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## PRODUCT MANIFOLDS AND THE CURVATURE OPERATOR OF THE SECOND KIND


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# PRODUCT MANIFOLDS AND THE CURVATURE OPERATOR OF THE SECOND KIND

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**We investigate the curvature operator of the second kind on product Riemannian manifolds and obtain some optimal rigidity results. For instance, we prove that the universal cover of an $n$-dimensional nonflat complete locally reducible Riemannian manifold with $\left(n+\frac{n-2}{n}\right)$-nonnegative (respectively, $\left(n+\frac{n-2}{n}\right)$-nonpositive) curvature operator of the second kind must be isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$ (respectively, $\mathbb{H}^{n-1} \times \mathbb{R}$) up to scaling. We also prove analogous optimal rigidity results for $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ and $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$, $n_1, n_2 \geq 2$, among product Riemannian manifolds, as well as for $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$ and $\mathbb{CH}^{m_1} \times \mathbb{CH}^{m_2}$, $m_1, m_2 \geq 1$, among product Kähler manifolds. The approach is pointwise and algebraic.**

## 1. Introduction

On a Riemannian manifold $(M^n, g)$, the *curvature operator of the second kind* refers to the symmetric bilinear form $\mathring{R} : S^2_0(T_pM) \times S^2_0(T_pM) \to \mathbb{R}$ defined by

$$\mathring{R}(\varphi, \psi) = R_{ijkl}\varphi_{il}\psi_{jk},$$

where $S^2_0(T_pM)$ is the space of traceless symmetric two-tensors on $T_pM$. The terminology is due to Nishikawa [1986]. Early works studying this notion of curvature operator include [Calabi and Vesentini 1960; Berger and Ebin 1969; Bourguignon and Karcher 1978; Koiso 1979a; 1979b; Ogiue and Tachibana 1979; Nishikawa 1986; Kashiwada 1993].

Recently, the curvature operator of the second kind has received much attention; see [Cao et al. 2023; Li 2022; 2023a; 2023b; 2024; Nienhaus et al. 2023a; 2023b; Fluck and Li 2024; Dai and Fu 2024; Dai et al. 2024]. In particular, the longstanding conjecture of Nishikawa [1986], which asserts that a closed Riemannian manifold with positive curvature operator of the second kind is diffeomorphic to a spherical

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space form and a closed Riemannian manifold with nonnegative curvature operator of the second kind is diffeomorphic to a Riemannian locally symmetric space, has been resolved by Cao, Gursky and Tran [Cao et al. 2023], Li [2024], and Nienhaus, Petersen, and Wink [Nienhaus et al. 2023a], under weaker assumptions but with stronger conclusions. More precisely, it is known now that:

**Theorem 1.1** [Cao et al. 2023; Li 2024; Nienhaus et al. 2023a]. *Let $(M^n, g)$ be a closed Riemannian manifold of dimension $n \geq 3$.*

(1) *If $(M^n, g)$ has three-positive curvature operator of the second kind, then $M$ is diffeomorphic to a spherical space form.*

(2) *If $(M^n, g)$ has three-nonnegative curvature operator of the second kind, then $M$ is either flat or diffeomorphic to a spherical space form.*

The key observation made by Cao, Gursky, and Tran in [2023] is that two-positive curvature operator of the second kind implies strictly PIC1 (i.e., $M \times \mathbb{R}$ has positive isotropic curvature). This is sufficient to solve the positive case of Nishikawa's conjecture, as one can appeal to a result of Brendle [2008] stating that the normalized Ricci flow on a compact manifold starting with a strictly PIC1 metric exists for all time and converges to a limit metric with constant positive sectional curvature. Shortly after, the author showed that strictly PIC1 is implied by three-positivity of the curvature operator of the second kind; thus getting an immediate improvement of the result in [Cao et al. 2023]. To deal with the nonnegative case, the author [2024] reduces the problem to the locally irreducible case by proving that a complete $n$-dimensional Riemannian manifold with $n$-nonnegative curvature operator of the second kind is either flat or locally irreducible (see also Theorem 1.6 below for an optimal improvement of this result). Finally, nonflat Kähler manifolds are ruled out using [Li 2024, Theorem 1.9] (see also [Li 2023a] for an optimal improvement of it) and compact irreducible symmetric spaces are ruled out by Nienhaus, Petersen, and Wink [2023a, Theorem A]. We refer the reader to [Li 2022] or [Li 2023a] for a detailed account of the notion of the curvature operator of the second kind, as well as some recent developments.

We aim to study the curvature operator of the second kind on product Riemannian manifolds and obtain some optimal rigidity results. We first recall the following definition. Let $N := \frac{(n-1)(n+2)}{2}$ denote the dimension of $S^2_0(T_pM)$. For $\alpha \in [1, N]$, we say $(M^n, g)$ has $\alpha$-positive (respectively, $\alpha$-nonnegative) curvature operator of the second kind if for any $p \in M$ and any orthonormal basis $\{\varphi_i\}_{i=1}^N$ of $S^2_0(T_pM)$,

$$\sum_{i=1}^{\lfloor\alpha\rfloor} \mathring{R}(\varphi_i, \varphi_i) + (\alpha - \lfloor\alpha\rfloor)\mathring{R}(\varphi_{\lfloor\alpha\rfloor+1}, \varphi_{\lfloor\alpha\rfloor+1}) > 0 \ \text{(respectively, } \geq 0). \tag{1-1}$$

Here and in the rest of this article, $\lfloor x \rfloor$ denotes the floor function defined by

$$\lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \le x\}.$$

When $\alpha = k$ is an integer, this reduces to the usual definition, which means the sum of the smallest $k$ eigenvalues of the matrix $\mathring{R}(\varphi_i, \varphi_j)$ is positive (respectively, nonnegative) for any orthonormal basis $\{\varphi_i\}_{i=1}^N$ of $S_0^2(T_pM)$. Similarly, $(M^n, g)$ is said to have $\alpha$-negative (respectively, $\alpha$-nonpositive) curvature operator of the second kind if the direction of the inequality (1-1) is reversed.

Our first main result is the following rigidity result for $\mathbb{S}^{n-1} \times \mathbb{R}$ and $\mathbb{H}^{n-1} \times \mathbb{R}$, where $\mathbb{S}^n$ and $\mathbb{H}^n$, $n \ge 2$, denote the $n$-dimensional sphere and hyperbolic space with constant sectional curvature 1 and $-1$, respectively.

**Theorem 1.2.** *Let $(M^n, g)$ be a nonflat complete locally reducible Riemannian manifold of dimension $n \ge 4$.*

(1) *If $M$ has $\left(n+\frac{n-2}{n}\right)$-nonnegative curvature operator of the second kind, then the universal cover of $M$ is, up to scaling, isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$.*

(2) *If $M$ has $\left(n+\frac{n-2}{n}\right)$-nonpositive curvature operator of the second kind, then the universal cover of $M$ is, up to scaling, isometric to $\mathbb{H}^{n-1} \times \mathbb{R}$.*

Closely related is the following holonomy restriction theorem in the spirit of [Nienhaus et al. 2023b].

**Theorem 1.3.** *Let $(M^n, g)$ be a (not necessarily complete) Riemannian manifold of dimension $n \ge 3$. Suppose that $(M, g)$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < n + \frac{n-2}{n}$. Then either $M$ is flat or the restricted holonomy of $M$ is $\mathsf{SO}(n)$.*

Theorems 1.2 and 1.3 improve previous results obtained in [Li 2024] and [Nienhaus et al. 2023b]. The author [2024, Theorem 1.8] proved that an $n$-dimensional complete Riemannian manifold with $n$-nonnegative curvature operator of the second kind is either flat or locally reducible. This result plays a significant role in resolving the nonnegative part of Nishikawa's conjecture in [Li 2024], as it allows one to reduce the problem to the locally irreducible setting. A slight modification of the proof yields the same conclusion under $n$-nonpositive curvature operator of the second kind. The method used in [Li 2024] is pointwise and algebraic. In [Nienhaus et al. 2023b], it is shown that if the curvature operator of the second kind of an $n$-dimensional Riemannian manifold, not necessarily complete, is $n$-nonnegative or $n$-nonpositive, then either the restricted holonomy of $M$ is $\mathsf{SO}(n)$ or $M$ is flat. This is a generalization of the author's result in [Li 2024] mentioned above. The approach of [Nienhaus et al. 2023b] is local. The key idea is that, unless the restricted holonomy is generic, there exists a parallel form, at least locally on the manifold. However, the Bochner technique with the curvature assumption implies that no such local parallel form exists unless the manifold is flat.

We would like to point out that the number $n+\frac{n-2}{n}$ in Theorems 1.2 and 1.3 is optimal in all dimensions, since $\mathbb{S}^{n-1}\times\mathbb{R}$ and $\mathbb{H}^{n-1}\times\mathbb{R}$ have $\left(n+\frac{n-2}{n}\right)$-nonnegative and $\left(n+\frac{n-2}{n}\right)$-nonpositive curvature operator of the second kind, respectively, and they both have restricted holonomy $\mathsf{SO}(n-1)$. In dimension four, $\mathbb{CP}^2$ and $\mathbb{CH}^2$ have $4\frac{1}{2}$-nonnegative and $4\frac{1}{2}$-nonpositive curvature operator of the second kind, respectively, and they both have restricted holonomy $\mathsf{U}(2)$.

Theorem 1.3 can also be viewed as supporting evidence to the author's conjecture in [Li 2022]: a closed $n$-dimensional Riemannian manifold with $\left(n+\frac{n-2}{n}\right)$-positive curvature operator of the second kind is diffeomorphic to a spherical space form.

As a generalization of Theorem 1.1, the author proved in [Li 2022] that a closed Riemannian manifold of dimension $n\geq 4$ with $4\frac{1}{2}$-positive curvature operator of the second kind is homeomorphic to a spherical space form. This is obtained by showing that $4\frac{1}{2}$-positive curvature operator of the second kind implies positive isotropic curvature and $\left(n+\frac{n-2}{n}\right)$-positive curvature operator of the second kind implies positive Ricci curvature, and then making use of the work of Micallef and Moore [1988]. A classification result of closed manifolds with $4\frac{1}{2}$-nonnegative curvature operator of the second kind was also obtained in [Li 2022, Theorem 1.4]. Using Theorem 1.2, together with [Li 2023a, Theorem 1.2] and [Nienhaus et al. 2023a, Theorem B], we get an improvement of [Li 2022, Theorem 1.4].

**Theorem 1.4.** *Let $(M^n, g)$ be a closed nonflat Riemannian manifold of dimension $n\geq 4$. Suppose that $M$ has $4\frac{1}{2}$-nonnegative curvature operator of the second kind. Then one of the following statements holds*:

(1) *$M$ is homeomorphic (diffeomorphic if either $n=4$ or $n\geq 12$) to a spherical space form.*

(2) *$n=4$ and $M$ is isometric to $\mathbb{CP}^2$ with Fubini–Study metric up to scaling.*

(3) *$n=4$ and the universal cover of $M$ is isometric to $\mathbb{S}^3\times\mathbb{R}$ up to scaling.*

Our second main result is the rigidity of $\mathbb{S}^{n_1}\times\mathbb{S}^{n_2}$ and $\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}$ among product Riemannian manifolds.

**Theorem 1.5.** *Let $(M_i^{n_i}, g_i)$ be a Riemannian manifold of dimension $n_i\geq 2$ for $i=1,2$, and let $(M^{n_1+n_2}, g)=(M_1^{n_1}\times M_2^{n_2}, g_1\oplus g_2)$. Set*

$$A_{n_1,n_2} := 1+n_1n_2+\frac{n_1(n_2-1)+n_2(n_1-1)}{n_1+n_2}. \tag{1-2}$$

*Then*:

(1) *If $M$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < A_{n_1,n_2}$, then $M$ is flat.*

(2) *If $M$ has $A_{n_1,n_2}$-nonnegative curvature operator of the second kind*, *then both $M_1$ and $M_2$ have constant sectional curvature $c \geq 0$.*

(3) *If $M$ has $A_{n_1,n_2}$-nonpositive curvature operator of the second kind*, *then both $M_1$ and $M_2$ have constant sectional curvature $c \leq 0$.*

*If $M$ is further assumed to be complete and nonflat*, *then the universal cover of $M$ is isometric to $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ in part* (2) *and $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ in part* (3), *up to scaling.*

The author [2024, Proposition 5.1] proved that an $n$-manifold with $(k(n-k)+1)$-nonnegative curvature operator of the second kind cannot split off a $k$-dimensional factor with $1 \leq k \leq \frac{n}{2}$, unless it is flat. The number $k(n-k)+1$ is only optimal for some special $n$ and $k$. Combining Theorems 1.2 and 1.5, we get the following generalization, which is optimal for any $n$ and $1 \leq k \leq \frac{n}{2}$.

**Theorem 1.6.** *An $n$-dimensional Riemannian manifold with $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some*

$$\alpha < k(n-k) + \frac{2k(n-k)}{n}$$

*cannot locally split off a $k$-dimensional factor with $1 \leq k \leq \frac{n}{2}$, unless it is flat.*

In another direction, the curvature operator of the second kind has been investigated for Kähler manifolds in [Bourguignon and Karcher 1978; Li 2023a; 2023b; 2024; Nienhaus et al. 2023b]. For instance, it was shown in [Li 2023a] that an $m$-dimensional Kähler manifold with $\frac{3}{2}(m^2-1)$-nonnegative (respectively, $\frac{3}{2}(m^2-1)$-nonpositive) curvature operator of the second kind has constant nonnegative (respectively, nonpositive) holomorphic sectional curvature, and a closed $m$-dimensional Kähler manifold with $\left(\frac{3m^3-m+2}{2m}\right)$-positive curvature operator of the second kind has positive orthogonal bisectional curvature; thus being biholomorphic to $\mathbb{CP}^m$. Here we prove the following rigidity result for $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$ and $\mathbb{CH}^{m_1} \times \mathbb{CH}^{m_2}$ (all equipped with their standard metrics) among product Kähler manifolds.

**Theorem 1.7.** *Let $(M_i^{m_i}, g_i)$ be a Kähler manifold of complex dimension $m_i \geq 1$ for $i = 1, 2$, and let $(M^{m_1+m_2}, g) = (M_1^{m_1} \times M_2^{m_2}, g_1 \oplus g_2)$. Set*

$$B_{m_1,m_2} := 4m_1m_2 + \frac{3}{2}(m_1^2 + m_2^2) + \frac{m_1m_2}{m_1+m_2}. \tag{1-3}$$

*Then*:

(1) *If $M$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < B_{m_1,m_2}$, then $M$ is flat.*

(2) *If $M$ has $B_{m_1,m_2}$-nonnegative curvature operator of the second kind, then both $M_1$ and $M_2$ have constant holomorphic sectional curvature $c \geq 0$.*

(3) *If $M$ has $B_{m_1,m_2}$-nonpositive curvature operator of the second kind, then both $M_1$ and $M_2$ have constant holomorphic sectional curvature $c \leq 0$.*

*If $M$ is further assumed to be complete and nonflat, then the universal cover of $M$ is isometric to $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$ in part* (2) *and $\mathbb{CH}^{m_1} \times \mathbb{CH}^{m_2}$ in part* (3), *up to scaling.*

Our investigation of the curvature operator of the second kind on product manifolds is motivated not only by the above mentioned optimal rigidity results but also by the fact that the spectrum of $\mathring{R}$ is known only for a few examples: space forms with constant sectional curvature, Kähler and quaternion-Kähler space forms [Bourguignon and Karcher 1978], $\mathbb{S}^2 \times \mathbb{S}^2$ [Cao et al. 2023], $\mathbb{S}^{n-1} \times \mathbb{R}$ [Li 2024], $\mathbb{S}^p \times \mathbb{S}^q$ [Nienhaus et al. 2023b]. We determine the spectrum of $\mathring{R}$ for a class of product manifolds by proving the following theorem.

**Theorem 1.8.** *Let $(M_i, g_i)$ be an $n_i$-dimensional Einstein manifold with $\mathrm{Ric}(g_i) = \rho_i g_i$ and $n_i \geq 1$ for $i = 1, 2$. Denote by $\mathring{R}_i$ the curvature operator of the second kind of $M_i$ for $i = 1, 2$, and $\mathring{R}$ the curvature operator of the second kind of the product manifold*

$$(M^{n_1+n_2}, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 \oplus g_2).$$

*Then the eigenvalues of $\mathring{R}$ are precisely those of $\mathring{R}_1$ and $\mathring{R}_2$, and $0$ with multiplicity $n_1 n_2$, and $-\frac{n_1\rho_2+n_2\rho_1}{n_1+n_2}$ with multiplicity one.*

Theorem 1.8 enables us to determine the spectrum of the curvature operator of the second kind on $(M_1, g_1) \times (M_2, g_2)$, with $(M_i, g_i)$ being either a space form with constant sectional curvature or a Kähler space form with constant holomorphic sectional curvature for $i = 1, 2$. Examples are listed at the end of Section 2. More generally, Theorem 1.8 can be applied repeatedly to calculate the spectrum of $\mathring{R}$ for product manifolds of the form $(M_1, g_1) \times \cdots \times (M_k, g_k)$, provided that each $(M_i, g_i)$ is Einstein and the eigenvalues of the curvature operator of the second kind are known on $M_i$.

Let's discuss the strategy of our proofs. The key idea to prove Theorems 1.2, 1.5 and 1.7 is to use the corresponding borderline example, such as $\mathbb{S}^{n-1} \times \mathbb{R}$, $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ or $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$, as a model space and apply $\mathring{R}$ to the eigenvectors of the curvature operator of the second kind on the model space. This idea has been successfully employed in [Li 2022] with $\mathbb{CP}^2$ and $\mathbb{S}^3 \times \mathbb{R}$ as model spaces, in [Li 2023b] with $\mathbb{S}^2 \times \mathbb{S}^2$ as the model space and in [Li 2023a] with $\mathbb{CP}^m$ and $\mathbb{CP}^{m-1} \times \mathbb{CP}^1$ as model spaces. With the right choice of model space, this strategy leads to optimal results as the inequalities are all achieved as equalities on the model space. Theorem 1.6 is essentially a consequence of Theorems 1.2 and 1.5. The proof of Theorem 1.3 uses Berger's classification of restricted holonomy groups, together with Propositions 3.1 and 4.1, and results in [Li 2023a] and [Nienhaus et al. 2023b]. The proof of Theorem 1.8 relies on the fact that when both factors are Einstein, we can choose an orthonormal basis of the space of traceless symmetric two-tensors that diagonalizes the curvature operator of the second kind on the product manifold.

At last, we emphasize that our approach is pointwise, and, therefore, many of our results are of a pointwise nature, and the completeness of the metric is not required. Another feature is that our proofs are purely algebraic and work equally well for nonpositivity conditions on $\mathring{R}$.

The article is organized as follows. In Section 2, we study the curvature operator of the second kind on product Riemannian manifolds and prove Theorem 1.8. We present the proofs of Theorems 1.2 and 1.4 in Section 3. The proofs of Theorems 1.5 and 1.6 are given in Section 4. In Section 5, we prove Theorem 1.3. Section 6 is devoted to the proof of Theorem 1.7.

## 2. Product manifolds

We study the curvature operator of the second kind on product Riemannian manifolds and prove Theorem 1.8.

Recall that for Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$, the product metric $g_1 \oplus g_2$ on $M_1 \times M_2$ is defined by

$$g(X_1 + X_2, Y_1 + Y_2) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$$

for $X_i, Y_i \in T_{p_i} M_i$ under the natural identification

$$T_{(p_1, p_2)}(M_1 \times M_2) = T_{p_1} M_1 \oplus T_{p_2} M_2.$$

Let $R$ denote the Riemann curvature tensor of $M = M_1 \times M_2$, and $R_1$ and $R_2$ denote the Riemann curvature tensor of $M_1$ and $M_2$, respectively. Then one can relate $R$, $R_1$ and $R_2$ by

$$R(X_1+X_2, Y_1+Y_2, Z_1+Z_2, W_1+W_2) = R_1(X_1, Y_1, Z_1, W_1) + R_2(X_2, Y_2, Z_2, W_2),$$

where $X_i, Y_i, Z_i, W_i \in TM_i$ for $i = 1, 2$. As the reader will see, the above equation, which is a consequence of the product structure, plays a significant role in this section.

From now on, let's focus on a single point in a product manifold and work in a purely algebraic way. For $i = 1, 2$, let $(V_i, g_i)$ be a Euclidean vector space of dimension $n_i \geq 1$. The product space $V = V_1 \times V_2$ will be naturally identified with $V_1 \oplus V_2$ via the isomorphism $(X_1, X_2) \to X_1 + X_2$ for $X_i \in V_i$. The product metric on $V$, denoted by $g = g_1 \oplus g_2$, is defined by

$$g(X_1 + X_2, Y_1 + Y_2) = g_1(X_1, Y_1) + g_2(X_2, Y_2) \tag{2-1}$$

for $X_i, Y_i \in V_i$.

Denote by $S^2_B(\Lambda^2 V)$ the space of algebraic curvature operators on $(V, g)$. That is to say, $R \in S^2_B(\Lambda^2 V)$ is a symmetric two-tensor on the space of two-forms $\Lambda^2 V$ on $V$ and $R$ also satisfies the first Bianchi identity. Given $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$, we define $R \in S^2_B(\Lambda^2 V)$ by

$$\begin{aligned} R(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2, W_1 + W_2) \\ = R_1(X_1, Y_1, Z_1, W_1) + R_2(X_2, Y_2, Z_2, W_2), \end{aligned} \tag{2-2}$$

for $X_i, Y_i, Z_i, W_i \in V_i$. Throughout this paper, we simply write

$$R = R_1 \oplus R_2$$

whenever $R$, $R_1$ and $R_2$ are related by (2-2). We denote by $\mathring{R}$, $\mathring{R}_1$ and $\mathring{R}_2$ the associated curvature operator of the second kind for $R = R_1 \oplus R_2$, $R_1$ and $R_2$, respectively.

The key result of this section is the following proposition.

**Proposition 2.1.** *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \geq 1$ and let $R = R_1 \oplus R_2$. If $\mathrm{Ric}(R_i) = \rho_i g_i$ for $i = 1, 2$, then the eigenvalues of $\mathring{R}$ are precisely those of $\mathring{R}_1$ and $\mathring{R}_2$, together with $0$ with multiplicity $n_1 n_2$ and $-\frac{n_2\rho_1 + n_1\rho_2}{n_1 + n_2}$ with multiplicity one.*

In the rest of this section, $\mathring{R}$ acts on the space of symmetric two-tensors $S^2(V)$ via

$$\mathring{R}(\varphi)_{ij} = \sum_{k,l=1}^{n} R_{iklj}\varphi_{kl}.$$

Note that the curvature operator of the second kind (defined as a symmetric bilinear form in the Introduction) is equivalent to the symmetric bilinear form associated with the self-adjoint operator $\pi \circ \mathring{R} : S^2_0(V) \to S^2_0(V)$, where $\pi : S^2(V) \to S^2_0(V)$ is the projection map. This can be seen as

$$\mathring{R}(\varphi, \psi) = \langle \mathring{R}(\varphi), \psi \rangle = \langle (\pi \circ \mathring{R})(\varphi), \psi \rangle = (\pi \circ \mathring{R})(\varphi, \psi)$$

for $\varphi, \psi \in S^2_0(V)$. Thus, the spectrum of the curvature operator of the second kind $\mathring{R}$ (as a bilinear form) is the same as the spectrum of the self-adjoint operator $\pi \circ \mathring{R}$.

We will present the proof of Proposition 2.1 after we establish the following three lemmas. First of all, standard calculations using (2-2) show that zero is an eigenvalue of $\mathring{R}$ with multiplicity (at least) $n_1 n_2$.

**Lemma 2.2.** *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \geq 1$ and let $R = R_1 \oplus R_2$. Let $E$ be the subspace of $S^2_0(V_1 \times V_2)$ given by*

$$E = \operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\},$$

*where $u \odot v = u \otimes v + v \otimes u$ is the symmetric product. Then $E$ lies in the kernel of $\mathring{R}$. In particular, $0$ is an eigenvalue of $\mathring{R}$ with multiplicity (at least) $n_1 n_2$.*

*Proof.* This is observed in [Nienhaus et al. 2023b, Lemma 2.1]. For the convenience of the reader, we give a detailed proof below. We start by constructing an orthonormal basis of $E$. Let $\{e_i\}_{i=1}^{n_1}$ be an orthonormal basis of $V_1$ and $\{e_i\}_{i=n_1+1}^{n_1+n_2}$ be an orthonormal basis of $V_2$. Then $\{e_i\}_{i=1}^{n_1+n_2}$ is an orthonormal basis of $V = V_1 \times V_2$. Define

$$\xi_{pq} = \frac{1}{\sqrt{2}} e_p \odot e_q,$$

for $1 \leq p \leq n_1$ and $n_1 + 1 \leq q \leq n_1 + n_2$. Then one can verify that the $\xi_{pq}$'s are traceless symmetric two-tensors on $V_1 \times V_2$ and they form an orthonormal basis of $E$. In particular, $\dim(E) = n_1 n_2$.

To prove that $E$ lies in the kernel of $\mathring{R}$, it suffices to show that $\mathring{R}(\xi_{pq}) = 0$. We first observe that (2-2) implies that

$$R(e_i, e_j, e_k, e_l) = \begin{cases} R_1(e_i, e_j, e_k, e_l), & i, j, k, l \in \{1, \ldots, n_1\}, \\ R_2(e_i, e_j, e_k, e_l), & i, j, k, l \in \{n_1+1, \ldots, n_1+n_2\}, \\ 0, & \text{otherwise}. \end{cases} \tag{2-3}$$

We then compute, using $(e_p \odot e_q)(e_j, e_k) = (\delta_{pj}\delta_{qk} + \delta_{qj}\delta_{pk})$, that

$$\begin{aligned} \mathring{R}(\xi_{pq})(e_i, e_l) &= \sum_{j,k=1}^{n} R(e_i, e_j, e_k, e_l)\xi_{pq}(e_j, e_k) \\ &= \frac{1}{\sqrt{2}} \sum_{j,k=1}^{n} R(e_i, e_j, e_k, e_l)(\delta_{pj}\delta_{qk} + \delta_{qj}\delta_{pk}) \\ &= \frac{1}{\sqrt{2}} \sum_{j,k=1}^{n_1} (R(e_i, e_p, e_q, e_l) + R(e_i, e_q, e_p, e_l)) \\ &= 0, \end{aligned}$$

where the last step is because of (2-3) and the fact that $1 \leq p \leq n_1$ and $n_1 + 1 \leq q \leq n_1 + n_2$. Thus we have proved that $0$ is an eigenvalue of $\mathring{R}$ with multiplicity (at least) $n_1 n_2$. □

Next, we show that the eigenvalues of $R_1$ and $R_2$ are also eigenvalues of $R = R_1 \oplus R_2$, provided that both $R_1$ and $R_2$ are Einstein.

**Lemma 2.3.** *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \geq 1$ and let $R = R_1 \oplus R_2$. If $R_1$ (respectively, $R_2$) is Einstein, then the eigenvalues of $\mathring{R}_1$ (respectively, $\mathring{R}_2$) are also eigenvalues of $\mathring{R}$.*

*Proof.* It suffices to prove the statement for $R_1$. Since $R_1$ is Einstein, we have that $\mathring{R}_1 : S^2_0(V_1) \to S^2_0(V_1)$ is a self-adjoint operator. We can then choose an orthonormal basis $\{\varphi_p\}_{p=1}^{N_1}$ of $S^2_0(V_1)$ such that

$$\mathring{R}_1(\varphi_p) = \lambda_p \varphi_p,$$

where $N_1 = \frac{(n_1-1)(n_1+2)}{2}$ is the dimension of $S^2_0(V_1)$. We may also view the $\varphi_p$'s as elements in $S^2_0(V_1 \times V_2)$ via zero extension, namely,

$$\varphi_p(X_1 + X_2, Y_1 + Y_2) = \varphi_p(X_1, Y_1),$$

for $X_i, Y_i \in V_i$. Then we have

$$\varphi_p(e_j, e_k) = \begin{cases} \varphi_p(e_j, e_k), & j, k \in \{1, \dots, n_1\}, \\ 0, & \text{otherwise}, \end{cases} \tag{2-4}$$

where $\{e_i\}_{i=1}^{n_1+n_2}$ is the same basis of $V$ in Lemma 2.2.

Next, we calculate using (2-4) that, for $1 \leq i, l \leq n_1$,

$$\begin{aligned} \mathring{R}(\varphi_p)(e_i, e_l) &= \sum_{j,k=1}^{n_1+n_2} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \sum_{j,k=1}^{n_1} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \sum_{j,k=1}^{n_1} R_1(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \lambda_p \varphi_p(e_i, e_l), \end{aligned}$$

and, for $n_1 + 1 \leq i, l \leq n_1 + n_2$,

$$\begin{aligned} \mathring{R}(\varphi_p)(e_i, e_l) &= \sum_{j,k=1}^{n_1+n_2} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \sum_{j,k=1}^{n_1} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \lambda_p \varphi_p(e_i, e_l) \\ &= 0. \end{aligned}$$

Therefore, we have proved $\mathring{R}(\varphi_p) = \lambda_p \varphi_p$ for $1 \le p \le N_1$. Hence the eigenvalues of $\mathring{R}_1$ are also eigenvalues of $\mathring{R}$ with the same eigenvectors. □

Finally, we prove:

**Lemma 2.4.** *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \ge 1$ and let $R = R_1 \oplus R_2$. If $\operatorname{Ric}(R_i) = \rho_i g_i$ for $i = 1, 2$, then $-\frac{n_2\rho_1 + n_1\rho_2}{n_1+n_2}$ is an eigenvalue of $\mathring{R}$ with eigenvector $n_2 g_1 - n_1 g_2$.*

*Proof.* As in the proof of Lemma 2.3, we may also view $g_1$ and $g_2$ as elements in $S^2(V_1 \times V_2)$ via zero extension. Clearly, $\operatorname{tr}(n_2 g_1 - n_1 g_2) = n_2 n_1 - n_1 n_2 = 0$. So we have $n_2 g_1 - n_1 g_2 \in S^2_0(V_1 \times V_2)$.

We then compute that

$$\begin{aligned} \mathring{R}(n_2 g_1 - n_1 g_2) &= n_2 \mathring{R}(g_1) - n_1 \mathring{R}(g_2) \\ &= n_2 \mathring{R}_1(g_1) - n_1 \mathring{R}_2(g_2) \\ &= -n_2 \operatorname{Ric}(R_1) + n_1 \operatorname{Ric}(R_2) \\ &= -n_2 \rho_1 g_1 + n_1 \rho_2 g_2, \end{aligned}$$

where we have used $\mathring{R}_i(g_i) = -\operatorname{Ric}(R_i) = -\rho_i g_i$ for $i = 1, 2$.

Using

$$\operatorname{tr}(-n_2\rho_1 g_1 + n_1 \rho_2 g_2) = -n_1 n_2(\rho_1 - \rho_2)$$

and $\mathring{R}(g_i) = -\rho_i g_1$ for $i = 1, 2$, we then obtain that

$$\begin{aligned} (\pi \circ \mathring{R})(n_2 g_1 - n_1 g_2) &= \pi(n_2 \mathring{R}(g_1) - n_1 \mathring{R}(g_2)) \\ &= \pi(n_2 \rho_1 g_1 + n_1 \rho_2 g_2) \\ &= -n_2\rho_1 g_1 + n_1 \rho_2 g_2 - \frac{-n_1 n_2(\rho_1 - \rho_2)}{n_1 + n_2}(g_1 + g_2) \\ &= -n_2 g_1 \left(\rho_1 - \frac{n_1(\rho_1 - \rho_2)}{n_1 + n_2}\right) + n_1 g_2 \left(\rho_2 + \frac{n_2(\rho_1 - \rho_2)}{n_1 + n_2}\right) \\ &= -\left(\frac{n_1\rho_2 + n_2\rho_1}{n_1 + n_2}\right)(n_2 g_1 - n_1 g_2). \end{aligned}$$

Thus, we see that $-\frac{n_1\rho_2 + n_2\rho_1}{n_1+n_2}$ is an eigenvalue of $\mathring{R}$ with eigenvector $n_2 g_1 - n_1 g_2$. The proof is now complete. □

*Proof of Proposition 2.1.* Let $\{e_i\}_{i=1}^{n_1+n_2}$ be an orthonormal basis of $V$, where $e_1, \dots, e_{n_1} \in V_1$ and $e_{n_1+1}, \dots, e_{n_1+n_2} \in V_2$. Let $\{\varphi_p\}_{p=1}^{N_1}$ be an orthonormal basis of $S^2_0(V_1)$ such that $\mathring{R}_1(\varphi_p) = \lambda_p \varphi_p$ and $\{\psi_q\}_{q=1}^{N_2}$ be an orthonormal basis of $S^2_0(V_2)$

such that $\mathring{R}_2(\psi_q) = \mu_q \psi_q$, where the dimension of $S^2_0(V_i)$ for $i = 1, 2$ is $N_i = \frac{(n_i-1)(n_i+2)}{2}$. We then define, on $V$, the traceless symmetric two-tensors

$$\xi_{pq} = \frac{1}{\sqrt{2}} e_p \odot e_q$$

for $1 \le p \le n_1$ and $n_1 + 1 \le q \le n_1 + n_2$, and

$$\zeta = \frac{1}{\sqrt{n_1 n_2 (n_1 + n_2)}} (n_2 g_1 - n_1 g_2).$$

Then one can verify, via straightforward computations, that

$$\{\varphi_p\}_{p=1}^{N_1} \cup \{\psi_q\}_{q=1}^{N_2} \cup \{\xi_{pq}\}_{1 \le p \le n_1, n_1+1 \le q \le n_1+n_2} \cup \{\zeta\}$$

forms an orthonormal basis of $S^2_0(V)$.

According to Lemma 2.2, 2.3 and 2.4, the above basis diagonalizes $\mathring{R}$ as

$$\left(\begin{array}{ccc|ccc|ccc|c}
\lambda_1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & \lambda_{N_1} & & & & & & & \\
\hline
& & & \mu_1 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & \mu_{N_2} & & & & \\
\hline
& & & & & & 0 & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 0 & \\
\hline
& & & & & & & & & -\frac{n_2\rho_1 + n_1\rho_2}{n_1+n_2}
\end{array}\right).$$

□

Theorem 1.8 now follows immediately from Proposition 2.1, since on a product manifold the product metric satisfies (2-1) and the Riemann curvature tensor satisfies (2-2).

Since the spectrum of $\mathring{R}$ is known on space forms with constant sectional curvature and Kähler space forms with constant holomorphic sectional curvature, we can use Theorem 1.8 or Proposition 2.1 to determine the eigenvalues of the curvature operator of the second kind on their product.

In the rest of this section, we use the following notation:

- $\mathbb{S}^n(\kappa)$ and $\mathbb{H}^n(-\kappa)$, $n \ge 2$ and $\kappa > 0$, denote the $n$-dimensional simply connected space form with constant sectional curvature $\kappa$ and $-\kappa$, respectively.

- $\mathbb{CP}^m(\kappa)$ and $\mathbb{CH}^m(-\kappa)$, $m \geq 1$ and $\kappa > 0$, denote the (complex) $m$-dimensional simply connected Kähler space form with constant holomorphic sectional curvature $4\kappa$ and $-4\kappa$, respectively.

**Example 2.5.** $\mathring{R} = \kappa \, \mathrm{id}_{S^2_0}$ *on* $\mathbb{S}^n(\kappa)$. $\mathring{R} = -\kappa \, \mathrm{id}_{S^2_0}$ *on* $\mathbb{H}^n(-\kappa)$.

**Example 2.6.** *$\mathring{R}$ has two distinct eigenvalues on $\mathbb{CP}^m(\kappa)$: $-2\kappa$ with multiplicity $(m-1)(m+1)$ and $4\kappa$ with multiplicity $m(m+1)$. $\mathring{R}$ has two distinct eigenvalues on $\mathbb{CH}^m(-\kappa)$: $2\kappa$ with multiplicity $(m-1)(m+1)$ and $-4\kappa$ with multiplicity $m(m+1)$. See* [Bourguignon and Karcher 1978].

**Example 2.7.** *Let $M = \mathbb{S}^{n_1}(\kappa_1) \times \mathbb{S}^{n_2}(\kappa_2)$. Then the curvature operator of the second kind of $M$ has eigenvalues: $\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $\kappa_2$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, $0$ with multiplicity $n_1 n_2$ and $-\frac{n_1(n_2-1)\kappa_2+n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

**Example 2.8.** *Let $M = \mathbb{H}^{n_1}(-\kappa_1) \times \mathbb{H}^{n_2}(-\kappa_2)$. Then the curvature operator of the second kind of $M$ has eigenvalues: $-\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $-\kappa_2$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, $0$ with multiplicity $n_1 n_2$ and $\frac{n_1(n_2-1)\kappa_2+n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

**Example 2.9.** *Let $M = \mathbb{S}^{n_1}(\kappa_1) \times \mathbb{R}^{n_2}$. Then the curvature operator of the second kind of $M$ has eigenvalues: $\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $0$ with multiplicity $n_1 n_2 + \frac{(n_2-1)(n_2+2)}{2}$ and $-\frac{n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

**Example 2.10.** *Let $M = \mathbb{H}^{n_1}(-\kappa_1) \times \mathbb{R}^{n_2}$. Then the curvature operator of the second kind of $M$ has eigenvalues: $-\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $0$ with multiplicity $n_1 n_2 + \frac{(n_2-1)(n_2+2)}{2}$ and $\frac{n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

**Example 2.11.** *Let $M = \mathbb{S}^{n_1}(\kappa_1) \times \mathbb{H}^{n_2}(-\kappa_2)$. Then the curvature operator of the second kind of $M$ has eigenvalues: $\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $-\kappa_2$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, $0$ with multiplicity $n_1 n_2$ and $-\frac{n_1 n_2(\kappa_1-\kappa_2)+n_1\kappa_2-n_2\kappa_1}{n_1+n_2}$ with multiplicity one.*

**Example 2.12.** *Let $M = \mathbb{CP}^{m_1}(\kappa_1) \times \mathbb{CP}^{m_2}(\kappa_2)$. Then the curvature operator of the second kind of $M$ has eigenvalues: $-2\kappa_1$ with multiplicity $(m_1-1)(m_1+1)$, $-2\kappa_2$ with multiplicity $(m_2-1)(m_2+1)$, $4\kappa_1$ with multiplicity $m_1(m_1+1)$, $4\kappa_2$ with multiplicity $m_2(m_2+1)$, $0$ with multiplicity $4m_1m_2$, and $-\frac{2m_1(m_2+1)\kappa_2+2m_2(m_1+1)\kappa_1}{m_1+m_2}$ with multiplicity one.*

**Example 2.13.** *Let $M = \mathbb{CH}^{m_1}(-\kappa_1) \times \mathbb{CH}^{m_2}(-\kappa_2)$. Then the curvature operator of the second kind of $M$ has eigenvalues: $2\kappa_1$ with multiplicity $(m_1-1)(m_1+1)$, $2\kappa_2$ with multiplicity $(m_2-1)(m_2+1)$, $-4\kappa_1$ with multiplicity $m_1(m_1+1)$, $-4\kappa_2$ with* *multiplicity* $m_2(m_2+1)$, $0$ *with multiplicity* $4m_1m_2$, *and* $\frac{2m_1(m_2+1)\kappa_2+2m_2(m_1+1)\kappa_1}{m_1+m_2}$ *with multiplicity one.*

**Example 2.14.** *Let* $M=\mathbb{CP}^{m_1}(\kappa_1)\times\mathbb{C}^{m_2}$. *Then the curvature operator of the second kind of* $M$ *has eigenvalues*: $-2\kappa_1$ *with multiplicity* $(m_1-1)(m_1+1)$, $4\kappa_1$ *with multiplicity* $m_1(m_1+1)$, $0$ *with multiplicity* $4m_1m_2+(2m_2-1)(m_2+1)$, *and* $-\frac{2m_2(m_1+1)\kappa_1}{m_1+m_2}$ *with multiplicity one.*

**Example 2.15.** *Let* $M=\mathbb{CH}^{m_1}(-\kappa_1)\times\mathbb{C}^{m_2}$. *Then the curvature operator of the second kind of* $M$ *has eigenvalues*: $2\kappa_1$ *with multiplicity* $(m_1-1)(m_1+1)$, $-4\kappa_2$ *with multiplicity* $m_1(m_1+1)$, $0$ *with multiplicity* $4m_1m_2+(2m_2-1)(m_2+1)$, *and* $\frac{2m_2(m_1+1)\kappa_1}{m_1+m_2}$ *with multiplicity one.*

**Example 2.16.** *Let* $M=\mathbb{CP}^{m_1}(\kappa_1)\times\mathbb{CH}^{m_2}(-\kappa_2)$. *Then the curvature operator of the second kind of* $M$ *has eigenvalues*: $-2\kappa_1$ *with multiplicity* $(m_1-1)(m_1+1)$, $4\kappa_2$ *with multiplicity* $m_1(m_1+1)$, $2\kappa_2$ *with multiplicity* $(m_2-1)(m_2+1)$, $-4\kappa_2$ *with multiplicity* $m_2(m_2+1)$, $0$ *with multiplicity* $4m_1m_2$, *and* $-\frac{2m_1m_2(\kappa_1-\kappa_2)+2m_2\kappa_1-2m_1\kappa_2}{m_1+m_2}$ *with multiplicity one.*

In particular, we have the following observation, which will be needed later on.

**Proposition 2.17.** *For* $n_1,n_2\geq 2$, $m_1,m_2\geq 1$, $\kappa_1,\kappa_2>0$, *we have the following*:

(1) $\mathbb{S}^{n_1}(\kappa_1)\times\mathbb{S}^{n_2}(\kappa_2)$ *has* $A_{n_1,n_2}$*-nonnegative curvature operator of the second kind if and only if* $\kappa_1=\kappa_2>0$.

(2) $\mathbb{H}^{n_1}(-\kappa_1)\times\mathbb{H}^{n_2}(-\kappa_2)$ *has* $A_{n_1,n_2}$*-nonpositive curvature operator of the second kind if and only if* $\kappa_1=\kappa_2>0$.

(3) $\mathbb{CP}^{m_1}(\kappa_1)\times\mathbb{CP}^{m_2}(\kappa_2)$ *has* $B_{m_1,m_2}$*-nonnegative curvature operator of the second kind if and only if* $\kappa_1=\kappa_2>0$.

(4) $\mathbb{CH}^{m_1}(-\kappa_1)\times\mathbb{CH}^{m_2}(-\kappa_2)$ *has* $B_{m_1,m_2}$*-nonpositive curvature operator of the second kind if and only if* $\kappa_1=\kappa_2<0$.

## 3. Rigidity of cylinders

We prove Theorem 1.2. The key result of this section is the following proposition.

**Proposition 3.1.** *Let* $(V,g)$ *be a Euclidean vector space of dimension* $n-1$ *with* $n\geq 2$ *and let* $R_1\in S^2_B(\Lambda^2 V)$.

(1) *Suppose that* $R=R_1\oplus 0\in S^2_B(\Lambda^2(V\times\mathbb{R}))$ *has* $\left(n+\frac{n-2}{n}\right)$*-nonnegative curvature operator of the second kind. Then* $R_1$ *has constant nonnegative sectional curvature.*

(2) *Suppose that* $R=R_1\oplus 0\in S^2_B(\Lambda^2(V\times\mathbb{R}))$ *has* $\left(n+\frac{n-2}{n}\right)$*-nonpositive curvature operator of the second kind. Then* $R_1$ *has constant nonpositive sectional curvature.*

(3) *Suppose that* $R = R_1 \oplus 0 \in S^2_B(\Lambda^2(V \times \mathbb{R}))$ *has* $\alpha$*-nonnegative or* $\alpha$*-nonpositive curvature operator of the second kind for some* $\alpha < n + \frac{n-2}{n}$. *Then* $R$ *is flat.*

*Proof.* (1) Let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis of $V$ and let $e_n$ be a unit vector in $\mathbb{R}$. Then $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of $V \times \mathbb{R} \cong V \oplus \mathbb{R}$. Next, we define, on $V \oplus \mathbb{R}$, the symmetric two-tensors

$$\begin{aligned}\xi_i &= \frac{1}{\sqrt{2}} e_i \odot e_n \quad \text{for } 1 \le i \le n-1,\\ \varphi_{kl} &= \frac{1}{\sqrt{2}} e_k \odot e_l \quad \text{for } 1 \le k < l \le n-1,\\ \zeta &= \frac{1}{2\sqrt{n(n-1)}}\left(\sum_{p=1}^{n-1} e_p \odot e_p - (n-1) e_n \odot e_n\right).\end{aligned}$$

One easily verifies that $\{\xi_i\}_{i=1}^{n-1} \cup \{\varphi_{kl}\}_{1 \le k<l \le n-1} \cup \{\zeta\}$ forms an orthonormal subset of $S^2_0(\Lambda^2(V \oplus \mathbb{R}))$.

Since $R = R_1 \oplus 0$, we have by (2-2) that

$$R(e_i, e_j, e_k, e_l) = \begin{cases} R_1(e_i, e_j, e_k, e_l), & i, j, k, l \in \{1, \dots, n-1\},\\ 0, & \text{otherwise.}\end{cases} \tag{3-1}$$

In particular, we have $R_{njnj} = 0$ for $1 \le j \le n-1$.

Direct calculation using the identity

$$\mathring{R}(e_i \odot e_j, e_k \odot e_l) = 2(R_{iklj} + R_{ilkj})$$

shows that

$$\begin{aligned}\mathring{R}(\xi_i, \xi_i) &= 0 && \text{for } 1 \le i \le n-1,\\ \mathring{R}(\varphi_{kl}, \varphi_{kl}) &= (R_1)_{klkl} && \text{for } 1 \le k < l \le n-1,\\ \mathring{R}(\zeta, \zeta) &= -\frac{1}{n(n-1)} S_1,\end{aligned}$$

where $S_1$ is the scalar curvature of $R_1$. Note that $S_1 \ge 0$ since $S_1$ is also equal to the scalar curvature of $R$, which must be nonnegative since $R$ has $\left(n + \frac{n-2}{n}\right)$-nonnegative curvature operator of the second kind; see, e.g., [Li 2024, Proposition 4.1, part (1)].

Since $R$ has $\left(n + \frac{n-2}{n}\right)$-nonnegative curvature operator of the second kind, we get that, for any $1 \le k < l \le n-1$,

$$\begin{aligned}0 &\le \mathring{R}(\zeta, \zeta) + \sum_{i=1}^{n-1} \mathring{R}(\xi_i, \xi_i) + \frac{n-2}{n} \mathring{R}(\varphi_{kl}, \varphi_{kl})\\ &= -\frac{1}{n(n-1)} S_1 + \frac{n-2}{n}(R_1)_{klkl} = \frac{n-2}{n}\left((R_1)_{klkl} - \frac{S_1}{(n-1)(n-2)}\right).\end{aligned}$$

Summing over $1 \le k < l \le n-1$ yields

$$S_1 \le \sum_{1\le k<l\le n-1} (R_1)_{klkl}.$$

On the other hand,

$$S_1 = \sum_{1\le k<l\le n-1} (R_1)_{klkl}.$$

Therefore, we must have $(R_1)_{klkl} = \frac{S_1}{(n-1)(n-2)}$ for all $1 \le k < l \le n-1$. Since the orthonormal basis $\{e_1, \dots, e_{n-1}\}$ is arbitrary, we conclude that $R_1$ has constant nonnegative sectional curvature.

(2) Apply (1) to $-R$.

(3) By (1) and (2), we have $R = cI_{n-1} \oplus 0$ for some $c \in \mathbb{R}$, where $I_{n-1}$ is the Riemann curvature tensor of $\mathbb{S}^{n-1}$. However, $R = cI_{n-1} \oplus 0$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < n + \frac{n-2}{n}$ if and only if $c = 0$. Therefore, $R$ is flat. □

We now present the proof of Theorem 1.2.

*Proof of Theorem 1.2.* (1) Recall that we say that $(M^n, g)$ is locally reducible if there exists a nontrivial subspace of $T_pM$ which is invariant under the action of the restricted holonomy group. By a theorem of de Rham, a complete Riemannian manifold is locally reducible if and only if its universal cover is isometric to the product of two Riemannian manifolds of lower dimension.

Denote by $(\widetilde{M}, \tilde{g})$ the universal cover of $M$ with the lifted metric $\tilde{g}$. Since $M$ is locally reducible, $(\widetilde{M}, \tilde{g})$ is isometric to a product of the form $(M_1^k, g_1)\times(M_2^{n-k}, g_2)$, where $1 \le k \le \frac{n}{2}$. Note that $k \ge 2$ implies

$$k(n-k) + 1 \ge n + \frac{n-2}{n},$$

so $\widetilde{M}$ must be flat if $k \ge 2$, according to [Li 2024, Proposition 5.1] (or its improvement Theorem 1.6). Thus we must have $k = 1$ and $\widetilde{M}$ is isometric to $N^{n-1} \times \mathbb{R}$. By part (1) of Proposition 3.1, $N$ has pointwise constant nonnegative sectional curvature. Since $n-1 \ge 3$, Schur's lemma implies that $N$ must have constant nonnegative sectional curvature. Therefore, $M$ is either flat or its universal cover is isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$ up to scaling.

(2) This is similar to the proof of (1), by noticing that [Li 2024, Proposition 5.1] is valid for the nonpositivity condition (alternatively, one can use Theorem 1.6). □

*Proof of Theorem 1.4.* Let $(M^n, g)$ be a closed nonflat Riemannian manifold of dimension $n \geq 4$ and suppose that $M$ has $4\frac{1}{2}$-nonnegative curvature operator of the second kind. It was shown in [Li 2022] that one of the following statements holds:

(a) $M$ is homeomorphic (diffeomorphic if $n = 4$ or $n \geq 12$) to a spherical space form.

(b) $n = 2m$ and the universal cover of $M$ is a Kähler manifold biholomorphic to $\mathbb{CP}^m$.

(c) $n = 4$ and the universal cover of $M$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{R}$.

(d) $n \geq 5$ and $M$ is isometric to a quotient of a compact irreducible symmetric space.

By Theorem 1.2 in [Li 2023a], the Kähler manifold in part (2) is either flat or isometric to $\mathbb{CP}^2$ with the Fubini–Study metric, up to scaling. In part (c), the manifold is reducible and we conclude using Theorem 1.2 that the universal cover of $M$ is isometric to $\mathbb{S}^3 \times \mathbb{R}$, up to scaling. Part (d) can be ruled out using [Nienhaus et al. 2023a, Theorem B], as the manifold is either flat or a homology sphere. □

## 4. Rigidity of product of spheres and hyperbolic spaces

We prove Theorem 1.5. The key result of this section is the following proposition. In this section, $I_n$, $n \geq 2$, denotes the Riemann curvature tensor of the $n$-sphere with constant sectional curvature 1.

**Proposition 4.1.** *For $i = 1, 2$, let $(V_i, g_i)$ be a Euclidean vector space of dimension $n_i$ with $n_i \geq 2$. Let $R_i \in S^2_B(\Lambda^2 V_i)$ and $R = R_1 \oplus R_2 \in S^2_B(\Lambda^2(V_1 \times V_2))$.*

(1) *Suppose that $R$ has $A_{n_1,n_2}$-nonnegative curvature operator of the second kind. Then $R = c(I_{n_1} \oplus I_{n_2})$ for some $c \geq 0$.*

(2) *Suppose that $R$ has $A_{n_1,n_2}$-nonpositive curvature operator of the second kind. Then $R = c(I_{n_1} \oplus I_{n_2})$ for some $c \leq 0$.*

(3) *Suppose that $R$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < A_{n_1,n_2}$. Then $R$ is flat.*

We need an elementary lemma, which can be found in [Li 2023a, Lemma 5.1].

**Lemma 4.2.** *Let $N$ be a positive integer and $A$ be a collection of $N$ real numbers. Denote by $a_i$ the $i$-th smallest number in $A$ for $1 \leq i \leq N$. Define a function $f(A, x)$ by*

$$f(A, x) = \sum_{i=1}^{\lfloor x \rfloor} a_i + (x - \lfloor x \rfloor) a_{\lfloor x \rfloor + 1},$$

*for* $x \in [1, N]$. *Then we have*

$$f(A, x) \le x\bar{a}, \tag{4-1}$$

*where* $\bar{a} := \frac{1}{N}\sum_{i=1}^{N} a_i$ *is the average of all numbers in* $A$. *The equality holds for some* $x \in [1, N)$ *if and only if* $a_i = \bar{a}$ *for all* $1 \le i \le N$.

*Proof of Proposition 4.1.* (1) Let $\{e_i\}_{i=1}^{n_1}$ be an orthonormal basis of $V_1$ and let $\{e_i\}_{i=n_1+1}^{n_1+n_2}$ be an orthonormal basis of $V_2$. Then $\{e_i\}_{i=1}^{n_1+n_2}$ is an orthonormal basis of $V_1 \times V_2 \cong V_1 \oplus V_2$.

We construct an orthonormal basis of $S_0^2(V_1 \times V_2)$ as follows. Choose an orthonormal basis $\{\varphi_i\}_{i=1}^{N_1}$ of $S_0^2(V_1)$ and an orthonormal basis $\{\psi_i\}_{i=1}^{N_2}$ of $S_0^2(V_2)$, where $N_i = \dim(S_0^2(V_i)) = \frac{(n_i-1)(n_i+2)}{2}$ for $i = 1, 2$. Note that $h \in S_0^2(V_1)$ can be identified with the element $\pi^* h$ in $S_0^2(V_1 \times V_2)$ via

$$(\pi^* h)(X_1 + X_2, Y_1 + Y_2) = h(X_1, X_2),$$

where $X_i, Y_i \in V_i$ for $i = 1, 2$. We shall simply write $\pi^* h$ as $h$. Similarly, $S_0^2(V_2)$ can be identified with a subspace of $S_0^2(V_1 \times V_2)$. Next, we define, on $V_1 \times V_2$, the symmetric two-tensors

$$\begin{aligned} \xi_{kl} &= \frac{1}{\sqrt{2}} e_k \odot e_l \quad \text{for } 1 \le k \le n_1,\ n_1 + 1 \le l \le n_1 + n_2, \\ \zeta &= \frac{1}{\sqrt{n_1 n_2 (n_1 + n_2)}}(n_2 g_1 - n_1 g_2). \end{aligned}$$

One verifies that

$$\{\varphi_i\}_{i=1}^{N_1} \cup \{\psi_i\}_{i=1}^{N_2} \cup \{\xi_{kl}\}_{1 \le k \le n_1, n_1+1 \le l \le n_1+n_2} \cup \{\zeta\}$$

forms an orthonormal basis of $S_0^2(V_1 \times V_2)$. This corresponds to the orthogonal decomposition

$$S_0^2(V_1 \times V_2) = S_0^2(V_1) \oplus S_0^2(V_2) \oplus \operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\} \oplus \mathbb{R}\zeta.$$

The next step is to calculate some diagonal elements of the matrix representing $\mathring{R}$ with respect to the above basis. Since $R = R_1 \oplus R_2$, we have by (2-2) that

$$R(e_i, e_j, e_k, e_l) = \begin{cases} R_1(e_i, e_j, e_k, e_l), & i, j, k, l \in \{1, \dots, n_1\}, \\ R_2(e_i, e_j, e_k, e_l), & i, j, k, l \in \{n_1 + 1, \dots, n_1 + n_2\}, \\ 0, & \text{otherwise}. \end{cases} \tag{4-2}$$

In particular, we have $R_{klkl} = 0$ if $1 \le k \le n_1$ and $n_1 \le l \le n_1 + n_2$. Using the identity

$$\mathring{R}(e_i \odot e_j, e_k \odot e_l) = 2(R_{iklj} + R_{ilkj}),$$

we get

$$\sum_{\substack{1\le k\le n_1\\ n_1+1\le l\le n_1+n_2}} \mathring{R}(\xi_{kl},\xi_{kl}) = \sum_{\substack{1\le k\le n_1\\ n_1+1\le l\le n_1+n_2}} R_{klkl} = 0. \tag{4-3}$$

We also calculate

$$\begin{aligned}\mathring{R}(\zeta,\zeta) &= \frac{1}{n_1n_2(n_1+n_2)}(n_2^2\mathring{R}(g_1,g_1)+n_1^2\mathring{R}(g_2,g_2)+2n_1n_2\mathring{R}(g_1,g_2))\\ &= \frac{1}{n_1n_2(n_1+n_2)}(n_2^2\mathring{R}_1(g_1,g_1)+n_1^2\mathring{R}_2(g_2,g_2))\\ &= -\frac{n_2^2S_1+n_1^2S_2}{n_1n_2(n_1+n_2)},\end{aligned}$$

where $S_i$ denotes the scalar curvature of $R_i$ for $i=1,2$.

Let $A$ be the collection of the values of $\mathring{R}(\varphi_i,\varphi_i)$ for $1\le i\le N_1$ and let $B$ be the collection of the values of $\mathring{R}(\psi_i,\psi_i)$ for $1\le i\le N_2$. Denote by $\bar{a}$ and $\bar{b}$ the average of all numbers in $A$ and $B$, respectively. Then

$$\begin{aligned}\bar{a} &= \frac{1}{N_1}\sum_{i=1}^{N_1}\mathring{R}(\varphi_i,\varphi_i) = \frac{1}{N_1}\sum_{i=1}^{N_1}\mathring{R}_1(\varphi_i,\varphi_i) = \frac{S_1}{n_1(n_1-1)},\\ \bar{b} &= \frac{1}{N_2}\sum_{i=1}^{N_2}\mathring{R}(\psi_i,\psi_i) = \frac{1}{N_2}\sum_{i=1}^{N_2}\mathring{R}_2(\psi_i,\psi_i) = \frac{S_2}{n_2(n_2-1)},\end{aligned}$$

where we have used

$$\sum_{i=1}^{N_1}\mathring{R}_1(\psi_i,\psi_i) = \frac{n_1+2}{2n_1}S_1 \quad\text{and}\quad \sum_{i=1}^{N_2}\mathring{R}_2(\psi_i,\psi_i) = \frac{n_2+2}{2n_2}S_2.$$

For simplicity, we write

$$A_1 = \frac{n_2(n_1-1)}{n_1+n_2} \quad\text{and}\quad A_2 = \frac{n_1(n_2-1)}{n_1+n_2}.$$

Notice that we have $A_1 < N_1$, $A_2 < N_2$ and

$$A_{n_1,n_2} = 1 + n_1n_2 + A_1 + A_2. \tag{4-4}$$

Also, the expression for $\mathring{R}(\zeta,\zeta)$ can be written as

$$\mathring{R}(\zeta,\zeta) = -A_1\bar{a} - A_2\bar{b}. \tag{4-5}$$

Since $R$ has $A_{n_1,n_2}$-nonnegative curvature operator of the second kind, we get using (4-3), (4-4) and (4-5) that

$$\begin{aligned}
-\mathring{R}(\zeta,\zeta) &\le f(A,\lfloor A_1\rfloor)+f(B,A_1+A_2-\lfloor A_1\rfloor)\\
&\le \lfloor A_1\rfloor\bar{a}+(A_1+A_2-\lfloor A_1\rfloor)\bar{b}\\
&= A_1\bar{a}+A_2\bar{b}+(A_1-\lfloor A_1\rfloor)(\bar{b}-\bar{a}),
\end{aligned}\tag{4-6}$$

where $f$ is the function defined in Lemma 4.2 and we have used Lemma 4.2 in estimating $f$. Similarly, we also have

$$\begin{aligned}
-\mathring{R}(\zeta,\zeta) &\le f(A,A_1+A_2-\lfloor A_2\rfloor)+f(B,\lfloor A_2\rfloor),\\
&\le (A_1+A_2-\lfloor A_2\rfloor)\bar{a}+\lfloor A_2\rfloor)\bar{b}\\
&= A_1\bar{a}+A_2\bar{b}+(A_2-\lfloor A_2\rfloor)(\bar{a}-\bar{b}).
\end{aligned}\tag{4-7}$$

Therefore, by (4-5), we get from (4-6) if $\bar{a}\ge\bar{b}$ and from (4-7) if $\bar{a}\le\bar{b}$ that

$$A_1\bar{a}+A_2\bar{b}=-\mathring{R}(\zeta,\zeta)\le A_1\bar{a}+A_2\bar{b}.$$

This implies that, either in (4-6) or (4-7), we must have equalities in the inequalities used for $f$. We then get from Lemma 4.2, that all the values in $A$ are equal to $\bar{a}$ and all the values in $B$ are equal to $\bar{b}$. Hence, both $R_1$ and $R_2$ have constant sectional curvature, that is to say, $R=c_1I_{n_1}\oplus c_2I_{n_2}$ for $c_1,c_2\in\mathbb{R}$.

Finally, we must have $c_1=c_2\ge 0$, as $R=c_1I_{n_1}\oplus c_2I_{n_2}$ has $A_{n_1,n_2}$-nonnegative curvature operator of the second kind if and only if $c_1=c_2\ge 0$ by Proposition 2.17.

(2) Apply (1) to $-R$.

(3) This follows from the fact that $R=c(I_{n_1}\oplus I_{n_2})$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha<A_{n_1,n_2}$ if and only if $c=0$. □

At last, we give the proof of Theorem 1.5.

*Proof of Theorem 1.5.* (1) This is an immediate consequence of part (3) of Proposition 4.1.

(2) Let $(p_1,p_2)\in M_1\times M_2$. By part (2) of Proposition 4.1, we have

$$R(p_1,p_2)=c(p_1,p_2)(I_{n_1}\oplus I_{n_2})$$

with $c(p_1,p_2)\ge 0$. If both $n_1$ and $n_2$ are at least 3, then Schur's lemma implies that $c(p_1,p_2)\equiv c\ge 0$. Below we provide an argument that works whenever $n_1,n_2\ge 2$.

Note that both $(M_1,g_1)$ and $(M_2,g_2)$ have pointwise constant sectional curvature. By Proposition 2.1, the eigenvalues of $\mathring{R}$ at $(p_1,p_2)$ are given by $\frac{\rho_1(p_1)}{n_1-1}$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $\frac{\rho_2(p_2)}{n_2-1}$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, 0 with multiplicity $n_1n_2$, and $-\frac{n_2\rho_1(p_1)+n_1\rho_2(p_2)}{n_1+n_2}$ with multiplicity one. Here $\rho_i(p_i)$ is the Einstein constant of $M_i$ at $p_i$, i.e., $\mathrm{Ric}(g_i)(p_i) = \rho_i(p_i)g_i$ at $p_i$ for $i = 1, 2$. Using the assumption that $M_1 \times M_2$ has $A_{n_1,n_2}$-nonnegative curvature operator of the second kind, we obtain

$$-\frac{n_2\rho_1(p_1)+n_1\rho_2(p_2)}{n_1+n_2}+\frac{n_1(n_2-1)+n_2(n_1-1)}{n_1+n_2}\frac{\rho_1(p_1)}{n_1-1}\geq 0$$

and

$$-\frac{n_2\rho_1(p_1)+n_1\rho_2(p_2)}{n_1+n_2}+\frac{n_1(n_2-1)+n_2(n_1-1)}{n_1+n_2}\frac{\rho_2(p_2)}{n_2-1}\geq 0.$$

The two inequalities force

$$(n_2-1)\rho_1(p_1) = (n_1-1)\rho_2(p_2).$$

Fixing $p_1$ while letting $p_2$ vary in $M_2$ shows that $\rho_2(p_2)$ is independent of $p_2$. Similarly, $\rho_1(p_1)$ is independent of $p_1$. Since $\rho_i(p_i) = (n_i-1)c(p_1, p_2)$ for $i = 1, 2$, we conclude that $c(p_1, p_2) \equiv c \geq 0$. Therefore, both $(M_1, g_1)$ and $(M_2, g_2)$ have constant sectional curvature $c \geq 0$.

If $M$ is further assumed to be complete, then $M$ is either flat or the universal cover of $M$ is isometric to $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$, up to scaling.

(3) Similar to the proof of (2). □

*Proof of Theorem 1.6*. Suppose that $(M^n, g)$ splits locally near $q \in M$ as a Riemannian product $(M_1^k \times M_2^{n-k}, g_1 \oplus g_2)$ with $1 \leq k \leq \frac{n}{2}$. Then the Riemann curvature tensor $R$ of $M$ satisfies $R = R_1 \oplus R_2$ near $q$, where $R_i$ denotes the Riemann curvature tensor of $M_i$ for $i = 1, 2$.

By part (3) of Proposition 3.1 if $k = 1$ and part (3) of Proposition 4.1 if $2 \leq k \leq \frac{n}{2}$, the assumption

$$\alpha < k(n-k) + \frac{2k(n-k)}{n}$$

implies that $M$ must be flat near $q$. Since the restricted holonomy does not depend on $q \in M$, we conclude that $M$ is flat. □

## 5. Holonomy restriction

*Proof of Theorem 1.3*. Suppose that $(M^n, g)$ splits locally near $q \in M$ as a Riemannian product $(M_1^k \times M_2^{n-k}, g_1 \oplus g_2)$ with $2 \leq k \leq \frac{n}{2}$. Then the Riemann curvature tensor $R$ of $M$ satisfies $R = R_1 \oplus R_2$ near $q$, where $R_i$ denotes the Riemann curvature tensor of $M_i$ for $i = 1, 2$.

Noticing that

$$\alpha < n + \frac{n-2}{n} \leq A_{k,n-k} = k(n-k) + \frac{2k(n-k)}{n}$$

for any $1 \leq k \leq \frac{n}{2}$, we conclude from part (3) of Propositions 3.1 if $k = 1$ and part (3) of Proposition 4.1 if $2 \leq k \leq \frac{n}{2}$ that $M$ is locally flat. Since the restricted holonomy does not depend on $q \in M$, we conclude that $M$ is flat. Therefore, $M$ is either locally irreducible or flat.

If $n = 3$, then the holonomy of $M$ must be $\mathsf{SO}(3)$ as $M$ is locally irreducible. So we may assume $n \geq 4$ below.

If $M$ is an irreducible locally symmetric space, then it is Einstein. Since

$$\alpha < n + \frac{n-2}{n} \leq \frac{3n}{2}\frac{n+2}{n+4}$$

for any $n \geq 4$, we get from [Nienhaus et al. 2023b, Theorem B] that either $M$ is flat or the restricted holonomy of $M$ is $\mathsf{SO}(n)$.

So we may assume that $M$ is not locally symmetric with irreducible holonomy representation. Then the restricted holonomy of $M$ is contained in Berger's list of holonomy groups [1955]: $\mathsf{SO}(n)$, $\mathsf{U}\left(\frac{n}{2}\right)$, $\mathsf{SU}\left(\frac{n}{2}\right)$, $\mathsf{Sp}\left(\frac{n}{4}\right)\mathsf{Sp}(1)$, $\mathsf{Sp}\left(\frac{n}{4}\right)$, $\mathsf{G}_2$ and $\mathsf{Spin}(7)$. Note that if its restricted holonomy is $\mathsf{SU}\left(\frac{n}{2}\right)$, $\mathsf{Sp}\left(\frac{n}{4}\right)$, $\mathsf{G}_2$ or $\mathsf{Spin}(7)$, then $M$ must be Ricci flat and thus flat.

If the restricted holonomy of $M$ is $\mathsf{Sp}\left(\frac{n}{4}\right)\mathsf{Sp}(1)$, then $M$ is quaternion-Kähler and it is also Einstein in this case. Thus, either the restricted holonomy of $M$ is $\mathsf{SO}(n)$ or $M$ is flat by [Nienhaus et al. 2023b, Theorem B].

If the restricted holonomy of $M$ is $\mathsf{U}\left(\frac{n}{2}\right)$, then $M$ is Kähler. Noticing that

$$\alpha < n + \frac{n-2}{n} \leq \frac{3}{2}\left(\frac{n^2}{4} - 1\right)$$

for any $n \geq 4$, $M$ must be flat by [Li 2023a, Therorem 1.2].

Overall, either the restricted holonomy of $M$ is $\mathsf{SO}(n)$ or $M$ is flat. □

## 6. Kähler manifolds

We prove Theorem 1.7. The proof shares the same idea as in Section 4, but we use the orthonormal basis of the space of traceless symmetric two-tensors on a complex Euclidean space constructed in [Li 2023a].

In the following, $B_{m_1,m_2}$ is the expression defined in (1-3) and $R_{\mathbb{CP}^m}$ denotes the Riemann curvature tensor of the complex projective space with constant holomorphic sectional curvature 4. We establish the following proposition.

**Proposition 6.1.** *For $i = 1, 2$, let $(V_i, g_i, J_i)$ be a complex Euclidean vector space of complex dimension $m_i \geq 1$. Let $R_i \in S^2_B(\Lambda^2 V_i)$ and $R = R_1 \oplus R_2 \in S^2_B(\Lambda^2(V_1 \times V_2))$.*

(1) *Suppose that $R$ has $B_{m_1,m_2}$-nonnegative curvature operator of the second kind. Then $R = c(R_{\mathbb{CP}^{m_1}} \oplus R_{\mathbb{CP}^{m_2}})$ for some $c \geq 0$.*

(2) *Suppose that $R$ has $B_{m_1,m_2}$-nonpositive curvature operator of the second kind. Then $R = c(R_{\mathbb{CP}^{m_1}} \oplus R_{\mathbb{CP}^{m_2}})$ for some $c \le 0$.*

(3) *Suppose that $R$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < B_{m_1,m_2}$. Then $R$ is flat.*

*Proof.* (1) Let

$$\{e_1, \dots, e_{m_1}, J_1e_1, \dots, J_1e_{m_1}\}$$

be an orthonormal basis of $(V_1, g_1, J_1)$ and

$$\{e_{m_1+1}, \dots, e_{m_1+m_2}, J_2e_{m_1+1}, \dots, J_2e_{m_1+m_2}\}$$

be an orthonormal basis of $(V_2, g_2, J_2)$.

As in Section 4, we have the orthogonal decomposition

$$S^2_0(V_1 \times V_2) = S^2_0(V_1) \oplus S^2_0(V_2) \oplus \operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\} \oplus \mathbb{R}\zeta,$$

where

$$\zeta = \frac{1}{\sqrt{2m_1m_2(m_1+m_2)}}(m_2g_1 - m_1g_2).$$

The same computation as in Section 4 gives that

$$\mathring{R}(\zeta, \zeta) = -\frac{m_2^2S_1 + m_1^2S_2}{2m_1m_2(m_1+m_2)}, \tag{6-1}$$

where $S_i$ denotes the scalar curvature of $R_i$ for $i = 1, 2$.

By Lemma 2.2, the subspace $\operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\}$ lies in the kernel of $\mathring{R}$ and its real dimension is $4m_1m_2$.

For $S^2_0(V_1)$ and $S^2_0(V_2)$, we use the orthonormal bases constructed in Section 4 of [Li 2023a]. More precisely, the following traceless symmetric two-tensors form an orthonormal basis of $S^2_0(V_1)$:

$$\begin{aligned}
\varphi^{1,\pm}_{ij} &= \frac{1}{2}(e_i \odot e_j \mp J_1e_i \odot J_1e_j) && \text{for } 1 \le i < j \le m_1,\\
\psi^{1,\pm}_{ij} &= \frac{1}{2}(e_i \odot J_1e_j \pm J_1e_i \odot e_j) && \text{for } 1 \le i < j \le m_1,\\
\alpha^1_i &= \frac{1}{2\sqrt{2}}(e_i \odot e_i - J_1e_i \odot Je_i) && \text{for } 1 \le i \le m_1,\\
\alpha^1_{m_1+i} &= \frac{1}{\sqrt{2}}(e_i \odot J_1e_i) && \text{for } 1 \le i \le m_1,\\
\eta^1_k &= \frac{k}{\sqrt{8k(k+1)}}(e_{k+1} \odot e_{k+1} + J_1e_{k+1} \odot J_1e_{k+1})\\
&\qquad - \frac{1}{\sqrt{8k(k+1)}}\sum_{i=1}^{k}(e_i \odot e_i + J_1e_i \odot J_1e_i) && \text{for } 1 \le k \le m_1 - 1.
\end{aligned}$$

Similarly, the traceless symmetric two-tensors

$$\begin{aligned}
\varphi_{ij}^{2,\pm} &= \frac{1}{2}(e_i \odot e_j \mp J_2 e_i \odot J_2 e_j) && \text{for } m_1+1 \le i < j \le m_1+m_2,\\
\psi_{ij}^{2,\pm} &= \frac{1}{2}(e_i \odot J_2 e_j \pm J_2 e_i \odot e_j) && \text{for } m_1+1 \le i < j \le m_1+m_2,\\
\alpha_i^2 &= \frac{1}{2\sqrt{2}}(e_i \odot e_i - J_1 e_i \odot J e_i) && \text{for } m_1+1 \le i \le m_1+m_2,\\
\alpha_{m_2+i}^2 &= \frac{1}{\sqrt{2}}(e_i \odot J_1 e_i) && \text{for } m_1+1 \le i \le m_1+m_2,\\
\eta_k^2 &= \frac{k}{\sqrt{8k(k+1)}}(e_{k+1} \odot e_{k+1} + J_2 e_{k+1} \odot J_2 e_{k+1})\\
&\qquad - \frac{1}{\sqrt{8k(k+1)}} \sum_{i=1}^{k} (e_i \odot e_i + J_2 e_i \odot J_2 e_i) && \text{for } m_1+1 \le k \le m_1+m_2-1
\end{aligned}$$

form an orthonormal basis for $S_0^2(V_2)$. Here the superscripts 1 and 2 indicate that these are quantities associated with the space $V_1$ and $V_2$, respectively.

By Lemma 4.3 in [Li 2023a], we have

$$\sum_{1 \le i < j \le m_1} (\mathring{R}(\varphi_{ij}^{1,-}, \varphi_{ij}^{1,-}) + \mathring{R}(\psi_{ij}^{1,-}, \psi_{ij}^{1,-})) + \sum_{k=1}^{m_1-1} \mathring{R}(\eta_k, \eta_k) = -\frac{m_1-1}{2m_1} S_1 \tag{6-2}$$

and

$$\sum_{m_1+1 \le i < j \le m_1+m_2} (\mathring{R}(\varphi_{ij}^{2,-}, \varphi_{ij}^{2,-}) + \mathring{R}(\psi_{ij}^{2,-}, \psi_{ij}^{2,-})) + \sum_{k=m_1+1}^{m_1+m_2-1} \mathring{R}(\eta_k, \eta_k) = -\frac{m_2-1}{2m_2} S_2. \tag{6-3}$$

Let $A$ be the collection of the values $\mathring{R}(\alpha_i^1, \alpha_i^1)$ for $1 \le i \le 2m_1$, $\mathring{R}(\varphi_{ij}^{1,+}, \varphi_{ij}^{1,+})$ and $\mathring{R}(\psi_{ij}^{1,+}, \psi_{ij}^{1,+})$ for $1 \le i < j \le m$. By Lemma 4.3 in [Li 2023a], we know that $A$ contains two copies of $R(e_i, J_1 e_i, e_i, J_1 e_i)$ for each $1 \le i \le m_1$ and two copies of $2R(e_i, J_1 e_i, e_j, J_1 e_j)$ for each $1 \le i < j \le m_1$. Therefore, the sum of all values in $A$ is equal to $S_1$, the scalar curvature of $R_1$, and $\bar{a}$, the average of all values in $A$, is given by

$$\bar{a} = \frac{S_1}{m_1(m_1+1)}.$$

Let $B$ be the collection of the values $\mathring{R}(\alpha_i^2, \alpha_i^2)$ for $m_1+1 \le i \le m_1+2m_2$, $\mathring{R}(\varphi_{ij}^{2,+}, \varphi_{ij}^{2,+})$ and $\mathring{R}(\psi_{ij}^{2,+}, \psi_{ij}^{2,+})$ for $m_1+1 \le i < j \le m_1+m_2$. By Lemma 4.3 in [Li 2023a], we know that $B$ contains two copies of $R(e_i, J_2 e_i, e_i, J_2 e_i)$ for each $m_1+1 \le i \le m_1+m_2$ and two copies of $2R(e_i, J_2 e_i, e_j, J_2 e_j)$ for each

$m_1+1 \le i < j \le m_1+m_2$. Therefore, the sum of all values in $B$ is equal to $S_2$, the scalar curvature of $R_2$, and $\bar{b}$, the average of all values in $B$, is given by

$$\bar{b} = \frac{S_2}{m_2(m_2+1)}.$$

Combining (6-1), (6-2) and (6-3) together yields

$$\begin{aligned}
\sum_{1\le i<j\le m_1} &(\mathring{R}(\varphi_{ij}^{1,-}, \varphi_{ij}^{1,-}) + \mathring{R}(\psi_{ij}^{1,-}, \psi_{ij}^{1,-})) \\
&+ \sum_{m_1+1\le i<j\le m_1+m_2} (\mathring{R}(\varphi_{ij}^{2,-}, \varphi_{ij}^{2,-}) + \mathring{R}(\psi_{ij}^{2,-}, \psi_{ij}^{2,-})) \\
&+ \sum_{k=1}^{m_1-1} \mathring{R}(\eta_k, \eta_k) + \sum_{k=m_1+1}^{m_1+m_2-1} \mathring{R}(\eta_k, \eta_k) + \mathring{R}(\zeta, \zeta) \\
&\qquad = -\frac{m_1-1}{2m_1} S_1 - \frac{m_2-1}{2m_2} S_2 + \mathring{R}(\zeta, \zeta) \\
&\qquad = -\frac{1}{2}(m_1^2-1)\bar{a} - \frac{1}{2}(m_2^2-1)\bar{b} - \frac{m_2^2 S_1 + m_1^2 S_2}{2m_1 m_2(m_1+m_2)} \\
&\qquad = -B_1\bar{a} - B_2\bar{b},
\end{aligned}$$

where we have introduced

$$B_1 = \frac{1}{2}(m_1^2-1) + \frac{(m_1+1)m_2}{2(m_1+m_2)} \quad \text{and} \quad B_2 = \frac{1}{2}(m_2^2-1) + \frac{(m_2+1)m_1}{2(m_1+m_2)}$$

for simplicity of notation. Note that $-B_1\bar{a} - B_2\bar{b}$ is the sum of

$$1 + 4m_1m_2 + (m_1^2-1) + (m_2^2-1)$$

many diagonal elements of the matrix representation of $\mathring{R}$ with respect to the orthonormal basis of $S^2_0(V_1 \times V_2)$ constructed above (here one can pick any orthonormal basis for the subspace $\operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\}$ as it is in the kernel of $\mathring{R}$).

Noticing that

$$B_{m_1,m_2} = 1 + (m_1^2-1) + (m_2^2-1) + 4m_1m_2 + B_1 + B_2,$$

the assumption $R$ has $B_{m_1,m_2}$-nonnegative curvature operator of the second kind implies that

$$\begin{aligned}
B_1\bar{a} + B_2\bar{b} &\le f(A, \lfloor B_1 \rfloor) + f(B, B_1+B_2-\lfloor B_1 \rfloor) \\
&\le \lfloor B_1 \rfloor \bar{a} + (B_1+B_2-\lfloor B_1 \rfloor)\bar{b} \\
&= B_1\bar{a} + B_2\bar{b} + (B_1-\lfloor B_1 \rfloor)(\bar{b}-\bar{a})
\end{aligned} \tag{6-4}$$

and

$$B_1\bar{a}+B_2\bar{b} \le f(A, B_1+B_2-\lfloor B_2\rfloor)+f(B, \lfloor B_2\rfloor) \tag{6-5}$$
$$\le (B_1+B_2-\lfloor B_2\rfloor)\bar{a}+\lfloor B_2\rfloor\bar{b}$$
$$= B_1\bar{a}+B_2\bar{b}+(B_2-\lfloor B_2\rfloor)(\bar{a}-\bar{b}),$$

where $f$ is the function defined in Lemma 4.2 and we have used Lemma 4.2 to estimate $f$. So we get from (6-4) if $\bar{a}\ge\bar{b}$ and from (6-5) if $\bar{a}\le\bar{b}$ that

$$B_1\bar{a}+B_2\bar{b} \le B_1\bar{a}+B_2\bar{b}.$$

Therefore, either in (6-4) or (6-5), we must have equalities in the inequalities used for $f$. By Lemma 4.2, we get that all the values in $A$ are equal to $\bar{a}$ and all the values in $B$ are equal to $\bar{b}$. Hence, both $R_1$ and $R_2$ have constant holomorphic sectional curvature, that is to say, $R = c_1 R_{\mathbb{CP}^{m_1}} \oplus c_2 R_{\mathbb{CP}^{m_2}}$ for $c_1, c_2 \in \mathbb{R}$.

Finally, we must have $c_1 = c_2 \ge 0$, as $R = c_1 R_{\mathbb{CP}^{m_1}} \oplus c_2 R_{\mathbb{CP}^{m_2}}$ has $B_{m_1,m_2}$-nonnegative curvature operator of the second kind if and only if $c_1 = c_2 \ge 0$ by Proposition 2.17.

(2) Apply (1) to $-R$.

(3) This follows from the fact that $R = c(R_{\mathbb{CP}^{m_1}} \oplus R_{\mathbb{CP}^{m_2}})$ has $\alpha$-nonnegative or $\alpha$-nonpositive curvature operator of the second kind for some $\alpha < B_{m_1,m_2}$ if and only if $c = 0$. □

*Proof of Theorem 1.7.* Once we have Proposition 6.1, this is similar to the proof of Theorem 1.5 and we omit the details. □

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