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LAGRANGIAN COBORDISM OF POSITROID LINKS

JOHAN ASPLUND, YOUNGJIN BAE, ORSOLA CAPOVILLA-SEARLE, MARCO CASTRONOVO, CAITLIN LEVERSON AND ANGELA WU

Positroid strata of the complex Grassmannian can be realized as augmentation varieties of Legendrians called positroid links. We prove that the partial order on strata induced by Zariski closure also has a symplectic interpretation, given by exact Lagrangian cobordism.

1. Introduction

Positroid varieties are irreducible subvarieties of the complex Grassmannian that were first introduced in the study of total positivity [Lusztig 1998; Postnikov 2006; Rietsch 2006], and Poisson geometry [Brown et al. 2006]. Open positroid varieties provide a stratification of the complex Grassmannian, and they can be enumerated by a handful of different combinatorial objects.

Positroid varieties are known to admit cluster structures, which have also been found on the coordinate rings of many algebraic varieties arising in representation theory, including double Bruhat cells [Fomin and Zelevinsky 2002], double Bott–Samelson cells [Shen and Weng 2021], positroid strata [Galashin and Lam 2023], and certain Richardson strata [Casals et al. 2022; Galashin et al. 2022; 2023]. Geometrically, this allows one to think of such varieties as the result of gluing algebraic tori along birational mutation maps, and their coordinate rings carry bases whose structure constants are positive integers counting tropical curves [Fock and Goncharov 2009; Gross et al. 2018].

Legendrian links in $\mathbb{R}^3$ are smooth links that are everywhere tangent to the plane field $\ker(dz - y dx)$ which is called the standard contact structure of $\mathbb{R}^3$. Their interpolating objects are exact Lagrangian cobordisms in the symplectization of $\mathbb{R}^3$. Legendrian links and exact Lagrangian cobordisms between them can be studied via the general framework of symplectic field theory [Eliashberg et al. 2000] which aims to use counts of pseudoholomorphic curves to define invariants of contact manifolds and the symplectic cobordisms between them. One such invariant is the Chekanov–Eliashberg differential graded algebra associated to a Legendrian link Λ,

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whose homology is a Legendrian isotopy invariant [Chekanov 2002; Eliashberg et al. 2000].

Under favorable circumstances, the space of augmentations of the Chekanov–Eliashberg dg-algebra forms an algebraic variety $\mathrm{Aug}(\Lambda)$. Exact Lagrangian fillings (cobordisms from the empty set to Λ) induce augmentations.

Augmentations have been shown to be related to simple microlocal sheaves associated to Λ [Shende et al. 2017; 2019; Ng et al. 2020], leading to the idea that augmentation varieties should have cluster structures, with torus charts corresponding to embedded exact Lagrangian fillings of Λ and mutations arising from Lagrangian surgery [Polterovich 1991]. So far this idea has been explored mainly for Legendrian links in the standard contact $\mathbb{R}^3$; see [Casals and Weng 2024]. This bridge between contact geometry and cluster algebras has been fruitful in both directions, having been instrumental in resolving long-standing conjectures on the abundance of Lagrangian fillings of Legendrian links [Casals and Gao 2022] and on the existence of cluster structures on spaces of interest in representation theory [Casals et al. 2022].

We explore this idea further, predicating that when augmentation varieties have compactifications stratified by augmentation varieties of smaller dimension, then the Legendrian submanifolds corresponding to adjacent strata should be related by exact Lagrangian cobordisms. We establish this in the simplest class of examples: positroid strata of complex Grassmannians [Knutson et al. 2013]. It is known that all positroid strata are isomorphic to the moduli space of simple microlocal sheaves of certain Legendrian links Λ in $\mathbb{R}^3$ with framings (marked points) [Shende et al. 2019] and to augmentation varieties of Λ [Casals et al. 2021; 2020]. The top positroid stratum was one of the key motivating examples for the development of cluster algebras [Fomin and Zelevinsky 2002; Scott 2006], while cluster structures on strata of lower dimension were described more recently [Galashin and Lam 2023].

1.1. ***Result.*** The positroid strata $\Pi^\circ \subset \mathrm{Gr}(k,n)$ of complex Grassmannians are disjoint Zariski locally closed sets; see Definition 4.5. There is a distinguished class of Legendrian links Λ_{Π° in the standard contact $\mathbb{R}^3$, referred to as positroid links whose augmentation varieties are related to the strata by an algebraic isomorphism

$$\Pi^\circ \cong \mathrm{Aug}(\Lambda_{\Pi^\circ}) \times (\mathbb{C}^*)^{N(\Pi^\circ)}, \tag{1}$$

where $N(\Pi^\circ) \in \mathbb{Z}_{\geq 0}$ is a nonnegative integer depending on Π°; see Section 5 for a precise statement. The positroid link Λ_{Π° is the Legendrian (-1)-closure of a braid on k-strands associated to Π°, known as a juggling braid (see Definition 5.4). The positroid strata can be enumerated by bounded affine permutations (see Section 3) and for each pair of integers $1 \leq k < n$, the set of positroid strata of $\mathrm{Gr}(k,n)$ is partially ordered by declaring $\Pi_f^\circ \leq \Pi_g^\circ$ if and only if $\Pi_f^\circ \subset \mathrm{cl}(\Pi_g^\circ)$. Our main result is the following.

Theorem 1.1 (Theorems 6.1 and 6.3). *Given two comparable positroid strata $\Pi_f^\circ \leq \Pi_g^\circ$ in $\mathrm{Gr}(k,n)$, their associated Legendrian links $\Lambda_{\Pi_f^\circ}$ and $\Lambda_{\Pi_g^\circ}$ are related by an exact Lagrangian cobordism from $\Lambda_{\Pi_f^\circ}$ to $\Lambda_{\Pi_g^\circ}$ whose Euler characteristic is*

$$\dim(\Pi_g^\circ) - \dim(\Pi_f^\circ) + \#\,\mathrm{Fix}(g) - \#\,\mathrm{Fix}(f),$$

where $\#\,\mathrm{Fix}$ denotes the number of fixed points (see Definition 3.9).

Remark 1.2. (1) We defer experts to Theorem 5.8 and Remark 6.2 for a discussion on marked points placed on the positroid links.

(2) Two positroid links being exact Lagrangian cobordant does not imply that the corresponding positroid strata are comparable in the partial order; see Example 7.2 and Remark 7.3.

Note that even for small k and n, complex Grassmannians have many positroid strata, and their partial order is quite complicated; see Example 4.9 and Figure 5. The exact Lagrangian cobordism in Theorem 1.1 is constructed by pinching contractible Reeb chords, which is a well-known technique in contact geometry. We establish that any chain connecting Π_f° and Π_g° in the partial order produces a sequence of pinch moves.

If $\Pi_f^\circ < \Pi_g^\circ$ then from Theorem 1.1 there is an exact Lagrangian cobordism from $\Lambda_{\Pi_f^\circ}$ to $\Lambda_{\Pi_g^\circ}$ consisting of pinch moves. Let r be the number of such moves. From [Pan 2017; Gao et al. 2024] it follows that there is an open embedding relating the augmentation varieties of the ends:

$$\mathrm{Aug}(\Lambda_{\Pi_f^\circ}) \times (\mathbb{C}^*)^r \hookrightarrow \mathrm{Aug}(\Lambda_{\Pi_g^\circ}).$$

This means that if the bounded affine permutations f and g are related by r affine transpositions, under the identification between positroid strata and augmentation varieties in (1) we get an open embedding

$$\Pi_f^\circ \times (\mathbb{C}^*)^{r+N(\Pi_g^\circ)} \hookrightarrow \Pi_g^\circ \times (\mathbb{C}^*)^{N(\Pi_f^\circ)}.$$

As pointed out to us by a referee it is an interesting question whether such embeddings admit a description purely in terms of algebraic combinatorics, i.e., without using the connection with the topology of Legendrians.

Outline. In Section 2 we provide the necessary background on Legendrian links and exact Lagrangian cobordisms. In Section 3 we provide the relevant definitions and properties of bounded affine permutations. We recall the definition of positroid strata of complex Grassmannians in Section 4. In Section 5 we describe positroid links via juggling braids coming from bounded affine permutations. Our main theorem Theorem 1.1 is proven in Section 6. In Section 7 we discuss examples.

2. Background on contact geometry

We briefly review some basic facts on Legendrian links and exact Lagrangian cobordisms. See [Etnyre and Ng 2022] for a more thorough introduction, and [Geiges 2008] for background on contact geometry.

2.1. *Legendrian links.* The *standard contact structure* on $\mathbb{R}^3$ is the plane field $\xi := \ker(dz - y\,dx)$. A smooth embedding of circles $\Lambda \subset \mathbb{R}^3$ is called a *Legendrian link* if $T_x\Lambda \subset \xi_x$ for all $x \in \Lambda$. Two Legendrian links are *Legendrian isotopic* if they are smoothly isotopic through Legendrian links. The maps $\pi_L(x, y, z) = (x, y)$ and $\pi_F(x, y, z) = (x, z)$ are called the *Lagrangian projection* and *front projection*, respectively. The Lagrangian projection of a Legendrian link is an immersed curve with zero oriented area. The front projection of a Legendrian link does not have any vertical tangencies but instead has cusp and crossing singularities. Conversely, any immersed disjoint union of circles with cusp and crossing singularities and no vertical tangencies lifts uniquely to a Legendrian link in $\mathbb{R}^3$; see Figure 1 for an example of both projections.

The *Thurston–Bennequin number* $\operatorname{tb}(\Lambda) \in \mathbb{Z}$ of a Legendrian link $\Lambda \subset \mathbb{R}^3$ measures how much the contact structure ξ rotates along Λ, and is defined as the linking number of Λ and its push-off in any direction transverse to ξ. This number is easily computed from a front projection as

$$\operatorname{tb}(\Lambda) = \#\text{positive crossings of } \pi_F(\Lambda) - \#\text{negative crossings of } \pi_F(\Lambda) - \#\text{right cusps of } \pi_F(\Lambda).$$

A *Reeb chord* of Λ is a trajectory of the vector field ∂_z that begins and ends on Λ. Note that the Lagrangian projection induces a bijection between Reeb chords and double points of $\pi_L(\Lambda)$. A Reeb chord of Λ is *contractible* if there exists a smooth homotopy of Λ through Legendrian immersions (such that the Lagrangian projections only have transverse double points throughout the homotopy) that shrinks the length of the Reeb chord to zero; see [Ekholm et al. 2016, Definition 6.13].

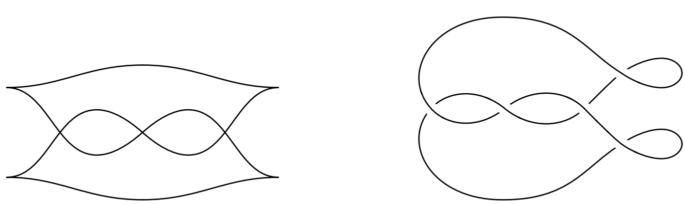

Figure 1. Front (left) and Lagrangian (right) projections of a Legendrian trefoil.

2.2. *Exact Lagrangian cobordisms.* The *symplectization* of the standard contact $\mathbb{R}^3$ is defined as $\mathbb{R} \times \mathbb{R}^3$ equipped with the closed nondegenerate 2-form $\omega = d\lambda$ where $\lambda = e^t(dz - y\,dx)$. A *Lagrangian cobordism* from a Legendrian link Λ_- to a Legendrian link Λ_+ is a smooth embedding of a surface $L \subset \mathbb{R} \times \mathbb{R}^3$ such that $\omega|_{TL} = 0$ and such that L is a cylinder over $\Lambda_\pm$ at infinity but is otherwise compact, i.e., there is some $T > 0$ for which $L \cap [-T, T]$ is compact,

$$\mathcal{E}_-(L) := L \cap ((-\infty, -T) \times \mathbb{R}^3) = (-\infty, -T) \times \Lambda_-,$$
$$\mathcal{E}_+(L) := L \cap ((T, \infty) \times \mathbb{R}^3) = (T, \infty) \times \Lambda_+.$$

A Lagrangian cobordism L is *exact* if there is a smooth function $f : L \to \mathbb{R}$ such that $df = \lambda|_L$ and $f|_{\mathcal{E}_\pm(L)}$ is constant. A *Lagrangian filling* is a Lagrangian cobordism with $\Lambda_- = \varnothing$. Exact Lagrangian cobordisms give a reflexive and transitive relation, but not a symmetric one [Chantraine 2015]. All known examples of exact Lagrangian cobordisms between Legendrians with maximal Thurston–Bennequin numbers arise from

- Legendrian isotopy,
- the unique exact Lagrangian disk filling of an unlinked unknot component with maximal Thurston–Bennequin number, and
- pinching a contractible Reeb chord.

The *pinch move* is a local modification of $\Lambda \subset \mathbb{R}^3$, depicted in Figure 2. A pinch move induces an exact Lagrangian cobordism in the symplectization of $\mathbb{R}^3$ from the knot after a pinch move to the knot before the pinch move. When a pinch move is performed, the number of components of the Legendrian link either increases or decreases by one, so the resulting exact Lagrangian cobordism is topologically a pair of pants, and it is often called a saddle cobordism. See Figure 12 for an example of an exact Lagrangian saddle cobordism between two Legendrians.

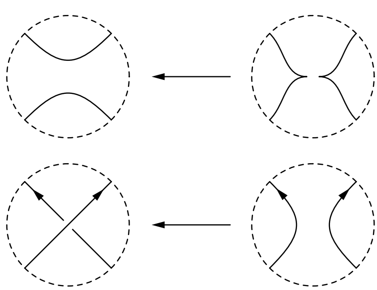

Figure 2. A pinch move in the front (left) and the Lagrangian (right) projection. The arrows show the direction of the induced exact Lagrangian cobordism.

3. Bounded affine permutations

We review bounded affine permutations, the affine analogs of ordinary permutations. See [Knutson et al. 2013] for a more thorough introduction and an interpretation in terms of juggling. Throughout this section let $k, n \in \mathbb{Z}_{\geq 1}$ with $k \leq n$.

Definition 3.1. An *affine permutation of size n* is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(i+n) = f(i)+n$ for all $i \in \mathbb{Z}$. In addition, it is *k-bounded* if

(1) $i \leq f(i) \leq i+n$, and

(2) $\sum_{i=1}^{n}(f(i)-i) = nk$.

Denote the set of k-bounded affine permutations of size n by $\mathrm{Bound}(k,n)$, and a k-bounded affine permutation $f : \mathbb{Z} \to \mathbb{Z}$ by $[f(1), f(2), \ldots, f(n)]$.

Lemma 3.2. *A bijection f is a k-bounded affine permutation of size n if and only if $g := -(-f)^{-1}$ is a k-bounded affine permutation of size n.*

Proof. Let $i \in \mathbb{Z}$ and define $j = -f(i)$. Then $f(i+n) = f(i)+n$ is equivalent to $i+n = -g(-f(i)-n)$, and hence $-g(j)+n = -g(j-n)$. Note that $i \leq f(i) \leq i+n$ is equivalent to $-g((-f)(i)) \leq f(i) \leq -g((-f)(i+n))$. We can rewrite these inequalities as $g(j) \geq j$ and $j \geq g(j-n) = g(j)-n$, which is equivalent to $j \leq g(j) \leq j+n$. Finally, $\sum_{i=1}^{n}(f(i)-i) = nk$ is equivalent to $\sum_{j=1}^{n}(-j+g(j)) = nk$. □

A bounded affine permutation $f \in \mathrm{Bound}(k,n)$ can be visualized in the plane as the set of line segments in $\mathbb{R}^2$ from $(i,1)$ to $(f(i),0)$ for all $i \in \mathbb{Z}$; see Figure 3 for an example. Note that once $f(i+n) = f(i)+n$ for all $i \in \mathbb{Z}$, the picture is fully determined by the region in the red dashed box in Figure 3.

Definition 3.3. For a bounded affine permutation $f \in \mathrm{Bound}(k,n)$, a pair $(i,j) \in \{1,\ldots,n\}^2$ is an *affine inversion* if $i < j$ and either $f(i) > f(j)$ or $f(i) < f(j)-n$. The *length* of an affine permutation f, $\ell(f) \in \mathbb{Z}_{\geq 0}$, is the number of affine inversions of f.

Example 3.4. For the bounded affine permutation $f = [3,5,6,4]$ depicted in Figure 3, $(2,4)$ and $(3,4)$ are the only affine inversions, and thus $\ell(f) = 2$. These

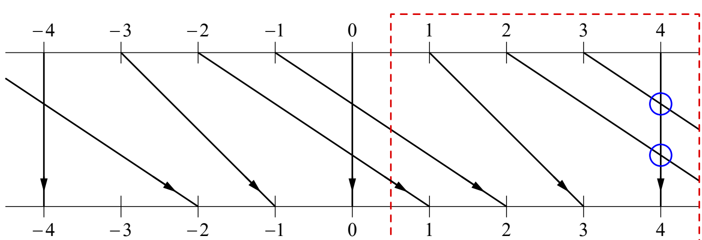

Figure 3. The bounded affine permutation $f = [3,5,6,4] \in \mathrm{Bound}(2,4)$.

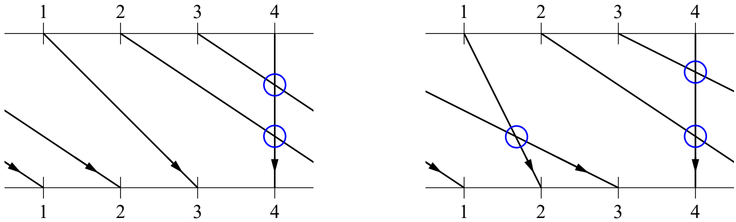

Figure 4. Left: $[3, 5, 6, 4] \in \mathrm{Bound}(2, 4)$. Right: $[2, 5, 7, 4] \in \mathrm{Bound}(2, 4)$.

two affine inversions correspond to the two circles in Figure 3. Similarly, we have $\ell([3, 4, 6, 5]) = 1$ and $\ell([2, 5, 7, 4]) = 3$.

We now equip $\mathrm{Bound}(k, n)$ with a partial order.

Definition 3.5. An affine permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ of size n is a *transposition* if $\sigma(k) = \sigma_i(k)$ for some $i \in \mathbb{Z}$, where

$$\sigma_i(k) := \begin{cases} k+1 & \text{if } k = i \pmod n, \\ k-1 & \text{if } k = i+1 \pmod n, \\ k & \text{if } k \neq i, i+1 \pmod n. \end{cases}$$

Definition 3.6. Let $f, f' \in \mathrm{Bound}(k, n)$. Declare $f \lessdot f'$ if and only if $\ell(f) < \ell(f')$ and there exists an affine transposition σ_i of size n such that $f' = f \circ \sigma_i$ or $f' = \sigma_i \circ f$. Define a relation $<$ on $\mathrm{Bound}(k, n)$ as the transitive closure of the relation $\lessdot$.

It follows from the definition that $(\mathrm{Bound}(k, n), <)$ is a partially ordered set.

Example 3.7. Note that

$$\ell([3, 4, 6, 5]) < \ell([3, 5, 6, 4]) \quad \text{and} \quad [3, 5, 6, 4] = \sigma_4 \circ [3, 4, 6, 5];$$

thus we have $[3, 4, 6, 5] \lessdot [3, 5, 6, 4]$. Similarly

$$\ell([3, 5, 6, 4]) < \ell([2, 5, 7, 4]) \quad \text{and} \quad [2, 5, 7, 4] = \sigma_2 \circ [3, 5, 6, 4],$$

so that $[3, 5, 6, 4] \lessdot [2, 5, 7, 4]$; see Figure 4. Then the induced partial order satisfies $[3, 4, 6, 5] < [2, 5, 7, 4]$. See Figure 5 for the Hasse diagram of the partial order $<$ on $\mathrm{Bound}(2, 4)$.

As in the case of ordinary permutations, one can define cycles of bounded affine permutations.

Lemma 3.8. *Let $f \in \mathrm{Bound}(k, n)$. Then f induces a bijection $\bar{f} : \mathbb{Z}/n \to \mathbb{Z}/n$ defined by $\bar{f}([i]) = [f(i)]$ for all $[i] \in \mathbb{Z}/n$.*

Proof. As f is a bounded affine permutation, for all $i, t \in \mathbb{Z}$, we know $f(i + tn) = f(i) + tn = f(i) \pmod n$. Thus, $\bar{f}$ is well-defined. Moreover, we have a well-defined inverse function $\bar{f}^{-1}$ given by $\bar{f}^{-1}([i]) = [f^{-1}(i)]$. So $\bar{f}$ is a bijection. □

Definition 3.9. A *cycle of length* t of $f \in \text{Bound}(k, n)$ is a tuple $(i_1, \ldots, i_t) \in (\mathbb{Z}/n)^t$ up to cyclic permutation such that

$$\bar{f}: i_1 \mapsto i_2 \mapsto \cdots \mapsto i_t \mapsto i_1,$$

where $i_1, \ldots, i_t$ are all distinct. A cycle of length 1 is called a *fixed point* of f.

Example 3.10. The affine permutation $f = [3, 5, 6, 4]$ has one cycle of length three being $(1, 3, 2)$ and one cycle of length one being (4).

4. Positroid strata

We collect definitions and known properties of positroid strata of the complex Grassmannian from the literature [Knutson et al. 2013; Galashin and Lam 2021].

4.1. ***Complex Grassmannians.*** Fix $k, n \in \mathbb{Z}_{\geq 1}$ such that $k \leq n$, and write $\text{Mat}(k, n)$ for the set of $k \times n$ matrices with complex entries.

Definition 4.1. The *complex Grassmannian of k-planes* in $\mathbb{C}^n$ is

$$\text{Gr}(k, n) = \{M \in \text{Mat}(k, n) \mid \text{rk}(M) = k\}/\text{row operations}.$$

Complex Grassmannians are smooth projective varieties. A widely used projective embedding of $\text{Gr}(k, n)$ in $\mathbb{P}^{\binom{n}{k}-1}$ is given by Plücker coordinates; see [Harris 1992, Lecture 6].

Definition 4.2. Given $M \in \text{Mat}(k, n)$ with column vectors $M_1, \ldots, M_n$ and $1 \leq i_1 < \cdots < i_k \leq n$, we define the *Plücker coordinates* $\Delta_{i_1,\ldots,i_k}(M)$ to be

$$\Delta_{i_1,\ldots,i_k}(M) = \det[M_{i_1}, M_{i_2}, M_{i_3}, \ldots, M_{i_k}].$$

Example 4.3. For $1 \leq i_1 < i_2 \leq 4$, label the $\binom{4}{2} = 6$ homogeneous coordinates of $\mathbb{P}^5$ by Δ_{i_1,i_2}. The corresponding Plücker coordinates on $\text{Gr}(2, 4)$ give a projective embedding of $\text{Gr}(2, 4)$ as a hypersurface in $\mathbb{P}^5$, whose equation is

$$\Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}.$$

For example, the matrix

$$M = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

has $\Delta_{1,2}(M) = 1$, $\Delta_{1,3}(M) = -1$, $\Delta_{1,4}(M) = 1$, $\Delta_{2,3}(M) = -2$, $\Delta_{2,4}(M) = 1$, and $\Delta_{3,4}(M) = 1$. Note that row operations on M change all Plücker coordinates by a common factor, which is immaterial once one thinks of them as homogeneous coordinates on $\mathbb{P}^5$.

4.2. ***Positroid strata.*** The complex Grassmannian $\mathrm{Gr}(k,n)$ decomposes into disjoint subsets Π°_f labeled by bounded affine permutations $f \in \mathrm{Bound}(k,n)$; see [Knutson et al. 2013]. Any $M \in \mathrm{Mat}(k,n)$ with columns $M_1, \dots, M_n$ extends periodically to a matrix with infinitely many columns, by setting $M_{i+n} = M_i$ for all $i \in \mathbb{Z}$. Define an associated function $f_M : \mathbb{Z} \to \mathbb{Z}$ by

$$f_M(i) = \min\{j \geq i \mid M_i \in \mathrm{Span}(M_{i+1}, \dots, M_j)\}.$$

If $M \in \mathrm{Mat}(k,n)$ has rank k, then $f_M : \mathbb{Z} \to \mathbb{Z}$ is a k-bounded affine permutation of size n that depends only on $[M] \in \mathrm{Gr}(k,n)$.

Example 4.4. The matrix $M \in \mathrm{Mat}(2,4)$ from Example 4.3 extends periodically to a matrix with infinitely many columns

$$\begin{bmatrix} \cdots & 0 & 1 & \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{1} & 1 & 2 & \cdots \\ \cdots & -1 & 1 & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{1} & 0 & 1 & \cdots \end{bmatrix},$$

and the corresponding bounded affine permutation $f_M : \mathbb{Z} \to \mathbb{Z}$ of type $(2,4)$ is $f_M = [3,4,5,6]$.

Definition 4.5. The *positroid stratum* associated to $f \in \mathrm{Bound}(k,n)$ is defined as

$$\Pi^\circ_f := \{[M] \in \mathrm{Gr}(k,n) \mid f_M = f\}.$$

The adjective positroid comes from the fact that the closure of a stratum is defined by the vanishing of Plücker coordinates $\Delta_{i_1,\dots i_k}$ whose indexing sets $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ form a particular class of matroids [Postnikov 2006]. The term strata refers to the following property.

Theorem 4.6 (Knutson–Lam–Speyer [Knutson et al. 2013, Theorems 5.9 and 5.10]). *Each positroid stratum is locally closed in the Zariski topology, and has closure*

$$\mathrm{cl}(\Pi^\circ_f) = \bigcup_{f' \geq f} \Pi^\circ_{f'}.$$

Definition 4.7 (partial order on positroid strata). Define $\Pi^\circ_1 \leq \Pi^\circ_2$ if and only if $\Pi^\circ_1 \subset \mathrm{cl}(\Pi^\circ_2)$.

It follows immediately that $\leq$ defines a partial order on the set of positroid strata of $\mathrm{Gr}(k,n)$.

Theorem 4.8 [Knutson et al. 2013, Theorem 5.9]. *The codimension of* $\Pi^\circ_f \subset \mathrm{Gr}(k,n)$ *is equal to* $\ell(f)$.

Example 4.9. There are 33 positroid strata $\Pi^\circ_f \subset \mathrm{Gr}(2,4)$: one of dimension 4, four of dimension 3, ten of dimension 2, twelve of dimension 1, and six of dimension 0. Each dimension corresponds to a row in the Hasse diagram of Figure 5, with the bottom row containing the only top-dimensional stratum.

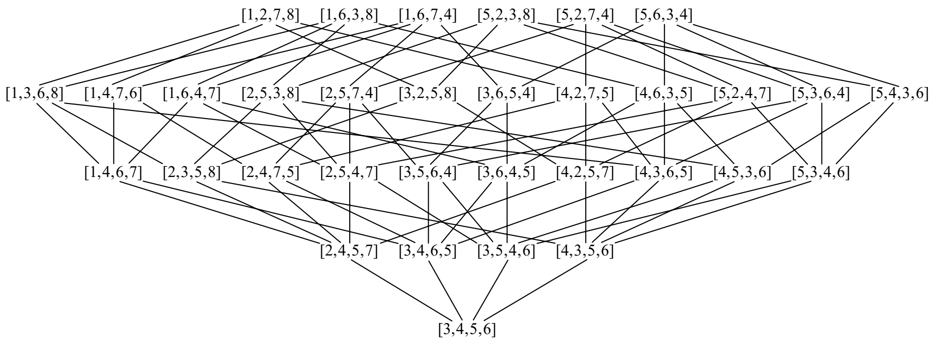

Figure 5. The Hasse diagram of the partial order on Bound(2, 4).

5. Positroid links

We follow Casals, Gorsky, Gorsky and Simental [Casals et al. 2021] and associate a Legendrian link to a bounded affine permutation $f \in \text{Bound}(k, n)$ (see Section 3).

Definition 5.1. Let $f \in \text{Bound}(k, n)$ be a bounded affine permutation. For each $i \in \{1, \ldots, n\}$, let

$$A_i(f) := \left\{(x, y) \in \mathbb{R}^2 \mid (2x - f(i) - i)^2 + 4y^2 = (f(i) - i)^2\right\} \cap \{y \geq 0\} \subset \mathbb{R}^2$$

be the upper semicircle of a circle intersecting the x-axis in the points $(i, 0)$ and $(f(i), 0)$. We define the *juggling diagram* associated to f to be the subset $\bigcup_{i=1}^n A_i(f) \subset \mathbb{R}^2$.

Definition 5.2. Let $f \in \text{Bound}(k, n)$ be a bounded affine permutation. After modifying the associated juggling diagram with the moves shown in Figure 6, we obtain a tangle diagram. After enumerating the strands of the tangle diagram from top to bottom we can describe the tangle diagram with a braid word that we denote by $J_k(f)$ and call the *juggling braid* of f.

Remark 5.3. By the definition of a bounded affine permutation, $J_k(f)$ is a positive braid on k strands.

See Figure 7 for examples of juggling diagrams and their corresponding juggling braids.

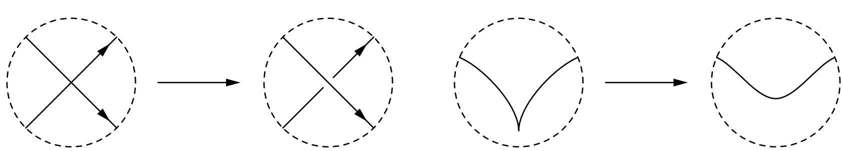

Figure 6. Converting from a juggling diagram to a braid via specified smoothings of cusps and crossings.

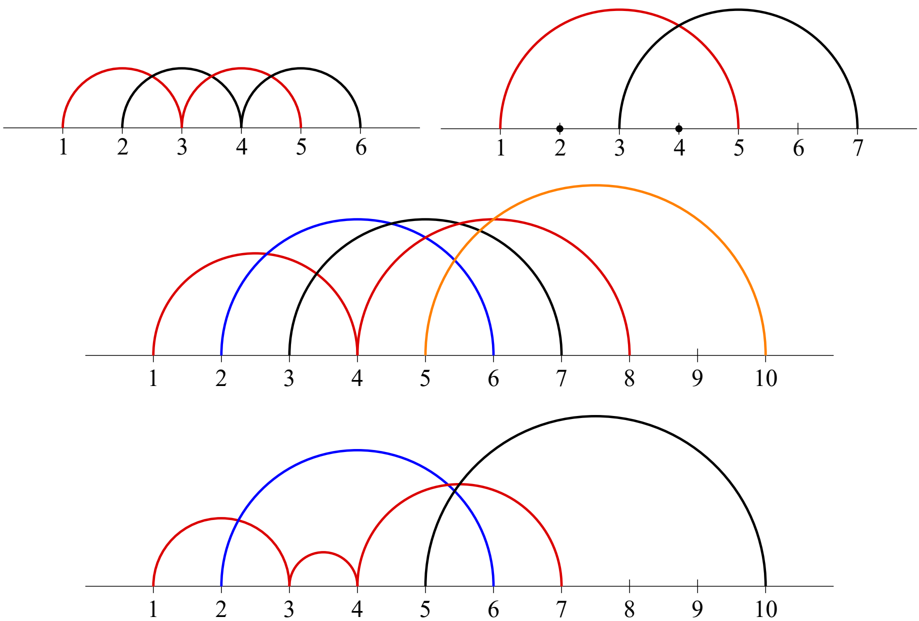

Figure 7. Examples of juggling diagrams and their corresponding juggling braid words. Top left: $J_2([3, 4, 5, 6]) = \sigma_1^3$. Top right: $J_2([5, 2, 7, 4]) = \sigma_1$. Middle: $J_4([4, 6, 7, 8, 10]) = \sigma_1\sigma_2\sigma_1\sigma_3(\sigma_2\sigma_1)^2$. Bottom: $J_3([3, 6, 4, 7, 10]) = \sigma_1\sigma_2\sigma_1^2$. Note that the dots in the top right indicate fixed points of the bounded affine permutation. The strands are different colors to increase visual clarity.

We will now set some notation. Let $\sigma_1, \ldots, \sigma_{k-1}$ denote the Artin generators of the braid group and let Br_k^+ be the submonoid of the braid group generated by non-negative powers of the Artin generators. We let $\Delta_k = (\sigma_1)(\sigma_2\sigma_1)\cdots(\sigma_{k-1}\cdots\sigma_1)$ denote the positive half twist. Let w_0 denote the image of Δ_k in the projection from the braid group to the symmetric group.

Definition 5.4 [Casals et al. 2021, Definition 3.3]. Let $f \in \mathrm{Bound}(k, n)$ be a bounded affine permutation, and let $J_k(f) \in \mathrm{Br}_k^+$ be its associated juggling braid. We define the *positroid link* of f, denoted by Λ_f, to be the Legendrian (-1)-closure (see Figure 8) of the positive braid $J_k(f)\Delta_k \in \mathrm{Br}_k^+$ with the orientation induced by giving all strands of $J_k(f) \in \mathrm{Br}_k^+$ the same orientation.

Remark 5.5. In [Casals et al. 2021], there are other (Legendrian isotopic) descriptions of Λ_f, using other enumerations of positroid strata of the complex Grassmannian such as pairs of permutations (satisfying some properties), Le diagrams, and cyclic rank matrices. For the scope of this article, it suffices to consider juggling braids.

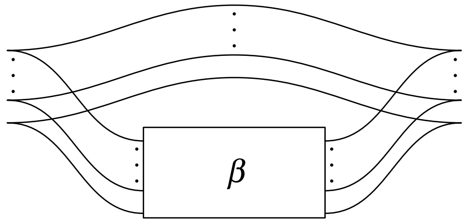

Figure 8. The front diagram of the Legendrian (-1)-closure of the positive braid word $\beta \in \mathrm{Br}_k^+$.

Lemma 5.6. *Let $f \in \mathrm{Bound}(k,n)$. The Thurston–Bennequin number of Λ_f is given by*

$$\mathrm{tb}(\Lambda_f) = |J_k(f)| - \frac{k(k+1)}{2},$$

where $|J_k(f)|$ denotes the length of the braid word $J_k(f) \in \mathrm{Br}_k^+$.

Proof. Recall that the Thurston–Bennequin number of a Legendrian link is given by the writhe minus the number of right cusps of a front diagram; see (2). There are $|J_k(f)| + |\Delta_k|$ positive crossings coming from the crossings in $\beta = J_k(f)\Delta_k$ and k right cusps in the positroid link of f. Note that $|\Delta_k| = \frac{k(k-1)}{2}$. There are $2|\Delta_k| = k(k-1)$ negative crossings coming from the portion of the positroid link of f outside of $\beta = J_k(f)\Delta_k$ in the Legendrian (-1)-closure diagram. The sum of these contributions is

$$\mathrm{tb}(\Lambda_f) = |J_k(f)| + \frac{k(k-1)}{2} - k(k-1) - k = |J_k(f)| - \frac{k(k+1)}{2}. \qquad \square$$

Corollary 5.7. *Let $f \in \mathrm{Bound}(k,n)$. The Thurston–Bennequin number of Λ_f is given by*

$$\mathrm{tb}(\Lambda_f) = \dim \Pi_f^\circ + \#\,\mathrm{Fix}(f) - n,$$

where $\mathrm{Fix}(f) = \{i \in \{1, \dots, n\} \mid f(i) = i\}$.

Proof. The statement of Lemma 3.10 in the first arXiv version of [Casals et al. 2021] states that

$$|J_k(f)| = |w| - |u| + \binom{k}{2} - (n-k) + \#\,\mathrm{Fix}(f),$$

where (u, w) is a pair of permutations called the positroid pair corresponding to the bounded affine permutation f (see [Casals et al. 2021, Definition 2.2] and [Knutson et al. 2013, Proposition 3.15]). It is well-known that $\dim \Pi_f^\circ = |w| - |u|$ (see, e.g., [Knutson et al. 2014, Corollary 3.2]); hence

$$(2) \qquad |J_k(f)| = \dim \Pi_f^\circ + \binom{k}{2} - (n-k) + \#\,\mathrm{Fix}(f).$$

Therefore we get

$$\begin{aligned}\operatorname{tb}(\Lambda_f) &= |J_k(f)| - \frac{k(k+1)}{2} \\ &= \dim \Pi_f^\circ + \binom{k}{2} - (n-k) + \#\operatorname{Fix}(f) - \frac{k(k+1)}{2} \\ &= \dim \Pi_f^\circ + \#\operatorname{Fix}(f) - n,\end{aligned}$$

where the first equality is by Lemma 5.6 and the second equality is by (2). □

The main motivation for calling Λ_f a positroid link is the following connection with positroid strata of the complex Grassmannian. To call upon this result, we follow the convention of placing a marked point on each strand in the braid $J_k(f)\Delta_k$, and place each to the right of all crossings in $J_k(f)\Delta_k$ on the respective strand when defining the augmentation variety associated to Λ_f, as in [Casals et al. 2020, Section 2.6].

Theorem 5.8 (Casals–Gorsky–Gorsky–Simental [Casals et al. 2020; 2021]). *Let $f \in \operatorname{Bound}(k,n)$ be a bounded affine permutation, and consider its positroid link Λ_f. Then, there is an algebraic isomorphism*

$$\Pi_f^\circ \cong \operatorname{Aug}(\Lambda_f) \times (\mathbb{C}^*)^{n-\#\operatorname{Fix} f - k}.$$

Proof. We have one marked point in Λ_f for each strand in the braid $J_k(f)\Delta_k$. Then by [Casals et al. 2020, Theorem 2.30] we have $\operatorname{Aug}(\Lambda_f) \cong X_0(J_k(f); w_0)$, where X_0 denotes the *braid variety* as defined in [Casals et al. 2020]. Then, by [Casals et al. 2021, Theorem 1.3] we have

$$\Pi_f^\circ \cong X_0(J_k(f); w_0) \times (\mathbb{C}^*)^{n-\#\operatorname{Fix} f - k},$$

which gives the result. □

Proposition 5.9. *For $f \in \operatorname{Bound}(k,n)$, the number of components of the link Λ_f is given by the number of cycles of f of length at least 2 (see Definition 3.9).*

Proof. Consider a cyclic juggling diagram of f which can be obtained from a juggling diagram by first restricting $\bigcup_{i=1}^n A_i(f) \subset \mathbb{R}^2$ to $\{1 \le x \le n\}$ and then extending each arc cyclically. More precisely, we first arrange the juggling diagram of f so that no crossing of $\bigcup_{i=1}^n A_i(f)$ belongs to $\{x \ge n\} \subset \mathbb{R}^2$ by a smooth isotopy of $\bigcup_{i=1}^n A_i(f)$ which leaves the braid word $J_k(f)$ unaffected (up to braid moves); see [Casals et al. 2021, Lemma 2.19]. Then we define the cyclic juggling diagram of f to be the subset

$$\bar{A}(f) := \left(\bigcup_{i=1}^n A_i(f) \cap \{1 \le x \le n\}\right) \cup \left(\bigcup_{\{i \mid f(i)>n\}} A_i^{\text{shift}}(f) \cap \{1 \le x \le n\}\right) \subset \mathbb{R}^2$$

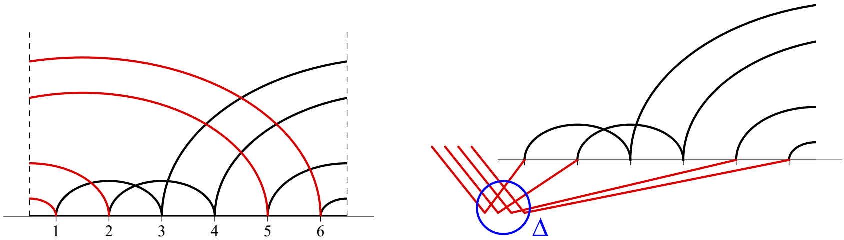

Figure 9. Left: Extending the arcs of [3, 4, 12, 11, 8, 7] cyclically to construct a cyclic juggling diagram. Right: The result of a smooth isotopy from the left diagram, where all the arcs are pulled downwards, demonstrating the half twist obtained from the added arcs of the cyclic juggling diagram.

where

$$A_i^{\text{shift}}(f) := \left\{(x, y) \in \mathbb{R}^2 \mid (2(x+n-1) - f(i) - i)^2 + 4y^2 = (f(i) - i)^2\right\} \cap \{y \geq 0\} \subset \mathbb{R}^2$$

is the arc $A_i(f)$ shifted to the left by $n-1$. Then, each cycle of f corresponds to a sequence of arcs that closes up onto itself in the cyclic juggling diagram. See Figure 9, left, for an example. We turn the cyclic juggling diagram $\bar{A}(f)$ into a braid by using the smoothing modifications of Figure 6 to obtain a cyclic juggling braid $\bar{J}_k(f)$. We take the (-1)-closure of the $\bar{J}_k(f)$. Then, the resulting link is smoothly isotopic to the (-1)-closure of the juggling braid $\Delta_k J_k(f)$, i.e., the link Λ_f; see Figure 9, right. Thus the number of components of Λ_f is exactly the number of cycles of f. □

6. Construction of the Lagrangian cobordisms

We say that there is a *path* from f to g in Bound(k, n) if there is a sequence of affine bounded permutations $(h_1, \ldots, h_k)$ (this sequence might be empty) such that

$$f \lessdot h_1 \lessdot \cdots \lessdot h_k \lessdot g.$$

Theorem 6.1. *Given any path from f to g in* Bound(k, n), *there is an exact Lagrangian cobordism from Λ_g to Λ_f.*

Proof. Recall from Definition 5.2 that any bounded affine permutation f corresponds to a juggling braid $J_k(f)$ which corresponds by Definition 5.4 to a Legendrian Λ_f given by the (-1)-closure of the positive braid $J_k(f)\Delta_k$. There is a convenient Lagrangian projection of Λ_f; see [Casals and Ng 2022, Figure 8]. Since the positive braid $J_k(f)\Delta_k$ contains a positive half twist Δ_k, every crossing in the Lagrangian projection of Λ_f corresponds to a contractible Reeb chord; see [Casals and Ng 2022, Proposition 2.8]. If two affine permutations f and g have the same juggling

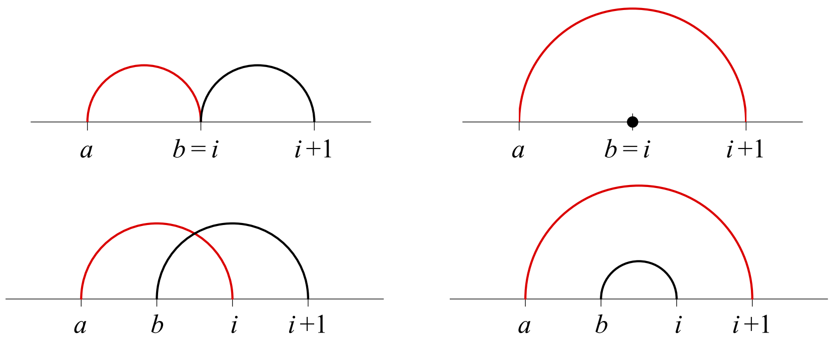

Figure 10. Top: Arcs in the juggling diagrams of f (left) and g (right) when $a < b = i$ where $f \lessdot g$. Bottom: Arcs in the juggling diagrams of f (left) and g (right) when $a < b < i$, where $f \lessdot g$.

braids $J_k(f) = J_k(g)$, then $\Lambda_f = \Lambda_g$ by definition. Suppose now that two affine permutations f and g have juggling braids $J_k(f)$ and $J_k(g)$ such that $J_k(f)$ has one more positive crossing x than $J_k(g)$. By Ng's resolution procedure, the crossing x in the front projection corresponds to a contractible Reeb chord of Λ_f, and we can perform a pinch move at x as in Figure 2 to obtain a Lagrangian saddle cobordism from Λ_g to Λ_f.

Let $f, g \in \mathrm{Bound}(k, n)$ with $f < g$. It suffices to assume $f \lessdot g$, that is, $g = \sigma_i \circ f$ or $g = f \circ \sigma_i$. Recall from Lemma 3.2 that h is a bounded affine permutation if and only if $-(-h)^{-1}$ is. Thus, because $g = f \circ \sigma_i$ is equivalent to $-(-f)^{-1} = \sigma_i \circ (-(-g)^{-1})$, it suffices to consider the case $g = \sigma_i \circ f$.

Namely, assume $g(a) = i+1$, $g(b) = i$, $f(a) = i$ and $f(b) = i+1$ for some $a, b, i \in \mathbb{Z}_{\geq 1}$ such that $a < b$. We show that $J_k(f)$ has one more positive crossing than $J_k(g)$ does or $J_k(f) = J_k(g)$. Therefore, there is either an orientable exact Lagrangian saddle cobordism from Λ_g to Λ_f, or the Legendrian links Λ_g and Λ_f are Legendrian isotopic and so are related by a trivial exact Lagrangian cobordism.

Since $g(b) = i$, we know $b \leq i$ so as $a < b$, we have $a < b \leq i$. If $b = i$, the respective juggling diagrams of f and g contain the arcs shown in Figure 10, top. Thus we see that the juggling braids $J_k(f)$ and $J_k(g)$ are equal as braids. If $b < i$, the juggling diagrams of f and g contain the arcs shown in Figure 10, bottom, from which we can immediately conclude that the juggling braid $J_k(f)$ has one more crossing than the juggling braid $J_k(g)$. □

Remark 6.2. In view of Theorem 5.8, a discussion on marked points in the construction of the exact Lagrangian cobordisms in the proof of Theorem 6.1 is warranted. Since both $J_k(f)\Delta_k$ and $J_k(g)\Delta_k$ are k-stranded braids, their Legendrian (-1)-closures are decorated with one marked point per strand of the underlying braid. Any trivial exact Lagrangian cobordism remains trivial when taking marked points into account. Any saddle cobordism induced by a pinch move will involve newly created marked points in order to retain functoriality of the associated Chekanov–Eliashberg dg-algebras with coefficients in $\mathbb{C}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$; see [Casals and Ng 2022, Section 3.5] and [Gao et al. 2024, Section 2.4]. For our purpose, we will ignore the marked points created by pinch moves by evaluating them to 1.

Theorem 6.3. *Given any path γ from f to g in* $\mathrm{Bound}(k, n)$, *the corresponding exact Lagrangian cobordism $L_\gamma(g, f)$ from the proof of Theorem 6.1 satisfies*

$$\chi(L_\gamma(g, f)) = \dim(\Pi_g^\circ) - \dim(\Pi_f^\circ) + \#\,\mathrm{Fix}(g) - \#\,\mathrm{Fix}(f),$$

where Π_f° is the open positroid stratum associated to f.

Proof. Let $\gamma = (f_1, \dots, f_t)$ be a sequence of bounded affine permutations such that each pair of adjacent bounded affine permutations are related by an affine transposition, in other words, $f_i \lessdot f_{i-1}$ for all $1 < i \le t$. So $L_\gamma(f_t, f_1)$ is the corresponding Lagrangian cobordism. For each i, the dimensions of the respective positroid strata differ by 1. Then following the construction, the exact Lagrangian cobordism corresponding to $f_i \lessdot f_{i-1}$ is one of two things:

1. A trivial cobordism, when the change in the juggling diagrams $J_{f_{i-1}}$ to J_{f_i} is the creation of a fixed point. This contributes 1 to $\#\,\mathrm{Fix}(f_t) - \#\,\mathrm{Fix}(f_1)$.
2. A saddle cobordism corresponding to a single pinch move, when the change is the removal of a crossing. This contributes 1 to $\chi(L(\gamma))$.

Thus,

$$\dim(\Pi_{f_1}^\circ) - \dim(\Pi_{f_t}^\circ) = \chi(L(\gamma)) + \#\,\mathrm{Fix}(f_t) - \#\,\mathrm{Fix}(f_1). \qquad \square$$

Remark 6.4. Theorem 6.3 can also be proved using Corollary 5.7 and the work of Chantraine [2010, Theorem 1.2] which provides the change in the Thurston–Bennequin number for Legendrians related by an exact orientable Lagrangian cobordism.

7. Examples

In Example 7.1 we provide an example of Theorem 1.1 and in Example 7.2 a counterexample to the converse of Theorem 1.1.

Example 7.1. We consider a path $f_1 \lessdot \cdots \lessdot f_7$ in the poset $\mathrm{Bound}(3, 8)$ where the bounded affine permutations $f_1, \dots, f_7$ are defined as

$$f_1 = [5, 4, 7, 6, 8, 9, 10, 11], \quad f_2 = [5, 4, 8, 6, 7, 9, 10, 11],$$
$$f_3 = [5, 4, 8, 7, 6, 9, 10, 11], \quad f_4 = [6, 4, 8, 7, 5, 9, 10, 11],$$
$$f_5 = [6, 3, 8, 7, 5, 9, 10, 12], \quad f_6 = [6, 2, 8, 7, 5, 9, 11, 12],$$
$$f_7 = [7, 2, 8, 6, 5, 9, 11, 12].$$

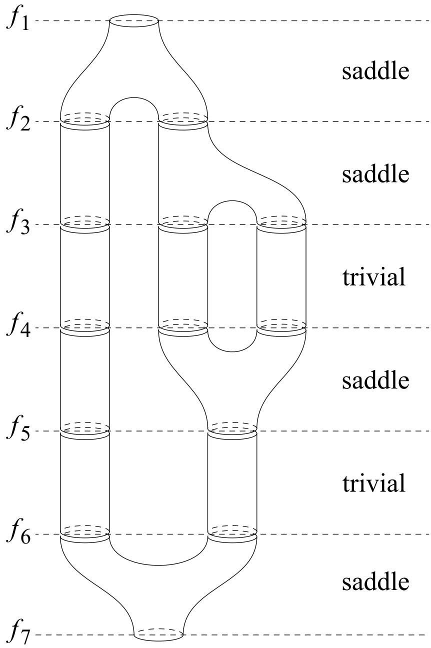

Figure 11. The exact Lagrangian cobordism from Λ_{f_7} to Λ_{f_1} corresponding to $f_1 \lessdot \cdots \lessdot f_7$.

Each f_i corresponds to a positroid stratum of Gr(3, 8) of codimension $i+1$ and a Legendrian link Λ_{f_i}, each of which is the (-1)-closure of the corresponding juggling braid $J_3(f_i)$. The corresponding juggling braids are

$$\begin{aligned} J_3(f_1) &= (\sigma_1\sigma_2)^4\sigma_2\sigma_1\sigma_2, & J_3(f_2) &= (\sigma_1\sigma_2)^3\sigma_2^2\sigma_1\sigma_2, \\ J_3(f_3) &= (\sigma_1\sigma_2)^3\sigma_2\sigma_1\sigma_2, & J_3(f_4) &= (\sigma_1\sigma_2)^3\sigma_2\sigma_1\sigma_2, \\ J_3(f_5) &= (\sigma_1\sigma_2)^3\sigma_2\sigma_1, & J_3(f_6) &= (\sigma_1\sigma_2)^3\sigma_2\sigma_1, \\ J_3(f_7) &= (\sigma_1\sigma_2)^3\sigma_1. \end{aligned}$$

We have depicted the corresponding composition of exact Lagrangian cobordisms in Figure 11.

As noted above we have $\operatorname{codim} \Pi^\circ_{f_i} = i+1$. Because we see that $\#\operatorname{Fix}(f_1) = 0$ and $\#\operatorname{Fix}(f_7) = 2$, Theorem 6.3 gives $\chi(L) = -4$, which correctly predicts that the exact Lagrangian cobordism depicted in Figure 11 has genus 2.

Example 7.2. We now show that the positroid links corresponding to two incomparable positroid strata can still be exact Lagrangian cobordant; this is the converse to Theorem 1.1.

Consider the two bounded affine permutations $f_1, f_2 \in \operatorname{Bound}(2, 6)$ defined by

$$f_1 := [3, 4, 5, 7, 8, 6] \quad \text{and} \quad f_2 := [3, 4, 7, 5, 6, 8].$$

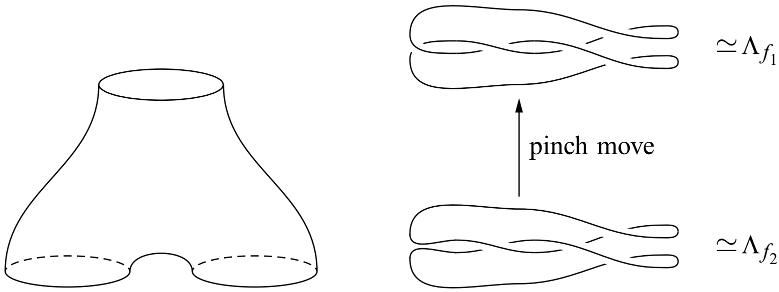

Figure 12. Pinch move giving a saddle cobordism from the Hopf link to the trefoil.

The corresponding juggling braids are $J_2(f_1) = \sigma_1^4$ and $J_2(f_2) = \sigma_1^3$. The two corresponding positroid links are the trefoil and the Hopf link, respectively. The two bounded affine permutations f_1 and f_2 correspond to two different positroid strata in $\mathrm{Gr}(2, 6)$ of dimension 6 and are therefore incomparable. However, there is an exact Lagrangian cobordism from Λ_{f_2} to Λ_{f_1} given by a saddle cobordism obtained by performing a pinch move at one of the crossings; see Figure 12.

Remark 7.3. In Example 7.2 we show that the braids σ_1^4 and σ_1^3 may appear as juggling braids of two incomparable bounded affine permutations. They also appear as the juggling braids $J_2(g_1)$ and $J_2(g_2)$, respectively, for $g_1, g_2 \in \mathrm{Bound}(2, 5)$ defined as

$$g_1 := [3, 4, 5, 6, 7] \quad \text{and} \quad g_2 := [4, 3, 5, 6, 7],$$

which *are* comparable. Namely $g_1 \lessdot g_2$ since $g_2 = g_1 \circ \sigma_1$ and $\ell(g_1) = 0 < 1 = \ell(g_2)$.

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LIOUVILLE EQUATIONS ON COMPLETE SURFACES WITH NONNEGATIVE GAUSS CURVATURE

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We study finite total curvature solutions of the Liouville equation $\Delta u + e^{2u} = 0$ on a complete surface (M, g) with nonnegative Gauss curvature. It turns out that the asymptotic behavior of the solution separates into two extremal cases: on the one end, if the solution decays not too fast, then (M, g) must be isometric to the standard Euclidean plane; on the other end, if (M, g) is isometric to the flat cylinder $S^1 \times \mathbb{R}$, then solutions must decay linearly and can be completely classified.

1. Introduction

In their seminal work [1991], Chen and Li obtained the radial symmetry of the solution of

$$\Delta u + e^{2u} = 0 \tag{1-1}$$

on $\mathbb{R}^2$, provided that $\int_{\mathbb{R}^2} e^{2u}\, dx < \infty$. Putting the center of symmetry at the origin and up to a rescaling, we have

$$u(x) = \ln \frac{2}{1+|x|^2}.$$

The geometric meaning of above equation is that the conformal metric $g = e^{2u} g_0$ has constant Gauss curvature 1. It is tempting to think that g is isometric to the standard round sphere. It is indeed true as the solution is the pull back of the round metric via stereographic projection. Nevertheless this line of reasoning is valid only if one establishes the precise asymptotic behavior of u at ∞, so that the metric extends to a smooth metric on the sphere from $\mathbb{R}^2$. The readers are referred to [Li and Tang 2020] for this line of reasoning; see also [Gui and Li 2021] regarding metric completion of solutions to more general equations.

The assumption $\int_{\mathbb{R}^2} e^{2u}\, dx < \infty$ is natural since there are infinitely many solutions to (1-1) with $\int_{\mathbb{R}^2} e^{2u}\, dx = \infty$. One way to obtain such a solution is to pull back the

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spherical metric via a univalent holomorphic map from $\mathbb{C}$ to $\overline{\mathbb{C}}$. Recently, there appeared some interesting studies on (1-1) subject to $\int_{\mathbb{R}^2} e^{2u}\,dx = \infty$. Eremenko, Gui, Li and Xu [Eremenko et al. 2022] give a complete classification of solutions of (1-1) which are bounded from above. We also refer to [Gui and Li 2023; Bergweiler et al. 2023; Lytchak 2023] for some studies on (1-1) from a geometric point of view.

The story in higher dimensions was accomplished even earlier. For $n \geq 3$, let u be a positive solution of

$$\Delta u + u^{(n+2)/(n-2)} = 0. \tag{1-2}$$

We refer to it as the scalar curvature equation as the conformal metric $g = u^{4/(n-2)} g_0$ has positive constant scalar curvature. Gidas, Ni and Nirenberg [Gidas et al. 1981] first proved the radial symmetry of the solutions under the assumption $u(x) \sim O(|x|^{2-n})$ as $|x| \to \infty$. This can be viewed as an analytical proof of a famous result of Obata on the classification of constant scalar curvature metrics which are conformal to an Einstein metric. In a remarkable paper [Caffarelli et al. 1989], Caffarelli, Gidas and Spruck established the radial symmetry of the solution without any assumption on the asymptotic behavior of u.

The scalar curvature equation for conformal metrics has critical Sobolev power. In the subcritical case,

$$\Delta u + u^p = 0, \qquad 1 < p < \frac{n+2}{n-2}. \tag{1-3}$$

Gidas and Spruck [1981] showed that any nonnegative solution must be trivial. Recently, Catino and Monticelli [2024] carried out a systematic study of (1-1)–(1-3) on complete manifolds with nonnegative Ricci curvature. Among many results, one particular case is a full extension of Caffarelli, Gidas and Spruck's result in dimension three to complete manifolds with nonnegative Ricci curvature.

Inspired by Catino and Monticelli's work, we study the Liouville equation (1-1) on complete surfaces with nonnegative Gauss curvature; in particular, we are able to connect the asymptotic behavior of the solution with the underlying manifold.

To be more precise, let (M, g) be a complete surface with nonnegative Gauss curvature. We study the Liouville equation

$$\Delta_g u + e^{2u} = 0 \tag{1-4}$$

on M. A solution is said to have *finite total curvature* if $\int_M e^{2u}\,dg < \infty$.

In view of the Cohn-Vossen splitting theorem, a complete surface (M, g) with nonnegative Gauss curvature is

- either isometric to the flat cylinder $S^1 \times \mathbb{R}$ (orientable case) or the flat Möbius band (nonorientable case),
- or diffeomorphic to $(\mathbb{R}^2, g_0)$.

In the latter case, by Huber's theorem [1957], (M, g) is conformal to $(\mathbb{R}^2, g_0)$.

Without loss of generality, we assume from now on that M is orientable. In the former case, we have the following classification of solutions to (1-4).

Theorem 1. *Let u be a solution of (1-4) with finite total curvature on the flat cylinder $(S^1 \times \mathbb{R}, g_{\text{prod}})$. Then there exists $\mu \in [0, \infty)$ and $\beta \in (-1, \infty)$, such that either β is an integer or $\mu = 0$, and up to a rescaling, we have*

$$e^{2u(z)} = \frac{(2\beta+2)^2|z|^{2\beta+2}}{(|1+\mu z^{\beta+1}|^2 + |z|^{2\beta+2})^2} \quad \text{on } \left(\mathbb{C} - \{0\}, \frac{1}{|z|^2} g_0\right).$$

The classification result is not new. Since the Gauss curvature for the flat cylinder is identically zero, (1-4) has a geometric meaning that the conformal metric $e^{2u} g_{\text{prod}}$ has Gauss curvature 1. Note the flat cylinder is conformal to $(\mathbb{R}^2 \setminus \{0\}, g_0)$; thus (1-4) can be translated to the Liouville equation on $\mathbb{R}^2 \setminus \{0\}$. Then the theorem follows from a combination of results of Chou and Wan [1994, Theorem 5], Chen and Li [1995, Theorem 3.1] and Troyanov [1989, Theorem II].

Our main theorem is the following rigidity result.

Theorem 2. *Let u be a solution of (1-4) with finite total curvature on a complete surface (M, g) with nonnegative Gauss curvature. Let $r(x)$ be the distance function on M with respect to a fixed point. If $u(x) \geq -2\ln r(x) + o(\ln r(x))$, for $r(x)$ large, then (M, g) must be isometric to $(\mathbb{R}^2, g_0)$. Moreover, -2 is optimal in the sense that there exists nonflat (M, g) which admits solutions satisfying $u(x) \sim \gamma \ln r(x)$ for any $\gamma < -2$.*

A similar result has been proved in [Catino and Monticelli 2024, Theorem 1.10]. Our contribution here has two-folds. On the one hand, our assumption on u is weaker than that in [Catino and Monticelli 2024] and our treatment emphasizes the analysis of asymptotic behavior of the solution which helps to identify the threshold where the rigidity occurs. On the other hand, by setting the stage on the complete surfaces with nonnegative Gauss curvature, we unite two works of Chen and Li [1991; 1995].

The strategy of our proof is study the asymptotic behavior of the solution. If (M, g) is conformal to $(\mathbb{R}^2, g_0)$, we write $g = e^{2f} g_0$. Then (1-4) becomes

$$\Delta u + e^{2f} e^{2u} = 0 \quad \text{on } \mathbb{R}^2. \tag{1-5}$$

This is the so-called prescribing Gauss curvature equation on $\mathbb{R}^2$, which has been investigated intensively over the past few decades. Under a suitable decay assumption of e^{2f} near infinity, Cheng and Lin [1997, Theorem 1.1] showed that the solution u of (1-5) has the asymptotic behavior

$$\lim_{x\to\infty} \frac{u(x)}{\ln|x|} = -\frac{1}{2\pi}\left(\int_{\mathbb{R}^2} e^{2f} e^{2u}\, dx\right)$$

if and only if $\int_{\mathbb{R}^2} e^{2f} e^{2u}\, dx < \infty$. However, a priori, there is not any decay control for e^{2f}. In fact, f satisfies the similar equation

$$\Delta f + K_g e^{2f} = 0,$$

where K_g is the Gauss curvature of g. The only information here is that $K_g \geq 0$. Nevertheless, using Arsove and Huber's result [1973], there exists an $m \in [0, 1]$ and an exceptional set E which is thin at infinity such that

$$\lim_{\substack{x\to\infty \\ x\notin E}} \frac{f(x)}{\ln|x|} = \liminf_{x\to\infty} \frac{f(x)}{\ln|x|} = -m. \tag{1-6}$$

Here the thinness of a set at infinity is a concept concerning the logarithmic capacity. For a complete conformal metric $e^{2f} g_0$ on $\mathbb{R}^n$ $(n \geq 3)$ with nonnegative Ricci curvature, Ma and Qing [2021] obtained a similar asymptotic behavior for the conformal factor f.

While Cheng and Lin's and Arsove and Huber's works are the main analytical inspirations for us, we also benefit from two interesting geometric ingredients: the first is Li and Tam's work [1991] on a comparison between the intrinsic distance and the Euclidean distance on $(\mathbb{R}^2, e^{2f} g_0)$ (see Lemma 2.2) and the second is an isoperimetric inequality on complete surfaces with nonnegative Gauss curvature established recently by Brendle [2023] (see Lemma 2.3).

We shall present proofs of Theorems 1 and 2 in the next section.

2. Proof of the main theorem

Proof of Theorem 1. The flat cylinder $\mathbb{S}^1 \times \mathbb{R}$ is conformal to $(\mathbb{R}^2 \setminus \{0\}, g_0)$ since

$$dt^2 + d\theta^2 = \frac{1}{r^2} dr^2 + d\theta^2 = \frac{1}{r^2} g_0,$$

by setting $t = \ln r$. Let $e^{2w(x)} = (1/|x|^2) e^{2u(x)}$. Then $\Delta_g u + e^{2u} = 0$ is equivalent to

$$\begin{cases} \Delta w + e^{2w} = 0 & \text{on } \mathbb{R}^2 \setminus \{0\}, \\ \displaystyle\int_{\mathbb{R}^2} e^{2w(x)}\, dx < \infty. \end{cases} \tag{2-1}$$

Chou and Wan's complex analysis argument [1994, Theorem 5] shows that

$$w(x) = \beta_1 \ln|x| + O(1) \quad \text{as } x \to 0, \text{ for some } \beta_1 > -1.$$

Let $\widetilde{w}(x) = w(x/|x|^2) - 2\ln|x|$, it is easy to see that $\widetilde{w}$ satisfies

$$\begin{cases} \Delta \widetilde{w} + e^{2\widetilde{w}} = 0 & \text{on } \mathbb{R}^2 \setminus \{0\}, \\ \displaystyle\int_{\mathbb{R}^2} e^{2\widetilde{w}(x)}\, dx < \infty. \end{cases}$$

Applying Chou and Wan's asymptotic result [1994, Theorem 5] to $\widetilde{w}$ and tracing back to w, we get

$$w(x) = \beta_2 \ln|x| + O(1) \quad \text{as } x \to \infty, \text{ for some } \beta_2 < -1.$$

Therefore, $w(x)$ is a solution of (2-1) with conical singularities at $x = 0$ and $x = \infty$. Hence the classification result of Troyanov [1989, Theorem II] yields that there exists $\mu \in [0, \infty)$ and $\beta \in (-1, \infty)$ such that either β is an integer or $\mu = 0$, and up to a rescaling, we have

$$e^{2w(z)} = \frac{(2\beta+2)^2 |z|^{2\beta}}{(|1+\mu z^{\beta+1}|^2 + |z|^{2\beta+2})^2} \quad \text{on } \mathbb{C} - \{0\}.$$

Then the desired result follows since $e^{2u(z)} = |z|^2 e^{2w(z)}$. Note if both cone angles are less than 2π ($\beta \in (-1, 0)$), Chen and Li [1995, Theorem 3.1] also obtained such a classification. □

Next, we give the complete proof of Theorem 2.

First we exclude the case of a flat cylinder in Theorem 2: Suppose u is a finite total curvature solution of (1-4) on the flat cylinder. Then Theorem 1 implies

$$u(x) \sim -(\beta+1) r(x) \quad \text{for } r(x) \text{ large},$$

where $\beta > -1$ is a constant. This is a contradiction with the assumption that $u(x) \geq -2\ln r(x) + o(\ln r(x))$ for $r(x)$ large. In conclusion, (M, g) cannot be the flat cylinder and thus is conformal to $(\mathbb{R}^2, g_0)$ by the Cohn-Vossen splitting theorem and Huber's theorem.

We write $g = e^{2f} g_0$. Then the finite total curvature solution u of (1-4) becomes

$$\begin{cases} \Delta u + e^{2f} e^{2u} = 0 & \text{on } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} e^{2f+2u}\, dx < \infty. \end{cases} \tag{2-2}$$

To fix the notation, we consider the quantity

$$\alpha := -\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{2f+2u}\, dx. \tag{2-3}$$

The strategy of our proof is as follows: using the asymptotic lower bound assumption of the solution u, we establish a lower bound of α by analyzing carefully the asymptotic upper bound of the solution to (2-2). On the other hand, with the help of Brendle's isoperimetric inequality, we prove that the reversed inequality still holds. Hence the equality is obtained and the rigidity part of the isoperimetric inequality brings the rigidity of the underlying manifold.

First, we aim at getting the lower bound of α. It is tempting to obtain a pointwise upper bound of the solution u to (2-2) in terms of α so that the lower bound assumption on u could imply immediately the lower bound of α. However, due to the lack of a uniform asymptotic behavior of the conformal factor f, it's impossible to derive such a pointwise bound for u. Instead, we shall give an upper bound of the integral average of u on small balls. The argument is based on that of [Cheng and Lin 1997].

Lemma 2.1. *Let* $(M, g) = (\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative Gauss curvature. Assume* $u \in C^2(\mathbb{R}^2)$ *is a solution to* (2-2). *Then for any* $\epsilon > 0$, $\sigma > 0$, *there exists* $R > 0$ *such that for* $|x| \geq R$ *and* $\rho = |x|^{-\sigma}$, *there holds*

$$\frac{1}{\pi\rho^2}\int_{B_\rho(x)} u(y)\,dy \leq (\alpha+\epsilon)\ln|x| + C,$$

where α *is given by* (2-3) *and* C *is a constant depending on* ϵ, σ, R.

Proof. Construct an auxiliary function

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \psi(y)\ln\frac{|x-y|}{|y|}\,dy,$$

where $\psi(y) = e^{2f(y)+2u(y)}$.

The proof consists of three claims:

(1) $v(x) \leq -\alpha\ln|x| + C$ for $|x| \geq 2$.

(2) $u + v$ is a constant.

(3) For any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$,

$$u(x) \leq (\alpha+\epsilon)\ln|x| + \frac{1}{2\pi}\int_{B_\rho(x)} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + C. \tag{2-4}$$

Proof of claim (1): For fixed x with $|x| \geq 2$,

$$\begin{aligned}2\pi v(x) &= \int_{T_1}\psi(y)\ln\frac{|x-y|}{|y|}\,dy + \int_{T_2}\psi(y)\ln\frac{|x-y|}{|y|}\,dy + \int_{T_3}\psi(y)\ln\frac{|x-y|}{|y|}\,dy\\ &\overset{\text{def}}{=} I_1 + I_2 + I_3,\end{aligned}$$

where

$$\begin{aligned}T_1 &= \{y : |y| \leq 2\},\\ T_2 &= \left\{y : |y-x| \leq \frac{|x|}{2},\ |y| \geq 2\right\},\\ T_3 &= \left\{y : |y-x| \geq \frac{|x|}{2},\ |y| \geq 2\right\}.\end{aligned}$$

For $|x| \ge 2$ and $y \in T_1$, we have $\ln|x-y| \le \ln(|x|+2) \le \ln|x| + \ln 2$. Thus

$$\begin{aligned} I_1 &= \int_{T_1} \psi(y)\ln|x-y|\,dy - \int_{T_1} \psi(y)\ln|y|\,dy \\ &\le \int_{T_1} \psi(y)(\ln|x|+\ln 2)\,dy - \int_{T_1} \psi(y)\ln|y|\,dy \\ &= (\ln|x|)\int_{T_1} \psi(y)\,dy + C. \end{aligned}$$

Now for $y \in T_2$, we have $|x-y| \le |x|/2 \le |y|$. Thus

$$I_2 \le 0.$$

For $y \in T_3$ and $|x| \ge 2$, there holds $|x-y| \le |x| + |y| \le |x|\,|y|$. Therefore

$$I_3 \le (\ln|x|)\int_{T_3} \psi(y)\,dy.$$

We conclude that

$$2\pi v(x) = I_1 + I_2 + I_3 \le -2\pi\alpha \ln|x| + C.$$

The proof of claim (1) is finished.

Proof of claim (2): It is easy to see that $\Delta v = e^{2f+2u}$ and $u+v$ is a harmonic function on $\mathbb{R}^2$. Hence there exists an entire function $f(z)$ such that $\operatorname{Re} f = 2(u+v)$. Let $F(z) = e^{f(z)}$. Clearly, by claim (1) we get

$$|F(z)| = e^{2u+2v} \le C|z|^{-2\alpha} e^{2u},$$

for $|z| \ge 2$. Using the lower bound (1-6) for the conformal factor f ($e^{2f} \ge |z|^{-2m}$), we get that for some R_0 large enough,

$$\int_{|z|\ge R_0} |F(z)|\,|z|^{2\alpha}|z|^{-2m}\,dx \le C\int_{|z|\ge R_0} e^{2u}e^{2f}\,dx < \infty.$$

Let $M(\rho) = \max_{|z|=\rho} |F(z)|$. We shall show that $M(\rho) \le C\rho^{2m-2\alpha}$ for $\rho \ge R_0 + 1$. In fact, assume $|z_0| = \rho$ and $M(\rho) = |F(z_0)|$. The mean value property implies

$$|F(z_0)| \le \frac{1}{\pi}\int_{B_1(z_0)} |F(z)|\,dx \le \frac{1}{\pi}\int_{\rho-1\le|z|\le\rho+1} |F(z)|\,dx.$$

Hence we get

$$\begin{aligned} M(\rho)\rho^{2\alpha-2m} &\le \frac{1}{\pi}\int_{\rho-1\le|z|\le\rho+1} |F(z)|\rho^{2\alpha-2m}\,dx \\ &\le \frac{1}{\pi}\int_{\rho-1\le|z|\le\rho+1} |F(z)|\,|z|^{2\alpha-2m}\left(\frac{\rho}{\rho+1}\right)^{2\alpha-2m}\,dx \\ &\le \frac{2^{2m-2\alpha}}{\pi}\int_{|z|\ge\rho-1} |F(z)|\,|z|^{2\alpha-2m}\,dx < \infty. \end{aligned}$$

Therefore, the order of the entire function $F(z)$ is

$$\lambda := \limsup_{\rho\to\infty} \frac{\ln\ln M(\rho)}{\ln\rho} = 0.$$

By a theorem of Hadamard (see Theorem 8 on p. 209 in [Ahlfors 1978]), we conclude that the genus of $F(z)$ is zero and $F(z)$ is a constant since F has no zeros. The proof of claim (2) is completed.

Proof of claim (3): For any $\epsilon > 0$, $\sigma > 0$, choose $R > 0$ large enough such that

$$(\sigma+1)\int_{|y|\ge R} \psi(y)\,dy \le \pi\epsilon,$$

where $\psi(y) = e^{2f(y)+2u(y)}$. By claim (2), we have

$$\begin{aligned} 2\pi u(x) &= C + \int_{\mathbb{R}^2} \psi(y)\ln\frac{|y|}{|x-y|}\,dy \\ &= C + \int_{\widetilde{T}_1} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{\widetilde{T}_2} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{\widetilde{T}_3} \psi(y)\ln\frac{|y|}{|x-y|}\,dy \\ &\overset{\text{def}}{=} \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3, \end{aligned}$$

where

$$\begin{aligned} \widetilde{T}_1 &= \{y : |y| \le R\}, \\ \widetilde{T}_2 &= \left\{y : |y-x| \le \frac{|x|}{2},\ |y| \ge R\right\}, \\ \widetilde{T}_3 &= \left\{y : |y-x| \ge \frac{|x|}{2},\ |y| \ge R\right\}. \end{aligned}$$

Now for $|x| \ge R^2/(R-1)$ and $y \in \widetilde{T}_1$, we have $\ln|x-y| \ge \ln(|x|-R) \ge \ln|x| - \ln R$. Thus

$$\begin{aligned} \tilde{I}_1 &= \int_{\widetilde{T}_1} \psi(y)\ln|y|\,dy - \int_{\widetilde{T}_1} \psi(y)\ln|x-y|\,dy \\ &\le C - (\ln|x|)\int_{\widetilde{T}_1} \psi(y)\,dy + (\ln R)\int_{\widetilde{T}_1} \psi(y)\,dy \\ &\le -(\ln|x|)\int_{\widetilde{T}_1} \psi(y)\,dy + C. \end{aligned}$$

To estimate $\tilde{I}_2$, let $\widetilde{T}^\sigma = \{y : |y-x| \le |x|^{-\sigma},\ |y| \ge R\}$. Then we have

$$\begin{aligned} \tilde{I}_2 &= \int_{\widetilde{T}^\sigma} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{|x|^{-\sigma}\le|y-x|\le|x|/2,\,|y|\ge R} \psi(y)\ln\frac{|y|}{|x-y|}\,dy \\ &\le \int_{\widetilde{T}^\sigma} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + \int_{|y|\ge R} \psi(y)\ln\frac{\frac{3}{2}|x|}{|x|^{-\sigma}}\,dy \\ &\le \int_{|y-x|\le|x|^{-\sigma}} \psi(y)\ln\frac{|y|}{|x-y|}\,dy + (\sigma+1)\int_{|y|\ge R} \psi(y)\,dy + C. \end{aligned}$$

Now for $y \in \widetilde{T}_3$, one easily gets $|y| \le 4|x-y|$. Therefore

$$\tilde{I}_3 = \int_{\widetilde{T}_3} \psi(y) \ln \frac{|y|}{|x-y|}\, dy \le (\ln 4) \int_{\widetilde{T}_3} \psi(y)\, dy \le C.$$

In conclusion, for $|x| \ge R^2/(R-1)$, there holds

$$\begin{aligned}
2\pi u(x) &= \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 \\
&\le C - (\ln|x|) \int_{|y|\le R} \psi(y)\, dy + \int_{|y-x|\le |x|^{-\sigma}} \psi(y) \ln \frac{|y|}{|x-y|}\, dy \\
&\qquad\qquad + (\sigma+1) \int_{|y|\ge R} \psi(y)\, dy \\
&\le C + 2\pi(\alpha+\epsilon) \ln|x| + \int_{|y-x|\le |x|^{-\sigma}} \psi(y) \ln \frac{|y|}{|x-y|}\, dy.
\end{aligned}$$

The proof of claim (3) is completed.

Finally, we give the upper bound of the integral average of u. By Green's formula,

$$u(x) = \frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy + \frac{1}{2\pi} \int_{B_\rho(x)} \psi(y) \ln \frac{\rho}{|x-y|}\, dy,$$

for every $x \in \mathbb{R}^2$ and $\rho > 0$. Combined with (2-4), we have for any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \ge R$ and $\rho = |x|^{-\sigma}$,

$$\frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy \le (\alpha+\epsilon) \ln|x| + \frac{1}{2\pi} \int_{B_\rho(x)} \psi(y) \ln \frac{|y|}{\rho}\, dy + C. \tag{2-5}$$

Since $|y|/\rho \le (|x|+\rho)/\rho = |x|^{\sigma+1} + 1 \le |x|^{\sigma+2}$ for $|x|$ large enough, the second term in the right-hand side of (2-5) could be estimated as

$$\frac{1}{2\pi} \int_{B_\rho(x)} \psi(y) \ln \frac{|y|}{\rho}\, dy \le \frac{\sigma+2}{2\pi} (\ln|x|) \int_{|y|\ge R/2} \psi(y)\, dy \le \epsilon \ln|x|,$$

for $|x| \ge R$ provided R is large enough. Inserting this into (2-5), the proof of the lemma is completed. □

To derive the lower bound of α, we need a lower bound of u in terms of the Euclidean distance $\ln|x|$ rather than the intrinsic distance $\ln r(x)$ that appeared in the hypotheses of Theorem 2. Fortunately, the comparison of these two distances is established by Li and Tam [1991, Corollary 3.3]. Hartman [1964, Theorem 7.1] revealed the connection between this limit and the asymptotic volume ratio of the manifold. Their results are combined as follows.

Lemma 2.2 (Hartman, Li–Tam). *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete manifold with nonnegative Gauss curvature* K. *Then*

$$\lim_{x\to\infty} \frac{\ln r(x)}{\ln |x|} = 1 - \frac{1}{2\pi} \int_{\mathbb{R}^2} K\, dg = \beta,$$

where

$$\beta := \lim_{t\to\infty} \frac{\operatorname{Area}(B(p,t))}{\pi t^2} \in [0, 1]$$

is the asymptotic volume ratio of the manifold $(\mathbb{R}^2, e^{2f} g_0)$.

Given this asymptotic behavior of $r(x)$, the a priori assumption on u could be applied to obtain the lower bound of α in terms of the asymptotic volume ratio.

Proposition 2.1. *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative Gauss curvature. Let* u *be a solution of* (2-2). *Assume*

$$u(x) \geq -2\ln r(x) + o(\ln r(x)),$$

for $r(x)$ *large. Then*

$$\alpha \geq -2\beta,$$

where α *is given by* (2-3) *and* β *is the asymptotic volume ratio of* $(\mathbb{R}^2, e^{2f} g_0)$.

Proof. By our assumption on u and Lemma 2.2, we get for any $\epsilon > 0$, there exists $R > 0$ such that for $r(x) \geq R$,

$$u(x) \geq -2\ln r(x) + o(\ln r(x)) \geq (-2\beta - 2\epsilon)\ln |x| + o(\ln |x|).$$

Lemma 2.1 yields that for any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$,

$$\frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy \leq (\alpha + \epsilon)\ln |x| + C,$$

where C is a constant depending on ϵ, σ, R.

We conclude that for any $\epsilon > 0$, $\sigma > 0$, there exists $R > 0$ such that for $|x| \geq R$ and $\rho = |x|^{-\sigma}$,

$$\begin{aligned}(\alpha + \epsilon)\ln |x| + C &\geq \frac{1}{\pi\rho^2} \int_{B_\rho(x)} u(y)\, dy \\ &\geq (-2\beta - 2\epsilon)\frac{1}{\pi\rho^2} \int_{B_\rho(x)} \ln |y|\, dy + o(\ln |x|) \\ &\geq (-2\beta - 2\epsilon)(\ln |x| - \epsilon) + o(\ln |x|).\end{aligned}$$

Letting $x \to \infty$, we get $\alpha + \epsilon \geq -2\beta - 2\epsilon$. Since ϵ could be arbitrarily small, we get

$$\alpha \geq -2\beta. \qquad \square$$

We shall see that the reversed inequality also holds, and thus the equality is obtained. For this, we need the isoperimetric inequality on nonnegatively curved surfaces established by Brendle [2023, Corollary 1.3], and it also helps to get the rigidity of the underlying manifold in our setting.

Lemma 2.3 (Brendle). *Let (M^2, g) be a complete noncompact manifold with nonnegative Gauss curvature. Let D be a compact domain in M with boundary ∂D. Then*

$$L(\partial D)^2 \geq 4\pi\beta A(D),$$

where $L(\partial D)$ and $A(D)$ represent the length of ∂D and the area of D, respectively, and β is the asymptotic volume ratio of (M, g). The equality holds if and only if (M, g) is isometric to Euclidean space and D is a ball.

Now with the help of Lemma 2.3, one could mimic the argument in [Chen and Li 1991] to give the upper bound of α.

Proposition 2.2. *Let $(\mathbb{R}^2, e^{2f} g_0)$ be a complete surface with nonnegative Gauss curvature. Let u be a solution of (2-2). Then*

$$\alpha \leq -2\beta,$$

where α is given by (2-3) and β is the asymptotic volume ratio of $(\mathbb{R}^2, e^{2f} g_0)$.

Proof. Consider $F(t) := \int_{\Omega_t} e^{2u}\, dg$, where $\Omega_t = \{x : u(x) > t\}$ is the upper level set of u.

The finite total curvature assumption $\int_M e^{2u}\, dg < \infty$ implies $A(\Omega_t) < \infty$, where $A(\Omega_t)$ represents the area of Ω_t in $(\mathbb{R}^2, g = e^{2f} g_0)$.

It follows from (1-4) and the divergence theorem that

$$F(t) = \int_{\Omega_t} e^{2u}\, dg = -\int_{\Omega_t} \Delta u\, dg = -\int_{\partial\Omega_t} \langle \nabla u, \eta\rangle\, dS_g = \int_{\partial\Omega_t} |\nabla u|\, dS_g.$$

By the coarea formula,

$$F'(t) = -\int_{\partial\Omega_t} \frac{e^{2u}}{|\nabla u|}\, dS_g = -e^{2t} \int_{\partial\Omega_t} \frac{1}{|\nabla u|}\, dS_g.$$

Then the Hölder inequality and the isoperimetric inequality (Lemma 2.3) imply

$$\begin{aligned}(F^2(t))' &= -2e^{2t} \int_{\partial\Omega_t} |\nabla u|\, dS_g \int_{\partial\Omega_t} \frac{1}{|\nabla u|}\, dS_g \\ &\leq -2e^{2t} L(\partial\Omega_t)^2 \\ &\leq -8\pi\beta\, e^{2t} A(\Omega_t).\end{aligned} \tag{2-6}$$

Note that the isoperimetric inequality still holds for noncompact regions whose area are finite, since the length of its boundary must be infinite by the completeness of $(\mathbb{R}^2, e^{2f} g_0)$.

Finally integrating (2-6) from $-\infty$ to ∞ yields

$$\begin{aligned}-\left(\int_M e^{2u}\, dg\right)^2 &\le -8\pi\beta \int_{-\infty}^{\infty} e^{2t} A(\{x : e^{2u(x)} > e^{2t}\})\, dt \\ &= -4\pi\beta \int_0^{\infty} A(\{x : e^{2u(x)} > \lambda\})\, d\lambda \\ &= -4\pi\beta \int_M e^{2u}\, dg.\end{aligned}$$

Thus the desired inequality holds. □

Proof of Theorem 2. By Propositions 2.1 and 2.2, we get

$$\alpha = -2\beta.$$

Inspecting the proof of Proposition 2.2 shows that $L(\partial\Omega_t)^2 = 4\pi\beta A(\Omega_t)$ for every $t \in \mathbb{R}$. Hence Lemma 2.3 tells us $(\mathbb{R}^2, e^{2f} g_0)$ must be isometric to the Euclidean space $(\mathbb{R}^2, g_0)$.

To see the sharpness of the -2 in the assumption $u(x) \ge -2\ln r(x) + o(\ln r(x))$, consider the following examples.

Let $e^{2f(x)} = \gamma/(1+|x|^2)^{2-2\gamma}$. Then for $\gamma \in \left[\frac{1}{2}, 1\right]$, $(\mathbb{R}^2, g = e^{2f} g_0)$ is a complete surface with nonnegative Gauss curvature $K_g = 4(1-\gamma)/\big(\gamma(1+|x|^2)^{2\gamma}\big)$.

Taking $e^{2u(x)} = 4/(1+|x|^2)^{2\gamma}$, it is easy to see that $\Delta u + e^{2f} e^{2u} = 0$, that is,

$$\Delta_g u + e^{2u} = 0.$$

Moreover, $\int_{\mathbb{R}^2} e^{2u}\, dg = \int_{\mathbb{R}^2} 4\gamma/(1+|x|^2)^2\, dx = 4\pi\gamma < \infty$.

Direct computation shows

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln r(x)}{\ln|x|} &= 2\gamma - 1 \quad \text{for } \gamma \in \left(\frac{1}{2}, 1\right], \\ \lim_{x\to\infty} \frac{r(x)}{\ln|x|} &= 1 \qquad\qquad \text{for } \gamma = \frac{1}{2}.\end{aligned}$$

Thus for $\gamma \in \left(\frac{1}{2}, 1\right)$, we have

$$u(x) \sim -2\gamma \ln|x| \sim -\frac{2\gamma}{2\gamma-1} \ln r(x),$$

where $-2\gamma/(2\gamma-1) \in (-\infty, -2)$.

In conclusion, for any $k < -2$, there exists a complete surface $(\mathbb{R}^2, e^{2f} g_0)$ with nonnegative Gauss curvature which admits a finite total curvature solution u of (1-4) with $u(x) \sim k \ln r(x)$. □

We conclude this paper with the following remark. Theorem 2 states the rigidity of the underlying manifold under the assumption $u(x) \geq -2\ln r(x) + o(\ln r(x))$. However, on the other end, we cannot expect such rigidity as illustrated by examples above. More precisely, when $\gamma = \frac{1}{2}$, it readily follows that the solution u decays linearly with respect to the distance induced by the metric. Hence one cannot distinguish the flat cylinder by imposing linear decay conditions on the solution.

Nevertheless, when the solution decays sufficiently fast, we can get the volume growth control of the underlying manifold. We record this as a result of independent interest.

Proposition 2.3. *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative Gauss curvature. Let u be a solution of (2-2) satisfying*

$$\liminf_{x\to\infty} \frac{u(x)}{\ln r(x)} = -\infty.$$

Then the asymptotic volume ratio of $(\mathbb{R}^2, e^{2f} g_0)$ *is zero.*

Proof. Suppose the asymptotic volume ratio β is positive. According to the claims (1) and (2) in Lemma 2.1, there holds

$$u(x) \geq \alpha \ln|x| + C \quad \text{for } |x| \geq 2,$$

where $\alpha = -(1/(2\pi)) \int_{\mathbb{R}^2} e^{2f} e^{2u}\, dx$. Combined with Lemma 2.2 one gets

$$\liminf_{x\to\infty} \frac{u(x)}{\ln r(x)} = \liminf_{x\to\infty} \frac{u(x)}{\ln|x|} \frac{\ln|x|}{\ln r(x)} \geq \frac{\alpha}{\beta} > -\infty.$$

This contradicts the hypothesis. Hence we get $\beta = 0$. ☐

We also have a partial converse to Proposition 2.3.

Proposition 2.4. *Let* $(\mathbb{R}^2, e^{2f} g_0)$ *be a complete surface with nonnegative and bounded Gauss curvature. Suppose the asymptotic volume ratio β equals* 0. *Then there exists a solution of (2-2) satisfying*

$$\lim_{x\to\infty} \frac{u(x)}{\ln r(x)} = -\infty.$$

Proof. Recall that f satisfies

$$\Delta f + K e^{2f} = 0,$$

where $0 \leq K \leq C$ by assumption. Based on a work of Taliaferro [1999], Bonini, Ma and Qing [Bonini et al. 2018, Lemma 4.2] showed that

$$e^{2f} \sim |x|^{-2(1-\beta)} = |x|^{-2} \quad \text{as } |x| \to \infty.$$

In view of the existence theorem of McOwen [1985, Theorem 1], for any $\alpha \in (-2, 0)$, there exists a solution u of (2-2) satisfying

$$u(x) \sim \alpha \ln |x| + O(1) \quad \text{at } \infty.$$

Since Lemma 2.2 still holds for $\beta = 0$, the conclusion readily follows. □

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ON MODULI AND ARGUMENTS OF ROOTS OF COMPLEX TRINOMIALS

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Root properties of a general complex trinomial have been explored in numerous papers. Two questions have attracted a significant attention: the relationships between the moduli of these roots and the trinomial's entries, and the location of the roots in the complex plane. We consider several particular problems connected with these topics, and provide new insights into them. As two main results, we describe the set of all trinomials having a root with a given modulus, and derive explicit formula for calculations of the arguments of such roots. In this fashion, we obtain a comprehensive characterization of these roots. In addition, we develop a procedure enabling us to compute moduli and arguments of all roots of a general complex trinomial with arbitrary precision. This procedure is based on the derivation of a family of real transcendental equations for the roots' moduli, and it is supported by the formula for their arguments. All our findings are compared with the existing results.

1. Introduction

We consider a trinomial of the form

$$p(z) = z^k + az^\ell + b,$$

where z, a, b are complex numbers, and $k > \ell$ are positive integers. Because of a lack of formula expressing the roots of p in terms of its entries, many theoretical works analyzed the relationship between the k roots of p, and the values a, b, k, ℓ. More precisely, dependence of moduli and arguments of these roots on a, b, k, ℓ was investigated, and, vice versa, for a given configuration of roots of p, the corresponding parameter space of coefficients was studied.

The list of particular problems discussed in this connection is pretty long. Among others, it includes the following fundamental questions on moduli of the roots of p (we still assume here that a, b are complex numbers, and $k > \ell$ are positive integers):

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Keywords: complex trinomial, root, location, modulus, argument.

(A) What is, for given a, b, k, ℓ and a positive real ϱ, the number of roots of p with moduli lower than ϱ? Alternatively, the inverse of this problem is, for a given ϱ and an integer n, $0 \le n \le k$, to describe the set of all a, b, k, ℓ such that p has just n roots with modulus lower than ϱ.

(B) What is, for given b, k, ℓ, and a positive real ϱ, the geometric structure of the set of all complex numbers a such that p has a root with modulus ϱ?

(C) What is, for given b, k, ℓ, the geometric structure of the set of all complex numbers a such that p has two (or more) roots with the same modulus? Alternatively, problems (B) and (C) can be considered for a fixed a instead of b.

(D) What is, for given a, b, k, ℓ, the geometric description of the location of k roots of p in the complex plane?

Problem (A) has an interesting history. It was completely answered in [Bohl 1908]. However, perhaps due to language reasons ([Bohl 1908] was written in German), this result remained nearly unnoticed by the mathematical community. Some of its particular cases were later rediscovered, among other studied things, for example, in [Brilleslyper and Schaubroeck 2014; 2018; Dilcher et al. 1992; Howell and Kyle 2018].

The inverse of problem (A) is of a great importance in the stability and asymptotic theory of difference equations. Although it is closely related to problem (A) itself (in fact, its solution can be deduced directly from Bohl's result), its various particular cases were investigated in dozens of works (see, for example, [Dannan 2004; Kipnis and Nigmatullin 2016; Kuruklis 1994; Matsunaga and Hajiri 2010; Papanicolaou 1996; Čermák and Jánský 2015]) — again independently of the existence of Bohl's result. Only recently, [Bohl 1908] has attracted attention corresponding to its relevance (see, for example, [Barrera et al. 2022; 2023a; Theobald and de Wolff 2016; Čermák and Fedorková 2023]).

Problems (B) and (C) were formulated and answered in [Theobald and de Wolff 2016], along with a comprehensive historical survey. Using the amoeba theory, these answers revealed a nice geometric and topological structure of the parameter space of trinomials p with respect to the moduli of their roots.

Problem (D) on locating and describing the geometry of roots of p in the complex plane is a classical matter. Starting with [Nekrassoff 1887], a series of papers on sectors in the complex plane, each containing a root of p, appeared (see, for example, [Egerváry 1930]). The strongest results in this sense, namely disjoint annular sectors smaller than those described in previous works, were obtained in [Melman 2012]. In these investigations, Rouché's theorem and other tools of complex analysis turned out to be very useful. For other relevant results on roots of complex trinomials, we refer to [Barrera et al. 2023b; Botta and da Silva 2019; Fell 1980; Szabó 2010; Čermák et al. 2022].

Our main goal is twofold. First, we wish to present new insights into problems (A)–(D), and offer alternate answers to some of them. Second, we aim to learn more about arguments of roots of p as well. Keeping in mind these outlines, the paper is organized as follows.

In Section 2, we recall Bohl's result answering problem (A). Using this result, we discuss an inverse version of problem (A), namely characterization of all couples (a, b) such that p has a prescribed number of roots whose modulus is lower than a prescribed real number. Section 3 deals with problem (B), and formulates explicit necessary and sufficient conditions guaranteeing that p has a root with a given modulus. A formula for calculation of the arguments of such roots is derived as well. Considerations performed in Section 4 are motivated by problem (D), and result in theoretical justification of an algorithm that enables us to localize all k roots of p with arbitrary precision. This algorithm is based on the derivation of k (real) transcendental equations for moduli of these roots, supported by a formula for calculation of their arguments. In Section 5, we illustrate our results and compare them with the existing ones. Doing so, we consider assertions and examples from earlier papers, and clarify contributions of our results to the existing theory on complex trinomials. The final section summarizes the key parts of the paper, outlines possible applications, and poses some open problems.

The main results of this paper are contained in Sections 3 and 4. Our intention was to derive them without any support of advanced theoretical tools, using only some basic facts from linear algebra, mathematical analysis and number theory.

Throughout the text, the following simplifications and notation are utilized. Without loss of generality, we assume that the integers k, ℓ are coprime (the opposite case can be easily reduced to this one), and the complex numbers a, b are nonzero (the opposite case is trivial). Further, we assume that the arguments of complex numbers are taken from the interval $(-\pi, \pi]$, and introduce the notation $\theta_a = \arg(a)$, $\theta_b = \arg(b)$,

$$\alpha_\varrho = \arccos \frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2}{2|ab|\varrho^\ell}, \quad \beta_\varrho = \arccos \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}, \tag{1-1}$$

$$\theta = k\theta_a - (k-\ell)\theta_b + (k-\ell)\pi, \tag{1-2}$$

and

$$\tau_\varrho^\pm = \frac{\theta}{2\pi} \pm \frac{k\alpha_\varrho + \ell\beta_\varrho}{2\pi}. \tag{1-3}$$

Also, we utilize the notation

$$\varphi \equiv \psi \pmod{2\pi}$$

for the arguments $\varphi \in (-\pi, \pi]$ of appropriate complex numbers, meaning that the difference between φ and a real number ψ is an integer multiple of 2π.

Finally, we call the roots of p with modulus lower than ϱ (or equal to ϱ) ϱ-interior (or ϱ-modular, respectively). If $\varrho = 1$, then the ϱ-modular roots of p are called unimodular.

2. A number of ϱ-interior roots of p

Let n_ϱ be the number of ϱ-interior roots of p. As we have already mentioned, the problem of finding an explicit formula for n_ϱ with respect to given a, b, k, ℓ and ϱ was solved in [Bohl 1908]. Since the original formulation of this result has a rather geometric character, we use here its equivalent analytical reformulation (see also [Čermák and Fedorková 2023]).

Theorem 2.1. *Let a, b be nonzero complex numbers, $k > \ell$ be coprime positive integers, ϱ be a positive real number, and let θ be given by (1-2).*

(i) *If $|b| \geq \varrho^k + |a|\varrho^\ell$ then $n_\varrho = 0$.*

(ii) *If $\varrho^k > |a|\varrho^\ell + |b|$ then $n_\varrho = k$.*

(iii) *If $|a|\varrho^\ell > \varrho^k + |b|$ then $n_\varrho = \ell$.*

(iv) *If $|a|\varrho^\ell = \varrho^k + |b| < \frac{k}{\ell}\varrho^k$ and $\frac{\theta}{\pi} + \ell$ is an even integer, then $n_\varrho = \ell$.*

(v) *If*

$$|b| < \varrho^k + |a|\varrho^\ell, \quad \varrho^k \leq |a|\varrho^\ell + |b|, \quad |a|\varrho^\ell \leq \varrho^k + |b|, \tag{2-1}$$

and at least one of the assumptions of (iv) *does not hold, then*

$$n_\varrho = \lceil \tau_\varrho^+ \rceil - \lfloor \tau_\varrho^- \rfloor - 1, \tag{2-2}$$

where $\tau_\varrho^\pm$ is given by (1-3), and the symbols $\lceil \cdot \rceil$, $\lfloor \cdot \rfloor$ mean the upper and lower integer part.

Remark 2.2. The conditions (i)–(v) of Theorem 2.1 cover all (nontrivial) possibilities for the complex coefficients a, b and exponents k, ℓ of an arbitrary trinomial p. One can observe some interesting geometric connections hidden behind the inequalities forming these conditions. In fact, the conditions (i)–(iv) reflect a dominance of monomials z^k, az^ℓ, b in the sense that one of them exceeds or equals to (in modulus) the sum of the remaining ones. The condition (v) is related to the opposite situation when there exists a triangle with edges of lengths ϱ^k, $|a|\varrho^\ell$ and $|b|$. In this geometric interpretation, the values α_ϱ and β_ϱ are nothing more than the angles between the edges of lengths $|a|\varrho^\ell$, $|b|$ and ϱ^k, $|b|$, respectively.

Also note that the above stated dominance of monomials z^k, az^ℓ, b is closely related to the essential concepts of tropical geometry. In particular, using some basic tools of tropical geometry, the problem of finding the roots of a tropical polynomial (composed of the monomials z^k, az^ℓ, b such that one of them is dominant) inside the circle of radius ϱ simultaneously provides the number of ϱ-interior roots of p (see, for example, [Brugallé et al. 2015; Viro 2011]).

Now we consider an opposite problem: for a given real $\varrho > 0$ and a given integer $n \geq 0$, we search for the set of all a, b, k, ℓ such that $n = n_\varrho$, i.e., p has just n ϱ-interior roots.

Since the relationship between n_ϱ and a, b, k, ℓ is elementary in the properties (i)–(iv) of Theorem 2.1, it is enough to analyze the formula (2-2) forming the core of the property (v). On this account, we introduce the function $\omega = \omega(x)$ to be a 2π-periodic extension of $\omega^*(x) = |x|$, $x \in [-\pi, \pi]$. Then the following holds:

Corollary 2.3. *Let $\varrho > 0$ be a real number, n be a nonnegative integer, let a, b be nonzero complex numbers, and $k > \ell$ be coprime positive integers. Further, assume that (2-1) holds, whereas at least one of the assumptions of the property (iv) of Theorem 2.1 does not hold. Then p has just n ϱ-interior roots if and only if either*

$$n \text{ is even} \quad \text{and} \quad n\pi - \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq n\pi + \omega(\theta), \tag{2-3}$$

or

$$n \text{ is odd} \quad \text{and} \quad (n-1)\pi + \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq (n+1)\pi - \omega(\theta), \tag{2-4}$$

where α_ϱ, β_ϱ and θ are given by (1-1) and (1-2), respectively.

Proof. First, we assume that

$$2m_1\pi \leq |\theta| < (2m_1 + 1)\pi$$

for a nonnegative integer m_1. Then, using

$$\theta = (2m_1\pi + \omega(\theta)) \operatorname{sgn}(\theta),$$

(2-2) implies that $n = n_\varrho$ just when

$$n = \left\lceil \frac{\omega(\theta)\operatorname{sgn}(\theta) + k\alpha_\varrho + \ell\beta_\varrho}{2\pi} \right\rceil - \left\lfloor \frac{\omega(\theta)\operatorname{sgn}(\theta) - (k\alpha_\varrho + \ell\beta_\varrho)}{2\pi} \right\rfloor - 1. \tag{2-5}$$

Now we distinguish two cases leading to (2-3) and (2-4). If

$$2m_2\pi - \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq 2m_2\pi + \omega(\theta), \quad \omega(\theta) \neq 0, \tag{2-6}$$

for an integer m_2, then (2-5) becomes

$$n = m_2 + 1 + m_2 - 1 = 2m_2 \qquad (\text{if } \theta > 0),$$
$$n = m_2 - (-m_2 - 1) - 1 = 2m_2 \quad (\text{if } \theta < 0).$$

Then, it is enough to substitute $2m_2 = n$ in (2-6) to get (2-3).

Similarly, if

$$2m_2\pi + \omega(\theta) < k\alpha_\varrho + \ell\beta_\varrho \leq (2m_2 + 2)\pi - \omega(\theta), \quad \omega(\theta) \neq \pi, \tag{2-7}$$

for an integer m_2, then

$$n = m_2 + 1 + m_2 + 1 - 1 = 2m_2 + 1.$$

Consequently, $2m_2 = n - 1$, and (2-7) yields (2-4).

Now, we assume that

$$(2m_1 - 1)\pi \le |\theta| < 2m_1\pi$$

for a positive integer m_1. Then

$$\theta = (2m_1\pi - \omega(\theta))\,\mathrm{sgn}(\theta),$$

and $n = n_\varrho$ is equivalent to

$$n = \left\lceil \frac{-\omega(\theta)\,\mathrm{sgn}(\theta) + k\alpha_\varrho + \ell\beta_\varrho}{2\pi} \right\rceil - \left\lfloor \frac{-\omega(\theta)\,\mathrm{sgn}(\theta) - (k\alpha_\varrho + \ell\beta_\varrho)}{2\pi} \right\rfloor - 1$$

due to (2-2). Thus, using the same line of arguments as given above, we arrive at (2-3) and (2-4). □

3. Existence of ϱ-modular roots of p

We formulate easily applicable conditions verifying whether p has a root with a prescribed modulus. In the affirmative case, we find explicit formulae for arguments of such roots. Thus, we find an effective answer to a more general version of problem (B).

We start with recapitulation of some useful facts from elementary number theory. Let $k > \ell$ be coprime positive integers. For a given integer τ, we consider the linear Diophantine equation in two integer variables u, v

(3-1) $$ku + (k - \ell)v = \tau.$$

If we put

(3-2) $$u = \tau u_0, \quad v = \tau v_0,$$

then (3-1) can be reduced to

(3-3) $$ku_0 + (k - \ell)v_0 = 1,$$

whose integer solutions (u_0, v_0) are so-called *Bézout coefficients* for a couple $(k, k-\ell)$. It is well known that (3-3) admits infinitely many integer solutions; indeed, if (u_0^*, v_0^*) are Bézout coefficients for $(k, k - \ell)$, then all integer solutions (u_0, v_0) of (3-3) can be written as

(3-4) $$u_0 = u_0^* + (k - \ell)m, \quad v_0 = v_0^* - km, \qquad m \in \mathbb{Z}.$$

There are several algorithms to determine a couple (u_0, v_0) satisfying (3-3); the most often used is the extended Euclidean algorithm applied to $(k, k-\ell)$ (see, for example, [Fuhrmann 2012]).

Now we come back to problem (B). The next assertion presents a simple condition verifying that p admits a ϱ-modular root. Moreover, for all a, b, k, ℓ meeting this condition, we give an explicit evaluation of arguments of ϱ-modular roots. Thus, we are able to provide their complete identification.

Theorem 3.1. *Let a, b be nonzero complex numbers, let $k > \ell$ be coprime positive integers, and let $\tau_\varrho^\pm$ be given by (1-3). Then p has a ϱ-modular root $z = \varrho \exp(\mathrm{i}\varphi)$, $\varphi \in (-\pi, \pi]$, if and only if*

$$|b| \le \varrho^k + |a|\varrho^\ell, \quad \varrho^k \le |a|\varrho^\ell + |b|, \quad |a|\varrho^\ell \le \varrho^k + |b|, \tag{3-5}$$

and at least one of the values $\tau_\varrho^\pm$ is an integer.

An explicit dependence of φ on ϱ can be expressed by the formula

$$\varphi = \varphi^\pm \equiv \begin{cases} \dfrac{(2v_0\tau_\varrho^+ - 1)\pi + \beta_\varrho + \theta_b}{k} \pmod{2\pi} & \text{if } \tau_\varrho^+ \text{ is an integer,} \\ \dfrac{(2v_0\tau_\varrho^- - 1)\pi - \beta_\varrho + \theta_b}{k} \pmod{2\pi} & \text{if } \tau_\varrho^- \text{ is an integer,} \end{cases} \tag{3-6}$$

where v_0 is the second component of a couple of Bézout coefficients (u_0, v_0) satisfying (3-3), and β_ϱ is given by (1-1).

Remark 3.2. (a) If just one of the values $\tau_\varrho^\pm$ is an integer, then there exists a unique ϱ-modular root of p (whose argument is $\varphi = \varphi^+$, or $\varphi = \varphi^-$ if τ_ϱ^+ is an integer, or τ_ϱ^- is an integer, respectively). If $\tau_\varrho^\pm$ are two distinct integers, then they generate, along with two arguments $\varphi = \varphi^+$ and $\varphi = \varphi^-$, two distinct ϱ-modular roots of p. Note that this situation occurs just when

$$|b| \ne \varrho^k + |a|\varrho^\ell, \quad k\theta_a - (k-\ell)\theta_b = j_1\pi, \quad k\alpha_\varrho + \ell\beta_\varrho = j_2\pi \tag{3-7}$$

for a couple of integers (j_1, j_2) satisfying $(-1)^{j_1+k} = (-1)^{j_2+\ell}$.

(b) All ϱ-modular roots of p described in Theorem 3.1 are simple except for those generated by the conditions

$$|a|\varrho^\ell = \varrho^k + |b|, \quad |a|\,|b|^{(\ell-k)/k} = \frac{k}{k-\ell}\left(\frac{k-\ell}{\ell}\right)^{\ell/k}, \quad \frac{\theta}{2\pi} + \frac{\ell}{2} \in \mathbb{Z} \tag{3-8}$$

with θ given by (1-2). In this case, p has a double ϱ-modular root z_d with the argument $\varphi = \varphi^+ = \varphi^-$ (in fact, the values φ^+ and φ^- coincide in such a case). Note that this property was already known before since it is a special instance of classical $\mathcal{A}$-discriminant theory. Indeed, applying the formula (1.38), p. 406, of [Gelfand et al. 1994], it is easy to verify that the discriminant of p vanishes just when (3-8) holds. The formula (1.28), p. 404, of [Gelfand et al. 1994] yields an algorithm for direct detection of the double root z_d of p (more precisely, k and ℓ powers of z_d can be expressed as linear rational functions of the coefficients of p). In general, $\mathcal{A}$-discriminants are useful to explain why many trinomial properties can be described explicitly (contrary to polynomials with more than three terms).

(c) The formula for argument φ can be equivalently expressed in the form

$$\varphi=\varphi^{\pm}\equiv\begin{cases}\dfrac{(2u_0\tau_\varrho^{+}+2v_0\tau_\varrho^{+}-1)\pi-\alpha_\varrho-\theta_a+\theta_b}{\ell} \pmod{2\pi} & \text{if } \tau_\varrho^{+}\in\mathbb{Z},\\ \dfrac{(2u_0\tau_\varrho^{-}+2v_0\tau_\varrho^{-}-1)\pi+\alpha_\varrho-\theta_a+\theta_b}{\ell} \pmod{2\pi} & \text{if } \tau_\varrho^{-}\in\mathbb{Z},\end{cases} \tag{3-9}$$

where u_0 is the first component of a couple of Bézout coefficients (u_0, v_0), and α_ϱ is given by (1-1). Because of (3-4), both the formulae (3-6) and (3-9) are independent of a concrete choice of Bézout coefficients.

Proof of Theorem 3.1 and Remark 3.2. Let $z=\varrho\exp(\mathrm{i}\varphi)$, $\varphi\in(-\pi,\pi]$, be a ϱ-modular root of p. Then

$$\varrho^k\exp(\mathrm{i}k\varphi)+|a|\varrho^\ell\exp(\mathrm{i}(\ell\varphi+\theta_a))+|b|\exp(\mathrm{i}\theta_b)=0,$$

i.e.,

$$\begin{aligned}|a|\varrho^\ell\cos(\ell\varphi+\theta_a)+|b|\cos(\theta_b)&=-\varrho^k\cos(k\varphi),\\ |a|\varrho^\ell\sin(\ell\varphi+\theta_a)+|b|\sin(\theta_b)&=-\varrho^k\sin(k\varphi).\end{aligned} \tag{3-10}$$

We solve (3-10) with respect to (positive real) unknowns $|a|$, $|b|$.

First, let the system matrix be singular, that is, $\sin(\ell\varphi+\theta_a-\theta_b)=0$. Then (3-10) has a solution $|a|$, $|b|$ if and only if

$$k\varphi-\theta_b=j_1\pi,\quad (k-\ell)\varphi-\theta_a=j_2\pi$$

for suitable integers j_1, j_2. Equivalently,

$$\varphi=\frac{j_1\pi+\theta_b}{k}=\frac{(j_1-j_2)\pi-\theta_a+\theta_b}{\ell}, \tag{3-11}$$

which implies

$$\frac{\theta}{\pi}=-j_2k+(j_1+1)(k-\ell) \tag{3-12}$$

due to (1-2). Substituting (3-11) into (3-10) one gets

$$\begin{aligned}|a|\varrho^\ell\cos(\theta_b+(j_1-j_2)\pi)+|b|\cos(\theta_b)&=-\varrho^k\cos(\theta_b+j_1\pi),\\ |a|\varrho^\ell\sin(\theta_b+(j_1-j_2)\pi)+|b|\sin(\theta_b)&=-\varrho^k\sin(\theta_b+j_1\pi),\end{aligned}$$

i.e.,

$$(-1)^{j_1-j_2}|a|\varrho^\ell\cos(\theta_b)+|b|\cos(\theta_b)=(-1)^{j_1+1}\varrho^k\cos(\theta_b),$$
$$(-1)^{j_1-j_2}|a|\varrho^\ell\sin(\theta_b)+|b|\sin(\theta_b)=(-1)^{j_1+1}\varrho^k\sin(\theta_b).$$

This yields

$$(-1)^{j_1}\varrho^k+(-1)^{j_1-j_2}|a|\varrho^\ell+|b|=0. \tag{3-13}$$

The case when both j_1 and j_2 are even cannot occur in (3-13). We explore the remaining parity variants.

If j_1 is odd, j_2 even, then (3-13) becomes $|b|=\varrho^k+|a|\varrho^\ell$, i.e., $\alpha_\varrho=\beta_\varrho=0$. Moreover, the right-hand side of (3-12) is even, which implies the first couple of conditions for the existence of a ϱ-modular root of p in the form

$$|b|=\varrho^k+|a|\varrho^\ell,\quad \tau_\varrho^\pm=\frac{\theta}{2\pi}\text{ is an integer.} \tag{3-14}$$

In addition, (3-12) is the linear Diophantine equation (3-1) with

$$u=-\frac{j_2}{2},\quad v=\frac{j_1+1}{2},\quad\text{and}\quad \tau=\tau_\varrho^\pm=\frac{\theta}{2\pi}.$$

Then (3-2) and (3-4) imply

$$j_1=2(v_0-km)\tau_\varrho^\pm-1,\quad j_2=-2(u_0+(k-\ell)m)\tau_\varrho^\pm \tag{3-15}$$

for an integer m. Now it is enough to substitute $(3\text{-}15)_1$ into $(3\text{-}11)_1$ to obtain

$$\varphi\equiv\frac{(2v_0\tau_\varrho^\pm-1)\pi+\theta_b}{k}\pmod{2\pi}. \tag{3-16}$$

If both j_1 and j_2 are odd, then (3-13) yields $\varrho^k=|a|\varrho^\ell+|b|$, that is, $\alpha_\varrho=\pi$, $\beta_\varrho=0$. Also, (3-12) can be written as

$$\frac{\theta}{\pi}+k=(1-j_2)k+(j_1+1)(k-\ell). \tag{3-17}$$

Since the right-hand side of (3-17) is even, we get another couple of conditions

$$\varrho^k=|a|\varrho^\ell+|b|,\quad \tau_\varrho^+=\frac{\theta}{2\pi}+\frac{k}{2}\text{ is an integer.} \tag{3-18}$$

Furthermore, (3-17) is (3-1) with

$$u=\frac{1-j_2}{2},\quad v=\frac{j_1+1}{2},\quad\text{and}\quad \tau=\tau_\varrho^+=\frac{\theta}{2\pi}+\frac{k}{2}.$$

Then (3-2) and (3-4) yield

$$j_1=2\tau_\varrho^+(v_0-km)-1,\quad j_2=1-2\tau_\varrho^+(u_0+(k-\ell)m)$$

for an integer m; hence (3-11)$_1$ becomes

$$\varphi \equiv \frac{(2v_0\tau_\varrho^+ - 1)\pi + \theta_b}{k} \pmod{2\pi}. \tag{3-19}$$

If j_1 is even, j_2 odd, then (3-13) and (3-12) imply $|a|\varrho^\ell = \varrho^k + |b|$, that is, $\alpha_\varrho = 0$, $\beta_\varrho = \pi$, and

$$\frac{\theta}{\pi} + \ell = (1 - j_2)k + j_1(k - \ell), \tag{3-20}$$

respectively. Since the right-hand side of (3-20) is even, we get the third couple of conditions for the existence of a ϱ-modular root of p, namely

$$|a|\varrho^\ell = \varrho^k + |b|, \quad \tau_\varrho^+ = \frac{\theta}{2\pi} + \frac{\ell}{2} \text{ is an integer.} \tag{3-21}$$

Obviously, (3-20) is (3-1) with

$$u = \frac{1 - j_2}{2}, \quad v = \frac{j_1}{2}, \quad \text{and} \quad \tau = \tau_\varrho^+ = \frac{\theta}{2\pi} + \frac{\ell}{2},$$

which along with (3-2) and (3-4) yields

$$j_1 = 2\tau_\varrho^+(v_0 - km), \quad j_2 = 1 - 2\tau_\varrho^+(u_0 + (k - \ell)m) \tag{3-22}$$

for an integer m. Then we substitute (3-22)$_1$ into (3-11)$_1$ to get

$$\varphi \equiv \frac{2v_0\tau_\varrho^+\pi + \theta_b}{k} \pmod{2\pi}. \tag{3-23}$$

Now we assume the system matrix of (3-10) to be regular, in other words, $\sin(\ell\varphi + \theta_a - \theta_b) \neq 0$. In this case, the solution of (3-10) is given by

$$|a| = -\varrho^{k-\ell}\frac{\sin(k\varphi - \theta_b)}{\sin(\ell\varphi + \theta_a - \theta_b)}, \quad |b| = \varrho^k\frac{\sin((k-\ell)\varphi - \theta_a)}{\sin(\ell\varphi + \theta_a - \theta_b)}. \tag{3-24}$$

At the same time, we solve (3-10) with respect to φ. To do this, we square and sum (3-10) to obtain

$$\cos(\ell\varphi + \theta_a - \theta_b) = \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} - |b|^2}{2|ab|\varrho^\ell}. \tag{3-25}$$

Alternatively, we can write (3-10) as

$$\varrho^k\cos(k\varphi) + |b|\cos(\theta_b) = -|a|\varrho^\ell\cos(\ell\varphi + \theta_a),$$
$$\varrho^k\sin(k\varphi) + |b|\sin(\theta_b) = -|a|\varrho^\ell\sin(\ell\varphi + \theta_a),$$

where repeated squaring and summation yield

$$\cos(k\varphi - \theta_b) = -\frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}. \tag{3-26}$$

Now we analyze (3-25) and (3-26). Since the values of their right-hand sides have to lie between -1 and 1, straightforward calculations imply

$$|b| < \varrho^k + |a|\varrho^\ell, \quad \varrho^k < |a|\varrho^\ell + |b|, \quad |a|\varrho^\ell < \varrho^k + |b| \tag{3-27}$$

(the strict inequalities occur here due to $\sin(\ell\varphi + \theta_a - \theta_b) \neq 0$, $\sin(k\varphi - \theta_b) \neq 0$). Further, to express φ from (3-25) and (3-26), we discuss the arguments of appropriate goniometric functions appearing here.

Let $\sin(\ell\varphi + \theta_a - \theta_b) > 0$. Then $|a|$, $|b|$ given by (3-24) are positive if and only if

$$\sin(k\varphi - \theta_b) < 0, \quad \sin((k-\ell)\varphi - \theta_a) > 0.$$

Equivalently, there exist some integers u, v such that

$$\begin{aligned} (2v-1)\pi &< k\varphi - \theta_b < 2v\pi, \\ -2u\pi &< (k-\ell)\varphi - \theta_a < (-2u+1)\pi, \\ (2u+2v-2)\pi &< \ell\varphi + \theta_a - \theta_b < (2u+2v-1)\pi. \end{aligned} \tag{3-28}$$

Then, using (3-28), we can rewrite (3-25) and (3-26) into

$$\ell\varphi + \theta_a - \theta_b - (2u+2v-2)\pi = \arccos \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} - |b|^2}{2|ab|\varrho^\ell}$$

and

$$2v\pi - (k\varphi - \theta_b) = \arccos \frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} - |b|^2}{2|b|\varrho^k},$$

respectively. From here,

$$\varphi = \frac{(2v-1)\pi + \beta_\varrho + \theta_b}{k} = \frac{(2u+2v-1)\pi - \alpha_\varrho - \theta_a + \theta_b}{\ell}. \tag{3-29}$$

The equality of two ratios in (3-29) is equivalent to

$$ku + (k-\ell)v = \frac{\theta}{2\pi} + \frac{k\alpha_\varrho + \ell\beta_\varrho}{2\pi}. \tag{3-30}$$

This particularly implies that

$$\tau_\varrho^+ = \frac{\theta}{2\pi} + \frac{k\alpha_\varrho + \ell\beta_\varrho}{2\pi} \text{ is an integer.} \tag{3-31}$$

Moreover, (3-30) is the Diophantine equation (3-1) with $\tau = \tau_\varrho^+$, which along with (3-2), (3-4) and $(3\text{-}29)_1$ yields

$$\varphi = \varphi^+ \equiv \frac{(2v_0\tau_\varrho^+ - 1)\pi + \beta_\varrho + \theta_b}{k} \pmod{2\pi}. \tag{3-32}$$

Now let $\sin(\ell\varphi+\theta_a-\theta_b)<0$. An analogous argumentation yields

$$\begin{aligned}(2v-2)\pi &< k\varphi-\theta_b<(2v-1)\pi,\\ (-2u-1)\pi &< (k-\ell)\varphi-\theta_a<-2u\pi,\\ (2u+2v-1)\pi &< \ell\varphi+\theta_a-\theta_b<(2u+2v)\pi\end{aligned}$$

for some integers u, v, and

$$\varphi=\frac{(2v-1)\pi-\beta_\varrho+\theta_b}{k}=\frac{(2u+2v-1)\pi+\alpha_\varrho-\theta_a+\theta_b}{\ell}. \tag{3-33}$$

This implies

$$ku+(k-\ell)v=\frac{\theta}{2\pi}-\frac{k\alpha_\varrho+\ell\beta_\varrho}{2\pi}. \tag{3-34}$$

Therefore,

$$\tau_\varrho^-=\frac{\theta}{2\pi}-\frac{k\alpha_\varrho+\ell\beta_\varrho}{2\pi}\ \text{is an integer,} \tag{3-35}$$

and (3-29)$_1$ supported by (3-34), (3-2), (3-4) results in

$$\varphi=\varphi^-\equiv\frac{(2v_0\tau_\varrho^--1)\pi-\beta_\varrho+\theta_b}{k}\pmod{2\pi}. \tag{3-36}$$

If we summarize the conditions (3-14), (3-18), (3-21), (3-27), (3-31), and (3-35), then we get the first assertion of Theorem 3.1. The formula (3-6) for arguments of appropriate ϱ-modular roots then follows from (3-16), (3-19), (3-23), (3-32), and (3-36).

Finally, we verify the observations mentioned in Remark 3.2. The part (a) follows directly from the above proof procedures. We confirm the conditions for the appearance of a ϱ-modular double root of p stated in the part (b). Let $z=\varrho\exp(i\varphi)$ be such a root. Then substitution into $p(z)=p'(z)=0$ (supported by some straightforward calculations) yields

$$|a|\varrho^\ell=\varrho^k+|b|=\frac{k}{\ell}\varrho^k. \tag{3-37}$$

This implies $\alpha_\varrho=0$, $\beta_\varrho=\pi$ and $\theta/(2\pi)+\ell/2$ is an integer. It is easy to verify that (3-37) is equivalent with (3-8)$_1$ and (3-8)$_2$. Also,

$$\varphi^+\equiv\frac{2v_0\tau_\varrho^+\pi+\theta_b}{k}\pmod{2\pi},\qquad \varphi^-\equiv\frac{(2v_0\tau_\varrho^--2)\pi+\theta_b}{k}\pmod{2\pi}$$

due to (3-6). Since

$$\tau_\varrho^+=\frac{\theta}{2\pi}+\frac{\ell}{2},\qquad \tau_\varrho^-=\frac{\theta}{2\pi}-\frac{\ell}{2},$$

we get $\varphi^+ = \varphi^-$ by use of (3-3). Regarding the observation (c), the formula (3-9) follows from alternate expressions of φ in (3-11), (3-29) and (3-33). More precisely, while the use of first expressions in (3-11), (3-29) and (3-33) results into (3-6), the use of the latter ones implies (3-9). □

Based on Theorem 3.1 and Remark 3.2, it is possible to deduce some other basic root properties of complex trinomials. For example, assume that p has a couple of complex conjugate imaginary roots $z = \varrho \exp(\pm i\varphi) = \varrho \exp(i\varphi^{\pm})$, i.e., $\varphi^+ = -\varphi^-$. Then, in addition to (3-7), the condition

$$\frac{(2v_0\tau_\varrho^+ - 1)\pi + \beta_\varrho + \theta_b}{k} + \frac{(2v_0\tau_\varrho^- - 1)\pi - \beta_\varrho + \theta_b}{k} = 2j\pi,$$

or equivalently,

$$(k\theta_a - (k-\ell)\theta_b)v_0 + ((k-\ell)v_0 - 1)\pi + \theta_b = kj\pi, \tag{3-38}$$

has to be met for an integer j. Obviously, (3-7) and (3-38) imply $\theta_a, \theta_b \in \{0, \pi\}$. Thus we get:

Corollary 3.3. *Suppose p admits a pair of complex conjugate imaginary roots. Then its coefficients a, b have to be real numbers.*

To summarize our previous investigations: Theorem 3.1 is effective in the sense that, for any $\varrho > 0$, it enables us to describe the set of all a, b, k, ℓ such that p admits a ϱ-modular root. Moreover, we are able to specify arguments of such roots. Then a crucial question arises, namely whether (and possibly how) this conclusion can contribute to discussions on location of trinomial roots in the complex plane (see problem (D)). In the next section, we are going to discuss this matter in more detail.

4. Calculating moduli and arguments of roots of *p*

We consider the trinomial p with arbitrary but fixed nonzero complex numbers a, b and coprime positive integers $k > \ell$. Let $z = \varrho \exp(i\varphi)$, $\varphi \in (-\pi, \pi]$, be a root of p. Our task is to find conditions on ϱ and φ expressed in terms of entry parameters a, b, k, ℓ.

By Theorem 3.1, z is a root of p if and only if (3-5) is true and

$$\theta + k\alpha_\varrho + \ell\beta_\varrho = 2s\pi \quad \text{or} \quad \theta - (k\alpha_\varrho + \ell\beta_\varrho) = 2s\pi \qquad \text{for an integer } s, \tag{4-1}$$

α_ϱ, β_ϱ and θ being given by (1-1) and (1-2), respectively. To analyze (4-1), we introduce the function

$$F(\varrho) = k\alpha_\varrho + \ell\beta_\varrho = k \arccos \frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2}{2|ab|\varrho^\ell} + \ell \arccos \frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}$$

whose domain is described just by the triplet of inequalities (3-5). Then (4-1) becomes

$$F(\varrho) = |\theta - 2s\pi|, \quad s \text{ is an integer.} \tag{4-2}$$

In the sequel, we describe some basic properties of F, namely its domain $D(F)$, image $H(F)$ and monotony.

To clarify the domain of F with respect to (3-5), we need to perform a proper sign analysis of real trinomials

$$Q_1(\varrho) = \varrho^k + |a|\varrho^\ell - |b|, \quad Q_2(\varrho) = \varrho^k - |a|\varrho^\ell - |b|, \quad Q_3(\varrho) = \varrho^k - |a|\varrho^\ell + |b|$$

considered for $\varrho \geq 0$. On this account, we put

$$\sigma(k, \ell) = \frac{k}{k-\ell}\left(\frac{k-\ell}{\ell}\right)^{\ell/k}. \tag{4-3}$$

Then, based on elementary calculations, the following observations hold.

Proposition 4.1. (i) *There is a unique positive root ξ_L of Q_1 such that $Q_1(\varrho) < 0$ for all $0 \leq \varrho < \xi_L$, and $Q_1(\varrho) > 0$ for all $\varrho > \xi_L$.*

(ii) *There is a unique positive root ξ_R of Q_2 such that $Q_2(\varrho) < 0$ for all $0 \leq \varrho < \xi_R$, and $Q_2(\varrho) > 0$ for all $\varrho > \xi_R$.*

(iii) *If $|a|\,|b|^{(\ell-k)/k} < \sigma(k, \ell)$, then Q_3 is positive for all $\varrho \geq 0$.*

(iv) *If $|a|\,|b|^{(\ell-k)/k} = \sigma(k, \ell)$, then a unique positive double root ξ_M of Q_3 appears* (*and Q_3 is positive otherwise*).

(v) *If $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$, then Q_3 has a couple of positive roots $\xi_{M_1} < \xi_{M_2}$ such that $Q_3(\varrho) > 0$ if either $0 \leq \varrho < \xi_{M_1}$, or $\varrho > \xi_{M_2}$, and $Q_3(\varrho) < 0$ whenever $\xi_{M_1} < \varrho < \xi_{M_2}$.*

(vi) *If the assumption of the property* (*iii*), *or* (*iv*), *or* (*v*) *holds*, *then*

$$\xi_L < \xi_R, \quad \text{or} \quad \xi_L < \xi_M < \xi_R, \quad \text{or} \quad \xi_L < \xi_{M_1} < \xi_{M_2} < \xi_R,$$

respectively.

Remark 4.2. In a slightly different context, the properties (i)–(vi) were described for a triplet of related trinomials Φ, χ, Ψ in [Melman 2012].

Thus, keeping in mind (3-5), we get the following description of the domain of F.

Lemma 4.3. *Let ξ_L, ξ_R and ξ_{M_1}, ξ_{M_2} be positive roots of Q_1, Q_2 and Q_3, respectively, whose existence is guaranteed by the conditions of Proposition 4.1.*

(i) *If $|a|\,|b|^{(\ell-k)/k} \leq \sigma(k, \ell)$, then $D(F) = [\xi_L, \xi_R]$.*

(ii) *If $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$, then $D(F) = [\xi_L, \xi_{M_1}] \cup [\xi_{M_2}, \xi_R]$.*

The next assertion reveals other important properties of F (we still assume that ξ_L, ξ_R and ξ_{M_1}, ξ_{M_2} are positive roots of Q_1, Q_2 and Q_3, respectively).

Lemma 4.4. *The function F defines a strictly increasing mapping of $D(F)$ onto $H(F) = [0, k\pi]$. This mapping is continuous on $[\xi_L, \xi_R]$ provided $|a|\,|b|^{(\ell-k)/k} \le \sigma(k, \ell)$, and it is continuous on $[\xi_L, \xi_{M_1}]$ and $[\xi_{M_2}, \xi_R]$ provided $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. In this case, $F(\xi_{M_1}) = F(\xi_{M_2}) = \ell\pi$.*

Proof. Obviously, the values of F are nonnegative. More precisely, $F(\xi_L) = 0$, and $F(\varrho) > 0$ for all $\varrho \in D(F)$, $\varrho > \xi_L$. Direct calculations confirm that $F(\xi_R) = k\pi$ and $F(\xi_{M_1}) = F(\xi_{M_2}) = \ell\pi$. It remains to show that F is strictly increasing on its domain.

After some straightforward calculations and using the relations

$$(2|ab|\varrho^\ell)^2 - (-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2)^2 = -Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\varrho^\ell + |b|),$$
$$(2|b|\varrho^k)^2 - (\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2)^2 = -Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\varrho^\ell + |b|),$$

one gets the derivative of F with respect to ϱ in the form

$$F'(\varrho) = \frac{-G(\varrho)}{\sqrt{-Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\varrho^\ell + |b|)}},$$

where

$$G(\varrho) = 2k|ab|\varrho^\ell\left(\frac{-\varrho^{2k} + |a|^2\varrho^{2\ell} + |b|^2}{2|ab|\varrho^\ell}\right)' + 2\ell|b|\varrho^k\left(\frac{\varrho^{2k} - |a|^2\varrho^{2\ell} + |b|^2}{2|b|\varrho^k}\right)'.$$

Then

$$F'(\varrho) = \frac{2(k^2\varrho^{2k} - k\ell\varrho^{2k} - k\ell|a|^2\varrho^{2\ell} + \ell^2|a|^2\varrho^{2\ell} + k\ell|b|^2)}{\varrho\sqrt{-Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\varrho^\ell + |b|)}},$$

i.e.,

$$F'(\varrho) = \frac{2(-k\varrho^k + \ell|a|\varrho^\ell)^2 - 2k\ell Q_2(\varrho)Q_3(\varrho)}{\varrho\sqrt{-Q_1(\varrho)Q_2(\varrho)Q_3(\varrho)(\varrho^k + |a|\varrho^\ell + |b|)}}, \qquad Q_1(\varrho)Q_2(\varrho)Q_3(\varrho) \neq 0.$$

Since $Q_2(\varrho)Q_3(\varrho) \le 0$ on $D(F)$, F' is positive on (ξ_L, ξ_R) provided $|a|\,|b|^{(\ell-k)/k} \le \sigma(k, \ell)$, and it is positive on $(\xi_L, \xi_{M_1}) \cup (\xi_{M_2}, \xi_R)$ provided $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. Since $F(\xi_{M_1}) = F(\xi_{M_2})$, we can conclude that F is strictly increasing on $D(F)$. □

Now we come back to the analysis of (4-2). Based on Lemma 4.4, its geometric interpretation is the following: we search for intersections ϱ of the function F (whose values are monotonically varying on $H(F) = [0, k\pi]$), and the modulus of the value θ moved via an integer multiple s of 2π.

To ensure the existence of such an intersection (and thus also solvability of (4-2)), we have to require

$$-k\pi \le \theta - 2s\pi \le k\pi.$$

This condition generates k integer values of s up to the case when $\theta/\pi + k$ is an even integer. In this case, there are $k+1$ integer values of s meeting the previous inequality; in particular, it is satisfied in the form of equality for two values of s, namely for $s = \theta/(2\pi) - k/2$ and $s = \theta/(2\pi) + k/2$. Substitution of both these values into (4-2) yields the same equation $F(\varrho) = k\pi$ having the unique root ξ_R. Also, (3-6) yields the same value of the argument φ for both the values. Altogether, these two values of s generate the same (simple) root of p; hence we can restrict to

$$-k\pi < \theta - 2s\pi \le k\pi,$$

i.e.,

$$s = \left\lceil \frac{\theta}{2\pi} - \frac{k}{2} \right\rceil, \left\lceil \frac{\theta}{2\pi} - \frac{k}{2} \right\rceil + 1, \dots, \left\lceil \frac{\theta}{2\pi} + \frac{k}{2} \right\rceil - 1. \tag{4-4}$$

Now let $(s_1, \dots, s_k)$ be a permutation of the k-tuple of integers from (4-4) such that

$$|\theta - 2s_j\pi| \le |\theta - 2s_{j+1}\pi| \quad \text{for all } j = 1, \dots, k-1. \tag{4-5}$$

For the sake of uniqueness, if the equality sign occurs here (it happens just when θ is an integer multiple of π), then we assume $\theta - 2s_j\pi > 0$ and $\theta - 2s_{j+1}\pi < 0$. To get a more explicit prescription for s_j, it is enough to rewrite (4-5) as

$$\left| \frac{\theta}{2\pi} - s_j \right| \le \left| \frac{\theta}{2\pi} - s_{j+1} \right| \quad \text{for all } j = 1, \dots, k-1,$$

i.e., values s_j are ordered with respect to their distance from $\theta/(2\pi)$. Using this geometric interpretation, it is easy to check that

$$s_1 = \operatorname{round}\left(\frac{\theta}{2\pi}\right), \qquad s_j = s_1 + \kappa \left\lfloor \frac{j}{2} \right\rfloor, \quad j = 2, \dots, k, \tag{4-6}$$

where round$(\,\cdot\,)$ means the nearest integer value (if θ/π is an odd integer, then we put $s_1 = \theta/(2\pi) - \frac{1}{2}$), and $\kappa = (-1)^j$ if $s_1 < \theta/(2\pi)$, or $\kappa = (-1)^{j+1}$ if $s_1 \ge \theta/(2\pi)$.

Now we are ready to formulate an algorithm for computations of moduli and arguments of roots of a given trinomial.

Theorem 4.5. *Let $z_j = \varrho_j \exp(\mathrm{i}\varphi_j)$, $\varphi_j \in (-\pi, \pi]$, $j = 1, \dots, k$, be roots of p, where a, b are nonzero complex numbers and $k > \ell$ are coprime positive integers. Further, let s_j, $j = 1, \dots, k$, be given by (4-6). Then ϱ_j are (unique) roots of*

$$F(\varrho) = |\theta - 2s_j\pi|, \quad j = 1, \dots, k, \tag{4-7}$$

and

$$\varphi_j \equiv \begin{cases} \dfrac{(2v_0 s_j - 1)\pi + \beta_{\varrho_j} + \theta_b}{k} \pmod{2\pi} & \text{if } \theta - 2s_j\pi \le 0, \\[2ex] \dfrac{(2v_0 s_j - 1)\pi - \beta_{\varrho_j} + \theta_b}{k} \pmod{2\pi} & \text{if } \theta - 2s_j\pi > 0. \end{cases} \tag{4-8}$$

Here, v_0 is the second component of a couple of Bézout coefficients (u_0, v_0) satisfying (3-3), and β_{ϱ_j} are given by (1-1) with $\varrho = \varrho_j$.

Remark 4.6. Theorem 4.5 offers a computational procedure for finding moduli ϱ_j and arguments φ_j of all roots of p. Based on this procedure, we are able to find all k moduli ϱ_j as roots of k (real) transcendental equations (4-7) with appropriate integer values s_j, $j = 1, \dots, k$, given by (4-6). In particular, we can a priori determine s_j such that (4-7) generates the maximal modulus. Then, a deeper analysis of such an equation may result in strong bounds of the maximal modulus. Note that this matter is crucial (and still insufficiently analyzed) in the frame of asymptotic theory for autonomous difference equations.

Despite a lot of literature on solving sparse polynomials (see, for example, [Tonelli-Cueto and Tsigaridas 2023]), we believe that Theorem 4.5 can provide a new insight into the distribution problem of trinomial roots. By (4-7), the moduli ϱ_j are the intersections of the transcendental function F (depending on moduli of a, b), and a constant function (depending on arguments of a, b) that is moved (in modulus) via an integer multiple of 2π. Furthermore, (4-8) yields the exact formula for the dependence of arguments φ_j on moduli ϱ_j. Besides its direct meaning, this formula might be useful in discussions on argument discrepancies and related equidistribution properties of roots of complex trinomials (see, for example, [D'Andrea et al. 2014] and [Erdős and Turán 1950]).

Some other comments on the application of Theorem 4.5 are presented in Example 5.3.

Proof of Theorem 4.5. The part describing calculations of the moduli ϱ_j follows from observations preceding this assertion. The formula (4-8) for the values of arguments φ_j is a direct consequence of (3-6). □

Now we present two consequences of Theorem 4.5. The first assertion answers problem (C), that is, formulates conditions under which p has two roots with the same modulus.

Corollary 4.7. *Let ϱ_j, $j = 1, \dots, k$, be moduli of roots of p labeled with respect to (4-7). Further, let ξ_L, ξ_R and ξ_M, ξ_{M_1}, ξ_{M_2} be positive roots of Q_1, Q_2 and Q_3, respectively (their existence is guaranteed by Proposition 4.1). Finally, let θ and $\sigma(k, \ell)$ be given by (1-2) and (4-3), respectively. We distinguish two cases:*

(i) *If θ/π is not an integer, then the ϱ_j satisfy the strict inequality*

$$\xi_L < \varrho_1 < \varrho_2 < \cdots < \varrho_k < \xi_R.$$

(ii) *If θ/π is an integer, then the ordering of ϱ_j is summarized in Table 1. Here, we use the symbols E or O if the appropriate integer values are even or odd, respectively, and put $\Sigma = \Sigma(a, b, k, \ell) = |a|\,|b|^{(\ell-k)/k}/\sigma(k, \ell)$.*

θ/π	k	ℓ	Σ	moduli ϱ_j $(j=1,\dots,k)$: their ordering and specific values
E	O	O		$\xi_L=\varrho_1<\varrho_2=\varrho_3<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
O	O	E		$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_{k-2}=\varrho_{k-1}<\varrho_k=\xi_R$
O	E	O	<1	$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_\ell$ $=\varrho_{\ell+1}<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
			$=1$	$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_\ell=\xi_M$ $=\varrho_{\ell+1}<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
			>1	$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_\ell$ $=\xi_{M_1}<\xi_{M_2}=\varrho_{\ell+1}<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
E	E	O		$\xi_L=\varrho_1<\varrho_2=\varrho_3<\dots<\varrho_{k-2}=\varrho_{k-1}<\varrho_k=\xi_R$
E	O	E	<1	$\xi_L=\varrho_1<\varrho_2=\varrho_3<\dots<\varrho_\ell$ $=\varrho_{\ell+1}<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
			$=1$	$\xi_L=\varrho_1<\varrho_2=\varrho_3<\dots<\varrho_\ell=\xi_M$ $=\varrho_{\ell+1}<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
			>1	$\xi_L=\varrho_1<\varrho_2=\varrho_3<\dots<\varrho_\ell$ $=\xi_{M_1}<\xi_{M_2}=\varrho_{\ell+1}<\dots<\varrho_{k-1}=\varrho_k<\xi_R$
O	O	O	<1	$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_\ell$ $=\varrho_{\ell+1}<\dots<\varrho_{k-2}=\varrho_{k-1}<\varrho_k=\xi_R$
			$=1$	$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_\ell=\xi_M$ $=\varrho_{\ell+1}<\dots<\varrho_{k-2}=\varrho_{k-1}<\varrho_k=\xi_R$
			>1	$\xi_L<\varrho_1=\varrho_2<\dots<\varrho_\ell=\xi_{M_1}<\xi_{M_2}$ $=\varrho_{\ell+1}<\dots<\varrho_{k-2}=\varrho_{k-1}<\varrho_k=\xi_R$

Table 1. Ordering of moduli ϱ_j $(j=1,\dots,k)$ provided θ/π is an integer (for several specific values of k and ℓ, some parts of the presented inequalities lose a formal sense; in such cases, these inequalities need to be simplified appropriately).

Proof. If θ/π is not an integer, then the strict ordering of moduli ϱ_j follows from the strict monotony of the sequence $(|\theta-2s_j\pi|)_{j=1}^k$. If θ/π is an integer, then $|\theta-2s_j\pi|=|\theta-2s_{j+1}\pi|$; hence $\varrho_j=\varrho_{j+1}$ for some $j=1,\dots,k-1$. Furthermore, θ/π is an integer if and only if at least one of the numbers

$$\frac{\theta}{\pi},\qquad \frac{\theta}{\pi}+k,\qquad \frac{\theta}{\pi}+\ell$$

is even (note that all three values cannot be simultaneously even). This implies six parity variants concerning θ/π, k and ℓ that produce slightly different conclusions on the ordering of the ϱ_j presented in Table 1 (this ordering reflects a type of monotony of the sequence $(|\theta - 2s_j\pi|)_{j=1}^{k}$, and its derivation is quite straightforward for any of the six variants). □

Remark 4.8. If θ/π is an integer, then Table 1 immediately implies the following location of moduli ϱ_j, $j = 1, \dots, k$ (we still assume that the ϱ_j are labeled with respect to (4-7)): Let $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. Then

(4-9) $\xi_L \le \varrho_j \le \xi_{M_1}, \quad j = 1, \dots, \ell, \quad$ and $\quad \xi_{M_2} \le \varrho_j \le \xi_R, \quad j = \ell+1, \dots, k.$

In fact, it is easy to check that (4-9) holds for noninteger values of θ/π as well. Indeed, by Proposition 4.1 (the case (vi)) and Lemma 4.4, it is enough to search for a number of solutions of (4-2), where $\xi_L \le \varrho \le \xi_{M_1}$, i.e., $0 \le F(\varrho) \le \ell\pi$. Thus, we need to find all s_j from (4-6) such that $\ell\pi \ge |\theta - 2s_j\pi|$. Equivalently,

$$\frac{\theta}{2\pi} - \frac{\ell}{2} < s_j < \frac{\theta}{2\pi} + \frac{\ell}{2}.$$

Exactly ℓ values of s_j from (4-6) (namely $s_1, \dots, s_\ell$) satisfy these inequalities.

As the second consequence of Theorem 4.5, we state some interesting connections between moduli and arguments of roots z_ℓ, $z_{\ell+1}$ of p. In particular, we describe the situation when p has two roots with the same argument.

Corollary 4.9. *Let ϱ_j and φ_j, $j = \ell, \ell+1$, be moduli and arguments of roots z_ℓ, $z_{\ell+1}$ of p labeled with respect to* (4-7) *and* (4-8), *respectively. Let θ and $\sigma(k, \ell)$ be given by* (1-2) *and* (4-3), *respectively, and let θ/π and ℓ have the same parity. Then three qualitatively different relations between moduli and arguments of z_ℓ, $z_{\ell+1}$ can occur*:

(i) *If $|a|\,|b|^{(\ell-k)/k} < \sigma(k, \ell)$, then $\varrho_\ell = \varrho_{\ell+1}$ and $\varphi_\ell \ne \varphi_{\ell+1}$ (i.e., we have two distinct simple roots $z_\ell \ne z_{\ell+1}$ with the same moduli and different arguments).*

(ii) *If $|a|\,|b|^{(\ell-k)/k} = \sigma(k, \ell)$, then $\varrho_\ell = \varrho_{\ell+1}$ and $\varphi_\ell = \varphi_{\ell+1}$ (i.e., we have a double root $z_\ell = z_{\ell+1}$).*

(iii) *If $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$, then $\varrho_\ell < \varrho_{\ell+1}$ and $\varphi_\ell = \varphi_{\ell+1}$ (i.e., we have two distinct simple roots $z_\ell \ne z_{\ell+1}$ with the same arguments and different moduli).*

Proof. Since θ/π and ℓ have the same parity, the properties (i) and (ii) follow from Table 1 (with respect to the conditions of Remark 3.2 (b)).

Let $|a|\,|b|^{(\ell-k)/k} > \sigma(k, \ell)$. Then $\varrho_\ell = \xi_{M_1} < \xi_{M_2} = \varrho_{\ell+1}$. Because of (3-6) and the fact that $\alpha_{\varrho_\ell} = \alpha_{\varrho_{\ell+1}} = 0$, $\beta_{\varrho_\ell} = \beta_{\varrho_{\ell+1}} = \pi$, the arguments φ_ℓ and $\varphi_{\ell+1}$ are generated by the formula

$$\varphi^{\pm} \equiv \frac{(\theta \pm \ell\pi)v_0 - \pi \pm \pi + \theta_b}{k} \pmod{2\pi}.$$

Since

$$\frac{(\theta - \ell\pi)v_0 - 2\pi + \theta_b}{k} = \frac{(\theta + \ell\pi)v_0 + \theta_b}{k} - \frac{\ell v_0 + 1}{k} 2\pi,$$

both the values $\varphi^{\pm}$ coincide, that is, $\varphi_\ell = \varphi_{\ell+1}$. □

5. Some comparisons with existing results

We compare conclusions of our main results (Theorems 3.1 and 4.5) with previous answers to problems (B)–(D).

5.1. ***Problem (B) — Theorem 3.1 and some earlier results.*** We start with comparisons between our Theorem 3.1 and Theorem 4.1 of [Theobald and de Wolff 2016] solving problem (B). Its formulation uses a roulette curve called a hypotrochoid which depends on three general positive real constants r, R, d with $r < R$. In the Gauss a-plane, this curve can be described by the parametric equation

$$\Re(a) + \mathrm{i}\Im(a) = (R-r)\exp(\mathrm{i}t) + d\exp\left(\mathrm{i}\frac{r-R}{r}t\right),$$

where t is a real parameter. If R/r is a rational number, then the hypotrochoid is a closed curve (for more interesting properties of this curve see [Lockwood 2007]). The assertion of Theorem 4.1 of [Theobald and de Wolff 2016] now can be formulated as follows:

> *Let b be a nonzero complex number, $k > \ell$ be coprime positive integers, and let ϱ be a positive real number. Then p has a ϱ-modular root z if and only if its complex coefficient a is located on a hypotrochoid up to a rotation with parameters*
>
> $$R = \frac{k\varrho^{k-\ell}}{\ell}, \quad r = \frac{(k-\ell)\varrho^{k-\ell}}{\ell}, \quad d = |b|\,\varrho^{-\ell}.$$

Equivalently, this condition says that p has a ϱ-modular root z if and only if its complex coefficient a satisfies

$$\begin{aligned} \Re(a) &= -(R-r)\cos\left(t + \frac{r}{R}\theta_b\right) - d\cos\left(\frac{r-R}{r}t + \frac{r}{R}\theta_b\right), \\ \Im(a) &= -(R-r)\sin\left(t + \frac{r}{R}\theta_b\right) - d\sin\left(\frac{r-R}{r}t + \frac{r}{R}\theta_b\right) \end{aligned} \tag{5-1}$$

for a suitable $t \in (-(k-\ell)\pi, (k-\ell)\pi]$.

To compare this result with Theorem 3.1, we rearrange the conclusions of Theorem 3.1 in the following way: if b is considered to be fixed, then we need to find all complex values a such that

$$\left|\varrho^{k-\ell} - |b|\,\varrho^{-\ell}\right| \le |a| \le \varrho^{k-\ell} + |b|\,\varrho^{-\ell}, \tag{5-2}$$

and at least one of the values $\tau_\varrho^{\pm}$ is an integer. Equivalently,

$$\theta_a = \pm\left(\alpha_\varrho + \frac{\ell}{k}\beta_\varrho\right) + \frac{(k-\ell)(\theta_b - \pi)}{k} + \frac{2m\pi}{k}, \tag{5-3}$$

where m is a positive integer such that $-\pi < \theta_a \le \pi$ (it is easy to check that there exist at most $2k$ integer values of m with this property). Thus, (5-3) offers a polar representation of all coefficients a such that p has a ϱ-modular root z. Notice that this representation provides an explicit dependence of the argument θ_a on modulus $|a|$ (we recall that $|a|$ is involved in α_ϱ and β_ϱ introduced by (1-1)). In the Gauss a-plane, (5-3) defines a closed curve whose parts are generated by the above specified integer values of m.

To demonstrate the usefulness of this polar representation, we utilize Example 4.2 of [Theobald and de Wolff 2016] serving as an illustration of Theorem 4.1 of the same article.

Example 5.1. We describe the set of all complex numbers a such that the trinomial

$$f(z) = z^5 + az + 1$$

has a unimodular root. By Theorem 4.1 of [Theobald and de Wolff 2016], this occurs if and only if the complex coefficient a is located on the trajectory of the hypotrochoid with parameters $R = 5$, $r = 4$, $d = 1$, or equivalently (according to (5-1)), a satisfies the equations

$$\Re(a) = -\cos(t) - \cos\left(\frac{t}{4}\right), \quad \Im(a) = -\sin(t) + \sin\left(\frac{t}{4}\right),$$

where $t \in (-4\pi, 4\pi]$. Now we apply conclusions of our previous considerations following from Theorem 3.1. If we put $b = 1$, $k = 5$, $\ell = 1$ and $\varrho = 1$, then

$$\alpha_1 = \arccos\frac{|a|}{2}, \quad \beta_1 = \arccos\frac{2-|a|^2}{2} = \pi - 2\arccos\frac{|a|}{2},$$

and (5-2), (5-3) become

$$0 \le |a| \le 2, \quad \theta_a = \pm\frac{1}{5}\left(\pi + 3\arccos\frac{|a|}{2}\right) + \frac{2m\pi - 4\pi}{5}. \tag{5-4}$$

In the case of the plus variant, (5-4) can be rewritten (using straightforward calculations) as

$$|a| = 2\cos\left(\frac{5}{3}\theta_a + s^+\pi\right), \quad \theta_a \in \left[\frac{-6s^+\pi}{10}, \frac{3\pi - 6s^+\pi}{10}\right], \tag{5-5}$$

where $s^+ = \frac{5}{3}, 1, \frac{1}{3}, -\frac{1}{3}, -1$. (To be consistent with the assumption $-\pi < \theta_a \le \pi$, we formally remove the left endpoint from this interval if $s^+ = \frac{5}{3}$.) Similarly, if the minus sign is considered in (5-4), then we get the remaining set of conditions, namely

$$|a| = 2\cos\left(\frac{5}{3}\theta_a + s^-\pi\right), \quad \theta_a \in \left[\frac{-3\pi - 6s^-\pi}{10}, \frac{-6s^-\pi}{10}\right], \tag{5-6}$$

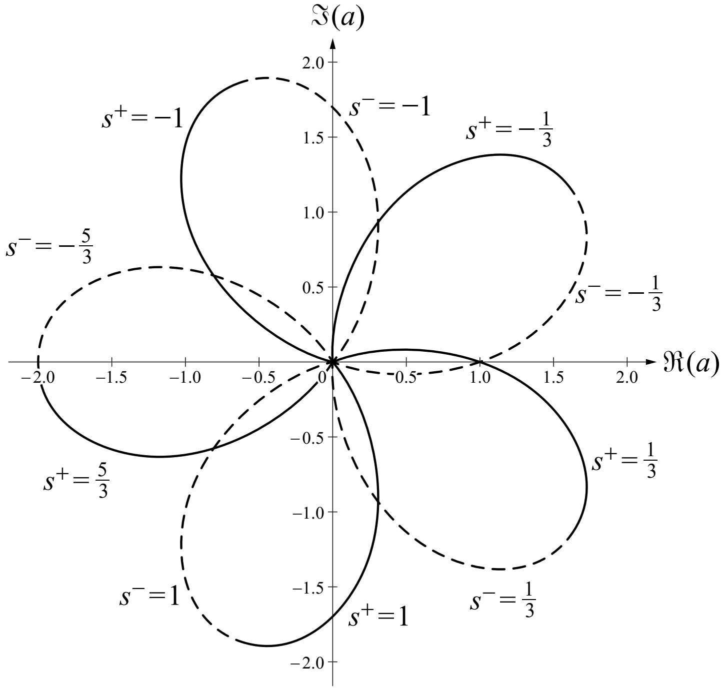

Figure 1. Parts of hypotrochoid described by (5-5) (curves labeled with the corresponding values of s^+) and (5-6) (dashed curves labeled with the corresponding values of s^-).

where $s^- = 1, \frac{1}{3}, -\frac{1}{3}, -1, -\frac{5}{3}$. Thus, (5-5) and (5-6) yield polar descriptions of all coefficients a such that f has a unimodular root. The corresponding curves are depicted in the a-plane in Figure 1.

Analogously, we can proceed in a more general case when the powers $k = 5$ and $\ell = 1$ in f are replaced by general coprime integers $k > \ell$. In this case, the trinomial

$$g(z) = z^k + az^\ell + 1$$

has a unimodular root if and only if

$$\text{(5-7)}\quad 0 \le |a| \le 2, \quad \theta_a = \pm\frac{1}{k}\left(\ell\pi + (k-2\ell)\arccos\frac{|a|}{2}\right) - \frac{(k-\ell)\pi}{k} + \frac{2m\pi}{k},$$

where m is a positive integer such that $-\pi < \theta_a \le \pi$. Moreover, (5-7) can possibly be converted into polar forms analogous to (5-5) and (5-6).

Such polar representations can offer a better insight into the structure of all complex numbers a such that p has a root with a given modulus. Among others, they enable us to decide immediately whether a given complex number a has this property. If the answer is affirmative, then Theorem 3.1 provides an additional benefit, namely calculation of the argument of such a root.

Example 5.2. We consider the trinomial f with $a = 1 + \sqrt{2}/2 + (\sqrt{2}/2)\mathrm{i}$, and discuss the existence of its unimodular root. First, we apply directly Theorem 3.1. After checking (3-5), one gets $\tau_1^+ = 3$, $\tau_1^- = 1.625$, which implies that f (with the above specified coefficient a) actually has a unique unimodular root. The Bézout coefficients for the couple $(5, 4)$ are $u_0 = 1$, $v_0 = -1$ due to (3-3). Then, using (3-6) with $\beta_1 = 3\pi/4$ and $\theta_b = 0$, one can find the exact form of this unimodular root, namely $z = \exp(-5\pi\mathrm{i}/4)$. Of course, verification of the existence of this unimodular root of f can be equivalently done by (5-5), (5-6).

On the other hand, application of Theorem 4.1 of [Theobald and de Wolff 2016] to this problem is more complicated. In general, numerical solution of a nonlinear equation with unknown parameter t is required.

5.2. ***Problem (D) — Theorem 4.5 and some earlier results.*** Now we turn our attention to Theorem 4.5 and its relevance with respect to previous results dealing with problem (D). In [Avendaño et al. 2018], the explicit metric bounds, circumscribing the annuli where log-moduli of roots of p cluster, are derived by use of a concept of Archimedean tropical variety. However, currently strongest bounds on moduli and arguments of roots of p were presented in [Melman 2012]; hence we shortly comment on the main results of this paper. Here, k disjoint annular sectors, each containing just one root of p were derived. More precisely, [Melman 2012] analyzes location of roots of p with $b = -1$ (which can be done without loss of generality). On this account, we involve this formal simplification in our next considerations as well. Forms of these sectors slightly differ with respect to the cases $|a| > \sigma(k, \ell)$ and $|a| < \sigma(k, \ell)$. In the sequel, we comment on the first case (discussions of the latter one are analogous).

If $|a| > \sigma(k, \ell)$, then Theorem 4.1 of [Melman 2012] describes several bounds on moduli and arguments of roots $z_j = \varrho_j \exp(\mathrm{i}\varphi_j)$ of p labeled so that $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, k-1$. The strongest bounds on the ϱ_j following from Theorem 4.1 and Remarks 4.2 of [Melman 2012] are given by (4-9) (thus, our observations made in Remark 4.8 confirm these bounds). Similarly, the arguments φ_j are located in k disjoint intervals whose lengths again depend on whether j belongs to the set $\{1, \dots, \ell\}$, or $\{\ell+1, \dots, k\}$ (for more details, including bounds not utilizing ξ_L, ξ_R, ξ_{M_1}, ξ_{M_2}, see [Melman 2012]).

Now we clarify the position of Theorem 4.5 with respect to problem (D) and results from [Melman 2012]. Theorem 4.5 enables us to calculate moduli ϱ_j of the k roots of p as a numerical solution of the k transcendental equations (4-7) which differ from each other only by an additive constant appearing on their right-hand side. Also, the interval $[\xi_L, \xi_R]$ was introduced here as a localization interval containing just one root for each of these equations. Notice that this interval can be slightly precised due to (4-9). Then, using an appropriate procedure (such as the bisection method) applied to (4-7), lengths of these localization intervals can be made arbitrarily small, and moduli ϱ_j of all roots of p can be computed with any prescribed precision. Finally, having at our disposal moduli of roots, their arguments are given directly by (4-8).

We demonstrate this algorithm via Example 4.4 of [Melman 2012] that illustrated bounds derived in Theorem 4.1 of that same article.

Example 5.3. We consider the trinomial

$$h(z) = z^{10} - (1.6+\mathrm{i})z^7 - 1.$$

In this case, $a = -1.6 - \mathrm{i}$, $b = -1$, $k = 10$ and $\ell = 7$; hence $|a| = \sqrt{89}/5 = 1.8868$ and $\sigma(10, 7) = 1.8420$. Let $z_j = \varrho_j \exp(\mathrm{i}\varphi_j)$, $j = 1, \dots, 10$, be roots of h labeled in a way so that $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, 9$. Using tropical methods, one can obtain that the z_j cluster around the circles with radii $r_1 = 0.9133$ and $r_2 = 1.2357$ for $j = 1, \dots, 7$ and $j = 8, 9, 10$, respectively. The explicit bounds given by Theorem 1.5 of [Avendaño et al. 2018] are nevertheless too wide, namely

$$0.4566 \le \varrho_j \le 2.4714, \quad j = 1, \dots, 10.$$

Example 4.4 of [Melman 2012] presents several tighter bounds for ϱ_j; the best of them, based on (4-9), yield

$$\begin{aligned} 0.8746 \le \varrho_j \le 1.0389, &\quad j = 1, \dots, 7, \\ 1.1438 \le \varrho_j \le 1.2744, &\quad j = 8, 9, 10. \end{aligned} \tag{5-8}$$

We employ Theorem 4.5 to specify values of these moduli. Doing this, we follow the algorithm summarized in Remark 4.6. First, (4-6) with $\theta = -25.8299$ yields

$$\begin{aligned} &s_1 = -4, \quad s_2 = -5, \quad s_3 = -3, \quad s_4 = -6, \quad s_5 = -2, \\ &s_6 = -7, \quad s_7 = -1, \quad s_8 = -8, \quad s_9 = 0, \quad s_{10} = -9. \end{aligned}$$

Then, by (4-7), moduli ϱ_j of roots z_j are solutions of

$$\begin{aligned} 10 \arccos(-0.2650\varrho^{13} + 0.9434\varrho^7 + 0.2650\varrho^{-7}) \\ + 7 \arccos(0.5\varrho^{10} - 1.78\varrho^4 + 0.5\varrho^{-10}) \\ = |25.8299 + 2s_j\pi|, \quad j = 1, \dots, 10. \end{aligned} \tag{5-9}$$

A clear geometric interpretation of these equations, depicted in Figure 2, offers a better understanding of the location of all the moduli (obviously, $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, 9$). As localization intervals for the use of appropriate numerical calculations, the bounds (5-8) can be used (of course, lengths of these intervals can be made

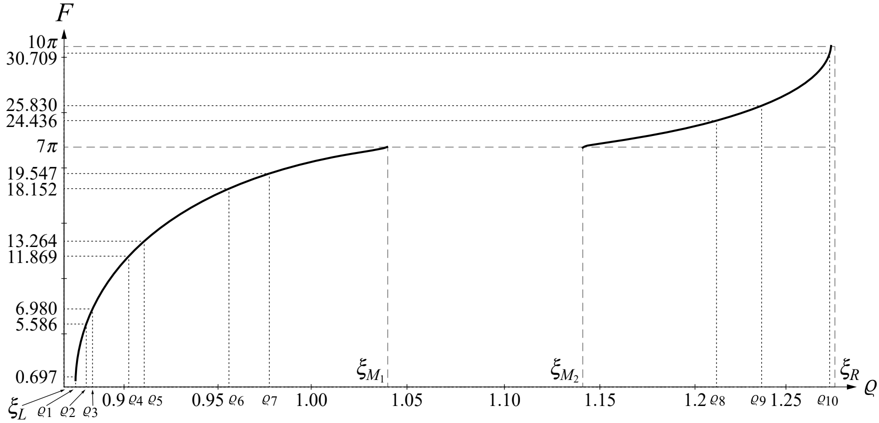

Figure 2. Geometric illustration of (5-9), and specification of some significant values.

arbitrarily small due to (5-9)). MATLAB's `fzero` routine[1] provides (in its default setting) the roots of (5-9) as

$$\varrho_1 = 0.8746, \quad \varrho_2 = 0.8803, \quad \varrho_3 = 0.8836, \quad \varrho_4 = 0.9027, \quad \varrho_5 = 0.9108,$$
$$\varrho_6 = 0.9557, \quad \varrho_7 = 0.9771, \quad \varrho_8 = 1.2140, \quad \varrho_9 = 1.2379, \quad \varrho_{10} = 1.2739.$$

Knowing moduli of all 10 roots of h now enables us, along with the formula (4-8), to directly compute their arguments as

$$\varphi_1 = 1.2632, \quad \varphi_2 = 3.0882, \quad \varphi_3 = -0.5613, \quad \varphi_4 = -1.3737, \quad \varphi_5 = -2.3808,$$
$$\varphi_6 = 0.4322, \quad \varphi_7 = 2.1059, \quad \varphi_8 = 2.3238, \quad \varphi_9 = 0.1467, \quad \varphi_{10} = -1.9025.$$

5.3. ***Problem (C) — Corollaries 4.7, 4.9, and some earlier results.*** Finally, we consider some consequences of Theorem 4.5 (discussed in Corollaries 4.7, 4.9, and summarized in Table 1) confirming and extending a series of assertions from [Theobald and de Wolff 2016] that are related to problem (C). In particular, our previous discussions confirmed that at most two roots of p have the same modulus (see also Proposition 4.3 of [Theobald and de Wolff 2016]), and formulated conditions under which two roots of p share the same modulus (see also Theorems 4.4 and 4.9 of [Theobald and de Wolff 2016]). Also, we clarified whether equality or strict inequality occurs between moduli ϱ_ℓ and $\varrho_{\ell+1}$ of roots z_ℓ and $z_{\ell+1}$ of a given trinomial p (see also Corollary 4.13 of [Theobald and de Wolff 2016]).

[1]The `fzero` algorithm uses a combination of bisection, secant, and inverse quadratic interpolation methods — the so-called Brent's method. It is known that the order of convergence is superlinear for well-behaved functions.

θ_a	$\lvert a\rvert$	$j = 1, \dots, 5$
$\pi/2$	6	ϱ_j: $0.5426 < 0.5498 < 0.5587 < 2.4397 < 2.4593$ φ_j: $0.5312,\ -1.5875,\ 2.6274,\ 2.3521,\ -0.7815$
0	6	ϱ_j: $0.5416 < 0.5546 = 0.5546 < 2.4498 = 2.4498$ φ_j: $\pi,\ 1.0318 \neq -1.0318,\ -1.5765 \neq 1.5765$
π	1.95	ϱ_j: $0.7534 = 0.7534 < 1.0842 = 1.0842 < 1.4990$ φ_j: $2.0297 \neq -2.0297,\ -0.0415 \neq 0.0415,\ \pi$
	$\sigma(5, 3)$	ϱ_j: $0.7524 = 0.7524 < 1.0845 = 1.0845 < 1.5018$ φ_j: $2.0301 \neq -2.0301,\ 0 = 0,\ \pi$
	1.97	ϱ_j: $0.7516 = 0.7516 < 1.0413 < 1.1301 < 1.5045$ φ_j: $2.0304 \neq -2.0304,\ 0 = 0,\ \pi$

Table 2. Moduli ϱ_j and arguments φ_j of roots z_j of q with given a, $j = 1, \dots, 5$.

We illustrate these observations by an extended version of a part of Example 4.10 of [Theobald and de Wolff 2016] that supported theoretical results obtained in that same article.

Example 5.4. We consider the trinomial

$$q(z) = z^5 + az^3 + 1.$$

Again, let $z_j = \varrho_j \exp(i\varphi_j)$ be roots of q labeled so that $\varrho_j \le \varrho_{j+1}$, $j = 1, \dots, 4$. Note that q with $a = 6$ was considered in Example 4.10 of [Theobald and de Wolff 2016] where the resulting relations between moduli ϱ_j appeared in the form

$$\varrho_1 < \varrho_2 = \varrho_3 < \varrho_4 = \varrho_5.$$

Based on Corollaries 4.7, 4.9 and Table 1, we perform the same discussion on ϱ_j with respect to the variable complex number a considered in q. Also, we state relations between some values of the arguments φ_j. Doing this, we first notice that considering the trinomial q we have $k = 5$, $\ell = 3$, $b = 1$, $\theta = 5\theta_a + 2\pi$ and $\sigma(5, 3) = 1.9601$. Then observations made in Corollaries 4.7, 4.9 and summarized in Table 1 imply (see Table 2):

- Let $\theta_a \neq j\pi/5$ for any integer j. Then

$$\varrho_1 < \varrho_2 < \varrho_3 < \varrho_4 < \varrho_5.$$

- Let $\theta_a = j\pi/5$ for some $j = 0, \pm 2, \pm 4$ (we note that this case includes the above choice $a = 6$). Then

$$\varrho_1 < \varrho_2 = \varrho_3 < \varrho_4 = \varrho_5, \quad \varphi_2 \neq \varphi_3,\ \varphi_4 \neq \varphi_5.$$

- Let $\theta_a = j\pi/5$ for some $j = \pm 1, \pm 3, 5$, and let $|a| < \sigma(5,3)$. Then
$$\varrho_1 = \varrho_2 < \varrho_3 = \varrho_4 < \varrho_5, \quad \varphi_1 \neq \varphi_2,\ \varphi_3 \neq \varphi_4.$$
- Let $\theta_a = j\pi/5$ for some $j = \pm 1, \pm 3, 5$, and let $|a| = \sigma(5,3)$. Then
$$\varrho_1 = \varrho_2 < \varrho_3 = \varrho_4 < \varrho_5, \quad \varphi_1 \neq \varphi_2,\ \varphi_3 = \varphi_4.$$
- Let $\theta_a = j\pi/5$ for some $j = \pm 1, \pm 3, 5$, and let $|a| > \sigma(5,3)$. Then
$$\varrho_1 = \varrho_2 < \varrho_3 < \varrho_4 < \varrho_5, \quad \varphi_1 \neq \varphi_2,\ \varphi_3 = \varphi_4.$$

6. Concluding remarks

We focused on several basic questions concerning moduli and arguments of roots of complex trinomials. Keeping in mind that similar problems were topics of many earlier investigations, we aimed to offer new views and new answers to these questions.

Our two main results analyzed problems (B) and (D) stated in the introduction. The assertion of Theorem 3.1 enabled us to describe the set of all entry parameters of a general trinomial p such that p has a root with a prescribed modulus. We were able to calculate the arguments of such roots, and thus obtain their complete identification. Theorem 4.5 described a procedure how to localize and compute moduli and arguments of roots of complex trinomials with arbitrary precision.

We believe that these results and their consequences can contribute not only to trinomial theory itself, but also to other areas connected with questions we discussed. In particular, conclusions of Corollary 2.3 and Theorem 3.1 have a considerable application potential towards qualitative theory of autonomous difference equations (stability of their equilibria, existence of periodic solutions, asymptotic bounds of solutions). Also, the comparisons performed in Section 5 indicate alternate possibilities of numerical evaluation of roots of complex trinomials, and new insights into analytic descriptions of some roulette curves (hypotrochoids, epitrochoids). Of course, investigations of problems (A)–(D) in the context of polynomials with more than three terms remain the main (and probably very difficult) challenge.

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ON THE TRANSIENT NUMBER OF A KNOT

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The transient number of a knot K, denoted $\mathrm{tr}(K)$, is the minimal number of simple arcs that have to be attached to K, in order for K to be homotoped to a trivial knot in a regular neighborhood of the union of K and the arcs. We give a lower bound for $\mathrm{tr}(K)$ in terms of the rank of the first homology group of the double branched cover of K. In particular, if $\mathrm{tr}(K) = 1$, then the first homology group of the double branched cover of K is cyclic. Using this, we can calculate the transient number of many knots in the tables and show that there are knots with arbitrarily large transient number.

1. Introduction

Let K be a knot in the 3-sphere and let M be a submanifold of S^3 containing K. We say that K is transient in M if K can be homotoped within M to the trivial knot in S^3; otherwise K is called persistent. For example, K is persistent in a regular neighborhood $\mathcal{N}(K)$ of K, but it is transient in a 3-ball B containing K. Yuya Koda and Makoto Ozawa [2] proved that every knot is transient in a submanifold M if and only if M is unknotted, that is, its complement in S^3 is a union of handlebodies. Then Koda and Ozawa [2] introduced a new invariant of knots, called the transient number of K, which somehow measures, starting with $\mathcal{N}(K)$, how large must be a submanifold in which K is transient.

The transient number is defined as follows: given a knot K in S^3, there is a collection of arcs $\{\tau_1, \tau_2, \dots, \tau_n\}$, disjointly embedded in S^3, each τ_i intersecting K exactly at its endpoints, such that K can be homotoped in a regular neighborhood of K union the arcs, $T = \mathcal{N}(K \cup \tau_1 \cup \cdots \cup \tau_n)$, into the trivial knot. That is, we perform crossing changes and isotopies inside T, until we get the trivial knot K'. Note that any knot K' obtained from K in this way is not trivial in T, i.e., it cannot bound a disk contained in T, but it can be trivial in S^3. The transient number of K, $\mathrm{tr}(K)$, is then defined as the minimal number of arcs needed in such a system of arcs. The transient number is related to other knot invariants, namely $\mathrm{tr}(K) \le u(K)$, where $u(K)$ is the unknotting number, and $\mathrm{tr}(K) \le t(K)$, where $t(K)$ is the tunnel number. It is easy to check these inequalities. For the unknotting number, given

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a sequence of crossing changes that unknot K, consider for each crossing change an arc with endpoints in K that guides the crossing change, such that a regular neighborhood of the arc encapsulates the crossing change. Then clearly K can be made trivial in a neighborhood T of K union the arcs. For the case of the tunnel number, consider a tunnel system and a neighborhood T of the union of K and the arcs, so that the exterior is a handlebody. Isotope T so that it looks like a standard handlebody in S^3. Then K can be projected to the intersection of a plane with T, and guided by this projection to a plane, crossing changes can be performed to K inside T to get the trivial knot.

There is a knot K such that $\operatorname{tr}(K) = 1$ whereas $u(K)$ and $t(k)$ are larger than one. Some examples with this property are given in [2]. However, in that paper no example is given of a knot K with $\operatorname{tr}(K) > 1$. Homology groups of branched covers have been used to bound invariants like $u(K)$ and $t(K)$, which goes back to the work of Wendt [13]. In fact, it is well known that if $\Sigma[K]$ denotes the double branched cover of K, then the rank of the group $H_1(\Sigma[K])$ gives a lower bound for $u(K)$; see [13] or [4]. It is also not difficult to show that the rank of $H_1(\Sigma[K])$ is at most $2t(K)+1$; in particular it is known that if $t(K) = 1$ then $H_1(\Sigma[K])$ is a cyclic group (though not explicitly stated, this follows from the computations of homology of cyclic covers done in [1], or from [8]).

We prove that the rank of the first homology group of a cyclic branched cover of a knot gives lower bounds for the transient number. By using the Montesinos trick, it can be shown that if K is a knot with $u(K) = n$, then $\Sigma[K]$ can be obtained by Dehn surgery on an n-component link in S^3, which implies then the bound for $u(K)$. We do a kind of generalized Montesinos trick. Our main results are the following.

Theorem 1.1. *If K is a knot in S^3 such that* $\operatorname{tr}(K) = n$, *then the first homology group of the double branched cover of K has a presentation with at most $2n+1$ generators.*

Theorem 1.2. *If K is a knot in S^3 such that* $\operatorname{tr}(K) = n$, *then the first homology group of the p-fold cyclic branched cover of K has a presentation with at most $pn+1$ generators.*

These results imply that $\operatorname{rank}\bigl(H_1(\Sigma[K])\bigr) \leq 2\operatorname{tr}(K)+1$. If $\Sigma_p[K]$ denotes the p-fold cylic branched cover of K, it follows that $\operatorname{rank}\bigl(H_1(\Sigma_p[K])\bigr) \leq p\operatorname{tr}(K)+1$.

For the case that $\operatorname{tr}(K) = 1$, we can get a better bound. In fact, by doing a careful calculation of the first homology group of $\Sigma[K]$, we get the following result.

Theorem 1.3. *If K is a knot in S^3 such that* $\operatorname{tr}(K) = 1$, *then the first homology group of the double branched cover of K is cyclic.*

Of course, these results may not be sharp. It would be interesting to find sharp bounds for these inequalities. It would also be interesting to find bounds for the transient number depending on other classical invariants of knots.

Given any knot invariant, it is always interesting to study its behavior under connected sums of knots. We have the following:

Theorem 1.4. *Let K_1, K_2 be knots in S^3. Then* $\mathrm{tr}(K_1 \# K_2) \leq \mathrm{tr}(K_1) + \mathrm{tr}(K_2) + 1$.

The paper is organized as follows. In Section 2 we sketch a proof that the unknotting number and tunnel number are bounded below by the rank of the first homology group of double branched covers. Then we prove the main results. As part of the proofs, we show also that if $t(K) = 1$, then $H_1(\Sigma[K])$ is cyclic; this claim is used to prove Theorem 1.3. In Section 3 we give examples of knots with large transient number and explore the transient number of knots in the tables of KnotInfo [3]. In Section 4 we consider the transient number of a connected sum of knots, prove some facts and propose some problems.

We work in the piecewise linear category. To avoid cumbersome notation we use expressions like the double branched cover of a knot to mean the double cover of S^3 branched along the knot. If Λ is a simple closed curve in the boundary of a 3-manifold M, we say adding a 2-handle along Λ, to mean that we attach a 2-handle $D^2 \times I$ to M, such that $\partial D^2 \times I$ is identified with a regular neighborhood of Λ in ∂M, which is an annulus. Also, if M and T are compact 3-manifolds, with $T \subset M$, then by $M \setminus T$ we mean M minus the interior of T, or the closure in M of $M - T$. If X is a topological space, $|X|$ denotes its number of components.

2. Transient number and double branched covers

This section is inspired by an idea that is used to build the double branched cover of a knot with unknotting number equal to one. Consider a knot K in S^3 with unknotting number equal to one. Let α be an arc embedded in S^3, with endpoints in K, such that a regular neighborhood of it encapsulates the crossing change. So there is a homotopy in $\mathcal{N}(K \cup \alpha)$ between the knot K and the trivial knot, which is denoted by K'. Clearly this homotopy can be taken so that it is constant in $\mathcal{N}(K) \setminus \mathcal{N}(\alpha)$ and that the changes are occurring only in $\mathcal{N}(\alpha)$; so we assume that K' is obtained from K just by taking the two arcs $K \cap \mathcal{N}(\alpha)$ and passing one arc through the other, which would correspond to a crossing change in the corresponding knot diagram. Due to the above we have that $K \cap (S^3 \setminus \mathcal{N}(\alpha)) = K' \cap (S^3 \setminus \mathcal{N}(\alpha))$.

Let $\Sigma(K')$ be the double branched cover of the knot K', with covering function given by $p : \Sigma(K') \to S^3$. Now, since K' is the trivial knot, $\Sigma(K')$ is homeomorphic to S^3. We know that $\mathcal{N}(\alpha)$ is a 3-ball intersecting K' in two arcs, therefore $p^{-1}(\mathcal{N}(\alpha))$ is a solid torus, and $p^{-1}(\partial\mathcal{N}(\alpha))$ is a surface of genus one. Therefore, $S^3 \setminus p^{-1}(\mathcal{N}(\alpha))$ is a double cover of $S^3 \setminus \mathcal{N}(\alpha)$ branched along $K \cap (S^3 \setminus \mathcal{N}(\alpha))$. So to finish building the double branched cover of the knot K, all we have to do is to refill $S^3 \setminus p^{-1}(\mathcal{N}(\alpha))$ appropriately.

Note that there exists a compressing disk for $\partial(\mathcal{N}(\alpha))\setminus K$ contained in $\mathcal{N}(\alpha)\setminus K$; we denote this disk by D. As $K\cap D=\varnothing$, $|K'\cap D|$ is an even number, so the curve ∂D is lifted by p into two curves in $p^{-1}(\partial\mathcal{N}(\alpha))$; we denote these curves by Λ_1 and Λ_2. Let Σ' be the 3-manifold obtained by adding two 2-handles to the 3-manifold $S^3\setminus p^{-1}(\mathcal{N}(\alpha))$, attached along the curves Λ_1 and Λ_2; we denote these 2-handles by $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$, respectively. So $\Sigma'=\big[S^3\setminus p^{-1}(\mathcal{N}(\alpha))\big]\cup\overline{\Lambda}_1\cup\overline{\Lambda}_2$.

We know that $\Lambda_1\cup\Lambda_2$ is a double cover of ∂D with covering function given by $p|_{\Lambda_1\cup\Lambda_2}$. So we can extend the function $p|_{\Lambda_1\cup\Lambda_2}$ to $\overline{\Lambda}_1\cup\overline{\Lambda}_2$, to get that $\overline{\Lambda}_1\cup\overline{\Lambda}_2$ is a double cover of $\mathcal{N}(D)$. Thus, Σ' is a double cover of $\big[S^3\setminus p^{-1}(\mathcal{N}(\alpha))\big]\cup\mathcal{N}(D)$ branched along two arcs of K.

We have that $\partial\big([S^3\setminus\mathcal{N}(\alpha)]\cup\mathcal{N}(D)\big)$ consists of two 2-spheres and $\partial\Sigma'$ also consists of two 2-spheres. The 2-spheres of $\partial\Sigma'$ are a double cover of the two spheres of $\partial\big([S^3\setminus\mathcal{N}(\alpha)]\cup\mathcal{N}(D)\big)$ branched over the points $K\cap\partial\big([S^3\setminus\mathcal{N}(\alpha)]\cup\mathcal{N}(D)\big)$.

Now we can fill the sphere boundary components of Σ' with 3-balls, and extend the function p to these 3-balls in order to get the double covering of S^3 branched along the knot K.

The idea described above is known as the Montesinos trick. Similar to the previous construction, we will build the double branched covers of knots for which we know the tunnel number or the transient number. For the case of the tunnel number, note that if K has tunnel number n, then K is contained in a genus $n+1$ handlebody V, such that its complement is another genus $n+1$ handlebody W. By taking $\Sigma[K]$, V and W lift to genus $2n+1$ handlebodies, that is, give a genus $2n+1$ Heegaard decomposition of $\Sigma[K]$. This shows that $H_1(\Sigma[K])$ is an abelian group of rank at most $2n+1$.

The following lemma is a general result of coverings which we will use often. The proof is a standard argument, so we omit it.

Lemma 2.1. *Let M be a manifold. Let Σ be a double cover of M with covering function $p:\Sigma\to M$; and let $C\subset M$. If M is path connected and $p^{-1}(C)$ is connected then Σ is connected.*

The following theorem is our first important result of this section. We will see that if we are given a transient system of a knot we can construct the double branched cover of this knot and from there calculate its first homology group.

Theorem 2.2. *If K is a knot in S^3 such that $\operatorname{tr}(K)=n$, then the first homology group of the double branched cover of K has a presentation with at most $2n+1$ generators.*

Proof. Let K be a knot in S^3 such that $\operatorname{tr}(K)=n$, let $\{\tau_1,\tau_2,\dots,\tau_n\}$ be a transient system for K, and let $T=\mathcal{N}(K\cup\tau_1\cup\tau_2\cup\cdots\cup\tau_n)$, this is a genus $n+1$ handlebody. Let $K'\subset T$ be the trivial knot, such that K' is homotopic to K in T.

Let us define a family of compressing disks for ∂T properly embedded in T, say $\{D_1, D_2, \ldots, D_n, D_{n+1}\}$, which satisfy the following properties:

(1) For each $i \in \{1, 2, \ldots, n\}$ the disk D_i is properly embedded in $\mathcal{N}(\tau_i)$.

(2) The disk D_{n+1} is properly embedded in $\mathcal{N}(K)$ and is a compression disk for it.

All of these disks are properly embedded in T, so we can deduce that:

(1) The family $\{D_1, D_2, \ldots, D_n, D_{n+1}\}$ is pairwise disjoint.

(2) For each $i \in \{1, 2, \ldots, n\}$, $|D_i \cup K| = 0$.

(3) $|D_{n+1} \cap K| = 1$.

Let $\Sigma[K']$ be the double branched cover of K' with covering function given by $p : \Sigma[K'] \to S^3$. Note that $\Sigma[K']$ is homeomorphic to S^3.

Claim 2.3. For each $i \in \{1, 2, \ldots, n\}$, $p^{-1}(\partial D_i)$ has exactly two connected components, where each connected component is a simple closed curve in $p^{-1}(\partial T)$; whereas $p^{-1}(\partial D_{n+1})$ is a single simple closed curve in $p^{-1}(\partial T)$. Also, all these curves are disjoint in $p^{-1}(\partial T)$.

Proof. We know that $|D_{n+1} \cap K| = 1$ and $|D_i \cap K| = 0$ for all $i \in \{1, 2, \ldots, n\}$. As K' is homotopic to K in T, $|D_{n+1} \cap K'|$ is an odd integer and $|D_i \cap K'|$ is an even integer for all $i \in \{1, 2, \ldots, n\}$. Therefore, for each $i \in \{1, 2, \ldots, n\}$ we have that $p^{-1}(\partial D_i)$ has exactly two connected components in $p^{-1}(\partial T)$, where each connected component is a simple closed curve; and $p^{-1}(\partial D_{n+1})$ is a simple closed connected curve in $p^{-1}(\partial T)$. Now, since the disks of the family $\{D_1, D_2, \ldots, D_{n+1}\}$ are pairwise disjoint, we have that all the curves are pairwise disjoint. □

Claim 2.4. $p^{-1}(\partial T)$ is a connected, orientable surface with Euler characteristic $-4n$ (and genus $2n + 1$) contained in $\Sigma[K']$.

Proof. Note that ∂T is a genus $n + 1$ surface, then $\chi(\partial T) = -2n$, and therefore $\chi(p^{-1}(\partial T))) = 2\chi(\partial T) = -4n$. Since ∂T is connected, $p^{-1}(\partial T)$ is a double cover of ∂T, $\partial D_{n+1} \subset \partial T$ and $p^{-1}(\partial D_{n+1})$ is a connected curve on $p^{-1}(\partial T)$. Then by Lemma 2.1 we have that $p^{-1}(\partial T)$ is connected. Therefore $p^{-1}(\partial T)$ is a connected orientable surface of Euler characteristic $-4n$ (and of genus $2n + 1$). □

Claim 2.5. $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$ is connected.

Proof. Clearly $\partial T \setminus \bigcup_{j=1}^{n} \partial D_j$ is connected. We have that $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$ is a double cover of $\partial T \setminus \bigcup_{j=1}^{n} \partial D_j$, that $\partial D_{n+1} \subset \partial T \setminus \bigcup_{j=1}^{n} \partial D_j$ and that $p^{-1}(\partial D_{n+1})$ is a connected curve on $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$. Then using Lemma 2.1 we have that $p^{-1}\big(\partial T \setminus \bigcup_{j=1}^{n} \partial D_j\big)$ is connected. □

By Claim 2.3 we know that for each $i \in \{1, 2, \ldots, n\}$ the curve ∂D_i lifts, under p, to exactly two simple closed curves in $p^{-1}(\partial T)$. Let us denote by Λ_1^i and Λ_2^i the two liftings of ∂D_i in $p^{-1}(\partial T)$, so $\{\Lambda_1^1, \Lambda_2^1, \Lambda_1^2, \Lambda_2^2, \ldots, \Lambda_1^n, \Lambda_2^n\}$ is a pairwise disjoint collection of simple closed curves in $p^{-1}(\partial T)$. Also, $\Lambda_1^i \cup \Lambda_2^i$ is a double cover of ∂D_i with $p|_{\Lambda_1^i \cup \Lambda_2^i}$ the corresponding covering function. Then the functions $p|_{\Lambda_1^i} : \Lambda_1^i \to \partial D_i$ and $p|_{\Lambda_2^i} : \Lambda_2^i \to \partial D_i$ are homeomorphisms.

By Claim 2.3 we have that $p^{-1}(D_{n+1})$ is a simple closed curve on $p^{-1}(\partial T)$. Let us denote by Λ the curve $p^{-1}(\partial D_{n+1})$. So Λ is a double cover for ∂D_{n+1} with covering function $p|_\Lambda : \Lambda \to \partial D_{n+1}$.

Let us introduce the notation

- $\text{Ext}(T) := S^3 \setminus T$,
- $\Sigma[\text{Ext}(T)] := \Sigma[K'] \setminus p^{-1}(T)$.

Note that $\Sigma[\text{Ext}(T)]$ is a double cover of $\text{Ext}(T)$, and $\partial\Sigma[\text{Ext}(T)] = p^{-1}(\partial T)$.

Let $\Sigma[\text{Ext}(K)]$ be the 3-manifold obtained from $\Sigma[\text{Ext}(T)]$ by adding a 2-handle along each of the members of the family of curves $\{\Lambda_1^1, \Lambda_2^1, \Lambda_1^2, \Lambda_2^2, \dots, \Lambda_1^n, \Lambda_2^n\}$. Since the functions $p|_{\Lambda_r^i}$ are homeomorphisms for each $i \in \{1, 2, \dots, n\}$ and $r \in \{1, 2\}$, we can extend each of these homeomorphisms to a homeomorphism whose domain is a disk whose boundary is the curve Λ_r^i, and which maps to the disk D_i. We then extend these last homeomorphisms to homeomorphisms from the 2-handle added along Λ_r^i to $\mathcal{N}(D_i)$. With this we conclude that $\Sigma[\text{Ext}(K)]$ is a double cover of $\text{Ext}(T) \cup \left(\bigcup_{j=1}^n \mathcal{N}(D_j)\right)$. Recall that the family of disks $\{D_1, D_2, \dots, D_n\}$ was chosen such that $\text{Ext}(T) \cup \left(\bigcup_{j=1}^n \mathcal{N}(Dj)\right)$ is homeomorphic to $\text{Ext}(K)$. Therefore $\Sigma[\text{Ext}(K)]$ is a double cover of $\text{Ext}(K)$.

On the other hand, from Claim 2.4 we know that $p^{-1}(\partial T)$ is an orientable connected surface of genus $2n+1$ and by Claim 2.5 we know that $p^{-1}\left(\partial T \setminus \bigcup_{j=1}^n \partial D_j\right)$ is connected. Since $\{\Lambda_1^1, \Lambda_2^1, \Lambda_1^2, \Lambda_2^2, \dots, \Lambda_1^n, \Lambda_2^n\}$ consist of $2n$ curves and

$$p^{-1}\left(\partial T \setminus \bigcup_{j=1}^n \partial D\right) = p^{-1}(\partial T) \setminus \bigcup_{\substack{i\in\{1,2,\dots,n\}\\ r\in\{1,2\}}} \Lambda_r^i,$$

$\partial\Sigma[\text{Ext}(K)]$ is an orientable surface of genus one.

Now, note that $\partial D_{n+1} \subset \partial\text{Ext}(K)$ since $\partial D_{n+1} \subset \partial\mathcal{N}(K)$ and $D_{n+1} \cap D_i = \varnothing$ for all $i \in \{1, 2, \dots, n\}$. Therefore we also have $\Lambda \subset \partial\Sigma[\text{Ext}(K)]$.

Let us define the 3-manifold $\Sigma[K]$ obtained from $\Sigma[\text{Ext}(K)]$ by adding a 2-handle along Λ on $\partial\Sigma[\text{Ext}(K)]$, and then complete with a 3-ball so that $\Sigma[K]$ is a closed 3-manifold. Since $p|_\Lambda$ is a two-to-one covering function then we can extend this function to a function that goes from a disk, whose boundary is Λ, to the disk D_{n+1}, where this extension is two-to-one branched at the point $K \cap D_{n+1}$. This last function is then extended to a function that goes from the 2-handle added along Λ to $\mathcal{N}(D_{n+1})$, where this function is two-to-one branched along the arc $K \cap \mathcal{N}(D_{n+1})$. Finally, this last function is extended to the added 3-ball, thus obtaining a function that goes from $\Sigma[K]$ to S^3 which is two-to-one branched along the knot K. From the above we conclude that $\Sigma[K]$ is the double branched cover of K.

By Claim 2.4, $p^{-1}(\partial T)$ is an orientable connected surface of genus $2n+1$ contained in S^3. Since $\partial\Sigma[\mathrm{Ext}(T)] = p^{-1}(\partial T)$ and $\Sigma[\mathrm{Ext}(T)] \subset \Sigma[K'] = S^3$, $H_1\big(\Sigma[\mathrm{Ext}(T)]\big)$ is a free abelian group of rank $2n+1$. So, let $H_1\big(\Sigma[\mathrm{Ext}(T)]\big) = \langle \theta_1, \theta_2, \ldots, \theta_{2n+1}\rangle$, where θ_i for $i \in \{1, 2, \ldots, 2n+1\}$ are generators.

Thus, $H_1(\Sigma[K]) = \langle \theta_1, \theta_2, \ldots, \theta_{2n+1} \mid \lambda_1^1, \lambda_1^2, \lambda_2^1, \lambda_2^2, \ldots, \lambda_n^1, \lambda_n^2, \lambda\rangle$, where λ and the λ_r^j, for $j \in \{1, 2, \ldots, n\}$ and $r \in \{1, 2\}$, correspond to the homology classes in $H_1\big(\Sigma[\mathrm{Ext}(T)]\big)$ of the respective curves Λ and Λ_r^j. □

In the proof of Theorem 2.2, besides from proving the result, we construct the double cover of S^3 branched along the knot for which we know the transient number. This construction will continue to be repeated throughout this work. Theorem 2.2 can be generalized to p-fold cyclic branched covers, with a similar proof.

Theorem 2.6. *If K is a knot in S^3 such that* $\mathrm{tr}(K) = n$, *then the first homology group of the p-fold cyclic branched cover of K has a presentation with at most $pn+1$ generators.*

The next lemma is a general result of the algebra of groups, which we will use for the proof of Theorems 2.8 and 2.10.

Lemma 2.7. *Let G_1 and G_2 be abelian groups such that*

$$G_1 = \langle \theta_1, \theta_2, \theta_3 : \lambda_1, \lambda_2, \lambda_3\rangle \quad \textit{and} \quad G_2 = \langle \beta_1, \beta_2 : \delta_1, \delta_2\rangle.$$

Suppose that there exist homomorphisms $\Psi : \langle \theta_1, \theta_2, \theta_3\rangle \to \langle \theta_1, \theta_2, \theta_3\rangle$ and $\Phi : \langle \theta_1, \theta_2, \theta_3\rangle \to \langle \beta_1, \beta_2\rangle$ between free abelian groups such that

$$\begin{aligned}
&\Psi(\theta_1) = \theta_2, && \Psi(\theta_2) = \theta_1, && \Psi(\theta_3) = \theta_3,\\
&\Psi(\lambda_1) = \lambda_2, && \Psi(\lambda_2) = \lambda_1, && \Psi(\lambda_3) = \lambda_3,\\
&\Phi(\theta_1) = \beta_1, && \Phi(\theta_2) = \beta_1, && \Phi(\theta_3) = 2\beta_2,\\
&\Phi(\lambda_1) = \delta_1, && \Phi(\lambda_2) = \delta_1, && \Phi(\lambda_3) = 2\delta_2.
\end{aligned}$$

If $\lambda_1 = x\theta_1 + y\theta_2 + z\theta_3$ and G_2 is the trivial group, then G_1 is isomorphic to the finite cyclic group Z_{x-y}.

Proof. Let a_{ij} be integers, with $i, j \in \{1, 2, 3\}$, such that

$$\begin{aligned}
\lambda_1 &= a_{11}\theta_1 + a_{12}\theta_2 + a_{13}\theta_3,\\
\lambda_2 &= a_{21}\theta_1 + a_{22}\theta_2 + a_{23}\theta_3,\\
\lambda_3 &= a_{31}\theta_1 + a_{32}\theta_2 + a_{33}\theta_3.
\end{aligned} \tag{1}$$

Applying the homomorphism Ψ, on both sides of the previous system of equations, we obtain

$$\begin{aligned}
\lambda_2 &= \Psi(\lambda_1) = \Psi(a_{11}\theta_1 + a_{12}\theta_2 + a_{13}\theta_3) = a_{11}\theta_2 + a_{12}\theta_1 + a_{13}\theta_3,\\
\lambda_1 &= \Psi(\lambda_2) = \Psi(a_{21}\theta_1 + a_{22}\theta_2 + a_{23}\theta_3) = a_{21}\theta_2 + a_{22}\theta_1 + a_{23}\theta_3,\\
\lambda_3 &= \Psi(\lambda_3) = \Psi(a_{31}\theta_1 + a_{32}\theta_2 + a_{33}\theta_3) = a_{31}\theta_2 + a_{32}\theta_1 + a_{33}\theta_3.
\end{aligned} \tag{2}$$

From the systems (1) and (2) we get

$$
\begin{aligned}
0 &= (a_{11}-a_{22})\theta_1 + (a_{12}-a_{21})\theta_2 + (a_{13}-a_{23})\theta_3,\\
0 &= (a_{12}-a_{21})\theta_1 + (a_{11}-a_{22})\theta_2 + (a_{13}-a_{23})\theta_3,\\
0 &= (a_{31}-a_{32})\theta_1 + (a_{32}-a_{31})\theta_2.
\end{aligned} \tag{3}
$$

Since $\langle\theta_1,\theta_2,\theta_3\rangle$ is a free abelian group, from the system in (3) we have

$$a_{11}=a_{22}, \quad a_{12}=a_{21}, \quad a_{13}=a_{23}, \quad a_{31}=a_{32}.$$

Then the system (1) can be rewritten as

$$
\begin{aligned}
\lambda_1 &= a_1\theta_1 + a_2\theta_2 + a_3\theta_3,\\
\lambda_2 &= a_2\theta_1 + a_1\theta_2 + a_3\theta_3,\\
\lambda_3 &= a_4\theta_1 + a_4\theta_2 + a_5\theta_3,
\end{aligned} \tag{4}
$$

where $a_1=a_{11}$, $a_2=a_{12}$, $a_3=a_{23}$, $a_4=a_{31}$ and $a_5=a_{33}$. Applying the homomorphism Φ to the system (4) we obtain

$$
\begin{aligned}
\delta_1 &= \Phi(\lambda_1) = \Phi(a_1\theta_1 + a_2\theta_2 + a_3\theta_3) = (a_1+a_2)\beta_1 + 2a_3\beta_2,\\
\delta_1 &= \Phi(\lambda_2) = \Phi(a_2\theta_1 + a_1\theta_2 + a_3\theta_3) = (a_2+a_1)\beta_1 + 2a_3\beta_2,\\
2\delta_2 &= \Phi(\lambda_3) = \Phi(a_4\theta_1 + a_4\theta_2 + a_5\theta_3) = 2a_4\beta_1 + 2a_5\beta_2.
\end{aligned} \tag{5}
$$

By properties of free abelian groups, we obtain from the last equation of the system (5) that

$$\delta_2 = a_4\beta_1 + a_5\beta_2.$$

So the system in (5) can be rewritten as

$$
\begin{aligned}
\delta_1 &= (a_1+a_2)\beta_1 + 2a_3\beta_2,\\
\delta_2 &= a_4\beta_1 + a_5\beta_2.
\end{aligned} \tag{6}
$$

From the system (6) we see that the matrix A, given by

$$A = \begin{pmatrix} a_1+a_2 & 2a_3\\ a_4 & a_5 \end{pmatrix}$$

is the representation matrix of the group $G_2 = \langle\beta_1,\beta_2 : \delta_1,\delta_2\rangle$. From the system in (4), doing an operation on rows, we see that the matrix $\tilde{A}$, given by

$$\tilde{A} = \begin{pmatrix} a_1 & a_2 & a_3\\ a_1+a_2 & a_1+a_2 & 2a_3\\ a_4 & a_4 & a_5 \end{pmatrix}$$

is a representation matrix of the group G_1.

By the Smith normal form theorem, there exists matrices S_1 and S_2 of order 2×2, invertible and with integer entries such that the matrix S_1AS_2 is a diagonal matrix with integer entries. From the Smith normal form theorem it is also known that the inverse matrices of S_1 and S_2 have integer entries, therefore $\det S_1 = \pm 1$ and $\det S_2 = \pm 1$. Now, since G_2 is the trivial group, $\det A = \pm 1$. So the matrix S_1AS_2 is of the form

$$S_1AS_2 = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{7}$$

From (7) we can ensure that there is a matrix S of order 2×2, invertible and with integer entries that satisfies

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Let us define the matrix

$$\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{S} \end{pmatrix}.$$

Clearly the matrix $\tilde{S}$ has integer entries and using the result in (8) we have

$$\tilde{S}\tilde{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Using elementary operations, from the matrix in (9) we obtain

$$\begin{pmatrix} a_1 - a_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

From the above matrix we conclude that the group $\langle \theta_1, \theta_2, \theta_3 : \lambda_1, \lambda_2, \lambda_3 \rangle$ is isomorphic to $Z_{a_1 - a_2}$, therefore the group G_1 is isomorphic to $Z_{a_1 - a_2}$. □

The following result is well known to experts. We include a proof for completeness and because it will help us as a lemma in the proof of Theorem 2.10.

Theorem 2.8. *If K is a knot in S^3 such that $t(K) = 1$, then the first homology group of the double branched cover of K is cyclic.*

Proof. Let K be a knot in S^3 such that $t(K) = 1$, and let τ be an unknotting tunnel for K. Let $T = \mathcal{N}(K \cup \tau)$ and $\mathrm{Ext}(T) = S^3 \setminus T$, so $\mathrm{Ext}(T)$ is a genus two handlebody. Since $\mathrm{Ext}(T)$ is a handlebody, we can ensure that there exists a knot $K' \subset T$ such that K' is a trivial knot in S^3 and it is homotopic to the knot K in T. Let $\Sigma(K')$ be the double branched cover of the knot K' and let $p : \Sigma(K') \to S^3$ be the associated covering function. It is easy to notice, for the way it is defined T, that there are meridian disks D_1 and D_2 in T such that $|D_1 \cap K| = 0$ and $|D_2 \cap K| = 1$. Since K' is homotopic to K in T, $|D_1 \cap K'|$ is an even integer and $|D_2 \cap K'|$ is an odd integer. Therefore ∂D_1 lifts, under p, to two simple closed curves; while ∂D_2 lifts to exactly a single simple closed curve. Let us denote by Λ_1 and Λ_2 the liftings of ∂D_1 and by Λ_3 the lifting of ∂D_2. For each $i \in \{1, 2, 3\}$ we attach a 2-handle to $p^{-1}(\mathrm{Ext}(T))$ along $\Lambda_i \subset \partial\big(p^{-1}(\mathrm{Ext}(T))\big)$; let us denote the 2-handle attached along Λ_i by $\overline{\Lambda}_i$. Let Σ be the 3-manifold obtained by attaching to $p^{-1}(\mathrm{Ext}(T))$ the 2-handles $\overline{\Lambda}_i$, that is: $\Sigma := p^{-1}(\mathrm{Ext}(T)) \cup \big(\bigcup_{i=1}^{3} \overline{\Lambda}_i\big)$.

Let us note the following observations:

(1) $\partial p^{-1}(\mathrm{Ext}(T))$ is a genus three connected surface.

(2) $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$.

(3) The function p can be extended to Σ, such that $\overline{\Lambda}_1 \cup \overline{\Lambda}_2$ is a double covering of $\mathcal{N}(D_1)$ and Λ_3 is a double covering of $\mathcal{N}(D_2)$ branched along $K \cap \mathcal{N}(D_2)$.

(4) $\partial\Sigma$ is a 2-sphere.

Let $\Sigma(K)$ be the 3-manifold obtained by attaching a 3-ball to Σ along its boundary. So, we can extend the covering function $p|_{p^{-1}(\mathrm{Ext}(T))} : p^{-1}(\mathrm{Ext}(T)) \to \mathrm{Ext}(T)$ to a covering function $p' : \Sigma(K) \to S^3$ which branches along the knot K. Therefore $\Sigma(K)$ is the double covering of S^3 branched along K with covering function given by p'.

We know that $\mathrm{Ext}(T)$ is a genus two handlebody, therefore $H_1(\mathrm{Ext}(T))$ is a free abelian group in two generators. Note that $p^{-1}(\mathrm{Ext}(T))$ is a genus three handlebody, therefore $H_1(p^{-1}(\mathrm{Ext}(T)))$ is a free abelian group in three generators.

Claim 2.9. There are two connected simple closed curves in $\mathrm{Ext}(T)$, denoted by B_1 and B_2, such that B_1 lifts, by p, to two closed and connected simple curves, denoted by Θ_1 and Θ_2; while B_2 lifts, by p, to exactly one simple curve closed, denoted by Θ_3. If β_j is the homology class of B_j in $H_1(\mathrm{Ext}(T))$ and θ_i is the homology class of Θ_i in $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ for all $j \in \{1, 2\}$ and $i \in \{1, 2, 3\}$, then $H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 \rangle$, $H_1\big(p^{-1}(\mathrm{Ext}(T))\big) = \langle \theta_1, \theta_2, \theta_3 \rangle$.

Proof. Note that $\mathrm{Ext}(T)$ is a genus two handlebody, call it V. Let D be a disk in V which splits it in two solid tori V_1 and V_2. Note that $p^{-1}(V_i)$ double covers V_i; thus, it is either a set of two solid tori or a solid torus that coves V_i two-to-one. There are two possibilities:

(1) V_1 is covered by two solid tori, say V_1^1 and V_1^2, and V_2 is covered two-to-one by a solid torus V_2'. See Figure 1.

(2) V_1 and V_2 are covered both two-to-one by solid tori V_1' and V_2'. See Figure 2.

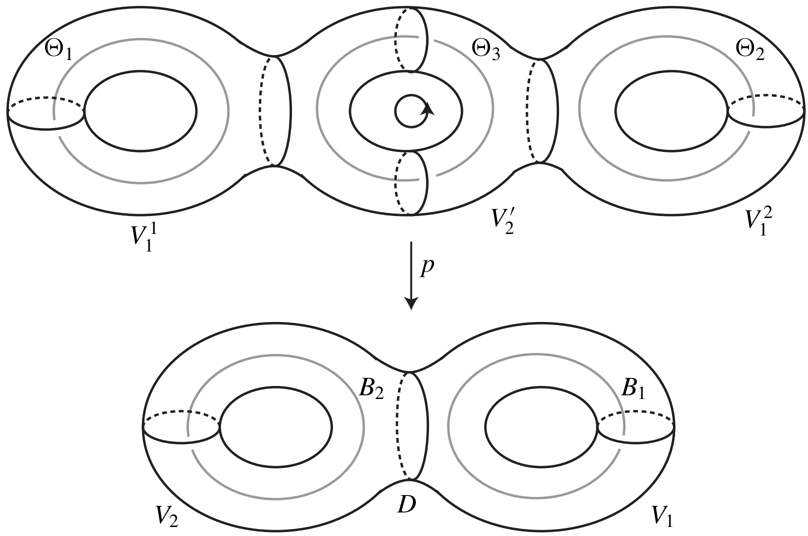

Figure 1. V_1 is covered by two solid tori, say V_1^1 and V_1^2, and V_2 is covered two-to-one by a solid torus V_2'.

In case (1), take as B_i, $i = 1, 2$, a core of the solid tori V_i. Clearly B_1 lifts to two simple closed curves Θ_1 and Θ_2, which are a core of the solid tori V_1^1 and V_1^2, and B_2 lifts to a simple closed curve Θ_3 which is a core of the solid tori V_2', and which cover two-to-one the curve B_2. In this case it is clear that the homology classes of the curves satisfy the required properties. See Figure 1.

In case (2), take as B_1 a curve that goes once around each of the cores of V_1 and V_2 and intersects D in two points. In this case B_1 lifts to two simple closed curves Θ_1 and Θ_2, each of which goes once around V_1^1 and V_1^2. Take as B_2 a core of V_1, then clearly it lifts to a curve Θ_3 which covers B_2 two-to-one. It is clear that the homology classes of the curves satisfy the required properties. See Figure 2. □

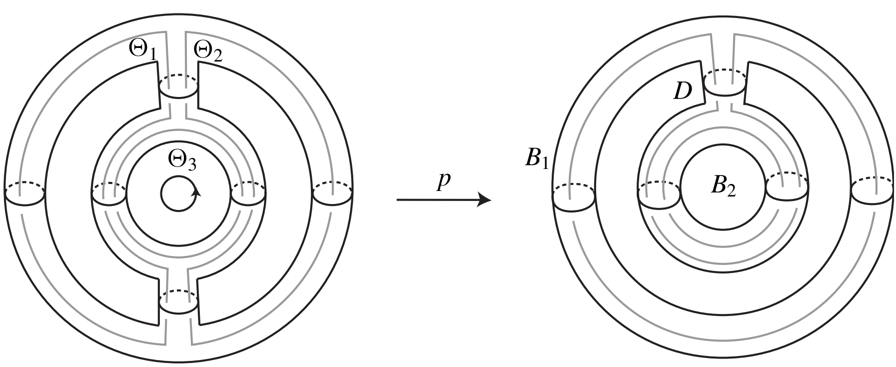

Figure 2. V_1 and V_2 are covered both two-to-one by solid tori V_1' and V_2'.

We know that $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$, with covering function given by the restriction of p. Let $p_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1(\mathrm{Ext}(T))$ be the homomorphism associated with the restriction of p. For each $i \in \{1, 2, 3\}$, let us denote by λ_i the homology class in $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ associated to the curve Λ_i. Note that $H_1(\Sigma[K]) = \langle \theta_1, \theta_2, \theta_3 : \lambda_1, \lambda_2, \lambda_3\rangle$. For each $j \in \{1, 2\}$, let us denote by δ_j the homology class in $H_1(\mathrm{Ext}(T))$ associated to the curve ∂D_j. We have that $H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 : \delta_1, \delta_2\rangle$. By choosing orientations conveniently, assume that

$$p_*(\lambda_1) = \delta_1, \quad p_*(\lambda_2) = \delta_1, \quad p_*(\lambda_3) = 2\delta_2. \tag{10}$$

According to Claim 2.9, we have that

$$p_*(\theta_1) = \beta_1, \quad p_*(\theta_2) = \beta_1, \quad p_*(\theta_3) = 2\beta_2. \tag{11}$$

Let $q : p^{-1}(\mathrm{Ext}(T)) \to p^{-1}(\mathrm{Ext}(T))$ be the nontrivial covering transformation associated to the covering function $p|_{p^{-1}(\mathrm{Ext}(T))}$. Let $q_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ be the homomorphism induced by the covering transformation q. By Claim 2.9 we have that

$$\begin{aligned} q_*(\theta_1) &= \theta_2, \quad q_*(\theta_2) = \theta_1, \quad q_*(\theta_3) = \theta_3, \\ q_*(\lambda_1) &= \lambda_2, \quad q_*(\lambda_2) = \lambda_1, \quad q_*(\lambda_3) = \lambda_3. \end{aligned} \tag{12}$$

Then, applying Lemma 2.7 directly we have that $H_1(\Sigma_2(K)) = \mathbb{Z}_{x-y}$, where $\lambda_1 = x\theta_1 + y\theta_2 + z\theta_3$. □

Now we prove our main result.

Theorem 2.10. *If K is a knot in S^3 such that* $\mathrm{tr}(K) = 1$ *then the first homology group of the double branched cover of K is cyclic.*

Proof. Let K be a knot in S^3 such that $\mathrm{tr}(K) = 1$, and let $\{\tau\}$ be a transient system for the knot K. Let $T = \mathcal{N}(K \cup \tau)$ and let $K' \subset T$ be a trivial knot in S^3 such that K' is homotopic to K in T. Define also the 3-manifold $\mathrm{Ext}(T)$ as $\mathrm{Ext}(T) := S^3 \setminus \mathrm{Int}(T)$.

As ∂T is a genus two surface in the exterior of the knot K', which is trivial, it follows that ∂T is compressible in $\mathrm{Ext}(K')$, that is, there is a compression disk E_1 for ∂T disjoint from K'.

There are two possibilities for the disk E_1:

(1) The disk E_1 is a compression disk for ∂T lying in the interior of T.

(2) The disk E_1 is a compression disk for ∂T lying in the exterior of T.

Suppose first that we have case (1), that is, E_1 lies in the interior of T. If E_1 separates T, then by cutting along E_1 we get two solid tori, one of them contains K', and then there is a compression disk in the other solid tori which is nonseparating in T. So, we can assume that there is a compression disk E_1 for ∂T, lying in T, and which does not separate T.

Claim 2.11. There exist a knot K'' and a disk E_2 in T such that:

(1) E_2 is a compression disk for ∂T which is properly embedded in T.

(2) K'' is a trivial knot in S^3 and it is homotopic to K in T.

(3) $|E_2 \cap K''| = 1$.

Proof. By cutting T along E_1, we get a solid torus V. The knot K' lies in V, and as K' represents a primitive element in $\pi_1(T)$, it must be homotopic to the core of V. If V is knotted, then ∂V is incompressible in $\mathrm{Ext}(K')$, which is not possible, for K' is the trivial knot. Then V must be a standard solid torus in S^3. Then K' can be further homotoped to the core of V, which is a trivial knot in the 3-sphere. Then there is a disk E_2 in V such that $|E_2 \cap K''| = 1$. □

Let $\Sigma[K'']$ be the double cover of S^3 branched along K'' with covering function given by $p : \Sigma[K''] \to S^3$. The disks E_1 and E_2 form a meridian disk system for T, and as K'' is disjoint from E_1 and intersects E_2 in one point, it follows that $p^{-1}(T)$ is a genus three handlebody, $p^{-1}(E_1)$ consists of two disks and $p^{-1}(E_2)$ consists of a single disk which covers two-to-one the disk E_2. Note that these disks form a meridian system for $p^{-1}(T)$. Let $B_i = \partial E_i$, $i = 1, 2$. Denote by Θ_1 and Θ_2 the two components of $p^{-1}(B_1)$, and let $\Theta_3 = p^{-1}(B_2)$. As $\Sigma[K'']$ is the 3-sphere, and $p^{-1}(T)$ is a genus three handlebody, it follows that the homology classes of the curves Θ_i, $i = 1, 2, 3$, generate $H_1(p^{-1}(\mathrm{Ext}(T))$.

Let D_1 and D_2 be compression disks in the interior of T such that D_1 is properly embedded in $\mathcal{N}(\tau)$ and D_2 is properly embedded in $\mathcal{N}(K)$, such that $|D_1 \cap K| = 0$ and $|D_2 \cap K| = 1$. Note that the disks D_1 and D_2 do not separate T. As K'' is homotopic to K in T, $|D_1 \cap K''|$ is an even number and $|D_2 \cap K''|$ is an odd number. Therefore ∂D_1 lifts, under p, to two simple closed curves, while ∂D_2 lifts exactly to a single simple closed curve. Denote by Λ_1 and Λ_2 the liftings of ∂D_1 and by Λ_3 the lifting of ∂D_2. Attach 2-handles to the 3-manifold $p^{-1}(\mathrm{Ext}(T))$ along the curves Λ_i, note that these curves lie in $\partial(p^{-1}(\mathrm{Ext}(T)))$, and denote the 2-handle attached along Λ_i by $\overline{\Lambda}_i$. Let Σ be the 3-manifold obtained by attaching to $p^{-1}(\mathrm{Ext}(T))$ the 2-handles $\overline{\Lambda}_i$.

Note that $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$, with covering function p' given by $p' = p|_{p^{-1}(\mathrm{Ext}(T))}$. The function p' can be extended to a function $p' : \Sigma \to \mathrm{Ext}(T) \cup N(D_1) \cup N(D_2)$, such that $\overline{\Lambda}_1 \cup \overline{\Lambda}_2$ is a double covering of $\mathcal{N}(D_1)$ and $\overline{\Lambda}_3$ is a double covering of $\mathcal{N}(D_2)$ branched along $K \cap \mathcal{N}(D_2)$.

Note that $\partial\Sigma$ is a 2-sphere. Let $\Sigma(K)$ be the 3-manifold obtained by attaching a 3-ball to Σ along its boundary. We can extend the covering function p' to a covering function $\hat{p} : \Sigma(K) \to S^3$, which branches along K. Therefore $\Sigma(K)$ is the double cover of S^3 branched along K with covering function given by $\hat{p}$.

As $p^{-1}(\mathrm{Ext}(T))$ is a double covering of $\mathrm{Ext}(T)$, with covering function given by the restriction of p, let $p_*: H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1(\mathrm{Ext}(T))$ be the homomorphism induced by p. For each $i \in \{1, 2, 3\}$ denote by λ_i the homology class in $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ associated to the curve Λ_i. For each $j \in \{1, 2\}$ denote by δ_j the homology class in $H_1(\mathrm{Ext}(T))$ associated to the curve ∂D_j. Then

$$p_*(\lambda_1) = \delta_1, \quad p_*(\lambda_2) = \delta_1, \quad p_*(\lambda_3) = 2\delta_2. \tag{13}$$

Note that $H_1(\mathrm{Ext}(T))$ is a free abelian group in two generators, generated by the homology classes of the curves B_1 and B_2, which we denote by β_1 and β_2. As we said before, $H_1\big(p^{-1}(\mathrm{Ext}(T))\big)$ is a free abelian group in three generators, generated by the homology classes of the curves Θ_i, which we denote by θ_i, $i = 1, 2, 3$. We have that

$$H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 \rangle, \quad H_1(p^{-1}(\mathrm{Ext}(T))) = \langle \theta_1, \theta_2, \theta_3 \rangle \tag{14}$$

We also obtain that

$$H_1(\mathrm{Ext}(T)) = \langle \beta_1, \beta_2 : \delta_1, \delta_2 \rangle, \quad H_1(\Sigma[K]) = \langle \theta_1, \theta_2, \theta_3; \lambda_1, \lambda_2, \lambda_3 \rangle, \tag{15}$$

$$p_*(\theta_1) = \beta_1, \quad p_*(\theta_2) = \beta_1, \quad p(\theta_3) = 2\beta_2. \tag{16}$$

Let $q : p^{-1}(\mathrm{Ext}(T)) \to p^{-1}(\mathrm{Ext}(T))$ be the nontrivial covering transformation, associated to the covering function p. Let $q_* : H_1\big(p^{-1}(\mathrm{Ext}(T))\big) \to H_1\big(p{-}1(\mathrm{Ext}(T))\big)$ be the homomorphism associated to the covering transformation q. By the way that θ_i and the λ_i were defined we have that

$$\begin{aligned} q_*(\theta_1) = \theta_2, \quad q_*(\theta_2) = \theta_1, \quad q_*(\theta_3) = \theta_3, \\ q_*(\lambda_1) = \lambda_2, \quad q_*(\lambda_2) = \lambda_1, \quad q_*(\lambda_3) = \lambda_3. \end{aligned} \tag{17}$$

Applying Lemma 2.7 we have that $H_1(\Sigma(K)) = \mathbb{Z}_{x-y}$, where $\lambda_1 = x\theta_1 + y\theta_2 + z\theta_3$. So, we have proved that if the compression disk E_1 is contained in T, then the homology group of the double branched cover of K is cyclic.

Now suppose that we have case (2), that is, the compression disk E_1 is contained in $\mathrm{Ext}(T)$. In this situation we can suppose that $\mathrm{Ext}(T)$ is not a handlebody, for otherwise we have that $t(K) = 1$ and by Theorem 2.8 we get the desired result. Suppose first that the disk E_1 does not separate $\mathrm{Ext}(T)$. Define $\Gamma = T \cup \mathcal{N}(E_1)$. As E_1 does not divide ∂T then $\partial\Gamma$ is a connected genus one surface, and it must bound a solid torus. Then Γ is a solid torus, for otherwise $\mathrm{Ext}(T)$ will be a genus two handlebody. So, Γ is a knotted solid torus and K' lies on it. As K' is a trivial knot, it must lie in a 3-ball contained in Γ, for otherwise there will be an incompressible torus in $\mathrm{Ext}(K)$. In particular, K' has winding number zero in Γ. Then K is also of winding number zero in Γ, as it is homotopic to K' in $T \subset \Gamma$. Embed Γ in S^3 so that it is an standard solid torus V, and that a preferred longitude of Γ goes to a preferred longitude of V. Let $\overline{K}$ be the image of K in V. Then K is a satellite knot with pattern given by $\overline{K}$. As $\overline{K}$ has winding number zero in V, it follows that $H_1(\Sigma[\overline{K}])$ is isomorphic to $H_1(\Sigma[K])$, by [12]. Let $\overline{T}$ be the image of T in V, clearly $\overline{T}$ is the neighborhood of $\overline{K}$ union a transient arc, and the exterior of $\overline{T}$ is the exterior of V, which is a solid torus, union a 1-handle given by the image of the disk E_1. This shows $\overline{K}$ is a tunnel number one knot and then $H_1(\Sigma[\overline{K}])$ is a cyclic group, which implies then that $H_1(\Sigma[K])$ is also cyclic.

Suppose now that the disk E_1 separates Ext(T) and that there is no nonseparating compression disk in Ext(T). Let $\Gamma = T \cup \mathcal{N}(E_1)$. As E_1 is separating, $\partial\Gamma$ consist of two tori, say S_1 and S_2. Then S_1 bounds a solid torus V_1 which contains Γ, and also contains S_2. Then V_1 is a knotted solid torus, and as K' is contained in V_1, it must lie inside a 3-ball, and then as in the previous case, K has winding number zero in V_1. Embed V_1 in S^3 so that it is an standard solid torus V_2, and such that a preferred longitude of V_1 goes to a preferred longitude of V_2. Let $\overline{K}$ be the image of K in V_2. Then K is a satellite knot with pattern given by $\overline{K}$. As $\overline{K}$ has winding number zero in V, it follows that $H_1(\Sigma[\overline{K}])$ is isomorphic to $H_1(\Sigma[K])$, by [12]. Let $\overline{T}$ be the image of T in V, clearly $\overline{T}$ is the neighborhood of $\overline{K}$ union a transient arc, and the exterior of $\overline{T}$ is the exterior of V, which is a solid torus union a manifold bounded by the image of S_2 plus 1-handle given by the image of the disk E_1. It follows that $\overline{K}$ is a transient number one knot such that the exterior of the knot union a transient arc is compressible, and it has a nonseparating compression disk. By the previous case, $H_1(\Sigma[\overline{K}])$ is a cyclic group, which implies then that $H_1(\Sigma[K])$ is also cyclic. □

3. Knots with large transient number

By the results of Section 2, we can now estimate the transient number of some knots.

Theorem 3.1. *Let K be a knot such that its double branched cover is not a homology sphere, that is, $H_1(\Sigma[K])$ is not trivial. Then*

(1) $\operatorname{tr}(K \# K) \geq 2$;

(2) $\operatorname{tr}(K_n) \geq (n-1)/2$, *where $K_n = K \# K \# \cdots \# K$, is the connected sum of n copies of K.*

Proof. It is known that $\Sigma[K_n] = \Sigma[K] \# \Sigma[K] \# \cdots \# \Sigma[K]$, the connected sum of n copies of $\Sigma[K]$. As $H_1(\Sigma[K])$ is not trivial, $H_1(\Sigma[K_n])$ has rank at least n. By Theorem 2.2, $\operatorname{tr}(K_n) \geq (n-1)/2$, this shows (2). In particular $H_1(\Sigma[K_2]) = H_1(\Sigma[K]) + H_1(\Sigma[K])$, which is not cyclic, and this implies (1). □

This shows that there are knots with arbitrarily large transient number, which answers a question of Koda and Ozawa [2].

Now we concentrate on the tables of knots up to crossing number 10.

10_{99}	{2, {9, 9}}, {6, {2, 2, 6, 6, 0, 0, 0, 0}}
10_{123}	{2, {11, 11}}, {5, {2, 2, 2, 2, 2, 2, 2, 2}}
$12a_{427}$	{2, {15, 15}}, {4, {3, 3, 3, 3, 15, 15}}, {6, {4, 4, 20, 20, 0, 0, 0, 0}}
$12a_{435}$	{2, {3, 75}}, {6, {2, 2, 8, 200, 0, 0, 0, 0}}
$12a_{465}$	{6, {2, 2, 2, 2, 2, 2, 38, 9158}}
$12a_{466}$	{6, {2, 2, 2, 2, 2, 2, 26, 5434}}
$12a_{475}$	{6, {2, 2, 2, 10, 20, 340, 0, 0}}
$12a_{647}$	{2, {3, 51}}, {6, {2, 2, 2, 34, 0, 0, 0, 0}}
$12a_{868}$	{5, {2, 2, 2, 2, 8, 8, 88, 88}}
$12a_{975}$	{2, {5, 45}}, {4, {5, 5, 5, 5, 5, 45}}
$12a_{990}$	{2, {3, 75}} {6, {2, 2, 8, 200, 0, 0, 0, 0}}
$12a_{1019}$	{2, {19, 19}}, {5, {6, 6, 6, 6, 6, 6, 6, 6}}
$12a_{1102}$	{6, {2, 2, 2, 2, 2, 2, 112, 34160}}
$12a_{1105}$	{2, {17, 17}}, {6, {2, 2, 2, 2, 10, 10, 170, 170}}
$12a_{1167}$	{5, {2, 2, 2, 2, 2, 2, 82, 82}}
$12a_{1229}$	{5, {2, 2, 2, 2, 8, 8, 8, 8}}
$12a_{1288}$	{2, {3, 39}}, {6, {2, 2, 2, 26, 0, 0, 0, 0}}
$12n_{518}$	{2, {3, 21}}, {6, {2, 2, 4, 28, 0, 0, 0, 0}}
$12n_{533}$	{6, {2, 2, 2, 2, 2, 42, 0, 0}}
$12n_{604}$	{2, {3, 27}}, {6, {2, 2, 2, 18, 0, 0, 0, 0}}
$12n_{605}$	{2, {3, 3}}, {6, {2, 2, 2, 2, 0, 0, 0, 0}}
$12n_{706}$	{2, {7, 7}}, {5, {3, 3, 3, 3, 3, 3, 3, 3}}, {6, {2, 2, 2, 2, 2, 2, 14, 14}}
$12n_{840}$	{6, {2, 2, 2, 2, 2, 2, 10, 1190}}
$12n_{879}$	{5, {2, 2, 2, 2, 4, 4, 4, 4}}
$12n_{888}$	{2, {3, 15}}, {6, {2, 2, 2, 10, 0, 0, 0, 0}}

Table 1. List of the knots with the corresponding homology group needed for the proof of Theorem 3.3.

Theorem 3.2. (1) *These knots have transient number* 2: 8_{18}, 9_{35}, 9_{37}, 9_{40}, 9_{41}, 9_{46}, 9_{47}, 9_{48}, 9_{49}, 10_{74}, 10_{75}, 10_{98}, 10_{99}, 10_{103}, 10_{123}, 10_{155}, 10_{157}.

(2) *These knots have transient number at most* 2: 8_{16}, 9_{29}, 9_{32}, 9_{38}, 10_{61}, 10_{62}, 10_{63}, 10_{64}, 10_{65}, 10_{66}, 10_{67}, 10_{68}, 10_{69}, 10_{79}, 10_{80}, 10_{81}, 10_{83}, 10_{85}, 10_{86}, 10_{87}, 10_{89}, 10_{90}, 10_{92}, 10_{93}, 10_{94}, 10_{96}, 10_{97}, 10_{100}, 10_{101}, 10_{105}, 10_{106}, 10_{108}, 10_{109}, 10_{110}, 10_{111}, 10_{112}, 10_{115}, 10_{116}, 10_{117}, 10_{120}, 10_{121}, 10_{122}, 10_{140}, 10_{142}, 10_{144}, 10_{148}, 10_{149}, 10_{150}, 10_{151}, 10_{152}, 10_{153}, 10_{154}, 10_{158}, 10_{160}, 10_{162}, 10_{163}, 10_{165}.

(3) *Any other knot of crossing number at most* 10 *has transient number one.*

Proof. According to the information given in KnotInfo [3], the knots in (1) and (2) are precisely the knots with crossing number up to 10, whose unknotting number and tunnel number are both larger that 1. So, any other knot has unknotting number or tunnel number equal to 1, and then has transient number 1. The knots in (1) are precisely the knots whose double branched cover has noncyclic first homology group, and furthermore these knots have tunnel number 2. Therefore its transient number must be two. The knots in (2) have tunnel number two but their double branched cover has cyclic first homology group; hence we cannot detect the transient number yet. □

A similar result can be done for the knots of crossing number 11 or 12.

The following knots are interesting, for we use the homology of p-branched covers of a knot to determine the transient number.

Theorem 3.3. *These knots have transient number* 2: 10_{99}, 10_{123}, $12a_{427}$, $12a_{435}$, $12a_{465}$, $12a_{466}$, $12a_{475}$, $12a_{647}$, $12a_{742}$, $12a_{801}$, $12a_{868}$, $12a_{975}$, $12a_{990}$, $12a_{1019}$, $12a_{1102}$, $12a_{1105}$, $12a_{1167}$, $12a_{1206}$, $12a_{1229}$, $12a_{1288}$, $12n_{518}$, $12n_{533}$, $12n_{604}$, $12n_{605}$, $12n_{642}$, $12n_{706}$, $12n_{840}$, $12n_{879}$, $12n_{888}$.

Proof. According to Theorem 2.6, if K has $\mathrm{tr}(K)=1$, then $\mathrm{rank}\big(H_1(\Sigma_p[K])\big) \leq p+1$. Using this and the information given in KnotInfo [3], we show that these knots cannot have transient number one. As they have tunnel number two, they must also have transient number two. In Table 1, there is a list of the knots with the corresponding homology group needed for the proof. For some of them, it is enough to use the homology of $\Sigma[K]$, but not for all. A symbol {6, {2, 2, 2, 10, 20, 340, 0, 0}} means that $H_1(\Sigma_6[K]) = \mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_2+\mathbb{Z}_{10}+\mathbb{Z}_{20}+\mathbb{Z}_{340}+\mathbb{Z}+\mathbb{Z}$. □

4. Transient number and connected sums

It is natural to consider the behavior of a knot invariant with respect to connected sums. It is easy to see that

$$u(K_1 \# K_2) \leq u(K_1)+u(K_2),$$

and equality is conjectured. It is also not difficult to see that

$$t(K_1 \# K_2) \leq t(K_1)+t(K_2)+1.$$

There are known examples of knots with $t(K_1 \# K_2) = t(K_1)+t(K_2)+1$ [6], examples with $t(K_1 \# K_2) = t(K_1)+t(K_2)$, and examples with $t(K_1 \# K_2) < t(K_1)+t(K_2)$ [4]. So, we can expect a similar inequality for the transient number.

Theorem 4.1. *Let K_1, K_2 be knots in S^3. Then*

$$\mathrm{tr}(K_1 \# K_2) \leq \mathrm{tr}(K_1)+\mathrm{tr}(K_2)+1.$$

Proof. Let K_1 be a knot with transient number $\mathrm{tr}(K)=n$, and let $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be a system of transient arcs for K_1. Let $T_1 = \mathcal{N}(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$. Then T_1 is a genus $n+1$ handlebody with the property that K_1 can be homotoped in the interior of T_1 to the trivial knot in S^3. We can assume that the homotopy that transform K_1 into the trivial knot can be realized by a sequence of ambient isotopies of K_1 and crossing changes. So, suppose that after making isotopies, all crossing changes are performed simultaneously. Suppose r crossing changes are performed, numbered $1, 2, \dots, r$, and for each crossing change let α_i be an arc with endpoints in K_1 which remembers the crossing change, that is, if B_i is a regular neighborhood of α_i, in fact a 3-ball that intersects K_1 in two unknotted arcs, then a crossing change can be performed inside each B_i to get the trivial knot. Make an isotopy to move K_1 to its original position, and then $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ is a collection of disjoint arcs with endpoints in K_1 contained in T_1. Let δ_1 be an arc in T_1 with an endpoint in K_1 and the other in $\partial N(K_1)$, such that δ_1 is disjoint from the arcs α_i.

If K_2 is knot with $\mathrm{tr}(K_2) = m$, then as above there is a genus $m+1$ handlebody that is the neighborhood of K union a system of transient arcs $\{\gamma_1', \gamma_2', \dots, \gamma_m'\}$, and there is a collection of arcs $\{\beta_1, \dots, \beta_s\}$ that determines crossing changes that unknot K_2. Let δ_2 be an arc in T_2 with an endpoint in K_2 and the other in $\partial N(K_2)$, such that δ_2 is disjoint from the arcs β_i.

Suppose that T_1 and T_2 lie in disjoint 3-balls C_1 and C_2 contained in S^3. Suppose that $\partial T_i \cap \partial C_i$ consists of a disk D_i, such that the endpoint of δ_i lying in ∂T_i, it lies in D_i, for $i = 1, 2$. Do a disk sum of T_1 and T_2, identifying D_1 and D_2, such that the endpoints of δ_1 and δ_2 coincide. Let $\delta = \delta_1 \cup \delta_2$, this is an arc with an endpoints in K_1 and K_2. Following δ, do a band sum of K_1 and K_2. As K_1 and K_2 lie in disjoint 3-balls, this band sum is in fact a connected sum $K_1 \# K_2$. Let $T = T_1 \cup T_2$, this is a genus $n+m+2$ handlebody, and $K_1 \# K_2$ can be homotoped to the trivial knot inside it, to see that just do crossing changes following the arcs α_i and β_j. Now note that T is the regular neighborhood of $K_1 \# K_2$ and a system of $n+m+1$ arcs, that is, the n arcs for a system of K_1, the m arcs for a system of K_2, plus one more arc which is dual to the band used to perform the connected sum of K_1 and K_2; see Figure 3. This shows that the transient number of $K_1 \# K_2$ is at most $n+m+1$. □

In many cases we can ensure that $\mathrm{tr}(K_1 \# K_2)$ is at most $\mathrm{tr}(K_1) + \mathrm{tr}(K_2)$. For example, if the arc systems that unknot K_1 and K_2 are disjoint from a meridian disk E_1 for $N(K_1)$ and a meridian disk E_2 for $N(K_2)$, then at most $\mathrm{tr}(K_1) + \mathrm{tr}(K_2)$ arcs are needed to unknot $K_1 \# K_2$. To see this consider handlebodies T_1 and T_2 as above, and disjoint 3-balls C_1 and C_2 that contain them, such that T_i and C_i intersect in a disk D_i. We can suppose that the boundary of the disk E_i intersects D_i in a single arc. Instead of doing a band sum of T_1 and T_2, cut T_i along E_i, and identify the two copies of E_1 with the corresponding copies for E_2, this is like doing a connected sum $T_2 \# T_2$ between T_1 and T_2. We get a genus $n+m+1$ handlebody T'. Note that $K_1 \# K_2$ is contained in the handlebody T'. Now consider the arcs α_i and β_j as in the above proof. These arcs are disjoint from the meridian

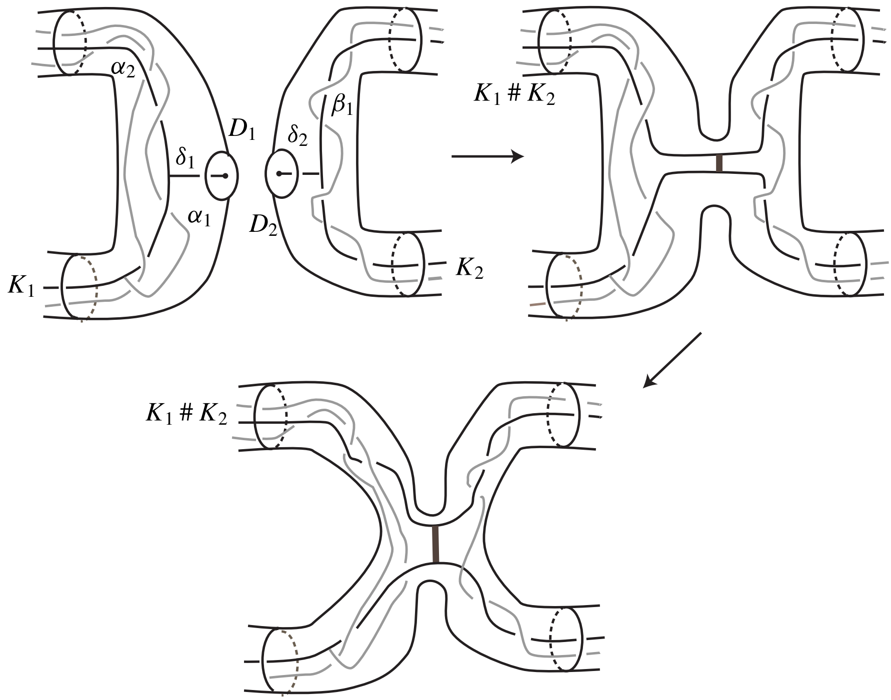

Figure 3. The arcs and sum $K_1 \# K_2$ described in the proof of Theorem 4.1.

disks E_i, and then they are contained in T'. Then these arcs can be used to unknot $K_1 \# K_2$, which then have transient number at most $n+m = \mathrm{tr}(K_1)+\mathrm{tr}(K_2)$.

There are examples of knots K_1 and K_2, such that $t(K_1) = 1 = t(K_2)$, but $t(K_1 \# K_2) = 3$ [6]. For these examples, it is clear that

$$\mathrm{tr}(K_1) = 1 = \mathrm{tr}(K_2),$$

but it is not clear what is $\mathrm{tr}(K_1 \# K_2)$.

There are also examples of knots K_1 and K_2, such that $t(K_1) = 2$, $t(K_2) = 1$, but $t(K_1 \# K_2) = 2$ [5]. In this case

$$\mathrm{tr}(K_2) = 1 \quad \text{and} \quad \mathrm{tr}(K_1 \# K_2) \leq 2,$$

but it is not clear whether $\mathrm{tr}(K_1) = 1$ or 2.

It is well known that knots with unknotting number one or tunnel number one are prime, but the proofs are not so easy. The first proof that knots K with $u(K) = 1$ are prime [10] uses heavy combinatorial arguments, a second proof uses sutured manifold theory [11], and a third proof depends on double branched covers and deep results on Dehn surgery on knots [14]. There are also two proofs that tunnel number one knots are prime, one uses combinatorial group theory [7], and the other uses combinatorial arguments [9]. A proof that transient number one knots are prime would imply both, that unknotting number one and tunnel number one knots are prime, so it may not be easy to prove that. However it seems reasonable to conjecture the following.

Conjecture 4.2. If K is a knot with $\mathrm{tr}(K) = 1$ then K is prime.

Theorem 3.1(1) gives some evidence for this conjecture.

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PRESERVATION OF ELEMENTARITY BY TENSOR PRODUCTS OF TRACIAL VON NEUMANN ALGEBRAS

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Tensoring with type I algebras preserves elementary equivalence in the category of tracial von Neumann algebras. The proof involves a novel and general Feferman–Vaught-type theorem for direct integrals of metric structures.

In [19] it was proven that reduced products and their generalizations preserve elementary equivalence, in the sense that the first-order theory of the product can be computed from the theories of the factors and information about the ideal (if any) used to form the reduced product. The question of preservation of elementarity by tensor products and free products is a bit subtler. Somewhat surprisingly, free products preserve elementary equivalence both in the case of groupoids [28] and, by a deep result of Sela [33], in the case of groups. It is not known whether this is the case with tracial von Neumann algebras [25, Question 5.3].

On the other hand, tensor products in general do not preserve elementary equivalence in the category of modules [27] and in the category of C*-algebras [11; 16]. David Jekel [25, Section 5.1] asked whether tensor products of tracial von Neumann algebras preserve elementary equivalence. We give partial positive answers to this question (see Section 1.2 for the notation and terminology).

Theorem 1. *If M and N are tracial von Neumann algebras at least one of which is type* I, *then the theory of their tensor product depends only on theories of M and N. In other words, if $M_1 \equiv M$ and $N_1 \equiv N$ then $M_1 \bar{\otimes} N_1 \equiv M \bar{\otimes} N$. More precisely, if $M_1 \preceq M$ and $N_1 \preceq N$ then (with the natural identification) $M_1 \bar{\otimes} N_1 \preceq M \bar{\otimes} N$.*

In the course of proving Theorem 1 we prove a Feferman–Vaught-type theorem for direct integrals of metric structures (Theorem 3.3). This proof roughly follows the lines of the proof of the Feferman–Vaught theorem for metric reduced products given in [22] (see also [10, Section 16]). Also, standard results imply that among McDuff factors, tensoring with the hyperfinite II_1 factor preserves elementarity (see Section 5). Unlike C*-algebras, among tracial von Neumann algebras there is no known example of a failure of preservation of elementarity by tensor products.

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1. Preliminaries

Good general references are, for operator algebras [8], for type II_1 factors [1], and for continuous model theory [6; 23; 24].

1.1. ***Direct integrals.*** Our original definition of measurable fields of metric structures and direct integrals of metric structures was analogous to measurable fields and direct integrals of Hilbert spaces and of von Neumann algebras [34, Definitions IV.8.9, IV.8.15, and IV.8.17]. Here we include the more polished definition from [36, Section 8].[1] Although this definition can be extended to the case of nonseparable metric spaces, we will consider only the separable case. For simplicity, we will restrict our attention to the case when $\mathscr{L}$ is a single-sorted language; the definition is generalized to multi-sorted languages by making obvious modifications.

Definition 1.1 (measurable fields and direct integrals of metric structures). Suppose that $\mathscr{L}$ is a continuous language, $(\Omega, \mathscr{B}, \mu)$ is a separable probability measure space, and (M_ω, d_ω) for $\omega \in \Omega$, are separable $\mathscr{L}$-structures. Assume that e_n for $n \in \mathbb{N}$, is a sequence in $\prod_{\omega\in\Omega} M_\omega$ such that the following two conditions hold:

(1) For every ω, the set $\{e_n(\omega) \mid n \in \mathbb{N}\}$ is dense in M_ω.

(2) For every predicate $R(\bar{x})$ in $\mathscr{L}$ and every tuple $e_{\bar{n}} = \langle e_{n(0)}, \dots, e_{n(j-1)}\rangle$ of the appropriate sort, the function $\omega \mapsto R^{M_\omega}(e_{\bar{n}}(\omega))$ is measurable.

The structures M_ω, together with the functions e_n, form a *measurable field* of $\mathscr{L}$-structures.

The *direct integral* of M_ω, for $\omega \in \Omega$, is the structure denoted

$$M = \int_\Omega^\oplus M_\omega \, d\mu(\omega)$$

and defined as follows. Consider the set $\mathcal{M}$ of all $a \in \prod_{\omega\in\Omega} M_\omega$ such that the functions

$$\omega \mapsto d_\omega(a(\omega), e_n(\omega))$$

are measurable for all n. On $\mathcal{M}$ consider the pseudometric

$$d^M(a, b) = \int_\Omega d_\omega(a(\omega), b(\omega)) \, d\mu(\omega).$$

Then the domain of M is defined to be the set of equivalence classes of functions in $\mathcal{M}$ with respect to the equivalence relation defined by $a \sim b$ if $d^M(a, b) = 0$, with the quotient metric d.

[1] A visible, but ultimately insubstantial, technical difference between this definition and the standard definition of a direct integral of tracial von Neumann algebras will be discussed in Section 4.

Lemma 1.2. *If F is an n-ary function symbol and $\bar{a} \in \mathcal{M}^n$, then the interpretation $\omega \mapsto F^{M_\omega}(\bar{a})$ is a measurable function. If R is an n-ary predicate symbol and $\bar{a} \in \mathcal{M}^n$, then the interpretation $\omega \mapsto F^{M_\omega}(\bar{a})$ is a measurable function.*

Proof. We will prove the second assertion. Fix an n-ary predicate symbol R and an n-tuple $\bar{a}$ in $\mathcal{M}$. Given $\varepsilon > 0$, the syntax requires that there is $\delta = \delta(\varepsilon) > 0$ such that for n-tuples $\bar{x}$ and $\bar{y}$ in any $\mathcal{L}$-structure N we have that $d(\bar{a}, \bar{b}) < \delta$ implies $|R^N(\bar{x}) - R^N(\bar{y})| < \varepsilon$.

Since $e_n(\omega)$ for $n \in \mathbb{N}$, is dense in every M_ω, there is a partition $\Omega = \bigsqcup_i X_i$ into measurable sets and a function $h : \mathbb{N} \times n \to \mathbb{N}$ such that for all i and j we have $d^{M_\omega}(a_j(\omega) - e_{h(i,j)}(\omega)) < \delta$ for all $\omega \in X_i$. Writing $\bar{e}(i)$ for the n-tuple in $\mathcal{M}$ whose j-th coordinate agrees with $e_{h(i,j)}(\omega)$, the interpretation $\omega \mapsto R^{M_\omega}(\bar{a})$ is uniformly ε-approximated by

$$\omega \mapsto \sum_i \chi_{X_i} R^{M_\omega}(\bar{e}(i)).$$

By (2) of Definition 1.1, this function is measurable. Therefore the evaluation of R at $\bar{a}$ is, as a uniform limit of measurable functions, measurable.

The proof in case of function symbols is analogous. □

On M the interpretations of function symbols and predicate symbols in $\mathcal{L}$ are defined in the natural way. The interpretation of a function symbol $F(\bar{x})$ in $\mathcal{L}$, for a tuple $\bar{a}$ in M of appropriate sort, is the equivalence class $F^M(\bar{a})$ of the function on Ω such that

$$\omega \mapsto F^{M_\omega}(\bar{a}_\omega).$$

If $R(\bar{x})$ is a relation symbol in $\mathcal{L}$ and $\bar{a}$ in M is of appropriate sort, then

$$R^M(\bar{a}) = \int_\Omega R^{M_\omega}(\bar{a}_\omega)\, d\mu(\omega).$$

Lemma 1.3. *If $\mathcal{L}$ is a continuous language and M_ω, for $\omega \in \Omega$, is a measurable field of $\mathcal{L}$-structures, then $\int^\oplus M_\omega \, d\mu(\omega)$ is an $\mathcal{L}$-structure.* □

Proof. It is straightforward to verify that M is complete with respect to d and that the interpretation of each function and predicate symbol in a direct integral is continuous with respect to d, with the modulus of continuity as required by the syntax in $\mathcal{L}$. The conclusion follows. □

A remark on randomizations. Keisler's randomizations of discrete structures [26] as well as their metric analog [3, Definition 3.4; 5] are closely related to direct integrals of measurable fields of structures. Unlike direct integrals, randomizations are presented in an expanded two-sorted language. Precise relation between randomizations (of both discrete structures and continuous structures whose theory has an additional property, that the space of quantifier-free n-types is a Bauer simplex for all $n \geq 1$) is discussed in detail in [36, Section 21]. Note that the quantifier elimination results for randomizations (see [26, Theorem 3.6], [5, Theorem 2.9], and [3, Corollary 3.33]) refer to the expanded language. It is not difficult to see that, for example, a nontrivial direct integral of a measurable field of II_1 factors does not admit quantifier elimination in the language of tracial von Neumann algebras. As a matter of fact, no type II_1 tracial von Neumann algebra admits quantifier elimination by the main result of [12]. This result has been improved further in [17] where it was shown that these theories are not even model complete.

1.2. ***Elementarity.*** The model theory of tracial von Neumann algebras and C*-algebras were introduced in [14]. For every tuple $\bar{x} = \langle x_0, \ldots, x_{n-1}\rangle$ of variables ($n \geq 0$, allowing for the empty tuple) one associates the algebra of formulas $\mathfrak{F}^{\bar{x}}$ with free variables included in $\bar{x}$. If $\varphi(\bar{x})$ is a formula with free variables included in $\bar{x}$, (N, τ) is a tracial von Neumann algebra, then the interpretation $\varphi^N(\bar{a})$ is defined for every tuple $\bar{a}$ in N of the appropriate sort. Sentences are formulas with no free variables.

To every tracial von Neumann algebra N one defines a seminorm $\|\cdot\|_N$ on $\mathfrak{F}^{\bar{x}}$ by

$$\|\varphi(\bar{x})\|_N = \sup \varphi^{N^{\mathcal{U}}}(\bar{a}).$$

Here $\bar{a}$ ranges over all n-tuples in the unit ball of N. (The standard definition of $\|\varphi(\bar{x})\|$ takes supremum over all structures M elementarily equivalent to N and all n-tuples in M of the appropriate sort, but the two seminorms coincide.)

Definition 1.4. Suppose that M and N are tracial von Neumann algebras. They are said to be *elementarily equivalent*, $M \equiv N$, if every sentence φ satisfies $\varphi^M = \varphi^N$. An *elementary embedding* $\Psi : M \to N$ is an embedding such that $\varphi^M(\bar{a}) = \varphi^N(\Psi(\bar{a}))$ for every $\varphi(\bar{x})$ and every $\bar{a}$ of the appropriate sort.

If M is a subalgebra of N and the identity map is an elementary embedding, then M is called an *elementary submodel* of N, in symbols $M \preceq N$.

The diagonal embedding of M into its ultrapower $M^{\mathcal{U}}$ is elementary (Łoś's theorem). If $M \preceq N$ and $N \preceq P$, then $M \preceq P$. If $M \preceq P$, $N \preceq P$, and $M \subseteq N$, then $M \preceq N$. However, $M \preceq P$, $M \preceq N$, and $P \subseteq N$ does not in general imply $P \preceq N$.

2. Definability in probability measure algebras

Here we start the proof of our Feferman–Vaught theorem stated and proven in Section 3, by laying down some general definability results following a request of one of the referees. Let $\mathscr{L}_{\mathrm{MBA}}$ denote the language of probability measure algebras as in [7]. In addition to Boolean operations, this language is equipped with a predicate for a probability measure and metric derived from it. Thus if $(\mathscr{B}, \mu)$ is a measure algebra, the distance is given by $d_\mu(A, B) = \mu(A \Delta B)$, and the language includes (symbols for) the Boolean operations. For definiteness, if $(\mathscr{B}, \mu)$ is a measure algebra then on $\mathscr{B}^n$ we consider the max distance, $d_\mu(\bar{A}, \bar{B}) = \max_{i<n} \mu(A_i \Delta B_i)$.

The simple fact stated in Lemma 2.1 below is a warm up for Lemma 2.2 used in proof of Theorem 3.3.

Lemma 2.1. *Each of the following sets is definable in every measure algebra* $(\mathcal{B}, \mu)$.

(1) *The set* $\mathcal{X}_1 = \{(A_1, A_2) \mid A_1 \subseteq A_2\}$.

(2) *The set* $\mathcal{X}_2 = \{(A, B, C) \mid A \cap B = C\}$.

(3) *For* $A \in \mathcal{B}$, *the set* $\mathcal{X}_A = \{B \in \mathcal{B} \mid B \subseteq A\}$ *is definable with* A *as a parameter.*

Proof. In each of the instances, we need a formula that bounds the distance of an element (or a tuple of the appropriate sort) to the set in question (see [6, Section 9] and more specifically [6, Definition 9.16]).

(1) Let $\varphi(X_1, X_2) = \mu(X_1 \setminus X_2)$. It is clear that $\mathcal{X}_1$ is the zero set of φ. Moreover, for every $\bar{A} \in \mathcal{B}^2$ we have that the pair $\bar{B} = (A_1 \cap A_2, A_2)$ is in $\mathcal{X}_1$ and that $d_\mu(\bar{A}, \bar{B}) = \mu(A_1 \Delta (A_1 \cap A_2)) = \varphi(A_1, A_2)$.

(2) Let $\psi(X_1, X_2, X_3) = \mu((X_1 \cap X_2) \Delta X_3)$. Clearly $\mathcal{X}_2$ is the zero set of ψ is $\mathcal{X}_2$. Also, for every $\bar{A} \in \mathcal{B}^3$ the triple $\bar{B} = (A_1, A_2, A_1 \cap A_2)$ is in $\mathcal{X}_2$ and $d_\mu(\bar{A}, \bar{B}) = \psi(\bar{A})$, as required.

(3) Follows from (1). □

Lemmas 2.2 and 2.3 below would be consequences of Lemma 2.1 if it only were the case that an intersection of definable sets is definable. (Counterexamples can be found in as of yet unpublished papers [4] and [18].) The notation in these two lemmas is chosen to comply with the natural notation in the proof of Theorem 3.3 at the point when Lemma 2.3 is being invoked.

Lemma 2.2. *Suppose that* $\ell \geq 1$ *and* $\bar{U} = (U_j)_{j<\ell}$ *is a tuple in a measure algebra* $(\mathcal{B}, \mu)$ *such that* $U_0 \geq U_1 \geq \cdots \geq U_{\ell-1}$. *Then the set*

$$\mathcal{X}[\bar{U}] = \Big\{\bar{Y} \in \mathcal{B}^\ell \mid Y_j \leq U_j \bigcap_{i<j} Y_i \text{ for } j < \ell\Big\}$$

is a definable set with parameter $\bar{U}$.

Proof. As in Lemma 2.1, it suffices to find a formula $\varphi_{\bar{U}}$ such that its zero set is $\mathcal{X}[\bar{U}]$ and for every $\bar{X} = (X_i : i < \ell)$ in $\mathcal{B}^\ell$ the distance from $\bar{X}$ to $\mathcal{X}[\bar{U}]$ is at most $\varphi_{\bar{U}}(\bar{X})$. For an ℓ-tuple $\bar{U}$ that satisfies $U_0 \geq U_1 \geq \cdots \geq U_{\ell-1}$ let

$$\varphi_{\bar{U}}(\bar{X}) = \max_{1 \leq m < \ell} \Big(\mu\Big(X_m \setminus \bigcap_{j<m} X_j\Big) + \mu(X_m \setminus U_m)\Big).$$

Clearly $\mathcal{X}[\bar{U}]$ is the zero set of $\varphi_{\bar{U}}$. Fix an ℓ-tuple $\bar{X}$ in $\mathcal{B}^\ell$. Let

$$Y_m = \bigcap_{j \leq m} X_j \cap U_m \quad \text{for } 1 \leq m < \ell.$$

Then $Y_m \subseteq \bigcap_{j<m} Y_j \cap U_m$ for all $1 \leq m < \ell$, and hence $\bar{Y} \in \mathcal{X}[\bar{U}]$.

To estimate $d_\mu(\overline{X},\overline{Y})$, note that since $Y_m \Delta X_m \subseteq (X_m \setminus \bigcap_{j<m} X_j) \cup (X_m \setminus U_m)$, we also have $d_\mu(X_m, Y_m) \le \varphi_{\overline{U}}(\overline{X})$, as required. □

A little bit of natural (albeit slightly cumbersome) notation will be helpful. Suppose $\mathbb{F}$ is a finite set, $\ell(\zeta) \ge 1$ for $\zeta \in \mathbb{F}$, and we have

$$\overline{U} = (U_j^\zeta)_{\zeta\in\mathbb{F}, j<\ell(\zeta)}.$$

Then we write $\overline{U}^\zeta = (U_j^\zeta)_{j<\ell(\zeta)}$ for $\zeta \in \mathbb{F}$.

As pointed out above, an intersection of definable sets is not necessarily definable. However, in the following lemma we are dealing with a product of definable sets, not an intersection.

Lemma 2.3. *Suppose that* $\mathbb{F}$ *is a finite set and* $\overline{U}^\zeta = (U_j^\zeta)_{j<\ell(\zeta)}$ *for* $\zeta \in \mathbb{F}$ *and* $\ell(\zeta) \ge 1$, *is a tuple in a measure algebra* $(\mathcal{B}, \mu)$ *such that* $U_0^\zeta \ge U_1^\zeta \ge \dots \ge U_{\ell(\zeta)-1}^\zeta$. *Then the set*

$$\mathcal{Y}[\overline{U}] = \{Y_j^\zeta \mid \zeta \in \mathbb{F},\ j < \ell(\zeta),\ \textit{and}\ (Y_j^\zeta)_{j<\ell(\zeta)} \in \mathcal{X}[\overline{U}^\zeta]\}$$

is definable with parameter $\overline{U} = (\overline{U}^\zeta)_{\zeta\in\mathbb{F}}$.

Proof. With $\varphi_{\overline{U}}$ as in the proof of Lemma 2.2, let $\psi_{\overline{U}}(\overline{X}) = \max_{\zeta\in\mathbb{F}} \varphi_{\overline{U}^\zeta}(\overline{X}^\zeta)$. As there is no interaction between $\overline{Y}^\zeta$ for different $\zeta \in \mathbb{F}$, and as $\mathcal{Y}[\overline{U}]$ is equal to $\{\overline{Y} \mid \overline{Y}^\zeta \in \mathcal{X}[\overline{U}^\zeta] \text{ for all } \zeta \in \mathbb{F}\}$, $\psi_{\overline{U}}$ witnesses that $\mathcal{Y}[\overline{U}]$ is definable, as required. □

3. The Feferman–Vaught-type theorem for direct integrals

Throughout this section we fix an arbitrary metric language $\mathcal{L}$ and let $\mathcal{L}_{\mathrm{MBA}}$ be the language of probability measure algebras studied in Section 2.

Definition 3.1. An $\mathcal{L}_{\mathrm{MBA}}$-formula $G(\overline{X})$ in m variables $\overline{X} = \langle X_1, \dots X_m\rangle$ is *coordinatewise increasing* if for every measure algebra $(\mathcal{B}, \mu)$ and every pair of m-tuples $\overline{A} = \langle A_i\rangle$ and $\overline{A}' = \langle A_i'\rangle$ in it, if $A_i \le A_i'$ for all $i \le m$ then $G^{(\mathcal{B},\mu)}(\overline{A}) \le G^{(\mathcal{B},\mu)}(\overline{A}')$.

Definition 3.2 and Theorem 3.3 are stated for $\mathcal{L}$-formulas whose ranges are included in [0, 1]. Since the range of every $\mathcal{L}$-formula $\varphi(\bar{x})$ is a bounded interval, the range of $r(\varphi(\bar{x}) - t)$ is [0, 1] for appropriately chosen real numbers r and t, and this assumption will not result in loss of applicability of the theorem. In particular, the conclusion of Theorem 3.3 holds for tracial von Neumann algebras.

Given a probability space $(\Omega, \mathcal{B}, \mu)$ and a measurable field M_ω for $\omega \in \Omega$, of $\mathcal{L}$-structures, for an $\mathcal{L}$-formula $\varphi \in \mathfrak{F}^{\bar{x}}$, $\bar{a}$ of the appropriate sort, and $t \in [0, 1]$ we define

$$Z_t^\zeta[\bar{a}] = \{\omega \in \Omega : \zeta(\bar{a}_\omega)^{M_\omega} > t\}. \tag{3-1}$$

Definition 3.2. An $\mathcal{L}$-formula $\varphi(\bar{x})$ whose range is included in [0, 1] is determined in *direct integrals* of $\mathcal{L}$-structures, if the following objects exist.

(D1) A finite set $\mathbb{F}[\varphi]$ of $\mathscr{L}$-formulas whose free variables are included in the free variables of $\varphi(\bar{x})$ and whose ranges are included in $[0, 1]$.

(D2) For every $k \geq 2$, an integer $l(k, \varphi, \zeta) \geq 1$ and a coordinatewise increasing $\mathscr{L}_{\mathrm{MBA}}$-formula $G_{\varphi,k}(\bar{X})$ with $\sum_{\zeta \in \mathbb{F}[\varphi]} l(k, \varphi, \zeta)$ many variables $X^{\zeta}_{i/l(k,\varphi,\zeta)}$ for $\zeta \in \mathbb{F}[\varphi]$ and $i < l(k, \varphi, \zeta)$.

These objects are required to be such that for every probability space $(\Omega, \mathscr{B}, \mu)$ and a measurable field M_ω for $\omega \in \Omega$, of $\mathscr{L}$-structures, and $\bar{a}$ of the appropriate sort the following hold. (Writing $M = \int_{\Omega}^{\oplus} M_\omega \, d\mu(\omega)$.)

(D3) $\varphi(\bar{a})^M > t/k$ implies

$$G_{\varphi,k}\big(Z^{\zeta}_{i/l(k,\varphi,\zeta)}[\bar{a}],\ i < l(k, \varphi, \zeta),\ \zeta \in \mathbb{F}[\varphi]\big) > (t-1)/k.$$

(D4) $G_{\varphi,k}\big(Z^{\zeta}_{i/l(k,\varphi,\zeta)}[\bar{a}],\ i < l(k, \varphi, \zeta),\ \zeta \in \mathbb{F}[\varphi]\big) > t/k$ implies

$$\varphi(\bar{a})^M > (t-1)/k.$$

Similarly, if (D1) holds and (D2)–(D4) hold for a specific value of k, we say that φ is *k-determined*.

In particular, Definition 3.2 asserts that the value of $\varphi(\bar{a})$ is determined up to $2/k$ by the value of $G_{\varphi,k}$, which is in turn determined by the distribution of the evaluations of formulas ζ in the finite set $\mathbb{F}[\varphi]$, up to (roughly) $1/l(k, \varphi, \zeta)$ in the measurable field M_ω.

On the set of all $\mathscr{L}$-formulas consider the natural uniform metric

$$d(\varphi(\bar{x}), \psi(\bar{x})) = \sup|\varphi^M(\bar{a}) - \psi^M(\bar{a})|,$$

where the supremum is taken over all $\mathscr{L}$-structures M and all tuples $\bar{a}$ of the appropriate sort in M.

Theorem 3.3. *For every metric language $\mathscr{L}$ the set of all determined formulas is dense in the set of all $\mathscr{L}$-formulas.*

Proof. The proof proceeds by induction on complexity of φ, simultaneously for all $k \geq 2$. It will be clear from the proof that the set $\mathbb{F}[\varphi]$ does not depend on the choice of k. This is essentially the set of all subformulas of φ.

By [6, Proposition 6.6], any set of $\mathscr{L}$-formulas that includes the atomic formulas and is closed under multiplication by $\frac{1}{2}$, the operation $\varphi \dot{-} \psi = \max(0, \varphi - \psi)$, and quantifiers inf and sup is dense in the set of all $\mathscr{L}$-formulas. It therefore suffices to prove that the sets of all k-determined formulas satisfy four closure properties:

(1) All atomic formulas are k-determined.

(2) If φ is k-determined, so is $\frac{1}{2}\varphi$.

(3) If φ and ψ are $3k$-determined, then $\varphi \dot{-} \psi$ is k-determined.

(4) If φ is k-determined, so are $\sup_x \varphi$ and $\inf_x \varphi$ for every variable x.

For readability of the ongoing proof, presented by induction on the complexity of φ simultaneously for all $k \geq 2$, we combine the recursive construction of $\mathbb{F}[\varphi]$, $l(k, \varphi, \zeta)$ for $\zeta \in \mathbb{F}[\varphi]$, and $G_{\varphi,k}$ with a proof that these objects have the required properties for an arbitrary probability space $(\Omega, \mathcal{B}, \mu)$, a measurable family of $\mathcal{L}$-structures $(M_\omega)_{\omega\in\Omega}$, and its direct integral

$$M = \int_\Omega^\oplus M_\omega \, d\mu(\omega).$$

(Needless to say, the constructed objects will not depend on the choice of the measure space or the measurable family.)

(1) Suppose that $\varphi(\bar{x})$ is atomic or constant (i.e., a scalar) and that $\bar{a} \in M$ is of the appropriate sort. Let $\mathbb{F}[\varphi] = \{\varphi\}$, $l(k, \varphi) = k$, and define

$$G_{\varphi,k}(\bar{X}) = \frac{1}{k}\sum_{i=1}^{k-1} \mu(X_i).$$

It is clear that this formula is coordinatewise increasing. Then for $0 \leq i < k$, with

$$Z^\varphi_{i/k}[\bar{a}] = \{\omega \in \Omega \mid \varphi(\bar{a}_\omega)^{M_\omega} > i/k\},$$

we have (for simplicity we will write $\bar{Z}^\varphi = (Z^\varphi_0[\bar{a}], \dots, Z^\varphi_{(k-1)/k}[\bar{a}])$). From the layer cake decomposition formula for the integral of a nonnegative function we obtain

$$G_{\varphi,k}(\bar{Z}^\varphi) \leq \int_\Omega \varphi(\bar{a}_\omega)^{M_\omega} \, d\mu(\omega) \leq \frac{1}{k}\sum_{i=0}^{k-1} \mu(Z^\varphi_{i/k}[\bar{a}]) \leq G_{\varphi,k}(\bar{Z}^\varphi) + \frac{1}{k}.$$

Thus the conditions (D3) and (D4) are clearly satisfied.

(2) Suppose that $\varphi(\bar{x}) = \frac{1}{2}\psi(\bar{x})$ and that ψ is k-determined. Let $\mathbb{F}[\varphi] = \mathbb{F}[\psi]$, $l(k, \varphi, \zeta) = l(k, \psi, \zeta)$ (we could have taken $l(k, \varphi, \zeta)$ to be $\lceil \frac{1}{2} l(k, \psi, \zeta) \rceil$, but there is no reason to be frugal) and define

$$G_{\varphi,k}(\bar{X}) = \tfrac{1}{2} G_{\psi,k}(\bar{X}).$$

These objects satisfy the requirements by the definitions.

(3) Suppose that $\varphi = \psi \dot{-} \eta$ and each one of ψ and η is $3k$-determined. In order to prove that φ is k-determined let

$$\mathbb{F}[\varphi] = \mathbb{F}[\psi] \cup \{1 - \zeta : \zeta \in \mathbb{F}[\eta]\}.$$

(If $\zeta \in \mathbb{F}[\eta]$, then its range is included in $[0, 1]$, and hence the range of $1 - \zeta$ is also included in $[0, 1]$.) Also, let $l(k, \varphi, \zeta) = l(3k, \psi, \zeta)$ for $\zeta \in \mathbb{F}[\psi]$, and $l(k, \varphi, 1 - \zeta) = l(3k, \eta, \zeta)$ for $\zeta \in \mathbb{F}[\eta]$. To define $G_{\varphi,k}$, we need an additional bit of notation.

For $\zeta \in \mathbb{F}[\eta]$ and $s \in [0,1]$ let (writing $\ell = l(3k, \eta, \zeta)$ for readability)

$$\widetilde{Z}_s^{1-\zeta} = \{\omega \in \Omega : 1 - \zeta(\bar{a}_\omega)^M \geq s\}.$$

For a tuple $\bar{Z}^{1-\zeta} = (Z_0^{1-\zeta}, \dots, Z_{(\ell-1)/\ell}^{1-\zeta})$ let

$$\overleftarrow{Z}^{1-\zeta} = (\widetilde{Z}_{(\ell-1)/\ell}^{1-\zeta}, \dots, \widetilde{Z}_0^{1-\zeta}).$$

First, note that for every $0 \leq i \leq \ell$ and $\zeta \in \mathbb{F}[\eta]$ we have ($Y^{\complement}$ denotes the complement of Y, applied pointwise if Y is a tuple)

$$\begin{aligned} (\widetilde{Z}_{(\ell-i)/\ell}^{1-\zeta}[\bar{a}])^{\complement} &= \left\{\omega \in \Omega : (1 - \zeta(\bar{a}_\omega))^{M_\omega} \geq \frac{\ell - i}{\ell}\right\}^{\complement} \\ &= \left\{\omega \in \Omega : \zeta(\bar{a}_\omega)^{M_\omega} \leq \frac{i}{\ell}\right\}^{\complement} \\ &= \left\{\omega \in \Omega : \zeta(\bar{a}_\omega)^{M_\omega} > \frac{i}{\ell}\right\} = Z_{i/\ell}^{\zeta}[\bar{a}]. \end{aligned}$$

This shows that $(\overleftarrow{Z^{1-\zeta}})^{\complement} = \bar{Z}^{\zeta}$. Define[2]

$$\begin{aligned} &G_{\varphi,k}(\bar{Z}^{\xi}, \bar{Z}^{1-\zeta}, \xi \in \mathbb{F}[\psi], \zeta \in \mathbb{F}[\eta]) \\ &\qquad = G_{\psi,3k}(\bar{Z}^{\xi}, \xi \in \mathbb{F}[\psi]) \dot{-} G_{\eta,3k}((\overleftarrow{Z}^{1-\zeta})^{\complement}, \zeta \in \mathbb{F}[\eta]). \end{aligned}$$

Then $G_{\varphi,k}$ is coordinatewise increasing since the same is true for $G_{\psi,3k}$ and $G_{\eta,3k}$.

Claim 3.4. *The formula $G_{\varphi,k}$ satisfies (D3) and (D4) of Definition 3.2.*

Proof. Suppose $\varphi^M(\bar{a}) > t/k$ for $M = \int_\Omega^{\oplus} M_\omega \, d\mu(\omega)$ and $\bar{a}$ in M of the appropriate sort. There exists $m < 3(k-t)$ such that

- $\psi(\bar{a})^M > (3t+m)/3k$ and
- $\eta(\bar{a})^M \leq (m+1)/3k$.

By the induction hypothesis,

(IH1) $G_{\psi,3k}(\bar{Z}^{\xi}, \xi \in \mathbb{F}[\psi]) > (3t+m-1)/3k$ and

(IH2) $G_{\eta,3k}(\bar{Z}^{\zeta}, \zeta \in \mathbb{F}[\eta]) \leq (m+2)/3k$.

[2]For simplicity, we present $G_{\varphi,k}$ not as a formula in a tuple $\bar{X}$ abstract variables, but in terms of the intended values for these variables. Note that, since $(\overleftarrow{Z}^{1-\zeta})^{\complement}$ consists of the complements of sets in $\bar{Z}^{1-\zeta}$, $G_{\varphi,k}$ depends on the correct choice of variables, $\bar{Z}^{\zeta}$ for $\zeta \in \mathbb{F}[\varphi]$.

By (IH1) and (IH2), we have

$$\begin{aligned}
&G_{\varphi,k}(\bar{Z}^{\xi}, \bar{Z}^{1-\zeta}, \ \xi \in \mathbb{F}[\psi], \zeta \in \mathbb{F}[\eta]) \\
&\qquad = G_{\psi,3k}(\bar{Z}^{\xi}, \ \xi \in \mathbb{F}[\psi]) \dot{-} G_{\eta,3k}((\overleftarrow{Z^{1-\xi}})^{\complement}, \ \zeta \in \mathbb{F}[\eta]) \\
&\qquad \geq G_{\psi,3k}(\bar{Z}^{\xi}, \ \xi \in \mathbb{F}[\psi]) \dot{-} G_{\eta,3k}(\bar{Z}^{\zeta}, \ \zeta \in \mathbb{F}[\eta]) \\
&\qquad > \frac{3t+m-1}{3k} - \frac{m+2}{3k} = \frac{t-1}{k}.
\end{aligned}$$

This completes the proof of (D3).

To prove (D4), suppose that $G_{\varphi,k}(\bar{Z}^{\xi}, \bar{Z}^{1-\zeta}, \ \xi \in \mathbb{F}[\psi], \zeta \in \mathbb{F}[\eta]) > t/k$. Then by the definition of $G_{\varphi,k}$, for some $m < 3(k-t)$ we have

$$G_{\psi,3k}(\bar{Z}^{\xi}, \ \xi \in \mathbb{F}[\psi]) > \frac{3t+m}{3k} \quad \text{and} \quad G_{\eta,3k}(\bar{Z}^{\zeta}, \ \zeta \in \mathbb{F}[\eta]) \leq \frac{m+1}{3k}.$$

By the induction hypothesis this implies

(IH3) $\psi(\bar{a})^M > (3t+m-1)/3k$ and

(IH4) $\eta(\bar{a})^M \leq (m+2)/3k$.

Conditions (IH3) and (IH4) immediately imply that $\varphi(\bar{a})^M > (t-1)/k$. □

(4) Suppose that $\varphi(\bar{x}) = \sup_y \psi(\bar{x}, y)$ and ψ is k-determined. Let

$$\mathcal{R}[\zeta] = \{i/l(k,\psi,\zeta) \mid i < l(k,\psi,\zeta)\} \quad \text{for } \zeta \in \mathbb{F}[\psi],$$
$$\mathcal{C} = \Big\{\alpha \mid \text{there is a nonempty } \mathbb{F} \subseteq \mathbb{F}[\psi] \text{ such that } \alpha \in \prod_{\zeta\in\mathbb{F}} \mathcal{R}[\zeta]\Big\}.$$

One may think of $\alpha \in \mathcal{C}$ as a function from $\mathbb{F}$ into $\mathbb{Q} \cap [0,1]$. The point of specifying $\alpha(\zeta) \in \mathcal{R}[\zeta]$ is that, because each $\mathcal{R}[\zeta]$ is finite, the set $\mathcal{C}$ is finite as well.

For $\alpha \in \mathcal{C}$ define the $\mathcal{L}$-formula

$$\xi_\alpha(\bar{x}) = \sup_y \min_{\zeta\in\mathrm{dom}(\alpha)} (\zeta(\bar{x}, y) - \alpha(\zeta)). \tag{3-2}$$

Then for every $\alpha \in \mathcal{C}$, and $\bar{a}$ in M we have

$$\begin{aligned}
Z_0^{\xi_\alpha}[\bar{a}] = \{\omega \mid \xi_\alpha(\bar{a}_\omega)^{M_\omega} > 0\} &= \Big\{\omega \mid \sup_{y\in M_\omega} \min_{\zeta\in\mathrm{dom}(\alpha)} (\zeta(\bar{a}, y) - \alpha(\zeta)) > 0\Big\} \\
&\subseteq \bigcap_{\zeta\in\mathrm{dom}(\alpha)} \Big\{\omega \mid \sup_{y\in M_\omega} \zeta(\bar{a}, y) > \alpha(\zeta)\Big\} \\
&= \bigcap_{\zeta\in\mathrm{dom}(\alpha)} Z^{\xi_\zeta}_{\alpha(\zeta)}[\bar{a}].
\end{aligned}$$

Let

$$\mathbb{F}[\varphi] = \{\xi_\alpha \mid \alpha \in \mathcal{C}\},$$
$$l(k, \xi_\alpha, \varphi) = \max\{l(k,\zeta,\psi) \mid \zeta \in \mathrm{dom}(\alpha)\} \quad \text{for } k \geq 2 \text{ and } \alpha \in \mathcal{C}.$$

For simplicity of notation we will denote $Z_0^{\xi_\alpha}[\bar{a}]$ with $Z_0^{\xi_\alpha}$ and, more generally, $Z_r^\eta[\bar{a}]$ with Z_r^η, whenever there is no ambiguity. It will also be helpful to introduce an abbreviation and write, for $\zeta \in \mathbb{F}[\psi]$,

$$\ell(\zeta) = l(k, \psi, \zeta).$$

Prior to defining the $\mathscr{L}_{\mathrm{MBA}}$-formula $G_{\varphi,k}$, we note that every variable occurring in $G_{\psi,k}$ is of the form $Z^\zeta_{i/\ell(\zeta)}$ for some $\zeta \in \mathbb{F}[\psi]$ and $i < \ell(\zeta)$. Let

$$\bar{Z} = (Z_0^{\xi_\alpha} \mid \alpha \in \mathscr{C}). \tag{3-3}$$

Claim 3.5. *With the notation from the previous paragraph, the set $\mathscr{Y}[\bar{Z}]$ of all $\bar{Y} = (Y_i^\zeta \mid i < \ell(\zeta),\ \zeta \in \mathbb{F}[\psi])$ that satisfy conditions*

(i) $Y_i^\zeta \subseteq Z_0^{\xi_\alpha}$ *for all* $\alpha \in \mathscr{C}$ *such that* $\zeta \in \mathrm{dom}(\alpha)$ *and* $\alpha(\zeta) = \frac{i}{\ell(\zeta)}$,

(ii) $Y_i^\zeta \subseteq Y_{i-1}^\zeta$ *if* $i \geq 1$

is definable with parameters $\bar{Z}$ as in (3-3).

Proof. For $m < \ell(\zeta)$, let

$$U_m^\zeta = \bigcap \left\{ Z_0^{\xi_\alpha} \mid \alpha \in \mathscr{C},\ \zeta \in \mathrm{dom}(\alpha),\ \alpha(\zeta) \leq \frac{m}{\ell(\zeta)} \right\}.$$

Then $U_0^\zeta \supseteq U_1^\zeta \supseteq \cdots \supseteq U^\zeta_{\ell(\zeta)-1}$ for every ζ. Let

$$\mathscr{X}[\bar{U}^\zeta] = \left\{ \bar{Y}^\zeta \in \mathscr{B}^{\ell(\zeta)} \mid Y_j^\zeta \leq U_j^\zeta \cap \bigcap_{i<j} Y_i^\zeta \text{ for } j < \ell(\zeta) \right\}.$$

Then $\mathscr{Y}[\bar{U}] = \{Y_j^\zeta \mid \zeta \in \mathbb{F}[\psi],\ j < \ell(\zeta) \text{ and } (Y_j^\zeta)_{j<\ell(\zeta)} \in \mathscr{X}[\bar{U}^\zeta]\}$, as considered in Lemma 2.3 is definable. This set is equal to $\mathscr{Y}[\bar{Z}]$ and it is definable with parameters $\bar{Z}$. □

Therefore $G_{\varphi,k}$ as defined below is a formula (on the right-hand side, in $G_{\psi,k}$ the variable $Z^\zeta_{i/l(\zeta)}$ is replaced with Y_i^ζ for all $\zeta \in \mathbb{F}[\psi]$ and $i < \ell(\zeta)$):

$$G_{\varphi,k}(\bar{Z}^\xi : \xi \in \mathbb{F}[\varphi]) = \sup_{\bar{Y} \in \mathscr{Y}[\bar{Z}]} G_{\psi,k}(Y_i^\zeta\ \zeta \in \mathbb{F}[\psi],\ i < l(\zeta)).$$

Clearly, $G_{\varphi,k}$ is coordinatewise increasing since $G_{\psi,k}$ has this property and since the set $\mathscr{Y}[\bar{Z}]$ is also increasing in $\bar{Z}$ (in the sense that $\bar{Z} \leq \bar{Z}'$ implies $\mathscr{Y}[\bar{Z}] \subseteq \mathscr{Y}[\bar{Z}']$). It remains to prove that $G_{\varphi,k}$ satisfies the requirements of Definition 3.2.

To prove (D3), suppose $\varphi(\bar{a})^M > t/k$ for $M = \int_\Omega^\oplus M_\omega \, d\mu(\omega)$ and $\bar{a}$ in M of the appropriate sort. Pick $b \in M$ such that $\psi(\bar{a}, b)^M > t/k$. Then, by the induction hypothesis we have

$$G_{\psi,k}(\bar{Z}^\zeta[\bar{a}, b],\ \zeta \in \mathbb{F}[\psi]) > \frac{t-1}{\ell(k)}. \tag{3-4}$$

For $\zeta \in \mathbb{F}[\psi]$ and $i < \ell(\zeta)$ let

$$Y_i^\zeta = Z^\zeta_{i/l(\zeta)}[\bar{a}, b].$$

We claim that $\bar{Y}$ defined in this manner belongs to the set $\mathcal{Y}[\bar{Z}]$ as in Claim 3.5. Condition (ii) is clearly satisfied and condition (i) is satisfied because for all $\zeta \in \mathbb{F}[\psi]$ and $i < l(\zeta)$, we have that

$$\begin{aligned} Y_i^\zeta &= \{\omega \in \Omega\} \mid \zeta(\bar{a}_\omega, b_\omega)^{M_\omega} > i/\ell(\zeta)\} \\ &\subseteq \bigcap_{\substack{\alpha\in\mathscr{C},\, \zeta\in\mathrm{dom}(\alpha) \\ \alpha(\zeta)\le i/\ell(\zeta)}} \Big\{\omega \in \Omega \mid \sup_{y\in M_\omega} \min_{\zeta\in\mathrm{dom}(\alpha)} \eta(\bar{a}_\omega, y)^{M_\omega} - \alpha(\zeta) > 0\Big\} \\ &= \bigcap_{\substack{\alpha\in\mathscr{C},\, \zeta\in\mathrm{dom}(\alpha) \\ \alpha(\zeta)\le i/\ell(\zeta)}} Z_0^{\xi_\alpha}. \end{aligned}$$

By (3-4) we have

$$G_{\varphi,k}(\bar{Z}^\xi,\ \xi \in \mathbb{F}[\varphi]) \ge G_{\psi,k}(Y_i^\zeta,\ \zeta \in \mathbb{F}[\psi],\ i < \ell(\zeta)) > \frac{t-1}{\ell(k)}.$$

This completes the proof of (D3).

To prove (D4), assume $G_{\varphi,k}(\bar{Z}^{\xi_\zeta}[\bar{a}],\ \zeta \in \mathbb{F}[\psi]) > t/k$. Then there are measurable sets (as before, $\bar{a}$ is suppressed for readability) Y_i^ζ for $\zeta \in \mathbb{F}[\psi]$ and $i < \ell(\zeta)$ satisfying (i) and (ii) such that

$$G_{\psi,k}(Y_i^\zeta,\ \zeta \in \mathbb{F}[\psi],\ i < \ell(\zeta)) > t/k. \tag{3-5}$$

For each $\omega \in \Omega$ let

$$D_\omega = \{\zeta \in \mathbb{F}[\psi] : \omega \in Y_0^\zeta\}.$$

Define $\alpha_\omega \in \mathscr{C}$ with $\mathrm{dom}(\alpha_\omega) = D_\omega$ by

$$\alpha_\omega(\zeta) = \max\{i/\ell(\zeta) : \omega \in Y_i^\zeta\}. \tag{3-6}$$

If $D_\omega \neq \varnothing$ then $\omega \in \bigcap_{\zeta\in D_\omega} Y^\zeta_{\alpha_\omega(\zeta)}$ and (i) implies that $Y^\zeta_{\alpha_\omega(\zeta)} \subseteq Z_0^{\xi_{\alpha_\omega}}$ for every $\zeta \in \mathrm{dom}(\alpha_\omega)$, and hence we have $\bigcap_{\zeta\in D_\omega} Y^\zeta_{\alpha_\omega(\zeta)} \subseteq Z_0^{\xi_{\alpha_\omega}}$ and $\omega \in Z_0^{\xi_{\alpha_\omega}}$. Therefore,

$$\sup_y \min_{\zeta\in D_\omega} \zeta(\bar{a}_\omega, y) > \alpha_\omega(\zeta).$$

Recall from Definition 1.1 that all M_ω for $\omega \in \Omega$, are separable and that e_n for $n \in \mathbb{N}$, enumerate a subset of M such that $e_n(\omega)$ for $n \in \mathbb{N}$, form a dense subset of M_ω for every $\omega \in \Omega$. Thus, if $D_\omega \neq \varnothing$ there exists $n \in \mathbb{N}$ such that

$$\min_{\zeta\in D_\omega} \zeta^{M_\omega}(\bar{a}_\omega, e_n(\omega)) > \alpha_\omega(\zeta).$$

Let $n(\omega)$ be the minimal n with this property. For each $n \in \mathbb{N}$, let

$$\Omega_n = \{\omega \in \Omega \mid n(\omega) = n\}$$

and $\Omega_\infty = \{\omega \in \Omega \mid D_\omega = \varnothing\}$. Note that $\Omega_\infty = \Omega \setminus \bigcup_{\zeta \in \mathbb{F}[\psi]} Y_0^\zeta$. Therefore, $\{\Omega_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ is a partition of Ω into measurable sets. The function b defined on Ω by

$$b_\omega = \begin{cases} e_n(\omega) & \text{if } \omega \in \Omega_n, \\ e_0(\omega) & \text{if } \omega \in \Omega_\infty \end{cases}$$

is a measurable field of elements and it therefore defines an element of M. By the choice of $n(\omega)$, if $D_\omega \neq \varnothing$, then

$$\min_{\zeta \in D_\omega} \zeta(\bar{a}_\omega, b_\omega)^{M_\omega} > \alpha_\omega(\zeta). \tag{3-7}$$

Claim 3.6. *We have $Y_i^\zeta \subseteq Z^\zeta_{i/\ell(\zeta)}[\bar{a}, b]$ for all $\zeta \in \mathbb{F}[\psi]$ and $i < \ell(\zeta)$.*

Proof. Suppose $\omega \in Y_i^\zeta$. Then, from (3-6) we have $i/\ell(\zeta) \leq \alpha_\omega(\zeta)$. By the choice of b, we have

$$\zeta(\bar{a}_\omega, b_\omega)^{M_\omega} > \alpha_\omega(\zeta) \geq i/\ell(\zeta),$$

which means, $\omega \in Z^\zeta_{i/\ell(\zeta)}[\bar{a}, b]$. □

Since $G_{\psi,k}$ is coordinatewise increasing, Claim 3.6 and (3-5) together imply

$$G_{\psi,k}(\bar{Z}^\zeta[\bar{a}, b],\ \zeta \in \mathbb{F}[\psi]) > t/k.$$

The inductive hypothesis implies that $\psi(\bar{a}, b)^M > (t-1)/k$, and therefore we have $\varphi(\bar{a})^M > (t-1)/k$ as required.

Since $\inf_y \psi(\bar{x}, y) = 1 - \sup_y(1 - \psi(\bar{x}, y))$, the case when $\varphi(\bar{x}) = \inf_y \psi(\bar{x}, y)$ for some ψ that satisfies the inductive assumption follows from the previous case. This completes the proof by induction on complexity of φ. □

What makes Theorem 3.3 (or in general, Feferman–Vaught-type theorems) "effective" is that the objects in (D1)-(D2) of Definition 3.2 can be recursively obtained from only the syntax of φ, as the proof shows. In order to state Corollary 3.8 we need a definition.

Definition 3.7 (measurable fields and direct integrals of metric structures). Suppose that $\mathcal{L}$ is a continuous language, $(\Omega, \mathcal{B}, \mu)$ is a separable probability measure space, (N_ω, d_ω) for $\omega \in \Omega$, are separable $\mathcal{L}$-structures and M_ω is a substructure of N_ω for a set of ω of full measure. Assume that e_n for $n \in \mathbb{N}$, is a sequence in $\prod_{\omega \in \Omega} N_\omega$ such that the following two conditions hold.

(1) For every ω the set $\{e_{2n}(\omega) \mid n \in \mathbb{N}\}$ is dense in M_ω and the set $\{e_n(\omega) \mid n \in \mathbb{N}\}$ is dense in N_ω.

(2) For every predicate $R(\bar{x})$ in $\mathcal{L}$ and every tuple $e_{2\bar{n}} = \langle e_{2n(0)}, \ldots, e_{2n(j-1)} \rangle$ of the appropriate sort, the function $\omega \mapsto R^{M_\omega}(e_{2\bar{n}}(\omega))$ is measurable.

(3) For every predicate $R(\bar{x})$ in $\mathscr{L}$ and every tuple $e_{\bar{n}} = \langle e_{n(0)}, \dots, e_{n(j-1)} \rangle$ of the appropriate sort, the function $\omega \mapsto R^{N_\omega}(e_{\bar{n}}(\omega))$ is measurable.

As in Definition 1.1, the structures N_ω, together with the functions e_n, form a *measurable field* of $\mathscr{L}$-structures and the structures M_ω form a *measurable subfield* of this measurable field.

Corollary 3.8. *Suppose* $(\Omega, \mathscr{B}, \mu)$ *is a separable measure space, and* M_ω *and* N_ω *are measurable fields of structures of the same language, for all* $\omega \in \Omega$.

If $M_\omega \equiv N_\omega$ *for almost all* ω, *then*

$$\int_\Omega^\oplus M_\omega \, d\mu(\omega) \equiv \int_\Omega^\oplus N_\omega \, d\mu(\omega).$$

If M_ω *for* $\omega \in \Omega$ *is a measurable subfield of* N_ω *for* $\omega \in \Omega$ *and in addition* $M_\omega \preceq N_\omega$ *for almost all* ω, *then*

$$\int_\Omega^\oplus M_\omega \, d\mu(\omega) \preceq \int_\Omega^\oplus N_\omega \, d\mu(\omega).$$

Proof. We prove the second part. Fix a formula $\varphi(\bar{x})$ and $k \geq 2$. By Theorem 3.3 can be uniformly approximated by a formula that is determined. Therefore, without loss of generality we assume $\varphi(\bar{x})$ is determined. Let

$$M = \int_\Omega^\oplus M_\omega \, d\mu(\omega) \quad \text{and} \quad N = \int_\Omega^\oplus N_\omega \, d\mu(\omega).$$

For every $\bar{a}$ in M of the appropriate sort and every formula $\zeta(\bar{x}) \in \mathbb{F}[\varphi]$, the set of ω such that $\zeta(\bar{a}_\omega)^{N_\omega} = \zeta(\bar{a}_\omega)^{M_\omega}$ has full measure. That is, the sets of the form $Z_r^\zeta[\bar{a}]$, as in Definition 3.2, evaluated in structures M and N are the same. Therefore $|\varphi(\bar{a})^N - \varphi(\bar{a})^M| < 2/k$ and because k was arbitrary it follows that $\varphi(\bar{a})^N = \varphi(\bar{a})^M$. Since φ and $\bar{a}$ were arbitrary, $N_\omega \preceq M_\omega$.

Proof of the first part is analogous. □

As Itaï Ben Yaacov [5] pointed out, Corollary 3.8 can be proven using quantifier elimination in atomless randomizations. This result applies only to atomless measure spaces but is in this case even slightly stronger as it shows that the direct integrals are elementarily equivalent even as randomization structures.

A special case of Corollary 3.8 where all the fiber of the direct integrals are the same tracial von Neumann algebra leads to the following corollary.

Corollary 3.9. *Suppose* M *and* N *are elementarily equivalent tracial von Neumann algebras and* $(\Omega, \mathscr{B}, \mu)$ *is a separable measure space. Then*

$$M \bar{\otimes} L^\infty(\Omega, \mu) \equiv N \bar{\otimes} L^\infty(\Omega, \mu).$$

If $M \preceq N$ *then*

$$M \bar{\otimes} L^\infty(\Omega, \mu) \preceq N \bar{\otimes} L^\infty(\Omega, \mu).$$

4. Applications to tracial von Neumann algebras

In this section we prove Theorem 1, after discussing a technical point.

4.1. ***Two languages for tracial von Neumann algebra.*** Tracial von Neumann algebras are equipped with a distinguished tracial state τ, usually suppressed for the simplicity of notation.[3] In the literature tracial von Neumann algebras are usually considered with respect to the $\|\cdot\|_2$-norm:

$$\|a\|_2 = \tau(a^*a)^{1/2},$$

but in [36, Section 29] they are for convenience considered with respect to the $\|\cdot\|_1$ norm:

$$\|a\|_1 = \tau((a^*a)^{1/2}).$$

We will denote the corresponding languages $\mathscr{L}_{\|\cdot\|_2}$ and $\mathscr{L}_{\|\cdot\|_1}$, respectively. Since the syntax of continuous logic requires each function symbol to be equipped with a modulus of uniform continuity, the difference between these two languages is not only notational. By [36, Lemma 29.1], on operator norm-bounded balls the $\|\cdot\|_1$-norm is compatible with the strong operator topology (and therefore equivalent to the $\|\cdot\|_2$-norm). We thus have two competing languages and two competing axiomatizations (the standard one and the one in [36, Proposition 29.4]) of tracial von Neumann algebras in continuous logic. In order to facilitate the ongoing discussion, for $j = 1, 2$ we will refer to the axiomatization (formulas, definable predicates, etc.) using the $\|\cdot\|_j$-norm as j-axiomatization (j-formulas, j-definable predicates, etc.).

Lemma 4.1. (1) *The tracial state is a j-definable predicate for $J = 1, 2$.*

(2) *The norm $\|\cdot\|_2$ is a* 1*-definable predicate.*

(3) *The norm $\|\cdot\|_1$ is a* 2*-definable affine predicate.*

(4) *Every* 2*-definable predicates is a* 1*-definable predicate and vice versa.*

Proof. We prove that the tracial state is a 1-definable predicate by exhibiting a concrete defining formula. If $a = a^*$ and $\|a\| \leq n$ (data visible from the sort of a) then $|a+n| = a+n$ and $\tau(a) = \|a+n\|_1 - n$. Since a can be written as $a = a_0 + ia_1$ where $a_0 := \frac{1}{2}(a + a^*)$ and $a_1 := \frac{1}{2i}(a - a^*)$ are self-adjoint, we have that (still assuming $\|a\| \leq n$)

$$\tau(a) = \|a_0 + n\|_1 + i\|a_1 + n\| - (1+i)n.$$

Similarly, if $a = a^*$ and $\|a\| \leq n$ then $\tau(a) = \|a+n\|_2^2 - n$, which by the above argument shows that τ is 2-definable.

[3] For simplicity, in this proof we allow tracial von Neumann algebras whose distinguished trace is not normalized.

Since $\|a\|_2 = \tau(a^*a)^{1/2}$, the 2-norm is a 1-definable predicate.

The remaining parts of the lemma follow. □

Corollary 4.2. *Two tracial von Neumann algebras are* 1*-elementarily equivalent if and only if they are* 2*-elementarily equivalent.*

A class of tracial von Neumann algebras is 1*-axiomatizable* (*in continuous logic*) *if and only if it is* 2*-axiomatizable* (*in continuous logic*). □

4.2. ***Proof of Theorem 1.*** En route to the proof of Theorem 1 we prove the following (very likely well-known, yet not completely trivial) result. By Corollary 4.2, we do not need to indicate whether "elementarily equivalent" refers to the 1-logic or to the 2-logic as discussed in the previous subsection.

Proposition 4.3. *Suppose that M and M_1 are elementarily equivalent tracial von Neumann algebras and one of them is of type I. Then the other one is also of type* I.

If in addition both M and M_1 have separable predual, then they are isomorphic. In other words, the theory is a complete isomorphism invariant for separable tracial von Neumann algebras of type I.

Proposition 4.3 will follow from a more precise (and more obvious), statement (Lemma 4.4) given after a few clarifying remarks.

Note that being type I is not axiomatizable in language of tracial von Neumann algebras, since the category of type I tracial von Neumann algebras is not preserved under ultraproducts. (E.g., the ultraproduct of $\mathbb{M}_n(\mathbb{C})$ for $n \in \mathbb{N}$ associated with a nonprincipal ultrafilter on $\mathbb{N}$ is an interesting II_1 factor without property Γ.) However, every tracial von Neumann algebra elementarily equivalent to $\mathbb{M}_n(\mathbb{C})$ is isomorphic to it.

By the second part of Proposition 4.3, type I tracial von Neumann algebras behave similarly to compact metric structures, or to finite-dimensional C*-algebras (all of whose sorts are compact). More precisely, the second part of Proposition 4.3 is a poor man's version of the fact that $\mathbb{M}_n(\mathbb{C}) \equiv A$ implies $\mathbb{M}_n(\mathbb{C}) \cong A$: Every tracial von Neumann algebra with separable predual elementarily equivalent to a given type I tracial von Neumann algebra M with separable predual is isomorphic to it. In terminology of [16], being isomorphic to M is *separably axiomatizable*. In the standard model-theoretic terminology, the theory of M is $\aleph_0$-categorical (some authors write ω-categorical, as the ordinal ω is routinely identified with the cardinal $\aleph_0$).

Lemma 4.4. *If M is a tracial von Neumann algebra, then there is a unique function*

$$\rho_M : (\mathbb{N} \setminus \{0\}) \times \mathbb{N} \to [0, 1],$$

with the following properties.

(1) $\sum_{m,n} \rho_M(m, n) \leq 1$, *with the equality holding if and only if M has type* I.

(2) $\rho_M(m, n) \geq \rho_M(m, n+1)$ *whenever* $n \geq 1$.

(3) $M = \prod_{m\geq 1} \mathbb{M}_m(L^\infty(X_m, \mu_m))$, *where* (X_m, μ_m) *is a measure space which has atoms of measure* $\rho_M(m, n)$ *for* $n \geq 1$ (*with multiplicities*), *and diffuse part of measure* $\rho_M(m, 0)$ (*with* $\mu(X_m) = \sum_n \rho_M(m, n)$).

Moreover, the function ρ_M *is computable from the theory of* M.

Proof. By the type decomposition of finite von Neumann algebras [34, Section V], M is isomorphic to the direct sum $M_{\mathrm{I}} \oplus M_{\mathrm{II}}$ where M_{I} is of type I and M_{II} of type II.

By the same decomposition result, M_{I} is of the form $\prod_{m\geq 1} \mathbb{M}_m(L^\infty(X_m, \mu_m))$ with $\sum_m \mu_m(X_m) = 1$ (possibly with $\mu_m(X_m) = 0$ for some m). Since every finite measure space can be decomposed into diffuse and atomic part as specified, giving rise to ρ_M. Measures of the atoms are listed in decreasing order in order to assure (2), securing the uniqueness of the function ρ_M. To be precise, let $Y_{m,n}$ for $n \in \mathbb{N}$, enumerate all atoms in the measure space (X_m, μ_m), listed in order of decreasing measure, with multiplicities. If there are only k atoms, then let $Y_{m,n} = \varnothing$ for $n \geq k$. Finally let $\rho_M(m, n) = \mu_m(Y_{m,n})$. We therefore only need to explain how to determine ρ_M from the theory of M.

First we use the fact that the center $Z(M)$ of a tracial von Neumann algebra M is definable (this is essentially [13, Lemma 4.1]). The proof shows that the lattice of projections in the center is also definable (this is not an immediate consequence of the fact that the set of projections is also definable, since by an unpublished result of Henson in continuous logic the intersection of definable sets is not necessarily definable).

As observed in [16, Theorem 2.5.1], m-subhomogeneous C*-algebras are axiomatizable. This clearly extends to von Neumann algebras, by using the same formula. So for every $m \geq 1$ the set (by τ we denote the distinguished tracial state of M)

$$\{\tau(p) \mid p \in Z(M) \text{ is a projection and } pMp \text{ is } m\text{-subhomogeneous}\}$$

can be read off the theory of M. The supremum of this set is equal to

$$\sum_{j\leq m} \sum_n \rho_M(j, n).$$

Thus $\mu_m(X_m)$ is determined from $\mathrm{Th}(M)$.

It remains to compute the measures of the atoms of each μ_m. Again using the fact that the projections in $Z(M)$ form a definable set, these are the values of $\tau(p)$ where p is a central projection such that $pMp \cong \mathbb{M}_m(\mathbb{C})$. Multiplicities are handled similarly. □

Proof of Proposition 4.3. Suppose that M and M_1 are elementarily equivalent tracial von Neumann algebras and that M has type I. Lemma 4.4 implies that $\rho_M = \rho_{M_1}$. By (1) of Lemma 4.4 we have $\sum_{m,n} \rho_{M_1}(m, n) = \sum_{m,n} \rho_M(m, n) = 1$, and therefore M_1 is also of type I.

Note that $\rho_M = \rho_{M_1}$ implies that the atomic parts of M and M_1 are isomorphic. Since every diffuse abelian tracial von Neumann algebra is isomorphic to $L^\infty([0, 1])$, if in addition to being of type I and elementarily equivalent both M and M_1 have separable predual, then they are isomorphic. □

The following well-known lemma uses the notion of the eq of a metric structure, studied in [16, Section 3]. Briefly, if M is a metric structure then M^{eq} is formed from M by expanding it as follows (see [16, Section 3.3] for details). First, one adds all countable products of sorts in M and equips them with natural product metric. Second, one adds all definable subsets of such products. Third, one takes quotients by all definable equivalence relations on such definable sets. The structure obtained in this way is denoted M^{eq}. Its theory T^{eq} depends only $T = \mathrm{Th}(M)$, the category of models of T^{eq} is equivalent to the category of models of T, and it is abstractly characterized as the largest conservative extension of the category of models of T (this is [16, Theorem 3.3.5]). It is easier than its C*-algebraic analog, [16, Lemma 3.10.2].

Lemma 4.5. *If N and F are tracial von Neumann algebras and F is finite-dimensional, then $F\bar{\otimes}N$ is in N^{eq}. Thus, if M is also a tracial von Neumann algebra such that $M \equiv N$ ($M \preceq M$), then $F\bar{\otimes}M \equiv F\bar{\otimes}N$ ($F\bar{\otimes}M \preceq F\bar{\otimes}N$).*

Proof. We first consider the case $F = \mathbb{M}_m(\mathbb{C})$ for some $m \geq 1$. Then the distinguished tracial state of $F\bar{\otimes}N$ is $\sigma = \mathrm{tr_m} \otimes \tau$ (where $\mathrm{tr_m}$ is the normalized tracial state on $\mathbb{M}_m(\mathbb{C})$ and τ is the distinguished tracial state of N). Then the unit ball of $(F\bar{\otimes}N, \sigma)$ can be identified with a subset of the unit ball N^{m^2} with the naturally defined matrix arithmetic operations and tracial state $\sigma((a_{ij})_{i,j\leq m}) = \sum_{i\leq m} \tau(a_{ii})$.

For the general case, note that F is the direct sum of full matrix algebras, $F = \bigoplus_{j\leq k} \mathbb{M}_{l(j)}(\mathbb{C})$ and that its distinguished tracial state is a convex combination of $\mathrm{tr}_{l(j)}$ for $j \leq k$. By the argument similar to the one for $\mathbb{M}_m(\mathbb{C})$, $F\bar{\otimes}N$ can be identified with $N^{\sum_{j\leq k} l(j)^2}$ with the appropriately defined arithmetic operations and distinguished tracial state. □

Proof of Theorem 1. We will prove the theorem for $\preceq$. The proof for $\equiv$ is analogous, and alternatively it follows by the fact that elementarily equivalent structures can be elementarily embedded into the same structure.

The result follows from the conjunction of the following two statements.

(1) If M, N, N_1 are tracial von Neumann algebras, $N_1 \preceq N$, and M has type I, then $M\bar{\otimes}N_1 \preceq M\bar{\otimes}N$.

(2) If M, N, N_1 are tracial von Neumann algebras, $N_1 \preceq N$, and N has type I, then $M\bar{\otimes}N_1 \preceq M\bar{\otimes}N$.

(1) As in the proof of Proposition 4.3, we have $M = \prod_{m\geq 1} \mathbb{M}_m(L^\infty(X_m, \mu_m))$. By Corollary 3.9, $N_1 \preceq N$ implies that $N_1(m) = L^\infty(X_m, \mu_m)\bar{\otimes}N_1$ is an elementary submodel of $N(m) = L^\infty(X_m, \mu_m)\bar{\otimes}N$.

The matrix case of Lemma 4.5 implies $\mathbb{M}_m(N_1(m)) \preceq \mathbb{M}_m(N(m))$ for all m. Corollary 3.8 applied to the measure space $(\mathbb{N}, \mu)$ where μ is a probability Radon measure implies that $M \bar{\otimes} N_1 \preceq M \bar{\otimes} N$.

(2) Analogously to case (1), $N = \prod_{m \geq 1} \mathbb{M}_m(P_m)$ for some abelian von Neumann algebras P_m. By Proposition 4.3, $N_1 = \prod_{m \geq 1} \mathbb{M}_m(P_{1,m})$ with $P_{1,m} \equiv P_m$ for all m, and furthermore the algebras P_m and $P_{1,m}$ have atoms and diffuse parts of the same measure. This implies that P_m and $P_{1,m}$ are isomorphic. Therefore, $P_{1,m} \bar{\otimes} M \preceq P \bar{\otimes} M$. Lemma 4.5 implies that further tensoring with $\mathbb{M}_m(\mathbb{C})$ preserves elementarity, and hence Corollary 3.8 implies that $\prod_{m \geq 1} \mathbb{M}_m(P_{1,m} \bar{\otimes} M)$ is an elementary submodel of $\prod_{m \geq 1} \mathbb{M}_m(P_m \bar{\otimes} M)$. That is, $M \bar{\otimes} N_1 \preceq M \bar{\otimes} N$. □

5. Concluding remarks

In general, tensoring with strongly self-absorbing C*-algebras [35] does not preserve elementarity [11, Proposition 6.2]. All known examples of the failure of preservation of elementary equivalence by tensor products relied on failure of regularity properties of C*-algebras, such as definability of tracial states [11, Proposition 6.2] and stable rank being greater than 1 ([16, Corollary 3.10.4], using [32, Theorem 3.1]).

However, known results imply that tensoring with a strongly self-absorbing C*-algebra D preserves elementary equivalence in a large class of C*-algebras. It follows from [16, Corollary 2.7.2] that if A is tensorially D-absorbing, then $A \preceq A \otimes D$ via the map that sends a to $a \otimes 1_D$ for all $a \in A$ (since $D \cong D^{\otimes \mathbb{N}}$ by [35, Proposition 1.9]). Being tensorially D-absorbing is separably axiomatizable by [16, Theorem 2.5.2]. (A property is *separably axiomatizable* if there is a theory T such that all separable models of T satisfy this property.) Let $\mathbb{T}_D$ be the theory of separable D-absorbing C*-algebras (see [15, Section 2.1]). Models of $\mathbb{T}_D$ are called *potentially D-absorbing* [15, Definition 2.4], and they have the property that all of their separable elementary submodels are D-absorbing. The Downward Löwenheim–Skolem theorem thus implies that if A is potentially D-absorbing and $A \equiv B$, then $A \otimes D \equiv B \otimes D$.

If $A \preceq B$ and A is potentially D-absorbing, then by identifying A with $A \otimes 1_D$ in $A \otimes D$ we have $A \preceq B \otimes D$. A slightly finer analysis using [15, Lemma 1.4] applied to the inclusions $A \subseteq A \otimes D \subseteq B \otimes D$ (or directly using the Tarski–Vaught test) shows that $A \otimes D \preceq B \otimes D$.

Proposition 5.1. *If $A \preceq B$ and A is potentially D-absorbing, then $A \otimes D \preceq B \otimes D$ via the natural embedding that fixes $1 \otimes D$.*

By a result of Connes, the hyperfinite II_1 factor R is the only strongly self-absorbing II_1 factor. Potentially R-absorbing tracial von Neumann algebras are the McDuff factors, and an argument analogous to that of the previous paragraph shows that if $M \equiv N$ and M is McDuff, then $M\bar{\otimes}R \equiv N\bar{\otimes}R$ and if $M \preceq N$ and M is McDuff then $M\bar{\otimes}R \preceq N\bar{\otimes}R$.

It is not known whether tensor products of tracial von Neumann algebras preserve elementarity or not. The fact that theories of type II_1 tracial von Neumann algebras do not admit quantifier elimination (see [12] and [17] for a finer result) makes the question of preservation of elementarity more intricate.

Definition 5.2. If $M = \int_\Omega^\oplus M_\omega \, d\mu(x)$ is a direct integral, then the *distribution of the theories* in the measurable field M_ω for $\omega \in \Omega$, is the function α that to every $n \geq 1$ and every n-tuple of $\mathcal{L}$-sentences $\bar{\varphi} = \langle \varphi_j : j \leq n \rangle$, associates the distribution $\alpha_{\bar{\varphi}} : [0, 1]^n \to [0, 1]$ by

$$\alpha_{\bar{\varphi}}(\bar{r}) = \mu\{\omega \mid \varphi_j^{M_\omega} > r_j \text{ for all } j \leq n\}.$$

Theorem 3.3 (together with Lemma 4.1 that provides translation between languages $\mathcal{L}_{\|\cdot\|_2}$ and $\mathcal{L}_{\|\cdot\|_1}$) has the following natural corollary.

Corollary 5.3. *The theory of a direct integral $M = \int_\Omega^\oplus M_\omega \, d\mu(\omega)$ is uniquely determined by the distribution of the theories in the measurable field M_ω for $\omega \in \Omega$.* □

The converse to this corollary is in general false, it is not possible to disintegrate the theory of a direct integral to recover the theories of M_ω for almost all ω. However, every tracial von Neumann algebra with separable predual admits a disintegration into a measurable field of factors that is essentially unique [34, Section IV]. Confirming Conjecture 4.5 from the original version of the present paper, David Gao and David Jekel proved that if M is a direct integral of a measurable field of II_1 factors then its theory uniquely determines the distribution of theories of II_1 factors in this measurable field [21, Theorem A]. The proof uses a variant of [5]. Together with an easy Fubini-type argument, this theorem implies the following.

Corollary 5.4. *Tensor products of tracial von Neumann algebras preserve elementary equivalence if and only if tensor products of* II_1 *factors preserve elementary equivalence.* □

While a Feferman–Vaught-style theorem technically solves the problem of computing the theory of a given structure, it is desirable to have a more efficient procedure. In [30] and [31], Palyutin isolated a class of so-called *h-formulas* and shown that they satisfy a version of Łoś's theorem in every reduced product $M = \prod_{i \in I} M_i/\mathcal{I}$ and that if in addition the Boolean algebra $\mathcal{P}(I)/\mathcal{I}$ is atomless then every formula in the language of M is equivalent to a Boolean combination of h-formulas. This has, for example, been used to provide a much simpler proof of $\aleph_1$-saturation of reduced products associated with a countable ideal in [29]. The analog of Palyutin's theory for continuous logic has been developed in [20].

It would be desirable to develop analogous theory for direct integrals of structures in place of reduced products. We will now describe some partial results along these lines. In [2], Bagheri proved a preservation theorem for affine formulas under direct integrals. He introduced a variant of the continuous logic, nowadays known as the *affine logic*, a systematic study of which is in [36] . Affine formulas are defined recursively starting from atomic formulas. Logical connectives are restricted to affine functions, while the role of quantifiers is still played by supremum and infimum [36, Section 2]. Structures and interpretation of formulas are analogous to those in continuous logic. The operation of taking direct integrals of measurable fields of affine structures preserves the affine theory (note that for tracial von Neumann algebras this is true only if they are considered with respect to the $\|\cdot\|_1$ norm, see Lemma 4.1). A preservation under direct integrals of tracial von Neumann algebras for certain convex formulas had been proven in [18] in a work motivated by the need to systematize the theory of tracially complete C*-algebras [9].

Moving to the other important class of self-adjoint operator algebras, we ask whether there is a C*-algebraic analog of Theorem 1? Tensor products by finite-dimensional C*-algebras preserve elementary equivalence. By the C*-algebraic analog of Lemma 4.5, if A and F are C*-algebras such that F is finite-dimensional, then $A \otimes F$ belongs to A^{eq} [16, Section 3] and therefore the theory of $A \otimes F$ can be computed from the theory of A. However, tensoring by $C([0,1])$ do not preserve elementary equivalence [16, Corollary 3.10.4]. To the best of our current (lamentably limited) understanding, the following would be a plausible analog of Theorem 1.

Conjecture 5.5. *Suppose that A is a* C**-algebra all of whose irreducible representations are finite-dimensional and the Gelfand spectrum of its center is totally disconnected. If B and C are elementarily equivalent* C**-algebras, then $A \otimes B$ and $A \otimes C$ are elementarily equivalent.*

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EFFICIENT CYCLES OF HYPERBOLIC MANIFOLDS

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Let N be a complete finite-volume hyperbolic n-manifold. An efficient cycle for N is the limit (in an appropriate measure space) of a sequence of fundamental cycles whose ℓ^1-norm converges to the simplicial volume of N. Gromov and Thurston's smearing construction exhibits an explicit efficient cycle, and Jungreis and Kuessner proved that, in dimension $n \geq 3$, such a cycle actually is the unique efficient cycle for a huge class of finite-volume hyperbolic manifolds, including all the closed ones. We prove that, for $n \geq 3$, the class of finite-volume hyperbolic manifolds for which the uniqueness of the efficient cycle does not hold is exactly the commensurability class of the figure-8 knot complement (or, equivalently, of the Gieseking manifold).

Introduction

The simplicial volume is a homotopy invariant of manifolds introduced by Gromov in his pioneering paper [1982]. If N is a compact connected oriented n-manifold (possibly with boundary) the simplicial volume $\|N\|$ of N is defined by

$$\|N\| = \inf\left\{ \sum_{i=1}^{k} |a_i| : \left[\sum_{i=1}^{k} a_i \sigma_i \right] = [N] \in H_n(N, \partial N) \right\},$$

where $[N]$ denotes the real fundamental class of N, and $H_n(N, \partial N)$ denotes the relative singular homology module of the pair $(N, \partial N)$ with real coefficients.

Computing the simplicial volume is usually a very difficult task. Many vanishing theorems are available by now, but positive exact values of the simplicial volume are known only for a few classes of manifolds, such as complete finite-volume hyperbolic manifolds [Gromov 1982; Thurston 1979], closed manifolds isometrically covered by the product of two copies of the hyperbolic plane [Bucher-Karlsson 2008], some 3-manifolds with higher genus boundary [Bucher et al. 2015] and special families of 4-manifolds [Heuer and Löh 2021]. Even when the simplicial volume of a manifold N is known, characterizing (or, at least, exhibiting) *almost*

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minimal fundamental cycles (i.e., fundamental cycles whose norm is close to $\|N\|$) may be surprisingly difficult. For example, it is known that the simplicial volume of any closed simply connected manifold N vanishes, but there is no recipe, in general, which describes fundamental cycles of N of arbitrarily small norm; in a similar spirit, even if the value of the simplicial volume of the product $\Sigma \times \Sigma'$ of two hyperbolic surfaces has been computed in [Bucher-Karlsson 2008], exhibiting a sequence of fundamental cycles whose norm approximates $\|\Sigma \times \Sigma'\|$ seems very challenging [Marasco 2023].

The situation is better understood for hyperbolic manifolds: the computation by Gromov and Thurston of the simplicial volume of such manifolds explicitly constructs almost minimal cycles via an averaging operator called *smearing* [Thurston 1979]. A natural question is to which extent this construction is unique, that is, whether there exist sequences of almost minimal fundamental cycles which do not come from smearing: this problem has been partially addressed by Jungreis [1997] and Kuessner [2003].

We improve their results by showing that, in dimension $n \geq 3$, the unique hyperbolic manifolds admitting "exotic" almost minimal fundamental cycles are those which are commensurable with the Gieseking manifold (it is known that hyperbolic surfaces admit many almost minimal efficient cycles which do not come from smearing; see, e.g., [Jungreis 1997, Remark at page 647]).

In order to state more precisely our results, let us introduce some notation. Let N be a complete finite-volume hyperbolic n-manifold. If N is closed, we denote by $\|N\|$ its simplicial volume. If N is noncompact, it is the internal part of a compact manifold with boundary $\overline{N}$, and for the sake of simplicity we still denote by $\|N\|$ the simplicial volume of $(\overline{N}, \partial\overline{N})$. In fact, by replacing finite chains with locally finite ones, the definition of simplicial volume may be extended to open manifolds, and for finite-volume hyperbolic manifolds this notion of simplicial volume coincides with the simplicial volume of the compactification (see, for example, [Kim and Kuessner 2015]). In order to better compare our results with Kuessner's we prefer to work with the relative simplicial volume of compact manifolds with boundary rather than the simplicial volume of open manifolds, even if our proofs can be easily adapted to the latter framework.

Let c_i, $i \in \mathbb{N}$, be a sequence of (relative) fundamental cycles such that

$$\lim_{i \to +\infty} \|c_i\| = \|N\|.$$

Any possible limit μ of such a sequence naturally sits in the space $\mathcal{M}(\overline{S}_n^*(N))$ of signed measures on the space of (nondegenerate and possibly ideal) geodesic simplices in N, and will be called an *efficient cycle* for N: thus, an efficient cycle is a measure rather than a classical chain (we refer the reader to Section 1 for more details). In fact, it is not difficult to prove that an efficient cycle μ is supported on the subspace $\mathrm{Reg}(N)$ of *regular ideal* simplices, which may be identified with $\Gamma\backslash\mathrm{Isom}(\mathbb{H}^n)$, where Γ is the subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ such that $N = \Gamma\backslash\mathbb{H}^n$ (see Lemma 2.3). The Haar measure on $\mathrm{Isom}(\mathbb{H}^n)$ may then be exploited to define a uniformly distributed measure μ_{eq} on $\mathrm{Reg}(N)$, and Gromov and Thurston's smearing procedure constructs sequences of fundamental cycles converging exactly to a suitable multiple of μ_{eq}.

Jungreis and Kuessner provided a complete characterization of efficient cycles for all finite-volume hyperbolic n-manifolds, $n \geq 3$, except for the so-called *Gieseking-like* manifolds.

Definition 1. Let $N = \Gamma\backslash\mathbb{H}^3$ be a cusped hyperbolic 3-manifold. Let us fix an identification between $\partial\mathbb{H}^3$ and the space $\mathbb{C}\cup\{\infty\}$, and let $\mathcal{P} = \mathbb{Q}(e^{i\pi/3})\cup\{\infty\} \subseteq \partial\mathbb{H}^3$. Then N is *Gieseking-like* if there exists a conjugate Γ' of Γ in $\mathrm{Isom}(\mathbb{H}^3)$ such that $\mathcal{P}$ is contained in the set of parabolic fixed points of Γ'.

The well-known Gieseking manifold is indeed Gieseking-like. Being Gieseking-like is invariant with respect to commensurability; hence all hyperbolic 3-manifolds which are commensurable with the Gieseking manifold (like, for example, the figure-8 knot complement) are Gieseking-like. It is still unknown whether the class of Gieseking-like manifolds coincides with the commensurability class of the Gieseking manifold, or it is strictly larger (see [Long and Reid 2002]).

Let v_n be the volume of a regular ideal simplex in hyperbolic space $\mathbb{H}^n$. We are now ready to state Jungreis' and Kuessner's results:

Theorem 2 [Jungreis 1997]. *Let N be a closed orientable n-hyperbolic manifold with $n \geq 3$. Then N admits a unique efficient cycle, which is given by the measure*

$$\frac{1}{2v_n} \cdot \mu_{\mathrm{eq}}.$$

Theorem 3 [Kuessner 2003]. *Let N be a complete finite-volume n-hyperbolic manifold, $n \geq 3$, and suppose that N is not Gieseking-like (this condition is automatically satisfied if $n \geq 4$). Then every efficient cycle of N is a multiple of μ_{eq}.*

In fact, Kuessner [2003, Theorem 4.5] proved that any efficient cycle is a nonvanishing multiple of μ_{eq}, without explicitly computing the proportionality coefficient $1/(2v_n)$ appearing in Jungreis' theorem. When N is noncompact, the space of straight simplices in N is noncompact, which introduces some issues when dealing with the weak-* convergence of measures (namely, by passing to the limit there could be some loss of mass).

Let us say that a measure in $\mathcal{M}(\mathrm{Reg}(N))$ is *equidistributed* if it is a multiple of μ_{eq}. Our main results strengthen and clarify Kuessner's result in three directions:

(1) We prove that the total variation of any efficient cycle of a cusped hyperbolic manifold is equal to its simplicial volume, thus showing that, in the non-Gieseking like case, also for cusped manifolds the proportionality coefficient between any efficient cycle and μ_{eq} is equal to $1/(2v_n)$, as in Jungreis' theorem.

(2) We show that if a cusped 3-manifold N admits nonequidistributed efficient cycles, then it is commensurable with the Gieseking manifold (a condition which is potentially stronger than being Gieseking-like).

(3) For any such manifold we exhibit nonequidistributed efficient cycles, thus obtaining a complete characterization of hyperbolic manifolds with nonunique efficient cycles.

Let us state more precisely our results:

Theorem 4 (no loss of mass). *Let N be a complete finite-volume hyperbolic n-manifold, $n \geq 3$, and let μ be an efficient cycle for N. Then $\|\mu\| = \|N\|$.*

Theorem 5. *Let N be a complete finite-volume hyperbolic manifold. Then N admits nonequidistributed efficient cycles if and only if it is commensurable with the Gieseking manifold.*

Putting together Theorems 4 and 5 we can then deduce the following:

Theorem 6. *Let N be a complete finite-volume hyperbolic n-manifold with $n \geq 3$.*

(1) *If N is not commensurable with the Gieseking manifold and c_i, $i \in \mathbb{N}$, is any minimizing sequence for N, then*

$$\lim_{i\to+\infty} c_i = \frac{1}{2v_n}\mu_{\mathrm{eq}}.$$

(2) *If N is commensurable with the Gieseking manifold, then N admits nonequidistributed efficient cycles.*

We can be more precise. If N is commensurable with the Gieseking manifold, then a finite cover M of N admits a decomposition T into regular ideal tetrahedra. The triangulation T induces a measure cycle $\mu_T \in \mathcal{M}(\mathrm{Reg}(M))$ which is a finite sum of atomic measures supported on the regular ideal tetrahedra appearing in T (see Section 4.4 for the precise definition of μ_T). We then have the following:

Theorem 7. *Let M be a complete finite-volume 3-manifold admitting a decomposition T into regular ideal tetrahedra. Then μ_T is an efficient cycle for M.*

From the nonequidistributed efficient cycle μ_T for M one can then easily construct a nonequidistributed efficient cycle for N. The proof of Theorem 7 exploits a construction described in [Francaviglia et al. 2012, Section 5.4], which allows one to replace an ideal triangulation T of a cusped manifold with a classical triangulation of its compactification in a very controlled way. By applying this procedure to a suitably chosen tower of coverings of M and pushing-forward the resulting classical triangulations to M we obtain a minimizing sequence whose limit is equal to μ_T.

Plan of the paper. In Section 1 we recall the definition of simplicial volume, of minimizing sequence, of efficient cycle and of equidistributed efficient cycle. To this aim we also introduce the measure spaces we will exploit throughout the paper. Section 2 is devoted to the proof of some fundamental properties of efficient cycles, including Theorem 4. In Section 3 we prove that, if a complete finite-volume hyperbolic n-manifold, $n \geq 3$, admits a nonequidistributed efficient cycle, then it is necessarily commensurable with the Gieseking manifold, while Section 4 is devoted to the construction of nonequidistributed efficient cycles for manifolds which are commensurable with the Gieseking manifold.

1. Preliminaries

1.1. ***Simplicial volume.*** Let X be a topological space. For every $k \in \mathbb{N}$, we denote by $\widehat{S}_k(X)$ the set of singular k-simplices with values in X, and by $C_k(X)$ the chain module of singular k-chains with *real* coefficients, i.e., the real vector space with free basis $\widehat{S}_k(X)$. If $Y \subseteq X$, we denote by $C_*(X, Y)$ the chain complex of relative singular cochains with real coefficients, and by $H_*(X, Y)$ the corresponding homology module. We endow $C_*(X)$ with the ℓ^1-norm $\|\cdot\|$ defined by

$$\left\| \sum_{i=1} a_i \sigma_i \right\| = \sum_{i=1}^{k} |a_i|.$$

This norm descends to a norm on $C_*(X, Y)$ and, by taking the infimum over representatives, to a seminorm on $H_*(X, Y)$, still denoted by $\|\cdot\|$.

If N is a compact oriented n-dimensional manifold (possibly with boundary), then the singular homology module with integral coefficients $H_n(N, \partial N; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the *integral fundamental class* $[N]_{\mathbb{Z}} \in H_n(N, \partial N, \mathbb{Z})$. Under the change of coefficient homomorphism induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, the class $[N]_{\mathbb{Z}}$ is sent to the *real fundamental class* $[N] \in H_n(N, \partial N; \mathbb{R})$.

Definition 1.1 [Gromov 1982]. The simplicial volume of N is

$$\|N\| = \|[N]\|.$$

Henceforth, all the homology modules will be understood with real coefficients, and the coefficients will be omitted from our notation.

1.2. ***Straight chains on hyperbolic manifolds.*** Let $N = \Gamma\backslash\mathbb{H}^n$ be a cusped oriented hyperbolic n-manifold, where Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. For every $k \in \mathbb{N}$, if $\sigma : \Delta^k \to \mathbb{H}^n$ is a singular simplex, we denote by $\widetilde{\mathrm{str}}_k(\sigma)$ the *straightening* of σ, that is, the singular simplex obtained by suitably parametrizing the convex hull of the vertices of σ (see, for instance, [Frigerio 2017, Section 8.7] or [Martelli 2022, Chapter III.13]). We denote by $S_k(\mathbb{H}^n) \subseteq \widehat{S}_k(\mathbb{H}^n)$ the image of $\widetilde{\mathrm{str}}_k$, i.e., the subset of *straight* hyperbolic k-simplices, and observe that there is a natural identification

$$S_k(\mathbb{H}^n) = (\mathbb{H}^n)^{k+1}$$

sending a straight simplex to the (ordered) set of its vertices, which will be understood henceforth. With a slight abuse, we still denote by $\widetilde{\mathrm{str}}_k : C_k(\mathbb{H}^n) \to C_k(\mathbb{H}^n)$ the $\mathbb{R}$-linear extension of $\widetilde{\mathrm{str}}_k$ to the space of singular chains, and recall that $\widetilde{\mathrm{str}}_* : C_*(\mathbb{H}^n) \to C_*(\mathbb{H}^n)$ is in fact a chain map (see, for instance, [Frigerio 2017, Proposition 8.11]).

Being $\mathrm{Isom}(\mathbb{H}^n)$-equivariant (and equivariantly homotopic to the identity), the map $\widetilde{\mathrm{str}}_k$ is in particular Γ-equivariant, and induces a well-defined chain map $\mathrm{str}_* : C_*(N) \to C_*(N)$, which is chain homotopic to the identity. We denote by $S_k(N)$ the image of $\widehat{S}_k(N)$ via str_k, i.e., the set of straight simplices in N, and we observe that there is a natural identification

$$S_k(N) = \Gamma\backslash S_k(\mathbb{H}^n) = \Gamma\backslash(\mathbb{H}^n)^{k+1}.$$

A chain is called *straight* if it is supported on straight simplices or, equivalently, if it lies in the image of the chain map str_* (or $\widetilde{\mathrm{str}}_*$).

We denote by $\mathrm{SC}_*(\mathbb{H}^n) \subseteq C_*(\mathbb{H}^n)$ (resp. $\mathrm{SC}_*(N) \subseteq C_*(N)$) the complex of straight chains in $\mathbb{H}^n$ (resp. in N). By construction, under the above identification between the set of straight k-simplices and $(\mathbb{H}^n)^{k+1}$ (resp. $\Gamma\backslash(\mathbb{H}^n)^{k+1}$), the complex $\mathrm{SC}_*(\mathbb{H}^n)$ (resp. $\mathrm{SC}_*(N)$) is identified with the free vector space with basis $(\mathbb{H}^n)^{*+1}$ (resp. $\Gamma\backslash(\mathbb{H}^n)^{*+1}$), with boundary operators which linearly extend the maps

$$\partial_k(v_0, \dots, v_k) = \sum_{i=0}^{k}(-1)^i(v_0, \dots, \widehat{v_i}, \dots, v_k)$$

$\big($resp. $\partial_k[(v_0, \dots, v_k)] = \sum_{i=0}^{k}(-1)^i[(v_0, \dots, \widehat{v_i}, \dots, v_k)]\big)$.

If $\sigma = (v_0, \dots, v_k) \in (\mathbb{H}^n)^{k+1}$ is a straight simplex, the alternating chain associated to σ is defined by

$$\mathrm{alt}_k(\sigma) = \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_{k+1}} \varepsilon(\tau) \cdot (v_{\tau(0)}, \dots, v_{\tau(k)}),$$

where $\mathfrak{S}_{k+1}$ is the group of the permutations of the set $\{0, \dots, k\}$, and $\varepsilon(\tau) = \pm 1$ is the sign of τ for every $\tau \in \mathfrak{S}_{k+1}$. The maps alt_k linearly extend to a chain map $\mathrm{alt}_* : \mathrm{SC}_*(\mathbb{H}^n) \to \mathrm{SC}_*(\mathbb{H}^n)$ which is Γ-equivariant, and Γ-equivariantly homotopic to the identity (see, for example, [Fujiwara and Manning 2011, Appendix A]). In particular, alt_* induces a well-defined chain map $\mathrm{alt}_* : \mathrm{SC}_*(N) \to \mathrm{SC}_*(N)$, which is homotopic to the identity. A chain in $\mathrm{SC}_*(\mathbb{H}^n)$ (or in $\mathrm{SC}_*(N)$) is *alternating* if it lies in the image of alt_*.

1.3. ***Thick-thin decomposition of hyperbolic manifolds.*** For every $\varepsilon > 0$ we denote by N_ε the ε-thick part of N, that is, the set of points of N whose injectivity radius is not smaller than ε. We will always choose $\varepsilon > 0$ small enough so that N_ε is a compact submanifold with boundary of N, obtained from N by removing open neighborhoods of its cusps. We denote by $\overline{N}$ the natural compactification of N, which is diffeomorphic to N_ε. The inclusion $(N_\varepsilon, \partial N_\varepsilon) \to (N, N \setminus \mathrm{int}(N_\varepsilon))$ and the obvious deformation retraction $r : (N, N \setminus \mathrm{int}(N_\varepsilon)) \to (N_\varepsilon, \partial N_\varepsilon)$ are the homotopy inverses of each other, and they induce norm nonincreasing maps in homology. Therefore, in order to compute the simplicial volume of $\overline{N}$ we may consider relative fundamental cycles in $C_n(N, N \setminus \mathrm{int}(N_\varepsilon))$. The complement in $\mathbb{H}^n$ of the preimage of N_ε under the covering projection is an equivariant family of disjoint open horoballs. Since horoballs are convex in $\mathbb{H}^n$, the straightening operator induces a well-defined chain map on the relative chain complex $C_*(N, N \setminus \mathrm{int}(N_\varepsilon))$. Finally, since both the straightening and the alternating operators are obviously norm nonincreasing (and they induce the identity also on relative homology), in order to compute the simplicial volume of N it is not restrictive to consider only straight and alternating relative cycles in $C_*(N, N \setminus \mathrm{int}(N_\varepsilon))$.

1.4. ***Minimizing sequences and efficient cycles.*** We say that a sequence $c_i \in C_n(N)$ of chains is a *minimizing sequence* if the following conditions hold:

(1) Each c_i is straight and alternating.

(2) For all sufficiently large $i \in \mathbb{N}$, c_i is a relative cycle in $C_n(N_{2^{-i}}, N \setminus \mathrm{int}(N_{2^{-i}}))$.

(3) Under the identification $H_n(N_{2^{-i}}, N \setminus \mathrm{int}(N_{2^{-i}})) \cong H_n(\overline{N}, \partial \overline{N})$ described above, the relative cycle c_i represents the fundamental class of $(\overline{N}, \partial \overline{N})$.

(4) $\|c_i\| \leq \|N\| + 2^{-i}$ for all sufficiently large $i \in \mathbb{N}$.

Of course, in the definition of minimizing sequence the values 2^{-i} may be replaced by any infinitesimal sequence η_i; we decided to choose this specific sequence just to simplify the notation.

We now introduce the measure spaces we are interested in. Recall that $S_n(N) = \Gamma\backslash(\mathbb{H}^n)^{n+1}$ is the space of straight simplices with values in N. Of course, this space does not contain any ideal simplex; hence we need to enlarge it in order to construct a measure space which could support possible limits of minimizing sequences (recall from the introduction that efficient cycles are supported on regular ideal simplices). The natural space to look at is then $\overline{S}_n(N) = \Gamma\backslash(\overline{\mathbb{H}^n})^{n+1}$ but, unfortunately, the action of Γ on $\overline{S}_n(\mathbb{H}^n) = (\overline{\mathbb{H}^n})^{n+1}$ has not closed orbits, so that the quotient space is not Hausdorff. In order to avoid this inconvenience, and for other later purposes, we introduce the following:

Definition 1.2. A straight simplex in $\overline{S}_n(\mathbb{H}^n)$ is *degenerate* if its vertices (hence, its image) lie on (the closure at infinity of) a hyperplane of $\mathbb{H}^n$ or, equivalently, if its image has volume equal to 0. A straight simplex in $\overline{S}_n(N) = \Gamma\backslash\overline{S}_n(\mathbb{H}^n)$ is degenerate if it is the image of a degenerate simplex in $\overline{S}_n(\mathbb{H}^n)$.

We denote by $\overline{S}_n^*(\mathbb{H}^n)$ (resp. $\overline{S}_n^*(N)$) the set of nondegenerate straight simplices in $\overline{S}_n(\mathbb{H}^n)$ (resp. $\overline{S}_n(N)$).

It is not difficult to show that, when endowed with the quotient topology, the space $\overline{S}_n^*(N)$ is metrizable and locally compact (see, e.g., [Kuessner 2003, Lemma 2.6] for a similar result and the proof of Lemma 2.7 here below). We denote by $\mathcal{M}(\overline{S}_n^*(N))$ the space of signed regular measures on $\overline{S}_n^*(N)$. If $\sigma : \Delta_n \to N$ is a straight simplex, then we denote by $\delta_\sigma \in \mathcal{M}(\overline{S}_n^*(N))$ the atomic measure concentrated on σ. The map $\sigma \mapsto \delta_\sigma$ linearly extends to a map

$$\Theta : \mathrm{SC}_n(N) \to \mathcal{M}(\overline{S}_n^*(N)). \tag{1}$$

We are now ready to define the notion of efficient cycle for complete finite-volume hyperbolic manifolds:

Definition 1.3. A measure $\mu \in \mathcal{M}(\overline{S}_n^*(N))$ is an *efficient cycle* for N if there exists a minimizing sequence c_i, $i \in \mathbb{N}$, such that

$$\mu = \lim_{i\to+\infty} \Theta(c_i),$$

where the limit is taken with respect to the weak-* topology on $\mathcal{M}(\overline{S}_n^*(N))$.

1.5. ***Equidistributed measure cycles.*** As explained in the introduction, we are going to prove that, if N is *not* commensurable with the Gieseking manifold, then there exists a unique efficient cycle for N, which is concentrated on (classes of) regular ideal straight simplices, and is equidistributed on such simplices. Let us formally describe what we mean by equidistributed measure on (classes of) regular ideal straight simplices.

We define

$$\mathrm{Reg}(\mathbb{H}^n) = \{(v_0, \dots, v_n) \in (\partial\mathbb{H}^n)^{n+1} \mid v_0, \dots, v_n \text{ span a regular ideal straight simplex}\}$$

and we denote by $\mathrm{Reg}^+(\mathbb{H}^n)$ (resp. $\mathrm{Reg}^-(\partial\mathbb{H}^n)$) the subset of $\mathrm{Reg}(\mathbb{H}^n)$ corresponding to positively oriented (resp. negatively oriented) simplices. We then set

$$\mathrm{Reg}^\pm(N) = \Gamma\backslash\mathrm{Reg}^\pm(\mathbb{H}^n) \subseteq \overline{S}_n^*(N).$$

Since N is oriented, elements of Γ are orientation-preserving; hence the sets $\mathrm{Reg}^+(N)$ and $\mathrm{Reg}^-(N)$ are disjoint.

Let $\Delta_0 = (v_0, \dots, v_n) \in (\partial\mathbb{H}^n)^{n+1}$ be the (ordered) $(n+1)$-tuple of vertices of a fixed positively oriented regular ideal hyperbolic simplex. We then have bijections

$$\mathrm{Isom}^\pm(\mathbb{H}^n) \to \mathrm{Reg}^\pm(\mathbb{H}^n), \quad g \mapsto g \cdot \Delta_0 = (g(v_0), \dots, g(v_n)),$$

which induce bijections

$$\Gamma\backslash\mathrm{Isom}^\pm(\mathbb{H}^n) \to \mathrm{Reg}^\pm(N).$$

We denote by the symbol Haar the Haar measure on $\mathrm{Isom}(\mathbb{H}^n)$, normalized in such a way that, for every measurable subset $\Omega \subseteq \mathbb{H}^n$ and any $x_0 \in \mathbb{H}^n$,

$$\mathrm{Haar}\{g \in \mathrm{Isom}(\mathbb{H}^n) \mid g(x_0) \in \Omega\} = \mathrm{Vol}(\Omega).$$

Being bi-invariant, the Haar measure induces well-defined finite measures $\mathrm{Haar}_\pm$ on $\Gamma/\mathrm{Isom}^\pm(\mathbb{H}^n)$, hence on $\mathrm{Reg}^\pm(N)$ via the above identifications. We finally set

$$\mu_{\mathrm{eq}} = \mathrm{Haar}_+ - \mathrm{Haar}_- \in \mathcal{M}(\mathrm{Reg}(N)) \subseteq \mathcal{M}(\overline{S}_n^*(N)),$$

where the subscript "eq" stands for "equidistributed". Using again the bi-invariance of Haar one can easily check that the definition of $\mathrm{Haar}_\pm$ (hence of μ_{eq}) on $\mathrm{Reg}(\mathbb{H}^n)$ does not depend on the choice of Δ_0.

2. Some properties of efficient cycles

For every $\varepsilon > 0$ we denote by $\omega_\varepsilon : C_n(N, N \setminus \mathrm{int}(N_\varepsilon)) \to \mathbb{R}$ the restriction of the volume cochain to N_ε, i.e., the cochain such that

$$\omega_\varepsilon(c) = \int_c d\,\mathrm{Vol}_\varepsilon,$$

where $d\,\mathrm{Vol}_\varepsilon$ is the (discontinuous) n-form that coincides with the hyperbolic volume form on N_ε and is equal to 0 on $N \setminus N_\varepsilon$ (for our purposes, it is sufficient to define ω_ε on straight chains, which of course are C^1, so the integral above makes sense). If $c \in C_n(N)$ is a straight relative fundamental cycle for $(N, N \setminus \mathrm{int}(N_\varepsilon))$, then we have

$$\omega_\varepsilon(c) = \mathrm{Vol}(N_\varepsilon).$$

If σ is a straight simplex with values in N, then it is immediate to check that $\omega_\varepsilon(\sigma) = \pm\,\mathrm{Vol}(\widetilde{\sigma} \cap \widetilde{N}_\varepsilon)$, where $\widetilde{\sigma}$ is a lift of σ to $\mathbb{H}^n$, the space $\widetilde{N}_\varepsilon$ is the preimage of N_ε in $\mathbb{H}^n$, and the sign is positive (resp. negative) if σ is positively oriented (resp. negatively oriented).

The following lemma shows that, in a minimizing sequence, the orientation of simplices has to be coherent with the sign of their coefficients, at least asymptotically.

Lemma 2.1. *Let c_i, $i \in \mathbb{N}$, be a minimizing sequence, and for every $i \in \mathbb{N}$ let*

$$c_i = \sum_{k=1}^{n_i} a_{i,k}\sigma_{i,k}$$

be the reduced form of c_i (that is, $\sigma_{i,k} \neq \sigma_{i,k'}$ whenever $k \neq k'$, where the $\sigma_{i,k}$ are straight singular simplices and the $a_{i,k}$ are real coefficients). For every i, k, set $b_{i,k} = a_{i,k}$ if $a_{i,k} > 0$ and $\sigma_{i,k}$ is not positively oriented or $a_{i,k} < 0$ and $\sigma_{i,k}$ is not negatively oriented, and $b_{i,k} = 0$ otherwise. If $c'_i = \sum_{k=1}^{n_i} b_{i,k}\sigma_{i,k}$, then

$$\lim_{i\to+\infty} \|c'_i\| = 0.$$

Proof. By definition of minimizing sequence we have

$$\lim_{i\to+\infty} \omega_{2^{-i}}(c_i) = \lim_{i\to+\infty} \mathrm{Vol}(N_{2^{-i}}) = \mathrm{Vol}(N);$$

hence

$$\lim_{i\to+\infty} \frac{\omega_{2^{-i}}(c_i)}{v_n} = \frac{\mathrm{Vol}(N)}{v_n} = \|N\| = \lim_{i\to+\infty} \|c_i\|. \tag{2}$$

Since the hyperbolic volume of any straight simplex is at most v_n, we have

$$\omega_{2^{-i}}(c_i - c'_i) \leq \|c_i - c'_i\| \cdot v_n,$$

while our definition of c'_i readily implies that $\omega_{2^{-i}}(c'_i) \leq 0$. Therefore,

$$\frac{\omega_{2^{-i}}(c_i)}{v_n} = \frac{\omega_{2^{-i}}(c_i - c'_i) + \omega_{2^{-i}}(c'_i)}{v_n} \leq \|c_i - c'_i\| = \|c_i\| - \|c'_i\|,$$

where the last equality follows from the fact that, by construction, the set of simplices appearing in c'_i is disjoint from the set of simplices appearing in $c_i - c'_i$, so that

$\|c_i\| = \|(c_i - c_i') + c_i'\| = \|c_i - c_i'\| + \|c_i'\|$. The conclusion follows from this inequality and (2). □

A very similar argument shows that the volume of "most" simplices appearing in minimizing sequences converges to v_n. We properly state and prove this result, since we will need it later.

Lemma 2.2. *Let c_i, $i \in \mathbb{N}$, be a minimizing sequence, and for every $i \in \mathbb{N}$ let*

$$c_i = \sum_{k=1}^{n_i} a_{i,k}\sigma_{i,k}$$

be the reduced form of c_i, as in the previous lemma. Let $\varepsilon > 0$ be fixed, and, for every i, k, set $b_{i,k} = a_{i,k}$ if the hyperbolic volume of a lift of $\sigma_{i,k}$ to $\mathbb{H}^n$ is smaller that $v_n - \varepsilon$, and $b_{i,k} = 0$ otherwise. If $c_i' = \sum_{k=1}^{n_i} b_{i,k}\sigma_{i,k}$, then

$$\lim_{i\to+\infty} \|c_i'\| = 0.$$

Proof. Just as in the proof of the previous lemma we have $\|c_i - c_i'\| = \|c_i\| - \|c_i'\|$. Using this fact, since the hyperbolic volume of any straight simplex is at most v_n, our definition of c_i' implies that

$$|\omega_{2^{-i}}(c_i)| \le |\omega_{2^{-i}}(c_i - c_i')| + |\omega_{2^{-i}}(c_i')| \le \|c_i - c_i'\| \cdot v_n + \|c_i'\|(v_n - \varepsilon) = \|c_i\| v_n - \|c_i'\|\varepsilon,$$

whence

$$\|N\| = \lim_{i\to+\infty} \frac{|\omega_{2^{-i}}(c_i)|}{v_n} \le \lim_{i\to+\infty} \|c_i\| - \frac{\varepsilon}{v_n} \limsup_{i\to+\infty} \|c_i'\| = \|N\| - \frac{\varepsilon}{v_n} \limsup_{i\to+\infty} \|c_i'\|.$$

The conclusion follows. □

The previous lemma may be exploited to prove that efficient cycles are supported on regular ideal straight simplices:

Lemma 2.3 [Kuessner 2003, Lemma 3.5]. *Let μ be an efficient cycle for N. Then μ is supported on* $\mathrm{Reg}(N) \subseteq \overline{S}_n^*(N)$.

Therefore, we will consider μ both as an element of $\mathcal{M}(\overline{S}_n^*(N))$ and as an element of $\mathcal{M}(\mathrm{Reg}(N))$.

We are now going to prove that the total variation of an efficient cycle is equal to the simplicial volume of N (recall that the total variation is only lower semicontinuous with respect to weak-* convergence; hence the total variation of an efficient cycle could be strictly smaller than $\|N\|$ a priori).

To this aim we need the definition of incenter and inradius of a straight hyperbolic simplex. Consider a nondegenerate straight n-simplex $\Delta \in \overline{S}^*(\mathbb{H}^n)$ (recall that a straight simplex is nondegenerate if its image is not contained in a hyperplane). For every point $p \in \Delta \cap \mathbb{H}^n$ we denote by $r_\Delta(p)$ the radius of the maximal hyperbolic ball centered in p and contained in Δ. Since the volume of any straight n-simplex is smaller than v_n and the volume of hyperbolic balls diverges as the radius diverges, there exists a constant $r_n > 0$ such that $r_\Delta(p) \leqslant r_n$ for every $\Delta \in \overline{S}_n^*(\mathbb{H}^n)$ and $p \in \Delta$.

Definition 2.4. Take a nondegenerate straight simplex $\Delta \in \overline{S}_n^*(\mathbb{H}^n)$. The *inradius* $r(\Delta)$ of Δ is

$$r(\Delta) = \sup_{p \in \Delta \cap \mathbb{H}^n} r_\Delta(p) \in (0, r_n]$$

(observe that $r(\Delta) > 0$ since Δ is nondegenerate). The *incenter* $\mathrm{inc}(\Delta)$ is the unique point $p \in \Delta \cap \mathbb{H}^n$ such that $r_\Delta(p) = r(\Delta)$.

It is shown in [Francaviglia et al. 2012, Lemma 3.12] that the incenter is well-defined, and that the functions

$$\mathrm{inc} : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{H}^n, \quad r : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{R}$$

are continuous.

If Δ is a straight simplex in $\overline{S}_n^*(N)$, we define its inradius $r(\Delta)$ as the inradius of any lift of Δ to $\mathbb{H}^n$, and its incenter $\mathrm{inc}(\Delta)$ as the projection in N of the incenter of any lift of Δ to $\mathbb{H}^n$ (the fact that these notions are well-defined is easily checked).

Lemma 2.5. *Let $\bar{\delta}$ be the inradius of the n-dimensional regular ideal straight simplex, and let $\Delta_i \in \overline{S}_n^*(\mathbb{H}^n)$, $i \in \mathbb{N}$, be a sequence such that $\lim_{i \to +\infty} \mathrm{Vol}(\Delta_i) = v_n$. Then $\lim_{i \to +\infty} r(\Delta_i) = \bar{\delta}$.*

Proof. By [Francaviglia et al. 2012, Proposition 3.14], for every $i \in \mathbb{N}$ there exists an element $g_i \in \mathrm{Isom}(\mathbb{H}^n)$ such that $\lim_{i \to +\infty} g_i(\Delta_i) = \overline{\Delta}$, where $\overline{\Delta}$ is a regular ideal straight simplex. Since the map $r : \overline{S}_n^*(\mathbb{H}^n) \to \mathbb{R}$ is continuous, we thus get

$$\lim_{i \to +\infty} r(\Delta_i) = \lim_{i \to +\infty} r(g_i(\Delta_i)) = r(\overline{\Delta}) = \bar{\delta}. \qquad \square$$

Lemma 2.6. *Let $K \subseteq N$ be compact, and let $\delta_0 > 0$. Then the set*

$$\Lambda = \{\Delta \in \overline{S}_n^*(N) \mid \mathrm{inc}(\Delta) \in K,\ r(\Delta) \geq \delta_0\}$$

is compact.

Proof. Let $\widetilde{K} \subseteq \mathbb{H}^n$ be a compact subset such that $\pi(\widetilde{K}) = K$ (for example, if $\pi : \mathbb{H}^n \to N$ is the universal covering, then $\widetilde{K}$ may be chosen as the intersection between $\pi^{-1}(K)$ and a Dirichlet domain for the action of Γ on $\mathbb{H}^3$), and let

$$\widetilde{\Lambda} = \{\widetilde{\Delta} \in \overline{S}_n^*(\mathbb{H}^n) \mid \mathrm{inc}(\widetilde{\Delta}) \in \widetilde{K},\ r(\widetilde{\Delta}) \geq \delta_0\}.$$

Under the projection $\overline{S}^*_n(\mathbb{H}^n) \to \overline{S}^*_n(N)$, the set $\widetilde{\Lambda}$ is sent to Λ; hence in order to conclude it suffices to show that $\widetilde{\Lambda}$ is compact or, equivalently, sequentially compact (being a subset of $(\overline{\mathbb{H}^n})^{n+1}$, the space $\widetilde{\Lambda}$ is metrizable).

Let $\widetilde{\Delta}_i = (v^i_0, v^i_1, \dots, v^i_n) \in (\overline{\mathbb{H}}^n)^{n+1}$, $i \in \mathbb{N}$, be a sequence of elements in $\widetilde{\Lambda}$. Since $(\overline{\mathbb{H}}^n)^{n+1}$ is compact, up to passing to a subsequence we may assume that $\widetilde{\Delta}_i$ tends to $\widetilde{\Delta}_\infty \in (\overline{\mathbb{H}}^n)^{n+1}$. Since the maps $r : \overline{S}^*_n(\mathbb{H}^n) \to \mathbb{R}$ and $\mathrm{inc} : \overline{S}^*_n(\mathbb{H}^n) \to \mathbb{H}^n$ are continuous and $\widetilde{K}$ is closed, we have $r(\widetilde{\Delta}_\infty) \geq \delta_0$ and $\mathrm{inc}(\widetilde{\Delta}_\infty) \in \widetilde{K}$. Thus in order to conclude it is sufficient to observe that $\widetilde{\Delta}_\infty$ is nondegenerate, since it contains the hyperbolic ball of radius $r(\widetilde{\Delta}_\infty) > 0$ centered at $\mathrm{inc}(\widetilde{\Delta}_\infty)$; hence it cannot be contained in a hyperplane. □

Lemma 2.7. *Let $n = \dim N \geq 3$, and let Λ be a compact subset of $\overline{S}^*_n(N)$. Then, there exists a compactly supported continuous function $g : \overline{S}^*_n(N) \to [-1, 1]$ such that $g(\Delta) = 1$ for every positively oriented simplex in Λ and $g(\Delta) = -1$ for every negatively oriented simplex in Λ.*

Proof. Let us first prove that $\overline{S}^*_n(N)$ is metrizable. Let $SS_n(\mathbb{H}^n)$ be the space of (possibly ideal) straight simplices with pairwise distinct vertices, i.e.,

$$SS_n(\mathbb{H}^n) = \{(v_0, \dots, v_n) \in (\overline{\mathbb{H}}^n)^{n+1} \mid v_i \neq v_j \text{ for } i \neq j\}.$$

It is proved in [Kuessner 2003, Lemma 2.6] that the action of Γ on $SS_n(\mathbb{H}^n)$ is free and properly discontinuous, and that the quotient space $SS_n(N) = \Gamma \backslash SS_n(\mathbb{H}^n)$ is metrizable. But $\overline{S}^*_n(N)$ is clearly a subspace of $SS_n(N)$, and its topology as a quotient of $\overline{S}^*_n(\overline{\mathbb{H}}^n)$ coincides with the topology it inherits as a subspace of $SS_n(N)$. Therefore, $\overline{S}^*_n(N)$ is metrizable. Indeed, since the action of Γ on $\overline{S}^*_n(\mathbb{H}^n)$ is free and properly discontinuous, and the space $\overline{S}^*_n(\mathbb{H}^n)$ is a topological manifold with boundary (being an open subset of the topological manifold $(\overline{\mathbb{H}}^n)^{n+1}$)), also the space $\overline{S}^*_n(N)$ is a topological manifold. In particular, it is locally compact.

Let now $h : \overline{S}^*_n(N) \to [-1, 1]$ be such that $h(\Delta) = 1$ if Δ is positively oriented, and $h(\Delta) = -1$ if Δ is negatively oriented. Since the subspace of positively oriented (resp. negatively oriented) simplices in $\overline{S}^*_n(N)$ is clopen in $\overline{S}^*_n(N)$, the map h is continuous. Since $\overline{S}^*_n(N)$ is locally compact, we may choose a relatively compact open neighborhood U of Λ. By the Urysohn lemma, there exists a continuous function $\psi : \overline{S}^*_n(N) \to [0, 1]$ such that $\psi(\Delta) = 1$ for every $\Delta \in \Lambda$ and $\psi(\Delta) = 0$ for every $\Delta \notin U$. By construction, the function $g = f \cdot \psi$ is continuous and compactly supported, takes values in $[-1, 1]$ and is such that $g(\Delta) = 1$ for every positively oriented simplex in Λ and $g(\Delta) = -1$ for every negatively oriented simplex in Λ. □

We are now ready to prove Theorem 4 from the introduction, which we recall here for the convenience of the reader:

Theorem 4. *Let N be a complete finite-volume hyperbolic n-manifold, $n \geq 3$, and let μ be an efficient cycle for N. Then $\|\mu\| = \|N\|$.*

Proof. Let r_n be a universal upper bound for the inradius of any nondegenerate n-dimensional straight simplex, as above. For every $\varepsilon > 0$ we set

$$\mathrm{th}_\varepsilon = \{\Delta \in \overline{S}_n^*(N) \mid \mathrm{inc}(\Delta) \in B(N_\varepsilon, r_n)\},$$

where $B(N_\varepsilon, r_n)$ denotes the closed r_n-neighborhood of N_ε in N; in other words, th_ε denotes the set of nondegenerate straight simplices of N whose incenter lies in the closed r_n-neighborhood of the ε-thick part of N.

Let $\bar{\delta}$ be the inradius of the regular ideal straight n-simplex, and fix some constant $0 < \delta_0 < \bar{\delta}$. Also denote by V_0 the hyperbolic volume of a hyperbolic n-ball of radius δ_0, and set $\Omega_{\delta_0} = \{\Delta \in \overline{S}_n^*(N) \mid r(\Delta) \geq \delta_0\}$.

Let now c_i, $i \in \mathbb{N}$, be a minimizing sequence, and let us fix $\varepsilon > 0$. We choose $i_0 \in \mathbb{N}$ such that $\mathrm{Vol}(N \setminus N_{2^{-i_0}}) \leq \varepsilon v_n$. Let $i \geq i_0$, and consider the following partition of the space of nondegenerate straight simplices in N:

$$\Lambda_1 = \overline{S}_n^*(N) \setminus \Omega_{\delta_0}, \qquad \Lambda_2 = \Omega_{\delta_0} \cap \mathrm{th}_{2^{-i_0}}, \qquad \Lambda_3 = \Omega_{\delta_0} \setminus \mathrm{th}_{2^{-i_0}}.$$

We denote by $c_i = c_i^1 + c_i^2 + c_i^3$ the corresponding decomposition of c_i, i.e., we assume that the simplices appearing in c_i^j belong to Λ_j for $j = 1, 2, 3$. By Lemma 2.5, since δ_0 is smaller than the inradius of the regular ideal tetrahedron, the volume of the lifts to $\mathbb{H}^n$ of the simplices in Λ_1 is bounded above by a constant strictly smaller than v_n. By Lemma 2.2, we then have

$$\lim_{i \to +\infty} \|c_i^1\| = 0. \tag{3}$$

Let now $\Delta \in \Lambda_3$, i.e., suppose that $r(\Delta) \geq \delta_0$ and $\mathrm{inc}(\Delta) \notin B(N_{2^{-i_0}}, r_n)$. Since $\delta_0 \leq r_n$, the ball $B(\mathrm{inc}(\Delta), \delta_0) \subseteq \Delta$ does not intersect $N_{2^{-i_0}}$; hence $|\omega_{2^{-i_0}}(\Delta)| \leq v_n - V_0$. Thus

$$|\omega_{2^{-i_0}}(c_i^3)| \leq \|c_i^3\| \cdot (v_n - V_0).$$

Observe now that c_i, being a relative fundamental cycle for $N_{2^{-i}}$, is a fortiori a relative fundamental cycle for $N_{2^{-i_0}}$. Hence

$$\begin{aligned}\mathrm{Vol}(N_{2^{-i_0}}) = |\omega_{2^{-i_0}}(c_i)| &\leq |\omega_{2^{-i_0}}(c_i^1)| + |\omega_{2^{-i_0}}(c_i^2)| + |\omega_{2^{-i_0}}(c_i^3)| \\ &\leq \|c_i^1\| \cdot v_n + \|c_i^2\| \cdot v_n + \|c_i^3\| \cdot (v_n - V_0) = \|c_i\| \cdot v_n - \|c_i^3\| \cdot V_0.\end{aligned}$$

After dividing by v_n we obtain

$$\frac{\mathrm{Vol}(N_{2^{-i_0}})}{v_n} \leq \|c_i\| - \|c_i^3\| \cdot \frac{V_0}{v_n},$$

whence

$$\|N\|-\varepsilon=\frac{\mathrm{Vol}(N)}{v_n}-\varepsilon\leq\frac{\mathrm{Vol}(N_{2^{-i_0}})}{v_n}\leq\|c_i\|-\|c_i^3\|\cdot\frac{V_0}{v_n}\leq\|N\|+2^{-i}-\|c_i^3\|\cdot\frac{V_0}{v_n}$$

and

$$\|c_i^3\|\leq\frac{v_n}{V_0}(\varepsilon+2^{-i}).$$

In particular, we have

$$\limsup_{i\to+\infty}\|c_i^3\|\leq\frac{v_n}{V_0}\varepsilon. \tag{4}$$

Since $\|c_i\|\geq\|N\|$ we also have

$$\|c_i^2\|=\|c_i\|-\|c_i^1\|-\|c_i^3\|\geq\|N\|-\|c_i^1\|-\frac{v_n}{V_0}(\varepsilon+2^{-i});$$

hence (recalling that $\|c_i^1\|\to 0$ as $i\to+\infty$)

$$\liminf_{i\to+\infty}\|c_i^2\|\geq\|N\|-\varepsilon\frac{v_n}{V_0}. \tag{5}$$

Observe that, thanks to Lemma 2.6, the set Λ_2 is compact. Therefore, by Lemma 2.7 one may construct a compactly supported continuous function $g:\overline{S}^*(N)\to[-1,1]$ such that $g(\Delta)=1$ for every positively oriented $\Delta\in\Lambda_2$ and $g(\Delta)=-1$ for every negatively oriented $\Delta\in\Lambda_2$. Just as in Lemma 2.1, decompose c_i^2 (resp. c_i) as $c_i^2=(c_i^2-(c_i^2)')$ (resp. $c_i=(c_i-c_i')+c_i'$), where $(c_i^2)'$ (resp. c_i') is a linear combination of positively oriented simplices with negative coefficients and of negatively oriented simplices with positive coefficients. We know from Lemma 2.1 that $\lim_{i\to+\infty}\|c_i'\|=0$; hence, a fortiori,

$$\lim_{i\to+\infty}\|(c_i^2)'\|=0 \tag{6}$$

(hence, also $\liminf_{i\to+\infty}\|c_i^2-(c_i^2)'\|=\liminf_{i\to+\infty}\|c_i^2\|$).

Since the supports of $\Theta(c_1^i)$ and of $\Theta(c_3^i)$ are disjoint from Λ_2, we have

$$\int_{\Lambda_2}g\,d\Theta(c_i)=\int_{\Lambda_2}g\,d\Theta(c_i^2)=\int_{\Lambda_2}g\,d\Theta(c_i^2-(c_i^2)')+\int_{\Lambda_2}g\,d\Theta((c_i^2)'). \tag{7}$$

By definition, every simplex appearing in the chain c_i-c_i' has a positive coefficient if it is positively oriented, and a negative coefficient otherwise. Therefore, by the very definition of the function g, we have

$$\int_{\Lambda_2}g\,d\Theta(c_i-(c_i^2)')=\|c_i^2-(c_i^2)'\|. \tag{8}$$

Finally, since $\|g\|_\infty \le 1$, we have

(9) $$\left|\int_{\Lambda_2} g\, d\Theta((c_i^2)')\right| \le \|(c_i^2)'\|.$$

Putting together the (in)equalities (6)–(9), we then have

(10) $$\liminf_{i\to+\infty} \int_{\Lambda_2} g\, d\Theta(c_i) = \liminf_{i\to+\infty} \|c_i^2\|.$$

By definition of weak-* convergence, if $\mu = \lim_{i\to+\infty} \Theta(c_i)$, then from (3)–(5) and (10) (and the fact that $\|g\|_\infty \le 1$) we obtain

$$\begin{aligned}\left|\int_{\overline{S}^*(N)} g\, d\mu\right| &= \left|\lim_{i\to+\infty}\int_{\Lambda_1} g\, d\Theta(c_i) + \lim_{i\to+\infty}\int_{\Lambda_2} g\, d\Theta(c_i) + \lim_{i\to+\infty}\int_{\Lambda_3} g\, d\Theta(c_i)\right| \\ &\ge -\left|\lim_{i\to+\infty}\int_{\Lambda_1} g\, d\Theta(c_i)\right| + \left|\lim_{i\to+\infty}\int_{\Lambda_2} g\, d\Theta(c_i)\right| - \left|\lim_{i\to+\infty}\int_{\Lambda_3} g\, d\Theta(c_i)\right| \\ &\ge -\limsup_{i\to+\infty}\|c_i^1\| + \liminf_{i\to+\infty}\|c_i^2\| - \limsup_{i\to+\infty}\|c_i^3\| \\ &\ge \|N\| - \frac{2\varepsilon v_n}{V_0}.\end{aligned}$$

Since $\|g\|_\infty \le 1$, this inequality implies that the total variation of μ is not smaller than $\|N\| - 2\varepsilon v_n/V_0$. Due to the arbitrariness of ε, we may conclude that $\|\mu\| \ge \|N\|$. On the other hand, it is well known that the total variation is lower semicontinuous with respect to the weak-* convergence; hence $\|\mu\| \le \lim_{i\to+\infty} \|\Theta(c_i)\| = \|N\|$. This concludes the proof. □

Our normalization of the Haar measure implies that $\|\mu_{\mathrm{eq}}\| = 2\,\mathrm{Vol}(N)$. Therefore, Theorem 4 readily implies the following:

Corollary 2.8. *Let $k \in \mathbb{R}$ and suppose that the measure $\mu = k\mu_{\mathrm{eq}}$ is an efficient cycle for N. Then $k = 1/(2v_n)$.*

3. Manifolds admitting a unique efficient cycle

Jungreis and Kuessner proved that, if N is a non-Gieseking like hyperbolic manifold, then every efficient cycle of N is equidistributed. We strengthen this result by showing that the same conclusion holds under the supposedly less restrictive requirement that N be noncommensurable with the Gieseking manifold. We may concentrate our attention on the three-dimensional case, the higher-dimensional case being covered by the results proved in [Kuessner 2003]. Therefore, throughout this section we denote by N a complete finite-volume hyperbolic 3-manifold.

Henceforth we fix a regular ideal straight simplex (with ordered vertices) $\Delta_0 \in \mathrm{Reg}(\mathbb{H}^3)$, which we exploit to fix an identification $\mathrm{Reg}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3)$, as explained at the end of Section 1. For $i = 0, \dots, 3$, let $r_i \in \mathrm{Isom}^-(\mathbb{H}^3)$ be the hyperbolic reflection with respect to the plane containing the i-th face of Δ_0. Under the identification $\mathrm{Reg}(\mathbb{H}^3) \cong \mathrm{Isom}(\mathbb{H}^3)$, the *right* multiplication

$$\mathrm{Isom}(\mathbb{H}^3) \to \mathrm{Isom}(\mathbb{H}^3), \quad g \mapsto g \cdot r_i,$$

corresponds to the map $\rho_i : \mathrm{Reg}(\mathbb{H}^3) \to \mathrm{Reg}(\mathbb{H}^3)$ sending any simplex $\Delta \in \mathrm{Reg}(\mathbb{H}^3)$ to the simplex obtained by reflecting Δ with respect to the plane containing its i-th face. We denote by R the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ generated by the r_i, $i = 0, \dots, 3$, and we set $R^\pm = R \cap \mathrm{Isom}^\pm(\mathbb{H}^3)$. Observe that, since the (left) action of Γ and the (right) action of R on $\mathrm{Isom}(\mathbb{H}^3)$ commute, the groups R, R^+ also act on $\mathrm{Reg}(N)$.

Recall that any efficient cycle for N is supported on $\mathrm{Reg}(N)$, so that we can consider efficient cycles as elements of $\mathcal{M}(\mathrm{Reg}(N))$. The following result is proved in [Kuessner 2003, Lemmas 3.9 and 3.10] (see also [Jungreis 1997, Lemma 2.2]):

Lemma 3.1. *Let $\mu \in \mathcal{M}(\mathrm{Reg}(N))$ be any efficient cycle for N. Then μ is invariant with respect to the right action of R^+ on* $\mathrm{Reg}(N)$. *For every $r \in R^-$ we have $r_*(\mu) = -\mu$.*

If $\Delta = (v_0, v_1, v_2, v_3) \in \mathrm{Reg}(\mathbb{H}^3)$ is an arbitrary regular ideal straight simplex, we denote by $T_\Delta \subseteq \mathrm{Reg}(\mathbb{H}^3)$ the set defined as follows: $\Delta' = (v'_0, v'_1, v'_2, v'_3) \in \mathrm{Reg}(\mathbb{H}^3)$ belongs to T_Δ if and only if its vertices v'_0, v'_1, v'_2, v'_3 span a simplex of the unique tiling of $\mathbb{H}^3$ by regular ideal straight tetrahedra containing the simplex spanned by v_0, v_1, v_2, v_3. We also denote by $\mathrm{Aut}(T_\Delta) < \mathrm{Isom}(\mathbb{H}^3)$ the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ leaving T_Δ invariant. It is easy to check that $\mathrm{Aut}(T_\Delta)$ is discrete.

Recall that two subgroups Γ_1, Γ_2 of $\mathrm{Isom}(\mathbb{H}^3)$ are *commensurable* if there exists $g \in \mathrm{Isom}(\mathbb{H}^3)$ such that $(g\Gamma_1 g^{-1}) \cap \Gamma_2$ has finite index both in $g\Gamma_1 g^{-1}$ and in Γ_2. If Γ_1 and Γ_2 are discrete and torsion-free, this is equivalent to requiring that the hyperbolic manifolds $\Gamma_1 \backslash \mathbb{H}^3$ and $\Gamma_2 \backslash \mathbb{H}^3$ admit a common finite-sheeted Riemannian covering.

Lemma 3.2. *For every $\Delta \in \mathrm{Reg}(\mathbb{H}^3)$, the group $\mathrm{Aut}(T_\Delta)$ is commensurable with R. Both these groups are commensurable with the fundamental group of the Gieseking manifold.*

Proof. If $g \in \mathrm{Isom}(\mathbb{H}^3)$ is such that $g(\Delta_0) = \Delta$, then $g \cdot \mathrm{Aut}(T_{\Delta_0}) \cdot g^{-1} = \mathrm{Aut}(T_\Delta)$. Moreover, $R < \mathrm{Aut}(T_{\Delta_0})$ and the index of R is finite, since $\mathrm{Aut}(T_{\Delta_0})$ is discrete and R has finite covolume. This implies that $\mathrm{Aut}(T_\Delta)$ is commensurable with R.

Up to conjugacy, we may suppose that Δ_0 is a fundamental domain for the action of the fundamental group G of the Gieseking manifold on $\mathbb{H}^3$. Then $G < \mathrm{Aut}(T_{\Delta_0})$ and, as above, the index of G in $\mathrm{Aut}(T_{\Delta_0})$ is finite because G has finite covolume (in fact, this index is equal to $4! = 24$). This concludes the proof. □

Theorem 3.3. *Suppose there is a nonequidistributed efficient cycle* $\mu \in \mathcal{M}(\mathrm{Reg}(N))$. *Then* N *is commensurable with the Gieseking manifold.*

Proof. Let $N = \Gamma\backslash\mathbb{H}^3$. As proved in [Kuessner 2003, Section 4], the efficient cycle μ decomposes into the sum of a multiple of μ_{eq} and a measure $\mu' \in \mathrm{Reg}(N)$ which is supported on tetrahedra whose lifts in $\mathbb{H}^3$ have all their vertices in parabolic fixed points of Γ. Since μ is nonequidistributed, we may assume that $\mu' \neq 0$.

Since parabolic fixed points of Γ are in countable number, the support of μ' is also countable, and this implies in turn that μ' is purely atomic. Moreover, since $\mu' = \mu - k\mu_{\mathrm{eq}}$ for some $k \in \mathbb{R}$, the measure μ' also satisfies $r_*(\mu') = \mu'$ for every $r \in R^+$ and $r_*(\mu') = -\mu'$ for every $r \in R^-$. Let us set

$$\Omega = \big\{[\Delta] \in \mathrm{Reg}(N) \mid \mu'\big(\{[\Delta]\}\big) \neq 0\big\} \neq \varnothing.$$

Due to the R-equivariance of μ', the countable set Ω is R-invariant. Let us fix a nonempty R-orbit $\overline{\Omega} \subseteq \Omega$. The absolute value of the measure μ' is constant on elements of $\overline{\Omega}$. Since μ' has finite total variation, this implies that

$$\overline{\Omega} = \{[\Delta_1], \ldots, [\Delta_k]\}$$

is finite. For every $i = 1, \ldots, k$, let Δ_i be a lift of $[\Delta_n] \in \mathrm{Reg}(N)$ in $\mathrm{Reg}(\mathbb{H}^3)$. By looking at the definition of the actions of R and of Γ on $\mathrm{Reg}(\mathbb{H}^3)$, we deduce that the R-orbit of Δ_1 in $\mathrm{Reg}(\mathbb{H}^3)$ is contained in

$$\Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_k.$$

Observe now that the R-orbit of Δ_1 in $\mathrm{Reg}(\mathbb{H}^3)$ realizes a tiling of $\mathbb{H}^3$ by regular ideal tetrahedra. Therefore, up to adding to the Δ_j, $j = 1, \ldots, k$, all the straight simplices obtained by permuting their vertices (which are still in finite number), we may assume that

$$T_{\Delta_1} \subseteq \Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_k.$$

Moreover, up to reordering the Δ_j, we may assume that $T_{\Delta_1} \cap (\Gamma \cdot \Delta_j) \neq \varnothing$ for every $j = 1, \ldots, k'$ and $T_{\Delta_1} \cap (\Gamma \cdot \Delta_j) = \varnothing$ for every $j = k'+1, \ldots, k$, for some $k' \leq k$. By construction, we still have

$$T_{\Delta_1} \subseteq \Gamma \cdot \Delta_1 \cup \cdots \cup \Gamma \cdot \Delta_{k'}. \tag{11}$$

We are now going to show that the group $\Gamma\cap\mathrm{Aut}(T_{\Delta_1})$ has finite index in $\mathrm{Aut}(T_{\Delta_1})$. To this aim we will just exploit (11). For $j=1,\dots,k'$, up to replacing Δ_j with another simplex in its Γ-orbit, we suppose that $\Delta_j\in T_{\Delta_1}$. Observe now that T_{Δ_1} is the orbit of Δ_1 under the action of $\mathrm{Aut}(T_{\Delta_1})$; hence, thanks to (11), for every $j=1,\dots,k'$ there exists $g_j\in\mathrm{Aut}(T_{\Delta_1})$ such that $g_j\cdot\Delta_j=\Delta_1$.

Let us fix $g\in\mathrm{Aut}(T_{\Delta_1})$. Since $g\cdot\Delta_1\in T_{\Delta_1}\subseteq\Gamma\cdot\Delta_1\cup\cdots\cup\Gamma\cdot\Delta_{k'}$, there exist $\gamma\in\Gamma$, $j\in\{1,\dots,k'\}$ such that $g\cdot\Delta_1=\gamma\cdot\Delta_j$, whence $(\gamma^{-1}g)\cdot\Delta_1=\Delta_j$ and $(g_j\gamma^{-1}g)\cdot\Delta_1=g_j\cdot\Delta_j=\Delta_1$. However, since the unique hyperbolic isometry which fixes the vertices of a regular ideal tetrahedron is the identity, the stabilizer of Δ_1 in $\mathrm{Aut}(T_{\Delta_1})$ is trivial; hence $g_j\gamma^{-1}g=1$, i.e., $g=\gamma g_j^{-1}$ (and, in particular, $\gamma\in\Gamma\cap\mathrm{Aut}(T_{\Delta_1})$). We have thus shown that the set $\{g_1,\dots,g_{k'}\}$ contains a set of representatives for the set of right lateral classes of $\Gamma\cap\mathrm{Aut}(T_{\Delta_1})$ in $\mathrm{Aut}(T_{\Delta_1})$.

Since Γ is discrete and $\Gamma\cap\mathrm{Aut}(T_{\Delta_1})$ has finite covolume (being a finite index subgroup of $\mathrm{Aut}(T_{\Delta_1})$), the group $\Gamma\cap\mathrm{Aut}(T_{\Delta_1})$ has finite index also in Γ. Thus Γ is commensurable with $\mathrm{Aut}(T_{\Delta_1})$; hence N is commensurable with the Gieseking manifold by Lemma 3.2. □

Putting together Theorem 3.3 and Corollary 2.8 we obtain the following:

Corollary 3.4. *Let N be a complete finite-volume n-hyperbolic manifold*, $n\geq 3$, *and suppose that N is not commensurable with the Gieseking manifold. Then N admits a unique efficient cycle*, *which is given by the measure*

$$\frac{1}{2v_n}\cdot\mu_{\mathrm{eq}}.$$

4. Manifolds admitting nonequidistributed efficient cycles

We prove that manifolds that are commensurable with the Gieseking manifold admit nonequidistributed efficient cycles. We will first prove that this phenomenon occurs for manifolds admitting an ideal triangulation by regular ideal tetrahedra, and we will then deduce the general case from the fact that any manifold which is commensurable with the Gieseking manifold admits a finite covering with such a triangulation.

4.1. ***Triangulations and ideal triangulations.*** Let $\overline{N}$ be a compact 3-manifold with nonempty boundary made of tori. We recall the well-known notions of triangulation and ideal triangulation, widely used in 3-dimensional topology.

A *triangulation* of $\overline{N}$ is a realization of $\overline{N}$ via a simplicial face-pairing of finitely many tetrahedra. A triangulation of $\overline{N}$ naturally induces a triangulation of its boundary.

An *ideal triangulation* of $\overline{N}$ (or of N) is a realization of $N = \text{int}(\overline{N})$ as a simplicial face-pairing of finitely many tetrahedra, with all their vertices removed. The removed vertices are called *ideal* and they correspond to the boundary components of $\overline{N}$; the link of every ideal vertex is a triangulation of the corresponding boundary component of $\overline{N}$.

We say as usual that $\overline{N}$ is *hyperbolic* if its interior has a finite-volume complete hyperbolic metric. If $\overline{N}$ is hyperbolic, every geometric decomposition of N into hyperbolic ideal tetrahedra is an example of ideal triangulation, that we call a *geometric ideal triangulation* of $\overline{N}$ (or of N). We still do not know whether every hyperbolic 3-manifold has a geometric ideal triangulation, but we know it does so virtually [Luo et al. 2008].

We are interested here in transforming a geometric ideal triangulation into a triangulation in an efficient way. One method called *inflation* was introduced by Jaco and Rubinstein [2014]. Here we introduce a similar method where we employ the dual viewpoint of simple spines, as Matveev [1990; 2003], in a similar fashion as in [Francaviglia et al. 2012, Section 5.4].

Consider a geometric ideal triangulation T of N. We lift it to a geometric ideal triangulation $\widetilde{T}$ of the universal cover $\mathbb{H}^3$. We choose some disjoint cusp sections in N; their preimage consists of infinitely many disjoint horoballs in $\mathbb{H}^3$, centered at the vertices of $\widetilde{T}$.

For $\varepsilon > 0$ sufficiently small, the ε-thick part N_ε of N is obtained by removing from N sufficiently deep cusp sections, and it is homeomorphic to $\overline{N}$. The ideal triangulation T of N restricts to a decomposition of N_ε into truncated tetrahedra. To obtain a triangulation for N_ε would now suffice to take its barycentric subdivision; however, this operation is not useful for us because it produces too many tetrahedra: we are looking for a triangulation for N_ε which contains the same number of tetrahedra as T, plus only a few more.

We explain our request more precisely. We say that a triangulation T' of N_ε is *adapted* to the geometric ideal triangulation T if there is an injective map i from the set of ideal tetrahedra of T to the set of tetrahedra of T' such that for every tetrahedron Δ of T, every lift of $i(\Delta)$ is a tetrahedron in $\mathbb{H}^3$ whose vertices lie in the boundary of the 4 removed horoballs whose centers are the vertices of a lift of Δ. (We do not require the lift of $i(\Delta)$ to be a straight tetrahedron, only a topological one.) In some sense we require Δ and $i(\Delta)$ to be close. Every tetrahedron of T' that is not in the image of i is called *residual*.

We will need the following lemma, which says that for any hyperbolic manifold N with a geometric ideal triangulation T it is possible to construct a tower of finite

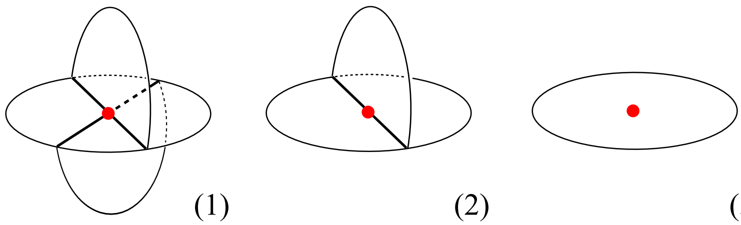

Figure 1. Neighborhoods of points in a simple polyhedron.

coverings, each equipped with an adapted triangulation T_i' whose residual tetrahedra grow sublinearly with respect to the degree of the cover.

Proposition 4.1. *Let N be a hyperbolic manifold equipped with a geometric ideal triangulation T. There is a tower of finite coverings $W_i \to \overline{N}$ of degree d_i such that the following holds: every W_i admits a triangulation T_i' adapted to the geometric ideal triangulation T_i obtained by lifting T, with r_i residual tetrahedra, such that*

$$\lim_{i\to\infty} \frac{r_i}{d_i} \to 0.$$

Subsections 4.2 and 4.3 are devoted to a proof of this proposition.

4.2. ***Construction of an adapted triangulation.*** We introduce an efficient method to transform a geometric ideal triangulation T of a hyperbolic manifold N into a triangulation T' that is adapted to T.

A compact 2-dimensional polyhedron X is *simple* if every point x of X has a star neighborhood PL-homeomorphic to one of the three models shown in Figure 1. Points of type (1) are called *vertices*. Points of type (2) and (3) form respectively some manifolds of dimension 1 and 2: their connected components are called respectively *edges* and *regions*. A simple polyhedron X is *special* if every edge is an open segment and every region is an open disc, so in particular it has a natural CW structure.

Let $\overline{N}$ be a compact 3-manifold with (possibly empty) boundary. A compact 2-dimensional subpolyhedron $X \subset N = \mathrm{int}(\overline{N})$ is a *spine* of $\overline{N}$ if $\overline{N} \setminus X$ consists of an open collar of $\partial \overline{N}$ and some (possibly none) open balls (the presence of some open balls is necessary when $\partial \overline{N} = \varnothing$).

Let $\overline{N}$ be a compact manifold with boundary made of tori. Suppose that N is hyperbolic and equipped with a geometric ideal triangulation T. We now describe a method to construct a triangulation T' for $\overline{N} \cong N_\varepsilon$ adapted to T.

First, we dualize the ideal triangulation T to get a special spine X of $\overline{N}$ with one vertex at the barycenter of each ideal tetrahedron as shown in Figure 2.

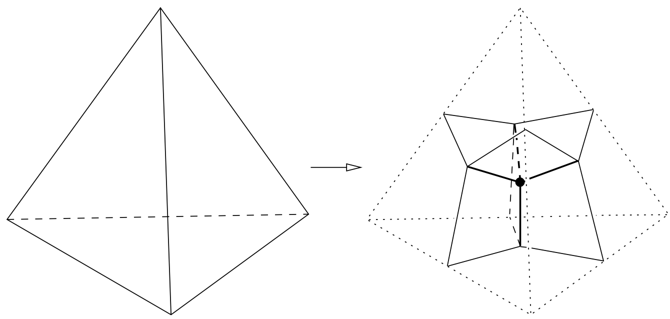

Figure 2. By dualizing an ideal triangulation we get a simple spine.

Second, we add some cells to X to obtain a new special polyhedron X', so that by dualizing X' back we will get our desired triangulation T'. We construct X' as follows. By construction $\overline{N} \setminus X$ consists of an open collar of $\partial\overline{N}$, that is a finite union of products $S \times (0, 1]$ where S is a torus and $S \times \{1\}$ is a boundary component of $\overline{N}$. Choose a θ-shaped graph $Y \subset S$ that is itself a spine of S, i.e., $S \setminus Y$ consists of an open disc. Add to X the polyhedron

$$Y \times (0, 1] \cup S \times \{1\}.$$

If we do this at each product $S \times (0, 1]$ in $\overline{N} \setminus X$, we obtain a 2-dimensional polyhedron $X' \subset \overline{N}$ that contains $\partial\overline{N}$. If Y is chosen generically, the polyhedron X' is special. The complement $\overline{N} \setminus X'$ consists of open balls, one for each boundary component of $\overline{N}$.

As we mentioned above, the triangulation T' for $\overline{N}$ is constructed by dualizing X' in the appropriate way. Every boundary torus S of $\overline{N}$ inherits from X' a cellularization with two vertices, three edges, and one disc (the cellularization depends on the chosen θ-shaped spine Y); this cellularization is dualized to a one-vertex triangulation for S. This triangulation extends from $\partial\overline{N}$ to $\overline{N}$ as follows: every disc, edge, and vertex of X' that is not adjacent to $\partial\overline{N}$ dualizes to an edge, a triangle, and a tetrahedron for T'.

The resulting triangulation T' has the smallest possible number of vertices: one for each boundary component. The tetrahedra of T are in natural 1-1 correspondence with the vertices of X. The tetrahedra of T' are in natural 1-1 correspondence with the vertices of X' that are not contained in $\partial\overline{N}$. Since every vertex of X is also a vertex of X' of this kind, we get a natural injection i from the set of tetrahedra of T into the set of tetrahedra of T'.

Lemma 4.2. *If T is a geometric ideal triangulation for $\overline{N}$, the triangulation T' is adapted to T.*

Proof. We fix some disjoint horocusp sections and truncate N along these, to obtain a smaller copy N_ε of $\overline{N}$. Their preimage in $\mathbb{H}^3$ consists of horospheres. When passing from X to X' we add the cusp sections ∂N_ε and some products $Y \times (0, 1]$. In $\mathbb{H}^3$ this corresponds to adding the horospheres and some products $\widetilde{Y} \times (0, 1]$. The resulting dual triangulation T' has all its vertices in the cusp sections, which lift to vertices in the horospheres. By construction for every ideal tetrahedron Δ in T the corresponding $i(\Delta)$ has its vertices in the same horospheres that are crossed by the edges of Δ. □

The residual tetrahedra correspond to the vertices of X' contained in the interior of $\overline{N}$ that were not themselves vertices of X, and that were created by attaching the products $Y \times (0, 1]$ along some generic map $Y \to X$. We now need to construct some tower of coverings where this kind of vertices grow sublinearly in number.

4.3. ***Characteristic coverings.*** We now build the tower of coverings for $\overline{N}$ needed in Proposition 4.1. We will use some results of Hamilton [Hamilton 2001] on coverings determined by characteristic subgroups. A similar construction was made in [Francaviglia et al. 2012, Section 5.3].

Recall that a *characteristic subgroup* of a group G is a subgroup $H < G$ which is invariant by any automorphism of G. For a natural number $x \in \mathbb{N}$, the *x-characteristic* subgroup of $\mathbb{Z} \times \mathbb{Z}$ is the subgroup $x(\mathbb{Z} \times \mathbb{Z})$ generated by $(x, 0)$ and $(0, x)$. It has index x^2 if $x > 0$ and ∞ if $x = 0$. The characteristic subgroups of $\mathbb{Z} \times \mathbb{Z}$ are precisely the x-characteristic subgroups with $x \in \mathbb{N}$. It is easy to prove that a subgroup of $\mathbb{Z} \times \mathbb{Z}$ of index x contains the x-characteristic subgroup.

A covering map $p : \widetilde{T} \to T$ of tori is *x-characteristic* if $p_*(\pi_1(\widetilde{T}))$ is the x-characteristic subgroup of $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$. A covering map $p : \widetilde{\overline{N}} \to \overline{N}$ of 3-manifolds bounded by tori is *x-characteristic* if the restriction of p to each boundary component of $\widetilde{\overline{N}}$ is x-characteristic.

Lemma 5 from [Hamilton 2001] implies the following.

Lemma 4.3 (E. Hamilton). *Let $\overline{N}$ be a hyperbolic compact, orientable 3-manifold with boundary tori. For every integer $i > 0$ there exist an integer $k > 0$ and a finite-index normal subgroup $K_i \triangleleft \pi_1(\overline{N})$ such that $K_i \cap \pi_1(T_j)$ is the characteristic subgroup of index $(ik)^2$ in $\pi_1(T_j)$, for each component T_j of $\partial\overline{N}$. Hence the covering $W_i \to \overline{N}$ corresponding to K_i is (ik)-characteristic.*

We can now prove Proposition 4.1. We restate it for the sake of clarity.

Proposition 4.1. *Let N be a hyperbolic manifold equipped with a geometric ideal triangulation T. There is a tower of finite coverings $W_i \to \overline{N}$ of degree d_i such that*

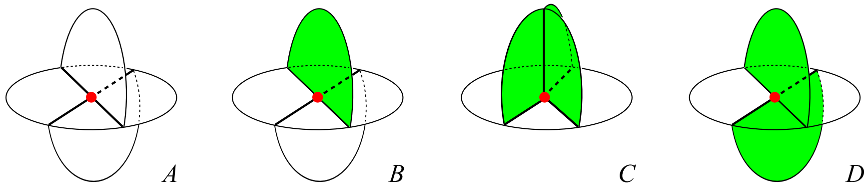

Figure 3. We color in green the regions of the inserted portions $Y \times (0, 1) \cup S \times \{1\}$. There are four types of vertices A, B, C, and D in the spine Q, according to the colors of the incident regions.

the following holds: every W_i admits a triangulation T_i' adapted to the geometric ideal triangulation T_i obtained by lifting T, with r_i residual tetrahedra, such that

$$\lim_{i\to\infty} \frac{r_i}{d_i} \to 0.$$

Proof. Let X be the spine dual to T. Following Section 4.2 we enlarge X to a special polyhedron X' by adding one piece

$$Y \times (0, 1] \cup S \times \{1\}$$

for each boundary torus S of $\bar{N}$, inside the corresponding collar $S \times (0, 1]$ in $\bar{N} \setminus X$. This operation depends on the choice of a generic θ-shaped spine $Y \subset S$.

The polyhedron X' has all the vertices of X, plus some additional ones that we now investigate carefully. The following discussion is similar to [Francaviglia et al. 2012, proof of Lemma 5.9]. Color in white the regions of X and in green the regions in the products $Y \times (0, 1] \cup S \times \{1\}$ that are attached to X. There are five types A, B, C, D, E of vertices in X' according to the colors of the incident regions: the vertices of type A, B, C, D are shown in Figure 3, while those of type E are those that lie in $\partial \bar{N}$ and that are incident to green regions only. The vertices of type A are precisely those of X. The vertices of type B, C, D are dual to the residual tetrahedra of T', and we want to control their number. Those of type E are not interesting here.

For every boundary torus S, the collar map $S \to X$ is a (possibly noninjective) immersion, and the cellularization of X pulls back to a cellularization of S, which is in fact dual to the triangulation link of the corresponding ideal vertex of T. The θ-shaped spine Y is generic, transverse to this cellularization as in Figure 4 (left). The four types of vertices A, B, C, D that may arise are shown in Figure 4 (right).

Let v_A, v_B, v_C, and v_D be the number of vertices of type A, B, C, and D in X'. The number of residual tetrahedra in T' is $v_B + v_C + v_D$.

We build the tower of coverings. By Lemma 4.3, for every integer $i \geq 1$, there are a $k_i > 0$ and an (ik_i)-characteristic covering $W_i \to \bar{N}$.

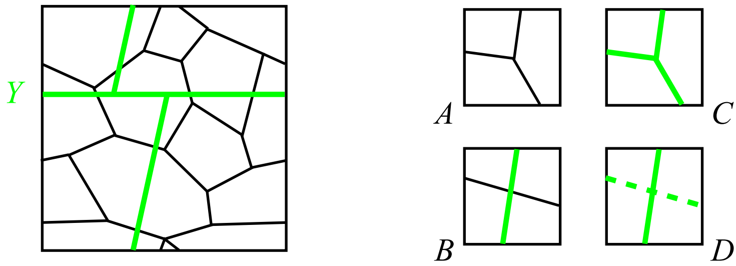

Figure 4. The cellularization of a boundary torus S induced by the collar map $S \to X$, and the θ-shaped spine Y of S colored in green (left). The four types of vertices A, B, C, D (right).

We now construct the triangulation T_i' adapted to the lifted geometric ideal triangulation T_i of W_i. The preimage of X is a spine X_i of W_i dual to T_i. To construct the adapted triangulation T_i', we choose an appropriate θ-shaped spine inside every boundary torus of W_i. We explain now how to make this choice.

Since the covering $W_i \to \overline{N}$ is (ik_i)-characteristic, every boundary torus $\widetilde{S}$ of W_i covers a torus S of $\overline{N}$ as an (ik_i)-characteristic covering. The case $ik_i = 3$ is shown in Figure 5. We have chosen in the previous paragraphs a spine Y for S; see Figure 4. As shown in Figure 5 (left), the preimage $\widetilde{Y}$ of Y in $\widetilde{S}$ is a spine of $\widetilde{S}$, whose complement in S consists of $(ik_i)^2$ discs. Figure 5 (right) shows that we can eliminate most vertices and edges of $\widetilde{Y}$ and obtain a simpler spine $\widetilde{Y}' \subset \widetilde{Y}$ of $\widetilde{S}$, whose complement in $\widetilde{S}$ consists of only one disc. This is the θ-shaped spine that we use on each boundary component $\widetilde{S}$ of W_i.

It remains to estimate the number r_i of residual tetrahedra in T_i'. Recall that

$$r_i = v_B^i + v_C^i + v_D^i,$$

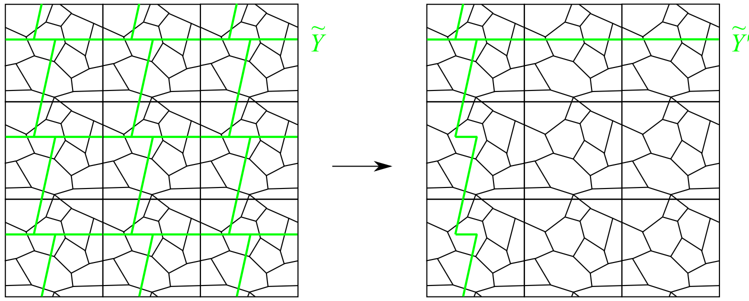

Figure 5. A 3-characteristic covering $\widetilde{S} \to S$. The spine Y of S lifts to the green spine $\widetilde{Y}$ shown in the left picture. We can eliminate most of its edges and still get a spine $\widetilde{Y}'$ of $\widetilde{S}$.

where v^i_B, v^i_C, v'_D are the numbers of vertices of type B, C, D in the dual polyhedron X'_i. The covering $W_i \to \overline{N}$ has degree

$$d_i = (ik_i)^2 h_i,$$

where h_i is the number of distinct boundary tori in ∂W_i that project to one boundary torus of $\overline{N}$. It is clear from Figure 5 that

$$v^i_B \le 2ik_ih_iv_B, \quad v^i_C \le 2v_C, \quad v^i_D \le 2ik_ih_iv_D.$$

Therefore

$$\frac{r_i}{d_i} = \frac{v^i_B + v^i_C + v^i_D}{(ik_i)^2 h_i} \le \frac{2ik_ih_i}{(ik_i)^2h_i}(v_B + v_C + v_D) \to 0$$

as $i \to \infty$. The proof is complete. □

4.4. ***Efficient cycles from regular ideal triangulations.*** We are now ready to show that if a hyperbolic 3-manifold N admits a geometric ideal triangulation T by regular ideal tetrahedra, then it also admits a nonequidistributed efficient cycle. Indeed, let $\Delta_1, \Delta_2, \dots, \Delta_h$ be the regular ideal tetrahedra of T, considered as subsets of N. For every $i = 1, \dots, k$ we denote by $\widetilde{\sigma}_i \in \mathrm{Reg}^+(\mathbb{H}^3) \subseteq (\mathbb{H}^3)^4$ a (positively oriented) ordering $(\tilde{v}_0, \dots, \tilde{v}_3)$ of the set of vertices of a lift of Δ_i to $\mathbb{H}^3$, and by σ_i the class of $\widetilde{\sigma}_i$ in $\mathrm{Reg}^+(N)$. Finally, we set

$$\mu_T = \Theta\left(\mathrm{alt}_3\left(\sum_{i=1}^{k} \sigma_i\right)\right). \tag{12}$$

(Strictly speaking, we defined the alternating operator only on straight simplices with vertices in $\mathbb{H}^3$, but of course it may be extended by the same formula also on ideal straight simplices).

The main result of this section is Theorem 7, which we recall here for the convenience of the reader:

Theorem 7. *Let M be a complete finite-volume 3-manifold admitting a decomposition T into regular ideal tetrahedra. Then μ_T is an efficient cycle for M.*

Proof. Let us fix some notation. As usual, for every sufficiently large $i \in \mathbb{N}$ we fix an identification $\overline{N} \cong N_{2^{-i}}$ between the natural compactification of N and the 2^{-i}-thick part of N. By Proposition 4.1, there is a tower of finite coverings $W_i \to N_{2^{-i}}$ of degree d_i such that the following holds: every W_i admits a triangulation T'_i adapted to the geometric ideal triangulation T_i obtained by lifting T, with r_i residual tetrahedra, such that

$$\lim_{i\to\infty} \frac{r_i}{d_i} \to 0. \tag{13}$$

For every sufficiently large $i \in \mathbb{N}$, we construct a relative fundamental cycle c_i for $N_{2^{-i}}$ as follows. The universal covering of W_i coincides with the universal covering of $N_{2^{-i}}$ (which is the complement of a collection of disjoint horoballs in $\mathbb{H}^3$); hence we may apply the straightening operator to any positively oriented parametrization of any simplex appearing in T_i'; after applying the alternating operator to the sum of the obtained straight tetrahedra, we get a relative fundamental cycle $\tilde{c}_i$ for W_i (more precisely, for the pair $(W_i', W_i' \setminus \operatorname{int}(W_i))$, where W_i' is the complete finite-volume hyperbolic manifold obtained from W_i by adding back the removed cusps). If $p_i : (W_i', W_i' \setminus \operatorname{int}(W_i)) \to (N, N \setminus \operatorname{int}(N_{2^{-i}}))$ is the covering projection, we then set

$$c_i = \frac{(p_i)_*(\tilde{c}_i)}{d_i}.$$

For simplicity, we will say that a simplex appearing in c_i is *nonresidual* if it is obtained (via $(p_i)_*$) from the alternation of the straightening of a nonresidual simplex of T_i'.

It is easy to check that c_i, $i \in \mathbb{N}$, is a minimizing sequence: if k is the number of the tetrahedra of T, then $\operatorname{Vol}(N) = k v_3$; hence $\|N\| = \operatorname{Vol}(N)/v_3 = k$. On the other hand, by construction the number of nonresidual simplices in $\tilde{c}_i$ is equal to kd_i and the alternating operator is norm nonincreasing; hence

$$\limsup_{i\to+\infty} \|c_i\| = \limsup_{i\to+\infty} \frac{\|(p_i)_*(\tilde{c}_i)\|}{d_i} \le \limsup_{i\to+\infty} \frac{\|\tilde{c}_i\|}{d_i} = \limsup_{i\to+\infty} \frac{kd_i + r_i}{d_i} = k,$$

and this proves that the sequence c_i, $i \in \mathbb{N}$, is minimizing.

In order to conclude we are then left to show that

$$\lim_{i\to+\infty} \Theta(c_i) = \mu_T,$$

where $\Theta(c_i)$ is the measure associated to the cycle c_i (see (1)) and μ_T is the measure associated to the triangulation T (see (12)). Let $\Delta_0 \in \operatorname{Reg}^+(N)$ be a (positively oriented representative of a) tetrahedron of T, and let $\widetilde{\Delta}_0 \in \operatorname{Reg}(\mathbb{H}^3)$ be a lift of Δ_0 to $\mathbb{H}^3$ with vertices (v_0, v_1, v_2, v_3). There exist pairwise disjoint open neighborhoods $U_0, \dots, U_3$ of $v_0, \dots, v_3$ in $\overline{\mathbb{H}}^3$ such that the following conditions hold: every straight tetrahedron having its i-th vertex in U_i is nondegenerate and positively oriented, and the tetrahedron $\widetilde{\Delta}_0 = (v_0, v_1, v_2, v_3)$ is the unique lift of the tetrahedron of T whose vertices lie (in the correct order) in $U_0, \dots, U_3$. We set

$$\widetilde{\Omega} = \{(v_0, v_1, v_2, v_3) \in \overline{S}_3^*(\mathbb{H}^3) \mid v_i \in U_i \text{ for every } i = 0, 1, 2, 3\}$$

and we let Ω be the projection of $\widetilde{\Omega}$ in $\overline{S}_3^*(N)$. Of course, $\widetilde{\Omega}$ is an open neighborhood of $\widetilde{\Delta}_0$ in $\overline{S}_3^*(\mathbb{H}^3)$, and since the projection $\overline{S}_3^*(\mathbb{H}^3) \to \overline{S}_3^*(N)$ is open, the set Ω is an open neighborhood of Δ_0 in $\overline{S}_3^*(N)$.

Let now $f : \overline{S}^*_3(N)$ be any continuous compactly supported function such that $f(\Delta_0) = 1$. Recall that the vertices of the lifts of nonresidual tetrahedra of T'_i lie on the boundary of (deeper and deeper, as $i \to +\infty$) removed horoballs centered at the ideal vertices of lifts of tetrahedra of T. We say that a simplex σ' appearing in the cycle c_i is a *relative* of Δ_0 if it is nonresidual and it admits a lift to $\mathbb{H}^3$ with vertices on horospheres centered at the ideal vertices of a lift of Δ_0 (in the correct order).

Thanks to our definition of Ω, we can choose $i \in \mathbb{N}$ such that, if σ is a nonresidual simplex appearing in c_i, then σ belongs to Ω if and only if it is a relative of Δ_0. Let us now decompose c_i as

$$c_i = c_i^0 + c_i^{\mathrm{nr}} + c_i^{\mathrm{r}},$$

where c_i^0 is supported on relatives of Δ_0, c_i^{nr} is supported on nonresidual simplices which are not relatives of Δ_0, and c_i^{r} is supported on residual simplices. Since the simplices appearing in c_i^{nr} cannot belong to Ω for i sufficiently large, we have

$$\lim_{i \to +\infty} \int_\Omega f \, d\Theta(c_i^{\mathrm{nr}}) = 0. \tag{14}$$

Recall now that the alternating operator associates to every simplex the average of 24 singular simplices, and that positively oriented simplices come with the coefficient $+\frac{1}{24}$. Therefore, c_i^0 is a linear combination of d_i simplices, each of which comes with the real coefficient $1/(24d_i)$. In particular, we have $\|c_i^0\| = \frac{1}{24}$. In the very same way, if one starts with a negatively oriented Δ_0, still $\|c_i^0\| = \frac{1}{24}$ but the coefficients appearing in c_i^0 are all negative. As a consequence, since $f(\Delta_0) = 1$ and the simplices appearing in c_i^0 are converging to Δ_0 in $\overline{S}^*_3(N)$ (and f is continuous),

$$\lim_{i \to +\infty} \int_\Omega f \, d\Theta(c_i^0) = \|c_i^0\| = \frac{1}{24} \tag{15}$$

$\big($while, if Δ_0 were negatively oriented, we would have $\lim_{i \to +\infty} \int_\Omega f \, d\Theta(c_i^0) = -\|c_i^0\| = -\frac{1}{24}\big)$.

Finally from (13) we deduce that $\lim_{i \to +\infty} \|c_i^{\mathrm{r}}\| = 0$; hence

$$\lim_{i \to +\infty} \int_\Omega f \, d\Theta(c_i^r) = 0. \tag{16}$$

Putting together (14)–(16) we then obtain

$$\lim_{i \to +\infty} \int_\Omega f \, d\Theta(c_i) = \pm \frac{1}{24},$$

where the sign depends on whether Δ_0 is positively or negatively oriented.

Let us now denote by μ the limit of $\Theta(c_i)$ (which we may assume to exist, up to passing to a subsequence; in fact, with a little more effort we could easily prove that the sequence $\Theta(c_i)$, $i \in \mathbb{N}$, is itself convergent). Due to the definition of weak-* convergence, we have thus proved that there exists a neighborhood Ω of Δ_0 such that, for every compactly supported $f : \overline{S}_3^*(N) \to \mathbb{R}$ with $f(\Delta_0) = 1$, we have

$$\int_\Omega f \, d\mu = \pm\frac{1}{24}.$$

This implies that $\mu(\{\Delta_0\}) = \pm\frac{1}{24}$.

We have thus shown that $\mu(\{\Delta_0\}) = \pm\frac{1}{24}$ for every tetrahedron $\Delta_0 \in \mathrm{Reg}(N)$ whose geometric realization is a tetrahedron of the ideal triangulation T we started with. But every ideal tetrahedron of T gives rise to 24 tetrahedra in $\mathrm{Reg}(N)$, and the simplicial volume $\|N\|$ is equal to the number of tetrahedra of T, hence the contribution to μ of the atomic measures supported by tetrahedra whose geometric realizations are in T has total variation equal to $\|N\|$. Since we already know from Theorem 4 that $\|\mu\| = \|N\|$, this finally implies that $\mu = \mu_T$, as desired. □

We can now conclude the proof of Theorems 5 and 6 by showing that, if N is commensurable with the Gieseking manifold, then it admits nonequidistributed efficient cycles.

4.5. ***Proof of Theorem 5.*** We have proved in Section 3 that, if N is not commensurable with the Gieseking manifold, then every efficient cycle for N is equidistributed.

Vice versa, if N is commensurable with the Gieseking manifold, then there exists a degree-d covering $p : \widehat{N} \to N$, where $\widehat{N}$ admits a triangulation $\widehat{T}$ by regular ideal tetrahedra. Let $\hat{c}_i$, $i \in \mathbb{N}$, be the relative fundamental cycles for $\widehat{N}$ constructed in the proof of Theorem 7, and for every $i \in \mathbb{N}$ let $c_i = p_*(c_i)/d$. The covering map p induces a continuous map $\overline{S}_3^*(\widehat{N}) \to \overline{S}_3^*(N)$, hence a map $\mathcal{M}(\overline{S}_3^*(\widehat{N})) \to \mathcal{M}(\overline{S}_3^*(N))$. The very same proof of Theorem 7 shows that the limit $\mu = \lim_{i\to+\infty} \Theta(c_i) \in \mathcal{M}(\overline{S}_3^*(N))$ is an efficient cycle for N, and is equal to the image of $\mu_{\widehat{T}}$ via the map $\mathcal{M}(\overline{S}_3^*(\widehat{N})) \to \mathcal{M}(\overline{S}_3^*(N))$. But the image of a purely atomic measure via a continuous map is itself purely atomic. In particular, μ is a nonequidistributed efficient cycle for N, and this concludes the proof.

4.6. ***Proof of Theorem 6.*** We are only left to show that, if N is not commensurable with the Gieseking manifold and c_i, $i \in \mathbb{N}$ is any minimizing sequence for N, then

$$\lim_{i\to+\infty} \Theta(c_i) = \frac{1}{2v_n}\mu_{\mathrm{eq}}.$$

Of course, it is sufficient to show that every subsequence of c_i, $i \in \mathbb{N}$ admits a subsequence whose image via Θ converges to $\mu_{\mathrm{eq}}/(2v_n)$. However, the total variation of the measures $\Theta(c_i)$ is uniformly bounded; hence by compactness of the unit ball in $\mathcal{M}(\overline{S}^*_n(N)$ every subsequence of $\Theta(c_i)$ admits a subsequence converging to some measure $\mu \in \mathcal{M}(\overline{S}^*_n(N)$. By Corollary 3.4 we must have $\mu = \mu_{\mathrm{eq}}/(2v_n)$, and this concludes the proof.

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ON DISJOINT STATIONARY SEQUENCES

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We answer a question of Krueger by obtaining disjoint stationary sequences on successive cardinals. The main idea is an alternative presentation of a mixed support iteration, using it more explicitly as a variant of Mitchell forcing. We also use a Mahlo cardinal to obtain a model in which $\aleph_2 \notin I[\aleph_2]$ and there is no disjoint stationary sequence on $\aleph_2$, answering a question of Gilton.

1. Introduction and background

In order to develop the theory of infinite cardinals, set theorists study a variety of objects that can potentially exist on these cardinals. The objects of interest for us are called *disjoint stationary sequences*. These were introduced by Krueger to answer a question of Abraham and Shelah about forcing clubs through stationary sets [2]. Beginning in joint work with Friedman, Krueger wrote a series of papers in this area, connecting a wide range of concepts and answering seemingly unrelated questions of Foreman and Todorčević [8; 17; 18; 19; 20; 21]. Our purpose is to further develop this area.

Krueger's new arguments generally hinged on the behavior of two-step iterations of the form $\mathrm{Add}(\tau) * \mathbb{P}$. In order to extend the application of these arguments as widely as possible, Krueger developed the notion of mixed support forcing [18; 21], which had apparently been part of the folklore for some time. These forcings are to some extent an analog of the forcing that Mitchell used to obtain the tree property at double successors of regular cardinals. Their most notable feature is the appearance of quotients insofar as the forcings took the form $\mathbb{M} \simeq \overline{\mathbb{M}} * \mathrm{Add}(\tau) * \mathbb{E}$ where $\overline{\mathbb{M}}$ is a partial mixed support iteration. The appearance of $\mathrm{Add}(\tau)$ after the initial component, together with the preservation properties of the quotient $\mathbb{E}$, allowed Krueger's new arguments to go through various complicated constructions. Mixed support iterations have found several applications since [10], particularly in regard to guessing models [22].

Our main idea is to use a version of Mitchell forcing to accomplish the task of a mixed support iteration. Specifically, we prove that this version of Mitchell

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forcing takes the form $\mathbb{M} \simeq \overline{\mathbb{M}} * \mathrm{Add}(\tau) * \mathbb{E}$.[1] The trick used to obtain this structural property goes back to Mitchell's thesis and is also reminiscent of the one used by Cummings, S. D. Friedman, Magidor, Rinot, and Sinapova [5] to demonstrate that subtle variations in the definitions of Mitchell forcing — up to merely shifting a Lévy collapse by a single coordinate — can substantially alter the properties of the forcing extension. The benefit of the forcing used here is that it comes with a projection analysis of the sort that Abraham used for Mitchell forcing [1]. Both the forcing itself and its quotients are projections of products of the form $\mathbb{A} \times \mathbb{T}$ where $\mathbb{A}$ has a good chain condition and $\mathbb{T}$ has a good closure property. This allows us to obtain preservation properties conveniently, without having to delve into too many technical details. Abraham in fact used this projection analysis to extend Mitchell's result to successive cardinals. This is exactly what we do here for disjoint stationary sequences, answering the first component of a question of Krueger [21, Question 12.8]:

Theorem 1. *Suppose $\lambda_1 < \lambda_2$ are two Mahlo cardinals in V. Then there is a forcing extension in which there are disjoint stationary sequences on $\aleph_2$ and $\aleph_3$.*

We lay out the basic definition and concepts in the following subsections and then develop the proof in Section 2. We also achieve one of Krueger's separations for successive cardinals, which answers a component of another one of his questions [21, Question 12.9]:

Theorem 2. *Suppose $\lambda_1 < \lambda_2$ are two Mahlo cardinals in V. Then there is a forcing extension in which for $\mu \in \{\aleph_1, \aleph_2\}$, there are stationarily many $N \in [H(\mu^+)]^\mu$ that are internally stationary but not internally club.*

The last main result is motivated by work of Gilton and Krueger, who answered a question from [5] by obtaining stationary reflection for subsets of $\aleph_2 \cap \mathrm{cof}(\omega)$ together with failure of approachability at $\aleph_2$ (i.e., $\aleph_2 \notin I[\aleph_2]$) using disjoint stationary sequences [10]. This result used the fact that the existence of a disjoint stationary sequence implies failure of approachability. Gilton asked for the exact consistency strength of the failure of approachability at $\aleph_2$ together with the nonexistence of a disjoint stationary sequence on $\aleph_2$ [9, Question 9.0.15]. (He pointed out that Cox found this separation using PFA [3].) It is known that the failure of approachability requires the consistency strength of a Mahlo cardinal since $\square_\tau$ holds if τ^+ is not Mahlo in L [16] and $\square_\tau$ implies the approachability property $\tau^+ \in I[\tau^+]$ [6]. In Section 3 we show that a Mahlo cardinal is sufficient for the separation:

Theorem 3. *Suppose that λ is Mahlo in V. Then there is a forcing extension in which $\aleph_2 \notin I[\aleph_2]$ and there is no disjoint stationary sequence on $\aleph_2$.*

[1] The extent to which all variations of these forcings are equivalent or not is left as a loose end. Here we only deal with the case where the two-step iteration $\mathrm{Add}(\tau) * \mathbb{P}$ takes the form $\mathrm{Add}(\tau) * \dot{\mathrm{Col}}(\mu, \delta)$.

Disjoint stationary sequences are known to be interpretable in terms of canonical structure (see Fact 6 below), and the main idea for Theorem 3 is a simple master condition argument that exploits this connection.

1.1. ***Basic definitions.*** We assume familiarity with the basics of forcing and large cardinals. We use the following conventions: If $\mathbb{P}$ is a forcing poset, then $p \leq q$ for $p, q \in \mathbb{P}$ means that p is stronger than q. We say that $\mathbb{P}$ is κ-closed if for all $\leq_{\mathbb{P}}$-decreasing sequences $\langle p_\xi : \xi < \tau \rangle$ with $\tau < \kappa$, there is a lower bound p, i.e., $p \leq p_\xi$ for all $\xi < \tau$. (Not all authors use this formulation of κ-closedness.) We say that $\mathbb{P}$ has the κ-chain condition if all antichains $A \subseteq \mathbb{P}$ have cardinality strictly less than κ. All posets considered will be separative. Now we give our main definitions:

Definition 4. Given a regular cardinal μ, a *disjoint stationary sequence* on μ^+ is a sequence $\langle \mathcal{S}_\alpha : \alpha \in S \rangle$ such that

- $S \subseteq \mu^+ \cap \mathrm{cof}(\mu)$ is stationary,
- $\mathcal{S}_\alpha$ is a stationary subset of $P_\mu(\alpha)$ for all $\alpha \in S$,
- $\mathcal{S}_\alpha \cap \mathcal{S}_\beta = \varnothing$ if $\alpha \neq \beta$.

We write $\mathsf{DSS}(\mu^+)$ to say that there is a disjoint stationary sequence on μ^+.

Definition 5. Given an uncountable regular κ and a set $N \in [H(\Theta)]^\kappa$,[2] we say:

- N is *internally unbounded* if for all $x \in P_\kappa(N)$, there is an $M \in N$ such that $x \subseteq M$.
- N is *internally stationary* if $P_\kappa(N) \cap N$ is stationary in $P_\kappa(N)$.
- N is *internally club* if $P_\kappa(N) \cap N$ is club in $P_\kappa(N)$.
- N is *internally approachable* if there is an increasing and continuous chain $\langle M_\xi : \xi < \kappa \rangle$ such that $|M_\xi| < \kappa$ and $\langle M_\eta : \eta < \xi \rangle \in M_{\xi+1}$ for all $\xi < \kappa$ such that $N = \bigcup_{\xi<\kappa} M_\xi$.

Although disjoint stationary sequences may seem unrelated to the separation of variants of internal approachability, there are deep connections here, for example:

Fact 6 (Krueger [21]). If μ is regular and $2^\mu = \mu^+$, then $\mathsf{DSS}(\mu^+)$ is equivalent to the existence of a stationary set $U \subseteq [H(\mu^+)]^\mu$ such that every $N \in U$ is internally unbounded but not internally club.

1.2. ***Projections and preservation lemmas.*** Technically speaking, our main goal is to show that certain forcing quotients behave nicely. We will make an effort to demonstrate the preservation properties of these quotients directly. These quotients will be defined in terms of projections:

[2] See Jech's book [15] for details on stationary sets.

Definition 7. If $\mathbb{P}_1$ and $\mathbb{P}_2$ are posets, a *projection* is an onto map $\pi : \mathbb{P}_1 \to \mathbb{P}_2$ such that

- $p \leq q$ implies that $\pi(p) \leq \pi(q)$,
- if $r \leq \pi(p)$, then there is some $q \leq p$ such that $\pi(q) = r$.

A projection is *trivial* if for all $p, q \in \mathbb{P}_1$, if $\pi(p)$ and $\pi(q)$ are compatible, then p and q are compatible.

Trivial projections are basically isomorphisms:

Fact 8. If $\pi : \mathbb{P}_1 \to \mathbb{P}_2$ is a trivial projection, then $\mathbb{P}_1 \simeq \mathbb{P}_2$, that is, $\mathbb{P}_1$ and $\mathbb{P}_2$ are forcing-equivalent.

For our purposes, we are interested in the preservation of stationary sets. The chain condition gives us preservation fairly straightforwardly. The following fact is implicit in parts of the literature, and a version of it can be found in the form of Proposition 26.

Fact 9. If $\mathbb{P}$ has the μ-chain condition and $S \subset P_\mu(X)$ is stationary, then $\mathbb{P}$ forces that S is stationary in $P_\mu(X)$.

However, we must place demands on our stationary sets in order for them to be preserved by closed forcings.

Definition 10. Consider a regular uncountable cardinal μ and a stationary set $S \subset P_\mu(X)$. We say that S is *internally approachable of length* τ if for all $N \in S$ with $N \prec H(X)$, there is a continuous chain of elementary submodels $\langle M_i : i < \tau \rangle$ such that: $N = \bigcup_{i<\tau} M_i$ and for all $i < \tau$, $\langle M_i : i < j \rangle \in N$. In this case we write $S \subseteq \mathcal{IA}(\tau)$.

Here we are following Krueger's convention [21], which withholds the requirement that $|M_i| < \mu$ for $i < \tau$.

Fact 11. If $S \subset P_\mu(X) \cap \mathcal{IA}(\tau)$ is an internally approachable stationary set, $\tau < \mu$, and $\mathbb{P}$ is μ-closed, then $\mathbb{P}$ forces that S is stationary in $P_\mu(X)^V$).[3]

1.3. ***Costationarity of the ground model.*** The notion of ground model costationarity is a key ingredient in arguments pertaining to disjoint stationary sequences. It will specifically give us the disjointness, since we will be picking stationary sets that are not added by initial segments of these forcings.

Gitik obtained the classical result:

Fact 12 (Gitik [12]). If $V \subset W$ are models of ZFC with the same ordinals, $W \setminus V$ contains a real, and κ is a regular cardinal in W such that $(\kappa^+)^W \leq \lambda$, then $P_\kappa^W(\lambda) \setminus V$ is stationary as a subset of $P_\kappa(\lambda)$ in the model W.

[3]See [7] for discussion of related facts.

Because we will need Fact 11, we will actually use Krueger's refinement of Gitik's theorem:

Fact 13 (Krueger [21]). Suppose $V \subset W$ are models of ZFC with the same ordinals, $W \setminus V$ contains a real, μ is a regular cardinal in W, and $X \in V$ is such that $(\mu^+)^W \subseteq X$, and that in W, Θ is a regular cardinal such that $X \subset H(\Theta)$. Then in W the set $\{N \in P_\mu(H(\Theta)) \cap \mathcal{IA}(\omega) : N \cap X \notin V\}$ is stationary.

2. The new Mitchell forcing

2.1. ***Defining the forcing.*** Here we will illustrate the basic idea of this paper by using our new take on Mitchell forcing to prove a known result:

Fact 14 (Krueger [21]). If λ is a Mahlo cardinal and $\mu < \lambda$ is regular, there is a forcing extension in which $2^\omega = \mu^+ = \lambda$ and there is a disjoint stationary sequence on λ.

Specifically, we will define a forcing $\mathbb{M}^+(\tau, \mu, \lambda)$ such that the model W in Fact 14 can be realized as an extension by $\mathbb{M}^+(\omega, \mu, \lambda)$.

For standard technical reasons, we define a poset isomorphic to $\text{Add}(\tau, \lambda)$:

Definition 15. Given a regular τ and a set of ordinals Y, we let $\text{Add}^*(\tau, Y)$ be the poset consisting of partial functions $p : \{\delta \in Y : \delta \text{ is inaccessible}\} \times \tau \to \{0, 1\}$ where $|\text{dom}\, p| < \tau$. We let $p \leq_{\text{Add}^*(\tau,Y)} q$ if and only if $p \supseteq q$.

In later subsections we will conflate $\text{Add}(\tau, \lambda)$ and $\text{Add}^*(\tau, \lambda)$ to simplify notation.

Definition 16. Let λ be inaccessible and let $\tau < \mu < \lambda$ be regular cardinals such that $\tau^{<\tau} = \tau$. We define a forcing $\mathbb{M}^+(\tau, \mu, \lambda)$ that consists of pairs (p, q) such that

(1) $p \in \text{Add}^*(\tau, \lambda)$;

(2) q is a function such that

(a) $\text{dom}\, q \subset \{\delta < \lambda : \delta \text{ is inaccessible}\}$,

(b) $|\text{dom}\, q| < \mu$,

(c) for all $\delta \in \text{dom}\, q$, $q(\delta)$ is an $\text{Add}^*(\tau, \delta + 1)$-name such that

$$p \restriction ((\delta + 1) \times \tau) \Vdash_{\text{Add}^*(\tau,\delta+1)} \text{“}q(\delta) \in \dot{\text{Col}}(\mu, \delta)\text{”}.$$

We let $(p, q) \leq (p', q')$ if and only if

(i) $p \leq_{\text{Add}^*(\tau,\lambda)} p'$,

(ii) $\text{dom}\, q \supseteq \text{dom}\, q'$,

(iii) for all $\delta \in \text{dom}\, q'$, $p \restriction ((\delta + 1) \times \tau) \Vdash_{\text{Add}^*(\tau,\delta+1)} \text{“}q(\delta) \leq_{\dot{\text{Col}}(\mu,\delta)} q'(\delta)\text{”}$.

First we go through the more routine properties that one would expect of this forcing.

Proposition 17. $\mathbb{M}^+(\tau, \mu, \lambda)$ *is* τ*-closed and* λ*-Knaster.*

Proof. Closure uses the facts that $\mathrm{Add}^*(\tau, \lambda)$ is τ-closed and $\Vdash_{\mathrm{Add}^*(\tau,\delta+1)}$ "$\dot{\mathrm{Col}}(\mu, \delta)$ is μ-closed" for all δ. For Knasterness: consider $\{(p_i, q_i) : i < \lambda\} \subseteq \mathbb{M}^+(\tau, \mu, \lambda)$, then fix a regular $\rho \in (\mu, \lambda)$ and find a stationary subset of $\lambda \cap \mathrm{cof}(\rho)$ on which $\mathrm{dom}\, p_i$, $\mathrm{dom}\, q_i$ are fixed, and then proceed with a standard delta system lemma argument. □

Crucially, we get a nice termspace:

Definition 18. Let $\mathbb{T} = \mathbb{T}(\mathbb{M}^+(\tau, \mu, \lambda))$ be the poset consisting of conditions q such that

(1) $\mathrm{dom}\, q \subset \{\delta < \lambda : \delta \text{ is inaccessible}\}$,

(2) $|\mathrm{dom}\, q| < \mu$,

(3) for all $\delta \in \mathrm{dom}\, q$, $\Vdash_{\mathrm{Add}^*(\tau,\delta+1)}$ "$q(\delta) \in \dot{\mathrm{Col}}(\mu, \delta)$".

Most importantly, we let $q \leq q'$ if and only if

(i) $\mathrm{dom}\, q \supseteq \mathrm{dom}\, q'$,

(ii) for all $\delta \in \mathrm{dom}\, q$, $\Vdash_{\mathrm{Add}^*(\tau,\delta+1)}$ "$q(\delta) \leq q'(\delta)$".

Proposition 19. *There is a projection* $\mathrm{Add}^*(\tau, \lambda) \times \mathbb{T}(\mathbb{M}^+(\tau, \mu, \lambda)) \twoheadrightarrow \mathbb{M}^+(\tau, \mu, \lambda)$.

Proof. We let π be the projection with the definition $\pi(p, q) = (p, q)$. This is automatically order-preserving because the ordering $\leq_{\mathrm{Add}^*(\tau,\lambda)\times\mathbb{T}}$ is coarser than the ordering $\leq_{\mathbb{M}^+(\tau,\mu,\lambda)}$. For obtaining the density condition, suppose $(r, s) \leq_{\mathbb{M}^+(\tau,\mu,\lambda)} (p_0, q_0)$. We want to find some (p_1, q_1) such that $(p_1, q_1) \leq_{\mathrm{Add}^*(\tau,\lambda)\times\mathbb{T}} (p_0, q_0)$ and $(p_1, q_1) \leq_{\mathbb{M}^+(\tau,\mu,\lambda)} (r, s)$. To do this, we first let $p_1 = r$, and then we define q_1 with $\mathrm{dom}\, q_1 = \mathrm{dom}\, r$ such that at each coordinate $\delta \in \mathrm{dom}\, q_1$, we use standard arguments on names to show that we can get both $p_0 \restriction ((\delta + 1) \times \tau) \Vdash_{\mathrm{Add}^*(\tau,\lambda)}$ "$q_1(\delta) \leq s(\delta)$" as well as $1_{\mathrm{Add}^*(\tau,\lambda)} \Vdash_{\mathrm{Add}^*(\tau,\lambda)}$ "$q_1(\delta) \leq q_0(\delta)$". □

Proposition 20. $\mathbb{T} = \mathbb{T}(\mathbb{M}^+(\tau, \mu, \lambda))$ *is* μ*-closed.*

Proof. This is an application of the mixing principle. Given a $\leq_{\mathbb{T}}$-decreasing sequence $\langle q_i : i < \tau \rangle$ with $\tau < \mu$ we let $d = \bigcup_{i<\tau} \mathrm{dom}\, q_i$. Then we define a lower bound $\bar{q}$ with domain d such that for all $\delta \in d$, $q(\delta)$ is a canonically defined name for a lower bound of the $q_i(\delta)$'s (where i is large enough that $\delta \in \mathrm{dom}\, q_i$). □

Then we get the standard consequences of the termspace analysis:

Proposition 21. *The following are true in any extension by* $\mathbb{M}^+(\tau, \mu, \lambda)$:

(1) *V-cardinals up to and including μ are cardinals.*

(2) *For all $\alpha < \lambda$, $|\alpha| = \mu$.*

(3) $\lambda = \mu^+$.

(4) $2^\tau = \lambda$.

Proof. (1) follows from the projection analysis and the fact that $\mathbb{T}$ is μ-closed and $\mathrm{Add}^*(\tau, \lambda)$ is τ^+-cc, and from τ-closure of $\mathbb{M}^+(\tau, \mu, \lambda)$. (2) follows from the fact that for all inaccessible $\delta < \lambda$, $\mathbb{M}^+(\tau, \mu, \lambda)$ projects onto $\mathrm{Add}^*(\tau, \delta) * \dot{\mathrm{C}}\mathrm{ol}(\mu, \delta)$. (3) follows from (1) and (2) plus λ-Knasterness. (4) follows from the fact that $\mathbb{M}^+(\mu, \lambda)$ projects onto $\mathrm{Add}^*(\tau, \lambda)$, so it forces that $2^\tau \geq \lambda$. Since the poset has size λ and λ is inaccessible, it also forces that $2^\tau \leq \lambda$. □

The following lemma is the crux of the new idea.

Lemma 22. *If $\delta_0 < \lambda$ is inaccessible, then there is a forcing equivalence*

$$\mathbb{M}^+(\tau, \mu, \lambda) \simeq \mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau) * \mathbb{E},$$

where $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$ forces that $\mathbb{E}$ is a projection of a product of a μ-closed forcing and a τ^+-cc forcing.

Proof. In particular, we will show that there is a forcing equivalence $\mathbb{M}^+(\tau, \mu, \lambda) \simeq \mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau) * (\mathbb{F} \times \mathbb{G})$ where, in the extension by $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$,

- $\mathbb{G}$ is a projection of a product of a μ-closed forcing and $\mathrm{Add}^*(\tau, \lambda)$, and
- $\mathbb{F}$ is μ-closed.

The statement of the lemma can then be obtained by merging $\mathbb{F}$ with the closed component of the product that projects onto $\mathbb{G}$.

First we describe $\mathbb{F}$ and $\mathbb{G}$. To do this, we fix some notation. Given $Y \subseteq \lambda$, we let π^Y_{Add} denote the projection $(p, q) \mapsto p \restriction (Y \times \tau)$ from $\mathbb{M}^+(\tau, \mu, \lambda)$ onto $\mathrm{Add}^*(\tau, Y)$. For any poset $\mathbb{P}$, we employ the convention that $\Gamma(\mathbb{P})$ denotes a canonical name for a $\mathbb{P}$-generic. If $X \subset \mathbb{P}$, then we use the notation $\uparrow X := \{q \in \mathbb{P} : \exists p \in X, p \leq q\}$.

We will let

$$\mathbb{F} := \mathrm{Col}(\mu, \delta_0)^{V[(\uparrow(\pi^{\delta_0}_{\mathrm{Add}}\text{“}\Gamma(\mathbb{M}^+(\tau,\mu,\delta_0)))) \times \Gamma(\mathrm{Add}(\tau))]}$$

if we are working in an extension by $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$. (In other words, the poset $\mathbb{F}$ will be the version of $\mathrm{Col}(\mu, \delta_0)$ as interpreted in the extension of V by $\mathrm{Add}^*(\tau, \delta_0 + 1)$ where the initial coordinates come from $\mathbb{M}^+(\tau, \mu, \delta_0)$ and the last coordinate comes from the additional copy of $\mathrm{Add}(\tau)$ that occupies the coordinate δ_0 in $\mathrm{Add}^*(\tau, \delta_0 + 1)$.)

Still working in an extension by $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau)$, the poset $\mathbb{G}$ consists of pairs (p, q) such that

(1) $p \in \mathrm{Add}^*(\tau, (\delta_0, \lambda))$;

(2) q is a function such that

(a) $\mathrm{dom}\, q \subset \{\delta \in (\delta_0, \lambda) : \delta \text{ is inaccessible}\}$,

(b) $|\mathrm{dom}\, q| < \mu$,

(c) for all $\delta \in \mathrm{dom}\, q$, $p \restriction \big((\delta_0, (\delta+1)) \times \tau\big) \Vdash_{\mathrm{Add}^*(\tau,(\delta_0,\delta+1))}$ "$q(\delta) \in \dot{\mathrm{C}}\mathrm{ol}(\mu, \delta)$".

The ordering is the one analogous to that of $\mathbb{M}^+(\tau, \mu, \lambda)$. An easy adaptation of the arguments for the projection analysis for $\mathbb{M}^+(\tau, \mu, \lambda)$ will then give a projection analysis for $\mathbb{G}$.

The rest of the proof of the lemma consists of verifying the more substantial claims.

Claim 23. $\mathbb{M}^+(\tau, \mu, \lambda) \simeq \mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau, 1) * (\mathbb{F} \times \mathbb{G})$.

Proof. We identify $\mathbb{M}^+(\tau, \mu, \delta_0) * \mathrm{Add}(\tau, 1) * (\mathbb{F} \times \mathbb{G})$ with the dense subset of conditions $((r, s), t, u, (\dot{r}', \dot{s}'))$ such that $\dot{s}'$ is forced to have a specific domain in V. The fact that this subset is dense follows from the fact that $\mathbb{M}^+(\tau, \mu, \lambda) * \mathrm{Add}(\tau, 1)$ has the μ-covering property.

We will argue that there is a trivial projection defined by

$$\pi : (p, q) \mapsto \Big(\underbrace{(p \restriction (\delta_0 \times \tau), q \restriction \delta_0)}_{\mathbb{M}^+(\mu,\delta_0)}, \underbrace{p \restriction (\{\delta_0\} \times \tau)}_{\mathrm{Add}(\tau)}, \underbrace{q^*(\delta_0)}_{\mathbb{F}}, \underbrace{(\bar{p}, \bar{q})}_{\mathbb{G}}\Big)$$

such that

- $\bar{p} := p \restriction ((\delta_0, \lambda) \times \tau)$;
- $q^*(\delta_0)$ is obtained by changing $q(\delta_0)$ from an $\mathrm{Add}^*(\tau, \delta_0 + 1)$-name to an $\mathrm{Add}(\tau)$-name as interpreted in the extension by the relevant generic, namely $\big(\uparrow\big(\pi^{\delta_0}_{\mathrm{Add}}``\Gamma(\mathbb{M}^+(\tau, \mu, \delta_0))\big)\big)$;
- $\bar{q}$ has domain (δ_0, λ), and for each $\delta \in (\delta_0, \lambda)$, $\bar{q}(\delta)$ has changes analogous to the changes made to $q^*(\delta_0)$.

It is clear that π is order-preserving. We also want to show that if

$$((r, s), t, u, (\dot{r}', \dot{s}')) \leq_{\mathbb{M}^+(\tau,\mu,\delta_0)*\mathrm{Add}(\tau)*(\mathbb{F}\times\mathbb{G})} \pi(p_0, q_0)$$

then there is some $(p_1, q_1) \leq_{\mathbb{M}^+(\mu,\lambda)} (p_0, q_0)$ such that we have $\pi(p_1, q_1) \leq ((r, s), t, u, (\dot{r}', \dot{s}'))$. This can be done by taking

- $p_1 = r^* \cup \tilde{t} \cup r'$ where $r^* \leq r$ decides t and $\dot{r}'$ and $\tilde{t}$ writes t as a partial function $\{\delta\} \times \tau \to \{0, 1\}$,
- $q_1 = s \cup \tilde{u} \cup \tilde{s}'$ where $\tilde{u}$ reinterprets u as an $\mathrm{Add}^*(\delta_0 + 1)$-name and for each $\delta \in \mathrm{dom}\, \dot{s}'$, $\tilde{s}'$ reinterprets $\dot{s}'(\delta)$ as an $\mathrm{Add}^*(\delta + 1)$-name.

Last, we argue that $\pi(p_0, q_0) = \pi(p_1, q_1)$ implies that (p_0, q_0) and (p_1, q_1) are compatible. Suppose that (p_0, q_0) and (p_1, q_1) are incompatible. If p_0 and p_1 are incompatible as elements of $\mathrm{Add}^*(\tau, \lambda)$, then one of $p_i \restriction (\delta_0 \times \tau)$, $p_i \restriction (\{\delta_0\} \times \tau)$, and $p_i \restriction ((\delta_0, \lambda) \times \tau)$ must be distinct for $i = 0$ and $i = 1$. Otherwise, there is some $p' \leq p_0, p_1$ and some $\delta \in \mathrm{dom}\, q_0 \cap \mathrm{dom}\, q_1$ inaccessible such that $p' \Vdash$ "$q_0(\delta) \perp q_1(\delta)$", which implies that $q_0(\delta) \neq q_1(\delta)$. Therefore, one of $q_i \restriction \delta_0$, $q_i(\delta_0)$, or $q_i \restriction (\delta_0, \lambda)$ is distinct for $i \in \{0, 1\}$. $\square$

Claim 24. $\Vdash_{\mathbb{M}^+(\tau,\mu,\delta_0)*\mathrm{Add}(\tau,1)}$ "$\mathbb{F}$ *is* μ*-closed*".

Proof. In fact, our argument will also show that

$$\Vdash_{\mathbb{M}^+(\tau,\mu,\delta_0)*\mathrm{Add}(\tau,1)} \text{“}\mathbb{F} = \mathrm{Col}(\mu, \delta_0)\text{”}.$$

We fix some arbitrary generics:

- G is $\mathbb{M}^+(\tau, \mu, \delta_0)$-generic over V,
- r is $\mathrm{Add}(\tau, 1)$-generic over $V[G]$,
- H is the $\mathrm{Add}^*(\tau, \delta_0)$-generic induced from G by $\pi^{\delta_0}_{\mathrm{Add}}$,
- K is the generic for the quotient of $\mathbb{M}^+(\tau, \mu, \delta_0)$ by $\mathrm{Add}^*(\tau, \delta_0)$, i.e., the generic such that $V[H][K] = V[G]$,
- T is the generic for the termspace forcing $\mathbb{T}(\mathbb{M}^+(\tau, \mu, \delta_0))$, so that $V[G] \subset V[T][H]$.

It is enough to argue that $V[G][r] \models$ "$\mathbb{F}$ is μ-closed" knowing that $V[H][r] \models$ "$\mathbb{F}$ is μ-closed". Because adjoining G does not change the definition of $\mathrm{Add}(\tau, 1)$, and because K is defined in terms of the subsets of τ adjoined by the filter H, we have $V[G][r] = V[H][K][r] = V[H][r][K]$. Therefore, it is enough to show that K does not add $<\mu$-sequences over $V[H][r]$, so that $V[H][r]$'s version of $\mathrm{Col}(\mu, \delta_0)$ remains μ-closed in $V[G][r]$. We have

$$V[H][r] \subset V[H][r][K] = V[H][K][r] = V[G][r] \subset V[T][H][r] = V[H][r][T].$$

Recall Easton's lemma, which states in part that if $\mathbb{A}$ is μ-cc and $\mathbb{B}$ is μ-closed, then $\Vdash_{\mathbb{A}}$ "$\mathbb{B}$ is μ-distributive". Easton's lemma implies that T does not add new $<\mu$-sequences over $V[H][r]$ since the forcing adjoining r is μ-cc over $V[H]$ and the forcing adjoining T is μ-closed over $V[H]$. Therefore K does not add new $<\mu$-sequences over $V[H][r]$ since it is an intermediate factor of the extension. $\square$

This completes the proof of the lemma. $\square$

Now we have an application for the case where $\tau = \omega$.

Proposition 25. *If λ is Mahlo, then* $\Vdash_{\mathbb{M}^+(\omega,\mu,\lambda)} \mathsf{DSS}(\lambda)$.

This basically repeats Krueger's argument for [21, Theorem 9.1].

Proof. Let G be $\mathbb{M}^+(\omega, \mu, \lambda)$-generic over V. The set of V-inaccessibles in λ will form the stationary set $S \subset \mu^+ \cap \mathrm{cof}(\mu)$ carrying the disjoint stationary sequence in the extension by $\mathbb{M}^+(\omega, \mu, \lambda)$. For every such $\delta \in S$, let $\overline{G}$ be the generic on $\mathbb{M}^+(\omega, \mu, \delta)$ induced by G and let r be the $\mathrm{Add}(\omega)$-generic induced by G via $\pi^{\{\delta\}}_{\mathrm{Add}}$. We use Fact 13 to obtain a stationary set $\mathcal{S}^*_\delta \subset P_\mu(H(\delta))^{V[\overline{G}][r]}$ such that for all $N \in \mathcal{S}^*_\delta$, $N \cap \delta \notin V[\bar{G}]$ and such that N is also internally approachable by a ω-sequence. Therefore we can apply Lemma 22 with Fact 11 and then Fact 9 to find that $\mathcal{S}^*_\delta$ is stationary in $V[G]$. We then let $\mathcal{S}_\delta = \{N \cap \delta : N \in \mathcal{S}^*_\delta\}$, and we see that $\langle \mathcal{S}_\delta : \delta \in S\rangle$ is a disjoint stationary sequence. □

2.2. ***Proving the main theorems.*** Now we will apply the new version of Mitchell forcing to answer Krueger's questions. We can readily prove Theorem 1, which states that we can obtain $\mathsf{DSS}(\aleph_2) \wedge \mathsf{DSS}(\aleph_3)$:

Proof of Theorem 1. Begin with a ground model V in which $\lambda_1 < \lambda_2$ and the λ's are Mahlo. Let $\mathbb{M}_1 = \mathbb{M}^+(\omega, \aleph_1, \lambda_1)$. (Any λ_1-sized forcing that turns λ_1 into $\aleph_2$ and adds a disjoint stationary sequence on $\aleph_2$ would work, so we could also use a more standard mixed support iteration.) Then let $\dot{\mathbb{M}}_2$ be an $\mathbb{M}_1$-name for $\mathbb{M}^+(\omega, \lambda_1, \lambda_2)$. We argue that if G_1 is $\mathbb{M}_1$-generic over V and G_2 is $\dot{\mathbb{M}}_2[G_1]$-generic over $V[G_1]$, then $V[G_1][G_2] \models$ "$\mathsf{DSS}(\lambda_1) \wedge \mathsf{DSS}(\lambda_2)$". We get $\mathsf{DSS}(\lambda_2)$ from the fact that λ_2 remains Mahlo in $V[G_1]$ together with Proposition 25, so we only need to argue that the disjoint stationary sequence $\vec{\mathcal{S}} := \langle \mathcal{S}_\alpha : \alpha \in S\rangle \in V[G_1]$ remains a disjoint stationary sequence in $V[G_1][G_2]$.

Working in $V[G_1]$, preservation of $\vec{\mathcal{S}}$ follows from the projection analysis: Let H_1 and H_2 be chosen so that H_1 is $\mathbb{T} := \mathbb{T}(\mathbb{M}_2)$-generic over $V[G_1]$, H_2 is $\mathrm{Add}(\omega, \lambda_2)^{V[G_1]}$-generic over $V[G_1][H_1]$, and $V[G_1][G_2] \subseteq V[G_1][H_1][H_2]$. Since $\mathbb{T}$ is λ_1-closed, it preserves stationarity of S and the $\mathcal{S}_\alpha$'s, and $\mathrm{Add}(\omega, \lambda_2)^{V[G_1]}$ still has the countable chain condition in $V[G_1][H_1]$. It follows that the stationarity of S is preserved in $V[G_1][H_1][H_2]$, as well as the stationarity of the $\mathcal{S}_\alpha$'s (by Fact 9). Therefore $\vec{\mathcal{S}}$ is a disjoint stationary sequence on λ_1 in $V[G_1][G_2]$. □

It will take a bit more work to show how to obtain Theorem 2 in the same model for Theorem 1. (Recall that Theorem 2 states that we can simultaneously separate internally stationary and internally club for $[H(\aleph_2)]^{\aleph_1}$ and $[H(\aleph_3)]^{\aleph_2}$.) Note that we cannot just apply Fact 6 because $2^\omega = \aleph_3$ in the model for Theorem 1, plus it is consistent that there can be a stationary set which is internally unbounded but not internally stationary [19].

We will give some facts on preservation of the distinction between stationary sets that are internally stationary but not internally club:

Proposition 26. *Suppose $\mathbb{P}$ is ν-closed and $S \subseteq [X]^\delta$ is a stationary set such that $|[X]^\delta| \leq \nu$ and $|X| > 1$. Then $\Vdash_{\mathbb{P}}$ "S is stationary in $[X]^\delta$".*

Proof. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in $[X]^\delta$ and let $\vec{x} = \langle x_\xi : \xi \leq \bar{\nu}\rangle$ enumerate $[X]^\delta$ (where $\bar{\nu} \leq \nu$). Note that we have $\delta < 2^\delta \leq |X|^\delta \leq \nu$, so conditions in $\mathbb{P}$ can decide names for elements of $\dot{C}$. We construct a sequence $\vec{z} = \langle z_\xi : \xi < \bar{\nu}\rangle \subseteq [X]^\delta$ and a $\leq_{\mathbb{P}}$-descending sequence $\langle p_\xi : \xi < \bar{\nu}\rangle$ using the closure of $\mathbb{P}$ such that for all $\xi < \bar{\nu}$, $p_\xi \Vdash$ "$x_\xi \subseteq z_\xi \in \dot{C}$" and $p_\xi \parallel$ "$x_\xi \in \dot{C}$".

Then let D be the set $\{x_\xi : \exists \zeta < \bar{\nu}, p_\zeta \Vdash \text{“}x_\xi \in \dot{C}\text{”}\}$. We can argue that D is a club: It is unbounded because of the sets chosen for z_ξ. It is closed because if $\langle x_{\xi_i} : i < \bar{\delta}\rangle \subseteq D$ (for $\bar{\delta} \leq \delta$) is an $\subseteq$-increasing sequence such that we have $p_{\zeta_i} \Vdash \text{“}x_{\xi_i} \in \dot{C}\text{”}$, and $\zeta^* = \sup_{i<\bar{\delta}} \zeta_i$, then $p_{\zeta^*} \Vdash \text{“}\bigcup_{i<\bar{\delta}} x_{\xi_i} \in \dot{C}\text{”}$.

There is some $w \in D \cap S$. If p_ξ is such that $p_\xi \Vdash \text{“}w \in \dot{C}\text{”}$, then we have $p_\xi \Vdash \text{“}\dot{C} \cap S \neq \varnothing\text{”}$. □

Proposition 27. *Suppose $|[H(\theta)]^\delta| \leq \nu$. Let $\mathbb{F}$ have the δ-chain condition and let $\mathbb{G}$ be ν-closed. If there is a stationary set $S \subseteq [H(\theta)]^\delta$ consisting of sets that are internally stationary but not internally club, then $\mathbb{F} \times \mathbb{G}$ forces that there is a stationary set consisting of sets that are internally stationary but not internally club.*

Proof. Since $\mathbb{G}$ preserves the chain condition of $\mathbb{F}$, we show that preservation of the distinction can be achieved by forcing with $\mathbb{G}$ and then $\mathbb{F}$. The poset $\mathbb{G}$ preserves the distinction by Proposition 26 and the fact that it does not change $H(\theta)$.

Now we argue that $\mathbb{F}$ preserves the distinction. Let S be the witnessing stationary set in V and let $X = H(\theta)^V$. If G is $\mathbb{F}$-generic over V, let $Y = H(\theta)^{V[G]}$ and let $S^* = \{M \in [Y]^\delta : M \cap X \in S\}$. We will argue that S^* witnesses the relevant statement in $V[G]$. Let $\dot{S}^*$ be a name for S^*.

To see that $\dot{S}^*$ is forced to be stationary, let $\dot{C}$ be a name for a club in $[\dot{Y}]^\delta$. Given $p \in \mathbb{F}$, let $D = \{z : \exists \dot{w}, p \Vdash \text{“}\dot{w} \in \dot{C}\text{”}, \dot{w} \cap X = z\}$. Then D is a club in $[X]^\delta$ as regarded in V, so there is some $z \in S$, and hence $p \Vdash \text{“}\dot{w} \in \dot{C} \cap \dot{S}^*\text{”}$.

Next we argue that members of $\dot{S}^*$ are forced not to be internally club. Suppose for contradiction, then, that p forces $\dot{M} \in \dot{S}^*$ to be internally club as witnessed by $\dot{c}$, and also that $N = \dot{M} \cap X$ where $N \in S$. Let $d = \{z : \exists \dot{w}, p \Vdash \text{“}\dot{w} \in \dot{c}\text{”}, \dot{w} \cap X = z\}$. Then d is a club in $P_\mu(N)$ since if $a \in z \subseteq N$ then $a \in N$ and if $p \Vdash \text{“}\dot{w} \cap X = z\text{”}$ then in particular $p \Vdash \text{“}\dot{w} \in \dot{M}\text{”}$, so $z \in N$. This contradicts the fact that N consists of sets that are not internally club.

Finally, we argue that $\dot{S}^*$ is forced to be internally stationary. Let $\dot{M}$ be forced by p to be in $\dot{S}^*$ and that $N = \dot{M} \cap X$. Let G be generic with $p \in G$ and work in $V[G]$. Then $\{w \subseteq P_\delta(M) : \exists z \in N, w = z[G]\}$ is a club as regarded in $V[G]$. As in the argument for stationarity, any name $\dot{c}$ for a club in $P_\delta(\dot{M})$ can produce a corresponding club d in the ground model. Then we can find some $z \in N \cap d$ and if G is generic with $p \in G$ then $z[G] \in c \cap M$. □

We use a concept from Harrington and Shelah to handle Mahlo cardinals [13]:

Definition 28. Let λ be Mahlo and let $\mathcal{N}$ be a model of some fragment of ZFC. We say that $\mathcal{M} \prec \mathcal{N}$ is *rich* if

(1) $\lambda \in \mathcal{M}$;

(2) $\bar{\lambda} := \mathcal{M} \cap \lambda \in \lambda$;

(3) $\bar{\lambda}$ is an inaccessible cardinal in $\mathcal{N}$;

(4) the size of $\mathcal{M}$ is $\bar{\lambda}$;

(5) $\mathcal{M}$ is closed under $<\bar{\lambda}$-sequences and $\bar{\lambda} < \lambda$.

Lemma 29. *If λ is Mahlo, then $\mathbb{M}^+(\omega, \mu, \lambda)$ forces that there are stationarily many $Z \in [\mu^+]^\mu$ which are internally stationary but not internally club.*

This follows Krueger's proof of [21, Theorem 10.1], making necessary changes for Mahlo cardinals, and including enough details to show that we can get the necessary preservation of stationarity simply from the projection analysis. We do not need guessing functions (which are used in Krueger's argument) because we are only obtaining one instance of separation per large cardinal.

Proof of Lemma 29. Define $\mathbb{M} := \mathbb{M}^+(\omega, \mu, \lambda)$ and let $\dot{C}$ be an $\mathbb{M}$-name for a club in $\left([H(\mu^+)]^\mu\right)^{V[\mathbb{M}]}$. We want to find an $\mathbb{M}$-name $\dot{Z}$ for an element of $\left([H(\mu^+)]^\mu\right)^{V[\mathbb{M}]} \cap \dot{C}$ that is internally stationary but not internally club. Let $\dot{F}$ be an $\mathbb{M}$-name for a function $(H(\mu^+)^{V[\mathbb{M}]})^{<\omega} \to H(\mu^+)^{V[\mathbb{M}]}$ with the property that all of its closure points are in $\dot{C}$. Let Θ be as large as needed for the following discussion and let $\mathcal{N}$ be the structure $(H(\Theta), \in, <_\Theta, \mathbb{M}, \dot{F}, \lambda, \mu)$ where $<_\Theta$ is a well-ordering of $H(\Theta)$.

Since λ is Mahlo, we can find some $\mathcal{K} \prec \mathcal{N}$ with $\mu \subset \mathcal{K}$ that is a rich submodel of cardinality $\bar{\lambda}$. Now set G to be $\mathbb{M}$-generic over V. Note that $H(\lambda)^{V[G]} = H(\lambda)[G]$ because $\mathbb{M}$ has the λ-chain condition and $\mathbb{M} \subset H(\lambda)$. We will argue that $Z := \mathcal{K}[G] \cap H(\lambda)[G]$ is what we are looking for.

Claim 30. $$Z \in C := \dot{C}[G].$$

Proof. We have $\bar{\lambda} \le |Z| \le |\mathcal{K}| \le \bar{\lambda}$ and $\bar{\lambda}$ has cardinality μ in $\mathcal{N}[G]$, so $Z \in [H(\lambda)^{V[G]}]^\mu$. If $a_1, \dots, a_n \in Z$, there are $\mathbb{M}$-names $\dot{b}_1, \dots, \dot{b}_n \in \mathcal{K} \cap H(\lambda)$ such that $a_i = \dot{b}_i^G$ for all $1 \le i \le n$. By elementarity, $\mathcal{K}$ contains the $<_\Theta$-least maximal antichain $A \subset \mathbb{M}$ of conditions deciding $\dot{F}(\dot{b}_1, \dots, \dot{b}_n)$. Since $|A| < \lambda$, we have $|A| \in \mathcal{K} \cap \lambda = \bar{\lambda}$, so it will follow that $A \subset \mathcal{K}$. Therefore if $p \in G \cap A$, then $p \in M$ in particular, so $p \Vdash \dot{F}(\dot{b}_1, \dots, \dot{b}_n) = \dot{b}_*$ for some $\dot{b}_* \in \mathcal{K} \cap H(\lambda)$ where we automatically get $\dot{b}_* \in H(\bar{\lambda})$, and therefore

$$F(a_1, \dots, a_n) = a_* := \dot{b}_*^G \in \mathcal{K}[G] \cap H(\lambda)[G] = Z$$

(where of course $F := \dot{F}[G]$). □

For the rest of the proof let $\overline{G} := \pi_{\mathcal{K}}(G)$ where $\pi_{\mathcal{K}}$ is the Mostowski collapse relative to $\mathcal{K}$. Since $\pi_{\mathcal{K}}(\mathbb{M}) = \mathbb{M}^+(\omega, \mu, \bar{\lambda})$, there is an extension $\pi_{\mathcal{K}} : \mathcal{K}[G] \cong \pi_{\mathcal{K}}(\mathcal{K})[\overline{G}]$. We also define $h := \pi_{\mathcal{K}}(H(\lambda)[G] \cap \mathcal{K}[G])$.

Claim 31. *Z is internally stationary.*

Proof. First, we argue that $S := P_\mu(h)^{\mathcal{N}[\overline{G}]}$ is stationary as a subset of $P_\mu(h)^{\mathcal{N}[G]}$ in $\mathcal{N}[G]$. By Lemma 22, the quotient $\mathbb{M}/\overline{G}$ is a projection of a forcing of the form

$\mathbb{A}_1 * (\dot{\mathbb{T}} \times \mathbb{A}_2)$ where $\mathbb{A}_1$ has the countable chain condition, $\dot{\mathbb{T}}$ is an $\mathbb{A}_1$-name for a μ-closed forcing, and $\mathbb{A}_2$ also has the countable chain condition. Let K_1, K_T, and K_2 be respective generics such that $V[G] \subseteq V[\bar{G}][K_1][K_T][K_2]$. Working in $\mathcal{N}[\bar{G}]$, note that $S' := S \cap \mathcal{IA}(\omega)$ is stationary, and therefore has its stationarity preserved in $V[\bar{G}][K_1]$ by Fact 9.

We must also show that the stationarity of S' will be preserved by countably closed forcings over $\mathcal{N}[\bar{G}][K_1]$. Suppose $\langle M_n : n < \omega\rangle$ witnesses internal approachability of some $N \in S'$ in $V[\bar{G}]$ with respect to the structure $H(\lambda^+)^{V[\bar{G}]}$, and let $M_\omega := \bigcup_{n<\omega} M_n$. Then we can see that $\langle M_n[K_1] : n < \omega\rangle$ is a chain of elementary submodels of $H(\lambda)[\bar{G}][K_1] = H(\lambda)^{V[\bar{G}][K_1]}$. We also have $M_n[K_1] \cap V[\bar{G}] = M$ and $M_\omega[K_1] \cap V[\bar{G}] = M_\omega \in S'$ with $M_\omega[K_1] \prec H(\lambda)^{V[\bar{G}][K_1]}$. If we choose the M_n's to be elementary substructures of $H(\lambda^+)^{V[\bar{G}]}(\in, <^*, \dot{C}, \ldots)$ where $<^*$ is a well-ordering and $\dot{C}$ is an $\mathbb{A}_1 * \dot{\mathbb{T}}$-name for a club, then an argument almost exactly like the one showing that internal approachability is preserved (i.e., the proof of Fact 11) will show that S' is stationary in $\mathcal{N}[\bar{G}][K_1][K_T]$.

Then the extension of $\mathcal{N}[\bar{G}][K_1][K_T][K_2]$ over $\mathcal{N}[\bar{G}][K_1][K_T]$ preserves the stationarity of S' by another application of Fact 9, so we get stationarity in $\mathcal{N}[G]$.

Now that we have established preservation of stationarity of S', we can finish the argument. Since $|h| = \mu$ in $\mathcal{N}[G]$, we can write $h = \bigcup_{i<\mu} x_i$ where $\langle x_i : i < \mu\rangle$ is a continuous and $\subset$-increasing chain of elements of $P_\mu(h)$. (This is *not* a chain through $P_\mu(h)^{\mathcal{N}[\bar{G}]}$.) The chain is a club in $P_\mu(h)^{\mathcal{N}[G]}$, in which S' is stationary, so there is a stationary $X \subseteq \mu$ such that $\{x_i : i \in X\} \subseteq S'$. Since $S' \subseteq S$, it follows that $i \in X$ implies that $x_i = \pi_{\mathcal{K}}(y_i)$ for some $y_i \in Z$. Therefore $\langle y_i : i < \mu\rangle$ is $\subset$-increasing with union Z, and in particular $\langle y_i : i \in X\rangle$ is stationary in Z. $\square$

Claim 32. *Z is* ***not*** *internally club.*

Proof. Suppose for contradiction that Z is internally club and hence that there is a $\subset$-increasing and continuous chain $\langle Z_i : i < \mu\rangle \in \mathcal{N}[G]$ with $|Z_i| < \mu$ for all $i < \mu$ and $\bigcup_{i<\mu} Z_i = Z$. So for all $i < \mu$, $Z_i \subset Z$, and so $\langle \pi_{\mathcal{K}}[Z_i] : i < \mu\rangle$ is an $\subset$-increasing and continuous chain with union h. If we let $W_i := \pi_{\mathcal{K}}[Z_i]$ for all $i < \mu$, then the fact that $|W_i| < \mu$ implies that $W_i = \pi_{\mathcal{K}}(Z_i) \in \mathcal{K}[\bar{G}]$. Therefore $\langle W_i : i < \mu\rangle$ is a continuous and $\subset$-increasing chain of sets in $P_\mu(h)$ with union h.

We define a set $U \in \mathcal{N}[\bar{G}][r]$ (where r is the generic induced by G from $\pi^{\{\bar{\lambda}\}}_{\mathrm{Add}}$) as

$$\left\{A \in P_\mu(H(\chi)) \cap \mathcal{IA}(\omega) : A \cap h \notin \mathcal{N}[\bar{G}]\right\}.$$

We have a real in $\mathcal{N}[\bar{G}][r] \setminus \mathcal{N}[\bar{G}]$ and $(\mu^+)^{\mathcal{N}[\bar{G}][r]} = \lambda \subset H(\lambda)$. Hence we apply Fact 13 to see that U is stationary in $\mathcal{N}[\bar{G}][r]$, and it remains stationary in $\mathcal{N}[G]$ by the preservation properties of the quotient (i.e., Lemma 22 combined with Facts 11 and 9). Therefore in $\mathcal{N}[G]$, since $h \subseteq H(\chi)^{\mathcal{N}[\bar{G}][r]}$, $\{A \cap h : A \in U\}$ is stationary in $P_\mu(h)$. Since $\langle W_i : i < \mu\rangle$ is club in h, there is some $i < \mu$ such that $W_i = A \cap h$

for some $A \in U$. But by definition, $A \cap h \notin \mathcal{N}[\overline{G}]$, but $W_i \in \mathcal{K}[\overline{G}] \subset \mathcal{N}[\overline{G}]$, so this is a contradiction. □

This completes the proof of the lemma. □

Proof of Theorem 2. Let $\mathbb{M}_1$ be any λ_1-sized forcing that turns λ_1 into $\aleph_2$ and adds stationarily many $N \in [H(\aleph_2)]^{\aleph_1}$ that are internally stationary but not internally club. Let $\dot{\mathbb{M}}_2$ be an $\mathbb{M}_1$-name for $\mathbb{M}^+(\omega, \lambda_1, \lambda_2)$, let G_1 be $\mathbb{M}_1$-generic over V, and let G_2 be $\dot{\mathbb{M}}_2[G_1]$-generic over $V[G_1]$. Then we can see that the theorem holds in $V[G_1][G_2]$: the distinction between internally stationary and internally club on $[H(\aleph_2)]^{\aleph_1}$ is preserved in $V[G_1][G_2]$ by Proposition 27, and we get a distinction between internally stationary and internally club for $[H(\aleph_3)]^{\aleph_2}$ by Lemma 29. □

3. A club forcing and a guessing sequence

3.1. ***A review of the tools.*** The main idea of the proof of Theorem 3 is to force a club through the complement of a canonical stationary set — that is, it is canonical in the sense that it is independent of a particular enumeration used to define it. This set is described as follows:

Fact 33 (Krueger [21]). Suppose μ is an uncountable regular cardinal and $\mu^{<\mu} \leq \mu^+$. Let $\underline{x} = \langle x_\alpha : \alpha < \mu^+ \rangle$ enumerate $[\mu^+]^{<\mu}$ and let

$$S(\underline{x}) := \{\alpha \in \mu^+ \cap \mathrm{cof}(\mu) : P_\mu(\alpha) \setminus \langle x_\beta : \beta < \alpha \rangle \text{ is stationary}\}.$$

Then $\mathsf{DSS}(\mu^+)$ holds if and only if $S(\underline{x})$ is stationary.

The natural thing to do is to define the following:

Definition 34. Let μ be an uncountable regular cardinal such that $\mu^{<\mu} = \mu^+$ and let $\underline{x}$ and $S(\underline{x})$ be defined as in Fact 33. Then let $\mathcal{C}(\underline{x})$ be the set of closed bounded subsets p of μ^+ such that $p \cap S(\underline{x}) = \varnothing$. We let $p' \leq p$ if and only if $p' \cap (\max p + 1) = p$.

Proposition 35. *Assuming $\mu^{<\mu} \leq \mu^+$, $\mathcal{C}(\underline{x})$ is μ^+-distributive.*

Sketch of proof. If $S(\underline{x})$ is nonstationary, then the result is trivial. If it is stationary, then $S(\underline{x})$ does not contain a stationary set of approachable points [17, Corollary 3.7]. Since $\mu^{<\mu} \leq \mu^+$ there is going to be a stationary set S^* of approachable points, which without loss of generality is disjoint from $S(\underline{x})$. Then a standard distributivity argument applies (see Cox's explanation [3]). □

We will also crucially need a characterization of diamonds. The following appears in joint work with Gilton and Stejskalová [11, Lemma 3.12].

Fact 36. The following are equivalent:

(1) λ is Mahlo and $\diamondsuit_\lambda(\mathrm{Reg})$ (where of course $\mathrm{Reg} = \{\tau < \lambda : \tau \text{ regular}\}$) holds.

(2) There is a function $\ell : \lambda \to V_\lambda$ such that for every transitive structure $\mathcal{N}$ satisfying a rich fragment of ZFC that is closed under λ^+-sequences in V, the following holds: for every $A \in \mathcal{N}$ with $A \in H(\lambda^+)$ and any $a \subset \mathcal{N}$ with $|a| < \lambda$, there is a rich $\mathcal{M} \prec \mathcal{N}$ with $a \cup \{\ell\} \cup \{A\} \subset \mathcal{M}$ such that $\ell(\bar{\lambda}) = \pi_{\mathcal{M}}(A)$ (where $\bar{\lambda} = \mathcal{M} \cap \lambda$ and $\pi_{\mathcal{M}}$ is the Mostowski collapse).[4]

We can always use such an ℓ assuming the consistency of a Mahlo cardinal: If λ is Mahlo in a model V, then it is Mahlo in Gödel's class L where $\Diamond_\lambda(S)$ holds for all regular λ and stationary $S \subset \lambda$.

Two other forcings will be used, mostly for their black-boxed properties:

Definition 37. If T is a wide Aronszajn tree[5] of cardinality $\aleph_1$, let $\mathbb{B}(T)$ be Baumgartner's forcing for specializing Aronszajn trees. It consists of finite functions $f : T \to \omega$ such that $f(x) \neq f(y)$ if $x \leq_T y$ or $y \leq_T x$. If $f, g \in \mathbb{B}(T)$, then $f \leq g$ if and only if $f \supseteq g$.

Definition 38. Let $S \subset [\aleph_2]^\omega$ be stationary. Then let $\mathbb{P}(S)$ be the forcing consisting of continuous, increasing, and countable chains $\langle M_\xi : \xi \leq \eta \rangle$ of elements of S. For $p, q \in \mathbb{P}(S)$, $p \leq q$ if and only if p end-extends q [8].

Fact 39. The following are true for these forcings:

(1) For Aronszajn trees T of cardinality $\aleph_1$, $\mathbb{B}(T)$ has the countable chain condition.

(2) For $S \subset [\aleph_2]^\omega$ stationary, $\mathbb{P}(S)$ adds a closed unbounded set in $[\aleph_2^V]^\omega$ through S.

(3) If $S \in V$, then $\mathrm{Add}(\omega) * \dot{\mathbb{P}}(S)$ has the weak ω_1-approximation property, that is, if $\dot{f}$ is an $\mathrm{Add}(\omega) * \dot{\mathbb{P}}(S)$-name for a function $\omega_1 \to \mathrm{ON}$ whose initial segments are in V, then $\dot{f}$ is forced to be in V [17].

(4) If $S \in V$, then $\mathrm{Add}(\omega) * \dot{\mathbb{P}}(S)$ is proper [17].

3.2. ***The proof.*** Now we prove Theorem 3. Fix λ Mahlo. We can assume that $\Diamond_\lambda(\mathrm{Reg})$ holds, so let ℓ witness Fact 36.

Let $\mathbb{I} = \langle \mathbb{I}_\alpha, \dot{\mathbb{J}}_\alpha : \alpha < \lambda \rangle$ be a countable-support iteration of length λ such that if $\ell(\delta)$ is an $\mathbb{I}_\delta$-name for a proper forcing then $\Vdash_{\mathbb{I}_\delta}$ "$\dot{\mathbb{J}}_\delta = \ell(\delta)$" and otherwise $\dot{\mathbb{J}}_\delta$ is forced to be the trivial forcing.[6] We will argue momentarily that we have $\Vdash_{\mathbb{I}}$ "$\aleph_1^{<\aleph_1} \leq \aleph_2 = \lambda$", so we fix an $\mathbb{I}$-name $\underset{\sim}{\dot{x}}$ of $[\aleph_2]^{<\aleph_1}$ in the extension by $\mathbb{I}$ as well as a sequence of names $\langle \dot{x}_\alpha : \alpha < \aleph_2 \rangle$ that canonically represent the elements listed by $\underset{\sim}{\dot{x}}$. Then let $\dot{\mathcal{C}}$

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[4]The original is stated with a different quantification — for all such $\mathcal{N}$, there exists a function, not the other way around. However, the proof works with the quantification used here.

[5]We say that T is a *wide Aronszajn tree* of cardinality $\aleph_1$ if it has no uncountable branches. This is meant to distinguish our situation from the case in which T has countable levels.

[6]See the work of Abraham [1] and Cummings and Foreman [4] for classical examples of forcings that use guessing functions in their definitions.

be an $\mathbb{I}$-name for $\mathcal{C}(\dot{\underline{x}})$. Let G be $\mathbb{I}$-generic over V and let H be $\mathcal{C} := \dot{\mathcal{C}}[G]$-generic over $V[G]$. Then the model in which the theorem is realized is $V[G][H]$.

Most of the desired properties of $V[G][H]$ follow easily. First of all, $V[G][H] \models \lambda = \aleph_2$: For all $\Theta < \lambda$ the forcing $\mathrm{Col}(\aleph_1, \Theta)$ appears in the iteration, $\mathbb{I}$ has the λ-chain condition because the iterands have size less than $< \lambda$, and $\mathbb{I}$ preserves $\aleph_1$ because it is proper. Then adjoining H preserves $\aleph_2$ by the distributivity property noted above. The fact that $V[G][H] \models \neg\mathsf{DSS}(\mu^+)$ follows from Fact 33 given that the generic object added by H is a club through the complement of the relevant stationary set. The main part of the work is to show that the approachability property fails.

If $\mathcal{M} \prec \mathcal{N}$ is rich and $\pi_{\mathcal{M}}$ is the Mostowski collapse relative to $\mathcal{M}$, we will typically denote $\pi_{\mathcal{M}}(a)$ as $\bar{a}$.

Lemma 40. $$V[G][H] \models \neg\mathsf{AP}(\aleph_2).$$

Proof. If $\mathsf{AP}(\aleph_2)$ holds then this is forced by some condition $z \in \mathbb{I} * \mathcal{C}$. Assuming this is the case, we can derive a contradiction.

Claim 41. *Let $\mathcal{M} \prec \mathcal{N}$ be a rich model chosen to witness Fact 36 in the sense of having the properties that $\mathcal{M} \cap \lambda = \bar{\lambda}$, $z \in \mathcal{M}$, and $\ell(\bar{\lambda})$ is an $\mathbb{I} \restriction \bar{\lambda}$-name for*

$$\pi_{\mathcal{M}}(\dot{\mathcal{C}}(\dot{\underline{x}}) * \mathrm{Add}(\omega) * \mathbb{P}(Y) * \mathbb{B}(Y)),$$

where $Y = ([\lambda]^{\omega})^{V[\mathbb{I}*\dot{\mathcal{C}}(\dot{\underline{x}})]}$, and $\mathbb{P}(Y)$ is defined with respect to the interpretation of Y as a stationary set and $\mathbb{B}(Y)$ is defined with respect to the interpretation of Y as a tree ordered by end-extension.

Suppose $\overline{G}_0 * \overline{H}_0$ is $\bar{\mathbb{I}} * \bar{\mathcal{C}}$-generic over V. Then there is a $G_0 * H_0$ which is $\mathbb{I} * \mathcal{C}(\dot{\underline{x}})$-generic over V such that if $j : \overline{\mathcal{M}} \to \mathcal{M} \subset \mathcal{N}$ is the inverse of the Mostowski collapse, then there is a lift $j : \overline{\mathcal{M}}[\overline{G}_0][\overline{H}_0] \to \mathcal{N}[G_0][H_0]$ with the property that $G_0 * H_0$ is an $\aleph_1$-preserving extension over $V[\overline{G}_0][\overline{H}_0][\overline{K}_0][\overline{K}_1][\overline{K}_2]$ where $\overline{K}_0 * \overline{K}_1 * \overline{K}_2$ is $\mathrm{Add}(\omega) * \overline{\mathbb{P}}(Y) * \overline{\mathbb{B}}(Y)$-generic.

Proof of Claim 41. We will lift the elementary embedding $j : \overline{\mathcal{M}} \to \mathcal{N}$ to $j : \overline{\mathcal{M}}[\overline{G}_0][\overline{H}_0] \to \mathcal{N}[G_0][H_0]$. We therefore fix the notation $\bar{\lambda} = \mathcal{M} \cap \lambda$, and we have an $\overline{\mathbb{M}}$-generic $\overline{G}_0$, so we let $\mathcal{C} = \dot{\mathcal{C}}(\dot{\underline{x}})[G_0]$.

To perform the lift, we need to show that we can absorb the generic $\overline{H}_0$. The first stage is for handling G_0. The forcing $\dot{\mathcal{C}}(\dot{\underline{x}}) * \mathrm{Add}(\omega) * \mathbb{P}(Y) * \mathbb{B}(Y)$ is an iteration of proper forcings and is therefore proper, and its image under $\pi_{\mathcal{M}}$ is proper for similar reasons. Hence, since it is also guessed, it is used in the iteration. Therefore G_0 takes the form $\overline{G}_0 * \overline{H}_0 * \overline{K}_0 * \overline{K}_1 * \overline{K}_2 * \overline{K}_3$ where $\overline{K}_3$ is just a remainder. The quotient preserves $\aleph_1$ since the whole forcing does.

To lift the embedding further, we use a master condition argument. Specifically, we want to show that $\cup \overline{H}_0 \cup \{\bar{\lambda}\}$ is a condition in $\mathcal{C}$. This follows because $\bar{\lambda} \notin S(\underline{x})$ as evaluated in $\mathcal{N}[G_0]$: Since $\overline{\mathcal{M}}^{<\bar{\lambda}} \subseteq \overline{\mathcal{M}}$ and $\mathbb{I} \restriction \bar{\lambda}$ has the $\bar{\lambda}$-chain condition, the evaluation $\langle x_\beta : \beta < \bar{\lambda} \rangle$ is equal to the countable subsets of $\bar{\lambda}$ in $\overline{\mathcal{M}}[\overline{G}_0]$. Therefore

$P_\mu(\bar{\lambda}) \setminus \langle x_\beta : \beta < \bar{\lambda}\rangle$ will be nonstationary because of the club added by $\mathbb{P}(Y)$. Hence we choose H_0 to be a generic containing $\cup \overline{H}_0 \cup \{\bar{\lambda}\}$. □

Suppose then that $z \in \mathbb{M} * \dot{\mathcal{C}}(\underline{x})$ forces that approachability holds. By the claim, there is an embedding $\overline{\mathcal{M}}[\overline{G}][\overline{H}] \to \mathcal{N}[G][H]$ such that $V[G * H]$ is an extension over $V[\overline{G}][\overline{H}][\overline{K}_0][\overline{K}_1][\overline{K}_2]$ that preserves $\aleph_1$ where $\overline{K}_0 * \overline{K}_1 * \overline{K}_2$ is generic for $\pi_{\mathcal{M}}(\mathrm{Add}(\omega) * \mathbb{P}(Y) * \mathbb{B}(Y))$. Since we are supposing that approachability holds, there is in $\mathcal{N}[G][H]$ a club $C \subseteq \aleph_2$ such that all of its points of cofinality $\aleph_1$ are approachable. By elementarity it follows that $\bar{\lambda} \in C$, so it is enough to show that $\bar{\lambda}$ cannot actually be an approachable point.

We need to show that $\overline{Y}$ does not have a cofinal branch. By the weak ω_1-approximation property of $\pi_{\mathcal{M}}(\mathrm{Add}(\omega) * \dot{\mathbb{P}}(S))$ (Fact 39), $\overline{Y}$ is a wide Aronszajn tree in $V[\overline{G}][\overline{H}][\overline{K}_0][\overline{K}_1]$ because no new cofinal branches are added. Moreover it has cardinality $\aleph_1$ in that model. If $D \subseteq \bar{\lambda}$ is a club of order-type ω_1 in $V[\overline{G}][\overline{H}][\overline{K}_0][\overline{K}_1]$, we can conflate $\overline{Y}$ with $\left\{x \in ([\bar{\lambda}]^\omega)^{V[\overline{G}][\overline{H}]} : \sup x \in D\right\}$ so that it has height ω_1. The forcing $\overline{K}_2$ adds a specializing function, therefore it remains a wide Aronszajn tree in any $\aleph_1$-preserving extension, so in particular this is true for $V[G][H]$.

If $\bar{\lambda}$ were an approachable point as witnessed by (without loss of generality) a club E, then for all $\alpha \in E \cap D$, we have $E \cap \alpha \in ([\bar{\lambda}]^\omega)^{V[\overline{G}][\overline{H}]}$. Hence it would be implied that $\overline{Y}$ has a cofinal branch, which is a contradiction. □

Remark 42. This master condition argument can also be used to show that $\mathcal{C}(\underline{x})$ is distributive over $V[\mathbb{I}]$.

Now we are finished with the proof of Theorem 3.

4. Further directions

We propose some other considerations along the lines of the question: Why did we have to do more work to get Theorem 2 after obtaining Theorem 1? Or rather, is the assumption $2^\mu = \mu^+$ necessary for Fact 6?

Question 1. Is it consistent for μ regular that exactly one of $\mathsf{DSS}(\mu^+)$ and "internally club and internally unbounded are distinct for $[H(\mu^+)]^\mu$" holds?[7]

On a similar note, the assumption that $2^\mu = |H(\mu^+)|$ is also used in a folklore result that assuming $2^\mu = \mu^+$, the distinction between internally unbounded and internally approachable for $[\mu^+]^\mu$ requires a Mahlo cardinal.

Question 2. What is the exact equiconsistency strength of the separation of internally approachable and internally unbounded for $[H(\mu^+)]^\mu$ for regular μ?

[7]This question was answered by Jakob after the previous version of this paper was released [14].

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PRODUCT MANIFOLDS AND THE CURVATURE OPERATOR OF THE SECOND KIND

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We investigate the curvature operator of the second kind on product Riemannian manifolds and obtain some optimal rigidity results. For instance, we prove that the universal cover of an n-dimensional nonflat complete locally reducible Riemannian manifold with $\left(n+\frac{n-2}{n}\right)$-nonnegative (respectively, $\left(n+\frac{n-2}{n}\right)$-nonpositive) curvature operator of the second kind must be isometric to $\mathbb{S}^{n-1}\times\mathbb{R}$ (respectively, $\mathbb{H}^{n-1}\times\mathbb{R}$) up to scaling. We also prove analogous optimal rigidity results for $\mathbb{S}^{n_1}\times\mathbb{S}^{n_2}$ and $\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}$, $n_1, n_2\geq 2$, among product Riemannian manifolds, as well as for $\mathbb{CP}^{m_1}\times\mathbb{CP}^{m_2}$ and $\mathbb{CH}^{m_1}\times\mathbb{CH}^{m_2}$, $m_1, m_2\geq 1$, among product Kähler manifolds. The approach is pointwise and algebraic.

1. Introduction

On a Riemannian manifold (M^n, g), the *curvature operator of the second kind* refers to the symmetric bilinear form $\mathring{R} : S^2_0(T_pM)\times S^2_0(T_pM)\to\mathbb{R}$ defined by

$$\mathring{R}(\varphi,\psi)=R_{ijkl}\varphi_{il}\psi_{jk},$$

where $S^2_0(T_pM)$ is the space of traceless symmetric two-tensors on T_pM. The terminology is due to Nishikawa [1986]. Early works studying this notion of curvature operator include [Calabi and Vesentini 1960; Berger and Ebin 1969; Bourguignon and Karcher 1978; Koiso 1979a; 1979b; Ogiue and Tachibana 1979; Nishikawa 1986; Kashiwada 1993].

Recently, the curvature operator of the second kind has received much attention; see [Cao et al. 2023; Li 2022; 2023a; 2023b; 2024; Nienhaus et al. 2023a; 2023b; Fluck and Li 2024; Dai and Fu 2024; Dai et al. 2024]. In particular, the longstanding conjecture of Nishikawa [1986], which asserts that a closed Riemannian manifold with positive curvature operator of the second kind is diffeomorphic to a spherical

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space form and a closed Riemannian manifold with nonnegative curvature operator of the second kind is diffeomorphic to a Riemannian locally symmetric space, has been resolved by Cao, Gursky and Tran [Cao et al. 2023], Li [2024], and Nienhaus, Petersen, and Wink [Nienhaus et al. 2023a], under weaker assumptions but with stronger conclusions. More precisely, it is known now that:

Theorem 1.1 [Cao et al. 2023; Li 2024; Nienhaus et al. 2023a]. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$.*

(1) *If (M^n, g) has three-positive curvature operator of the second kind, then M is diffeomorphic to a spherical space form.*

(2) *If (M^n, g) has three-nonnegative curvature operator of the second kind, then M is either flat or diffeomorphic to a spherical space form.*

The key observation made by Cao, Gursky, and Tran in [2023] is that two-positive curvature operator of the second kind implies strictly PIC1 (i.e., $M \times \mathbb{R}$ has positive isotropic curvature). This is sufficient to solve the positive case of Nishikawa's conjecture, as one can appeal to a result of Brendle [2008] stating that the normalized Ricci flow on a compact manifold starting with a strictly PIC1 metric exists for all time and converges to a limit metric with constant positive sectional curvature. Shortly after, the author showed that strictly PIC1 is implied by three-positivity of the curvature operator of the second kind; thus getting an immediate improvement of the result in [Cao et al. 2023]. To deal with the nonnegative case, the author [2024] reduces the problem to the locally irreducible case by proving that a complete n-dimensional Riemannian manifold with n-nonnegative curvature operator of the second kind is either flat or locally irreducible (see also Theorem 1.6 below for an optimal improvement of this result). Finally, nonflat Kähler manifolds are ruled out using [Li 2024, Theorem 1.9] (see also [Li 2023a] for an optimal improvement of it) and compact irreducible symmetric spaces are ruled out by Nienhaus, Petersen, and Wink [2023a, Theorem A]. We refer the reader to [Li 2022] or [Li 2023a] for a detailed account of the notion of the curvature operator of the second kind, as well as some recent developments.

We aim to study the curvature operator of the second kind on product Riemannian manifolds and obtain some optimal rigidity results. We first recall the following definition. Let $N := \frac{(n-1)(n+2)}{2}$ denote the dimension of $S^2_0(T_pM)$. For $\alpha \in [1, N]$, we say (M^n, g) has α-positive (respectively, α-nonnegative) curvature operator of the second kind if for any $p \in M$ and any orthonormal basis $\{\varphi_i\}_{i=1}^N$ of $S^2_0(T_pM)$,

$$(1\text{-}1) \qquad \sum_{i=1}^{\lfloor\alpha\rfloor} \mathring{R}(\varphi_i, \varphi_i) + (\alpha - \lfloor\alpha\rfloor)\mathring{R}(\varphi_{\lfloor\alpha\rfloor+1}, \varphi_{\lfloor\alpha\rfloor+1}) > 0 \ \text{(respectively, } \geq 0).$$

Here and in the rest of this article, $\lfloor x \rfloor$ denotes the floor function defined by

$$\lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \leq x\}.$$

When $\alpha = k$ is an integer, this reduces to the usual definition, which means the sum of the smallest k eigenvalues of the matrix $\mathring{R}(\varphi_i, \varphi_j)$ is positive (respectively, nonnegative) for any orthonormal basis $\{\varphi_i\}_{i=1}^N$ of $S_0^2(T_pM)$. Similarly, (M^n, g) is said to have α-negative (respectively, α-nonpositive) curvature operator of the second kind if the direction of the inequality (1-1) is reversed.

Our first main result is the following rigidity result for $\mathbb{S}^{n-1} \times \mathbb{R}$ and $\mathbb{H}^{n-1} \times \mathbb{R}$, where $\mathbb{S}^n$ and $\mathbb{H}^n$, $n \geq 2$, denote the n-dimensional sphere and hyperbolic space with constant sectional curvature 1 and -1, respectively.

Theorem 1.2. *Let (M^n, g) be a nonflat complete locally reducible Riemannian manifold of dimension $n \geq 4$.*

(1) *If M has $\left(n+\frac{n-2}{n}\right)$-nonnegative curvature operator of the second kind, then the universal cover of M is, up to scaling, isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$.*

(2) *If M has $\left(n+\frac{n-2}{n}\right)$-nonpositive curvature operator of the second kind, then the universal cover of M is, up to scaling, isometric to $\mathbb{H}^{n-1} \times \mathbb{R}$.*

Closely related is the following holonomy restriction theorem in the spirit of [Nienhaus et al. 2023b].

Theorem 1.3. *Let (M^n, g) be a (not necessarily complete) Riemannian manifold of dimension $n \geq 3$. Suppose that (M, g) has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha < n + \frac{n-2}{n}$. Then either M is flat or the restricted holonomy of M is $\mathsf{SO}(n)$.*

Theorems 1.2 and 1.3 improve previous results obtained in [Li 2024] and [Nienhaus et al. 2023b]. The author [2024, Theorem 1.8] proved that an n-dimensional complete Riemannian manifold with n-nonnegative curvature operator of the second kind is either flat or locally reducible. This result plays a significant role in resolving the nonnegative part of Nishikawa's conjecture in [Li 2024], as it allows one to reduce the problem to the locally irreducible setting. A slight modification of the proof yields the same conclusion under n-nonpositive curvature operator of the second kind. The method used in [Li 2024] is pointwise and algebraic. In [Nienhaus et al. 2023b], it is shown that if the curvature operator of the second kind of an n-dimensional Riemannian manifold, not necessarily complete, is n-nonnegative or n-nonpositive, then either the restricted holonomy of M is $\mathsf{SO}(n)$ or M is flat. This is a generalization of the author's result in [Li 2024] mentioned above. The approach of [Nienhaus et al. 2023b] is local. The key idea is that, unless the restricted holonomy is generic, there exists a parallel form, at least locally on the manifold. However, the Bochner technique with the curvature assumption implies that no such local parallel form exists unless the manifold is flat.

We would like to point out that the number $n+\frac{n-2}{n}$ in Theorems 1.2 and 1.3 is optimal in all dimensions, since $\mathbb{S}^{n-1}\times\mathbb{R}$ and $\mathbb{H}^{n-1}\times\mathbb{R}$ have $\left(n+\frac{n-2}{n}\right)$-nonnegative and $\left(n+\frac{n-2}{n}\right)$-nonpositive curvature operator of the second kind, respectively, and they both have restricted holonomy $\mathsf{SO}(n-1)$. In dimension four, $\mathbb{CP}^2$ and $\mathbb{CH}^2$ have $4\frac{1}{2}$-nonnegative and $4\frac{1}{2}$-nonpositive curvature operator of the second kind, respectively, and they both have restricted holonomy $\mathsf{U}(2)$.

Theorem 1.3 can also be viewed as supporting evidence to the author's conjecture in [Li 2022]: a closed n-dimensional Riemannian manifold with $\left(n+\frac{n-2}{n}\right)$-positive curvature operator of the second kind is diffeomorphic to a spherical space form.

As a generalization of Theorem 1.1, the author proved in [Li 2022] that a closed Riemannian manifold of dimension $n\geq 4$ with $4\frac{1}{2}$-positive curvature operator of the second kind is homeomorphic to a spherical space form. This is obtained by showing that $4\frac{1}{2}$-positive curvature operator of the second kind implies positive isotropic curvature and $\left(n+\frac{n-2}{n}\right)$-positive curvature operator of the second kind implies positive Ricci curvature, and then making use of the work of Micallef and Moore [1988]. A classification result of closed manifolds with $4\frac{1}{2}$-nonnegative curvature operator of the second kind was also obtained in [Li 2022, Theorem 1.4]. Using Theorem 1.2, together with [Li 2023a, Theorem 1.2] and [Nienhaus et al. 2023a, Theorem B], we get an improvement of [Li 2022, Theorem 1.4].

Theorem 1.4. *Let (M^n, g) be a closed nonflat Riemannian manifold of dimension $n\geq 4$. Suppose that M has $4\frac{1}{2}$-nonnegative curvature operator of the second kind. Then one of the following statements holds*:

(1) *M is homeomorphic (diffeomorphic if either $n=4$ or $n\geq 12$) to a spherical space form.*

(2) *$n=4$ and M is isometric to $\mathbb{CP}^2$ with Fubini–Study metric up to scaling.*

(3) *$n=4$ and the universal cover of M is isometric to $\mathbb{S}^3\times\mathbb{R}$ up to scaling.*

Our second main result is the rigidity of $\mathbb{S}^{n_1}\times\mathbb{S}^{n_2}$ and $\mathbb{H}^{n_1}\times\mathbb{H}^{n_2}$ among product Riemannian manifolds.

Theorem 1.5. *Let $(M_i^{n_i}, g_i)$ be a Riemannian manifold of dimension $n_i\geq 2$ for $i=1,2$, and let $(M^{n_1+n_2}, g)=(M_1^{n_1}\times M_2^{n_2}, g_1\oplus g_2)$. Set*

$$A_{n_1,n_2}:=1+n_1n_2+\frac{n_1(n_2-1)+n_2(n_1-1)}{n_1+n_2}. \tag{1-2}$$

Then:

(1) *If M has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha < A_{n_1,n_2}$, then M is flat.*

(2) *If M has A_{n_1,n_2}-nonnegative curvature operator of the second kind, then both M_1 and M_2 have constant sectional curvature $c \geq 0$.*

(3) *If M has A_{n_1,n_2}-nonpositive curvature operator of the second kind, then both M_1 and M_2 have constant sectional curvature $c \leq 0$.*

If M is further assumed to be complete and nonflat, then the universal cover of M is isometric to $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ in part (2) *and $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ in part* (3), *up to scaling.*

The author [2024, Proposition 5.1] proved that an n-manifold with $(k(n-k)+1)$-nonnegative curvature operator of the second kind cannot split off a k-dimensional factor with $1 \leq k \leq \frac{n}{2}$, unless it is flat. The number $k(n-k)+1$ is only optimal for some special n and k. Combining Theorems 1.2 and 1.5, we get the following generalization, which is optimal for any n and $1 \leq k \leq \frac{n}{2}$.

Theorem 1.6. *An n-dimensional Riemannian manifold with α-nonnegative or α-nonpositive curvature operator of the second kind for some*

$$\alpha < k(n-k) + \frac{2k(n-k)}{n}$$

cannot locally split off a k-dimensional factor with $1 \leq k \leq \frac{n}{2}$, unless it is flat.

In another direction, the curvature operator of the second kind has been investigated for Kähler manifolds in [Bourguignon and Karcher 1978; Li 2023a; 2023b; 2024; Nienhaus et al. 2023b]. For instance, it was shown in [Li 2023a] that an m-dimensional Kähler manifold with $\frac{3}{2}(m^2-1)$-nonnegative (respectively, $\frac{3}{2}(m^2-1)$-nonpositive) curvature operator of the second kind has constant nonnegative (respectively, nonpositive) holomorphic sectional curvature, and a closed m-dimensional Kähler manifold with $\left(\frac{3m^3-m+2}{2m}\right)$-positive curvature operator of the second kind has positive orthogonal bisectional curvature; thus being biholomorphic to $\mathbb{CP}^m$. Here we prove the following rigidity result for $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$ and $\mathbb{CH}^{m_1} \times \mathbb{CH}^{m_2}$ (all equipped with their standard metrics) among product Kähler manifolds.

Theorem 1.7. *Let $(M_i^{m_i}, g_i)$ be a Kähler manifold of complex dimension $m_i \geq 1$ for $i = 1, 2$, and let $(M^{m_1+m_2}, g) = (M_1^{m_1} \times M_2^{m_2}, g_1 \oplus g_2)$. Set*

$$B_{m_1,m_2} := 4m_1m_2 + \frac{3}{2}(m_1^2 + m_2^2) + \frac{m_1m_2}{m_1+m_2}. \tag{1-3}$$

Then:

(1) *If M has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha < B_{m_1,m_2}$, then M is flat.*

(2) *If M has B_{m_1,m_2}-nonnegative curvature operator of the second kind, then both M_1 and M_2 have constant holomorphic sectional curvature $c \geq 0$.*

(3) *If M has B_{m_1,m_2}-nonpositive curvature operator of the second kind, then both M_1 and M_2 have constant holomorphic sectional curvature $c \leq 0$.*

If M is further assumed to be complete and nonflat, then the universal cover of M is isometric to $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$ in part (2) and $\mathbb{CH}^{m_1} \times \mathbb{CH}^{m_2}$ in part (3), up to scaling.

Our investigation of the curvature operator of the second kind on product manifolds is motivated not only by the above mentioned optimal rigidity results but also by the fact that the spectrum of $\mathring{R}$ is known only for a few examples: space forms with constant sectional curvature, Kähler and quaternion-Kähler space forms [Bourguignon and Karcher 1978], $\mathbb{S}^2 \times \mathbb{S}^2$ [Cao et al. 2023], $\mathbb{S}^{n-1} \times \mathbb{R}$ [Li 2024], $\mathbb{S}^p \times \mathbb{S}^q$ [Nienhaus et al. 2023b]. We determine the spectrum of $\mathring{R}$ for a class of product manifolds by proving the following theorem.

Theorem 1.8. *Let (M_i, g_i) be an n_i-dimensional Einstein manifold with* $\mathrm{Ric}(g_i) = \rho_i g_i$ *and $n_i \geq 1$ for $i = 1, 2$. Denote by $\mathring{R}_i$ the curvature operator of the second kind of M_i for $i = 1, 2$, and $\mathring{R}$ the curvature operator of the second kind of the product manifold*

$$(M^{n_1+n_2}, g) = (M_1^{n_1} \times M_2^{n_2}, g_1 \oplus g_2).$$

Then the eigenvalues of $\mathring{R}$ are precisely those of $\mathring{R}_1$ and $\mathring{R}_2$, and 0 with multiplicity $n_1 n_2$, and $-\frac{n_1\rho_2+n_2\rho_1}{n_1+n_2}$ with multiplicity one.

Theorem 1.8 enables us to determine the spectrum of the curvature operator of the second kind on $(M_1, g_1) \times (M_2, g_2)$, with (M_i, g_i) being either a space form with constant sectional curvature or a Kähler space form with constant holomorphic sectional curvature for $i = 1, 2$. Examples are listed at the end of Section 2. More generally, Theorem 1.8 can be applied repeatedly to calculate the spectrum of $\mathring{R}$ for product manifolds of the form $(M_1, g_1) \times \cdots \times (M_k, g_k)$, provided that each (M_i, g_i) is Einstein and the eigenvalues of the curvature operator of the second kind are known on M_i.

Let's discuss the strategy of our proofs. The key idea to prove Theorems 1.2, 1.5 and 1.7 is to use the corresponding borderline example, such as $\mathbb{S}^{n-1} \times \mathbb{R}$, $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ or $\mathbb{CP}^{m_1} \times \mathbb{CP}^{m_2}$, as a model space and apply $\mathring{R}$ to the eigenvectors of the curvature operator of the second kind on the model space. This idea has been successfully employed in [Li 2022] with $\mathbb{CP}^2$ and $\mathbb{S}^3 \times \mathbb{R}$ as model spaces, in [Li 2023b] with $\mathbb{S}^2 \times \mathbb{S}^2$ as the model space and in [Li 2023a] with $\mathbb{CP}^m$ and $\mathbb{CP}^{m-1} \times \mathbb{CP}^1$ as model spaces. With the right choice of model space, this strategy leads to optimal results as the inequalities are all achieved as equalities on the model space. Theorem 1.6 is essentially a consequence of Theorems 1.2 and 1.5. The proof of Theorem 1.3 uses Berger's classification of restricted holonomy groups, together with Propositions 3.1 and 4.1, and results in [Li 2023a] and [Nienhaus et al. 2023b]. The proof of Theorem 1.8 relies on the fact that when both factors are Einstein, we can choose an orthonormal basis of the space of traceless symmetric two-tensors that diagonalizes the curvature operator of the second kind on the product manifold.

At last, we emphasize that our approach is pointwise, and, therefore, many of our results are of a pointwise nature, and the completeness of the metric is not required. Another feature is that our proofs are purely algebraic and work equally well for nonpositivity conditions on $\mathring{R}$.

The article is organized as follows. In Section 2, we study the curvature operator of the second kind on product Riemannian manifolds and prove Theorem 1.8. We present the proofs of Theorems 1.2 and 1.4 in Section 3. The proofs of Theorems 1.5 and 1.6 are given in Section 4. In Section 5, we prove Theorem 1.3. Section 6 is devoted to the proof of Theorem 1.7.

2. Product manifolds

We study the curvature operator of the second kind on product Riemannian manifolds and prove Theorem 1.8.

Recall that for Riemannian manifolds (M_1, g_1) and (M_2, g_2), the product metric $g_1 \oplus g_2$ on $M_1 \times M_2$ is defined by

$$g(X_1 + X_2, Y_1 + Y_2) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$$

for $X_i, Y_i \in T_{p_i} M_i$ under the natural identification

$$T_{(p_1, p_2)}(M_1 \times M_2) = T_{p_1} M_1 \oplus T_{p_2} M_2.$$

Let R denote the Riemann curvature tensor of $M = M_1 \times M_2$, and R_1 and R_2 denote the Riemann curvature tensor of M_1 and M_2, respectively. Then one can relate R, R_1 and R_2 by

$$R(X_1+X_2, Y_1+Y_2, Z_1+Z_2, W_1+W_2) = R_1(X_1, Y_1, Z_1, W_1) + R_2(X_2, Y_2, Z_2, W_2),$$

where $X_i, Y_i, Z_i, W_i \in TM_i$ for $i = 1, 2$. As the reader will see, the above equation, which is a consequence of the product structure, plays a significant role in this section.

From now on, let's focus on a single point in a product manifold and work in a purely algebraic way. For $i = 1, 2$, let (V_i, g_i) be a Euclidean vector space of dimension $n_i \geq 1$. The product space $V = V_1 \times V_2$ will be naturally identified with $V_1 \oplus V_2$ via the isomorphism $(X_1, X_2) \to X_1 + X_2$ for $X_i \in V_i$. The product metric on V, denoted by $g = g_1 \oplus g_2$, is defined by

$$g(X_1 + X_2, Y_1 + Y_2) = g_1(X_1, Y_1) + g_2(X_2, Y_2) \tag{2-1}$$

for $X_i, Y_i \in V_i$.

Denote by $S^2_B(\Lambda^2 V)$ the space of algebraic curvature operators on (V, g). That is to say, $R \in S^2_B(\Lambda^2 V)$ is a symmetric two-tensor on the space of two-forms $\Lambda^2 V$ on V and R also satisfies the first Bianchi identity. Given $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$, we define $R \in S^2_B(\Lambda^2 V)$ by

$$\begin{aligned} R(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2, W_1 + W_2) \\ = R_1(X_1, Y_1, Z_1, W_1) + R_2(X_2, Y_2, Z_2, W_2), \end{aligned} \tag{2-2}$$

for $X_i, Y_i, Z_i, W_i \in V_i$. Throughout this paper, we simply write

$$R = R_1 \oplus R_2$$

whenever R, R_1 and R_2 are related by (2-2). We denote by $\mathring{R}$, $\mathring{R}_1$ and $\mathring{R}_2$ the associated curvature operator of the second kind for $R = R_1 \oplus R_2$, R_1 and R_2, respectively.

The key result of this section is the following proposition.

Proposition 2.1. *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \geq 1$ and let $R = R_1 \oplus R_2$. If $\operatorname{Ric}(R_i) = \rho_i g_i$ for $i = 1, 2$, then the eigenvalues of $\mathring{R}$ are precisely those of $\mathring{R}_1$ and $\mathring{R}_2$, together with 0 with multiplicity $n_1 n_2$ and $-\frac{n_2\rho_1 + n_1\rho_2}{n_1 + n_2}$ with multiplicity one.*

In the rest of this section, $\mathring{R}$ acts on the space of symmetric two-tensors $S^2(V)$ via

$$\mathring{R}(\varphi)_{ij} = \sum_{k,l=1}^{n} R_{iklj}\varphi_{kl}.$$

Note that the curvature operator of the second kind (defined as a symmetric bilinear form in the Introduction) is equivalent to the symmetric bilinear form associated with the self-adjoint operator $\pi \circ \mathring{R} : S^2_0(V) \to S^2_0(V)$, where $\pi : S^2(V) \to S^2_0(V)$ is the projection map. This can be seen as

$$\mathring{R}(\varphi, \psi) = \langle \mathring{R}(\varphi), \psi \rangle = \langle (\pi \circ \mathring{R})(\varphi), \psi \rangle = (\pi \circ \mathring{R})(\varphi, \psi)$$

for $\varphi, \psi \in S^2_0(V)$. Thus, the spectrum of the curvature operator of the second kind $\mathring{R}$ (as a bilinear form) is the same as the spectrum of the self-adjoint operator $\pi \circ \mathring{R}$.

We will present the proof of Proposition 2.1 after we establish the following three lemmas. First of all, standard calculations using (2-2) show that zero is an eigenvalue of $\mathring{R}$ with multiplicity (at least) $n_1 n_2$.

Lemma 2.2. *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \geq 1$ and let $R = R_1 \oplus R_2$. Let E be the subspace of $S^2_0(V_1 \times V_2)$ given by*

$$E = \operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\},$$

where $u \odot v = u \otimes v + v \otimes u$ is the symmetric product. Then E lies in the kernel of $\mathring{R}$. In particular, 0 is an eigenvalue of $\mathring{R}$ with multiplicity (at least) $n_1 n_2$.

Proof. This is observed in [Nienhaus et al. 2023b, Lemma 2.1]. For the convenience of the reader, we give a detailed proof below. We start by constructing an orthonormal basis of E. Let $\{e_i\}_{i=1}^{n_1}$ be an orthonormal basis of V_1 and $\{e_i\}_{i=n_1+1}^{n_1+n_2}$ be an orthonormal basis of V_2. Then $\{e_i\}_{i=1}^{n_1+n_2}$ is an orthonormal basis of $V = V_1 \times V_2$. Define

$$\xi_{pq} = \frac{1}{\sqrt{2}} e_p \odot e_q,$$

for $1 \leq p \leq n_1$ and $n_1 + 1 \leq q \leq n_1 + n_2$. Then one can verify that the ξ_{pq}'s are traceless symmetric two-tensors on $V_1 \times V_2$ and they form an orthonormal basis of E. In particular, $\dim(E) = n_1 n_2$.

To prove that E lies in the kernel of $\mathring{R}$, it suffices to show that $\mathring{R}(\xi_{pq}) = 0$. We first observe that (2-2) implies that

$$R(e_i, e_j, e_k, e_l) = \begin{cases} R_1(e_i, e_j, e_k, e_l), & i, j, k, l \in \{1, \dots, n_1\}, \\ R_2(e_i, e_j, e_k, e_l), & i, j, k, l \in \{n_1+1, \dots, n_1+n_2\}, \\ 0, & \text{otherwise.} \end{cases} \tag{2-3}$$

We then compute, using $(e_p \odot e_q)(e_j, e_k) = (\delta_{pj}\delta_{qk} + \delta_{qj}\delta_{pk})$, that

$$\begin{aligned} \mathring{R}(\xi_{pq})(e_i, e_l) &= \sum_{j,k=1}^{n} R(e_i, e_j, e_k, e_l)\xi_{pq}(e_j, e_k) \\ &= \frac{1}{\sqrt{2}} \sum_{j,k=1}^{n} R(e_i, e_j, e_k, e_l)(\delta_{pj}\delta_{qk} + \delta_{qj}\delta_{pk}) \\ &= \frac{1}{\sqrt{2}} \sum_{j,k=1}^{n_1} (R(e_i, e_p, e_q, e_l) + R(e_i, e_q, e_p, e_l)) \\ &= 0, \end{aligned}$$

where the last step is because of (2-3) and the fact that $1 \leq p \leq n_1$ and $n_1 + 1 \leq q \leq n_1 + n_2$. Thus we have proved that 0 is an eigenvalue of $\mathring{R}$ with multiplicity (at least) $n_1 n_2$. □

Next, we show that the eigenvalues of R_1 and R_2 are also eigenvalues of $R = R_1 \oplus R_2$, provided that both R_1 and R_2 are Einstein.

Lemma 2.3. *Let* $R_i \in S^2_B(\Lambda^2 V_i)$ *for* $i = 1, 2$ *with* $\dim(V_i) = n_i \geq 1$ *and let* $R = R_1 \oplus R_2$. *If* R_1 (*respectively*, R_2) *is Einstein*, *then the eigenvalues of* $\mathring{R}_1$ (*respectively*, $\mathring{R}_2$) *are also eigenvalues of* $\mathring{R}$.

Proof. It suffices to prove the statement for R_1. Since R_1 is Einstein, we have that $\mathring{R}_1 : S^2_0(V_1) \to S^2_0(V_1)$ is a self-adjoint operator. We can then choose an orthonormal basis $\{\varphi_p\}_{p=1}^{N_1}$ of $S^2_0(V_1)$ such that

$$\mathring{R}_1(\varphi_p) = \lambda_p \varphi_p,$$

where $N_1 = \frac{(n_1-1)(n_1+2)}{2}$ is the dimension of $S^2_0(V_1)$. We may also view the φ_p's as elements in $S^2_0(V_1 \times V_2)$ via zero extension, namely,

$$\varphi_p(X_1 + X_2, Y_1 + Y_2) = \varphi_p(X_1, Y_1),$$

for $X_i, Y_i \in V_i$. Then we have

$$\varphi_p(e_j, e_k) = \begin{cases} \varphi_p(e_j, e_k), & j, k \in \{1, \dots, n_1\}, \\ 0, & \text{otherwise}, \end{cases} \tag{2-4}$$

where $\{e_i\}_{i=1}^{n_1+n_2}$ is the same basis of V in Lemma 2.2.

Next, we calculate using (2-4) that, for $1 \leq i, l \leq n_1$,

$$\begin{aligned} \mathring{R}(\varphi_p)(e_i, e_l) &= \sum_{j,k=1}^{n_1+n_2} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \sum_{j,k=1}^{n_1} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \sum_{j,k=1}^{n_1} R_1(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \lambda_p \varphi_p(e_i, e_l), \end{aligned}$$

and, for $n_1 + 1 \leq i, l \leq n_1 + n_2$,

$$\begin{aligned} \mathring{R}(\varphi_p)(e_i, e_l) &= \sum_{j,k=1}^{n_1+n_2} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \sum_{j,k=1}^{n_1} R(e_i, e_j, e_k, e_l)\varphi_p(e_j, e_k) \\ &= \lambda_p \varphi_p(e_i, e_l) \\ &= 0. \end{aligned}$$

Therefore, we have proved $\mathring{R}(\varphi_p) = \lambda_p \varphi_p$ for $1 \le p \le N_1$. Hence the eigenvalues of $\mathring{R}_1$ are also eigenvalues of $\mathring{R}$ with the same eigenvectors. □

Finally, we prove:

Lemma 2.4. *Let $R_i \in S^2_B(\Lambda^2 V_i)$ for $i = 1, 2$ with $\dim(V_i) = n_i \ge 1$ and let $R = R_1 \oplus R_2$. If $\operatorname{Ric}(R_i) = \rho_i g_i$ for $i = 1, 2$, then $-\frac{n_2\rho_1 + n_1\rho_2}{n_1+n_2}$ is an eigenvalue of $\mathring{R}$ with eigenvector $n_2 g_1 - n_1 g_2$.*

Proof. As in the proof of Lemma 2.3, we may also view g_1 and g_2 as elements in $S^2(V_1 \times V_2)$ via zero extension. Clearly, $\operatorname{tr}(n_2 g_1 - n_1 g_2) = n_2 n_1 - n_1 n_2 = 0$. So we have $n_2 g_1 - n_1 g_2 \in S^2_0(V_1 \times V_2)$.

We then compute that

$$\begin{aligned}
\mathring{R}(n_2 g_1 - n_1 g_2) &= n_2 \mathring{R}(g_1) - n_1 \mathring{R}(g_2) \\
&= n_2 \mathring{R}_1(g_1) - n_1 \mathring{R}_2(g_2) \\
&= -n_2 \operatorname{Ric}(R_1) + n_1 \operatorname{Ric}(R_2) \\
&= -n_2 \rho_1 g_1 + n_1 \rho_2 g_2,
\end{aligned}$$

where we have used $\mathring{R}_i(g_i) = -\operatorname{Ric}(R_i) = -\rho_i g_i$ for $i = 1, 2$.

Using

$$\operatorname{tr}(-n_2 \rho_1 g_1 + n_1 \rho_2 g_2) = -n_1 n_2 (\rho_1 - \rho_2)$$

and $\mathring{R}(g_i) = -\rho_i g_1$ for $i = 1, 2$, we then obtain that

$$\begin{aligned}
(\pi \circ \mathring{R})(n_2 g_1 - n_1 g_2) &= \pi(n_2 \mathring{R}(g_1) - n_1 \mathring{R}(g_2)) \\
&= \pi(n_2 \rho_1 g_1 + n_1 \rho_2 g_2) \\
&= -n_2 \rho_1 g_1 + n_1 \rho_2 g_2 - \frac{-n_1 n_2 (\rho_1 - \rho_2)}{n_1 + n_2}(g_1 + g_2) \\
&= -n_2 g_1 \left(\rho_1 - \frac{n_1(\rho_1 - \rho_2)}{n_1 + n_2} \right) + n_1 g_2 \left(\rho_2 + \frac{n_2(\rho_1 - \rho_2)}{n_1 + n_2} \right) \\
&= -\left(\frac{n_1 \rho_2 + n_2 \rho_1}{n_1 + n_2} \right)(n_2 g_1 - n_1 g_2).
\end{aligned}$$

Thus, we see that $-\frac{n_1\rho_2 + n_2\rho_1}{n_1+n_2}$ is an eigenvalue of $\mathring{R}$ with eigenvector $n_2 g_1 - n_1 g_2$. The proof is now complete. □

Proof of Proposition 2.1. Let $\{e_i\}_{i=1}^{n_1+n_2}$ be an orthonormal basis of V, where $e_1, \dots, e_{n_1} \in V_1$ and $e_{n_1+1}, \dots, e_{n_1+n_2} \in V_2$. Let $\{\varphi_p\}_{p=1}^{N_1}$ be an orthonormal basis of $S^2_0(V_1)$ such that $\mathring{R}_1(\varphi_p) = \lambda_p \varphi_p$ and $\{\psi_q\}_{q=1}^{N_2}$ be an orthonormal basis of $S^2_0(V_2)$

such that $\mathring{R}_2(\psi_q) = \mu_q \psi_q$, where the dimension of $S^2_0(V_i)$ for $i = 1, 2$ is $N_i = \frac{(n_i-1)(n_i+2)}{2}$. We then define, on V, the traceless symmetric two-tensors

$$\xi_{pq} = \frac{1}{\sqrt{2}} e_p \odot e_q$$

for $1 \leq p \leq n_1$ and $n_1 + 1 \leq q \leq n_1 + n_2$, and

$$\zeta = \frac{1}{\sqrt{n_1 n_2 (n_1 + n_2)}} (n_2 g_1 - n_1 g_2).$$

Then one can verify, via straightforward computations, that

$$\{\varphi_p\}_{p=1}^{N_1} \cup \{\psi_q\}_{q=1}^{N_2} \cup \{\xi_{pq}\}_{1 \leq p \leq n_1, n_1+1 \leq q \leq n_1+n_2} \cup \{\zeta\}$$

forms an orthonormal basis of $S^2_0(V)$.

According to Lemma 2.2, 2.3 and 2.4, the above basis diagonalizes $\mathring{R}$ as

$$\left(\begin{array}{ccc|ccc|ccc|c}
\lambda_1 & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & \lambda_{N_1} & & & & & & & \\
\hline
& & & \mu_1 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & \mu_{N_2} & & & & \\
\hline
& & & & & & 0 & & & \\
& & & & & & & \ddots & & \\
& & & & & & & & 0 & \\
\hline
& & & & & & & & & -\frac{n_2\rho_1 + n_1\rho_2}{n_1+n_2}
\end{array}\right).$$

□

Theorem 1.8 now follows immediately from Proposition 2.1, since on a product manifold the product metric satisfies (2-1) and the Riemann curvature tensor satisfies (2-2).

Since the spectrum of $\mathring{R}$ is known on space forms with constant sectional curvature and Kähler space forms with constant holomorphic sectional curvature, we can use Theorem 1.8 or Proposition 2.1 to determine the eigenvalues of the curvature operator of the second kind on their product.

In the rest of this section, we use the following notation:

- $\mathbb{S}^n(\kappa)$ and $\mathbb{H}^n(-\kappa)$, $n \geq 2$ and $\kappa > 0$, denote the n-dimensional simply connected space form with constant sectional curvature κ and $-\kappa$, respectively.

- $\mathbb{CP}^m(\kappa)$ and $\mathbb{CH}^m(-\kappa)$, $m \geq 1$ and $\kappa > 0$, denote the (complex) m-dimensional simply connected Kähler space form with constant holomorphic sectional curvature 4κ and -4κ, respectively.

Example 2.5. *$\mathring{R} = \kappa \, \mathrm{id}_{S^2_0}$ on $\mathbb{S}^n(\kappa)$. $\mathring{R} = -\kappa \, \mathrm{id}_{S^2_0}$ on $\mathbb{H}^n(-\kappa)$.*

Example 2.6. *$\mathring{R}$ has two distinct eigenvalues on $\mathbb{CP}^m(\kappa)$: -2κ with multiplicity $(m-1)(m+1)$ and 4κ with multiplicity $m(m+1)$. $\mathring{R}$ has two distinct eigenvalues on $\mathbb{CH}^m(-\kappa)$: 2κ with multiplicity $(m-1)(m+1)$ and -4κ with multiplicity $m(m+1)$. See [Bourguignon and Karcher 1978].*

Example 2.7. *Let $M = \mathbb{S}^{n_1}(\kappa_1) \times \mathbb{S}^{n_2}(\kappa_2)$. Then the curvature operator of the second kind of M has eigenvalues: κ_1 with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, κ_2 with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, 0 with multiplicity $n_1 n_2$ and $-\frac{n_1(n_2-1)\kappa_2+n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

Example 2.8. *Let $M = \mathbb{H}^{n_1}(-\kappa_1) \times \mathbb{H}^{n_2}(-\kappa_2)$. Then the curvature operator of the second kind of M has eigenvalues: $-\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $-\kappa_2$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, 0 with multiplicity $n_1 n_2$ and $\frac{n_1(n_2-1)\kappa_2+n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

Example 2.9. *Let $M = \mathbb{S}^{n_1}(\kappa_1) \times \mathbb{R}^{n_2}$. Then the curvature operator of the second kind of M has eigenvalues: κ_1 with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, 0 with multiplicity $n_1 n_2 + \frac{(n_2-1)(n_2+2)}{2}$ and $-\frac{n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

Example 2.10. *Let $M = \mathbb{H}^{n_1}(-\kappa_1) \times \mathbb{R}^{n_2}$. Then the curvature operator of the second kind of M has eigenvalues: $-\kappa_1$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, 0 with multiplicity $n_1 n_2 + \frac{(n_2-1)(n_2+2)}{2}$ and $\frac{n_2(n_1-1)\kappa_1}{n_1+n_2}$ with multiplicity one.*

Example 2.11. *Let $M = \mathbb{S}^{n_1}(\kappa_1) \times \mathbb{H}^{n_2}(-\kappa_2)$. Then the curvature operator of the second kind of M has eigenvalues: κ_1 with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $-\kappa_2$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, 0 with multiplicity $n_1 n_2$ and $-\frac{n_1 n_2(\kappa_1-\kappa_2)+n_1\kappa_2-n_2\kappa_1}{n_1+n_2}$ with multiplicity one.*

Example 2.12. *Let $M = \mathbb{CP}^{m_1}(\kappa_1) \times \mathbb{CP}^{m_2}(\kappa_2)$. Then the curvature operator of the second kind of M has eigenvalues: $-2\kappa_1$ with multiplicity $(m_1-1)(m_1+1)$, $-2\kappa_2$ with multiplicity $(m_2-1)(m_2+1)$, $4\kappa_1$ with multiplicity $m_1(m_1+1)$, $4\kappa_2$ with multiplicity $m_2(m_2+1)$, 0 with multiplicity $4m_1m_2$, and $-\frac{2m_1(m_2+1)\kappa_2+2m_2(m_1+1)\kappa_1}{m_1+m_2}$ with multiplicity one.*

Example 2.13. *Let $M = \mathbb{CH}^{m_1}(-\kappa_1) \times \mathbb{CH}^{m_2}(-\kappa_2)$. Then the curvature operator of the second kind of M has eigenvalues: $2\kappa_1$ with multiplicity $(m_1-1)(m_1+1)$, $2\kappa_2$ with multiplicity $(m_2-1)(m_2+1)$, $-4\kappa_1$ with multiplicity $m_1(m_1+1)$, $-4\kappa_2$ with* *multiplicity* $m_2(m_2+1)$, 0 *with multiplicity* $4m_1m_2$, *and* $\frac{2m_1(m_2+1)\kappa_2+2m_2(m_1+1)\kappa_1}{m_1+m_2}$ *with multiplicity one.*

Example 2.14. *Let* $M=\mathbb{CP}^{m_1}(\kappa_1)\times\mathbb{C}^{m_2}$. *Then the curvature operator of the second kind of M has eigenvalues*: $-2\kappa_1$ *with multiplicity* $(m_1-1)(m_1+1)$, $4\kappa_1$ *with multiplicity* $m_1(m_1+1)$, 0 *with multiplicity* $4m_1m_2+(2m_2-1)(m_2+1)$, *and* $-\frac{2m_2(m_1+1)\kappa_1}{m_1+m_2}$ *with multiplicity one.*

Example 2.15. *Let* $M=\mathbb{CH}^{m_1}(-\kappa_1)\times\mathbb{C}^{m_2}$. *Then the curvature operator of the second kind of M has eigenvalues*: $2\kappa_1$ *with multiplicity* $(m_1-1)(m_1+1)$, $-4\kappa_2$ *with multiplicity* $m_1(m_1+1)$, 0 *with multiplicity* $4m_1m_2+(2m_2-1)(m_2+1)$, *and* $\frac{2m_2(m_1+1)\kappa_1}{m_1+m_2}$ *with multiplicity one.*

Example 2.16. *Let* $M=\mathbb{CP}^{m_1}(\kappa_1)\times\mathbb{CH}^{m_2}(-\kappa_2)$. *Then the curvature operator of the second kind of M has eigenvalues*: $-2\kappa_1$ *with multiplicity* $(m_1-1)(m_1+1)$, $4\kappa_2$ *with multiplicity* $m_1(m_1+1)$, $2\kappa_2$ *with multiplicity* $(m_2-1)(m_2+1)$, $-4\kappa_2$ *with multiplicity* $m_2(m_2+1)$, 0 *with multiplicity* $4m_1m_2$, *and* $-\frac{2m_1m_2(\kappa_1-\kappa_2)+2m_2\kappa_1-2m_1\kappa_2}{m_1+m_2}$ *with multiplicity one.*

In particular, we have the following observation, which will be needed later on.

Proposition 2.17. *For* $n_1,n_2\geq 2$, $m_1,m_2\geq 1$, $\kappa_1,\kappa_2>0$, *we have the following*:

1. $\mathbb{S}^{n_1}(\kappa_1)\times\mathbb{S}^{n_2}(\kappa_2)$ *has* A_{n_1,n_2}*-nonnegative curvature operator of the second kind if and only if* $\kappa_1=\kappa_2>0$.
2. $\mathbb{H}^{n_1}(-\kappa_1)\times\mathbb{H}^{n_2}(-\kappa_2)$ *has* A_{n_1,n_2}*-nonpositive curvature operator of the second kind if and only if* $\kappa_1=\kappa_2>0$.
3. $\mathbb{CP}^{m_1}(\kappa_1)\times\mathbb{CP}^{m_2}(\kappa_2)$ *has* B_{m_1,m_2}*-nonnegative curvature operator of the second kind if and only if* $\kappa_1=\kappa_2>0$.
4. $\mathbb{CH}^{m_1}(-\kappa_1)\times\mathbb{CH}^{m_2}(-\kappa_2)$ *has* B_{m_1,m_2}*-nonpositive curvature operator of the second kind if and only if* $\kappa_1=\kappa_2<0$.

3. Rigidity of cylinders

We prove Theorem 1.2. The key result of this section is the following proposition.

Proposition 3.1. *Let* (V,g) *be a Euclidean vector space of dimension* $n-1$ *with* $n\geq 2$ *and let* $R_1\in S^2_B(\Lambda^2V)$.

(1) *Suppose that* $R=R_1\oplus 0\in S^2_B(\Lambda^2(V\times\mathbb{R}))$ *has* $\left(n+\frac{n-2}{n}\right)$*-nonnegative curvature operator of the second kind. Then* R_1 *has constant nonnegative sectional curvature.*

(2) *Suppose that* $R=R_1\oplus 0\in S^2_B(\Lambda^2(V\times\mathbb{R}))$ *has* $\left(n+\frac{n-2}{n}\right)$*-nonpositive curvature operator of the second kind. Then* R_1 *has constant nonpositive sectional curvature.*

(3) *Suppose that* $R = R_1 \oplus 0 \in S^2_B(\Lambda^2(V \times \mathbb{R}))$ *has* α*-nonnegative or* α*-nonpositive curvature operator of the second kind for some* $\alpha < n + \frac{n-2}{n}$. *Then* R *is flat.*

Proof. (1) Let $\{e_i\}_{i=1}^{n-1}$ be an orthonormal basis of V and let e_n be a unit vector in $\mathbb{R}$. Then $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of $V \times \mathbb{R} \cong V \oplus \mathbb{R}$. Next, we define, on $V \oplus \mathbb{R}$, the symmetric two-tensors

$$\xi_i = \frac{1}{\sqrt{2}} e_i \odot e_n \quad \text{for } 1 \le i \le n-1,$$
$$\varphi_{kl} = \frac{1}{\sqrt{2}} e_k \odot e_l \quad \text{for } 1 \le k < l \le n-1,$$
$$\zeta = \frac{1}{2\sqrt{n(n-1)}} \left(\sum_{p=1}^{n-1} e_p \odot e_p - (n-1) e_n \odot e_n \right).$$

One easily verifies that $\{\xi_i\}_{i=1}^{n-1} \cup \{\varphi_{kl}\}_{1 \le k < l \le n-1} \cup \{\zeta\}$ forms an orthonormal subset of $S^2_0(\Lambda^2(V \oplus \mathbb{R}))$.

Since $R = R_1 \oplus 0$, we have by (2-2) that

$$R(e_i, e_j, e_k, e_l) = \begin{cases} R_1(e_i, e_j, e_k, e_l), & i, j, k, l \in \{1, \dots, n-1\}, \\ 0, & \text{otherwise.} \end{cases} \tag{3-1}$$

In particular, we have $R_{njnj} = 0$ for $1 \le j \le n-1$.

Direct calculation using the identity

$$\mathring{R}(e_i \odot e_j, e_k \odot e_l) = 2(R_{iklj} + R_{ilkj})$$

shows that

$$\mathring{R}(\xi_i, \xi_i) = 0 \qquad \text{for } 1 \le i \le n-1,$$
$$\mathring{R}(\varphi_{kl}, \varphi_{kl}) = (R_1)_{klkl} \qquad \text{for } 1 \le k < l \le n-1,$$
$$\mathring{R}(\zeta, \zeta) = -\frac{1}{n(n-1)} S_1,$$

where S_1 is the scalar curvature of R_1. Note that $S_1 \ge 0$ since S_1 is also equal to the scalar curvature of R, which must be nonnegative since R has $\left(n + \frac{n-2}{n}\right)$-nonnegative curvature operator of the second kind; see, e.g., [Li 2024, Proposition 4.1, part (1)].

Since R has $\left(n + \frac{n-2}{n}\right)$-nonnegative curvature operator of the second kind, we get that, for any $1 \le k < l \le n-1$,

$$\begin{aligned} 0 &\le \mathring{R}(\zeta, \zeta) + \sum_{i=1}^{n-1} \mathring{R}(\xi_i, \xi_i) + \frac{n-2}{n} \mathring{R}(\varphi_{kl}, \varphi_{kl}) \\ &= -\frac{1}{n(n-1)} S_1 + \frac{n-2}{n} (R_1)_{klkl} = \frac{n-2}{n} \left((R_1)_{klkl} - \frac{S_1}{(n-1)(n-2)} \right). \end{aligned}$$

Summing over $1 \le k < l \le n-1$ yields

$$S_1 \le \sum_{1\le k<l\le n-1} (R_1)_{klkl}.$$

On the other hand,

$$S_1 = \sum_{1\le k<l\le n-1} (R_1)_{klkl}.$$

Therefore, we must have $(R_1)_{klkl} = \frac{S_1}{(n-1)(n-2)}$ for all $1 \le k < l \le n-1$. Since the orthonormal basis $\{e_1, \dots, e_{n-1}\}$ is arbitrary, we conclude that R_1 has constant nonnegative sectional curvature.

(2) Apply (1) to $-R$.

(3) By (1) and (2), we have $R = cI_{n-1} \oplus 0$ for some $c \in \mathbb{R}$, where I_{n-1} is the Riemann curvature tensor of $\mathbb{S}^{n-1}$. However, $R = cI_{n-1} \oplus 0$ has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha < n + \frac{n-2}{n}$ if and only if $c = 0$. Therefore, R is flat. □

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2. (1) Recall that we say that (M^n, g) is locally reducible if there exists a nontrivial subspace of T_pM which is invariant under the action of the restricted holonomy group. By a theorem of de Rham, a complete Riemannian manifold is locally reducible if and only if its universal cover is isometric to the product of two Riemannian manifolds of lower dimension.

Denote by $(\widetilde{M}, \tilde{g})$ the universal cover of M with the lifted metric $\tilde{g}$. Since M is locally reducible, $(\widetilde{M}, \tilde{g})$ is isometric to a product of the form $(M_1^k, g_1) \times (M_2^{n-k}, g_2)$, where $1 \le k \le \frac{n}{2}$. Note that $k \ge 2$ implies

$$k(n-k) + 1 \ge n + \frac{n-2}{n},$$

so $\widetilde{M}$ must be flat if $k \ge 2$, according to [Li 2024, Proposition 5.1] (or its improvement Theorem 1.6). Thus we must have $k = 1$ and $\widetilde{M}$ is isometric to $N^{n-1} \times \mathbb{R}$. By part (1) of Proposition 3.1, N has pointwise constant nonnegative sectional curvature. Since $n - 1 \ge 3$, Schur's lemma implies that N must have constant nonnegative sectional curvature. Therefore, M is either flat or its universal cover is isometric to $\mathbb{S}^{n-1} \times \mathbb{R}$ up to scaling.

(2) This is similar to the proof of (1), by noticing that [Li 2024, Proposition 5.1] is valid for the nonpositivity condition (alternatively, one can use Theorem 1.6). □

Proof of Theorem 1.4. Let (M^n, g) be a closed nonflat Riemannian manifold of dimension $n \geq 4$ and suppose that M has $4\frac{1}{2}$-nonnegative curvature operator of the second kind. It was shown in [Li 2022] that one of the following statements holds:

(a) M is homeomorphic (diffeomorphic if $n = 4$ or $n \geq 12$) to a spherical space form.

(b) $n = 2m$ and the universal cover of M is a Kähler manifold biholomorphic to $\mathbb{CP}^m$.

(c) $n = 4$ and the universal cover of M is diffeomorphic to $\mathbb{S}^3 \times \mathbb{R}$.

(d) $n \geq 5$ and M is isometric to a quotient of a compact irreducible symmetric space.

By Theorem 1.2 in [Li 2023a], the Kähler manifold in part (2) is either flat or isometric to $\mathbb{CP}^2$ with the Fubini–Study metric, up to scaling. In part (c), the manifold is reducible and we conclude using Theorem 1.2 that the universal cover of M is isometric to $\mathbb{S}^3 \times \mathbb{R}$, up to scaling. Part (d) can be ruled out using [Nienhaus et al. 2023a, Theorem B], as the manifold is either flat or a homology sphere. □

4. Rigidity of product of spheres and hyperbolic spaces

We prove Theorem 1.5. The key result of this section is the following proposition. In this section, I_n, $n \geq 2$, denotes the Riemann curvature tensor of the n-sphere with constant sectional curvature 1.

Proposition 4.1. *For $i = 1, 2$, let (V_i, g_i) be a Euclidean vector space of dimension n_i with $n_i \geq 2$. Let $R_i \in S^2_B(\Lambda^2 V_i)$ and $R = R_1 \oplus R_2 \in S^2_B(\Lambda^2(V_1 \times V_2))$.*

(1) *Suppose that R has A_{n_1,n_2}-nonnegative curvature operator of the second kind. Then $R = c(I_{n_1} \oplus I_{n_2})$ for some $c \geq 0$.*

(2) *Suppose that R has A_{n_1,n_2}-nonpositive curvature operator of the second kind. Then $R = c(I_{n_1} \oplus I_{n_2})$ for some $c \leq 0$.*

(3) *Suppose that R has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha < A_{n_1,n_2}$. Then R is flat.*

We need an elementary lemma, which can be found in [Li 2023a, Lemma 5.1].

Lemma 4.2. *Let N be a positive integer and A be a collection of N real numbers. Denote by a_i the i-th smallest number in A for $1 \leq i \leq N$. Define a function $f(A, x)$ by*

$$f(A, x) = \sum_{i=1}^{\lfloor x \rfloor} a_i + (x - \lfloor x \rfloor) a_{\lfloor x \rfloor + 1},$$

for $x \in [1, N]$. *Then we have*

$$f(A, x) \leq x\bar{a}, \tag{4-1}$$

where $\bar{a} := \frac{1}{N}\sum_{i=1}^{N} a_i$ *is the average of all numbers in* A. *The equality holds for some* $x \in [1, N)$ *if and only if* $a_i = \bar{a}$ *for all* $1 \leq i \leq N$.

Proof of Proposition 4.1. (1) Let $\{e_i\}_{i=1}^{n_1}$ be an orthonormal basis of V_1 and let $\{e_i\}_{i=n_1+1}^{n_1+n_2}$ be an orthonormal basis of V_2. Then $\{e_i\}_{i=1}^{n_1+n_2}$ is an orthonormal basis of $V_1 \times V_2 \cong V_1 \oplus V_2$.

We construct an orthonormal basis of $S^2_0(V_1 \times V_2)$ as follows. Choose an orthonormal basis $\{\varphi_i\}_{i=1}^{N_1}$ of $S^2_0(V_1)$ and an orthonormal basis $\{\psi_i\}_{i=1}^{N_2}$ of $S^2_0(V_2)$, where $N_i = \dim(S^2_0(V_i)) = \frac{(n_i-1)(n_i+2)}{2}$ for $i = 1, 2$. Note that $h \in S^2_0(V_1)$ can be identified with the element $\pi^* h$ in $S^2_0(V_1 \times V_2)$ via

$$(\pi^* h)(X_1 + X_2, Y_1 + Y_2) = h(X_1, X_2),$$

where $X_i, Y_i \in V_i$ for $i = 1, 2$. We shall simply write $\pi^* h$ as h. Similarly, $S^2_0(V_2)$ can be identified with a subspace of $S^2_0(V_1 \times V_2)$. Next, we define, on $V_1 \times V_2$, the symmetric two-tensors

$$\begin{aligned} \xi_{kl} &= \frac{1}{\sqrt{2}} e_k \odot e_l \quad \text{for } 1 \leq k \leq n_1,\ n_1 + 1 \leq l \leq n_1 + n_2, \\ \zeta &= \frac{1}{\sqrt{n_1 n_2 (n_1 + n_2)}} (n_2 g_1 - n_1 g_2). \end{aligned}$$

One verifies that

$$\{\varphi_i\}_{i=1}^{N_1} \cup \{\psi_i\}_{i=1}^{N_2} \cup \{\xi_{kl}\}_{1 \leq k \leq n_1, n_1+1 \leq l \leq n_1+n_2} \cup \{\zeta\}$$

forms an orthonormal basis of $S^2_0(V_1 \times V_2)$. This corresponds to the orthogonal decomposition

$$S^2_0(V_1 \times V_2) = S^2_0(V_1) \oplus S^2_0(V_2) \oplus \operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\} \oplus \mathbb{R}\zeta.$$

The next step is to calculate some diagonal elements of the matrix representing $\mathring{R}$ with respect to the above basis. Since $R = R_1 \oplus R_2$, we have by (2-2) that

$$R(e_i, e_j, e_k, e_l) = \begin{cases} R_1(e_i, e_j, e_k, e_l), & i, j, k, l \in \{1, \dots, n_1\}, \\ R_2(e_i, e_j, e_k, e_l), & i, j, k, l \in \{n_1 + 1, \dots, n_1 + n_2\}, \\ 0, & \text{otherwise}. \end{cases} \tag{4-2}$$

In particular, we have $R_{klkl} = 0$ if $1 \leq k \leq n_1$ and $n_1 \leq l \leq n_1 + n_2$. Using the identity

$$\mathring{R}(e_i \odot e_j, e_k \odot e_l) = 2(R_{iklj} + R_{ilkj}),$$

we get

$$\sum_{\substack{1\le k\le n_1\\ n_1+1\le l\le n_1+n_2}} \mathring{R}(\xi_{kl},\xi_{kl}) = \sum_{\substack{1\le k\le n_1\\ n_1+1\le l\le n_1+n_2}} R_{klkl}=0. \tag{4-3}$$

We also calculate

$$\begin{aligned}\mathring{R}(\zeta,\zeta) &= \frac{1}{n_1n_2(n_1+n_2)}(n_2^2\mathring{R}(g_1,g_1)+n_1^2\mathring{R}(g_2,g_2)+2n_1n_2\mathring{R}(g_1,g_2))\\ &= \frac{1}{n_1n_2(n_1+n_2)}(n_2^2\mathring{R}_1(g_1,g_1)+n_1^2\mathring{R}_2(g_2,g_2))\\ &= -\frac{n_2^2S_1+n_1^2S_2}{n_1n_2(n_1+n_2)},\end{aligned}$$

where S_i denotes the scalar curvature of R_i for $i=1,2$.

Let A be the collection of the values of $\mathring{R}(\varphi_i,\varphi_i)$ for $1\le i\le N_1$ and let B be the collection of the values of $\mathring{R}(\psi_i,\psi_i)$ for $1\le i\le N_2$. Denote by $\bar{a}$ and $\bar{b}$ the average of all numbers in A and B, respectively. Then

$$\begin{aligned}\bar{a} &= \frac{1}{N_1}\sum_{i=1}^{N_1}\mathring{R}(\varphi_i,\varphi_i) = \frac{1}{N_1}\sum_{i=1}^{N_1}\mathring{R}_1(\varphi_i,\varphi_i) = \frac{S_1}{n_1(n_1-1)},\\ \bar{b} &= \frac{1}{N_2}\sum_{i=1}^{N_2}\mathring{R}(\psi_i,\psi_i) = \frac{1}{N_2}\sum_{i=1}^{N_2}\mathring{R}_2(\psi_i,\psi_i) = \frac{S_2}{n_2(n_2-1)},\end{aligned}$$

where we have used

$$\sum_{i=1}^{N_1}\mathring{R}_1(\psi_i,\psi_i)=\frac{n_1+2}{2n_1}S_1 \quad \text{and} \quad \sum_{i=1}^{N_2}\mathring{R}_2(\psi_i,\psi_i)=\frac{n_2+2}{2n_2}S_2.$$

For simplicity, we write

$$A_1=\frac{n_2(n_1-1)}{n_1+n_2} \quad \text{and} \quad A_2=\frac{n_1(n_2-1)}{n_1+n_2}.$$

Notice that we have $A_1<N_1$, $A_2<N_2$ and

$$A_{n_1,n_2}=1+n_1n_2+A_1+A_2. \tag{4-4}$$

Also, the expression for $\mathring{R}(\zeta,\zeta)$ can be written as

$$\mathring{R}(\zeta,\zeta)=-A_1\bar{a}-A_2\bar{b}. \tag{4-5}$$

Since R has A_{n_1,n_2}-nonnegative curvature operator of the second kind, we get using (4-3), (4-4) and (4-5) that

$$\begin{aligned} -\mathring{R}(\zeta,\zeta) &\le f(A,\lfloor A_1\rfloor)+f(B,A_1+A_2-\lfloor A_1\rfloor) \\ &\le \lfloor A_1\rfloor\bar{a}+(A_1+A_2-\lfloor A_1\rfloor)\bar{b} \\ &= A_1\bar{a}+A_2\bar{b}+(A_1-\lfloor A_1\rfloor)(\bar{b}-\bar{a}), \end{aligned} \tag{4-6}$$

where f is the function defined in Lemma 4.2 and we have used Lemma 4.2 in estimating f. Similarly, we also have

$$\begin{aligned} -\mathring{R}(\zeta,\zeta) &\le f(A,A_1+A_2-\lfloor A_2\rfloor)+f(B,\lfloor A_2\rfloor), \\ &\le (A_1+A_2-\lfloor A_2\rfloor)\bar{a}+\lfloor A_2\rfloor)\bar{b} \\ &= A_1\bar{a}+A_2\bar{b}+(A_2-\lfloor A_2\rfloor)(\bar{a}-\bar{b}). \end{aligned} \tag{4-7}$$

Therefore, by (4-5), we get from (4-6) if $\bar{a}\ge\bar{b}$ and from (4-7) if $\bar{a}\le\bar{b}$ that

$$A_1\bar{a}+A_2\bar{b}=-\mathring{R}(\zeta,\zeta)\le A_1\bar{a}+A_2\bar{b}.$$

This implies that, either in (4-6) or (4-7), we must have equalities in the inequalities used for f. We then get from Lemma 4.2, that all the values in A are equal to $\bar{a}$ and all the values in B are equal to $\bar{b}$. Hence, both R_1 and R_2 have constant sectional curvature, that is to say, $R=c_1I_{n_1}\oplus c_2I_{n_2}$ for $c_1,c_2\in\mathbb{R}$.

Finally, we must have $c_1=c_2\ge 0$, as $R=c_1I_{n_1}\oplus c_2I_{n_2}$ has A_{n_1,n_2}-nonnegative curvature operator of the second kind if and only if $c_1=c_2\ge 0$ by Proposition 2.17.

(2) Apply (1) to $-R$.

(3) This follows from the fact that $R=c(I_{n_1}\oplus I_{n_2})$ has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha<A_{n_1,n_2}$ if and only if $c=0$. □

At last, we give the proof of Theorem 1.5.

Proof of Theorem 1.5. (1) This is an immediate consequence of part (3) of Proposition 4.1.

(2) Let $(p_1,p_2)\in M_1\times M_2$. By part (2) of Proposition 4.1, we have

$$R(p_1,p_2)=c(p_1,p_2)(I_{n_1}\oplus I_{n_2})$$

with $c(p_1,p_2)\ge 0$. If both n_1 and n_2 are at least 3, then Schur's lemma implies that $c(p_1,p_2)\equiv c\ge 0$. Below we provide an argument that works whenever $n_1,n_2\ge 2$.

Note that both (M_1,g_1) and (M_2,g_2) have pointwise constant sectional curvature. By Proposition 2.1, the eigenvalues of $\mathring{R}$ at (p_1,p_2) are given by $\frac{\rho_1(p_1)}{n_1-1}$ with multiplicity $\frac{(n_1-1)(n_1+2)}{2}$, $\frac{\rho_2(p_2)}{n_2-1}$ with multiplicity $\frac{(n_2-1)(n_2+2)}{2}$, 0 with multiplicity n_1n_2, and $-\frac{n_2\rho_1(p_1)+n_1\rho_2(p_2)}{n_1+n_2}$ with multiplicity one. Here $\rho_i(p_i)$ is the Einstein constant of M_i at p_i, i.e., $\mathrm{Ric}(g_i)(p_i) = \rho_i(p_i)g_i$ at p_i for $i = 1, 2$. Using the assumption that $M_1 \times M_2$ has A_{n_1,n_2}-nonnegative curvature operator of the second kind, we obtain

$$-\frac{n_2\rho_1(p_1)+n_1\rho_2(p_2)}{n_1+n_2}+\frac{n_1(n_2-1)+n_2(n_1-1)}{n_1+n_2}\frac{\rho_1(p_1)}{n_1-1}\geq 0$$

and

$$-\frac{n_2\rho_1(p_1)+n_1\rho_2(p_2)}{n_1+n_2}+\frac{n_1(n_2-1)+n_2(n_1-1)}{n_1+n_2}\frac{\rho_2(p_2)}{n_2-1}\geq 0.$$

The two inequalities force

$$(n_2-1)\rho_1(p_1) = (n_1-1)\rho_2(p_2).$$

Fixing p_1 while letting p_2 vary in M_2 shows that $\rho_2(p_2)$ is independent of p_2. Similarly, $\rho_1(p_1)$ is independent of p_1. Since $\rho_i(p_i) = (n_i-1)c(p_1, p_2)$ for $i = 1, 2$, we conclude that $c(p_1, p_2) \equiv c \geq 0$. Therefore, both (M_1, g_1) and (M_2, g_2) have constant sectional curvature $c \geq 0$.

If M is further assumed to be complete, then M is either flat or the universal cover of M is isometric to $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$, up to scaling.

(3) Similar to the proof of (2). □

Proof of Theorem 1.6. Suppose that (M^n, g) splits locally near $q \in M$ as a Riemannian product $(M_1^k \times M_2^{n-k}, g_1 \oplus g_2)$ with $1 \leq k \leq \frac{n}{2}$. Then the Riemann curvature tensor R of M satisfies $R = R_1 \oplus R_2$ near q, where R_i denotes the Riemann curvature tensor of M_i for $i = 1, 2$.

By part (3) of Proposition 3.1 if $k = 1$ and part (3) of Proposition 4.1 if $2 \leq k \leq \frac{n}{2}$, the assumption

$$\alpha < k(n-k)+\frac{2k(n-k)}{n}$$

implies that M must be flat near q. Since the restricted holonomy does not depend on $q \in M$, we conclude that M is flat. □

5. Holonomy restriction

Proof of Theorem 1.3. Suppose that (M^n, g) splits locally near $q \in M$ as a Riemannian product $(M_1^k \times M_2^{n-k}, g_1 \oplus g_2)$ with $2 \leq k \leq \frac{n}{2}$. Then the Riemann curvature tensor R of M satisfies $R = R_1 \oplus R_2$ near q, where R_i denotes the Riemann curvature tensor of M_i for $i = 1, 2$.

Noticing that

$$\alpha < n+\frac{n-2}{n} \leq A_{k,n-k} = k(n-k)+\frac{2k(n-k)}{n}$$

for any $1 \le k \le \frac{n}{2}$, we conclude from part (3) of Propositions 3.1 if $k = 1$ and part (3) of Proposition 4.1 if $2 \le k \le \frac{n}{2}$ that M is locally flat. Since the restricted holonomy does not depend on $q \in M$, we conclude that M is flat. Therefore, M is either locally irreducible or flat.

If $n = 3$, then the holonomy of M must be $\mathsf{SO}(3)$ as M is locally irreducible. So we may assume $n \ge 4$ below.

If M is an irreducible locally symmetric space, then it is Einstein. Since

$$\alpha < n + \frac{n-2}{n} \le \frac{3n}{2}\frac{n+2}{n+4}$$

for any $n \ge 4$, we get from [Nienhaus et al. 2023b, Theorem B] that either M is flat or the restricted holonomy of M is $\mathsf{SO}(n)$.

So we may assume that M is not locally symmetric with irreducible holonomy representation. Then the restricted holonomy of M is contained in Berger's list of holonomy groups [1955]: $\mathsf{SO}(n)$, $\mathsf{U}\left(\frac{n}{2}\right)$, $\mathsf{SU}\left(\frac{n}{2}\right)$, $\mathsf{Sp}\left(\frac{n}{4}\right)\mathsf{Sp}(1)$, $\mathsf{Sp}\left(\frac{n}{4}\right)$, G_2 and $\mathsf{Spin}(7)$. Note that if its restricted holonomy is $\mathsf{SU}\left(\frac{n}{2}\right)$, $\mathsf{Sp}\left(\frac{n}{4}\right)$, G_2 or $\mathsf{Spin}(7)$, then M must be Ricci flat and thus flat.

If the restricted holonomy of M is $\mathsf{Sp}\left(\frac{n}{4}\right)\mathsf{Sp}(1)$, then M is quaternion-Kähler and it is also Einstein in this case. Thus, either the restricted holonomy of M is $\mathsf{SO}(n)$ or M is flat by [Nienhaus et al. 2023b, Theorem B].

If the restricted holonomy of M is $\mathsf{U}\left(\frac{n}{2}\right)$, then M is Kähler. Noticing that

$$\alpha < n + \frac{n-2}{n} \le \frac{3}{2}\left(\frac{n^2}{4} - 1\right)$$

for any $n \ge 4$, M must be flat by [Li 2023a, Therorem 1.2].

Overall, either the restricted holonomy of M is $\mathsf{SO}(n)$ or M is flat. □

6. Kähler manifolds

We prove Theorem 1.7. The proof shares the same idea as in Section 4, but we use the orthonormal basis of the space of traceless symmetric two-tensors on a complex Euclidean space constructed in [Li 2023a].

In the following, B_{m_1,m_2} is the expression defined in (1-3) and $R_{\mathbb{CP}^m}$ denotes the Riemann curvature tensor of the complex projective space with constant holomorphic sectional curvature 4. We establish the following proposition.

Proposition 6.1. *For $i = 1, 2$, let (V_i, g_i, J_i) be a complex Euclidean vector space of complex dimension $m_i \ge 1$. Let $R_i \in S^2_B(\Lambda^2 V_i)$ and $R = R_1 \oplus R_2 \in S^2_B(\Lambda^2(V_1 \times V_2))$.*

(1) *Suppose that R has B_{m_1,m_2}-nonnegative curvature operator of the second kind. Then $R = c(R_{\mathbb{CP}^{m_1}} \oplus R_{\mathbb{CP}^{m_2}})$ for some $c \ge 0$.*

(2) *Suppose that R has B_{m_1,m_2}-nonpositive curvature operator of the second kind. Then $R = c(R_{\mathbb{CP}^{m_1}} \oplus R_{\mathbb{CP}^{m_2}})$ for some $c \le 0$.*

(3) *Suppose that R has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha < B_{m_1,m_2}$. Then R is flat.*

Proof. (1) Let

$$\{e_1, \dots, e_{m_1}, J_1e_1, \dots, J_1e_{m_1}\}$$

be an orthonormal basis of (V_1, g_1, J_1) and

$$\{e_{m_1+1}, \dots, e_{m_1+m_2}, J_2e_{m_1+1}, \dots, J_2e_{m_1+m_2}\}$$

be an orthonormal basis of (V_2, g_2, J_2).

As in Section 4, we have the orthogonal decomposition

$$S^2_0(V_1 \times V_2) = S^2_0(V_1) \oplus S^2_0(V_2) \oplus \operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\} \oplus \mathbb{R}\zeta,$$

where

$$\zeta = \frac{1}{\sqrt{2m_1m_2(m_1+m_2)}}(m_2g_1 - m_1g_2).$$

The same computation as in Section 4 gives that

$$\mathring{R}(\zeta,\zeta) = -\frac{m_2^2S_1 + m_1^2S_2}{2m_1m_2(m_1+m_2)}, \tag{6-1}$$

where S_i denotes the scalar curvature of R_i for $i = 1, 2$.

By Lemma 2.2, the subspace $\operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\}$ lies in the kernel of $\mathring{R}$ and its real dimension is $4m_1m_2$.

For $S^2_0(V_1)$ and $S^2_0(V_2)$, we use the orthonormal bases constructed in Section 4 of [Li 2023a]. More precisely, the following traceless symmetric two-tensors form an orthonormal basis of $S^2_0(V_1)$:

$$\begin{aligned}
\varphi^{1,\pm}_{ij} &= \frac{1}{2}(e_i \odot e_j \mp J_1e_i \odot J_1e_j) && \text{for } 1 \le i < j \le m_1,\\
\psi^{1,\pm}_{ij} &= \frac{1}{2}(e_i \odot J_1e_j \pm J_1e_i \odot e_j) && \text{for } 1 \le i < j \le m_1,\\
\alpha^1_i &= \frac{1}{2\sqrt{2}}(e_i \odot e_i - J_1e_i \odot Je_i) && \text{for } 1 \le i \le m_1,\\
\alpha^1_{m_1+i} &= \frac{1}{\sqrt{2}}(e_i \odot J_1e_i) && \text{for } 1 \le i \le m_1,\\
\eta^1_k &= \frac{k}{\sqrt{8k(k+1)}}(e_{k+1} \odot e_{k+1} + J_1e_{k+1} \odot J_1e_{k+1})\\
&\qquad - \frac{1}{\sqrt{8k(k+1)}}\sum_{i=1}^{k}(e_i \odot e_i + J_1e_i \odot J_1e_i) && \text{for } 1 \le k \le m_1 - 1.
\end{aligned}$$

Similarly, the traceless symmetric two-tensors

$$\begin{aligned}
\varphi_{ij}^{2,\pm} &= \frac{1}{2}(e_i\odot e_j \mp J_2e_i\odot J_2e_j) && \text{for } m_1+1\le i<j\le m_1+m_2,\\
\psi_{ij}^{2,\pm} &= \frac{1}{2}(e_i\odot J_2e_j \pm J_2e_i\odot e_j) && \text{for } m_1+1\le i<j\le m_1+m_2,\\
\alpha_i^2 &= \frac{1}{2\sqrt{2}}(e_i\odot e_i - J_1e_i\odot Je_i) && \text{for } m_1+1\le i\le m_1+m_2,\\
\alpha_{m_2+i}^2 &= \frac{1}{\sqrt{2}}(e_i\odot J_1e_i) && \text{for } m_1+1\le i\le m_1+m_2,\\
\eta_k^2 &= \frac{k}{\sqrt{8k(k+1)}}(e_{k+1}\odot e_{k+1}+J_2e_{k+1}\odot J_2e_{k+1})\\
&\qquad -\frac{1}{\sqrt{8k(k+1)}}\sum_{i=1}^{k}(e_i\odot e_i+J_2e_i\odot J_2e_i) && \text{for } m_1+1\le k\le m_1+m_2-1
\end{aligned}$$

form an orthonormal basis for $S_0^2(V_2)$. Here the superscripts 1 and 2 indicate that these are quantities associated with the space V_1 and V_2, respectively.

By Lemma 4.3 in [Li 2023a], we have

$$\sum_{1\le i<j\le m_1}(\mathring{R}(\varphi_{ij}^{1,-},\varphi_{ij}^{1,-})+\mathring{R}(\psi_{ij}^{1,-},\psi_{ij}^{1,-}))+\sum_{k=1}^{m_1-1}\mathring{R}(\eta_k,\eta_k) = -\frac{m_1-1}{2m_1}S_1 \tag{6-2}$$

and

$$\sum_{m_1+1\le i<j\le m_1+m_2}(\mathring{R}(\varphi_{ij}^{2,-},\varphi_{ij}^{2,-})+\mathring{R}(\psi_{ij}^{2,-},\psi_{ij}^{2,-}))+\sum_{k=m_1+1}^{m_1+m_2-1}\mathring{R}(\eta_k,\eta_k) = -\frac{m_2-1}{2m_2}S_2. \tag{6-3}$$

Let A be the collection of the values $\mathring{R}(\alpha_i^1,\alpha_i^1)$ for $1\le i\le 2m_1$, $\mathring{R}(\varphi_{ij}^{1,+},\varphi_{ij}^{1,+})$ and $\mathring{R}(\psi_{ij}^{1,+},\psi_{ij}^{1,+})$ for $1\le i<j\le m$. By Lemma 4.3 in [Li 2023a], we know that A contains two copies of $R(e_i, J_1e_i, e_i, J_1e_i)$ for each $1\le i\le m_1$ and two copies of $2R(e_i, J_1e_i, e_j, J_1e_j)$ for each $1\le i<j\le m_1$. Therefore, the sum of all values in A is equal to S_1, the scalar curvature of R_1, and $\bar{a}$, the average of all values in A, is given by

$$\bar{a} = \frac{S_1}{m_1(m_1+1)}.$$

Let B be the collection of the values $\mathring{R}(\alpha_i^2,\alpha_i^2)$ for $m_1+1\le i\le m_1+2m_2$, $\mathring{R}(\varphi_{ij}^{2,+},\varphi_{ij}^{2,+})$ and $\mathring{R}(\psi_{ij}^{2,+},\psi_{ij}^{2,+})$ for $m_1+1\le i<j\le m_1+m_2$. By Lemma 4.3 in [Li 2023a], we know that B contains two copies of $R(e_i, J_2e_i, e_i, J_2e_i)$ for each $m_1+1\le i\le m_1+m_2$ and two copies of $2R(e_i, J_2e_i, e_j, J_2e_j)$ for each

$m_1+1 \le i < j \le m_1+m_2$. Therefore, the sum of all values in B is equal to S_2, the scalar curvature of R_2, and $\bar{b}$, the average of all values in B, is given by

$$\bar{b} = \frac{S_2}{m_2(m_2+1)}.$$

Combining (6-1), (6-2) and (6-3) together yields

$$\begin{aligned}
\sum_{1\le i<j\le m_1} &(\mathring{R}(\varphi_{ij}^{1,-},\varphi_{ij}^{1,-}) + \mathring{R}(\psi_{ij}^{1,-},\psi_{ij}^{1,-})) \\
&+ \sum_{m_1+1\le i<j\le m_1+m_2} (\mathring{R}(\varphi_{ij}^{2,-},\varphi_{ij}^{2,-}) + \mathring{R}(\psi_{ij}^{2,-},\psi_{ij}^{2,-})) \\
&+ \sum_{k=1}^{m_1-1} \mathring{R}(\eta_k,\eta_k) + \sum_{k=m_1+1}^{m_1+m_2-1} \mathring{R}(\eta_k,\eta_k) + \mathring{R}(\zeta,\zeta) \\
&\qquad = -\frac{m_1-1}{2m_1} S_1 - \frac{m_2-1}{2m_2} S_2 + \mathring{R}(\zeta,\zeta) \\
&\qquad = -\frac{1}{2}(m_1^2-1)\bar{a} - \frac{1}{2}(m_2^2-1)\bar{b} - \frac{m_2^2 S_1 + m_1^2 S_2}{2m_1 m_2(m_1+m_2)} \\
&\qquad = -B_1\bar{a} - B_2\bar{b},
\end{aligned}$$

where we have introduced

$$B_1 = \frac{1}{2}(m_1^2-1) + \frac{(m_1+1)m_2}{2(m_1+m_2)} \quad \text{and} \quad B_2 = \frac{1}{2}(m_2^2-1) + \frac{(m_2+1)m_1}{2(m_1+m_2)}$$

for simplicity of notation. Note that $-B_1\bar{a} - B_2\bar{b}$ is the sum of

$$1 + 4m_1m_2 + (m_1^2-1) + (m_2^2-1)$$

many diagonal elements of the matrix representation of $\mathring{R}$ with respect to the orthonormal basis of $S_0^2(V_1 \times V_2)$ constructed above (here one can pick any orthonormal basis for the subspace $\operatorname{span}\{u \odot v : u \in V_1,\ v \in V_2\}$ as it is in the kernel of $\mathring{R}$).

Noticing that

$$B_{m_1,m_2} = 1 + (m_1^2-1) + (m_2^2-1) + 4m_1m_2 + B_1 + B_2,$$

the assumption R has B_{m_1,m_2}-nonnegative curvature operator of the second kind implies that

$$\begin{aligned}
B_1\bar{a} + B_2\bar{b} &\le f(A, \lfloor B_1 \rfloor) + f(B, B_1 + B_2 - \lfloor B_1 \rfloor) \\
&\le \lfloor B_1 \rfloor \bar{a} + (B_1 + B_2 - \lfloor B_1 \rfloor)\bar{b} \\
&= B_1\bar{a} + B_2\bar{b} + (B_1 - \lfloor B_1 \rfloor)(\bar{b} - \bar{a})
\end{aligned} \tag{6-4}$$

and

$$B_1\bar{a}+B_2\bar{b} \le f(A, B_1+B_2-\lfloor B_2\rfloor)+f(B, \lfloor B_2\rfloor) \le (B_1+B_2-\lfloor B_2\rfloor)\bar{a}+\lfloor B_2\rfloor\bar{b} = B_1\bar{a}+B_2\bar{b}+(B_2-\lfloor B_2\rfloor)(\bar{a}-\bar{b}), \tag{6-5}$$

where f is the function defined in Lemma 4.2 and we have used Lemma 4.2 to estimate f. So we get from (6-4) if $\bar{a}\ge\bar{b}$ and from (6-5) if $\bar{a}\le\bar{b}$ that

$$B_1\bar{a}+B_2\bar{b}\le B_1\bar{a}+B_2\bar{b}.$$

Therefore, either in (6-4) or (6-5), we must have equalities in the inequalities used for f. By Lemma 4.2, we get that all the values in A are equal to $\bar{a}$ and all the values in B are equal to $\bar{b}$. Hence, both R_1 and R_2 have constant holomorphic sectional curvature, that is to say, $R=c_1R_{\mathbb{CP}^{m_1}}\oplus c_2R_{\mathbb{CP}^{m_2}}$ for $c_1, c_2\in\mathbb{R}$.

Finally, we must have $c_1=c_2\ge 0$, as $R=c_1R_{\mathbb{CP}^{m_1}}\oplus c_2R_{\mathbb{CP}^{m_2}}$ has B_{m_1,m_2}-nonnegative curvature operator of the second kind if and only if $c_1=c_2\ge 0$ by Proposition 2.17.

(2) Apply (1) to $-R$.

(3) This follows from the fact that $R=c(R_{\mathbb{CP}^{m_1}}\oplus R_{\mathbb{CP}^{m_2}})$ has α-nonnegative or α-nonpositive curvature operator of the second kind for some $\alpha<B_{m_1,m_2}$ if and only if $c=0$. □

Proof of Theorem 1.7. Once we have Proposition 6.1, this is similar to the proof of Theorem 1.5 and we omit the details. □

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