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A NEW CONVERGENCE THEOREM FOR MEAN CURVATURE FLOW OF HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACES

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We investigate the smooth convergence of the mean curvature flow of hypersurfaces in the quaternionic projective spaces. We prove that if the initial hypersurface satisfies a new nonlinear curvature pinching condition, then the mean curvature flow converges smoothly to a round point in finite time. Our result improves a smooth convergence theorem due to Pipoli and Sinestrari (2017).

1. Introduction

There are many famous geometric evolution equations, such as the Ricci flow, the mean curvature flow and others. Huisken [9] studied the mean curvature flow from the perspective of partial differential equations, and he proved that convex hypersurface in the Euclidean space converges to a round point along the flow. Afterwards, Huisken [10; 11] obtained convergence results for mean curvature flow of convex hypersurfaces in Riemannian manifolds and pinched hypersurfaces in spheres. Following the argument of Huisken [9], Andrews and Baker [1] proved a convergence theorem for the mean curvature flow of closed submanifolds satisfying a suitable pinching condition in the Euclidean space. Later, Baker [2], Liu et al. [20] proved sharp convergence theorems for the mean curvature flow in the spheres and the hyperbolic spaces, respectively. Liu, Xu and Zhao [19] studied the mean curvature flow of arbitrary codimensional submanifolds in the Riemannian manifold and proved a smooth convergence theorem. Lei and Xu [15] verified an optimal convergence theorem for the mean curvature flow of submanifolds in hyperbolic spaces, which implies the first optimal differentiable sphere theorem for submanifolds with positive Ricci curvature. It should be remarked that an optimal topological sphere theorem for complete submanifolds with positive Ricci curvature in a space form with nonnegative curvature has been proved previously by Shiohama and Xu [26]. Lei and Xu [15] also proved sharp convergence theorems for the mean curvature

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flow of submanifolds in the sphere [13; 14], which also improve the convergence theorems due to Huisken [10] and Baker [2]. See [16; 18; 21] for recent progress in the smooth convergence theory for mean curvature flow of arbitrary codimensions. As consequences of these smooth convergence theorems, the submanifolds satisfying the initial curvature conditions are diffeomorphic to the standard sphere. We remark that some of these differentiable sphere theorems are also proved by using the Ricci flow, which has proven to be a very useful tool in understanding the topology of Riemannian manifolds, see [3; 4; 6; 7; 8; 22; 23; 24; 27; 28].

Pipoli and Sinestrari [25] obtained a convergence theorem for mean curvature flow of small codimension in the complex projective spaces. Later, Lei and Xu [17] investigated the smooth convergence of mean curvature flow of arbitrary codimensional submanifolds in the complex projective spaces, which improved and extended the convergence theorem due to Pipoli and Sinestrari [25]. In this paper, we investigate the mean curvature flow in the quaternionic projective spaces. We mainly consider the codimension-one case.

Let M be an n-dimensional closed manifold, and let $F : M^n \times [0, T) \to N^{n+1}$ be a one-parameter family of smooth hypersurfaces immersed in a Riemannian manifold (N, h). We say that $M_t = F_t(M)$ is a solution to the mean curvature flow if F_t satisfies

$$\begin{cases} \frac{\partial}{\partial t} F = -H\nu, \\ F(\,\cdot\,, 0) = F_0(\,\cdot\,), \end{cases} \tag{1-1}$$

where $F_t(\,\cdot\,) = F(\,\cdot\,, t)$, H and ν are the mean curvature of M and the unit outward normal vector of M respectively, such that $\vec{H} = -H\nu$ is the mean curvature vector of M.

Pipoli and Sinestrari [25] obtained a convergence theorem for the mean curvature flow of hypersurfaces in the quaternionic projective spaces, and the proof is the same as their convergence theorem for mean curvature flow of hypersurfaces in the complex projective spaces.

Theorem 1.1 [25]. *Let M^n $(n \geqslant 11)$ be a closed real hypersurface in quaternionic projective space $\mathbb{QP}^{(n+1)/4}(4)$, and M_t be the mean curvature flow starting from M. Assume that M satisfies the following pinching condition*:

$$|h|^2 < \frac{1}{n-1} H^2 + 2.$$

Then the flow has a smooth solution on the maximal time interval $[0, T)$ with $T < \infty$. Moreover, the pinching condition is preserved and M_t collapses to a round point as $t \to T$.

We note that here and in the remaining part of the paper, $n = 4m - 1$ for $m \geqslant 2$. The aim of the present paper is to prove the following refinement of Theorem 1.1.

Theorem 1.2. *Let M^n be an $n(\geqslant 7)$-dimensional closed real hypersurface in quaternionic projective space $\mathbb{QP}^{(n+1)/4}(4)$, and M_t be the mean curvature flow starting from M. Assume that M satisfies the following pinching condition*:

$$|h|^2 < \varphi(H^2).$$

Then the flow has a smooth solution on the maximal time interval $[0, T)$ with $T < \infty$. Moreover, the pinching condition is preserved and M_t collapses to a round point as $t \to T$.

In Theorem 1.2, $\varphi(H^2)$ is given by

$$\varphi(H^2) = 2 + a_n + \left(b_n + \frac{1}{n-1}\right)H^2 - \sqrt{b_n^2 H^4 + 2a_n b_n H^2}, \tag{1-2}$$

where

$$a_n = \sqrt{8(n-5)(n-1)\, b_n}, \quad b_n = \min\left\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\right\}.$$

Remark 1.3. By a computation, we have $\varphi(x) > \frac{x}{n-1} + 2$ for $x \geqslant 0$. So, Theorem 1.2 is an improvement of Theorem 1.1. Furthermore, we have $\varphi(x) \geqslant 4\sqrt{n-1} - 6$ for $7 \leqslant n \leqslant 17$, and $\varphi(x) > 2 + \frac{8\sqrt{2}}{5}\sqrt{n-5}$ for $n \geqslant 18$.

It is well known that $\mathbb{QP}^1$ is just the round sphere. By [11; 14], the similar smooth convergence theorem holds for mean curvature flow of closed hypersurfaces in $\mathbb{QP}^1$.

By Theorem 1.2, we have:

Corollary 1.4. *Let M^n be an $n(\geqslant 7)$-dimensional closed real hypersurface in quaternionic projective space $\mathbb{QP}^{(n+1)/4}(4)$. If $|h|^2 < \varphi(H^2)$, then M is diffeomorphic to the standard sphere.*

The rest of the paper is organized as follows. In Section 2, we introduce some notations, formulas and basic equations in submanifold theory, and prove a gradient inequality involving the second fundamental form and the mean curvature for hypersurfaces in the quaternionic projective spaces. We also recall some evolution equations along the mean curvature flow in this section. In Section 3, we show that the pinching condition $|h|^2 < \varphi(H^2)$ is preserved along the mean curvature flow. We also derive an evolution inequality of

$$f_\sigma = \frac{|\mathring{h}|^2}{(\varphi - H^2/n)^{1-\sigma}}.$$

A pinching estimate for the traceless second fundamental form is obtained in Section 4. We give an estimate of the gradient of the mean curvature in Section 5, which is used to compare the mean curvature at different points. In Section 6, we show that the hypersurface shrinks to a single round point in finite time.

2. Notations and formulas

Let $\mathbb{QP}^m$ be the m-dimensional quaternionic projective space with the Fubini–Study metric g_{FS}. Let $J_{k_0},\ k_0 = 1, 2, 3$ be complex structures on $\mathbb{QP}^m$. We denote by $\overline{\nabla}$ the Levi–Civita connection of $(\mathbb{QP}^m, g_{\mathrm{FS}})$. Since the Fubini–Study metric is a Kähler metric, we have $\overline{\nabla} J_{k_0} = 0$ for $k_0 = 1, 2, 3$. The curvature tensor $\bar{R}$ of $\mathbb{QP}^m$ can be written as

$$\begin{aligned}
&\bar{R}(X, Y, Z, W)\\
&= \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle\\
&\qquad + \sum_{k_0=1}^{3}\big(\langle X, J_{k_0}Z\rangle\langle Y, J_{k_0}W\rangle - \langle X, J_{k_0}W\rangle\langle Y, J_{k_0}Z\rangle + 2\langle X, J_{k_0}Y\rangle\langle Z, J_{k_0}W\rangle\big)
\end{aligned}$$

and $J_{k_0},\ k_0 = 1, 2, 3$ satisfies

$$J_{k_0}^2 = -\,\mathrm{Id},\quad J_1 J_2 = -J_2 J_1 = J_3,\quad J_1 J_3 = -J_3 J_1 = -J_2,\quad J_2 J_3 = -J_2 J_3 = J_1.$$

Let (M^n, g) be an n-dimensional Riemannian submanifold in $(\mathbb{QP}^m, g_{\mathrm{FS}})$. Let q be its codimension, i.e., $n + q = 4m$. At a point $p \in M$, let T_pM and N_pM be the tangent space and normal space, respectively. For a vector in $T_pM \oplus N_pM$, we denote by $(\,\cdot\,)^T$ and $(\,\cdot\,)^N$ its projections onto T_pM and N_pM, respectively. We use the symbols ∇ and $\nabla^{\perp}$ to represent the connections of tangent bundle TM and normal bundle NM. Denote by $\Gamma(E)$ the space of smooth sections of a vector bundle E. For $X, Y \in \Gamma(TM),\ \xi \in \Gamma(NM)$, the connections ∇ and $\nabla^{\perp}$ are given by $\nabla_X Y = (\overline{\nabla}_X Y)^T$ and $\nabla^{\perp}_X \xi = (\overline{\nabla}_X \xi)^N$. The second fundamental form of M is defined as $h(X, Y) = (\overline{\nabla}_X Y)^N$.

We mainly consider the codimension-one case. Throughout this paper, we shall make the following convention on indices:

$$1 \leqslant A, B, C, \cdots \leqslant n+1, \quad 1 \leqslant i, j, k, \cdots \leqslant n.$$

We choose a local orthonormal frame $\{e_i\}$ for the tangent bundle and let $\nu = e_{n+1}$ be the unit normal vector field. Denote by $\{\omega^i\}$ the dual frame of $\{e_i\}$. Let h and H denote the second fundamental form and the mean curvature given by

$$h = \sum_{i,j} h_{ij}\,\omega^i \otimes \omega^j \quad \text{and} \quad H = \sum_i h_{ii}.$$

Let $\mathring{h} = h - \frac{1}{n} H g$ be the traceless second fundamental form. We have the relations

$$|\mathring{h}|^2 = |h|^2 - \frac{1}{n}H^2, \quad |\nabla\mathring{h}|^2 = |\nabla h|^2 - \frac{1}{n}|\nabla H|^2.$$

Setting $J^{(k_0)}_{AB} = \langle e_A, J_{k_0} e_B\rangle$ for $k_0 = 1, 2, 3$, we have

$$J^{(k_0)}_{AB} = -J^{(k_0)}_{BA}, \quad \sum_B J^{(k_0)}_{AB} J^{(k_0)}_{BC} = J^{(k_0)}_{AC}, \quad \sum_B J^{(1)}_{AB} J^{(2)}_{BC} = J^{(3)}_{AC}.$$

Similarly, we have

$$\sum_B J^{(1)}_{AB} J^{(3)}_{BC} = -J^{(2)}_{AC}, \quad \sum_B J^{(2)}_{AB} J^{(3)}_{BC} = J^{(1)}_{AC}.$$

Also, $J^{(k_0)}_{AA} = 0$ for any A and k_0.

Let h_{ijk} and H_i denote the components of ∇h and ∇H, the covariant differentiations of h and H, respectively. We have the following sharp gradient inequality (see Remark 2.2).

Lemma 2.1. *For a hypersurface in $\mathbb{QP}^{(n+1)/4}$, we have*

$$|\nabla h|^2 \geqslant \frac{3}{n+2}|\nabla H|^2 + 6(n-3).$$

Proof. Set $S = \sum S_{ijk}\, \omega^i \otimes \omega^j \otimes \omega^k$, where $S_{ijk} = \frac{1}{3}(h_{ijk} + h_{jki} + h_{kij})$. Then S_{ijk} is totally symmetric for i, j, k. Using the same technique as in Lemma 2.2 in [9], we have

$$|S|^2 \geqslant \frac{3}{n+2} \sum_i \Big(\sum_k S_{kki}\Big)^2.$$

By the Codazzi equation, we have

$$\begin{aligned}\sum_k S_{kki} &= \tfrac{1}{3} \sum_k (h_{ikk} + h_{kki} + h_{kik}) \\ &= \tfrac{1}{3} \sum_k (h_{kki} + 2h_{kik}) \\ &= \tfrac{1}{3} \sum_k (h_{kki} + 2h_{kki} - 2\bar{R}_{n+1kik}) = H_i - \tfrac{2}{3} \sum_k \bar{R}_{n+1kik}.\end{aligned}$$

As

$$\bar{R}_{n+1kik} = \sum_{k_0=1}^{3} (J^{(k_0)}_{n+1i} J^{(k_0)}_{kk} - J^{(k_0)}_{n+1k} J^{(k_0)}_{ki} + 2J^{(k_0)}_{n+1k} J^{(k_0)}_{ik}),$$

one has

$$\begin{aligned}-\tfrac{2}{3} \sum_k \bar{R}_{n+1kik} &= -\tfrac{2}{3} \sum_k \sum_{k_0=1}^{3} (J^{(k_0)}_{n+1i} J^{(k_0)}_{kk} - J^{(k_0)}_{n+1k} J^{(k_0)}_{ki} + 2J^{(k_0)}_{n+1k} J^{(k_0)}_{ik}) \\ &= -\tfrac{2}{3} \sum_k \sum_{k_0=1}^{3} (-J^{(k_0)}_{n+1k} J^{(k_0)}_{ki} - 2J^{(k_0)}_{n+1k} J^{(k_0)}_{ki}) \\ &= 2 \sum_{k_0=1}^{3} \sum_k (J^{(k_0)}_{n+1k} J^{(k_0)}_{ki}).\end{aligned}$$

Then we get

$$\sum_k S_{kki} = H_i + 2 \sum_{k_0=1}^{3} \sum_k (J^{(k_0)}_{n+1k} J^{(k_0)}_{ki}).$$

This implies

$$\Big(\sum_k S_{kki}\Big)^2 = (H_i)^2 + 4 \sum_{k_0=1}^{3} \sum_k H_i (J^{(k_0)}_{n+1k} J^{(k_0)}_{ki}) + 4\Big[\sum_{k_0=1}^{3} \sum_k (J^{(k_0)}_{n+1k} J^{(k_0)}_{ki})\Big]^2.$$

Since

$$4\sum_{k_0=1}^{3}\Big[\sum_{i,k} H_i\, J_{n+1k}^{(k_0)}\, J_{ki}^{(k_0)}\Big]+4\sum_i\Big[\sum_{k_0=1}^{3}\sum_k J_{n+1k}^{(k_0)}\, J_{ki}^{(k_0)}\Big]^2$$
$$=4\sum_{k_0=1}^{3}\Big[\sum_i H_i \sum_A J_{n+1A}^{(k_0)}\, J_{Ai}^{(k_0)}\Big]+4\sum_i\Big[\sum_{k_0=1}^{3}\sum_A J_{n+1A}^{(k_0)}\, J_{Ai}^{(k_0)}\Big]^2$$
$$=4\sum_{k_0=1}^{3}\Big[\sum_i H_i\,\delta_{n+1i}\Big]+4\sum_i\Big[\sum_{k_0=1}^{3}\delta_{n+1i}\Big]^2=0,$$

one has

$$|S|^2 \geqslant \frac{3}{n+2}|\nabla H|^2. \tag{2-1}$$

On the other hand, by the Codazzi equation, we have

$$\begin{aligned}|S|^2=\sum(S_{ijk})^2 &= \tfrac{1}{9}\sum(h_{ikk}+h_{kki}+h_{kik})^2\\ &= \tfrac{1}{3}\sum(h_{ijk})^2+\tfrac{2}{3}\sum h_{ijk}\,h_{ikj}\\ &= \tfrac{1}{3}\sum(h_{ijk})^2+\tfrac{2}{3}\sum h_{ijk}(h_{ijk}+\bar{R}_{n+1ijk})\\ &= \sum(h_{ijk})^2+\tfrac{2}{3}\sum\bar{R}_{n+1jki}\,\bar{R}_{n+1ijk}\\ &= |\nabla h|^2+\tfrac{2}{3}\sum\bar{R}_{n+1jki}\,\bar{R}_{n+1ijk}.\end{aligned}$$

Since

$$\bar{R}_{ABCD}=\delta_{AC}\,\delta_{BD}-\delta_{AD}\,\delta_{BC}+\sum_{k_0=1}^{3}(J_{AC}^{(k_0)}J_{BD}^{(k_0)}-J_{AD}^{(k_0)}J_{BC}^{(k_0)}+2J_{AB}^{(k_0)}J_{CD}^{(k_0)}),$$

one has

$$\begin{aligned}\sum\bar{R}_{n+1jki}\,\bar{R}_{n+1ijk}=\sum_{i,j,k}&\Big[\sum_{k_0=1}^{3}(J_{n+1k}^{(k_0)}\,J_{ji}^{(k_0)}-J_{n+1i}^{(k_0)}\,J_{jk}^{(k_0)}+2J_{n+1j}^{(k_0)}\,J_{ki}^{(k_0)})\Big]\\ &\times\Big[\sum_{l_0=1}^{3}(J_{n+1j}^{(l_0)}\,J_{ik}^{(l_0)}-J_{n+1k}^{(l_0)}\,J_{ij}^{(l_0)}+2J_{n+1i}^{(l_0)}\,J_{jk}^{(l_0)})\Big].\end{aligned}$$

For each k_0, according to the special property of matrix $(J_{AB}^{(k_0)})$, by direct computation we have

$$\sum_k(J_{n+1k}^{(k_0)})^2=-\sum_k J_{n+1k}^{(k_0)}\,J_{kn+1}^{(k_0)}=-\sum_A J_{n+1A}^{(k_0)}\,J_{An+1}^{(k_0)}=\delta_{(n+1)(n+1)}=1,$$

and

$$\sum_k J_{n+1k}^{(k_0)}\,J_{Bk}^{(l_0)}=-\sum_k J_{n+1k}^{(k_0)}\,J_{kB}^{(l_0)}=-\sum_A J_{n+1A}^{(k_0)}\,J_{AB}^{(l_0)}=\pm J_{n+1B}^{(j_0)},$$

where $\pm$ depends on j_0, k_0, l_0. By some computations, we obtain

$$\sum\bar{R}_{n+1jki}\,\bar{R}_{n+1ijk}=-9(n-3).$$

Hence

$$|S|^2=|\nabla h|^2+\tfrac{2}{3}(-9(n-3))=|\nabla h|^2-6(n-3).$$

Combining this with (2-1) implies

$$|\nabla h|^2 \geqslant \frac{3}{n+2}|\nabla H|^2 + 6(n-3). \qquad \square$$

Remark 2.2. For hypersurface M^{4m-1} in $\mathbb{QP}^m$, one has

$$|\nabla h|^2 \geqslant \frac{3}{4m+1}|\nabla H|^2 + 24(m-1).$$

In particular, one has $|\nabla h|^2 \geqslant 24(m-1)$, which has been proved previously by Dong [5]. Dong also proved that a real hypersurface satisfying $|\nabla h|^2 = 24(m-1)$ is one of the generalized equators $M^Q_{p,q}$. See, e.g., [5; 12] for the detailed construction of generalized equators. From this we see that our gradient inequality is sharp.

Let $F : M \times [0, T) \to \mathbb{QP}^{(n+1)/4}$ be a mean curvature flow of hypersurface in the quaternionic projective space $\mathbb{QP}^{(n+1)/4}$. Set $M_t = F_t(M)$, where $F_t(\,\cdot\,) = F(\,\cdot\,, t)$. Following [1; 25], we have the evolution equations.

Lemma 2.3. *For mean curvature flow $F : M \times [0, T) \to \mathbb{QP}^{(n+1)/4}$, we have*

$$\frac{\partial}{\partial t}|h|^2 = \Delta|h|^2 - 2|\nabla h|^2 - 2n|h|^2 + 2|h|^4 + 18|h|^2 + 4H^2 + 12S_1,$$
$$\frac{\partial}{\partial t}H^2 = \Delta H^2 - 2|\nabla H|^2 + 2H^2(|h|^2 + n + 9),$$

where

$$S_1 = \sum_{k_0=1}^{3} \sum_{i,j,k,l} (\mathring{h}_{ij}\,\mathring{h}_{kl}\,J^{(k_0)}_{il}\,J^{(k_0)}_{jk} - \mathring{h}_{ik}\,\mathring{h}_{jk}\,J^{(k_0)}_{il}\,J^{(k_0)}_{jl}).$$

To do computations involving $(J^{(k_0)}_{AB})$ for $k_0 = 1, 2, 3$, the following well-known property of skew-symmetric matrix will be important.

Proposition 2.4. *Let A be a real skew-symmetric matrix. Then there exists an orthogonal matrix C, such that $C^{-1}AC$ takes the following form*:

$$\text{(2-2)} \qquad \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & & \\ & & 0 & \lambda_3 & & & & \\ & & -\lambda_3 & 0 & & & & \\ & & & & 0 & \lambda_5 & & \\ & & & & -\lambda_5 & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & \ddots \end{pmatrix}.$$

We use a notation

$$\tilde{i} = \begin{cases} i+1, & i \text{ is odd}, \\ i-1, & i \text{ is even}. \end{cases}$$

If a matrix (a_{ij}) takes the form as (2-2), then $a_{ij} = 0$, for all $j \neq \tilde{i}$.

3. Preservation of curvature pinching

For each fixed $k_0 \in \{1, 2, 3\}$, we choose a local orthonormal frame $\{e_i\}$ such that the matrix $(J_{ij}^{(k_0)})$ takes the form of (2-2). In fact, let $\{\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}\}$ be a local orthonormal frame on $\mathbb{QP}^{(n+1)/4}$ such that $\epsilon_1, \dots, \epsilon_n$ are tangent to M and ϵ_{n+1} is normal to M. Let $\tilde{J}_{AB}^{(k_0)} = \langle \epsilon_A, J^{(k_0)} \epsilon_B \rangle$. Since $(\tilde{J}_{ij})_{n\times n}$ is antisymmetric and n is odd, there is an orthonormal matrix $C = (c_{ij})_{n\times n}$, where c_{ij}'s are local functions, such that

$$(c_{ij}^{-1} \tilde{J}_{jk}^{(k_0)} c_{kl})_{n\times n} = \begin{pmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Here $(c_{ij}^{-1})_{n\times n} = (c_{ij})_{n\times n}^{-1}$. Set $e_i = \sum_{j=1}^n c_{ij}^{-1} \epsilon_j,\ e_{n+1} = \epsilon_{n+1}$. Then

$$\begin{aligned} J_{ij}^{(k_0)} &= \langle e_i, J^{(k_0)} e_j \rangle \\ &= \Big\langle \sum_k c_{ik}^{-1} \epsilon_k, J^{(k_0)} \Big(\sum_l c_{jl}^{-1} \epsilon_l \Big) \Big\rangle = \sum_{k,l} c_{ik}^{-1} \tilde{J}_{kl}^{(k_0)} c_{jl}^{-1} = \sum_{k,l} c_{ik}^{-1} \tilde{J}_{kl}^{(k_0)} c_{lj}. \end{aligned}$$

This implies

$$(J_{ij}^{(k_0)})_{n\times n} = \begin{pmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} \sum_{i,j,k,l} (\mathring{h}_{ij} \mathring{h}_{kl} J_{il}^{(k_0)} J_{jk}^{(k_0)} - \mathring{h}_{ik} \mathring{h}_{kj} J_{il}^{(k_0)} J_{jl}^{(k_0)}) &= \sum_{i,k} \big(-\mathring{h}_{i\tilde{k}} \mathring{h}_{k\tilde{i}} J_{i\tilde{i}}^{(k_0)} J_{k\tilde{k}}^{(k_0)} - (\mathring{h}_{\tilde{i}k} J_{i\tilde{i}}^{(k_0)})^2 \big) \\ &= -\tfrac{1}{2} \sum_{i,k} (\mathring{h}_{\tilde{i}k} J_{i\tilde{i}}^{(k_0)} + \mathring{h}_{i\tilde{k}} J_{k\tilde{k}}^{(k_0)})^2 \\ &\leqslant 0. \end{aligned}$$

Therefore,

$$S_1 = \sum_{k_0=1}^3 \Big[\sum_{i,j,k,l} (\mathring{h}_{ij} \mathring{h}_{kl} J_{il}^{(k_0)} J_{jk}^{(k_0)} - \mathring{h}_{ik} \mathring{h}_{jk} J_{il}^{(k_0)} J_{jl}^{(k_0)}) \Big] \leqslant 0.$$

So we get from Lemma 2.3 that

$$\frac{\partial}{\partial t} |\mathring{h}|^2 \leqslant \Delta |\mathring{h}|^2 - 2|\nabla \mathring{h}|^2 + 2|\mathring{h}|^2 (|h|^2 - n + 9). \tag{3-1}$$

For a real number $\varepsilon \in (0, 1)$, by the definition of φ, we define $\varphi_\varepsilon : [0, +\infty) \to \mathbb{R}$ by

$$\varphi_\varepsilon(x) = d_\varepsilon + c_\varepsilon x - \sqrt{b^2 x^2 + 2abx + e}, \tag{3-2}$$

where $a = \sqrt{8(n-5)(n-1)\,b}$, $b = \min\big\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\big\}$, $c_\varepsilon = b + \frac{1}{n-1+\varepsilon}$, $d_\varepsilon = 2 - 2\varepsilon + a$, $e = \sqrt{\varepsilon}$. We define $\varphi = \varphi_0$.

Lemma 3.1. *The function φ has the following property.*

(i) $\frac{x}{n-1}+2<\varphi(x)<\frac{x}{n-1}+n$,

(ii) $\varphi(x)>4\sqrt{n-1}-6$ *if* $n=7,\ 11,\ 15$ *and* $\varphi(x)>2+\frac{8\sqrt{2}}{5}\sqrt{n-5}$ *if* $n=4m-1$, $m\geqslant 5$.

Proof. By direct computations, we get

$$\varphi'(x)=c_0-\frac{bx+a}{\sqrt{x^2+2ax/b}},\qquad \varphi''(x)=\frac{a^2}{b(x^2+2ax/b)^{3/2}}.$$

Since $\left(\varphi(x)-\frac{x}{n-1}\right)''=\varphi(x)''>0$ and $\lim_{x\to\infty}\varphi'(x)=\frac{1}{n-1}$, we have $\varphi'(x)<\frac{1}{n-1}$. Hence we get

$$2=\lim_{x\to\infty}\left(\varphi(x)-\frac{x}{n-1}\right)<\varphi(x)-\frac{x}{n-1}\leqslant\varphi(0)=2+a<n.$$

We figure out that

$$\min_{x\geqslant 0}\varphi(x)=\varphi\left(\frac{ac_0}{b\sqrt{c_0^2-b^2}}-\frac{a}{b}\right)=d_0-\frac{ac_0}{b}+\frac{a}{b}\sqrt{c_0^2-b^2}.$$

If $n=7,\ 11,\ 15$, we have $\min_{x\geqslant 0}\varphi(x)=4\sqrt{n-1}-6$. If $n=4m-1,\ m\geqslant 5$, then we have

$$\min_{x\geqslant 0}\varphi(x)=2+\sqrt{\frac{8(n-5)}{2n-5}}(\sqrt{5n-8}-\sqrt{n+2})>2+\frac{8\sqrt{2}}{5}\sqrt{n-5}.\qquad\square$$

Let $\mathring{\varphi}_\varepsilon=\varphi_\varepsilon-\frac{1}{n}x$. We will prove the following lemma.

Lemma 3.2. *For sufficiently small ε, the function $\mathring{\varphi}_\varepsilon$ satisfies*

(i) $\mathring{\varphi}_\varepsilon'+2x\mathring{\varphi}_\varepsilon''<\frac{2(n-1)}{n(n+2)}$,

(ii) $\mathring{\varphi}_\varepsilon(x)(\varphi_\varepsilon(x)-n+9)-x\mathring{\varphi}_\varepsilon'(x)(\varphi_\varepsilon(x)+n+9)<6(n-3)$,

(iii) $\mathring{\varphi}_\varepsilon(x)-x\mathring{\varphi}_\varepsilon'(x)>1$.

Proof. By direct computations, we have

$$\mathring{\varphi}_\varepsilon'=c_\varepsilon-\frac{1}{n}-\frac{b^2x+ab}{\sqrt{b^2x^2+2abx+e}},$$
$$\mathring{\varphi}_\varepsilon''=\frac{(b^2x+ab)^2-b^2(b^2x^2+2abx+e)}{(b^2x^2+2abx+e)^{3/2}},$$
$$\mathring{\varphi}_\varepsilon'''=-\frac{3b^3(a^2-e)(bx+a)}{(b^2x^2+2abx+e)^{5/2}}.$$

Then we have

$$\mathring{\varphi}_\varepsilon' + 2x\mathring{\varphi}_\varepsilon'' = c_\varepsilon - \frac{1}{n} - \frac{b^3x^2(bx+3a)+eb(3bx+a)}{(b^2x^2+2abx+e)^{3/2}} < b + \frac{1}{n-1+\varepsilon} - \frac{1}{n} < \frac{2(n-1)}{n(n+2)},$$

as $b=\min\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\}$, so we get the inequality (i).

Setting

$$f(x) = \mathring{\varphi}_\varepsilon(\varphi_\varepsilon - n + 9) - x\mathring{\varphi}_\varepsilon'(\varphi_\varepsilon + n + 9).$$

Then

$$\begin{aligned} f(x) &= d_\varepsilon(d_\varepsilon - n + 9) + e_\varepsilon + (2 + ab + c_\varepsilon(d_\varepsilon - 2n))\, x \\ &\quad - (b^2x^2 + 2abx + e)^{-1/2} \\ &\qquad \times \big[b((d_\varepsilon - 2n)\, b + ac)\, x^2 + (3(d_\varepsilon - n + 3)\, ab + ec)\, x + e(2d_\varepsilon - n + 9)\big]. \end{aligned}$$

Then for ε small enough we get

$$\begin{aligned} \lim_{x\to+\infty} f(x) &= \frac{a^2c_\varepsilon}{b} + d_\varepsilon(d_\varepsilon - n + 9) + a(n - 2d_\varepsilon - 9) + e\left(1 - \frac{c_\varepsilon}{b}\right) \\ &= 6(n-3) + \frac{2\varepsilon(n^2 - (18-3\varepsilon)\, n + 33 - 15\varepsilon + 2\varepsilon^2)}{n-1+\varepsilon} + e\left(1 - \frac{c_\varepsilon}{b}\right) \\ &= 6(n-3) + \frac{2\varepsilon(n^2 - (18-3\varepsilon)\, n + 33 - 15\varepsilon + 2\varepsilon^2)}{n-1+\varepsilon} - \frac{\sqrt{\varepsilon}}{(n-1+\varepsilon)\, b} \\ &< 6(n-3), \end{aligned}$$

and

$$\begin{aligned} &f'(x) \\ &= 2 + ab + c_\varepsilon(d_\varepsilon - 2n) \\ &\quad - \frac{1}{(b^2x^2+2abx+e)^{3/2}}\big[b^3((d_\varepsilon - 2n)\, b + ac_\varepsilon)\, x^3 + 3ab^2((d_\varepsilon - 2n)\, b + ac_\varepsilon)\, x^2 \\ &\qquad + [3a^2b^2(d_\varepsilon - n + 3) - 3eb^2(n+3) + 3abc_\varepsilon\, e]\, x \\ &\qquad + eab(d_\varepsilon - 2n) + e^2c_\varepsilon\big]. \end{aligned}$$

Then we have

$$\lim_{x\to+\infty} f'(x) = 2 + ab + c_\varepsilon(d_\varepsilon - 2n) - (ac_\varepsilon + b(d_\varepsilon - 2n)) = 0$$

and

$$\begin{aligned} f''(x) &= \frac{3b^2(a^2-e)}{(b^2x^2+2abx+e)^{5/2}} \\ &\qquad \times \big[b(b(d_\varepsilon+6) - ac_\varepsilon)\, x^2 + (ab(d_\varepsilon - n + 3) - ec_\varepsilon)\, x - e(n+3)\big]. \end{aligned}$$

For $b=\min\{\frac{n-5}{8(n-1)}, \frac{2n-5}{(n+2)(n-1)}\}$, we obtain

$$b(d_\varepsilon + 6) - ac_\varepsilon < 0 \quad \text{and} \quad ab(d_\varepsilon - n + 3) - ec_\varepsilon < 0.$$

So $f''(x) < 0$. Then we have $f'(x) > 0$. From this we deduce that

$$f(x) < \lim_{x\to+\infty} f(x) < 6(n-3).$$

Thus, inequality (ii) is proved.

We have

$$\mathring{\varphi}_\varepsilon - x\mathring{\varphi}'_\varepsilon = d_\varepsilon - \frac{abx+e}{\sqrt{b^2x^2+2abx+e}} > d_\varepsilon - \frac{abx}{\sqrt{b^2x^2}} - \frac{e}{\sqrt{e}} = 2-2\varepsilon-\sqrt[4]{\varepsilon}.$$

This implies inequality (iii). □

Suppose that M_0 is an $n(\geqslant 7)$-dimensional closed hypersurface in $\mathbb{QP}^{(n+1)/4}$ satisfying $|h|^2 < \varphi(H^2)$. Let

$$F : M^n \times [0, T) \to \mathbb{QP}^{(n+1)/4}$$

be a mean curvature flow with initial value M_0. We will show that the pinching condition is preserved along the flow. For convenience, we denote $\mathring{\varphi}_\varepsilon(H^2)$, $\mathring{\varphi}'_\varepsilon(H^2)$, $\mathring{\varphi}''_\varepsilon(H^2)$ by $\mathring{\varphi}_\varepsilon$, $\mathring{\varphi}'_\varepsilon$, $\mathring{\varphi}''_\varepsilon$, respectively.

Theorem 3.3. *If the initial value M_0 satisfies $|h|^2 < \varphi(H^2)$, then there exists a small positive number ε, such that for all $t \in [0, T)$, we have $|h|^2 < \varphi(H^2) - \varepsilon H^2 - \varepsilon$.*

Proof. Since M_0 is closed, there exists a small positive number ε_1, such that M_0 satisfies $|\mathring{h}|^2 < \mathring{\varphi}_{\varepsilon_1}$.

From Lemma 3.2(i), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathring{\varphi}_{\varepsilon_1} = -2(\mathring{\varphi}'_{\varepsilon_1} + 2H^2\cdot\mathring{\varphi}''_{\varepsilon_1})|\nabla H|^2 + 2H^2\cdot\mathring{\varphi}'_{\varepsilon_1}(\varphi_{\varepsilon_1}+n+9)$$
$$\geqslant -\frac{4(n-1)}{n(n+2)}|\nabla H|^2 + 2H^2\cdot\mathring{\varphi}'_{\varepsilon_1}(\varphi_{\varepsilon_1}+n+9). \tag{3-3}$$

Let $U = |\mathring{h}|^2 - \mathring{\varphi}_{\varepsilon_1}$. We get

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} - \Delta\right)U \leqslant -|\nabla\mathring{h}|^2 + \frac{2(n-1)}{n(n+2)}|\nabla H|^2 + |\mathring{h}|^2(|h|^2-n+9) - H^2\cdot\mathring{\varphi}'_{\varepsilon_1}(|h|^2+n+9).$$

By Lemma 2.1, we have

$$-|\nabla\mathring{h}|^2 + \frac{2(n-1)}{n(n+2)}|\nabla H|^2 < -6(n-3).$$

At the point where $U = 0$, we get

$$\tfrac{1}{2}\left(\frac{\partial}{\partial t} - \Delta\right)U \leqslant -6(n-3) + \mathring{\varphi}_{\varepsilon_1}(\varphi_{\varepsilon_1}-n+9) - H^2\cdot\mathring{\varphi}'_{\varepsilon_1}(\varphi_{\varepsilon_1}+n+9) < 0.$$

Applying the maximum principle, we obtain $U < 0$ for all $t \in [0, T)$. Choose a suitable small positive number ε, we complete the proof of Theorem 3.3. □

Let

$$f_\sigma = \frac{|\mathring{h}|^2}{(\mathring{\varphi})^{1-\sigma}},$$

where $\sigma \in (0, \varepsilon^2)$ is a positive constant. The following lemma is very useful for deriving the pinching estimate for $|\mathring{h}|^2$.

Lemma 3.4. *If M_0 satisfies $|h|^2 < \varphi(H^2)$, then there exists a small positive number ε, such that the following inequality holds along the mean curvature flow*:

$$\frac{\partial}{\partial t} f_\sigma \leqslant \Delta f_\sigma + \frac{2}{\mathring{\varphi}} |\nabla f_\sigma|\,|\nabla \mathring{\varphi}| - \frac{2\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla \mathring{h}|^2 + 2\sigma |h|^2 f_\sigma - \frac{\varepsilon}{n} f_\sigma .$$

Proof. By a straightforward computation, we have

$$\begin{aligned}\Big(\frac{\partial}{\partial t} - \Delta\Big) f_\sigma = f_\sigma \bigg[\frac{1}{|\mathring{h}|^2}\Big(\frac{\partial}{\partial t} - \Delta\Big)|\mathring{h}|^2 - \frac{1-\sigma}{\mathring{\varphi}}\Big(\frac{\partial}{\partial t} - \Delta\Big)\mathring{\varphi}\bigg] \\ + 2(1-\sigma)\frac{\langle \nabla f_\sigma, \nabla \mathring{\varphi}\rangle}{\mathring{\varphi}} - \sigma(1-\sigma) f_\sigma \frac{|\nabla \mathring{\varphi}|^2}{|\mathring{\varphi}|^2}.\end{aligned}$$

Using (3-1) and (3-3), we have

$$\begin{aligned}\text{(3-4)}\qquad \Big(\frac{\partial}{\partial t} - \Delta\Big) f_\sigma \leqslant 2 f_\sigma \bigg[-\frac{|\nabla \mathring{h}|^2}{|\mathring{h}|^2} + \frac{2(n-1)}{n(n+2)}\frac{|\nabla H|^2}{\mathring{\varphi}}\bigg] \\ + 2 f_\sigma \bigg[|h|^2 + 9 - n - (1-\sigma)\frac{H^2 \cdot \mathring{\varphi}'}{\mathring{\varphi}}(|h|^2 + n + 9)\bigg] \\ + \frac{2}{\mathring{\varphi}}|\nabla f_\sigma|\,|\nabla \mathring{\varphi}|.\end{aligned}$$

From Lemma 3.2 and Theorem 3.3, we have

$$\begin{aligned}-\frac{|\nabla \mathring{h}|^2}{|\mathring{h}|^2} + \frac{2(n-1)}{n(n+2)}\frac{|\nabla H|^2}{\mathring{\varphi}} &\leqslant -\frac{|\nabla \mathring{h}|^2}{|\mathring{h}|^2} + \frac{|\nabla \mathring{h}|^2 - 6(n-3)}{\mathring{\varphi}} \\ &\leqslant \frac{|\mathring{h}|^2 - \mathring{\varphi}}{|\mathring{h}|^2 \mathring{\varphi}}|\nabla \mathring{h}|^2 - \frac{6(n-3)}{\mathring{\varphi}} \\ &\leqslant -\varepsilon \frac{|H|^2 + 1}{|\mathring{h}|^2 \mathring{\varphi}}|\nabla \mathring{h}|^2 - \frac{6(n-3)}{\mathring{\varphi}} \\ &\leqslant -\frac{\varepsilon}{n|\mathring{h}|^2}|\nabla \mathring{h}|^2 - \frac{6(n-3)}{\mathring{\varphi}}.\end{aligned}$$

From (ii) and (iii) of Lemma 3.2, we have

$$
\begin{aligned}
|h|^2+9-n-(1-\sigma)\frac{H^2\cdot\mathring{\varphi}'}{\mathring{\varphi}}(|h|^2+n+9) \\
&= \frac{1-\sigma}{\mathring{\varphi}}[(\mathring{\varphi}-H^2\cdot\mathring{\varphi}')\,|h|^2-H^2\cdot\mathring{\varphi}'(n+9)]-n+9+\sigma|h|^2 \\
&\leqslant \frac{1-\sigma}{\mathring{\varphi}}[(\mathring{\varphi}-H^2\cdot\mathring{\varphi}')(\varphi-\varepsilon H^2-\varepsilon)-H^2\cdot\mathring{\varphi}'(n+9)]-n+9+\sigma|h|^2 \\
&= \frac{1-\sigma}{\mathring{\varphi}}[(\mathring{\varphi}-H^2\cdot\mathring{\varphi}')\varphi-H^2\cdot\mathring{\varphi}'(n+9)]-n+9+\sigma|h|^2 \\
&\qquad\qquad -\frac{(1-\sigma)\,\varepsilon}{\mathring{\varphi}}(\mathring{\varphi}-H^2\cdot\mathring{\varphi}')(H^2+1) \\
&\leqslant (1-\sigma)\left[n-9+\frac{6(n-3)}{\mathring{\varphi}}\right]-n+9+\sigma|h|^2-\frac{(1-\sigma)\,\varepsilon}{\mathring{\varphi}}(H^2+1) \\
&\leqslant \sigma|h|^2+\frac{6(n-3)}{\mathring{\varphi}}-\frac{\varepsilon}{2n}.
\end{aligned}
$$

Inserting these two estimates into (3-4) will complete the proof. □

4. An estimate for traceless second fundamental form

Suppose that the initial value M_0 satisfies the condition in Theorem 1.2. For convenience, we put $W=\mathring{\varphi}$. By the conclusion of the previous section, there exists a sufficiently small positive number ε, such that for all $t\in[0,T)$, the following pinching condition holds:

$$
|\mathring{h}|^2 < W-\varepsilon H^2. \tag{4-1}
$$

From this inequality and the definition of W, we have $W<\frac{H^2}{n(n-1)}+n$.

We consider the auxiliary function

$$
f_\sigma=\frac{|\mathring{h}|^2}{W^{1-\sigma}}.
$$

In this section, we will show that f_σ decays exponentially.

Lemma 4.1. *There exist positive numbers ε and C_1 depending only on M_0, such that*

$$
\frac{\partial}{\partial t}f_\sigma \leqslant \Delta f_\sigma+\frac{2C_1}{|\mathring{h}|}|\nabla f_\sigma|\,|\nabla\mathring{h}|-\frac{\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2+2\sigma|h|^2 f_\sigma-\frac{\varepsilon}{n}f_\sigma. \tag{4-2}
$$

Proof. According to Lemma 3.4, we have the following inequality with some suitable small $\varepsilon>0$:

$$
\frac{\partial}{\partial t}f_\sigma \leqslant \Delta f_\sigma+\frac{2}{W}|\nabla f_\sigma|\,|\nabla W|-\frac{2\varepsilon f_\sigma}{n|\mathring{h}|^2}|\nabla\mathring{h}|^2+2\sigma|h|^2 f_\sigma-\frac{\varepsilon}{n}f_\sigma.
$$

By the definition of W, there exists a constant B_1, such that $|\nabla W| < B_1|\nabla H^2|$ and $|H| < B_1\sqrt{W}$. Let C_1 be a constant such that $2B_1^2|\nabla H| \leqslant C_1|\nabla \mathring{h}|$. From Lemma 2.1, we have

$$\frac{|\nabla W|}{W} \leqslant \frac{2B_1|H|\,|\nabla H|}{\sqrt{W}|\mathring{h}|} \leqslant \frac{2B_1^2|\nabla H|}{|\mathring{h}|} \leqslant \frac{C_1|\nabla \mathring{h}|}{|\mathring{h}|}. \tag{4-3}$$ □

We need the following estimate for the Laplacian of $|\mathring{h}|^2$.

Lemma 4.2. $\Delta|\mathring{h}|^2 \geqslant 2\langle \mathring{h}, \nabla^2 H\rangle + 2|\mathring{h}|^2(\varepsilon|h|^2 - 2n^2) - 18|\mathring{h}|\,|H|$.

Proof. We have

$$\Delta|\mathring{h}|^2 = 2|\nabla\mathring{h}|^2 + 2\mathring{h}\cdot\Delta\mathring{h} = 2|\nabla\mathring{h}|^2 + 2\sum_{i,j}\mathring{h}_{ij}\cdot\Delta h_{ij}$$

and

$$\begin{aligned}\sum_{i,j}\mathring{h}_{ij}\cdot\Delta h_{ij} &= \langle\mathring{h}, \nabla^2 H\rangle + \sum_{i,p,j} H h_{ip}\,h_{pj}\,h_{ij} - |h|^4 \\ &\quad + 3H\sum_{i,j}\sum_{k_0=1}^{3} J^{(k_0)}_{in+1}\,J^{(k_0)}_{jn+1}\,\mathring{h}_{ij} - (n+9)\,|\mathring{h}|^2 + 2n|\mathring{h}|^2 - 6S_1 \\ &\geqslant \langle\mathring{h}, \nabla^2 H\rangle + \sum_{i,p,j} H h_{ip}h_{pj}h_{ij} - |h|^4 + (n-9)\,|\mathring{h}|^2 - 9|\mathring{h}|\,|H|.\end{aligned}$$

It follows from the proof of the Lemma 4.2 in [17], we choose a local orthonormal frame such that

$$H = |H|\,e_{n+1} \quad \text{and} \quad \mathring{h} = \operatorname{diag}\{\mathring{\lambda}_1, \dots, \mathring{\lambda}_n\}.$$

So we have

$$\begin{aligned}\sum_{i,p,j} H h_{ip}\,h_{pj}\,h_{ij} - |h|^4 &= H\sum_i \mathring{\lambda}_i^3 + \frac{1}{n}H^2|\mathring{h}|^2 - |\mathring{h}|^4 \\ &\geqslant -|H|\frac{n-2}{\sqrt{n(n-1)}}|\mathring{h}|^3 + \frac{1}{n}H^2|\mathring{h}|^2 - |\mathring{h}|^4 \\ &= |\mathring{h}|^2\left(\frac{1}{n}H^2 - |\mathring{h}|^2 - \frac{n-2}{\sqrt{n(n-1)}}|\mathring{h}|\,|H|\right) \\ &\geqslant |\mathring{h}|^2\left[\frac{1}{n}H^2 - \left(\frac{H^2}{n(n-1)} + n - \varepsilon H^2\right) - (n-2)\left(\frac{H^2}{n(n-1)} + n\right)\right] \\ &= |\mathring{h}|^2(\varepsilon H^2 - n(n-1)) \\ &> |\mathring{h}|^2(\varepsilon|h|^2 - n^2),\end{aligned}$$

where we have used $|\mathring{h}|^2 < W - \varepsilon H^2$ and $W < \frac{H^2}{n(n-1)} + n$. □

From (4-3) and Lemma 4.2, we have

$$\begin{aligned}\Delta f_\sigma &= f_\sigma\left(\frac{\Delta|\mathring{h}|^2}{|\mathring{h}|^2}-(1-\sigma)\frac{\Delta W}{W}\right)-2(1-\sigma)\frac{\langle\nabla f_\sigma,\nabla W\rangle}{W}+\sigma(1-\sigma)\, f_\sigma\frac{|\nabla W|^2}{W^2}\\ &\geqslant f_\sigma\frac{\Delta|\mathring{h}|^2}{|\mathring{h}|^2}-(1-\sigma)\, f_\sigma\frac{\Delta W}{W}-\frac{2C_1|\nabla f_\sigma|\,|\nabla\mathring{h}|}{|\mathring{h}|}\\ &\geqslant \frac{2\langle\mathring{h},\nabla^2 H\rangle}{\mathring{W}^{1-\sigma}}+2f_\sigma(\varepsilon|h|^2-2n^2)-(1-\sigma)\frac{f_\sigma\Delta W}{W}-\frac{2C_1|\nabla f_\sigma||\nabla\mathring{h}|}{|\mathring{h}|}-\frac{18f_\sigma|H|}{|\mathring{h}|}.\end{aligned}$$

Multiplying both sides of the above inequality by f_σ^{p-1}, we get

$$\begin{aligned}2\varepsilon f_\sigma^p|h|^2 \leqslant f_\sigma^{p-1}\Delta f_\sigma+(1-\sigma)\frac{f_\sigma^p\Delta W}{W}-\frac{2f_\sigma^{p-1}\langle\mathring{h},\nabla^2H\rangle}{W^{1-\sigma}}\\ +\frac{2C_1f_\sigma^{p-1}|\nabla f_\sigma|\,|\nabla\mathring{h}|}{|\mathring{h}|}+4n^2f_\sigma^p+\frac{18f_\sigma|H|}{|\mathring{h}|}.\end{aligned}\tag{4-4}$$

Then integrate both sides of (4-4) over M_t. By the divergence theorem, we get

$$\int_{M_t} f_\sigma^{p-1}\Delta f_\sigma\, d\mu_t = -(p-1)\int_{M_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\, d\mu_t.\tag{4-5}$$

From (4-4), we have

$$\begin{aligned}\int_{M_t}\frac{f_\sigma^p}{W}\Delta W\, d\mu_t &= -\int_{M_t}\left\langle\nabla\left(\frac{f_\sigma^p}{W}\right),\nabla W\right\rangle d\mu_t\\ &=\int_{M_t}\left(-\frac{pf_\sigma^{p-1}}{W}\langle\nabla f_\sigma,\nabla W\rangle+\frac{f_\sigma^p}{W^2}|\nabla W|^2\right)d\mu_t\\ &\leqslant\int_{M_t}\left(\frac{C_1pf_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma|\,|\nabla\mathring{h}|+\frac{C_1^2f_\sigma^p}{|\mathring{h}|^2}|\nabla\mathring{h}|^2\right)d\mu_t.\end{aligned}\tag{4-6}$$

We also have

$$\begin{aligned}&-\int_{M_t}\frac{f_\sigma^{p-1}\langle\mathring{h},\nabla^2H\rangle}{W^{1-\sigma}}d\mu_t\\ &=\int_{M_t}\nabla_i\left(\frac{f_\sigma^{p-1}}{W^{1-\sigma}}\mathring{h}_{ij}\right)\nabla_j H\, d\mu_t\\ &=\int_{M_t}\left[\frac{(p-1)\,f_\sigma^{p-2}}{W^{1-\sigma}}\mathring{h}_{ij}\nabla_i f_\sigma-\frac{(1-\sigma)\,f_\sigma^{p-1}}{W^{2-\sigma}}\mathring{h}_{ij}\nabla_i W+\frac{f_\sigma^{p-1}}{W^{1-\sigma}}\nabla_i\mathring{h}_{ij}\right]\nabla_j H\, d\mu_t\\ &\leqslant\int_{M_t}\left[\frac{(p-1)\,f_\sigma^{p-1}}{|\mathring{h}|}|\nabla f_\sigma|+\frac{f_\sigma^{p-1}}{W^{2-\sigma}}|\mathring{h}|\,|\nabla W|+\frac{f_\sigma^{p-1}}{W^{1-\sigma}}n|\nabla\mathring{h}|\right]|\nabla H|\, d\mu_t\end{aligned}\tag{4-7}$$

$$\leqslant \int_{M_t} \left[\frac{(p-1)\, f_\sigma^{p-1}}{|\mathring{h}|} |\nabla f_\sigma| + \frac{C_1 f_\sigma^{p-1}}{W^{1-\sigma}} |\nabla \mathring{h}| + \frac{f_\sigma^{p-1}}{W^{1-\sigma}} n |\nabla \mathring{h}| \right] n |\nabla \mathring{h}| \, d\mu_t$$

$$\leqslant \int_{M_t} \left[\frac{n(p-1)\, f_\sigma^{p-1}}{|\mathring{h}|} |\nabla f_\sigma| \, |\nabla \mathring{h}| + \frac{(C_1 n + n^2)\, f_\sigma^{p}}{|\mathring{h}|^2} |\nabla \mathring{h}|^2 \right] d\mu_t.$$

Putting (4-4)–(4-7) together, we get

$$\int_{M_t} |h|^2 f_\sigma^p \, d\mu_t \leqslant C_2 \int_{M_t} \left[\frac{p f_\sigma^{p-1}}{|\mathring{h}|} |\nabla f_\sigma| \, |\nabla \mathring{h}| + \frac{f_\sigma^p}{|\mathring{h}|^2} |\nabla \mathring{h}|^2 + f_\sigma^p + \frac{f_\sigma^p |H|}{|\mathring{h}|} \right] d\mu_t,$$

where C_2 is a positive constant depending only on M_0.

Combining Lemma 4.2 and (4-2), we get

$$\begin{aligned}
\text{(4-8)}\quad & \frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \, d\mu_t \\
&= p \int_{M_t} f_\sigma^{p-1} \frac{\partial}{\partial t} f_\sigma \, d\mu_t - \int_{M_t} f_\sigma^p H^2 \, d\mu_t \\
&\leqslant p \int_{M_t} f_\sigma^{p-2} \bigg[-(p-1) |\nabla f_\sigma|^2 + (2C_1 + 2\sigma C_2 p) \frac{f_\sigma}{|\mathring{h}|} |\nabla f_\sigma| \, |\nabla \mathring{h}| \\
&\qquad\qquad - \left(\frac{\varepsilon}{2n} - 2\sigma C_2 \right) \frac{f_\sigma^2}{|\mathring{h}|^2} |\nabla \mathring{h}|^2 \bigg] d\mu_t \\
&\qquad\qquad - p \int_{M_t} f_\sigma^p \left(\frac{\varepsilon}{n} - 2\sigma C_2 + \frac{6(n-3)\,\varepsilon}{2n|\mathring{h}|^2} - \frac{2\sigma C_2 |H|}{|\mathring{h}|} + \frac{|H|^2}{p} \right) d\mu_t.
\end{aligned}$$

Now we will show that the L^p-form of f_σ decays exponentially.

Lemma 4.3. *There exist positive constants C_3, p_0, σ_0 depending only on M_0, such that for all $p \geqslant p_0$ and $\sigma \leqslant \sigma_0/\sqrt{p}$, we have*

$$\left(\int_{M_t} f_\sigma^p \, d\mu_t \right)^{1/p} < C_3 \, e^{-\varepsilon t}.$$

Proof. The expression in the square bracket of the right side of (4-8) is a quadratic polynomial. With p_0 large enough and σ_0 small enough, its discriminant satisfies

$$(2C_1 + 2\sigma C_2 p)^2 - 4(p-1) \left(\frac{\varepsilon}{2n} - 2\sigma C_2 \right) < 0 \quad \text{and} \quad \frac{12\varepsilon}{7} \geqslant p\sigma^2 C_2^2.$$

We have

$$\begin{aligned}
\frac{\varepsilon}{n} - 2\sigma C_2 + \frac{6(n-3)\,\varepsilon}{2n|\mathring{h}|^2} - \frac{2\sigma C_2 |H|}{|\mathring{h}|} + \frac{|H|^2}{p} &\geqslant \frac{\varepsilon}{n} - 2\sigma C_2 + \frac{12\varepsilon}{7|\mathring{h}|^2} - \frac{p\sigma^2 C_2^2}{|\mathring{h}|^2} \\
&\geqslant \frac{\varepsilon}{n} - 2\sigma C_2 > \frac{\varepsilon}{2n}.
\end{aligned}$$

Here we have used the inequality $\frac{\varepsilon}{2n} - 2\sigma C_2 > 0$, which is implied by the choices of p_0 and σ_0. Then we get

$$\frac{d}{dt}\int_{M_t} f_\sigma^p \, d\mu_t \leqslant -\frac{p\varepsilon}{2n}\int_{M_t} f_\sigma^p \, d\mu_t.$$

So we get $\int_{M_t} f_\sigma^p \, d\mu_t \leqslant e^{-p\varepsilon/2n}\int_{M_0} f_\sigma^p \, d\mu_0$, which completes the proof. □

Let $g_\sigma = f_\sigma e^{\varepsilon t/2}$. By the Sobolev inequality on submanifolds and a Stampacchia iteration procedure, we obtain that g_σ is uniformly bounded for all t (see [9] or [14] for the details). Then we obtain the following theorem.

Theorem 4.4. *There exist positive constants ε, σ and C_0 depending only on M_0, such that for all $t \in [0, T)$, we have*

$$|\mathring{h}|^2 \leqslant C_0(H^2+1)^{1-\sigma} e^{-\varepsilon t/2}.$$

5. A gradient estimate

We derive an estimate for $|\nabla H|^2$ along the mean curvature flow. Firstly, the same as Proposition 4.3 in [25], we have:

Lemma 5.1. *There exists a positive constants $C_4 > 1$ depending only on n, such that*

$$\frac{\partial}{\partial t}|\nabla H|^2 \leqslant \Delta|\nabla H|^2 + C_4(H^2+1)\,|\nabla h|^2.$$

Secondly, we need the following estimates.

Lemma 5.2. *Along the mean curvature flow, we have*

(i) $\frac{\partial}{\partial t} H^4 \geqslant \Delta H^4 - 12nH^2|\nabla h|^2 + \frac{4}{n}H^6$,

(ii) $\frac{\partial}{\partial t}|\mathring{h}|^2 \leqslant \Delta|\mathring{h}|^2 - \frac{1}{3}|\nabla h|^2 + C_5|\mathring{h}|^2(H^2+1)$,

(iii) $\frac{\partial}{\partial t}(H^2|\mathring{h}|^2) \leqslant \Delta(H^2|\mathring{h}|^2) - \frac{1}{6}H^2|\nabla h|^2 + C_6(H^2+1)^2|\mathring{h}|^2 + C_7|\nabla h|^2$,

where C_5, C_6, C_7 are sufficiently large constants.

Proof. (i) From Lemma 2.3, we derive that

$$\frac{\partial}{\partial t} H^4 = \Delta H^4 - 12H^2|\nabla H|^2 + 4H^4(|h|^2+n+9).$$

From Lemma 2.1, we have $12H^2|\nabla H|^2 \leqslant 12nH^2|\nabla h|^2$. Obviously, inequality (i) holds.

(ii) We have

$$\frac{\partial}{\partial t}|\mathring{h}|^2 = \Delta|\mathring{h}|^2 - 2|\nabla\mathring{h}|^2 + 2|h|^2|\mathring{h}|^2 + 18|\mathring{h}|^2 - 2n|\mathring{h}|^2 + 12S_1.$$

From Lemma 2.1, we get $|\nabla\mathring{h}|^2 \geqslant \frac{1}{6}|\nabla h|^2$. Choose a large constant C_5, we obtain inequality (ii).

(iii) It follows the evolution equation that

$$\frac{\partial}{\partial t}(H^2|\mathring{h}|^2) = \Delta(H^2|\mathring{h}|^2) - 2\langle\nabla H^2, \nabla|\mathring{h}|^2\rangle + 4|\mathring{h}|^2H^2|h|^2 - 2|\mathring{h}|^2|\nabla H|^2 - 2H^2|\nabla\mathring{h}|^2 + 36H^2|\mathring{h}|^2 + 12S_1H^2.$$

From Lemma 2.1, we get $-2H^2|\nabla\mathring{h}|^2 \leqslant -\frac{1}{3}H^2|\nabla h|^2$. From the preserved pinching condition $|\mathring{h}|^2 < W$, we have

$$4|\mathring{h}|^2H^2|h|^2 + 36H^2|\mathring{h}|^2 \leqslant C_6(H^2+1)^2|\mathring{h}|^2.$$

Using Theorem 4.4, we have

$$-2\langle\nabla H^2, \nabla|\mathring{h}|^2\rangle \leqslant 8|H|\,|\nabla H|\,|\mathring{h}|\,|\nabla h| \leqslant 8n\sqrt{C_0}|H|(H^2+1)^{(1-\sigma)/2}|\nabla h|^2.$$

By Young's inequality, there exists a positive constant C_7, such that

$$-2\langle\nabla H^2, \nabla|\mathring{h}|^2\rangle \leqslant \left(C_7 + \tfrac{1}{6}H^2\right)|\nabla h|^2. \qquad \square$$

Now we prove a gradient estimate for mean curvature.

Theorem 5.3. *For any $\eta \in (0, \sqrt{\varepsilon}/4\pi n)$, there exists a number $\Psi(\eta)$ depending only on η and M_0, such that*

$$|\nabla H|^2 < [(\eta H)^4 + \Psi^2(\eta)]\,e^{-\varepsilon t/4}.$$

Proof. Define a scalar function

$$f = (|\nabla H|^2 + B_1|\mathring{h}|^2 + B_2H^2|\mathring{h}|^2)\,e^{\varepsilon t/4} - (\eta H)^4,$$

where B_1, B_2 are two positive constants.

From Lemmas 5.1 and 5.2, we obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right) f \\
&= \frac{\varepsilon}{4}(|\nabla H|^2 + B_1|\mathring{h}|^2 + B_2H^2|\mathring{h}|^2)\,e^{\varepsilon t/4} \\
&\qquad + e^{\varepsilon t/4}\left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla H|^2 + B_1|\mathring{h}|^2 + B_2H^2|\mathring{h}|^2) - \eta^4\left(\frac{\partial}{\partial t} - \Delta\right)H^4 \\
&\leqslant \frac{\varepsilon}{4}(|\nabla H|^2 + B_1|\mathring{h}|^2 + B_2H^2|\mathring{h}|^2)\,e^{\varepsilon t/4} \\
&\qquad + e^{\varepsilon t/4}\big\{\big(C_4(H^2+1)|\nabla h|^2\big) + B_1\big(-\tfrac{1}{3}|\nabla h|^2 + C_5(H^2+1)|\mathring{h}|^2\big) \\
&\qquad\qquad + B_2\big(-\tfrac{1}{6}H^2|\nabla h|^2 + C_7|\nabla h|^2 + C_6(H^2+1)^2|\mathring{h}|^2\big)\big\} \\
&\qquad - \eta^4\left(-12nH^2|\nabla h|^2 + \frac{4}{n}H^6\right) \\
&= H^2|\nabla h|^2\left[e^{\varepsilon t/4}\left(C_4 - \frac{B_2}{6}\right) + 24n\eta^4\right] + e^{\varepsilon t/4}\left[|\nabla h|^2\left(C_4 - \frac{B_1}{3} + C_7B_2\right) + \frac{\varepsilon}{4}|\nabla H|^2\right] \\
&\qquad + e^{\varepsilon t/4}|\mathring{h}|^2\left[B_1C_5(H^2+1) + B_2C_6(H^2+1)^2 + \frac{\varepsilon}{4}(B_1 + B_2H^2)\right] - \frac{4\eta^4}{n}|H|^6.
\end{aligned}$$

Choose constants B_1 and B_2, such that $C_4 - \frac{B_2}{6} < -1$ and $C_4 - \frac{B_1}{3} + C_7 B_2 < -1$. Then applying Theorem 4.4, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leqslant e^{-\varepsilon t/4}\left[B_3(H^2+1)^2(H^2+1)^{1-\sigma} - \frac{4\eta^4}{n} H^6\right]. \tag{5-1}$$

Consider the expression in the bracket of (5-1). Since the coefficient of H^6 is negative, it has a upper bound $\Psi_2(\eta)$. Then we have $\left(\frac{\partial}{\partial t} - \Delta\right) f \leqslant e^{-\varepsilon t/4}\, \Psi_2(\eta)$. It follows from the maximum principle that f is bounded. This completes the proof of Theorem 5.3. □

6. Convergence

In order to estimate the diameter of M_t, we need the well-known Myers's theorem:

Theorem 6.1 (Myers's theorem). *Let Γ be a geodesic of length at least $\pi/\sqrt{K}$ on M. If the Ricci curvature satisfies* $\mathrm{Ric}(X) \geqslant (n-1)K$ *for each unit vector $X \in T_x M$, at any point $x \in \Gamma$, then Γ has conjugate points.*

Now we show that under the assumption of Theorem 1.2, the mean curvature flow converges to a round point.

Theorem 6.2. *If M_0 satisfies $|\mathring{h}|^2 < \mathring{\varphi}$, then $T < \infty$ and M_t converges to a round point as $t \to T$.*

Proof. Assume $T = \infty$. Let $|H|_{\min}(t) = \min_{M_t} |H|$, $|H|_{\max}(t) = \max_{M_t} |H|$.

We claim that $H^2 \cdot e^{\varepsilon t/8}$ is uniformly bounded on $[0, \infty)$. Suppose not, then there is a time τ such that $|H|^2_{\max}(\tau)\cdot e^{\varepsilon\tau/8} > \Psi/\eta^2$. By Theorem 5.3, for every small positive number η, there exists a positive number Ψ, such that $|\nabla H| < [(\eta H)^2 + \Psi] e^{-\varepsilon t/8}$. Then we have $|\nabla H| < 2\eta^2 |H|^2_{\max}$ on M_τ.

From Lemma 4.1 in [27], the sectional curvature K of M satisfies

$$K \geqslant \tfrac{1}{2}\left(2 + \frac{1}{n-1} H^2 - |h|^2\right). \tag{6-1}$$

By Theorem 4.4, we obtain

$$K \geqslant \tfrac{1}{2}\left(2 + \frac{1}{n(n-1)} H^2 - C_0 (H^2+1)^{1-\sigma} e^{-\varepsilon t/2}\right).$$

Hence, we can pick τ large enough such that $K \geqslant (1/2n^2) H^2$ on M_τ.

Let x be a point on M_τ where $|H|$ achieves its maximum. Consider all the geodesics of length at most $(4\eta |H|_{\max})^{-1}$ starting from x. As $|\nabla H^2| < 4\eta^2 |H|^3_{\max}$, we have

$$H^2 \geqslant |H|^2_{\max} - 4\eta^2 |H|^3_{\max} \cdot (4\eta |H|_{\max})^{-1} = (1-\eta)|H|^2_{\max}$$

along such geodesics. Since $|\nabla H| < 2\eta^2 |H|^2_{\max}$ and $K \geqslant (1/2n^2)\, H^2$ on M_τ, one has

$$K \geqslant \frac{1}{2n^2}(1-\eta)\,|H|^2_{\max}$$

along such geodesics provided

$$\eta \in \left(0, \min\left\{\frac{1}{32\pi n}, \frac{\sqrt{\varepsilon}}{4\pi n}\right\}\right).$$

By Myers's theorem, these geodesics can reach any point of M_τ. This implies

$$H^2 \geqslant (1-\eta)\,|H|^2_{\max} \quad \text{on } M_\tau.$$

Combining this inequality with $|H|^2_{\max}(\tau) \cdot e^{\varepsilon\tau/8} > \Psi/\eta^2$ and Theorem 5.3, we get

$$|\nabla H|^2 < (\eta H)^4 + \frac{\eta^4}{(1-\eta)^2}|H|^4_{\min}(\tau), \quad t \geqslant \tau.$$

From the evolution equation of H^2, we have that for $t \geqslant \tau$,

$$\left(\frac{\partial}{\partial t} - \Delta\right) H^2 \geqslant -2|\nabla H|^2 + \frac{2}{n}H^4 \geqslant \frac{1}{n}H^4 - \frac{1}{2n}|H|^4_{\min}(\tau) \tag{6-2}$$

for $\eta > 0$ sufficiently small. By the maximum principle, we get $H^2 \geqslant |H|^2_{\min}(\tau)$ for $t \geqslant \tau$. Then (6-2) yields

$$\left(\frac{\partial}{\partial t} - \Delta\right) H^2 \geqslant \frac{1}{2n}H^4, \quad t \geqslant \tau.$$

By the maximum principle, H^2 blows up in finite time. This contradicts the infinity of T. Therefore, we obtain $H^2 \leqslant Ce^{-\varepsilon t/8}$ for $t \in [0, \infty)$ for a uniform positive constant C. By Theorem 4.4, $|h|^2 = |\mathring{h}|^2 + \frac{1}{n}|H|^2 \leqslant Ce^{-\varepsilon t/8}$ for $t \in [0, \infty)$, which implies that M_t converges to a closed totally geodesic hypersurface M_∞ as $t \to \infty$. However, there is no closed totally geodesic hypersurface in $\mathbb{QP}^{(n+1)/4}$, see, e.g., Corollary 7.2 in [25]. Therefore, we get a contradiction, and hence $T < \infty$.

So T is finite, and $\max_{M_t} |h|^2$ blows up as $t \to T$. From the preserved pinching condition, $|H|_{\max}(t)$ also blows up as $t \to T$. By Theorem 5.3, for any $\eta \in (0, \sqrt{\varepsilon}/4n\pi)$, there exists a positive number $\Psi = \Psi(\eta) > 1$, such that

$$|\nabla H| < (\eta H)^2 + \Psi \quad \text{for } t \in [0, T).$$

Since $|H|_{\max}(t)$ blows up as $t \to T$, there exists a time τ_1 depending on η, such that

$$|H|^2_{\max} \geqslant \max\left\{\frac{2\Psi}{\eta^2}, \frac{8n}{\varepsilon}\right\} \quad \text{on } M_{\tau_1},$$

where $\varepsilon > 0$ is as in (4-1). Then we get $|\nabla H| \leqslant 2\eta^2 |H|^2_{\max}$ on M_{τ_1}. Similarly as above, we obtain

$$|H|^2_{\min} \geqslant (1-\eta)|H|^2_{\max} \quad \text{on } M_{\tau_1}. \tag{6-3}$$

By (4-1) and (6-1), one has $K \geqslant \frac{1}{2}(\varepsilon H^2 - n)$ for all $t \in [0, T)$. Hence for small $\eta > 0$, we have the following estimate at $t = \tau_1$:

$$K \geqslant \tfrac{1}{4}\varepsilon H^2 + \tfrac{1}{4}[\varepsilon(1-\eta)|H|^2_{\max} - 2n] \geqslant \tfrac{1}{4}\varepsilon H^2 + \tfrac{1}{8}(\varepsilon |H|^2_{\max} - 4n) \geqslant \tfrac{1}{16}\varepsilon |H|^2_{\max}.$$

This implies $\operatorname{diam}(M_{\tau_1}) \leqslant 4\pi/(\sqrt{\varepsilon}|H|_{\max})$.

Furthermore, by Theorem 5.3 and (6-3) one has that for $t \geqslant \tau_1$,

$$|\nabla H|^2 < 2(\eta H)^4 + 2\Psi^2 \leqslant 2(\eta H)^4 + \tfrac{1}{4}\eta^4 |H|^2_{\max}(\tau_1) \leqslant 2(\eta H)^4 + \tfrac{1}{2}\eta^4 |H|^2_{\min}(\tau_1).$$

Hence for $t \geqslant \tau_1$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) H^2 \geqslant -2|\nabla H|^2 + \frac{2}{n} H^4 \geqslant \frac{1}{n} H^4 - \frac{1}{2n} |H|^4_{\min}(\tau_1) \tag{6-4}$$

provided that $\eta > 0$ is sufficiently small. By the maximum principle, we get $H^2 \geqslant |H|^2_{\min}(\tau_1)$ for $t \geqslant \tau_1$. Then (6-4) yields

$$\left(\frac{\partial}{\partial t} - \Delta\right) H^2 \geqslant \frac{1}{2n} H^4, \quad t \geqslant \tau_1.$$

By the maximum principle, $|H|^2_{\min}(t)$ is increasing on $[\tau_1, T)$. So

$$|H|^2_{\max}(t) \geqslant |H|^2_{\min}(t) \geqslant |H|^2_{\min}(\tau_1) \geqslant \tfrac{1}{2}|H|^2_{\max}(\tau_1) \geqslant \max\left\{\frac{\Psi}{\eta^2}, \frac{4n}{\varepsilon}\right\}$$

for all $t \geqslant \tau_1$ and for every $\eta > 0$ sufficiently small. Hence $|\nabla H| \leqslant 2\eta^2 |H|^2_{\max}$ for all $t \geqslant \tau_1$. By a similar argument, we get $|H|^2_{\min} \geqslant (1-\eta)|H|^2_{\max}$ for all η sufficiently small and all $t \geqslant \tau_1$. This implies $|H|_{\min}/|H|_{\max} \to 1$ as $t \to T$.

Since for $t \geqslant \tau_1$,

$$K \geqslant \tfrac{1}{4}\varepsilon H^2 + \tfrac{1}{8}(\varepsilon |H|^2_{\max} - 4n) \geqslant \tfrac{1}{16}\varepsilon |H|^2_{\max},$$

we have

$$\operatorname{diam}(M_t) \leqslant \frac{4\pi}{\sqrt{\varepsilon}|H|_{\max}(t)}$$

for all $t \geqslant \tau_1$. So $\operatorname{diam}(M_t) \to 0$, and by a similar argument as in [10], M_t shrinks to a single point as $t \to T$.

Now we dilate the metric of the ambient space such that the hypersurface maintains its volume along the flow. Using the same method as in [19], we can prove that the sequence of time-slices of rescaled flow corresponding to any sequence of times that tends to infinity has a subsequence that converges to a round sphere. This proves that the limit point of the mean curvature flow is round. □

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References

[1] B. Andrews and C. Baker, "Mean curvature flow of pinched submanifolds to spheres", *J. Differential Geom.* **85**:3 (2010), 357–395. MR Zbl

[2] C. Baker, "The mean curvature flow of submanifolds of high codimension", preprint, 2011. arXiv

[3] S. Brendle, "A general convergence result for the Ricci flow in higher dimensions", *Duke Math. J.* **145**:3 (2008), 585–601. MR Zbl

[4] S. Brendle and R. Schoen, "Manifolds with 1/4-pinched curvature are space forms", *J. Amer. Math. Soc.* **22**:1 (2009), 287–307. MR Zbl

[5] Y. X. Dong, "Real hypersurfaces of a quaternionic projective space", *Chinese Ann. Math. Ser. A* **14**:1 (1993), 75–80. MR Zbl

[6] J.-R. Gu and H.-W. Xu, "The sphere theorems for manifolds with positive scalar curvature", *J. Differential Geom.* **92**:3 (2012), 507–545. MR Zbl

[7] R. S. Hamilton, "Three-manifolds with positive Ricci curvature", *J. Differential Geometry* **17**:2 (1982), 255–306. MR Zbl

[8] R. S. Hamilton, "The formation of singularities in the Ricci flow", pp. 7–136 in *Surveys in differential geometry*, *II* (Cambridge, MA, 1993), Int. Press, Cambridge, MA, 1995. MR Zbl

[9] G. Huisken, "Flow by mean curvature of convex surfaces into spheres", *J. Differential Geom.* **20**:1 (1984), 237–266. MR Zbl

[10] G. Huisken, "Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature", *Invent. Math.* **84**:3 (1986), 463–480. MR Zbl

[11] G. Huisken, "Deforming hypersurfaces of the sphere by their mean curvature", *Math. Z.* **195**:2 (1987), 205–219. MR Zbl

[12] H. B. Lawson, Jr., "Rigidity theorems in rank-1 symmetric spaces", *J. Differential Geometry* **4** (1970), 349–357. MR Zbl

[13] L. Lei and H. Xu, "Mean curvature flow of arbitrary codimension in spheres and sharp differentiable sphere theorem", preprint, 2015. arXiv

[14] L. Lei and H. Xu, "A new version of Huisken's convergence theorem for mean curvature flow in spheres", preprint, 2015. arXiv

[15] L. Lei and H. Xu, "An optimal convergence theorem for mean curvature flow of arbitrary codimension in hyperbolic spaces", preprint, 2015. arXiv

[16] L. Lei and H.-W. Xu, "New developments in mean curvature flow of arbitrary codimension inspired by Yau rigidity theory", pp. 327–348 in *Proceedings of the Seventh International Congress of Chinese Mathematicians*, *I*, Adv. Lect. Math. (ALM) **43**, Int. Press, Somerville, MA, 2019. MR Zbl

[17] L. Lei and H. Xu, "Mean curvature flow of arbitrary codimension in complex projective spaces", *Chinese Ann. Math. Ser. B* **44**:6 (2023), 857–892. MR Zbl

[18] K. Liu, H. Xu, and E. Zhao, "Deforming submanifolds of arbitrary codimension in a sphere", preprint, 2012. arXiv

[19] K. Liu, H. Xu, and E. Zhao, "Mean curvature flow of higher codimension in Riemannian manifolds", preprint, 2012. arXiv

[20] K. Liu, H. Xu, F. Ye, and E. Zhao, "Mean curvature flow of higher codimension in hyperbolic spaces", *Comm. Anal. Geom.* **21**:3 (2013), 651–669. MR Zbl

[21] K. Liu, H. Xu, F. Ye, and E. Zhao, "The extension and convergence of mean curvature flow in higher codimension", *Trans. Amer. Math. Soc.* **370**:3 (2018), 2231–2262. MR Zbl

[22] G. Perelman, "The entropy formula for the Ricci flow and its geometric applications", preprint, 2002. arXiv

[23] G. Perelman, "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds", preprint, 2003. arXiv

[24] G. Perelman, "Ricci flow with surgery on three-manifolds", preprint, 2003. arXiv

[25] G. Pipoli and C. Sinestrari, "Mean curvature flow of pinched submanifolds of $\mathbb{CP}^n$", *Comm. Anal. Geom.* **25**:4 (2017), 799–846. MR Zbl

[26] K. Shiohama and H. Xu, "The topological sphere theorem for complete submanifolds", *Compositio Math.* **107**:2 (1997), 221–232. MR Zbl

[27] H.-W. Xu and J.-R. Gu, "An optimal differentiable sphere theorem for complete manifolds", *Math. Res. Lett.* **17**:6 (2010), 1111–1124. MR Zbl

[28] H.-W. Xu and E.-T. Zhao, "Topological and differentiable sphere theorems for complete submanifolds", *Comm. Anal. Geom.* **17**:3 (2009), 565–585. MR Zbl

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