HOMOTOPY VERSUS ISOTOPY: 2-SPHERES IN 5-MANIFOLDS

DANICA KOSANOVIĆ, ROB SCHNEIDERMAN AND PETER TEICHNER

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We give a complete obstruction for two homotopic embeddings of a 2-sphere into a 5-manifold to be isotopic. The results are new even though the methods are classical, the main tool being the elimination of double points via a level preserving Whitney move in codimension 3. Moreover, we discuss how this recovers a particular case of a result of Dax on metastable homotopy groups of embedding spaces. It follows that "homotopy implies isotopy" for 2-spheres in simply connected 5-manifolds and for 2-spheres admitting algebraic dual 3-spheres.

1. Introduction and results

A curious consequence of our generalizations [Schneiderman and Teichner 2022; Kosanović and Teichner 2024a] of the 4-dimensional light bulb theorems of David Gabai [2020; 2021] is that homotopic 2-spheres R, $R' : S^2 \hookrightarrow M$, embedded in a 4-manifold M with a common dual sphere, are smoothly isotopic in M if and only if they are isotopic in the 5-manifold $M \times \mathbb{R}$, see [Schneiderman and Teichner 2022, Corollary 1.5]. The complete isotopy obstruction in [Schneiderman and Teichner 2022, Theorem 1.1] is given by the Freedman–Quinn invariant

$$\mathrm{fq}(R, R') := [\mu_3(H)] \in \frac{\mathbb{F}_2 T_M}{\mu_3(\pi_3 M)},$$

where $\mu_3(H)$ is Wall's self-intersection invariant of the track

$$H : S^2 \times [0, 1] \looparrowright M \times \mathbb{R} \times [0, 1]$$

of a generic homotopy between R and R' in $M \times \mathbb{R}$. Moreover, $\mathbb{F}_2 T_M$ is the $\mathbb{F}_2$-vector space with basis $T_M := \{g \in \pi_1 M \mid g^2 = 1 \neq g\}$, the set of involutions in $\pi_1 M$. It turns out that μ_3 also gives a homomorphism $\mu_3 : \pi_3 M \to \mathbb{F}_2 T_M$, whose cokernel eliminates the choice of homotopy in the definition of fq. Michael Freedman and Frank Quinn [1990, Chapter 10] introduced this invariant, while studying topological concordance classes of embedded 2-spheres in 4-manifolds.

This isotopy classification also follows from [Kosanović and Teichner 2024a, Theorem 1.1] via a more powerful invariant due originally to Jean-Pierre Dax [1972] which detects relative isotopy classes of neatly embedded 2-disks having a common dual in ∂M. Dax extends the parametrized double-point elimination method of André Haefliger [1961a; 1961b], which is in turn an extension of the Whitney trick [Whitney 1944]. Haefliger's results were used by Lawrence Larmore [1978, Theorem 6.0.1] to show a special case of Dax's result, see (1.14) below.

In the current paper we consider the "homotopy versus isotopy" question for 2-spheres in general 5-manifolds and show that there is again a self-intersection invariant of a homotopies which detects isotopy classes and takes values in a quotient of the group ring of the ambient fundamental group. The dimensions under consideration here are right at the transition between low- and high-dimensional topology, with successful Whitney moves generally available in the presence of vanishing algebraic obstructions.

With this in mind, our exposition will be aimed at describing this transition from the point of view of the low-dimensional topologist, rather than starting by presenting results in full generality. In particular, we will:

(1) Explain how the isotopy classification can be described by self-intersection invariants of homotopies, using a level-preserving Whitney trick.

(2) Explain how Dax's work recovers the same result, from the perspective of space level techniques and homotopy groups of embedding spaces.

Both approaches can be generalized to describe isotopy classifications for compact n-manifolds embedded in $(2n + 1)$-manifolds, as will be described in upcoming work [Kosanović et al. $\geq$ 2024] which recasts Dax [1972] in full generality in this language; see Theorem 4.1 below.

We next give a quick outline of the main results, and refer to the rest of the introduction for details. For any fixed embedded sphere $U : S^2 \hookrightarrow N^5$ in a 5-manifold N, we will define the set $\mathbb{A}_{U_*}$ as a certain quotient of the group ring $\mathbb{Z}\pi_1 N$, see Definition 1.10. The image in $\mathbb{A}_{U_*}$ of the self-intersections $\mu_3(H)$ of a generic track $H : S^2 \times I \looparrowright N \times I$ of a homotopy between U and $R : S^2 \hookrightarrow N$ will by design only depend on U and R, not on H. Here the notation U is meant to suggest that one can think of this fixed 2-sphere as an "unknot", although we emphasize that there are no restrictions on its homotopy class. Denoting this invariant by $\mathrm{fq}_{U_*}(R) := [\mu_3(H)] \in \mathbb{A}_{U_*}$, a basic statement of our main result is the following corollary of Theorem 1.11:

Corollary 1.1. *Homotopic spheres U and R are isotopic if and only if the formula $\mathrm{fq}_{U_*}(R) = 0 \in \mathbb{A}_{U_*}$ holds. Moreover, any element in $\mathbb{A}_{U_*}$ is realized as $\mathrm{fq}_{U_*}(R)$ for an embedded sphere R.*

We note that in this 5-dimensional setting, the result does not require any dual spheres (unlike in four dimensions), see Corollary 1.13. If $\pi_1 N$ is trivial, then $\mathbb{A}_{U_*} = \{0\}$ and we get:

Corollary 1.2. *Homotopy implies isotopy for 2-spheres in simply connected 5-manifolds.*

There is a straightforward proof of Corollary 1.2, using cusp-cancellation (in dimension 6) and the theorem of John Hudson [1970] that in codimension > 2 concordance implies isotopy. We will give a self-contained proof of the general classification result, by providing a level-preserving version of the Whitney move in codimension > 2 (Proposition 2.5).

If $\pi_1 N$ is not trivial then our classification result is similar to the 4-dimensional setting with common duals, namely fq_{U_*} gives the unique obstruction for homotopic embeddings to be isotopic. Our main work will be in spelling out the precise range of this obstruction and showing that all values in $\mathbb{A}_{U_*}$ are realized. A new issue that arises in the current setting is the distinction between based and free homotopies, whereas the assumption of common duals in the 4-dimensional setting essentially allowed for consideration of only based homotopies (see [Gabai 2020, Theorem 6.1; Schneiderman and Teichner 2022, Lemma 2.1]). We will occasionally emphasize this issue by applying the adjective "free" to the terms "homotopy" and "isotopy", even though by traditional definitions it would suffice to just omit the adjective "based".

1.1. *2-knots in 5-manifolds*. We now turn to precise formulations of our main results, working in the smooth oriented category throughout. Fixing a basepoint in such a 5-manifold N, and a basepoint in S^2, we have the following commutative diagram which will guide the discussion of our invariants:

$$
\begin{array}{ccc}
\pi_0 \, \mathrm{Emb}_*(S^2, N) & \xrightarrow{\;p_*\;} & \pi_0 \, \mathrm{Map}_*(S^2, N) \\
\Big\downarrow{\scriptstyle \mathrm{mod}\,\pi_1 N} & & \Big\downarrow{\scriptstyle \mathrm{mod}\,\pi_1 N} \\
\pi_0 \, \mathrm{Emb}(S^2, N) & \xrightarrow{\;p\;} & \pi_0 \, \mathrm{Map}(S^2, N)
\end{array}
$$

Here

$$\pi_0 \, \mathrm{Emb}_*(S^2, N) := \{\text{based embeddings } S^2 \hookrightarrow N\}/\text{based isotopy}$$

is the embedded version of $\pi_0 \, \mathrm{Map}_*(S^2, N) = \pi_2 N$, and

$$\pi_0 \, \mathrm{Emb}(S^2, N) := \{\text{embeddings } S^2 \hookrightarrow N\}/\text{free isotopy}$$

is the embedded version of $\pi_0 \, \mathrm{Map}(S^2, N) = [S^2, N]$. Both horizontal arrows forget the fact that we have embeddings, and the vertical arrows divide out the $\pi_1 N$-actions (using embedded tubes along closed paths at the basepoint on the

left-hand side). Both p and p_* are surjective (that is, any (based) map is homotopic to a (based) embedding) by general position: the dimension of the double point set is $5 - 6 = -1$ (so generically this set is empty), since codimensions of generic intersections add.

Along with using the label $*$ for based objects, our notational convention is to use a bracket to denote the homotopy class of an embedded object, which is otherwise considered up to isotopy. For example, the upper horizontal map p_* sends $R_* \in \pi_0 \operatorname{Emb}_*(S^2, N)$ to $[R_*] \in \pi_0 \operatorname{Map}_*(S^2, N)$, and the left vertical map sends R_* to $R \in \pi_0 \operatorname{Emb}(S^2, N)$. We are ultimately interested in the fibers of p, but it turns out to be convenient to first understand the fibers of p_*.

1.2. The based isotopy invariant. As recalled in (2.2) below, the quotient

$$\mathbb{A} := \frac{\mathbb{Z}\pi_1 N}{\langle g + g^{-1}, 1 \rangle}$$

of the integral fundamental group ring $\mathbb{Z}\pi_1 N \cong \mathbb{Z}\pi_1(N \times I)$ is the usual target for the self-intersection invariant

(1.3) $\mu_3 : \{\text{simply connected 3-manifolds immersed in the 6-manifold } N \times I\}$
$\to \mathbb{A}$,

which is invariant under homotopy rel boundary.

Let us fix a based embedding $U_* : S^2 \hookrightarrow N$, and define a homomorphism of abelian groups $\phi_{[U_*]} : \pi_3 N \to \mathbb{A}$ by

(1.4) $\phi_{[U_*]}(A) := \mu_3(A) + [\lambda_N(A, [U_*])]$,

where μ_3 denotes the self-intersection invariant on $\pi_3(N \times I) \cong \pi_3 N$, and λ_N is the intersection pairing between $\pi_3 N$ and $\pi_2 N$ taking values in $\mathbb{Z}\pi_1 N$.

Definition 1.5. $\mathbb{A}_{[U_*]} := \dfrac{\mathbb{A}}{\phi_{[U_*]}(\pi_3 N)}.$

Note that the abelian group $\mathbb{A}_{[U_*]}$ only depends on the based homotopy class $[U_*] \in \pi_2 N$. We now consider the fiber of p_* over $[U_*] \in \pi_2 N$:

$$p_*^{-1}([U_*]) = \{R_* : S^2 \hookrightarrow N \mid R_* \text{ is based homotopic to } U_*\}/(\text{based isotopy}).$$

Definition 1.6. For $R_* \in p_*^{-1}[U_*]$, let $H_* : S^2 \times I \looparrowright N \times I$ be a generic track of a based homotopy from U_* to R_*, and define

$$\operatorname{fq}_{[U_*]}(R_*) := [\mu_3(H_*)] \in \mathbb{A}_{[U_*]}.$$

By the following theorem, $\operatorname{fq}_{[U_*]}(R_*)$ does not depend on the choice of homotopy, and vanishes if and only if R_* and U_* are based isotopic.

Theorem 1.7. *The map* $\mathrm{fq}_{[U_*]} : p_*^{-1}[U_*] \to \mathbb{A}_{[U_*]}$ *is a bijection, whose inverse is given by a geometric action on* U_*.

The action of $g \in \pi_1 N$ on $U_* \in \pi_0 \mathrm{Emb}_*(S^2, N)$ is by a "finger move" along g, which in this setting is an ambient connected sum of U_* with its meridian sphere m_{U_*} along a tube following a loop representing g. Elements in the group ring act by multiple finger moves, which turn out to involve signs and preserve the relations in the quotient $\mathbb{A}$ of the group ring (see Section 2.3). The proof of Theorem 1.7 shows the following.

Corollary 1.8. *The abelian group* $\mathbb{A}$ *acts on* $\pi_0 \mathrm{Emb}_*(S^2, N)$ *compatibly with the* $\pi_1 N$-*actions, preserving* p_* *and acting transitively on its fibers, with the stabilizer of* U_* *equal to* $\phi_{[U_*]}(\pi_3 N)$.

1.3. *The free isotopy invariant.* Now consider *free* homotopy versus isotopy, i.e., the set

$$p^{-1}[U] := \{R : S^2 \hookrightarrow N \mid R \text{ is freely homotopic to } U\}/(\text{free isotopy})$$

for $U : S^2 \hookrightarrow N$ a fixed embedding in $\mathrm{Emb}(S^2, N) \subset \mathrm{Map}(S^2, N)$. Choose $U(e) \in N$ as the basepoint for N, where e denotes the basepoint for S^2.

To define the target of an invariant that characterizes $p^{-1}[U]$ we will define an affine action on $\mathbb{A}_{[U_*]}$ (the range of the bijection in the based setting of Theorem 1.7) by the group

$$\mathrm{Stab}[U_*] := \{s \in \pi_1 N : s \cdot [U_*] = [U_*]\},$$

that is, the stabilizer subgroup of $[U_*] \in \pi_2 N$ of the usual action of $\pi_1 N$ on $\pi_2 N$.

Recall that an *affine transformation* T of an abelian group A is given by an endomorphism ℓ and a translation a_0 of A, i.e., $T(a) = a_0 + \ell(a)$, where $a_0 = T(0)$. An *affine action* of a group on A is a homomorphism to the group of affine transformations of A. In our case, the linear action of $s \in \mathrm{Stab}[U_*]$ will be $a \mapsto sas^{-1}$, whereas the translational part will be given by U_s, both of which we explain next.

Firstly, we claim that the linear action $(s, a) \mapsto sas^{-1}$ of $\mathrm{Stab}[U_*]$ on $\mathbb{Z}\pi_1 N$ descends to $\mathbb{A}_{[U_*]}$: for $A \in \pi_3 N$ we have

$$\mu_3(g \cdot A) = g \mu_3(A) \, g^{-1} \quad \text{and} \quad \lambda_N(g \cdot A, [U_*]) = g \lambda_N(A, [U_*]),$$

so if $g \cdot [U_*] = [U_*]$ then the last expression also equals $g \lambda_N(A, [U_*]) g^{-1}$, implying

$$g \phi_{[U_*]}(A) g^{-1} = \phi_{[U_*]}(g \cdot A) \in \phi_{[U_*]}(\pi_3 N).$$

Secondly, for $s \in \mathrm{Stab}[U_*]$ there is a generic track $J_s : S^2 \times I \looparrowright N \times I$ of a free self-homotopy of U_* such that the projection of $J_s(e, -)$ to N represents s. It turns out that

$$U_s := [\mu_3(J_s)] \in \mathbb{A}_{[U_*]}$$

only depends on s and the isotopy class U_* (and not on J_s, see Lemma 3.1). It is easy to show that under concatenation, for $s, r \in \mathrm{Stab}[U_*]$ this behaves as

$$U_{s \cdot r} = U_s + sU_r\, s^{-1}.$$

This implies that the formula

(aff) $$^s a := U_s + sas^{-1}$$

satisfies

(1.9) $^{sr}a = U_{sr} + srar^{-1}s^{-1} = U_s + sU_r\, s^{-1} + srar^{-1}s^{-1} = {}^s(U_r + rar^{-1}) = {}^s({}^r a).$

In other words, the composition in the group $\mathrm{Stab}[U_*]$ turns into the composition of affine transformations, so $s \mapsto {}^s a$ defines an affine action of $s \in \mathrm{Stab}[U_*]$ on $a \in \mathbb{A}_{[U_*]}$. This action and the following definition will be examined and clarified below in Section 3.2.

Definition 1.10. Denote by $\mathbb{A}_{U_*}$ the quotient set of this affine action by $\mathrm{Stab}[U_*]$ on $\mathbb{A}_{[U_*]}$.

Note that the definition of $\mathbb{A}_{U_*}$ depends on the based isotopy class U_*, a fixed basing of U. However, the following result shows that it characterizes the set of *free* isotopy classes of embeddings homotopic to U.

Theorem 1.11. *There is a bijection*

$$\mathrm{fq}_{U_*} : p^{-1}[U] \to \mathbb{A}_{U_*}, \quad R \mapsto [\mu_3(H)],$$

where H is a generic track of any free homotopy from U to R.

Here both the computation of $\mu_3(H)$ and the definition of $\mathbb{A}_{U_*}$ use the same basing U_*, and the inverse of the bijection is defined using the same geometric action as in Corollary 1.8.

Curiously, in the based setting of Theorem 1.7, both the target $\mathbb{A}_{[U_*]}$ and the set $p_*^{-1}[U_*]$ only depend on the based homotopy class $[U_*] \in \pi_2 N$, whereas in the free setting of Theorem 1.11, the target $\mathbb{A}_{U_*}$ depends on the *based isotopy* class $U_* \in \pi_0 \mathrm{Emb}_*(S^2, N)$, while the set $p^{-1}[U]$ depends on the *free homotopy* class $[U] \in \pi_0 \mathrm{Map}(S^2, N)$ of the embedding U.

Example. If U_* is the trivial 2-sphere then $U_s = 0$ for all $s \in \mathrm{Stab}[U_*] = \pi_1 N$ because any self-homotopy J_s can be chosen to be a self-isotopy. As a consequence:

Corollary 1.12 (null-homotopic isotopy classes). *Null-homotopic free isotopy classes of 2-spheres in N are in bijection with $\mathbb{A}_{U_*} = \mathbb{A}_{[U_*]}/\pi_1 N$, the quotient of the abelian group*

$$\mathbb{A}_{[U_*]} = \frac{\mathbb{Z}[\pi_1 N]}{\langle 1,\, g + g^{-1},\, \mu_3(A) : g \in \pi_1 N,\, A \in \pi_3 N \rangle},$$

by the conjugation action of $\pi_1 N$.

Note that although the free self-isotopies of U have vanishing self-intersection invariants, they still contribute to the indeterminacy of the invariant fq_{U_*} by arbitrarily conjugating double-point group elements in the computation of $\mu_3(H)$.

Example. If $\lambda_N(G, [U_*]) = 1$ for some $G \in \pi_3 N$ such that $\mu_3(G) = 0 \in \mathbb{A}$, then $\phi_{[U_*]}(\pi_3 N) = \mathbb{A}$, since for any $\sum g_i \in \mathbb{Z}[\pi_1 N]$ we have

$$\phi_{[U_*]}(\textstyle\sum g_i \cdot G) = 0 + \big[\lambda_N(\textstyle\sum g_i \cdot G, [U_*])\big] = \big[\textstyle\sum g_i \cdot \lambda_N(G, [U_*])\big] = \big[\textstyle\sum g_i\big].$$

It follows that in this case $\mathbb{A}_{[U_*]}$ contains a single element, and hence so does $\mathbb{A}_{U_*}$.

Corollary 1.13 (5-dimensional light bulb theorem). ***Homotopy implies isotopy*** *for spheres* $U_* : S^2 \hookrightarrow N^5$ *admitting an* ***algebraic dual*** $G \in \pi_3 N$ *as above.*

See Section 3.3 for more examples.

1.4. *Isotopy classification via mapping spaces.* In Section 4 we present a slightly different perspective to the problem of isotopy classification. Namely, the fibers of the map $p : \pi_0 \operatorname{Emb}(S^2, N) \to \pi_0 \operatorname{Map}(S^2, N)$ can also be determined using the homotopy exact sequence associated to the inclusion of mapping spaces $\operatorname{Emb}(S^2, N) \subset \operatorname{Map}(S^2, N)$:

$$\pi_1(\operatorname{Map}(S^2, N), U)$$
$$\Big\downarrow{\scriptstyle j}$$
$$\pi_1(\operatorname{Map}(S^2, N), \operatorname{Emb}(S^2, N), U) \longrightarrow \pi_0 \operatorname{Emb}(S^2, N) \xrightarrow{\ p\ }\!\!\!\!\to \pi_0 \operatorname{Map}(S^2, N).$$

Here we picked an embedding $U : S^2 \hookrightarrow N$ as a basepoint, and the leftmost absolute π_1 is a group that acts on the relative π_1 (which is just a set) such that the quotient set is isomorphic to the fiber $p^{-1}[U]$ of p over $[U] \in \pi_0 \operatorname{Map}(S^2, N)$.

The relative π_1 is the first nonvanishing relative homotopy group, and that is exactly what was computed by Jean-Pierre Dax [1972]. He translated this (and also other relative homotopy groups in the "metastable range") to certain bordism groups. Computations of this bordism group (which is 0-dimensional in the first nonvanishing case) were carried out in [Kosanović and Teichner 2024b, Theorem 4.14], for the cases $(\operatorname{Imm}_\partial(V, X), \operatorname{Emb}_\partial(V, X))$ when embeddings have nonempty boundary condition, and V is 1-connected. In [Kosanović et al. $\geq$ 2024] we extend this to closed and disconnected manifolds, and compare to maps instead of immersions. Specializing [Kosanović et al. $\geq$ 2024] to $V = S^2$ and $d = 5$ leads to Theorem 4.1: there is a bijection

$$(1.14) \qquad \operatorname{Dax} : \pi_1(\operatorname{Map}(S^2, N), \operatorname{Emb}(S^2, N), U) \to \mathbb{A}, \quad h \mapsto \mu_3(H),$$

where H is a homotopy from U to an embedding that represents h, and $\mu_3(H) \in \mathbb{A}$ is the self-intersection invariant of a generic track H of the homotopy, as in (1.3).

In order to compute the set $p^{-1}[U]$ from this viewpoint it remains to understand the action of the absolute π_1 on the relative π_1 in the sequence displayed above. To the best of our current knowledge this does not seem to have been done previously, and guided by our classification using self-intersection invariants of homotopy tracks we proceed as follows: Firstly, we use the fibration sequence

$$\mathrm{Map}_*(S^2, N) \to \mathrm{Map}(S^2, N) \to N$$

to obtain exactness in the column of the following diagram, where the first map is the inclusion i and the second map evaluates at the basepoint $e \in S^2$:

$$\pi_3 N \cong \pi_1(\mathrm{Map}_*(S^2, N), U_*)$$

$$\downarrow i$$

$$\pi_1(\mathrm{Map}(S^2, N), U) \xrightarrow{\ \ j\ \ } \pi_1(\mathrm{Map}(S^2, N), \mathrm{Emb}(S^2, N), U) \xrightarrow[\cong]{\mathrm{Dax}} \mathbb{A}$$

$$\downarrow ev_e$$

$$\mathrm{Stab}[U_*]$$

By definition the composite $\mathrm{Dax} \circ j \circ i$ sends $\beta \in \pi_3 N$ to $\mathrm{Dax}(A)$ where A is a loop in $\mathrm{Map}_*(S^2, N)$, based at U. We will see that this precisely agrees with $\phi_{[U_*]}$ from (1.4).

Moreover, we will see that the induced action of $s \in \mathrm{Stab}[U_*]$ on the quotient of $\mathbb{A}$ by $\mathrm{Dax} \circ j \circ i(\pi_3 N)$ sends $a = \mathrm{Dax}(H)$ to

$$(1.15) \qquad\qquad \mathrm{Dax}(J_s) + sas^{-1},$$

where J_s is a free self-homotopy of U_* such that $ev_e(J_s) = s$, i.e., the projection of $J_s(e, -)$ to N represents s. This action is precisely (aff), so we recover

$$p^{-1}[U] \cong \mathbb{A}_{U_*}$$

as in Theorem 1.11, except that instead of fq_{U_*} this map is now naturally called Dax. This will be stated as Theorem 4.7.

Remark 1.16. Using the analogous fibration sequence

$$\mathrm{Emb}_*(S^2, N) \to \mathrm{Emb}(S^2, N) \to N,$$

we show in [Kosanović et al. $\geq$ 2024] that there is an isomorphism

$$i^{\mathrm{rel}}: \pi_1(\mathrm{Map}_*(S^2, N), \mathrm{Emb}_*(S^2, N), U_*) \xrightarrow{\ \cong\ } \pi_1(\mathrm{Map}(S^2, N), \mathrm{Emb}(S^2, N), U).$$

In Theorem 4.6 we will show that $\mathrm{Dax} \circ i^{\mathrm{rel}} \circ j_* = \phi_{[U_*]}$ precisely gives the indeterminacy for the based setting: $p_*^{-1}[U_*] \cong \mathbb{A}/\phi_{[U_*]}(\pi_3 N)$, as in Theorem 1.7. Moreover, this shows that $\mathrm{Dax} \circ i^{\mathrm{rel}} \circ j_* = \mathrm{Dax} \circ j \circ i$. Therefore, similarly to our

first approach, in this approach we see: a linear action for the based setting (just the quotient by the image of $\phi_{[U_*]}$), and an affine action for the free setting (the further quotient by the action (aff)).

2. Intersection invariants and homotopies

2.1. *3-manifolds in 6-manifolds.* Recall that for a smooth oriented 6-manifold P^6, the intersection and self-intersection invariants give maps

$$\lambda_3 : \pi_3 P \times \pi_3 P \to \mathbb{Z}\pi_1 P \quad \text{and} \quad \mu_3 : \pi_3 P \to \mathbb{Z}\pi_1 P / \langle g + g^{-1}, 1 \rangle.$$

To compute λ_3 geometrically, start by representing the two homotopy classes by transverse based maps $S^3 \to P$, and then count each intersection point p with a sign ϵ_p determined by orientations and a group element $g_p \in \pi_1 P$ represented by a sheet-changing loop through p. Here a *based map* is equipped with a *whisker*, which is an arc running between a basepoint on the image of the map and the basepoint of the ambient manifold P. Note that by general position a map of a manifold of codimension > 1 is ambient isotopic to a map whose basepoint is equal to the basepoint of the ambient manifold.

Similarly, for μ_3 one represents the homotopy class by a generic map $A : S^3 \looparrowright P$ and counts self-intersections, again with signs and group elements. In this dimension, switching the ordering of sheets at a double point p changes ε_p to $-\varepsilon_p$, and changes g_p to g_p^{-1}, explaining the relation $g + g^{-1} = 0$ in the range of μ_3. The relation $1 = 0$ makes $\mu_3(A)$ only depend on the homotopy class of A, since a cusp homotopy introduces a double point with arbitrary sign and trivial group element. Changing the whisker on A changes $\mu_3(A)$ by a conjugation, with the corresponding group element represented by the difference of the whiskers. The argument for homotopy invariance of μ_3 arises from considering the double-point arcs and circles of the track of a generic homotopy $S^3 \times I \looparrowright P^6 \times I$ of A.

Using the involution $\bar{g} := g^{-1}$ on $\mathbb{Z}\pi_1 P$, the "quadratic form" (λ_3, μ_3) satisfies

(2.1)
$$\mu_3(A + B) = \mu_3(A) + \mu_3(B) + [\lambda_3(A, B)],$$
$$\lambda_3(A, A) = \mu_3(A) - \overline{\mu_3(A)} \in \mathbb{Z}\pi_1 P,$$

where the second formula has no content for the coefficient at the trivial element in $\pi_1 P$: Since λ_3 is skew-hermitian, it vanishes on the left-hand side, whereas it is automatically zero on the right-hand side that is defined by picking a representative of $\mu_3(A) \in \mathbb{Z}\pi_1 P$ and then applying the involution to that specific choice. We will be interested in the case that $P = N \times I$ is the product of a 5-manifold N with an interval I, and we denote the target of μ_3 by

$$\mathbb{A} := \frac{\mathbb{Z}\pi_1 N}{\langle g + g^{-1}, 1 \rangle} \cong \frac{\mathbb{Z}\pi_1 P}{\langle g + g^{-1}, 1 \rangle}.$$

2.2. *The self-intersection invariant for homotopies of 2-spheres in 5-manifolds.*
The above descriptions of λ_3 and μ_3 can also be applied to properly immersed simply connected 3-manifolds with boundary in a 6-manifold. In this setting the invariants are computed just as above, by summing signed double point group elements, and are invariant under homotopies that restrict to isotopies on the boundary. For a smooth oriented 5-manifold N, and any homotopy $H : S^2 \times I \to N$ between embedded spheres, we define the self-intersection invariant

$$(2.2) \qquad\qquad \mu_3(H) \in \mathbb{A}$$

to be the self-intersection invariant of a generic track $S^2 \times I \looparrowright N \times I$ for H. We will sometimes use the same letter H to denote either the homotopy or its track when the context is clear.

The "time" parameter (the I-factor) of a homotopy will generally be the unit interval $I = [0, 1]$, although it will be frequently suppressed from notation or reparametrized without mention.

For the purpose of computing the self-intersection invariant $\mu_3(H)$, the whisker on the track of H will be taken at the "start" $H(S^2 \times 0) \subset N \times 0 \subset N \times I$ of the homotopy unless explicitly stated otherwise. So for a homotopy H from $U_* : S^2 \hookrightarrow N$ to an embedding, the whisker for U_* will generally be used to compute $\mu_3(H)$.

Note that choosing a whisker on the track of a homotopy to provide a "basing" for the purposes of computing an intersection invariant is different than saying that the homotopy is a "based homotopy", which is "a homotopy through based maps".

2.3. *Geometric action of $\mathbb{A}$.* For $g \in \pi_1 N$ and $U_* : S^2 \hookrightarrow N$, we define $g \cdot U_* : S^2 \hookrightarrow N$
as follows, see Figure 2.3 for several examples. Firstly, note that the normal bundle of U_* is 3-dimensional, so its meridian m_F is a 2-sphere; we choose it over a point near the basepoint $z := U_*(e)$ and orient it according to the orientations of S^2 and N. We then define $g \cdot U_*$ as an ambient connected sum of U_* with m_F along a tube following an arc representing g, where the arc starts and ends near z and has interior disjoint from U_*. This is well defined up to isotopy since removing a neighborhood of U_* does not change the fundamental group and "homotopy implies isotopy" for arcs in this dimension.

Similarly, $(-g) \cdot U_*$ is defined to be the connected sum of U_* with the oppositely oriented meridian sphere $-m_F$. Linear combinations $\sum_i n_i g_i$ act by multiple connected sums along g_i into copies of m_F for $n_i > 0$ respectively $-m_F$ for $n_i < 0$. It is not hard to check that the relations $g + g^{-1} = 0 = 1$ carry over to isotopies of these connected sums, see Figure 2.3. Therefore, we have an action of $\mathbb{A}$ on the set $\pi_0 \mathrm{Emb}_*(S^2, N)$.
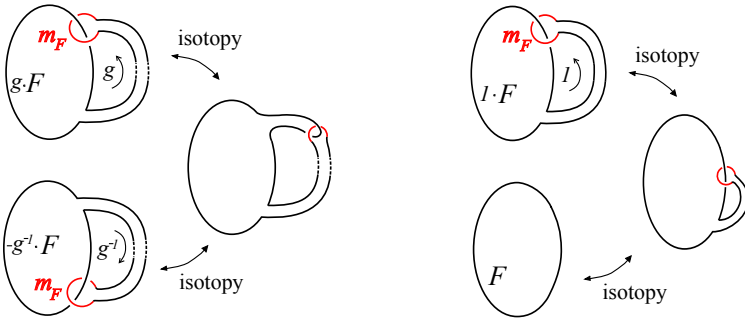
Figure 2.3. The relations $g = -g^{-1}$ and $1 = 0$ realized by isotopies.

Since each meridian sphere m_F bounds a normal 3-ball that intersects U_* exactly once, we get the following result.

Lemma 2.4. *For any $a \in \mathbb{A}$, shrinking the meridian spheres along their 3-balls gives a based homotopy H_a from $a \cdot U_*$ to U_* with $\mu_3(H_a) = a$.* ☐

2.4. *The level-preserving Whitney move.*

Proposition 2.5. *The track $H : S^2 \times I \to N^5 \times I$ of a homotopy between two embeddings is homotopic (rel boundary) to the track of an isotopy if and only if its self-intersection invariant $\mu_3(H) \in \mathbb{A}$ vanishes.*

It will follow from the proof that if the original homotopy is a based homotopy, then the resulting isotopy can be taken to be based. In fact, the construction given in the proof can be taken to be supported away from any I-family of whiskers.

We remark that since the classical Whitney move works for immersed 3-manifolds in 6-manifolds [Milnor 1965, Theorem 6.6], the vanishing of $\mu_3(H) \in \mathbb{A}$ immediately implies that the track $S^2 \times I \hookrightarrow N \times I$ is homotopic (rel boundary) to a concordance, so Proposition 2.5 would then follow from Hudson's theorem that concordance implies isotopy in codimensions ≥ 3 [Hudson 1970]. Rather than invoking Hudson's result, our proof of Proposition 2.5 will show that one can arrange for the Whitney moves to preserve I-levels in order to directly achieve an isotopy rather than just a concordance.

Proof of Proposition 2.5. The "only if" direction is clear since μ_3 is invariant under homotopy and vanishes on embeddings.

To prove the "if" direction, we first introduce some streamlined notation that will only be used in the proof of Proposition 2.5, including the ancillary Lemma 2.6.

Notation. For any subset $\sigma \subset I$, denote by $H_\sigma := H|_{S^2 \times \sigma}$ the restriction to $S^2 \times \sigma$ of the track $H : S^2 \times I \to N \times I$. By the standard abuse of the notation, $H_\sigma := H(S^2 \times \sigma)$ is also the image of this map, and is contained in the subset $N_\sigma := N \times \sigma \subset N \times I$.

Lemma 2.6. *For H as in Proposition 2.5 with $\mu_3(H) = 0 \in \mathbb{A}$, it may be arranged by a homotopy rel ∂ that there exist finitely many distinct points $c_i \in I$ such that the transverse self-intersections of H occur in pairs $\{p_i, q_i\} \subset H_{c_i}$ with $g_{p_i} = g_{q_i}$ and $\epsilon_{p_i} = -\epsilon_{q_i}$ for each i.*

Assuming Lemma 2.6 (which will be proved just below), Proposition 2.5 follows by a standard application of Whitney moves to eliminate each of the self-intersection pairs of Lemma 2.6 in a way that yields the track of an isotopy. We describe details here for completeness, with the key observation being that each Whitney disk can be chosen to be contained in a level.

Dropping the subscript i from the notation, let p and q be a pair of self-intersections of H_c as in Lemma 2.6. Since H_c is a map of a 2-sphere $S^2 \times c$, the self-intersections p and q are not transverse for H_c, but there exists some small $\delta > 0$ such that $\{p, q\} = H_{[c-\delta, c+\delta]} \pitchfork H_{[c-\delta, c+\delta]}$, the transverse self-intersections of the immersed 3-manifold $H_{[c-\delta, c+\delta]}$ in the 6-manifold $N_{[c-\delta, c+\delta]}$.

Since p and q have the same group elements $g_p = g_q$ and opposite signs $\epsilon_p = -\epsilon_q$, there exists a *Whitney disk* $W \subset N_{[c-\delta, c+\delta]}$ pairing p and q. By general position we may assume that W is embedded in the 5-dimensional slice $N_c \subset N_{[c-\delta, c+\delta]}$ with interior disjoint from H_c. The Whitney disk boundary $\partial W = \alpha \cup \beta$ is the union of embedded arcs α and β contained in H_c, with $\alpha \cap \beta = \{p, q\}$. Let $\bar{\alpha}$ and $\bar{\beta}$ be slightly longer arcs in H_c containing α and β, respectively, such that $\bar{\alpha}$ and $\bar{\beta}$ extend just beyond p and q.

Let $A, B \subset H_{[c-\delta, c+\delta]}$ denote regular 3-ball neighborhoods of $\bar{\alpha}$ and $\bar{\beta}$ in $H_{[c-\delta, c+\delta]}$. Each of A and B is "the image of a local sheet of a 2-sphere H_t moving in time", with A_t and B_t each embedded in H_t for $t \in [c-\delta, c+\delta]$, and $A \cap B = \{p, q\}$. It follows that $A : D^2 \times I \hookrightarrow N \times I$ is the track of an isotopy A_t, and similarly for B.

The Whitney move that eliminates p and q will be described using a particular choice of coordinates for an open neighborhood $V \subset N_{[c-\delta, c+\delta]}$ containing W.

By [Milnor 1965, Lemma 6.7], the neighborhood V may be chosen to be diffeomorphic to $\overline{W} \times \mathbb{R}_A^2 \times \mathbb{R}_B^2$, where:

- $\overline{W} \subset N_c$ is a smooth 2-disk formed from W by attaching a half-open collar to ∂W.
- $V \cap A = \bar{\alpha} \times \mathbb{R}_A^2 \times (0, 0)$ and $V \cap B = \bar{\beta} \times (0, 0) \times \mathbb{R}_B^2$.

Let $\bar{\alpha}(s)$ be a smooth isotopy of the arc $\bar{\alpha}$ in $\overline{W}$, for $0 \le s \le 1$, such that $\bar{\alpha}(0) = \bar{\alpha}$, and $\bar{\alpha}(1)$ passes just above $\beta \subset \bar{\beta}$ as in Figure 2.7. In particular, $\bar{\alpha}(s)$ is supported near $W \subset \overline{W}$ for all $0 \le s \le 1$, and $\bar{\alpha}(1)$ is disjoint from B.

Let $\rho : \mathbb{R}^2 \to [0, 1]$ be a smooth bump function $(u, v) \mapsto \rho(u, v)$ such that

- $\rho(u, v) = 1$ if $\sqrt{u^2 + v^2} \le 1$,
- $\rho(u, v) = 0$ if $\sqrt{u^2 + v^2} \ge 2$.
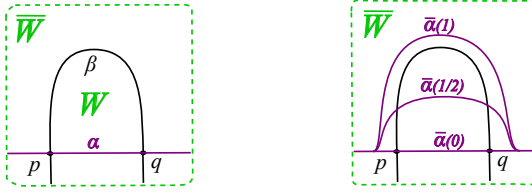
Figure 2.7. A smooth isotopy of the arc $\bar{\alpha}$ in $\overline{W}$.

Now we use $\bar{\alpha}(s)$ and ρ to define the Whitney move as the result of an isotopy $A(s)$, $0 \leq s \leq 1$ of A which fixes B, where (a_1, a_2) runs through $\mathbb{R}_A^2$:

$$A(s) = \bar{\alpha}(s\rho(a_1, a_2)) \times (a_1, a_2) \times (0, 0) \subset \overline{W} \times \mathbb{R}_A^2 \times \mathbb{R}_B^2,$$

Then $A(0) = A$, and the result of the W-Whitney move on A is $A' := A(1)$, so that $A' \cap B = \varnothing$.

Note that $A(s) = A$ near ∂V, and hence we can extend $A(s)$ to be the identity outside V. This defines a homotopy $H(s)$ of $H = H(0)$ such that $H' := H(1)$ satisfies $H' \pitchfork H' = (H \pitchfork H) - \{p, q\}$.

By construction $A(s)$ only moves points of A along the $\overline{W}$-factor, which is orthogonal to the I-factor of $N \times I$ since $\overline{W} \subset N_c$. This means that for each s, $A(s)$ consists of the track of an isotopy $A_t(s)$, and similarly for $B(s)$.

Performing Whitney moves on all the self-intersection pairs of Lemma 2.6 yields the track of an isotopy as desired.   □

Proof of Lemma 2.6. The condition $\mu_3 H = 0 \in \mathbb{A}$ means that the (finite) set of transverse self-intersections of the generic track $H : S^2 \times I \looparrowright N \times I$ can be decomposed into finitely many pairs $\{p_i, q_i\}$ with $g_{p_i} = g_{q_i}$ and $\epsilon_{p_i} = -\epsilon_{q_i}$ (for appropriately chosen sheets, and after perhaps performing a single cusp homotopy on H).

Suppose, for some i we have $p_i \in H_{t_1} \cap H_{t_1} \subset H \pitchfork H$ and $q_i \in H_{t_2} \cap H_{t_2} \subset H \pitchfork H$, with $t_1 < t_2$. We will describe how to change H by an isotopy rel boundary which "moves" q_i to $q_i' \in H_{t_1} \cap H_{t_1}$ and is supported away from all other self-intersections of H. The construction will show more generally that self-intersections of H can be arranged to occur at any chosen times, while preserving signs and group elements.

Special case. Consider the special case that H has just a single pair $\{p, q\} = H \pitchfork H$ of self-intersections with $p \in H_{t_1} \cap H_{t_1}$ and $q \in H_{t_2} \cap H_{t_2}$, and with $0 < t_1 < t_2 < 1$ in $I = [0, 1]$. Suppose that (x, t_2) and (y, t_2) are the two distinct preimages in $S^2 \times t_2 \subset S^2 \times I$ of $q = H_{t_2}((x, t_2)) = H_{t_2}((y, t_2))$. Define vertical arcs $a := x \times [t_1, t_2]$ and $b := y \times [t_1, t_2]$ in the domain $S^2 \times I$ (see Figure 2.8, left).

For $0 \leq s \leq 1$ we will define smooth isotopies $\psi_s : S^2 \times I \to S^2 \times I$ supported near $a \cup b$ and projecting to the identity on S^2 such that the smooth family of homotopies $H \circ \psi_s$ satisfies $H = H \circ \psi_0$ and $(H \circ \psi_1) \pitchfork (H \circ \psi_1) = \{p, q'\} \subset (H \circ \psi_1)_{t_1}$.
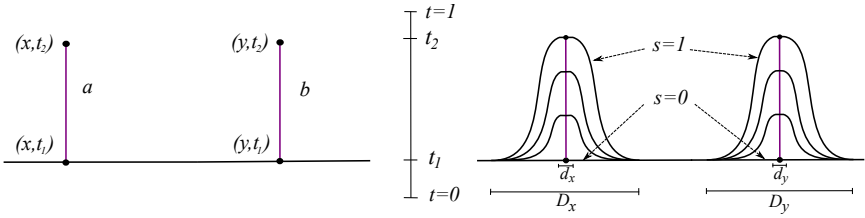
Figure 2.8. Schematic pictures in the domain $S^2 \times I$, with the I-factor running vertically from bottom to top. Left: the vertical arcs a and b, and a horizontal subarc of $S^2 \times t_1$. Right: images of the subarc of $S^2 \times t_1$ containing the nested disks $d_x \subset D_x$ and $d_y \subset D_y$ under the isotopy ψ_s for $s = 0$, $s = 1$, and for two other intermediate values of s.
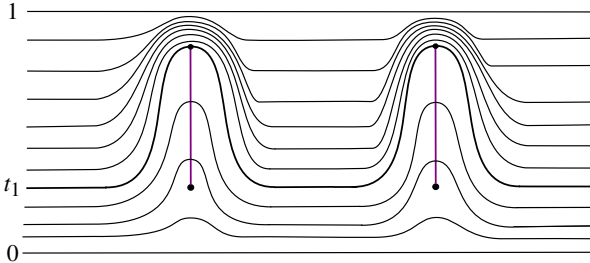
Figure 2.9. A schematic picture of the images under ψ_1 of some horizontal slices of $S^2 \times I$.

First we define ψ_s on $S^2 \times t_1$ as the sum of two local bump functions of height $s(t_2 - t_1)$ in the positive I-direction centered at (x, t_1) and (y, t_1) (see Figure 2.8, right). More specifically, let $x \in d_x \subset D_x \subset S^2$ and $y \in d_y \subset D_y \subset S^2$ be small concentric pairs of nested disks around x and y, respectively. Then

$$\psi_s(z, t_1) = \begin{cases} (z, t_1) & \text{if } z \notin D_x \cup D_y, \\ (z, t_1 + s(t_2 - t_1)) & \text{if } z \in d_x \cup d_y, \\ (z, t_1 + \text{sig}(z)\, s(t_2 - t_1)) & \text{if } z \in (D_x \setminus \text{int}(d_x)) \cup (D_y \setminus \text{int}(d_y)). \end{cases}$$

Here the sigmoid function $\text{sig}(z)$ smoothly interpolates between $\text{sig}(z) = 0$ for $z \in \partial D_x \cup \partial D_y$ and $\text{sig}(z) = 1$ for $z \in \partial d_x \cup \partial d_y$.

Now extend ψ_s to all of $S^2 \times I$ by tapering the bump functions down to zero as t moves away from t_1, so that $\psi_s(z, 0) = (z, 0)$ and $\psi_s(z, 1) = (z, 1)$ for all s. See Figure 2.9 for an illustration of the extended $\psi_1 : S^2 \times I \to S^2 \times I$.

Next we check that $H' := H \circ \psi_1$ has the desired properties. Observe that since ψ_s restricts to the identity map on the complement of $(D_x \cup D_y) \times I$, we have $H' = H$ when restricted to $(S^2 \setminus (D_x \cup D_y)) \times I$. In particular, $H'((S^2 \setminus (D_x \cup D_y)) \times I)$ has only the single transverse self-intersection point $p \in H \pitchfork H$ which occurs at t_1.

Now consider the restriction of H' to $(D_x \cup D_y) \times I$. We have that by construction, $H'((D_x \cup D_y) \times I)$ has just the single transverse self-intersection such that $q' = H'(d_x, t_1) \cap H'(d_y, t_1)$, which is the image of the projection to N_{t_1} of $q \in H \pitchfork H$.

Now it remains to check that there are no transverse intersections between $H'((D_x \cup D_y) \times I)$ and $H'((S^2 \setminus (D_x \cup D_y)) \times I)$. For each $t \in I$, suppose that $\mathrm{proj}_t : N \times I \to N_t$ is the projection map. By general position, for each $t \in I$ the image $\mathrm{proj}_t \circ H(a \cup b) \subset N_t$ is an embedded arc γ_t. Since the image $H'((D_x \cup D_y) \times I)$ in the 6-manifold $N \times I$ can be arranged to be contained in an arbitrarily small neighborhood of the 2-dimensional union $\cup_{t \in I} \gamma_t$, it follows by general position that $H'((D_x \cup D_y) \times I)$ has no transverse intersections with the 3-dimensional $H'((S^2 \setminus (D_x \cup D_y)) \times I)$.

General case. Since by general position any number of self-intersections can be assumed to have preimages projecting to distinct points in S^2, the above construction moving $q \in H_{t_2} \cap H_{t_2}$ to $q' \in H'_{t_1} \cap H'_{t_1}$ for $t_1 < t_2$ can be carried out iteratively (or even simultaneously) for any chosen subset of self-intersections while fixing the complementary subset. $\qquad\square$

2.5. *Based self-homotopies.* Recall from [Section 1.2](#) of the introduction that, for a fixed based embedding $U_* : S^2 \hookrightarrow N$, we denote by $\mathbb{A}_{[U_*]}$ the quotient of $\mathbb{A}$ by the image of the indeterminacy homomorphism $\phi_{[U_*]} : \pi_3 N \to \mathbb{A}$ defined by $A \mapsto \mu_3(A) + [\lambda_N(A, [U_*])]$.

The following lemma will be used to show that our invariants are well defined.

Lemma 2.10. *If $J_* : S^2 \times I \looparrowright N \times I$ is a generic track of a based self-homotopy of U_*, then*

$$\mu_3(J_*) = 0 \in \mathbb{A}_{[U_*]}.$$

Proof. Since J_* is a based self-homotopy, it agrees with $U_* \times I$ on the 2-skeleton $S^2 \times \{0, 1\} \cup e \times I$ of $S^2 \times I$, with $e \in S^2$ the basepoint. So they only differ on the 3-cell, where $U_* \times I$ is represented by $B_U := U_*(D^2) \times I$ (here D^2 is the complement in S^2 of a small disk around e) and J_* is represented by a generic 3-ball $B_J : D^3 \looparrowright (N \times I) \setminus \nu(U_*(e) \times I)$. By construction, the boundaries of these two 3-balls are parallel copies of an embedded 2-sphere in the boundary of a small neighborhood of $U_* \times \{0, 1\} \cup (U_*(e) \times I)$.

Gluing B_U and B_J together along a small embedded cylinder $S^2 \times I$ between their boundaries yields a map of a 3-sphere $A := B_J \cup (-B_U) : S^3 \to N \times I$. To prove the lemma we will show that $\mu_3(J_*) = \phi_{[U_*]}(A)$.

First note that on one hand, all contributions to $\mu_3(J_*)$ come from the self-intersections of the immersed 3-ball B_J. On the other hand, contributions to $\mu_3(A)$ come from the self-intersections of B_J and the intersections between B_J and

the embedded 3-ball $-B_U$. The latter intersections are precisely counted by $-\lambda_3(J_*, U_* \times I)$, see (2.1). Therefore,

$$\mu_3(J_*) - \mu_3(A) = \lambda_3(J_*, U_* \times I),$$

and since $\lambda_N(A, U_*) = \lambda_3(A, U_* \times I) = \lambda_3(J_*, U_* \times I)$, we obtain

$$\mu_3(J_*) = \mu_3(A) + \lambda_N(A, U_*) = \phi_{[U_*]}(A). \qquad \square$$

2.6. *From homotopy to isotopy by adding 3-spheres.* The following lemma will be used to show that our invariants are injective.

Lemma 2.11. *Suppose $H : S^2 \times I \hookrightarrow N \times I$ is a generic track of a homotopy between embeddings $R : S^2 \hookrightarrow N$ and $R' : S^2 \hookrightarrow N$ such that the homotopy restricts to an embedding $U : S^2 \hookrightarrow N$ for some point $t_0 \in I$. Then R is isotopic to R' if $\mu_3(H) = 0 \in \mathbb{A}_{[U_*]}$. Moreover, if H is a based homotopy, then the resulting isotopy may be taken to be based.*

Proof. Since $\mu_3(H) = 0 \in \mathbb{A}_{[U_*]}$, there exists $A \in \pi_3 N$ such that $\mu_3(H) = \phi_{[U_*]}(A)$. By using a small ambient isotopy we may assume that H restricts to a product $U \times [t_0 - \epsilon, t_0 + \epsilon]$ on a small interval around t_0. Represent A by a generic regular homotopy $f_t : S^2 \times [t_0 - \epsilon, t_0 + \epsilon] \to N$ from a local trivial sphere $f_{t_0 - \epsilon} = f_{t_0 + \epsilon}$ in N to itself via the isomorphism $\pi_3 N \cong \pi_1(\mathrm{Map}_*(S^2, N))$. Taking a smooth family of ambient connected sums of f_t with $U \times t \subset U \times [t_0 - \epsilon, t_0 + \epsilon]$ yields a self-homotopy J^A of U. We can assume the guiding paths for these connected sums have interiors disjoint from every f_t and $U \times t$, so that $\mu_3(J^A) = \phi_{[U_*]}(A)$. Reversing the t-parameter of J^A yields the track $-J^A$ of a self-homotopy of U with $\mu_3(-J^A) = -\phi_{[U_*]}(A)$.

Now, deleting $H \times [t_0 - \epsilon, t_0 + \epsilon]$ from $H \times I$ and gluing in $-J^A$ yields a based homotopy H^0 between R and R' with $\mu_3(H^0) = 0 \in \mathbb{A}$. It follows from Proposition 2.5 that H^0 is homotopic rel boundary to an isotopy R between R'. If H is a based homotopy, then this resulting isotopy inherits the extended whiskers from H. $\square$

3. Homotopy versus isotopy

In Section 3.1 we recall the statement of Theorem 1.7 describing the based setting, and give a proof. In Section 3.2 we clarify and prove Theorem 1.11, describing the free setting.

Our convention is to write concatenations of homotopies as unions from left to right, with a minus sign indicating that the orientation of the I-factor has been reversed. Recall (Section 2.2) that for the purposes of computing the self-intersection invariant $\mu_3(H)$ of a homotopy H the whisker on the track of H will be assumed to be taken at the "start" $H(S^2 \times 0) \subset N \times 0 \subset N \times I$ unless otherwise explicitly specified.

3.1. *The based setting.* Theorem 1.7 states that for a fixed based embedding $U_* : S^2 \hookrightarrow N^5$ the map

$$p_*^{-1}[U_*] \to \mathbb{A}_{[U_*]}, \quad R_* \mapsto \mathrm{fq}_{[U_*]}(R_*) := [\mu_3(H_*)]$$

for H_* any based homotopy from U_* to R_*, is a bijection.

Here we have $p_*^{-1}[U_*]$ the set of based isotopy classes of embedded spheres $R_* : S^2 \hookrightarrow N^5$ that are based homotopic to U_*. Moreover, the group $\mathbb{A}_{[U_*]}$ is the quotient of $\mathbb{A} := \mathbb{Z}\pi_1 N / \langle g + g^{-1}, 1 \rangle$ by the image of the indeterminacy homomorphism $\phi_{[U_*]} : \pi_3 N \to \mathbb{A}$ defined by

$$A \mapsto \mu_3(A) + [\lambda_N(A, [U_*])].$$

$\mathrm{fq}_{[U_*]}$ *is well defined.* It suffices to show that $\mathrm{fq}_{[U_*]}(R_*) \in \mathbb{A}_{[U_*]}$ is independent of the choice of H_*.

Taking the union along R_* of any two based homotopies H_*, H_*' from U_* to R_* gives a based self-homotopy $J_* = H_* \cup_{R_*} -H_*'$ of U_* such that

$$\mu_3(J_*) = \mu_3(H_* \cup_{R_*} -H_*') = \mu_3(H_*) - \mu_3(H_*'),$$

where we are using that μ_3 is additive under concatenations of based homotopies and changes sign under reversing the orientation of the time parameter. Since $\mu_3(J_*)$ lies in the image of $\phi_{[U_*]}$ by Lemma 2.10, we have $[\mu_3(H_*)] = [\mu_3(H_*')] \in \mathbb{A}_{[U_*]}$.

$\mathrm{fq}_{[U_*]}$ *is injective.* If $\mathrm{fq}_{[U_*]}(R_*) = \mathrm{fq}_{[U_*]}(R_*')$, then there exist based homotopies H_* and H_*' from U_* to R_* and R_*', such that $\mu_3(H_*) = \mu_3(H_*') \in \mathbb{A}_{[U_*]}$. Taking the union of these homotopies along U_* gives a based homotopy $H_*'' := H_* \cup_{U_*} -H_*'$ from R_* to R_*' with $\mu_3(H_*'') = \mu_3(H_*) - \mu_3(H_*') = 0 \in \mathbb{A}_{[U_*]}$. It follows from Lemma 2.11 that R_* is based isotopic to R_*'.

$\mathrm{fq}_{[U_*]}$ *is surjective.* Surjectivity follows directly from Lemma 2.4.

3.2. *The free setting.* This section clarifies the target of the invariants in the free setting, and proves Theorem 1.11, which we recall here for the reader's convenience: For U_* a fixed basing of an embedding $U : S^2 \hookrightarrow N^5$, the map

$$p^{-1}[U] \to \mathbb{A}_{U_*}, \quad R \mapsto \mathrm{fq}_{U_*}(R) := [\mu_3(H)],$$

where H is any free homotopy from U to R, is a bijection.

Here $p^{-1}[U]$ is the set of isotopy classes of embedded spheres $R : S^2 \hookrightarrow N^5$ that are freely homotopic to U. Moreover, the group $\mathbb{A}_{U_*}$ is the quotient set of the based target $\mathbb{A}_{[U_*]}$ of Theorem 1.7 by the affine action of $\mathrm{Stab}[U_*] < \pi_1 N$ given by $^s a = U_s + sas^{-1}$ for all $a \in \mathbb{A}_{[U_*]}$ and $s \in \mathrm{Stab}[U_*]$, with the definition of $U_s \in \mathbb{A}_{[U_*]}$ given just after Lemma 3.1 in the next subsection.

The affine action of $\mathrm{Stab}[U_*]$ *on* $\mathbb{A}_{[U_*]}$. For each $s \in \mathrm{Stab}[U_*]$ there is a track

$$J_s : S^2 \times I \to N \times I$$

of a free self-homotopy of U_* such that the projection of $J_s(e, -)$ represents s, where the basepoint $e \in S^2$ is the preimage of the basepoint of U_*. For instance, such a J_s can be taken to be a (level-preserving) subset of the track of any based homotopy between $s \cdot U_*$ and U_*. We say that J_s *represents* $s \in \mathrm{Stab}[U_*]$, and call s the *core* of J_s, frequently using the subscript notation to indicate this representation.

Note the following three properties of core elements:

(1) Any self-homotopy whose core is the trivial element of $\pi_1 N$ is homotopic rel boundary to a based self-homotopy.

(2) Concatenating self-homotopies multiplies the core elements: $J_{sr} = J_s \cup J_r$.

(3) Reversing a self-homotopy inverts its core: $-J_s = J_{s^{-1}}$.

It follows from these three properties that given two free self-homotopies J_s and J_s' of U_* representing the same element $s \in \mathrm{Stab}[U_*]$, we can form a based self-homotopy $J_1 = J_{ss^{-1}} := J_s \cup -J_s'$ of U_* representing the trivial element $1 \in \mathrm{Stab}[U_*]$, with $\mu_3(J_1) = \mu_3(J_s) - \mu_3(J_s')$. Together with Lemma 2.10 we immediately get:

Lemma 3.1. *If* J_s *and* J_s' *are two free self-homotopies of* U_* *representing the same element* $s \in \mathrm{Stab}[U_*]$, *then* $\mu_3(J_s) - \mu_3(J_s') = 0 \in \mathbb{A}_{[U_*]}$. □

As a result of Lemma 3.1, the element

$$U_s := [\mu_3(J_s)] \in \mathbb{A}_{[U_*]}$$

is well defined, and hence so is the affine action $^s a := U_s + s a s^{-1}$ of $\mathrm{Stab}[U_*]$ on $\mathbb{A}_{[U_*]}$. This clarifies Definition 1.10 of the target of the free isotopy invariant $\mathrm{fq}_{U_*} \in \mathbb{A}_{U_*}$ as the quotient set of the based isotopy target $\mathrm{fq}_{[U_*]} \in \mathbb{A}_{[U_*]}$ under this action.

The following lemma illustrates how the affine action describes the effect of free self-homotopies on the self-intersection invariant.

Lemma 3.2. *If* H *is a homotopy from* U_* *to an embedding* R, *and* J_s *is a free self-homotopy of* U_* *representing* $s \in \mathrm{Stab}[U_*]$, *then the free homotopy* $J_s \cup H$ *from* U *to* R *satisfies*

$$\mu_3(J_s \cup H) = \mu_3(J_s) + s \mu_3(H) s^{-1} \in \mathbb{A}_{[U_*]}.$$

Proof. It is clear that each double point of the track of $J_s \cup H$ is either a double point of J_s or H. By our convention, the computation of $\mu_3(J_s \cup H)$ uses the whisker for U_* at the start of J_s. Thus, double point loops of H get conjugated by representatives of s while traversing J_s, so all the double-point group elements of $\mu_3(H)$ get conjugated by s. □

fq_{U_*} *is well defined.* It suffices to show that $\mathrm{fq}_{U_*}(R) = [\mu_3(H)] \in \mathbb{A}_{U_*}$ is independent of the choice of H. For H and H' two choices of free homotopies from U_* to R, the concatenation $J_s := H \cup -H'$ is a self-homotopy of U_* representing some $s \in \mathrm{Stab}[U_*]$, and by Lemma 3.2 we have

$$\mu_3(J_s) = \mu_3(H) - s\mu_3(H')\,s^{-1} \in \mathbb{A}.$$

So

$$\mu_3(H) = \mu_3(J_s) + s\mu_3(H')\,s^{-1} = {}^s(\mu_3(H')) \in \mathbb{A}_{[U_*]},$$

which implies $[\mu_3(H)] = [\mu_3(H')] \in \mathbb{A}_{U_*}$, and hence $\mathrm{fq}_{U_*}(R)$ is well defined.

fq_{U_*} *is injective.* If $\mathrm{fq}_{U_*}(R) = \mathrm{fq}_{U_*}(R')$, then by the definition of the target $\mathbb{A}_{U_*}$ there exist homotopies H and H' from U_* to R and R', respectively, such that

$$\mu_3(H') = U_s + s\mu_3(H)\,s^{-1} \in \mathbb{A}_{[U_*]}$$

for some $s \in \mathrm{Stab}[U_*]$.

Consider the homotopy $H'' := -H' \cup J_s \cup H$ from R' to R, where J_s is any self-homotopy of U_* representing s. Using the whisker on U_* in $-H' \cap J_s \subset H''$ we have the following computation in $\mathbb{A}_{[U_*]}$:

$$\begin{aligned}
\mu_3(H'') &= \mu_3(-H') + \mu_3(J_s) + s\mu_3(H)\,s^{-1} \\
&= -\mu_3(H') + U_s + s\mu_3(H)\,s^{-1} \\
&= -(U_s + s\mu_3(H)\,s^{-1}) + U_s + s\mu_3(H)\,s^{-1} \\
&= 0.
\end{aligned}$$

It follows from Lemma 2.11 that R is isotopic to R'.

fq_{U_*} *is surjective.* Surjectivity follows directly from Lemma 2.4.

3.3. Examples. Recall that Corollary 1.12 states that free isotopy classes of null-homotopic 2-spheres in N are in bijection with $\mathbb{A}_{U_*} = \mathbb{A}_{[U_*]}/\pi_1 N$, where the action is by conjugation.

Here we examine some examples of free isotopy classes of essential 2-spheres:

Example. Consider $U_* = S^2 \times \{p\} \subset N = S^2 \times M^3$.

Then $U_s = 0$ for all $s \in \mathrm{Stab}[U_*] = \pi_1 N$, because any self-homotopy J_s can be chosen to be a self-isotopy which moves p around a loop representing s while fixing the S^2-factor.

So the affine action has trivial translations (see Section 1.3) and free isotopy classes of spheres homotopic to U_* are in bijection with $\mathbb{A}_{U_*} = \mathbb{A}_{[U_*]}/\pi_1 M$ (with conjugation action).

Example. Consider again $U_* = S^2 \times \{p\} \subset N = S^2 \times M^3$.

Assume $[g, h] \neq 1 \in \pi_1 M \cong \pi_1 N$. If U_*^g is the result of doing a g-finger move on U_*, then $U_s^g = g - sgs^{-1}$ for each $s \in \mathrm{Stab}[U_*^g] = \pi_1 N$. Here $U_s^g = \mu_3(J_s)$ where J_s is a self-homotopy of U_*^g that undoes the g-finger move, then moves p around a loop representing s while fixing S^2, and then redoes the g-finger move. In particular,

$$U_h^g = g - hgh^{-1} \neq 0 \in \mathbb{A}_{[U_*^g]} = \mathbb{A}_{[U_*]}$$

if $\pi_3 M = 0$. (To see that $\mathbb{A}_{[U_*^g]} = \mathbb{A}_{[U_*]}$ if $\pi_3 M = 0$, observe that any representative of a generator of $\pi_3(S^2 \times M) \cong \pi_3 S^2$ is homotopic to a generic immersion contained in the product of S^2 with a 3-ball, and hence has only double points with trivial group element, implying that μ_3 vanishes.)

So in this case the affine action $^h a = (g - hgh^{-1}) + hah^{-1}$ defining $\mathbb{A}_{U_*^g}$ as a quotient of $\mathbb{A}_{[U_*^g]} = \mathbb{A}_{[U_*]}$ has nontrivial translations, illustrating how the target of the free isotopy invariant depends in general on the isotopy class of the fixed embedding, and not just on its homotopy class.

This suggests the following questions: When does a homotopy class of 2-spheres in N contain an isotopy class such that the corresponding affine action has trivial translations? Are stabilizers of elements of $\pi_2 N$ always represented by some embedded $S^2 \times S^1 \subset N \times S^1$?

4. A space level approach following Dax

In this section we reprove our two main results, Theorems 1.7 and 1.11, using a space level approach given in [Dax 1972] and [Kosanović et al. $\geq$ 2024].

4.1. _The relative homotopy group._ Following [Kosanović and Teichner 2024b] and [Dax 1972], we compute in [Kosanović et al. $\geq$ 2024] the relative homotopy group $\pi_{d-2\ell}(\mathrm{Map}(V, X), \mathrm{Emb}(V, X), U)$ for any ℓ-manifold V and d-manifold X, and a fixed embedding $U : V \hookrightarrow X$ taken as the basepoint. In our case of interest, $V = S^2$ and $X = N$ of dimension $d = 5$, the relevant result is as follows.

Theorem 4.1 [Kosanović et al. $\geq$ 2024]. _Let N be an oriented connected 5-manifold and $U_* : S^2 \hookrightarrow N$ a smooth based embedding. Then there are bijections_

$$\mathrm{Dax} : \pi_1(\mathrm{Map}_*(S^2, N), \mathrm{Emb}_*(S^2, N), U_*)$$

$$\cong \Big\downarrow i^{\mathrm{rel}}$$

$$\pi_1(\mathrm{Map}(S^2, N), \mathrm{Emb}(S^2, N), U)$$

$$\cong \Big\downarrow$$

$$\mathbb{A}$$

given on a class $[H]$ as the sum over double points of the associated group elements of the track of $H : I \to \mathrm{Map}(S^2, N)$, defined by $I \times S^2 \to I \times N$, $(t, v) \mapsto (t, H_t(v))$.

In other words, $\mathsf{Dax}([H]) = \mu_3(H)$ *is precisely the self-intersection invariant from* (2.2).

Remark 4.2. Using Lemma 2.4 one can define an explicit inverse of Dax. This is completely analogous to the realization map $\mathfrak{r}$ in [Kosanović and Teichner 2024b; Kosanović et al. ≥ 2024].

Our main square from Section 1.1 extends to a commutative diagram:

(4.3)

$$
\begin{array}{ccc}
\pi_1(\mathrm{Map}_*(S^2, N), U_*) & \xrightarrow{\ i\ } & \pi_1(\mathrm{Map}(S^2, N), U) \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j} \\
\pi_1(\mathrm{Map}_*, \mathrm{Emb}_*, U_*) & \xrightarrow{\ i^{\mathrm{rel}}\ } & \pi_1(\mathrm{Map}, \mathrm{Emb}, U) \\
\downarrow & & \downarrow \\
\pi_0 \mathrm{Emb}_*(S^2, N) & \longrightarrow\!\!\!\!\!\rightarrow & \pi_0 \mathrm{Emb}(S^2, N) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p} \\
\pi_0 \mathrm{Map}_*(S^2, N) & \longrightarrow\!\!\!\!\!\rightarrow & \pi_0 \mathrm{Map}(S^2, N)
\end{array}
$$

The left column is the final part of the long exact sequence of the pair in the based case,

$$(\mathrm{Map}_*, \mathrm{Emb}_*) := (\mathrm{Map}_*(S^2, N), \mathrm{Emb}_*(S^2, N)),$$

whereas the right column is from the long exact sequence of the pair in the corresponding free case.

We use the following standard facts about homotopy groups of mapping spaces, see [Kosanović et al. ≥ 2024] for details.

Lemma 4.4. *There are isomorphisms*

$$\pi_k(\mathrm{Map}_*(S^2, N), U_*) \to \pi_{k+2}(N)$$

for $k \geq 0$, and a bijection

$$\pi_0 \mathrm{Map}(S^2, N) \cong \pi_2 N / \{\alpha - g\alpha\}$$

for the usual action of $g \in \pi_1 N$ on $\alpha \in \pi_2 N$. For any $\beta \in \mathrm{Map}_*(S^2, N)$ there is an exact sequence

$$\pi_3 N \cong \pi_1(\mathrm{Map}_*(S^2, N), \beta) \xrightarrow{\ i\ } \pi_1(\mathrm{Map}(S^2, N), \beta) \xrightarrow{\ ev_e\ }\!\!\!\!\!\rightarrow \mathrm{Stab}\,\beta,$$

where

$$\mathrm{Stab}\,\beta := \{g \in \pi_1 N : g\beta = \beta \in \pi_2 N\},$$

and ev_e is induced by the map $\mathrm{Map}(S^2, N) \to N$ given by $f \mapsto f(e)$.

Combining (4.3) and Lemma 4.4 with Theorem 4.1, and denoting by $[U_*]$ the class of U_* in $\pi_3 N \cong \pi_1 \operatorname{Map}_*(S^2, N)$, we have the commutative diagram:

$$
\begin{array}{ccccc}
\pi_3 N & \xrightarrow{\ i\ } & \pi_1(\operatorname{Map}(S^2, N), U) & \xrightarrow{\ ev_e\ } & \operatorname{Stab}[U_*] \\
\downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j} & & \\
\pi_1(\operatorname{Map}_*, \operatorname{Emb}_*, U_*) & \xrightarrow[\cong]{i^{\mathrm{rel}}} & \pi_1(\operatorname{Map}, \operatorname{Emb}, U) & \xrightarrow[\cong]{\operatorname{Dax}} & \mathbb{A} \\
\downarrow & & \downarrow & & \\
\pi_0 \operatorname{Emb}_*(S^2, N) & \longrightarrow\!\!\!\!\!\twoheadrightarrow & \pi_0 \operatorname{Emb}(S^2, N) & & \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p} & & \\
\pi_2 N & \longrightarrow\!\!\!\!\!\twoheadrightarrow & \pi_0 \operatorname{Map}(S^2, N) & &
\end{array}
$$

(4.5)

which will imply desired results, as explained next.

4.2. _The proofs._ The following recovers Theorem 1.7.

Theorem 4.6. _There is a short exact sequence of sets_

$$\mathbb{A}/(\operatorname{Dax} \circ i^{\mathrm{rel}} \circ j_*(\pi_3 N)) \rightarrowtail \pi_0 \operatorname{Emb}_*(S^2, N) \xrightarrow{\ p_*\ }\!\!\!\!\twoheadrightarrow \pi_0 \operatorname{Map}_*(S^2, N) \cong \pi_2 N$$

and $\operatorname{Dax} \circ i^{\mathrm{rel}} \circ j_* = \phi_{[U_*]}$ _from (1.4) of_ Section 1.2.

Proof. From diagram (4.5) we have $\ker(p_*) = \operatorname{coker}(j_*) \cong \operatorname{coker}(\operatorname{Dax} \circ i^{\mathrm{rel}} \circ j_*)$, so it only remains to identify the last homomorphism. And indeed, for a class $A \in \pi_3 N$ the element $j_*(A) : I \to \operatorname{Map}_*$ is a self-homotopy of U that represents A and $\operatorname{Dax}(j_*(A)) = \mu_3(j_*(A))$ by definition. Now, arguing as in the proof of Lemma 2.10 we see that the track of $j_*(A)$ has $\mu_3(j_*(A)) = \mu_3(A) + \lambda_N(A, U_*)$, therefore $\operatorname{Dax} \circ i^{\mathrm{rel}} \circ j_*(A) = \phi_{[U_*]}(A)$ as desired. $\qquad\square$

Similarly, the following recovers Theorem 1.11.

Theorem 4.7. _There is a short exact sequence of sets_

$$\left(\mathbb{A}/(\phi_{[U_*]}(\pi_3 N))\right)_{s \mapsto {}^s a} \rightarrowtail \pi_0 \operatorname{Emb}(S^2, N) \xrightarrow{\ p\ }\!\!\!\!\twoheadrightarrow \pi_0 \operatorname{Map}(S^2, N),$$

where on the left we take the quotient by the action $s \mapsto {}^s a$ _of_ $\operatorname{Stab}[U_*]$ _from (aff) of_ Section 1.3.

Proof. From diagram (4.5) we have $\ker(p) = \operatorname{coker}(j) \cong \operatorname{coker}(\operatorname{Dax} \circ j)$. Using the leftmost column we can compute $\operatorname{coker}(\operatorname{Dax} \circ j)$ in two steps:

(1) First take the cokernel of $\operatorname{Dax} \circ j \circ i$.

(2) Then mod out the induced action of $\operatorname{Stab}[U_*]$, using any section of ev_e.

Note that the action in (2) is well defined, and the set of coinvariants is independent of the section, since in $\mathrm{coker}(j \circ i)$ we have modded out $\ker(ev_e)$.

For (1), we simply note that $\mathrm{Dax} \circ j \circ i = \mathrm{Dax} \circ i^{\mathrm{rel}} \circ j_*$ by the commutativity of the leftmost square in (4.5), and this is equal to $\phi_{[U_*]}$ by Theorem 4.6.

For (2), to compute the action, we pick any section; by definition, this sends $s \in \mathrm{Stab}[U_*]$ to any

$$J_s \in \pi_1(\mathrm{Map}(S^2, N), U),$$

which we view as a free self-homotopy of U_*, for which $ev_e(J_s) = J_s(-, e)$ represents s.

Then $s \in \mathrm{Stab}[U_*]$ acts by sending $a = \mathrm{Dax}(H)$ to $\mathrm{Dax}(J_s \cup H)$. Since we have $\mathrm{Dax}(J_s \cup H) = \mathrm{Dax}(J_s) + s\,\mathrm{Dax}(H)\,s^{-1}$ by Lemma 3.2 (where μ_3 notation was used in place of Dax), we see that the action of s on a is given as claimed by

$$\mathrm{Dax}(J_s) + sas^{-1} = \mu_3(J_s) + sas^{-1} = {}^s a. \qquad \square$$

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DANICA KOSANOVIĆ
DEPARTMENT OF MATHEMATICS
ETH ZÜRICH
ZÜRICH
SWITZERLAND
danica.kosanovic@math.ethz.ch

ROB SCHNEIDERMAN
DEPARTMENT OF MATHEMATICS
LEHMAN COLLEGE
CITY UNIVERSITY OF NEW YORK
BRONX, NY
UNITED STATES
robert.schneiderman@lehman.cuny.edu

PETER TEICHNER
MAX-PLANCK-INSTITUT FÜR MATHEMATIK
BONN
GERMANY
teichner@mac.com

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