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**A NEW CONVERGENCE THEOREM  
FOR MEAN CURVATURE FLOW OF HYPERSURFACES  
IN QUATERNIONIC PROJECTIVE SPACES**

SHIYANG LI, HONGWEI XU AND ENTAO ZHAO



# A NEW CONVERGENCE THEOREM FOR MEAN CURVATURE FLOW OF HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACES

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**We investigate the smooth convergence of the mean curvature flow of hypersurfaces in the quaternionic projective spaces. We prove that if the initial hypersurface satisfies a new nonlinear curvature pinching condition, then the mean curvature flow converges smoothly to a round point in finite time. Our result improves a smooth convergence theorem due to Pipoli and Sinestrari (2017).**

## 1. Introduction

There are many famous geometric evolution equations, such as the Ricci flow, the mean curvature flow and others. Huisken [9] studied the mean curvature flow from the perspective of partial differential equations, and he proved that convex hypersurface in the Euclidean space converges to a round point along the flow. Afterwards, Huisken [10; 11] obtained convergence results for mean curvature flow of convex hypersurfaces in Riemannian manifolds and pinched hypersurfaces in spheres. Following the argument of Huisken [9], Andrews and Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a suitable pinching condition in the Euclidean space. Later, Baker [2], Liu et al. [20] proved sharp convergence theorems for the mean curvature flow in the spheres and the hyperbolic spaces, respectively. Liu, Xu and Zhao [19] studied the mean curvature flow of arbitrary codimensional submanifolds in the Riemannian manifold and proved a smooth convergence theorem. Lei and Xu [15] verified an optimal convergence theorem for the mean curvature flow of submanifolds in hyperbolic spaces, which implies the first optimal differentiable sphere theorem for submanifolds with positive Ricci curvature. It should be remarked that an optimal topological sphere theorem for complete submanifolds with positive Ricci curvature in a space form with nonnegative curvature has been proved previously by Shiohama and Xu [26]. Lei and Xu [15] also proved sharp convergence theorems for the mean curvature

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flow of submanifolds in the sphere [13; 14], which also improve the convergence theorems due to Huisken [10] and Baker [2]. See [16; 18; 21] for recent progress in the smooth convergence theory for mean curvature flow of arbitrary codimensions. As consequences of these smooth convergence theorems, the submanifolds satisfying the initial curvature conditions are diffeomorphic to the standard sphere. We remark that some of these differentiable sphere theorems are also proved by using the Ricci flow, which has proven to be a very useful tool in understanding the topology of Riemannian manifolds, see [3; 4; 6; 7; 8; 22; 23; 24; 27; 28].

Pipoli and Sinestrari [25] obtained a convergence theorem for mean curvature flow of small codimension in the complex projective spaces. Later, Lei and Xu [17] investigated the smooth convergence of mean curvature flow of arbitrary codimensional submanifolds in the complex projective spaces, which improved and extended the convergence theorem due to Pipoli and Sinestrari [25]. In this paper, we investigate the mean curvature flow in the quaternionic projective spaces. We mainly consider the codimension-one case.

Let  $M$  be an  $n$ -dimensional closed manifold, and let  $F : M^n \times [0, T) \rightarrow N^{n+1}$  be a one-parameter family of smooth hypersurfaces immersed in a Riemannian manifold  $(N, h)$ . We say that  $M_t = F_t(M)$  is a solution to the mean curvature flow if  $F_t$  satisfies

$$(1-1) \quad \begin{cases} \frac{\partial}{\partial t} F = -H\nu, \\ F(\cdot, 0) = F_0(\cdot), \end{cases}$$

where  $F_t(\cdot) = F(\cdot, t)$ ,  $H$  and  $\nu$  are the mean curvature of  $M$  and the unit outward normal vector of  $M$  respectively, such that  $\vec{H} = -H\nu$  is the mean curvature vector of  $M$ .

Pipoli and Sinestrari [25] obtained a convergence theorem for the mean curvature flow of hypersurfaces in the quaternionic projective spaces, and the proof is the same as their convergence theorem for mean curvature flow of hypersurfaces in the complex projective spaces.

**Theorem 1.1** [25]. *Let  $M^n$  ( $n \geq 11$ ) be a closed real hypersurface in quaternionic projective space  $\mathbb{Q}\mathbb{P}^{(n+1)/4}(4)$ , and  $M_t$  be the mean curvature flow starting from  $M$ . Assume that  $M$  satisfies the following pinching condition:*

$$|h|^2 < \frac{1}{n-1} H^2 + 2.$$

*Then the flow has a smooth solution on the maximal time interval  $[0, T)$  with  $T < \infty$ . Moreover, the pinching condition is preserved and  $M_t$  collapses to a round point as  $t \rightarrow T$ .*

We note that here and in the remaining part of the paper,  $n = 4m - 1$  for  $m \geq 2$ . The aim of the present paper is to prove the following refinement of Theorem 1.1.

**Theorem 1.2.** *Let  $M^n$  be an  $n(\geq 7)$ -dimensional closed real hypersurface in quaternionic projective space  $\mathbb{Q}\mathbb{P}^{(n+1)/4}(4)$ , and  $M_t$  be the mean curvature flow starting from  $M$ . Assume that  $M$  satisfies the following pinching condition:*

$$|h|^2 < \varphi(H^2).$$

*Then the flow has a smooth solution on the maximal time interval  $[0, T)$  with  $T < \infty$ . Moreover, the pinching condition is preserved and  $M_t$  collapses to a round point as  $t \rightarrow T$ .*

In Theorem 1.2,  $\varphi(H^2)$  is given by

$$(1-2) \quad \varphi(H^2) = 2 + a_n + \left(b_n + \frac{1}{n-1}\right) H^2 - \sqrt{b_n^2 H^4 + 2a_n b_n H^2},$$

where

$$a_n = \sqrt{8(n-5)(n-1)b_n}, \quad b_n = \min \left\{ \frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)} \right\}.$$

**Remark 1.3.** By a computation, we have  $\varphi(x) > \frac{x}{n-1} + 2$  for  $x \geq 0$ . So, Theorem 1.2 is an improvement of Theorem 1.1. Furthermore, we have  $\varphi(x) \geq 4\sqrt{n-1} - 6$  for  $7 \leq n \leq 17$ , and  $\varphi(x) > 2 + \frac{8\sqrt{2}}{5}\sqrt{n-5}$  for  $n \geq 18$ .

It is well known that  $\mathbb{Q}\mathbb{P}^1$  is just the round sphere. By [11; 14], the similar smooth convergence theorem holds for mean curvature flow of closed hypersurfaces in  $\mathbb{Q}\mathbb{P}^1$ .

By Theorem 1.2, we have:

**Corollary 1.4.** *Let  $M^n$  be an  $n(\geq 7)$ -dimensional closed real hypersurface in quaternionic projective space  $\mathbb{Q}\mathbb{P}^{(n+1)/4}(4)$ . If  $|h|^2 < \varphi(H^2)$ , then  $M$  is diffeomorphic to the standard sphere.*

The rest of the paper is organized as follows. In Section 2, we introduce some notations, formulas and basic equations in submanifold theory, and prove a gradient inequality involving the second fundamental form and the mean curvature for hypersurfaces in the quaternionic projective spaces. We also recall some evolution equations along the mean curvature flow in this section. In Section 3, we show that the pinching condition  $|h|^2 < \varphi(H^2)$  is preserved along the mean curvature flow. We also derive an evolution inequality of

$$f_\sigma = \frac{|\mathring{h}|^2}{(\varphi - H^2/n)^{1-\sigma}}.$$

A pinching estimate for the traceless second fundamental form is obtained in Section 4. We give an estimate of the gradient of the mean curvature in Section 5, which is used to compare the mean curvature at different points. In Section 6, we show that the hypersurface shrinks to a single round point in finite time.

## 2. Notations and formulas

Let  $\mathbb{Q}\mathbb{P}^m$  be the  $m$ -dimensional quaternionic projective space with the Fubini–Study metric  $g_{\text{FS}}$ . Let  $J_{k_0}$ ,  $k_0 = 1, 2, 3$  be complex structures on  $\mathbb{Q}\mathbb{P}^m$ . We denote by  $\bar{\nabla}$  the Levi–Civita connection of  $(\mathbb{Q}\mathbb{P}^m, g_{\text{FS}})$ . Since the Fubini–Study metric is a Kähler metric, we have  $\bar{\nabla} J_{k_0} = 0$  for  $k_0 = 1, 2, 3$ . The curvature tensor  $\bar{R}$  of  $\mathbb{Q}\mathbb{P}^m$  can be written as

$$\begin{aligned} & \bar{R}(X, Y, Z, W) \\ &= \langle X, Z \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, Z \rangle \\ & \quad + \sum_{k_0=1}^3 (\langle X, J_{k_0} Z \rangle \langle Y, J_{k_0} W \rangle - \langle X, J_{k_0} W \rangle \langle Y, J_{k_0} Z \rangle + 2\langle X, J_{k_0} Y \rangle \langle Z, J_{k_0} W \rangle) \end{aligned}$$

and  $J_{k_0}$ ,  $k_0 = 1, 2, 3$  satisfies

$$J_{k_0}^2 = -\text{Id}, \quad J_1 J_2 = -J_2 J_1 = J_3, \quad J_1 J_3 = -J_3 J_1 = -J_2, \quad J_2 J_3 = -J_3 J_2 = J_1.$$

Let  $(M^n, g)$  be an  $n$ -dimensional Riemannian submanifold in  $(\mathbb{Q}\mathbb{P}^m, g_{\text{FS}})$ . Let  $q$  be its codimension, i.e.,  $n + q = 4m$ . At a point  $p \in M$ , let  $T_p M$  and  $N_p M$  be the tangent space and normal space, respectively. For a vector in  $T_p M \oplus N_p M$ , we denote by  $(\cdot)^T$  and  $(\cdot)^N$  its projections onto  $T_p M$  and  $N_p M$ , respectively. We use the symbols  $\nabla$  and  $\nabla^\perp$  to represent the connections of tangent bundle  $TM$  and normal bundle  $NM$ . Denote by  $\Gamma(E)$  the space of smooth sections of a vector bundle  $E$ . For  $X, Y \in \Gamma(TM)$ ,  $\xi \in \Gamma(NM)$ , the connections  $\nabla$  and  $\nabla^\perp$  are given by  $\nabla_X Y = (\bar{\nabla}_X Y)^T$  and  $\nabla_X^\perp \xi = (\bar{\nabla}_X \xi)^N$ . The second fundamental form of  $M$  is defined as  $h(X, Y) = (\bar{\nabla}_X Y)^N$ .

We mainly consider the codimension-one case. Throughout this paper, we shall make the following convention on indices:

$$1 \leq A, B, C, \dots \leq n+1, \quad 1 \leq i, j, k, \dots \leq n.$$

We choose a local orthonormal frame  $\{e_i\}$  for the tangent bundle and let  $\nu = e_{n+1}$  be the unit normal vector field. Denote by  $\{\omega^i\}$  the dual frame of  $\{e_i\}$ . Let  $h$  and  $H$  denote the second fundamental form and the mean curvature given by

$$h = \sum_{i,j} h_{ij} \omega^i \otimes \omega^j \quad \text{and} \quad H = \sum_i h_{ii}.$$

Let  $\mathring{h} = h - \frac{1}{n} Hg$  be the traceless second fundamental form. We have the relations

$$|\mathring{h}|^2 = |h|^2 - \frac{1}{n} H^2, \quad |\nabla \mathring{h}|^2 = |\nabla h|^2 - \frac{1}{n} |\nabla H|^2.$$

Setting  $J_{AB}^{(k_0)} = \langle e_A, J_{k_0} e_B \rangle$  for  $k_0 = 1, 2, 3$ , we have

$$J_{AB}^{(k_0)} = -J_{BA}^{(k_0)}, \quad \sum_B J_{AB}^{(k_0)} J_{BC}^{(k_0)} = J_{AC}^{(k_0)}, \quad \sum_B J_{AB}^{(1)} J_{BC}^{(2)} = J_{AC}^{(3)}.$$

Similarly, we have

$$\sum_B J_{AB}^{(1)} J_{BC}^{(3)} = -J_{AC}^{(2)}, \quad \sum_B J_{AB}^{(2)} J_{BC}^{(3)} = J_{AC}^{(1)}.$$

Also,  $J_{AA}^{(k_0)} = 0$  for any  $A$  and  $k_0$ .

Let  $h_{ijk}$  and  $H_i$  denote the components of  $\nabla h$  and  $\nabla H$ , the covariant differentiations of  $h$  and  $H$ , respectively. We have the following sharp gradient inequality (see Remark 2.2).

**Lemma 2.1.** *For a hypersurface in  $\mathbb{Q}\mathbb{P}^{(n+1)/4}$ , we have*

$$|\nabla h|^2 \geq \frac{3}{n+2} |\nabla H|^2 + 6(n-3).$$

*Proof.* Set  $S = \sum S_{ijk} \omega^i \otimes \omega^j \otimes \omega^k$ , where  $S_{ijk} = \frac{1}{3}(h_{ijk} + h_{jki} + h_{kij})$ . Then  $S_{ijk}$  is totally symmetric for  $i, j, k$ . Using the same technique as in Lemma 2.2 in [9], we have

$$|S|^2 \geq \frac{3}{n+2} \sum_i \left( \sum_k S_{kki} \right)^2.$$

By the Codazzi equation, we have

$$\begin{aligned} \sum_k S_{kki} &= \frac{1}{3} \sum_k (h_{ikk} + h_{kki} + h_{kik}) \\ &= \frac{1}{3} \sum_k (h_{kki} + 2h_{kik}) \\ &= \frac{1}{3} \sum_k (h_{kki} + 2h_{kki} - 2\bar{R}_{n+1kik}) = H_i - \frac{2}{3} \sum_k \bar{R}_{n+1kik}. \end{aligned}$$

As

$$\bar{R}_{n+1kik} = \sum_{k_0=1}^3 (J_{n+1i}^{(k_0)} J_{kk}^{(k_0)} - J_{n+1k}^{(k_0)} J_{ki}^{(k_0)} + 2J_{n+1k}^{(k_0)} J_{ik}^{(k_0)}),$$

one has

$$\begin{aligned} -\frac{2}{3} \sum_k \bar{R}_{n+1kik} &= -\frac{2}{3} \sum_k \sum_{k_0=1}^3 (J_{n+1i}^{(k_0)} J_{kk}^{(k_0)} - J_{n+1k}^{(k_0)} J_{ki}^{(k_0)} + 2J_{n+1k}^{(k_0)} J_{ik}^{(k_0)}) \\ &= -\frac{2}{3} \sum_k \sum_{k_0=1}^3 (-J_{n+1k}^{(k_0)} J_{ki}^{(k_0)} - 2J_{n+1k}^{(k_0)} J_{ki}^{(k_0)}) \\ &= 2 \sum_{k_0=1}^3 \sum_k (J_{n+1k}^{(k_0)} J_{ki}^{(k_0)}). \end{aligned}$$

Then we get

$$\sum_k S_{kki} = H_i + 2 \sum_{k_0=1}^3 \sum_k (J_{n+1k}^{(k_0)} J_{ki}^{(k_0)}).$$

This implies

$$\left( \sum_k S_{kki} \right)^2 = (H_i)^2 + 4 \sum_{k_0=1}^3 \sum_k H_i (J_{n+1k}^{(k_0)} J_{ki}^{(k_0)}) + 4 \left[ \sum_{k_0=1}^3 \sum_k (J_{n+1k}^{(k_0)} J_{ki}^{(k_0)}) \right]^2.$$

Since

$$\begin{aligned}
& 4 \sum_{k_0=1}^3 \left[ \sum_{i,k} H_i J_{n+1k}^{(k_0)} J_{ki}^{(k_0)} \right] + 4 \sum_i \left[ \sum_{k_0=1}^3 \sum_k J_{n+1k}^{(k_0)} J_{ki}^{(k_0)} \right]^2 \\
&= 4 \sum_{k_0=1}^3 \left[ \sum_i H_i \sum_A J_{n+1A}^{(k_0)} J_{Ai}^{(k_0)} \right] + 4 \sum_i \left[ \sum_{k_0=1}^3 \sum_A J_{n+1A}^{(k_0)} J_{Ai}^{(k_0)} \right]^2 \\
&= 4 \sum_{k_0=1}^3 \left[ \sum_i H_i \delta_{n+1i} \right] + 4 \sum_i \left[ \sum_{k_0=1}^3 \delta_{n+1i} \right]^2 = 0,
\end{aligned}$$

one has

$$(2-1) \quad |S|^2 \geq \frac{3}{n+2} |\nabla H|^2.$$

On the other hand, by the Codazzi equation, we have

$$\begin{aligned}
|S|^2 &= \sum (S_{ijk})^2 = \frac{1}{9} \sum (h_{ikk} + h_{kki} + h_{kik})^2 \\
&= \frac{1}{3} \sum (h_{ijk})^2 + \frac{2}{3} \sum h_{ijk} h_{ikj} \\
&= \frac{1}{3} \sum (h_{ijk})^2 + \frac{2}{3} \sum h_{ijk} (h_{ijk} + \bar{R}_{n+1ijk}) \\
&= \sum (h_{ijk})^2 + \frac{2}{3} \sum \bar{R}_{n+1jki} \bar{R}_{n+1ijk} \\
&= |\nabla h|^2 + \frac{2}{3} \sum \bar{R}_{n+1jki} \bar{R}_{n+1ijk}.
\end{aligned}$$

Since

$$\bar{R}_{ABCD} = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + \sum_{k_0=1}^3 (J_{AC}^{(k_0)} J_{BD}^{(k_0)} - J_{AD}^{(k_0)} J_{BC}^{(k_0)} + 2J_{AB}^{(k_0)} J_{CD}^{(k_0)}),$$

one has

$$\begin{aligned}
\sum \bar{R}_{n+1jki} \bar{R}_{n+1ijk} &= \sum_{i,j,k} \left[ \sum_{k_0=1}^3 (J_{n+1k}^{(k_0)} J_{ji}^{(k_0)} - J_{n+1i}^{(k_0)} J_{jk}^{(k_0)} + 2J_{n+1j}^{(k_0)} J_{ki}^{(k_0)}) \right] \\
&\quad \times \left[ \sum_{l_0=1}^3 (J_{n+1j}^{(l_0)} J_{ik}^{(l_0)} - J_{n+1k}^{(l_0)} J_{ij}^{(l_0)} + 2J_{n+1i}^{(l_0)} J_{jk}^{(l_0)}) \right].
\end{aligned}$$

For each  $k_0$ , according to the special property of matrix  $(J_{AB}^{(k_0)})$ , by direct computation we have

$$\sum_k (J_{n+1k}^{(k_0)})^2 = - \sum_k J_{n+1k}^{(k_0)} J_{kn+1}^{(k_0)} = - \sum_A J_{n+1A}^{(k_0)} J_{An+1}^{(k_0)} = \delta_{(n+1)(n+1)} = 1,$$

and

$$\sum_k J_{n+1k}^{(k_0)} J_{Bk}^{(l_0)} = - \sum_k J_{n+1k}^{(k_0)} J_{kB}^{(l_0)} = - \sum_A J_{n+1A}^{(k_0)} J_{AB}^{(l_0)} = \pm J_{n+1B}^{(j_0)},$$

where  $\pm$  depends on  $j_0, k_0, l_0$ . By some computations, we obtain

$$\sum \bar{R}_{n+1jki} \bar{R}_{n+1ijk} = -9(n-3).$$

Hence

$$|S|^2 = |\nabla h|^2 + \frac{2}{3}(-9(n-3)) = |\nabla h|^2 - 6(n-3).$$



### 3. Preservation of curvature pinching

For each fixed  $k_0 \in \{1, 2, 3\}$ , we choose a local orthonormal frame  $\{e_i\}$  such that the matrix  $(J_{ij}^{(k_0)})$  takes the form of (2-2). In fact, let  $\{\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}\}$  be a local orthonormal frame on  $\mathbb{Q}\mathbb{P}^{(n+1)/4}$  such that  $\epsilon_1, \dots, \epsilon_n$  are tangent to  $M$  and  $\epsilon_{n+1}$  is normal to  $M$ . Let  $\tilde{J}_{AB}^{(k_0)} = \langle \epsilon_A, J^{(k_0)} \epsilon_B \rangle$ . Since  $(\tilde{J}_{ij}^{(k_0)})_{n \times n}$  is antisymmetric and  $n$  is odd, there is an orthonormal matrix  $C = (c_{ij})_{n \times n}$ , where  $c_{ij}$ 's are local functions, such that

$$(c_{ij}^{-1} \tilde{J}_{jk}^{(k_0)} c_{kl})_{n \times n} = \begin{pmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Here  $(c_{ij}^{-1})_{n \times n} = (c_{ij})_{n \times n}^{-1}$ . Set  $e_i = \sum_{j=1}^n c_{ij}^{-1} \epsilon_j$ ,  $e_{n+1} = \epsilon_{n+1}$ . Then

$$\begin{aligned} J_{ij}^{(k_0)} &= \langle e_i, J^{(k_0)} e_j \rangle \\ &= \left\langle \sum_k c_{ik}^{-1} \epsilon_k, J^{(k_0)} \left( \sum_l c_{jl}^{-1} \epsilon_l \right) \right\rangle = \sum_{k,l} c_{ik}^{-1} \tilde{J}_{kl}^{(k_0)} c_{jl}^{-1} = \sum_{k,l} c_{ik}^{-1} \tilde{J}_{kl}^{(k_0)} c_{lj}. \end{aligned}$$

This implies

$$(J_{ij}^{(k_0)})_{n \times n} = \begin{pmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} \sum_{i,j,k,l} (\dot{h}_{ij} \dot{h}_{kl} J_{il}^{(k_0)} J_{jk}^{(k_0)} - \dot{h}_{ik} \dot{h}_{kj} J_{il}^{(k_0)} J_{jl}^{(k_0)}) &= \sum_{i,k} (-\dot{h}_{i\bar{k}} \dot{h}_{k\bar{i}} J_{i\bar{i}}^{(k_0)} J_{k\bar{k}}^{(k_0)} - (\dot{h}_{i\bar{k}} J_{i\bar{i}}^{(k_0)})^2) \\ &= -\frac{1}{2} \sum_{i,k} (\dot{h}_{i\bar{k}} J_{i\bar{i}}^{(k_0)} + \dot{h}_{i\bar{k}} J_{k\bar{k}}^{(k_0)})^2 \\ &\leq 0. \end{aligned}$$

Therefore,

$$S_1 = \sum_{k_0=1}^3 \left[ \sum_{i,j,k,l} (\dot{h}_{ij} \dot{h}_{kl} J_{il}^{(k_0)} J_{jk}^{(k_0)} - \dot{h}_{ik} \dot{h}_{jk} J_{il}^{(k_0)} J_{jl}^{(k_0)}) \right] \leq 0.$$

So we get from Lemma 2.3 that

$$(3-1) \quad \frac{\partial}{\partial t} |\dot{h}|^2 \leq \Delta |\dot{h}|^2 - 2|\nabla \dot{h}|^2 + 2|\dot{h}|^2(|h|^2 - n + 9).$$

For a real number  $\varepsilon \in (0, 1)$ , by the definition of  $\varphi$ , we define  $\varphi_\varepsilon : [0, +\infty) \rightarrow \mathbb{R}$  by

$$(3-2) \quad \varphi_\varepsilon(x) = d_\varepsilon + c_\varepsilon x - \sqrt{b^2 x^2 + 2abx + e},$$

where  $a = \sqrt{8(n-5)(n-1)b}$ ,  $b = \min\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\}$ ,  $c_\varepsilon = b + \frac{1}{n-1+\varepsilon}$ ,  $d_\varepsilon = 2 - 2\varepsilon + a$ ,  $e = \sqrt{\varepsilon}$ . We define  $\varphi = \varphi_0$ .

**Lemma 3.1.** *The function  $\varphi$  has the following property.*

- (i)  $\frac{x}{n-1} + 2 < \varphi(x) < \frac{x}{n-1} + n$ ,
- (ii)  $\varphi(x) > 4\sqrt{n-1} - 6$  if  $n = 7, 11, 15$  and  $\varphi(x) > 2 + \frac{8\sqrt{2}}{5}\sqrt{n-5}$  if  $n = 4m - 1$ ,  $m \geq 5$ .

*Proof.* By direct computations, we get

$$\varphi'(x) = c_0 - \frac{bx + a}{\sqrt{x^2 + 2ax/b}}, \quad \varphi''(x) = \frac{a^2}{b(x^2 + 2ax/b)^{3/2}}.$$

Since  $(\varphi(x) - \frac{x}{n-1})'' = \varphi(x)'' > 0$  and  $\lim_{x \rightarrow \infty} \varphi'(x) = \frac{1}{n-1}$ , we have  $\varphi'(x) < \frac{1}{n-1}$ . Hence we get

$$2 = \lim_{x \rightarrow \infty} \left( \varphi(x) - \frac{x}{n-1} \right) < \varphi(x) - \frac{x}{n-1} \leq \varphi(0) = 2 + a < n.$$

We figure out that

$$\min_{x \geq 0} \varphi(x) = \varphi\left(\frac{ac_0}{b\sqrt{c_0^2 - b^2}} - \frac{a}{b}\right) = d_0 - \frac{ac_0}{b} + \frac{a}{b}\sqrt{c_0^2 - b^2}.$$

If  $n = 7, 11, 15$ , we have  $\min_{x \geq 0} \varphi(x) = 4\sqrt{n-1} - 6$ . If  $n = 4m - 1$ ,  $m \geq 5$ , then we have

$$\min_{x \geq 0} \varphi(x) = 2 + \sqrt{\frac{8(n-5)}{2n-5}}(\sqrt{5n-8} - \sqrt{n+2}) > 2 + \frac{8\sqrt{2}}{5}\sqrt{n-5}. \quad \square$$

Let  $\dot{\varphi}_\varepsilon = \varphi_\varepsilon - \frac{1}{n}x$ . We will prove the following lemma.

**Lemma 3.2.** *For sufficiently small  $\varepsilon$ , the function  $\dot{\varphi}_\varepsilon$  satisfies*

- (i)  $\dot{\varphi}'_\varepsilon + 2x\dot{\varphi}''_\varepsilon < \frac{2(n-1)}{n(n+2)}$ ,
- (ii)  $\dot{\varphi}_\varepsilon(x)(\varphi_\varepsilon(x) - n + 9) - x\dot{\varphi}'_\varepsilon(x)(\varphi_\varepsilon(x) + n + 9) < 6(n-3)$ ,
- (iii)  $\dot{\varphi}_\varepsilon(x) - x\dot{\varphi}'_\varepsilon(x) > 1$ .

*Proof.* By direct computations, we have

$$\begin{aligned} \dot{\varphi}'_\varepsilon &= c_\varepsilon - \frac{1}{n} - \frac{b^2x + ab}{\sqrt{b^2x^2 + 2abx + e}}, \\ \dot{\varphi}''_\varepsilon &= \frac{(b^2x + ab)^2 - b^2(b^2x^2 + 2abx + e)}{(b^2x^2 + 2abx + e)^{3/2}}, \\ \dot{\varphi}'''_\varepsilon &= -\frac{3b^3(a^2 - e)(bx + a)}{(b^2x^2 + 2abx + e)^{5/2}}. \end{aligned}$$

Then we have

$$\begin{aligned}\dot{\varphi}'_\varepsilon + 2x\dot{\varphi}''_\varepsilon &= c_\varepsilon - \frac{1}{n} - \frac{b^3x^2(bx+3a) + eb(3bx+a)}{(b^2x^2+2abx+e)^{3/2}} \\ &< b + \frac{1}{n-1+\varepsilon} - \frac{1}{n} < \frac{2(n-1)}{n(n+2)},\end{aligned}$$

as  $b = \min\left\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\right\}$ , so we get the inequality (i).

Setting

$$f(x) = \dot{\varphi}_\varepsilon(\varphi_\varepsilon - n + 9) - x\dot{\varphi}'_\varepsilon(\varphi_\varepsilon + n + 9).$$

Then

$$\begin{aligned}f(x) &= d_\varepsilon(d_\varepsilon - n + 9) + e_\varepsilon + (2 + ab + c_\varepsilon(d_\varepsilon - 2n))x \\ &\quad - (b^2x^2 + 2abx + e)^{-1/2} \\ &\quad \times [b((d_\varepsilon - 2n)b + ac)x^2 + (3(d_\varepsilon - n + 3)ab + ec)x + e(2d_\varepsilon - n + 9)].\end{aligned}$$

Then for  $\varepsilon$  small enough we get

$$\begin{aligned}\lim_{x \rightarrow +\infty} f(x) &= \frac{a^2c_\varepsilon}{b} + d_\varepsilon(d_\varepsilon - n + 9) + a(n - 2d_\varepsilon - 9) + e\left(1 - \frac{c_\varepsilon}{b}\right) \\ &= 6(n-3) + \frac{2\varepsilon(n^2 - (18-3\varepsilon)n + 33 - 15\varepsilon + 2\varepsilon^2)}{n-1+\varepsilon} + e\left(1 - \frac{c_\varepsilon}{b}\right) \\ &= 6(n-3) + \frac{2\varepsilon(n^2 - (18-3\varepsilon)n + 33 - 15\varepsilon + 2\varepsilon^2)}{n-1+\varepsilon} - \frac{\sqrt{\varepsilon}}{(n-1+\varepsilon)b} \\ &< 6(n-3),\end{aligned}$$

and

$$\begin{aligned}f'(x) &= 2 + ab + c_\varepsilon(d_\varepsilon - 2n) \\ &\quad - \frac{1}{(b^2x^2 + 2abx + e)^{3/2}} [b^3((d_\varepsilon - 2n)b + ac_\varepsilon)x^3 + 3ab^2((d_\varepsilon - 2n)b + ac_\varepsilon)x^2 \\ &\quad + [3a^2b^2(d_\varepsilon - n + 3) - 3eb^2(n + 3) + 3abc_\varepsilon e]x \\ &\quad + eab(d_\varepsilon - 2n) + e^2c_\varepsilon].\end{aligned}$$

Then we have

$$\lim_{x \rightarrow +\infty} f'(x) = 2 + ab + c_\varepsilon(d_\varepsilon - 2n) - (ac_\varepsilon + b(d_\varepsilon - 2n)) = 0$$

and

$$\begin{aligned}f''(x) &= \frac{3b^2(a^2 - e)}{(b^2x^2 + 2abx + e)^{5/2}} \\ &\quad \times [b(b(d_\varepsilon + 6) - ac_\varepsilon)x^2 + (ab(d_\varepsilon - n + 3) - ec_\varepsilon)x - e(n + 3)].\end{aligned}$$

For  $b = \min\left\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\right\}$ , we obtain

$$b(d_\varepsilon + 6) - ac_\varepsilon < 0 \quad \text{and} \quad ab(d_\varepsilon - n + 3) - ec_\varepsilon < 0.$$

So  $f''(x) < 0$ . Then we have  $f'(x) > 0$ . From this we deduce that

$$f(x) < \lim_{x \rightarrow +\infty} f(x) < 6(n-3).$$

Thus, inequality (ii) is proved.

We have

$$\dot{\varphi}_\varepsilon - x\dot{\varphi}'_\varepsilon = d_\varepsilon - \frac{abx + e}{\sqrt{b^2x^2 + 2abx + e}} > d_\varepsilon - \frac{abx}{\sqrt{b^2x^2}} - \frac{e}{\sqrt{e}} = 2 - 2\varepsilon - \sqrt[4]{\varepsilon}.$$

This implies inequality (iii). □

Suppose that  $M_0$  is an  $n(\geq 7)$ -dimensional closed hypersurface in  $\mathbb{Q}\mathbb{P}^{(n+1)/4}$  satisfying  $|h|^2 < \varphi(H^2)$ . Let

$$F : M^n \times [0, T) \rightarrow \mathbb{Q}\mathbb{P}^{(n+1)/4}$$

be a mean curvature flow with initial value  $M_0$ . We will show that the pinching condition is preserved along the flow. For convenience, we denote  $\dot{\varphi}_\varepsilon(H^2)$ ,  $\dot{\varphi}'_\varepsilon(H^2)$ ,  $\dot{\varphi}''_\varepsilon(H^2)$  by  $\dot{\varphi}_\varepsilon$ ,  $\dot{\varphi}'_\varepsilon$ ,  $\dot{\varphi}''_\varepsilon$ , respectively.

**Theorem 3.3.** *If the initial value  $M_0$  satisfies  $|h|^2 < \varphi(H^2)$ , then there exists a small positive number  $\varepsilon$ , such that for all  $t \in [0, T)$ , we have  $|h|^2 < \varphi(H^2) - \varepsilon H^2 - \varepsilon$ .*

*Proof.* Since  $M_0$  is closed, there exists a small positive number  $\varepsilon_1$ , such that  $M_0$  satisfies  $|\dot{h}|^2 < \dot{\varphi}_{\varepsilon_1}$ .

From Lemma 3.2(i), we have

$$(3-3) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)\dot{\varphi}_{\varepsilon_1} &= -2(\dot{\varphi}'_{\varepsilon_1} + 2H^2 \cdot \dot{\varphi}''_{\varepsilon_1})|\nabla H|^2 + 2H^2 \cdot \dot{\varphi}'_{\varepsilon_1}(\varphi_{\varepsilon_1} + n + 9) \\ &\geq -\frac{4(n-1)}{n(n+2)}|\nabla H|^2 + 2H^2 \cdot \dot{\varphi}'_{\varepsilon_1}(\varphi_{\varepsilon_1} + n + 9). \end{aligned}$$

Let  $U = |\dot{h}|^2 - \dot{\varphi}_{\varepsilon_1}$ . We get

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial}{\partial t} - \Delta\right)U &\leq -|\nabla \dot{h}|^2 + \frac{2(n-1)}{n(n+2)}|\nabla H|^2 + |\dot{h}|^2(|h|^2 - n + 9) - H^2 \cdot \dot{\varphi}'_{\varepsilon_1}(|h|^2 + n + 9). \end{aligned}$$

By Lemma 2.1, we have

$$-|\nabla \dot{h}|^2 + \frac{2(n-1)}{n(n+2)}|\nabla H|^2 < -6(n-3).$$

At the point where  $U = 0$ , we get

$$\frac{1}{2}\left(\frac{\partial}{\partial t} - \Delta\right)U \leq -6(n-3) + \dot{\varphi}_{\varepsilon_1}(\varphi_{\varepsilon_1} - n + 9) - H^2 \cdot \dot{\varphi}'_{\varepsilon_1}(\varphi_{\varepsilon_1} + n + 9) < 0.$$

Applying the maximum principle, we obtain  $U < 0$  for all  $t \in [0, T)$ . Choose a suitable small positive number  $\varepsilon$ , we complete the proof of Theorem 3.3.  $\square$

Let

$$f_\sigma = \frac{|\mathring{h}|^2}{(\mathring{\varphi})^{1-\sigma}},$$

where  $\sigma \in (0, \varepsilon^2)$  is a positive constant. The following lemma is very useful for deriving the pinching estimate for  $|\mathring{h}|^2$ .

**Lemma 3.4.** *If  $M_0$  satisfies  $|h|^2 < \varphi(H^2)$ , then there exists a small positive number  $\varepsilon$ , such that the following inequality holds along the mean curvature flow:*

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2}{\mathring{\varphi}} |\nabla f_\sigma| |\nabla \mathring{\varphi}| - \frac{2\varepsilon f_\sigma}{n|\mathring{h}|^2} |\nabla \mathring{h}|^2 + 2\sigma |h|^2 f_\sigma - \frac{\varepsilon}{n} f_\sigma.$$

*Proof.* By a straightforward computation, we have

$$\begin{aligned} \left( \frac{\partial}{\partial t} - \Delta \right) f_\sigma &= f_\sigma \left[ \frac{1}{|\mathring{h}|^2} \left( \frac{\partial}{\partial t} - \Delta \right) |\mathring{h}|^2 - \frac{1-\sigma}{\mathring{\varphi}} \left( \frac{\partial}{\partial t} - \Delta \right) \mathring{\varphi} \right] \\ &\quad + 2(1-\sigma) \frac{\langle \nabla f_\sigma, \nabla \mathring{\varphi} \rangle}{\mathring{\varphi}} - \sigma(1-\sigma) f_\sigma \frac{|\nabla \mathring{\varphi}|^2}{|\mathring{\varphi}|^2}. \end{aligned}$$

Using (3-1) and (3-3), we have

$$\begin{aligned} (3-4) \quad \left( \frac{\partial}{\partial t} - \Delta \right) f_\sigma &\leq 2f_\sigma \left[ -\frac{|\nabla \mathring{h}|^2}{|\mathring{h}|^2} + \frac{2(n-1)}{n(n+2)} \frac{|\nabla H|^2}{\mathring{\varphi}} \right] \\ &\quad + 2f_\sigma \left[ |h|^2 + 9 - n - (1-\sigma) \frac{H^2 \cdot \mathring{\varphi}'}{\mathring{\varphi}} (|h|^2 + n + 9) \right] \\ &\quad + \frac{2}{\mathring{\varphi}} |\nabla f_\sigma| |\nabla \mathring{\varphi}|. \end{aligned}$$

From Lemma 3.2 and Theorem 3.3, we have

$$\begin{aligned} -\frac{|\nabla \mathring{h}|^2}{|\mathring{h}|^2} + \frac{2(n-1)}{n(n+2)} \frac{|\nabla H|^2}{\mathring{\varphi}} &\leq -\frac{|\nabla \mathring{h}|^2}{|\mathring{h}|^2} + \frac{|\nabla \mathring{h}|^2 - 6(n-3)}{\mathring{\varphi}} \\ &\leq \frac{|\mathring{h}|^2 - \mathring{\varphi}}{|\mathring{h}|^2 \mathring{\varphi}} |\nabla \mathring{h}|^2 - \frac{6(n-3)}{\mathring{\varphi}} \\ &\leq -\varepsilon \frac{|H|^2 + 1}{|\mathring{h}|^2 \mathring{\varphi}} |\nabla \mathring{h}|^2 - \frac{6(n-3)}{\mathring{\varphi}} \\ &\leq -\frac{\varepsilon}{n|\mathring{h}|^2} |\nabla \mathring{h}|^2 - \frac{6(n-3)}{\mathring{\varphi}}. \end{aligned}$$

From (ii) and (iii) of Lemma 3.2, we have

$$\begin{aligned}
 & |h|^2 + 9 - n - (1 - \sigma) \frac{H^2 \cdot \dot{\varphi}'}{\dot{\varphi}} (|h|^2 + n + 9) \\
 &= \frac{1 - \sigma}{\dot{\varphi}} [(\dot{\varphi} - H^2 \cdot \dot{\varphi}') |h|^2 - H^2 \cdot \dot{\varphi}' (n + 9)] - n + 9 + \sigma |h|^2 \\
 &\leq \frac{1 - \sigma}{\dot{\varphi}} [(\dot{\varphi} - H^2 \cdot \dot{\varphi}') (\varphi - \varepsilon H^2 - \varepsilon) - H^2 \cdot \dot{\varphi}' (n + 9)] - n + 9 + \sigma |h|^2 \\
 &= \frac{1 - \sigma}{\dot{\varphi}} [(\dot{\varphi} - H^2 \cdot \dot{\varphi}') \varphi - H^2 \cdot \dot{\varphi}' (n + 9)] - n + 9 + \sigma |h|^2 \\
 &\qquad\qquad\qquad - \frac{(1 - \sigma) \varepsilon}{\dot{\varphi}} (\dot{\varphi} - H^2 \cdot \dot{\varphi}') (H^2 + 1) \\
 &\leq (1 - \sigma) \left[ n - 9 + \frac{6(n - 3)}{\dot{\varphi}} \right] - n + 9 + \sigma |h|^2 - \frac{(1 - \sigma) \varepsilon}{\dot{\varphi}} (H^2 + 1) \\
 &\leq \sigma |h|^2 + \frac{6(n - 3)}{\dot{\varphi}} - \frac{\varepsilon}{2n}.
 \end{aligned}$$

Inserting these two estimates into (3-4) will complete the proof. □

**4. An estimate for traceless second fundamental form**

Suppose that the initial value  $M_0$  satisfies the condition in Theorem 1.2. For convenience, we put  $W = \dot{\varphi}$ . By the conclusion of the previous section, there exists a sufficiently small positive number  $\varepsilon$ , such that for all  $t \in [0, T)$ , the following pinching condition holds:

$$(4-1) \qquad |\dot{h}|^2 < W - \varepsilon H^2.$$

From this inequality and the definition of  $W$ , we have  $W < \frac{H^2}{n(n-1)} + n$ .

We consider the auxiliary function

$$f_\sigma = \frac{|\dot{h}|^2}{W^{1-\sigma}}.$$

In this section, we will show that  $f_\sigma$  decays exponentially.

**Lemma 4.1.** *There exist positive numbers  $\varepsilon$  and  $C_1$  depending only on  $M_0$ , such that*

$$(4-2) \qquad \frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2C_1}{|\dot{h}|} |\nabla f_\sigma| |\nabla \dot{h}| - \frac{\varepsilon f_\sigma}{n |\dot{h}|^2} |\nabla \dot{h}|^2 + 2\sigma |h|^2 f_\sigma - \frac{\varepsilon}{n} f_\sigma.$$

*Proof.* According to Lemma 3.4, we have the following inequality with some suitable small  $\varepsilon > 0$ :

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2}{W} |\nabla f_\sigma| |\nabla W| - \frac{2\varepsilon f_\sigma}{n |\dot{h}|^2} |\nabla \dot{h}|^2 + 2\sigma |h|^2 f_\sigma - \frac{\varepsilon}{n} f_\sigma.$$

By the definition of  $W$ , there exists a constant  $B_1$ , such that  $|\nabla W| < B_1|\nabla H^2|$  and  $|H| < B_1\sqrt{W}$ . Let  $C_1$  be a constant such that  $2B_1^2|\nabla H| \leq C_1|\nabla \dot{h}|$ . From Lemma 2.1, we have

$$(4-3) \quad \frac{|\nabla W|}{W} \leq \frac{2B_1|H||\nabla H|}{\sqrt{W}|\dot{h}|} \leq \frac{2B_1^2|\nabla H|}{|\dot{h}|} \leq \frac{C_1|\nabla \dot{h}|}{|\dot{h}|}. \quad \square$$

We need the following estimate for the Laplacian of  $|\dot{h}|^2$ .

**Lemma 4.2.**  $\Delta|\dot{h}|^2 \geq 2\langle \dot{h}, \nabla^2 H \rangle + 2|\dot{h}|^2(\varepsilon|h|^2 - 2n^2) - 18|\dot{h}||H|$ .

*Proof.* We have

$$\Delta|\dot{h}|^2 = 2|\nabla \dot{h}|^2 + 2\dot{h} \cdot \Delta \dot{h} = 2|\nabla \dot{h}|^2 + 2 \sum_{i,j} \dot{h}_{ij} \cdot \Delta h_{ij}$$

and

$$\begin{aligned} \sum_{i,j} \dot{h}_{ij} \cdot \Delta h_{ij} &= \langle \dot{h}, \nabla^2 H \rangle + \sum_{i,p,j} H h_{ip} h_{pj} h_{ij} - |h|^4 \\ &\quad + 3H \sum_{i,j} \sum_{k_0=1}^3 J_{in+1}^{(k_0)} J_{jn+1}^{(k_0)} \dot{h}_{ij} - (n+9)|\dot{h}|^2 + 2n|\dot{h}|^2 - 6S_1 \\ &\geq \langle \dot{h}, \nabla^2 H \rangle + \sum_{i,p,j} H h_{ip} h_{pj} h_{ij} - |h|^4 + (n-9)|\dot{h}|^2 - 9|\dot{h}||H|. \end{aligned}$$

It follows from the proof of the Lemma 4.2 in [17], we choose a local orthonormal frame such that

$$H = |H| e_{n+1} \quad \text{and} \quad \dot{h} = \text{diag}\{\dot{\lambda}_1, \dots, \dot{\lambda}_n\}.$$

So we have

$$\begin{aligned} \sum_{i,p,j} H h_{ip} h_{pj} h_{ij} - |h|^4 &= H \sum_i \dot{\lambda}_i^3 + \frac{1}{n} H^2 |\dot{h}|^2 - |\dot{h}|^4 \\ &\geq -|H| \frac{n-2}{\sqrt{n(n-1)}} |\dot{h}|^3 + \frac{1}{n} H^2 |\dot{h}|^2 - |\dot{h}|^4 \\ &= |\dot{h}|^2 \left( \frac{1}{n} H^2 - |\dot{h}|^2 - \frac{n-2}{\sqrt{n(n-1)}} |\dot{h}||H| \right) \\ &\geq |\dot{h}|^2 \left[ \frac{1}{n} H^2 - \left( \frac{H^2}{n(n-1)} + n - \varepsilon H^2 \right) \right] - (n-2) \left( \frac{H^2}{n(n-1)} + n \right) \\ &= |\dot{h}|^2 (\varepsilon H^2 - n(n-1)) \\ &> |\dot{h}|^2 (\varepsilon |h|^2 - n^2), \end{aligned}$$

where we have used  $|\dot{h}|^2 < W - \varepsilon H^2$  and  $W < \frac{H^2}{n(n-1)} + n$ . □

From (4-3) and Lemma 4.2, we have

$$\begin{aligned} \Delta f_\sigma &= f_\sigma \left( \frac{\Delta |\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma) \frac{\Delta W}{W} \right) - 2(1-\sigma) \frac{\langle \nabla f_\sigma, \nabla W \rangle}{W} + \sigma(1-\sigma) f_\sigma \frac{|\nabla W|^2}{W^2} \\ &\geq f_\sigma \frac{\Delta |\mathring{h}|^2}{|\mathring{h}|^2} - (1-\sigma) f_\sigma \frac{\Delta W}{W} - \frac{2C_1 |\nabla f_\sigma| |\nabla \mathring{h}|}{|\mathring{h}|} \\ &\geq \frac{2\langle \mathring{h}, \nabla^2 H \rangle}{W^{1-\sigma}} + 2f_\sigma (\varepsilon |h|^2 - 2n^2) - (1-\sigma) \frac{f_\sigma \Delta W}{W} - \frac{2C_1 |\nabla f_\sigma| |\nabla \mathring{h}|}{|\mathring{h}|} - \frac{18f_\sigma |H|}{|\mathring{h}|}. \end{aligned}$$

Multiplying both sides of the above inequality by  $f_\sigma^{p-1}$ , we get

$$(4-4) \quad 2\varepsilon f_\sigma^p |h|^2 \leq f_\sigma^{p-1} \Delta f_\sigma + (1-\sigma) \frac{f_\sigma^p \Delta W}{W} - \frac{2f_\sigma^{p-1} \langle \mathring{h}, \nabla^2 H \rangle}{W^{1-\sigma}} + \frac{2C_1 f_\sigma^{p-1} |\nabla f_\sigma| |\nabla \mathring{h}|}{|\mathring{h}|} + 4n^2 f_\sigma^p + \frac{18f_\sigma |H|}{|\mathring{h}|}.$$

Then integrate both sides of (4-4) over  $M_t$ . By the divergence theorem, we get

$$(4-5) \quad \int_{M_t} f_\sigma^{p-1} \Delta f_\sigma d\mu_t = -(p-1) \int_{M_t} f_\sigma^{p-2} |\nabla f_\sigma|^2 d\mu_t.$$

From (4-4), we have

$$(4-6) \quad \begin{aligned} \int_{M_t} \frac{f_\sigma^p}{W} \Delta W d\mu_t &= - \int_{M_t} \left\langle \nabla \left( \frac{f_\sigma^p}{W} \right), \nabla W \right\rangle d\mu_t \\ &= \int_{M_t} \left( -\frac{p f_\sigma^{p-1}}{W} \langle \nabla f_\sigma, \nabla W \rangle + \frac{f_\sigma^p}{W^2} |\nabla W|^2 \right) d\mu_t \\ &\leq \int_{M_t} \left( \frac{C_1 p f_\sigma^{p-1}}{|\mathring{h}|} |\nabla f_\sigma| |\nabla \mathring{h}| + \frac{C_1^2 f_\sigma^p}{|\mathring{h}|^2} |\nabla \mathring{h}|^2 \right) d\mu_t. \end{aligned}$$

We also have

$$(4-7) \quad \begin{aligned} & - \int_{M_t} \frac{f_\sigma^{p-1} \langle \mathring{h}, \nabla^2 H \rangle}{W^{1-\sigma}} d\mu_t \\ &= \int_{M_t} \nabla_i \left( \frac{f_\sigma^{p-1}}{W^{1-\sigma}} \mathring{h}_{ij} \right) \nabla_j H d\mu_t \\ &= \int_{M_t} \left[ \frac{(p-1) f_\sigma^{p-2}}{W^{1-\sigma}} \mathring{h}_{ij} \nabla_i f_\sigma - \frac{(1-\sigma) f_\sigma^{p-1}}{W^{2-\sigma}} \mathring{h}_{ij} \nabla_i W + \frac{f_\sigma^{p-1}}{W^{1-\sigma}} \nabla_i \mathring{h}_{ij} \right] \nabla_j H d\mu_t \\ &\leq \int_{M_t} \left[ \frac{(p-1) f_\sigma^{p-1}}{|\mathring{h}|} |\nabla f_\sigma| + \frac{f_\sigma^{p-1}}{W^{2-\sigma}} |\mathring{h}| |\nabla W| + \frac{f_\sigma^{p-1}}{W^{1-\sigma}} n |\nabla \mathring{h}| \right] |\nabla H| d\mu_t \end{aligned}$$

$$\begin{aligned}
&\leq \int_{M_t} \left[ \frac{(p-1)f_\sigma^{p-1}}{|\dot{h}|} |\nabla f_\sigma| + \frac{C_1 f_\sigma^{p-1}}{W^{1-\sigma}} |\nabla \dot{h}| + \frac{f_\sigma^{p-1}}{W^{1-\sigma}} n |\nabla \dot{h}| \right] n |\nabla \dot{h}| d\mu_t \\
&\leq \int_{M_t} \left[ \frac{n(p-1)f_\sigma^{p-1}}{|\dot{h}|} |\nabla f_\sigma| |\nabla \dot{h}| + \frac{(C_1 n + n^2) f_\sigma^p}{|\dot{h}|^2} |\nabla \dot{h}|^2 \right] d\mu_t.
\end{aligned}$$

Putting (4-4)–(4-7) together, we get

$$\int_{M_t} |h|^2 f_\sigma^p d\mu_t \leq C_2 \int_{M_t} \left[ \frac{p f_\sigma^{p-1}}{|\dot{h}|} |\nabla f_\sigma| |\nabla \dot{h}| + \frac{f_\sigma^p}{|\dot{h}|^2} |\nabla \dot{h}|^2 + f_\sigma^p + \frac{f_\sigma^p |H|}{|\dot{h}|} \right] d\mu_t,$$

where  $C_2$  is a positive constant depending only on  $M_0$ .

Combining Lemma 4.2 and (4-2), we get

$$\begin{aligned}
(4-8) \quad &\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu_t \\
&= p \int_{M_t} f_\sigma^{p-1} \frac{\partial}{\partial t} f_\sigma d\mu_t - \int_{M_t} f_\sigma^p H^2 d\mu_t \\
&\leq p \int_{M_t} f_\sigma^{p-2} \left[ -(p-1) |\nabla f_\sigma|^2 + (2C_1 + 2\sigma C_2 p) \frac{f_\sigma}{|\dot{h}|} |\nabla f_\sigma| |\nabla \dot{h}| \right. \\
&\quad \left. - \left( \frac{\varepsilon}{2n} - 2\sigma C_2 \right) \frac{f_\sigma^2}{|\dot{h}|^2} |\nabla \dot{h}|^2 \right] d\mu_t \\
&\quad - p \int_{M_t} f_\sigma^p \left( \frac{\varepsilon}{n} - 2\sigma C_2 + \frac{6(n-3)\varepsilon}{2n|\dot{h}|^2} - \frac{2\sigma C_2 |H|}{|\dot{h}|} + \frac{|H|^2}{p} \right) d\mu_t.
\end{aligned}$$

Now we will show that the  $L^p$ -form of  $f_\sigma$  decays exponentially.

**Lemma 4.3.** *There exist positive constants  $C_3, p_0, \sigma_0$  depending only on  $M_0$ , such that for all  $p \geq p_0$  and  $\sigma \leq \sigma_0/\sqrt{p}$ , we have*

$$\left( \int_{M_t} f_\sigma^p d\mu_t \right)^{1/p} < C_3 e^{-\varepsilon t}.$$

*Proof.* The expression in the square bracket of the right side of (4-8) is a quadratic polynomial. With  $p_0$  large enough and  $\sigma_0$  small enough, its discriminant satisfies

$$(2C_1 + 2\sigma C_2 p)^2 - 4(p-1) \left( \frac{\varepsilon}{2n} - 2\sigma C_2 \right) < 0 \quad \text{and} \quad \frac{12\varepsilon}{7} \geq p\sigma^2 C_2^2.$$

We have

$$\begin{aligned}
\frac{\varepsilon}{n} - 2\sigma C_2 + \frac{6(n-3)\varepsilon}{2n|\dot{h}|^2} - \frac{2\sigma C_2 |H|}{|\dot{h}|} + \frac{|H|^2}{p} &\geq \frac{\varepsilon}{n} - 2\sigma C_2 + \frac{12\varepsilon}{7|\dot{h}|^2} - \frac{p\sigma^2 C_2^2}{|\dot{h}|^2} \\
&\geq \frac{\varepsilon}{n} - 2\sigma C_2 > \frac{\varepsilon}{2n}.
\end{aligned}$$

Here we have used the inequality  $\frac{\varepsilon}{2n} - 2\sigma C_2 > 0$ , which is implied by the choices of  $p_0$  and  $\sigma_0$ . Then we get

$$\frac{d}{dt} \int_{M_t} f_\sigma^p d\mu_t \leq -\frac{p\varepsilon}{2n} \int_{M_t} f_\sigma^p d\mu_t.$$

So we get  $\int_{M_t} f_\sigma^p d\mu_t \leq e^{-p\varepsilon/2n} \int_{M_0} f_\sigma^p d\mu_0$ , which completes the proof.  $\square$

Let  $g_\sigma = f_\sigma e^{\varepsilon t/2}$ . By the Sobolev inequality on submanifolds and a Stampacchia iteration procedure, we obtain that  $g_\sigma$  is uniformly bounded for all  $t$  (see [9] or [14] for the details). Then we obtain the following theorem.

**Theorem 4.4.** *There exist positive constants  $\varepsilon, \sigma$  and  $C_0$  depending only on  $M_0$ , such that for all  $t \in [0, T)$ , we have*

$$|\mathring{h}|^2 \leq C_0(H^2 + 1)^{1-\sigma} e^{-\varepsilon t/2}.$$

### 5. A gradient estimate

We derive an estimate for  $|\nabla H|^2$  along the mean curvature flow. Firstly, the same as Proposition 4.3 in [25], we have:

**Lemma 5.1.** *There exists a positive constants  $C_4 > 1$  depending only on  $n$ , such that*

$$\frac{\partial}{\partial t} |\nabla H|^2 \leq \Delta |\nabla H|^2 + C_4(H^2 + 1) |\nabla h|^2.$$

Secondly, we need the following estimates.

**Lemma 5.2.** *Along the mean curvature flow, we have*

- (i)  $\frac{\partial}{\partial t} H^4 \geq \Delta H^4 - 12nH^2|\nabla h|^2 + \frac{4}{n}H^6,$
- (ii)  $\frac{\partial}{\partial t} |\mathring{h}|^2 \leq \Delta |\mathring{h}|^2 - \frac{1}{3}|\nabla h|^2 + C_5|\mathring{h}|^2(H^2 + 1),$
- (iii)  $\frac{\partial}{\partial t} (H^2|\mathring{h}|^2) \leq \Delta(H^2|\mathring{h}|^2) - \frac{1}{6}H^2|\nabla h|^2 + C_6(H^2 + 1)^2|\mathring{h}|^2 + C_7|\nabla h|^2,$

where  $C_5, C_6, C_7$  are sufficiently large constants.

*Proof.* (i) From Lemma 2.3, we derive that

$$\frac{\partial}{\partial t} H^4 = \Delta H^4 - 12H^2|\nabla H|^2 + 4H^4(|h|^2 + n + 9).$$

From Lemma 2.1, we have  $12H^2|\nabla H|^2 \leq 12nH^2|\nabla h|^2$ . Obviously, inequality (i) holds.

(ii) We have

$$\frac{\partial}{\partial t} |\mathring{h}|^2 = \Delta |\mathring{h}|^2 - 2|\nabla \mathring{h}|^2 + 2|h|^2|\mathring{h}|^2 + 18|\mathring{h}|^2 - 2n|\mathring{h}|^2 + 12S_1.$$

From Lemma 2.1, we get  $|\nabla \mathring{h}|^2 \geq \frac{1}{6}|\nabla h|^2$ . Choose a large constant  $C_5$ , we obtain inequality (ii).

(iii) It follows the evolution equation that

$$\begin{aligned} \frac{\partial}{\partial t}(H^2|\dot{h}|^2) &= \Delta(H^2|\dot{h}|^2) - 2\langle \nabla H^2, \nabla|\dot{h}|^2 \rangle + 4|\dot{h}|^2 H^2|h|^2 - 2|\dot{h}|^2|\nabla H|^2 \\ &\quad - 2H^2|\nabla\dot{h}|^2 + 36H^2|\dot{h}|^2 + 12S_1H^2. \end{aligned}$$

From Lemma 2.1, we get  $-2H^2|\nabla\dot{h}|^2 \leq -\frac{1}{3}H^2|\nabla h|^2$ . From the preserved pinching condition  $|\dot{h}|^2 < W$ , we have

$$4|\dot{h}|^2 H^2|h|^2 + 36H^2|\dot{h}|^2 \leq C_6(H^2 + 1)^2|\dot{h}|^2.$$

Using Theorem 4.4, we have

$$-2\langle \nabla H^2, \nabla|\dot{h}|^2 \rangle \leq 8|H| |\nabla H| |\dot{h}| |\nabla h| \leq 8n\sqrt{C_0}|H|(H^2 + 1)^{(1-\sigma)/2}|\nabla h|^2.$$

By Young's inequality, there exists a positive constant  $C_7$ , such that

$$-2\langle \nabla H^2, \nabla|\dot{h}|^2 \rangle \leq (C_7 + \frac{1}{6}H^2)|\nabla h|^2. \quad \square$$

Now we prove a gradient estimate for mean curvature.

**Theorem 5.3.** *For any  $\eta \in (0, \sqrt{\varepsilon}/4\pi n)$ , there exists a number  $\Psi(\eta)$  depending only on  $\eta$  and  $M_0$ , such that*

$$|\nabla H|^2 < [(\eta H)^4 + \Psi^2(\eta)] e^{-\varepsilon t/4}.$$

*Proof.* Define a scalar function

$$f = (|\nabla H|^2 + B_1|\dot{h}|^2 + B_2H^2|\dot{h}|^2) e^{\varepsilon t/4} - (\eta H)^4,$$

where  $B_1, B_2$  are two positive constants.

From Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)f \\ &= \frac{\varepsilon}{4}(|\nabla H|^2 + B_1|\dot{h}|^2 + B_2H^2|\dot{h}|^2) e^{\varepsilon t/4} \\ &\quad + e^{\varepsilon t/4} \left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla H|^2 + B_1|\dot{h}|^2 + B_2H^2|\dot{h}|^2) - \eta^4 \left(\frac{\partial}{\partial t} - \Delta\right)H^4 \\ &\leq \frac{\varepsilon}{4}(|\nabla H|^2 + B_1|\dot{h}|^2 + B_2H^2|\dot{h}|^2) e^{\varepsilon t/4} \\ &\quad + e^{\varepsilon t/4} \left\{ (C_4(H^2 + 1)|\nabla h|^2) + B_1\left(-\frac{1}{3}|\nabla h|^2 + C_5(H^2 + 1)|\dot{h}|^2\right) \right. \\ &\quad \quad \quad \left. + B_2\left(-\frac{1}{6}H^2|\nabla h|^2 + C_7|\nabla h|^2 + C_6(H^2 + 1)^2|\dot{h}|^2\right) \right\} \\ &\quad - \eta^4 \left(-12nH^2|\nabla h|^2 + \frac{4}{n}H^6\right) \\ &= H^2|\nabla h|^2 \left[ e^{\varepsilon t/4} \left(C_4 - \frac{B_2}{6}\right) + 24n\eta^4 \right] + e^{\varepsilon t/4} \left[ |\nabla h|^2 \left(C_4 - \frac{B_1}{3} + C_7B_2\right) + \frac{\varepsilon}{4}|\nabla H|^2 \right] \\ &\quad + e^{\varepsilon t/4} |\dot{h}|^2 \left[ B_1C_5(H^2 + 1) + B_2C_6(H^2 + 1)^2 + \frac{\varepsilon}{4}(B_1 + B_2H^2) \right] - \frac{4\eta^4}{n}|H|^6. \end{aligned}$$

Choose constants  $B_1$  and  $B_2$ , such that  $C_4 - \frac{B_2}{6} < -1$  and  $C_4 - \frac{B_1}{3} + C_7 B_2 < -1$ . Then applying Theorem 4.4, we get

$$(5-1) \quad \left(\frac{\partial}{\partial t} - \Delta\right) f \leq e^{-\varepsilon t/4} \left[ B_3 (H^2 + 1)^2 (H^2 + 1)^{1-\sigma} - \frac{4\eta^4}{n} H^6 \right].$$

Consider the expression in the bracket of (5-1). Since the coefficient of  $H^6$  is negative, it has an upper bound  $\Psi_2(\eta)$ . Then we have  $\left(\frac{\partial}{\partial t} - \Delta\right) f \leq e^{-\varepsilon t/4} \Psi_2(\eta)$ . It follows from the maximum principle that  $f$  is bounded. This completes the proof of Theorem 5.3.  $\square$

### 6. Convergence

In order to estimate the diameter of  $M_t$ , we need the well-known Myers's theorem:

**Theorem 6.1** (Myers's theorem). *Let  $\Gamma$  be a geodesic of length at least  $\pi/\sqrt{K}$  on  $M$ . If the Ricci curvature satisfies  $\text{Ric}(X) \geq (n - 1)K$  for each unit vector  $X \in T_x M$ , at any point  $x \in \Gamma$ , then  $\Gamma$  has conjugate points.*

Now we show that under the assumption of Theorem 1.2, the mean curvature flow converges to a round point.

**Theorem 6.2.** *If  $M_0$  satisfies  $|\dot{h}|^2 < \dot{\varphi}$ , then  $T < \infty$  and  $M_t$  converges to a round point as  $t \rightarrow T$ .*

*Proof.* Assume  $T = \infty$ . Let  $|H|_{\min}(t) = \min_{M_t} |H|$ ,  $|H|_{\max}(t) = \max_{M_t} |H|$ .

We claim that  $H^2 \cdot e^{\varepsilon t/8}$  is uniformly bounded on  $[0, \infty)$ . Suppose not, then there is a time  $\tau$  such that  $|H|_{\max}^2(\tau) \cdot e^{\varepsilon \tau/8} > \Psi/\eta^2$ . By Theorem 5.3, for every small positive number  $\eta$ , there exists a positive number  $\Psi$ , such that  $|\nabla H| < [(\eta H)^2 + \Psi]e^{-\varepsilon t/8}$ . Then we have  $|\nabla H| < 2\eta^2 |H|_{\max}^2$  on  $M_\tau$ .

From Lemma 4.1 in [27], the sectional curvature  $K$  of  $M$  satisfies

$$(6-1) \quad K \geq \frac{1}{2} \left( 2 + \frac{1}{n-1} H^2 - |h|^2 \right).$$

By Theorem 4.4, we obtain

$$K \geq \frac{1}{2} \left( 2 + \frac{1}{n(n-1)} H^2 - C_0 (H^2 + 1)^{1-\sigma} e^{-\varepsilon t/2} \right).$$

Hence, we can pick  $\tau$  large enough such that  $K \geq (1/2n^2) H^2$  on  $M_\tau$ .

Let  $x$  be a point on  $M_\tau$  where  $|H|$  achieves its maximum. Consider all the geodesics of length at most  $(4\eta |H|_{\max})^{-1}$  starting from  $x$ . As  $|\nabla H^2| < 4\eta^2 |H|_{\max}^3$ , we have

$$H^2 \geq |H|_{\max}^2 - 4\eta^2 |H|_{\max}^3 \cdot (4\eta |H|_{\max})^{-1} = (1 - \eta) |H|_{\max}^2$$

along such geodesics. Since  $|\nabla H| < 2\eta^2 |H|_{\max}^2$  and  $K \geq (1/2n^2) H^2$  on  $M_\tau$ , one has

$$K \geq \frac{1}{2n^2} (1 - \eta) |H|_{\max}^2$$

along such geodesics provided

$$\eta \in \left( 0, \min \left\{ \frac{1}{32\pi n}, \frac{\sqrt{\varepsilon}}{4\pi n} \right\} \right).$$

By Myers's theorem, these geodesics can reach any point of  $M_\tau$ . This implies

$$H^2 \geq (1 - \eta) |H|_{\max}^2 \quad \text{on } M_\tau.$$

Combining this inequality with  $|H|_{\max}^2(\tau) \cdot e^{\varepsilon\tau/8} > \Psi/\eta^2$  and Theorem 5.3, we get

$$|\nabla H|^2 < (\eta H)^4 + \frac{\eta^4}{(1 - \eta)^2} |H|_{\min}^4(\tau), \quad t \geq \tau.$$

From the evolution equation of  $H^2$ , we have that for  $t \geq \tau$ ,

$$(6-2) \quad \left( \frac{\partial}{\partial t} - \Delta \right) H^2 \geq -2|\nabla H|^2 + \frac{2}{n} H^4 \geq \frac{1}{n} H^4 - \frac{1}{2n} |H|_{\min}^4(\tau)$$

for  $\eta > 0$  sufficiently small. By the maximum principle, we get  $H^2 \geq |H|_{\min}^2(\tau)$  for  $t \geq \tau$ . Then (6-2) yields

$$\left( \frac{\partial}{\partial t} - \Delta \right) H^2 \geq \frac{1}{2n} H^4, \quad t \geq \tau.$$

By the maximum principle,  $H^2$  blows up in finite time. This contradicts the infinity of  $T$ . Therefore, we obtain  $H^2 \leq C e^{-\varepsilon t/8}$  for  $t \in [0, \infty)$  for a uniform positive constant  $C$ . By Theorem 4.4,  $|h|^2 = |\mathring{h}|^2 + \frac{1}{n} |H|^2 \leq C e^{-\varepsilon t/8}$  for  $t \in [0, \infty)$ , which implies that  $M_t$  converges to a closed totally geodesic hypersurface  $M_\infty$  as  $t \rightarrow \infty$ . However, there is no closed totally geodesic hypersurface in  $\mathbb{Q}\mathbb{P}^{(n+1)/4}$ , see, e.g., Corollary 7.2 in [25]. Therefore, we get a contradiction, and hence  $T < \infty$ .

So  $T$  is finite, and  $\max_{M_t} |h|^2$  blows up as  $t \rightarrow T$ . From the preserved pinching condition,  $|H|_{\max}(t)$  also blows up as  $t \rightarrow T$ . By Theorem 5.3, for any  $\eta \in (0, \sqrt{\varepsilon}/4n\pi)$ , there exists a positive number  $\Psi = \Psi(\eta) > 1$ , such that

$$|\nabla H| < (\eta H)^2 + \Psi \quad \text{for } t \in [0, T).$$

Since  $|H|_{\max}(t)$  blows up as  $t \rightarrow T$ , there exists a time  $\tau_1$  depending on  $\eta$ , such that

$$|H|_{\max}^2 \geq \max \left\{ \frac{2\Psi}{\eta^2}, \frac{8n}{\varepsilon} \right\} \quad \text{on } M_{\tau_1},$$

where  $\varepsilon > 0$  is as in (4-1). Then we get  $|\nabla H| \leq 2\eta^2 |H|_{\max}^2$  on  $M_{\tau_1}$ . Similarly as above, we obtain

$$(6-3) \quad |H|_{\min}^2 \geq (1 - \eta) |H|_{\max}^2 \quad \text{on } M_{\tau_1}.$$

By (4-1) and (6-1), one has  $K \geq \frac{1}{2}(\varepsilon H^2 - n)$  for all  $t \in [0, T)$ . Hence for small  $\eta > 0$ , we have the following estimate at  $t = \tau_1$ :

$$K \geq \frac{1}{4}\varepsilon H^2 + \frac{1}{4}[\varepsilon(1 - \eta) |H|_{\max}^2 - 2n] \geq \frac{1}{4}\varepsilon H^2 + \frac{1}{8}(\varepsilon |H|_{\max}^2 - 4n) \geq \frac{1}{16}\varepsilon |H|_{\max}^2.$$

This implies  $\text{diam}(M_{\tau_1}) \leq 4\pi/(\sqrt{\varepsilon}|H|_{\max})$ .

Furthermore, by Theorem 5.3 and (6-3) one has that for  $t \geq \tau_1$ ,

$$|\nabla H|^2 < 2(\eta H)^4 + 2\Psi^2 \leq 2(\eta H)^4 + \frac{1}{4}\eta^4 |H|_{\max}^2(\tau_1) \leq 2(\eta H)^4 + \frac{1}{2}\eta^4 |H|_{\min}^2(\tau_1).$$

Hence for  $t \geq \tau_1$ , we have

$$(6-4) \quad \left(\frac{\partial}{\partial t} - \Delta\right) H^2 \geq -2|\nabla H|^2 + \frac{2}{n} H^4 \geq \frac{1}{n} H^4 - \frac{1}{2n} |H|_{\min}^4(\tau_1)$$

provided that  $\eta > 0$  is sufficiently small. By the maximum principle, we get  $H^2 \geq |H|_{\min}^2(\tau_1)$  for  $t \geq \tau_1$ . Then (6-4) yields

$$\left(\frac{\partial}{\partial t} - \Delta\right) H^2 \geq \frac{1}{2n} H^4, \quad t \geq \tau_1.$$

By the maximum principle,  $|H|_{\min}^2(t)$  is increasing on  $[\tau_1, T)$ . So

$$|H|_{\max}^2(t) \geq |H|_{\min}^2(t) \geq |H|_{\min}^2(\tau_1) \geq \frac{1}{2}|H|_{\max}^2(\tau_1) \geq \max\left\{\frac{\Psi}{\eta^2}, \frac{4n}{\varepsilon}\right\}$$

for all  $t \geq \tau_1$  and for every  $\eta > 0$  sufficiently small. Hence  $|\nabla H| \leq 2\eta^2 |H|_{\max}^2$  for all  $t \geq \tau_1$ . By a similar argument, we get  $|H|_{\min}^2 \geq (1 - \eta) |H|_{\max}^2$  for all  $\eta$  sufficiently small and all  $t \geq \tau_1$ . This implies  $|H|_{\min}/|H|_{\max} \rightarrow 1$  as  $t \rightarrow T$ .

Since for  $t \geq \tau_1$ ,

$$K \geq \frac{1}{4}\varepsilon H^2 + \frac{1}{8}(\varepsilon |H|_{\max}^2 - 4n) \geq \frac{1}{16}\varepsilon |H|_{\max}^2,$$

we have

$$\text{diam}(M_t) \leq \frac{4\pi}{\sqrt{\varepsilon}|H|_{\max}(t)}$$

for all  $t \geq \tau_1$ . So  $\text{diam}(M_t) \rightarrow 0$ , and by a similar argument as in [10],  $M_t$  shrinks to a single point as  $t \rightarrow T$ .

Now we dilate the metric of the ambient space such that the hypersurface maintains its volume along the flow. Using the same method as in [19], we can prove that the sequence of time-slices of rescaled flow corresponding to any sequence of times that tends to infinity has a subsequence that converges to a round sphere. This proves that the limit point of the mean curvature flow is round.  $\square$

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SHIYANG LI  
CENTER OF MATHEMATICAL SCIENCES  
ZHEJIANG UNIVERSITY  
HANGZHOU  
CHINA  
lisycms@zju.edu.cn

HONGWEI XU  
CENTER OF MATHEMATICAL SCIENCES  
ZHEJIANG UNIVERSITY  
HANGZHOU  
CHINA  
xuhw@zju.edu.cn

ENTAO ZHAO  
CENTER OF MATHEMATICAL SCIENCES  
ZHEJIANG UNIVERSITY  
HANGZHOU  
CHINA  
zhaoet@zju.edu.cn



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[lipshitz@uoregon.edu](mailto:lipshitz@uoregon.edu)

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Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Dimitri Shlyakhtenko  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[shlyakht@ipam.ucla.edu](mailto:shlyakht@ipam.ucla.edu)

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
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