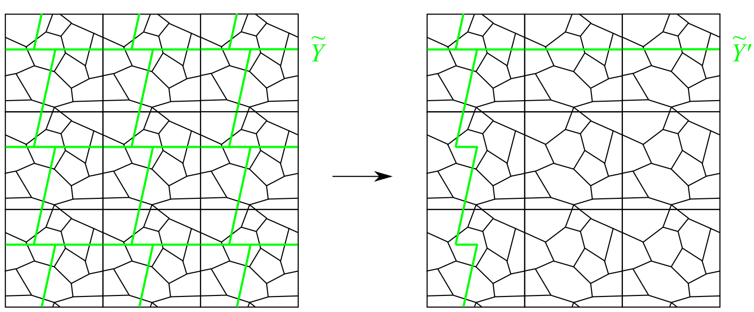

Figure 5. A 3-characteristic covering $\widetilde{S} \to S$. The spine Y of S lifts to the green spine $\widetilde{Y}$ shown in the left picture. We can eliminate most of its edges and still get a spine $\widetilde{Y}'$ of $\widetilde{S}$.

where v^i_B, v^i_C, v'_D are the numbers of vertices of type B, C, D in the dual polyhedron X'_i. The covering $W_i \to \overline{N}$ has degree

$$d_i = (ik_i)^2 h_i,$$

where h_i is the number of distinct boundary tori in ∂W_i that project to one boundary torus of $\overline{N}$. It is clear from Figure 5 that

$$v^i_B \le 2ik_i h_i v_B, \quad v^i_C \le 2v_C, \quad v^i_D \le 2ik_i h_i v_D.$$

Therefore

$$\frac{r_i}{d_i} = \frac{v^i_B + v^i_C + v^i_D}{(ik_i)^2 h_i} \le \frac{2ik_i h_i}{(ik_i)^2 h_i}(v_B + v_C + v_D) \to 0$$

as $i \to \infty$. The proof is complete. □

4.4. ***Efficient cycles from regular ideal triangulations.*** We are now ready to show that if a hyperbolic 3-manifold N admits a geometric ideal triangulation T by regular ideal tetrahedra, then it also admits a nonequidistributed efficient cycle. Indeed, let $\Delta_1, \Delta_2, \ldots, \Delta_h$ be the regular ideal tetrahedra of T, considered as subsets of N. For every $i = 1, \ldots, k$ we denote by $\widetilde{\sigma_i} \in \mathrm{Reg}^+(\mathbb{H}^3) \subseteq (\mathbb{H}^3)^4$ a (positively oriented) ordering $(\tilde{v}_0, \ldots, \tilde{v}_3)$ of the set of vertices of a lift of Δ_i to $\mathbb{H}^3$, and by σ_i the class of $\widetilde{\sigma_i}$ in $\mathrm{Reg}^+(N)$. Finally, we set

$$\mu_T = \Theta\left(\mathrm{alt}_3\left(\sum_{i=1}^{k} \sigma_i\right)\right). \tag{12}$$

(Strictly speaking, we defined the alternating operator only on straight simplices with vertices in $\mathbb{H}^3$, but of course it may be extended by the same formula also on ideal straight simplices).

The main result of this section is Theorem 7, which we recall here for the convenience of the reader:

Theorem 7. *Let M be a complete finite-volume 3-manifold admitting a decomposition T into regular ideal tetrahedra. Then μ_T is an efficient cycle for M.*

Proof. Let us fix some notation. As usual, for every sufficiently large $i \in \mathbb{N}$ we fix an identification $\overline{N} \cong N_{2^{-i}}$ between the natural compactification of N and the 2^{-i}-thick part of N. By Proposition 4.1, there is a tower of finite coverings $W_i \to N_{2^{-i}}$ of degree d_i such that the following holds: every W_i admits a triangulation T'_i adapted to the geometric ideal triangulation T_i obtained by lifting T, with r_i residual tetrahedra, such that

$$\lim_{i\to\infty} \frac{r_i}{d_i} \to 0. \tag{13}$$

For every sufficiently large $i \in \mathbb{N}$, we construct a relative fundamental cycle c_i for $N_{2^{-i}}$ as follows. The universal covering of W_i coincides with the universal covering of $N_{2^{-i}}$ (which is the complement of a collection of disjoint horoballs in $\mathbb{H}^3$); hence we may apply the straightening operator to any positively oriented parametrization of any simplex appearing in T_i'; after applying the alternating operator to the sum of the obtained straight tetrahedra, we get a relative fundamental cycle $\tilde{c}_i$ for W_i (more precisely, for the pair $(W_i', W_i' \setminus \mathrm{int}(W_i))$, where W_i' is the complete finite-volume hyperbolic manifold obtained from W_i by adding back the removed cusps). If $p_i : (W_i', W_i' \setminus \mathrm{int}(W_i)) \to (N, N \setminus \mathrm{int}(N_{2^{-i}}))$ is the covering projection, we then set

$$c_i = \frac{(p_i)_*(\tilde{c}_i)}{d_i}.$$

For simplicity, we will say that a simplex appearing in c_i is *nonresidual* if it is obtained (via $(p_i)_*$) from the alternation of the straightening of a nonresidual simplex of T_i'.

It is easy to check that c_i, $i \in \mathbb{N}$, is a minimizing sequence: if k is the number of the tetrahedra of T, then $\mathrm{Vol}(N) = k v_3$; hence $\|N\| = \mathrm{Vol}(N)/v_3 = k$. On the other hand, by construction the number of nonresidual simplices in $\tilde{c}_i$ is equal to $k d_i$ and the alternating operator is norm nonincreasing; hence

$$\limsup_{i\to+\infty} \|c_i\| = \limsup_{i\to+\infty} \frac{\|(p_i)_*(\tilde{c}_i)\|}{d_i} \le \limsup_{i\to+\infty} \frac{\|\tilde{c}_i\|}{d_i} = \limsup_{i\to+\infty} \frac{k d_i + r_i}{d_i} = k,$$

and this proves that the sequence c_i, $i \in \mathbb{N}$, is minimizing.

In order to conclude we are then left to show that

$$\lim_{i\to+\infty} \Theta(c_i) = \mu_T,$$

where $\Theta(c_i)$ is the measure associated to the cycle c_i (see (1)) and μ_T is the measure associated to the triangulation T (see (12)). Let $\Delta_0 \in \mathrm{Reg}^+(N)$ be a (positively oriented representative of a) tetrahedron of T, and let $\widetilde{\Delta}_0 \in \mathrm{Reg}(\mathbb{H}^3)$ be a lift of Δ_0 to $\mathbb{H}^3$ with vertices (v_0, v_1, v_2, v_3). There exist pairwise disjoint open neighborhoods $U_0, \dots, U_3$ of $v_0, \dots, v_3$ in $\overline{\mathbb{H}}^3$ such that the following conditions hold: every straight tetrahedron having its i-th vertex in U_i is nondegenerate and positively oriented, and the tetrahedron $\widetilde{\Delta}_0 = (v_0, v_1, v_2, v_3)$ is the unique lift of the tetrahedron of T whose vertices lie (in the correct order) in $U_0, \dots, U_3$. We set

$$\widetilde{\Omega} = \{(v_0, v_1, v_2, v_3) \in \overline{S}_3^*(\mathbb{H}^3) \mid v_i \in U_i \text{ for every } i = 0, 1, 2, 3\}$$

and we let Ω be the projection of $\widetilde{\Omega}$ in $\overline{S}_3^*(N)$. Of course, $\widetilde{\Omega}$ is an open neighborhood of $\widetilde{\Delta}_0$ in $\overline{S}_3^*(\mathbb{H}^3)$, and since the projection $\overline{S}_3^*(\mathbb{H}^3) \to \overline{S}_3^*(N)$ is open, the set Ω is an open neighborhood of Δ_0 in $\overline{S}_3^*(N)$.

Let now $f : \overline{S}^*_3(N)$ be any continuous compactly supported function such that $f(\Delta_0) = 1$. Recall that the vertices of the lifts of nonresidual tetrahedra of T'_i lie on the boundary of (deeper and deeper, as $i \to +\infty$) removed horoballs centered at the ideal vertices of lifts of tetrahedra of T. We say that a simplex σ' appearing in the cycle c_i is a *relative* of Δ_0 if it is nonresidual and it admits a lift to $\mathbb{H}^3$ with vertices on horospheres centered at the ideal vertices of a lift of Δ_0 (in the correct order).

Thanks to our definition of Ω, we can choose $i \in \mathbb{N}$ such that, if σ is a nonresidual simplex appearing in c_i, then σ belongs to Ω if and only if it is a relative of Δ_0. Let us now decompose c_i as

$$c_i = c_i^0 + c_i^{\mathrm{nr}} + c_i^{\mathrm{r}},$$

where c_i^0 is supported on relatives of Δ_0, c_i^{nr} is supported on nonresidual simplices which are not relatives of Δ_0, and c_i^{r} is supported on residual simplices. Since the simplices appearing in c_i^{nr} cannot belong to Ω for i sufficiently large, we have

$$\lim_{i\to+\infty} \int_\Omega f \, d\Theta(c_i^{\mathrm{nr}}) = 0. \tag{14}$$

Recall now that the alternating operator associates to every simplex the average of 24 singular simplices, and that positively oriented simplices come with the coefficient $+\frac{1}{24}$. Therefore, c_i^0 is a linear combination of d_i simplices, each of which comes with the real coefficient $1/(24d_i)$. In particular, we have $\|c_i^0\| = \frac{1}{24}$. In the very same way, if one starts with a negatively oriented Δ_0, still $\|c_i^0\| = \frac{1}{24}$ but the coefficients appearing in c_i^0 are all negative. As a consequence, since $f(\Delta_0) = 1$ and the simplices appearing in c_i^0 are converging to Δ_0 in $\overline{S}^*_3(N)$ (and f is continuous),

$$\lim_{i\to+\infty} \int_\Omega f \, d\Theta(c_i^0) = \|c_i^0\| = \frac{1}{24} \tag{15}$$

(while, if Δ_0 were negatively oriented, we would have $\lim_{i\to+\infty} \int_\Omega f \, d\Theta(c_i^0) = -\|c_i^0\| = -\frac{1}{24}$).

Finally from (13) we deduce that $\lim_{i\to+\infty} \|c_i^{\mathrm{r}}\| = 0$; hence

$$\lim_{i\to+\infty} \int_\Omega f \, d\Theta(c_i^r) = 0. \tag{16}$$

Putting together (14)–(16) we then obtain

$$\lim_{i\to+\infty} \int_\Omega f \, d\Theta(c_i) = \pm\frac{1}{24},$$

where the sign depends on whether Δ_0 is positively or negatively oriented.

Let us now denote by μ the limit of $\Theta(c_i)$ (which we may assume to exist, up to passing to a subsequence; in fact, with a little more effort we could easily prove that the sequence $\Theta(c_i)$, $i \in \mathbb{N}$, is itself convergent). Due to the definition of weak-* convergence, we have thus proved that there exists a neighborhood Ω of Δ_0 such that, for every compactly supported $f : \overline{S}_3^*(N) \to \mathbb{R}$ with $f(\Delta_0) = 1$, we have

$$\int_\Omega f \, d\mu = \pm\frac{1}{24}.$$

This implies that $\mu(\{\Delta_0\}) = \pm\frac{1}{24}$.

We have thus shown that $\mu(\{\Delta_0\}) = \pm\frac{1}{24}$ for every tetrahedron $\Delta_0 \in \mathrm{Reg}(N)$ whose geometric realization is a tetrahedron of the ideal triangulation T we started with. But every ideal tetrahedron of T gives rise to 24 tetrahedra in $\mathrm{Reg}(N)$, and the simplicial volume $\|N\|$ is equal to the number of tetrahedra of T, hence the contribution to μ of the atomic measures supported by tetrahedra whose geometric realizations are in T has total variation equal to $\|N\|$. Since we already know from Theorem 4 that $\|\mu\| = \|N\|$, this finally implies that $\mu = \mu_T$, as desired. □

We can now conclude the proof of Theorems 5 and 6 by showing that, if N is commensurable with the Gieseking manifold, then it admits nonequidistributed efficient cycles.

4.5. *Proof of Theorem 5.* We have proved in Section 3 that, if N is not commensurable with the Gieseking manifold, then every efficient cycle for N is equidistributed.

Vice versa, if N is commensurable with the Gieseking manifold, then there exists a degree-d covering $p : \widehat{N} \to N$, where $\widehat{N}$ admits a triangulation $\widehat{T}$ by regular ideal tetrahedra. Let $\hat{c}_i$, $i \in \mathbb{N}$, be the relative fundamental cycles for $\widehat{N}$ constructed in the proof of Theorem 7, and for every $i \in \mathbb{N}$ let $c_i = p_*(c_i)/d$. The covering map p induces a continuous map $\overline{S}_3^*(\widehat{N}) \to \overline{S}_3^*(N)$, hence a map $\mathcal{M}(\overline{S}_3^*(\widehat{N})) \to \mathcal{M}(\overline{S}_3^*(N))$. The very same proof of Theorem 7 shows that the limit $\mu = \lim_{i\to+\infty} \Theta(c_i) \in \mathcal{M}(\overline{S}_3^*(N))$ is an efficient cycle for N, and is equal to the image of $\mu_{\widehat{T}}$ via the map $\mathcal{M}(\overline{S}_3^*(\widehat{N})) \to \mathcal{M}(\overline{S}_3^*(N))$. But the image of a purely atomic measure via a continuous map is itself purely atomic. In particular, μ is a nonequidistributed efficient cycle for N, and this concludes the proof.

4.6. *Proof of Theorem 6.* We are only left to show that, if N is not commensurable with the Gieseking manifold and c_i, $i \in \mathbb{N}$ is any minimizing sequence for N, then

$$\lim_{i\to+\infty} \Theta(c_i) = \frac{1}{2v_n}\mu_{\mathrm{eq}}.$$

Of course, it is sufficient to show that every subsequence of c_i, $i \in \mathbb{N}$ admits a subsequence whose image via Θ converges to $\mu_{\mathrm{eq}}/(2v_n)$. However, the total variation of the measures $\Theta(c_i)$ is uniformly bounded; hence by compactness of the unit ball in $\mathcal{M}(\overline{S}^*_n(N)$ every subsequence of $\Theta(c_i)$ admits a subsequence converging to some measure $\mu \in \mathcal{M}(\overline{S}^*_n(N)$. By Corollary 3.4 we must have $\mu = \mu_{\mathrm{eq}}/(2v_n)$, and this concludes the proof.

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