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The nonvanishing of the first Fourier–Jacobi coefficient of a Siegel eigenform F gives us that the vanishing of its m-th Fourier–Jacobi coefficient $F \mid \rho_m$ implies the vanishing of its m-th eigenvalue $\lambda_F(m)$. Conversely, we prove that for any odd, squarefree m if $\lambda_F(m)$ is zero then $F \mid \rho_m$ vanishes. While investigating this converse question and its important consequences, we generalize certain existing results of Kohnen and Skoruppa (1989) for index 1 Jacobi cusp forms to any arbitrary index, which are also of independent interest.

1. Introduction

In [6], Kohnen and Skoruppa introduced a novel Dirichlet series attached to any two Siegel cusp forms of degree 2 involving their Fourier–Jacobi coefficients. More importantly they could connect the Dirichlet series attached to a Siegel eigenform and any Siegel cusp form in the Maass space to the spinor zeta function of the Siegel eigenform. In particular, this connection gives us that the image of the m-th Fourier–Jacobi coefficient under certain adjoint operator is same as the m-th eigenvalue times the first Fourier–Jacobi coefficient of the Siegel eigenform (see (1)). Formally this could be viewed as an analogue of the relation between Fourier coefficients and eigenvalues of the Hecke eigenforms in the degree 1 case. Therefore it is natural to explore the relation between Fourier–Jacobi coefficients and eigenvalues further. In this paper, we take up this problem and investigate it in detail.

To state our results precisely, let us first introduce some notation. Throughout this article, k stands for an even integer and $k \geqslant 4$. Let $S_k(\Gamma_2)$ be the space of Siegel cusp forms of weight k for the symplectic group $\Gamma_2 := \mathrm{Sp}_4(\mathbb{Z})$. Let $J_{k,m}^{\mathrm{cusp}}$ denote the space of Jacobi cusp forms of weight k and index m for the group $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$. For any $l \geqslant 1$, let $V_{m,l} : J_{k,m}^{\mathrm{cusp}} \to J_{k,ml}^{\mathrm{cusp}}$ be the linear operator defined by [3, page 41, (2)]

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and let $T_{J,l} : J_{k,m}^{\text{cusp}} \to J_{k,m}^{\text{cusp}}$ be the l-th Hecke operator on Jacobi forms defined by [3, page 41, (3)]. For $m = 1$, Kohnen and Skoruppa [6, page 549, Proposition (i)] calculated the Fourier coefficients of the adjoint operator $V_{1,l}^*$ of the operator $V_{1,l}$. For any $l \geqslant 1$, let $T_{s,l}$ denote the l-th Hecke operator on $S_k(\Gamma_2)$ and let ρ_l denote the l-th projection of any element in $S_k(\Gamma_2)$ to its l-th Fourier–Jacobi coefficient in $J_{k,l}^{\text{cusp}}$. By using a result of Kohnen and Skoruppa [6, page 541, Theorem 2], one gets the following interesting identity [7, Lemma 2.1]. For any $F \in S_k(\Gamma_2)$ and $l \geqslant 1$, we have

$$F \,|\, T_{s,l} | \rho_1 = F \,|\, \rho_\ell \,|\, V_{1,l}^*. \tag{1}$$

Note that the above identity gives the first Fourier–Jacobi coefficient of the image of the l-th Hecke operator $T_{s,l}$. Manickam [7] used this identity crucially to establish the nonvanishing of the first Fourier–Jacobi coefficient of any Siegel eigenform in $S_k(\Gamma_2)$. By using this nonvanishing result, the identity (1) gives us the following result. For any Siegel eigenform $F \in S_k(\Gamma_2)$ and any $l \geqslant 1$, we have

$$F \,|\, \rho_l = 0 \implies \lambda_F(l) = 0, \tag{2}$$

where $F \,|\, T_{s,l} = \lambda_F(l)\, F$. In this article, we investigate the converse of (2) and its interesting consequences through certain important generalizations.

We first calculate the Fourier coefficients of the adjoint operator $V_{m,l}^*$, which generalizes the above mentioned result of Kohnen and Skoruppa [6, page 549, Proposition (i)] to any index $m \geqslant 1$. Our approach is quite different from the one taken in the literature.

Proposition 1.1. *Let $\phi \in J_{k,ml}^{\text{cusp}}$ be a Jacobi cusp form with the Fourier series expansion*

$$\phi(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 < 4lmn}} c_\phi(n, r)\, q^n \xi^r, \quad q = e^{2\pi i \tau},\ \xi = e^{2\pi i z}.$$

Then we have

$$\phi \,|\, V_{m,l}^*(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 < 4mn}} c_{\phi | V_{m,l}^*}(n, r)\, q^n \xi^r,$$

where

$$c_{\phi | V_{m,l}^*}(n, r) := \sum_{d | l} d^{k-2} \sum_{\substack{s=0 \\ d \,|\, (ms^2 + rs + n)}}^{d-1} c_\phi\left(\frac{(ms^2 + rs + n)\, l}{d^2}, \frac{(r + 2ms)\, l}{d}\right).$$

Let $J_{k,m}^{\text{cusp, new}}$ denote the space of Jacobi cusp newforms of weight k and index m, considered and studied extensively in [11, page 138]. As a consequence of Proposition 1.1 we derive the following identity of the operators on $J_{k,m}^{\text{cusp, new}}$ which generalizes the result of Kohnen and Skoruppa [6, page 549, Proposition (ii)] in the index 1 case to any arbitrary index $m \geqslant 1$. We follow the steps of the proof sketched in the index 1 case with appropriate modifications. For the sake of completion and for the benefit of the readers we provide the proof in Section 3 highlighting the main steps.

Proposition 1.2. *Let $\phi \in J_{k,m}^{\text{cusp, new}}$ and l be any positive integer coprime to m. Then*

$$\phi \,|\, V_{m,l}\, V_{m,l}^{*} = \phi \Big| \sum_{d|l} d^{k-2}\, \psi(d)\, T_{J,(l/d)},$$

where $\psi(d) = d \prod_{p|d}\big(1+\frac{1}{p}\big)$.

Now, we generalize the identity (1) to get the m-th Fourier–Jacobi coefficient of the image of Siegel cusp forms under the Hecke operator $T_{s,p^{\delta}}$, where p is a prime and δ is a positive integer.

Theorem 1.3. *Let $F \in S_k(\Gamma_2)$ and p be any prime. Then for any two positive integers δ and m with $p \nmid m$, we have*

$$F \,|\, T_{s,p^{\delta}} \,|\, \rho_m = F \,|\, \rho_{mp^{\delta}} | V_{m,p^{\delta}}^{*}. \tag{3}$$

Also, we have the following two identities:

$$F \,|\, T_{s,p} \,|\, \rho_p = F \,|\, \rho_{p^2} \,|\, V_{p,p}^{*} + p^{k-2}\, F \,|\, \rho_1 \,|\, V_{1,p} \tag{4}$$

and

$$F \,|\, T_{s,p^2} \,|\, \rho_p = F \,|\, \rho_{p^3} | V_{p,p^2}^{*} + p^{k-2} F \,|\, \rho_p \,|\, T_{J,p} + p^{2k-4} F \,|\, \rho_p. \tag{5}$$

Note that the algebra of the Hecke operators acting on the space $S_k(\Gamma_2)$ is generated by $T_{s,p}$ and T_{s,p^2}, where p varies over primes. Using the fact that the operator $V_{1,p} : J_{k,1}^{\text{cusp}} \to J_{k,p}^{\text{cusp}}$ is injective together with the identity (4), we have:

Corollary 1.4. *Let p be any prime. For any Siegel eigenform $F \in S_k(\Gamma_2)$ at least one of the Fourier–Jacobi coefficients $F \,|\, \rho_p$ and $F \,|\, \rho_{p^2}$ is nonzero.*

For any Siegel eigenform $F \in S_k(\Gamma_2)$, we have $F \,|\, \rho_{p^2} \,|\, V_{1,p^2}^{*} = \lambda_F(p^2)\, F \,|\, \rho_1$ from (1). On the other hand, by applying $V_{1,p}^{*}$ on both sides of the identity (4) and then by using (1) together with Proposition 1.2, we get:

Corollary 1.5. *Let $F \in S_k(\Gamma_2)$ and p be any prime such that $F \,|\, T_{s,p} = \lambda_F(p)\, F$. Then we have $F \,|\, \rho_{p^2} \,|\, V_{p,p}^{*} V_{1,p}^{*} = (\lambda_F^2(p) - p^{2k-3} - p^{2k-4})\, F \,|\, \rho_1 - p^{k-2}\, F \,|\, \rho_1 \,|\, T_{J,p}$.*

Our next result shows that any nonzero Fourier–Jacobi coefficient of odd, squarefree index of a Siegel cusp form cannot be a newform. In particular, we prove the following theorem.

Theorem 1.6. *Let $F \in S_k(\Gamma_2)$ and $m \geqslant 3$ be any odd, squarefree integer. If $F \,|\, \rho_m \in J_{k,m}^{cusp, new}$ then $F \,|\, \rho_m = 0$.*

Our next result shows that the converse of (2) is also true for l odd, squarefree. More precisely, we prove:

Theorem 1.7. *Let $F \in S_k(\Gamma_2)$ be a Siegel eigenform with n-th eigenvalue $\lambda_F(n)$. Then for any odd, squarefree positive integer m, we have*

$$\lambda_F(m) = 0 \iff F \mid \rho_m = 0.$$

Remark 1.8. The reverse direction $\Leftarrow$ of Theorem 1.7 follows from [7] (see (2)) and that only direction $\Rightarrow$ is proved here. To establish $\Longrightarrow$ part for a given odd, squarefree positive integer m we require the Siegel cusp form to be eigenvector only for the Hecke operators $T_{s,l}$ with $l \mid m$. Also, we only need Proposition 1.1, the identity (3) of Theorem 1.3 and Theorem 1.6, not any other result stated above.

By using the multiplicative property of the eigenvalues of a Siegel eigenform together with Theorem 1.7, we get:

Corollary 1.9. *Let $F \in S_k(\Gamma_2)$ be a Siegel eigenform. Then for any odd prime p, we have*

$$F \mid \rho_p = 0 \implies F \mid \rho_m = 0$$

for any odd, squarefree positive integers m with $p \mid m$.

If m is any positive integer such that $\lambda_F(m) \neq 0$ then (2) implies the existence of infinitely many symmetric, half-integral, positive definite matrices T such that the quadratic form T represents m and $a_F(T) \neq 0$. Conversely, we establish the following two corollaries of Theorem 1.7 assuring the nonvanishing of certain eigenvalues.

Corollary 1.10. *Let $F \in S_k(\Gamma_2)$ be a Siegel eigenform with n-th eigenvalue $\lambda_F(n)$ and T be a symmetric, half-integral, positive definite matrix such that the T-th Fourier coefficient $a_F(T) \neq 0$. If m is any odd, squarefree, positive integer represented by the quadratic form T then $\lambda_F(m) \neq 0$.*

Corollary 1.11. *Let $F \in S_k(\Gamma_2)$ be a Siegel eigenform with n-th eigenvalue $\lambda_F(n)$. Then there exists a positive integer $1 \leqslant n \leqslant \frac{k}{2} - 2$ such that for any odd, squarefree, positive integer m of the form $x^2 + ny^2$ we have $\lambda_F(m) \neq 0$.*

Remark (concluding remark). One may ask more generally about the nonvanishing of the m-th eigenvalue $\lambda_F(m)$ of a Siegel eigenform F if its m-th Fourier–Jacobi coefficient $F \mid \rho_m$ is nonzero. In this paper, we answer it affirmatively for any odd, squarefree m but could not address this question for arbitrary m. However, the intermediate results obtained by us while addressing the question highlight the importance of the theory of Jacobi forms and provide better understanding of certain Hecke-type operators on Jacobi forms.

The question of nonvanishing of Fourier–Jacobi coefficients of Siegel cusp forms of arbitrary degree and eigenvalues of Siegel eigenforms of degree 2 is also considered in [2]. However, the results obtained there are of different nature and do not address the question asked here in this paper.

2. Prerequisites

We refer to [1], [3] and [10] for definitions and basic properties of Jacobi and Siegel modular forms. In this section we fix notation and recall certain results.

Jacobi forms. Let G^J be the group of triplets $[M, X, \xi]$, $M \in \mathrm{SL}_2(\mathbb{R})$, $X \in \mathbb{R}^2$, $\xi \in \mathbb{C}$ with $|\xi| = 1$, via the multiplication

$$[M, X, \xi][M', X', \xi'] = [MM', XM' + X', \xi\xi' e^{2\pi i \det\left(\begin{smallmatrix} XM' \\ X' \end{smallmatrix}\right)}].$$

The group G^J acts on the set of functions $\{\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}\}$ as

$$\phi|_{k,m}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu), \xi\right](\tau, z) = \xi^m (c\tau + d)^{-k} e^{2\pi i m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right).$$

We consider the action of the discrete subgroup $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$ of G^J on the set of functions on $\mathbb{H} \times \mathbb{C}$ by fixing $\xi = 1$. Let $J_{k,m}$ (resp. $J_{k,m}^{\mathrm{cusp}}$) denote the space of Jacobi forms (resp. Jacobi cusp forms) of weight k and index m for the group $\mathrm{SL}_2(\mathbb{Z}) \ltimes (\mathbb{Z} \times \mathbb{Z})$. For any $l \geqslant 1$, let U_l, V_l and T_l be the operators acting on $J_{k,m}$ defined and studied systematically in [3, Section 4]. We are denoting them respectively by $U_{m,l}$, $V_{m,l}$ and $T_{J,l}$ throughout the paper to avoid certain potential confusions. The operator $T_{J,l} : J_{k,m} \to J_{k,m}$ is called the l-th Hecke operator on Jacobi forms.

Any $\phi(\tau, z) \in J_{k,m}^{\mathrm{cusp}}$ with Fourier series expansion

$$\phi(\tau, z) = \sum_{n, r \in \mathbb{Z},\, r^2 < 4mn} c_\phi(n, r)\, q^n \xi^r, \qquad q = e^{2\pi i \tau},\ \xi = e^{2\pi i z},$$

admits the following theta decomposition [3, pages 58–59]:

$$\phi(\tau, z) = \sum_{\mu=0}^{2m-1} h_\mu(\tau)\, \theta_{m,\mu}(\tau, z), \tag{6}$$

where

$$h_\mu(\tau) := \sum_{\substack{N \geqslant 1 \\ N \equiv -\mu^2 (\mathrm{mod}\, 4m)}} c_\phi\left(\frac{N+\mu^2}{4m}, \mu\right) q^{N/4m}, \qquad \theta_{m,\mu}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu (\mathrm{mod}\, 2m)}} q^{r^2/4m} \xi^r.$$

By using the transformation law of the Jacobi form ϕ and the Jacobi theta functions $\theta_{m,\mu}$ with respect to the inversion $(\tau, z) \to \left(-\frac{1}{\tau}, \frac{z}{\tau}\right)$, we get

$$h_\mu\left(\frac{-1}{\tau}\right) = \frac{\tau^k}{\sqrt{2m\tau/i}} \sum_{\nu=0}^{2m-1} e^{\pi i \mu\nu/m}\, h_\nu(\tau). \tag{7}$$

Let $J_{k,m}^{\text{cusp, new}}$ be the space of Jacobi cusp newforms considered in [11, page 138] giving the direct sum decomposition

$$J_{k,m}^{\text{cusp}} = J_{k,m}^{\text{cusp, new}} \overset{\perp}{\bigoplus} \Bigg(\bigoplus_{\substack{l \geqslant 1,\, d \geqslant 1 \\ ld^2 \mid m,\, ld^2 > 1}} J_{k,(m/ld^2)}^{\text{cusp, new}} \,|\, U_{m/ld^2,\,d}\, V_{m/l,\,l} \Bigg). \tag{8}$$

Note that the first direct sum in the above decomposition is orthogonal. If m is squarefree, then for any divisor $l > 1$ of m there is only one copy of the newforms space of index m/l given by $J_{k,(m/l)}^{\text{cusp, new}} \,|\, V_{(m/l),l}$ in the oldforms direct sum decomposition. By using the Shimura correspondence and the Atkin–Lehner theory for modular forms on the congruence subgroups $\Gamma_0(N)$, we get that for squarefree index m all the direct sums in the above decomposition (8) are orthogonal with respect to the Petersson inner product. For a detailed proof of this fact we refer to [5, Lemma 4]. In [8, Section 5.1], the space of Jacobi cusp newforms has been defined differently but in [7, page 406] it is observed that this newforms space is same as the one considered earlier in [11]. To prove Theorem 1.6, we use an important property of newforms [8, Corollary 5.3] saying that the (n,r)-th Fourier coefficient $c_\phi(n,r)$ of a Jacobi cusp form $\phi \in J_{k,m}^{\text{cusp, new}}$ depends only on the discriminant $r^2 - 4mn$ and not on $r \pmod{2m}$.

Siegel modular forms. The real symplectic unimodular group of degree 2 is defined by

$$\mathrm{Sp}_4(\mathbb{R}) = \{M \in \mathrm{GL}_4(\mathbb{R}) : MJ\,{}^tM = J\},$$

where $J = \left(\begin{smallmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{smallmatrix}\right)$, ${}^tM$ denotes the transpose matrix of the matrix M, 0_2 is the 2×2 zero matrix and I_2 is the 2×2 identity matrix. Let $\Gamma_2 := \mathrm{Sp}_4(\mathbb{Z})$ be the subgroup of $\mathrm{Sp}_4(\mathbb{R})$ consisting of matrices with integer entries. Let

$$\mathbb{H}_2 := \{Z \in M_2(\mathbb{C}) : Z = {}^tZ,\ \mathrm{Im}(Z) > 0\}$$

be the Siegel upper half-space of degree 2. We denote the space of Siegel modular forms (resp. cusp forms) on $\mathbb{H}_2$ of weight k for the group Γ_2 by $M_k(\Gamma_2)$ (resp. $S_k(\Gamma_2)$). There is an algebra of Hecke operators acting on the space $M_k(\Gamma_2)$ which preserves $S_k(\Gamma_2)$. For any $l \geqslant 1$, let $T_{s,l}$ denote the l-th Hecke operator on $S_k(\Gamma_2)$. An element in $S_k(\Gamma_2)$ is called a *Siegel eigenform* if it is a common eigenvector of all the Hecke operators $T_{s,l},\ l \geqslant 1$. Note that the space $S_k(\Gamma_2)$ is a Hilbert space under the Petersson inner product.

Any $F \in S_k(\Gamma_2)$ has the Fourier series expansion of the form

$$F(Z) = \sum_T a_F(T)\, e^{2\pi i\, \mathrm{trace}(TZ)},$$

where the sum varies over the set of symmetric, half-integral, positive definite 2×2 matrices. Writing $Z = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathbb{H}_2$, where τ, τ' are in the complex upper half-plane $\mathbb{H}$ and $z \in \mathbb{C}$, we get the following Fourier–Jacobi decomposition [3, Theorem 6.1]:

$$F(Z) = F(\tau, z, \tau') = \sum_{m \geqslant 1} \phi_m(\tau, z)\, e^{2\pi i m \tau'},$$

where

$$\phi_m(\tau, z) := \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 < 4nm}} A_F \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} q^n \xi^r$$

belongs to the space $J_{k,m}^{\text{cusp}}$ and is called the m-th Fourier–Jacobi coefficient of F.

3. Proof of Propositions 1.1 and 1.2

Proof of Proposition 1.1. Let l, m be any two positive integers. Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. On the space $J_{k,m}^{\text{cusp}}$, the index changing Hecke operator $V_{m,l}$ is defined by

$$\begin{aligned} \phi \mid V_{m,l}(\tau, z) &:= l^{k-1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \backslash M_2(\mathbb{Z}) \\ ad - bc = l}} (c\tau + d)^{-k} e^{ml\left(\frac{-cz^2}{c\tau + d}\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{lz}{c\tau + d}\right) \\ &= l^{(k/2)-1} \sum_{\substack{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \\ ad = l,\, b \pmod d}} \phi_{\sqrt{l}} \Big|_{k,ml} \left[\frac{1}{\sqrt{l}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, (0, 0), 1\right](\tau, z), \end{aligned}$$

where $\phi_{\sqrt{l}}(\tau, z) := \phi(\tau, \sqrt{l}\, z)$. To prove our claim, first we calculate the image of Jacobi Poincaré series $P_{k,m;n,r}$, $n, r \in \mathbb{Z}$ with $r^2 - 4mn < 0$, under the operator $V_{m,l}$. By using the definition of Jacobi Poincaré series, we have

$$\begin{aligned} &P_{k,m;n,r} \mid V_{m,l} \\ &= \sum_{\substack{\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \\ \alpha\delta = l, \beta (\mathrm{mod}\, \delta)}} l^{(k/2)-1} \Bigg(\sum_{\substack{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma \\ \lambda \in \mathbb{Z}}} e(n\tau + rz) \Big|_{k,m} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda a, \lambda b), 1\right] \Bigg)_{\sqrt{l}} \Bigg|_{k,ml} \\ &\qquad \left[\frac{1}{\sqrt{l}} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, (0, 0), 1\right]. \end{aligned}$$

Using the definition of $\phi_{\sqrt{l}}(\tau, z) = \phi(\tau, \sqrt{l}\, z)$ and then adjusting the stroke operators in the inner sum, we obtain

$$\begin{aligned} P_{k,m;n,r} \mid V_{m,l} = &\sum_{\substack{\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \\ \alpha\delta = l,\, \beta\, (\mathrm{mod}\, \delta)}} l^{(k/2)-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma,\, \lambda \in \mathbb{Z}} e(n\tau + r\sqrt{l} z) \Big|_{k,ml} \\ &\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{\lambda a}{\sqrt{l}}, \frac{\lambda b}{\sqrt{l}}\right), 1\right] \left[\frac{1}{\sqrt{l}} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, (0, 0), 1\right]. \end{aligned}$$

In the Jacobi group $\mathrm{SL}_2(\mathbb{R}) \ltimes (\mathbb{R}^2 \times \mathbb{S}^1)$, where $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$, we have

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \left(\frac{\lambda a}{\sqrt{l}}, \frac{\lambda b}{\sqrt{l}}\right), 1\right]\left[\frac{1}{\sqrt{l}}\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, (0,0), 1\right] \\ = \left[\frac{1}{\sqrt{l}}\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix}, \left(\frac{\lambda\alpha'}{l}, \frac{\lambda\beta'}{l}\right), 1\right]\left[\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, (0,0), 1\right]$$

for some $\left(\begin{smallmatrix} \alpha' & \beta' \\ 0 & \delta' \end{smallmatrix}\right) \in M_2(\mathbb{Z})$ with $\alpha'\delta' = l$ and $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma$ with the following crucial property. These matrices vary over a complete set of representatives of the indexing sets in the above summation as the matrices $\left(\begin{smallmatrix} \alpha & \beta \\ 0 & \delta \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ vary over the same, respectively. Using this and interchanging the order of the summations, we get

$$P_{k,m;n,r} \,|\, V_{m,l} = \sum_{\substack{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma \\ \lambda \in \mathbb{Z}}} l^{(k/2)-1} \sum_{\substack{\left(\begin{smallmatrix} \alpha' & \beta' \\ 0 & \delta' \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \\ \alpha'\delta' = l,\, \beta' \,(\mathrm{mod}\ \delta)'}} e(n\tau + r\sqrt{l}z)\Big|_{k,ml} \\ \left[\frac{1}{\sqrt{l}}\begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix}, \left(\frac{\lambda\alpha'}{l}, \frac{\lambda\beta'}{l}\right), 1\right]\left[\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, (0,0), 1\right].$$

For any $\lambda \in \mathbb{Z}$, we write $\lambda = \lambda'\delta' + s$ with $s \pmod{\delta}'$. Then λ' varies over $\mathbb{Z}$ and s varies over a complete residue system mod δ'. Therefore, we have

$$P_{k,m;n,r} \,|\, V_{m,l} = \sum_{\substack{\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma \\ \lambda' \in \mathbb{Z}}} l^{(k/2)-1} \sum_{\substack{\delta' \,|\, l,\, \beta' \,(\mathrm{mod}\ \delta)' \\ s \,(\mathrm{mod}\ \delta)'}} e(n\tau + r\sqrt{l}z)\Big|_{k,ml} \\ \left[\frac{1}{\sqrt{l}}\begin{pmatrix} l/\delta' & \beta' \\ 0 & \delta' \end{pmatrix}, \left(\frac{s}{\delta'}, \frac{(s+\lambda'\delta')\beta'}{l}\right), 1\right]\left[\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, (\lambda' a', \lambda' b'), 1\right].$$

Let us first simplify the inner sum. We set

$$I_{k,m;l} := \sum_{\substack{\delta | l, \beta (\mathrm{mod}\ \delta) \\ s (\mathrm{mod}\ \delta)}} e(n\tau + r\sqrt{\ell}z)\Big|_{k,ml} \left[\frac{1}{\sqrt{l}}\begin{pmatrix} l/\delta & \beta \\ 0 & \delta \end{pmatrix}, \left(\frac{s}{\delta}, \frac{(s+\lambda'\delta)\beta}{l}\right), 1\right] \\ = l^{k/2} \sum_{\substack{\delta | l \\ s (\mathrm{mod}\ \delta)}} \delta^{-k} e\left(\left(\frac{l}{\delta^2}(ms^2 + rs + n)\,\tau\right) + \left(\frac{l}{\delta}(r + 2sm)\,z\right)\right) \\ \sum_{\beta (\mathrm{mod}\ \delta)} e\left(\frac{\beta}{\delta}(ms^2 + rs + n)\right) \\ = l^{k/2} \sum_{\substack{\delta \,|\, l, s (\mathrm{mod}\ \delta) \\ \delta \,|\, (ms^2 + rs + n)}} \delta^{-k+1} e\left(\left(\frac{l}{\delta^2}(ms^2 + rs + n)\,\tau\right) + \left(\frac{l}{\delta}(r + 2sm)\,z\right)\right)$$

Therefore, we have

$$
\begin{aligned}
&P_{k,m;n,r}\,|\,V_{m,l}\\
&=\sum_{\delta|l}\Big(\frac{l}{\delta}\Big)^{k-1}\sum_{\substack{s\,(\mathrm{mod}\ \delta)\\ \delta\,|\,(ms^2+rs+n)}}\ \sum_{\substack{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma\\ \lambda\in\mathbb{Z}}}\\
&\qquad e\bigg(\Big(\frac{l}{\delta^2}(ms^2+rs+n)\,\tau\Big)+\Big(\frac{l}{\delta}(r+2sm)\,z\Big)\bigg)\bigg|_{k,ml}\bigg[\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda a,\lambda b),1\bigg]\\
&=\sum_{\delta|l}\Big(\frac{l}{\delta}\Big)^{k-1}\sum_{\substack{s\,(\mathrm{mod}\ \delta)\\ \delta\,|\,(ms^2+rs+n)}} P_{k,ml;\frac{(ms^2+rs+n)\,l}{\delta^2},\frac{(r+2sm)\,l}{\delta}}.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&c_{\phi\,|\,V^*_{m,l}}(n,r)\\
&\quad=\frac{(2\pi(4mn-r^2))^{k-3/2}}{2^{k-5/2}\Gamma(k-3/2)\,m^{k-2}}\langle\phi\,|\,V^*_{m,l},P_{k,m;n,r}\rangle\\
&\quad=\frac{(2\pi(4mn-r^2))^{k-3/2}}{2^{k-5/2}\Gamma(k-3/2)\,m^{k-2}}\Big\langle\phi,\sum_{\delta|l}\Big(\frac{l}{\delta}\Big)^{k-1}\sum_{\substack{s\,(\mathrm{mod}\ \delta)\\ \delta\,|\,(ms^2+rs+n)}} P_{k,ml;\frac{(ms^2+rs+n)\,l}{\delta^2},\frac{(r+2sm)\,l}{\delta}}\Big\rangle\\
&\quad=\sum_{d|l}d^{k-2}\sum_{\substack{s\,(\mathrm{mod}\ d)\\ d\,|\,(ms^2+rs+n)}} c_\phi\Big(\frac{(ms^2+rs+n)\,l}{d^2},\frac{(r+2sm)\,l}{d}\Big).\qquad\square
\end{aligned}
$$

Proof of Proposition 1.2. For all the facts used in this proof about the operators $T_{J,l}$, $V_{m,l}$ and the space $J^{\text{cusp, new}}_{k,m}$, we refer to [3; 11]. Since l and m are coprime, the right-hand side operator

$$\mathbb{T}_{J,l}:=\sum_{d|l}d^{k-2}\psi(d)\,T_{J,(l/d)}$$

is multiplicative. Moreover, the operator $V_{m,l}$ is multiplicative and the Hecke operator $T_{J,n}$ commutes with the operator $V_{m,l}$ if $\gcd(n,lm)=1$. Therefore it is enough to establish the identity for prime powers, that is, $l=p^\alpha$, where p is a prime and α is any positive integer. Since the space $J^{\text{cusp, new}}_{k,m}$ has a basis of simultaneous eigenfunctions of all the Hecke operators $T_{J,n}$ with $\gcd(n,m)=1$, it is enough to check the identity for such eigenforms. Let $\varphi\in J^{\text{cusp, new}}_{k,m}$ be any such eigenform. The Hecke operators $T_{J,n}$ with $\gcd(n,m)=1$ are hermitian and commute with $T_{J,l'}$ and $V_{m,t}$ for $\gcd(nl',m)=1$ and $\gcd(n,mt)=1$. Therefore the Jacobi forms $\varphi\,|\,V_{m,l}\,V^*_{m,l}$ and $\varphi\,|\,\mathbb{T}_l$ are again simultaneous eigenfunctions of all the Hecke operators $T_{J,n}$ for $\gcd(n,lm)=1$ with eigenvalues same as of φ. By using multiplicity one result, we get that the Jacobi forms $\varphi \,|\, V_{m,l}\, V_{m,l}^{*}$ and $\varphi \,|\, \mathbb{T}_l$ both are constant multiples of φ. To show that both are same we compare their (n,r)-th Fourier coefficients with the condition that r^2-4mn is a fundamental discriminant and prove that they are equal. We have

$$c_{\varphi \,|\, V_{m,l}\, V_{m,l}^{*}}(n,r) = \sum_{t\,|\,l} t^{k-1} \Biggl(\sum_{d\,|\,t} \frac{1}{d} \sum_{\substack{s=0 \\ d\,|\,ms^2+rs+n}}^{d-1} 1 \Biggr) c_\varphi\Bigl(n\frac{l^2}{t^2}, r\frac{l}{t}\Bigr). \tag{9}$$

The cardinality of the set

$$\{s \pmod d : ms^2+rs+n \equiv 0 \pmod d\}$$

is same as the cardinality of the set $\{x (\mathrm{mod}\ 2d) : x^2 \equiv (r^2-4mn) \pmod{4d}\}$. Let us denote this cardinality by $N_d(r^2-4mn)$. By using [3, page 50, (16)], we have

$$c_\varphi\Bigl(n\frac{l^2}{t^2}, r\frac{l}{t}\Bigr) = \sum_{\delta\,|\,(l/t)} \mu(\delta)\, \chi_D(\delta)\, \delta^{k-2}\, c_{\varphi \,|\, T_{J,(l/\delta t)}}(n,r),$$

where $D=r^2-4mn$ and χ_D denotes the Dirichlet character $\bigl(\frac{D}{\cdot}\bigr)$. By using the above observations in (9), we see that it is sufficient to prove the following formal identity of the operators:

$$\sum_{t\,|\,l} t^{k-1} \sum_{d\,|\,t} \frac{N_d(D)}{d} \sum_{\delta\,|\,(l/t)} \mu(\delta)\, \chi_D(\delta)\, \delta^{k-2}\, T_{J,(l/\delta t)} = \sum_{t\,|\,l} t^{k-2} \psi(t)\, T_{J,(l/t)}. \tag{10}$$

Since D is a fundamental discriminant, by using [3, page 21, (6)] we get that

$$N_p(D) = (1+\chi_D(p)) \quad \text{and} \quad N_{p^a}(D) = N_p(D)$$

for any prime p, positive integer a. By using these facts we get that the coefficients of $T_{J,p^a},\ 1 \leqslant a \leqslant \alpha$, in both sides of (10) are equal. □

4. Proof of Theorem 1.3

We prove the identities by equating the Fourier coefficients on both sides. First let us write down the Fourier coefficients of $F \,|\, T_{s,p^\delta}$, where p is a prime and δ is a positive integer [9, Corollaries 2.2, 2.4 and 2.5]. For any positive integer l and any finite sequence of integers $\{a_1, a_2, \ldots, a_n\}$, we use the notation $\delta_l(a_1, a_2, \ldots, a_n)$ defined to be 1 if $l \,|\, \gcd(a_1, a_2, \ldots, a_n)$ and 0 otherwise. Let

$$F(Z) = \sum_{T=\left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right) > 0} A_F(n,r,m)\, e^{2\pi i\, \mathrm{trace}(TZ)},$$

where $T > 0$ indicates that T is positive definite. Then we have

$$(11)\quad A_{F|T_{s,p}}(n,r,m) = A_F(pn,pr,pm) + p^{k-2}\Bigg(\sum_{\substack{\alpha=0\\ p|(n+r\alpha+m\alpha^2)}}^{p-1} A_F\Big(\frac{n+r\alpha+m\alpha^2}{p}, r+2m\alpha, mp\Big) + \delta_p(m)\,A_F\Big(np,r,\frac{m}{p}\Big)\Bigg) + p^{2k-3}\delta_p(n,r,m)\,A_F\Big(\frac{n}{p},\frac{r}{p},\frac{m}{p}\Big).$$

Also, we have

$$\begin{aligned}(12)\quad & A_{F|T_{s,p^2}}(n,r,m)\\ &= A_F(p^2n,p^2r,p^2m) + p^{2k-3}\delta_p(n,r,m)\,A_F(n,r,m) + p^{4k-6}\delta_{p^2}(n,r,m)A_F\Big(\frac{n}{p^2},\frac{r}{p^2},\frac{m}{p^2}\Big)\\ &\quad + p^{k-2}\Bigg(\sum_{\substack{\alpha=0\\ p|(n+r\alpha+m\alpha^2)}}^{p-1} A_F\Big(p\Big(\frac{n+r\alpha+m\alpha^2}{p}, r+2m\alpha, mp\Big)\Big) + \delta_p(m)\,A_F\Big(p\Big(np,r,\frac{m}{p}\Big)\Big)\Bigg)\\ &\quad + p^{2k-4}\Bigg(\sum_{\substack{\alpha=0\\ p^2\,|\,(n+r\alpha+m\alpha^2)}}^{p^2-1} A_F\Big(\frac{n+r\alpha+m\alpha^2}{p^2}, r+2m\alpha, mp^2\Big) + \sum_{\substack{\beta=0\\ p^2\,|\,(rp\beta+m)}}^{p-1} A_F\Big(np^2, r+2np\beta, n\beta^2+\frac{rp\beta+m}{p^2}\Big)\Bigg)\\ &\quad + p^{3k-5}\Bigg(\sum_{\substack{\alpha=0\\ p^2\,|\,(n+r\alpha+m\alpha^2),\,p\,|\,r,\,p\,|\,m}}^{p-1} A_F\Big(\frac{n+r\alpha+m\alpha^2}{p^2}, \frac{r+2m\alpha}{p}, m\Big) + \delta_p(n,r)\,\delta_{p^2}(m)\,A_F\Big(n,\frac{r}{2p},\frac{m}{p^2}\Big)\Bigg).\end{aligned}$$

If $p \nmid m$ then we have

$$\begin{aligned}(13)\quad & A_{F|T_{s,p^\delta}}(n,r,m)\\ &= A_F(p^\delta(n,r,m)) + \sum_{\beta=1}^{\delta} p^{(k-2)\beta}\Bigg(\sum_{\substack{\alpha=0\\ p^\beta|(n+r\alpha+m\alpha^2)}}^{p^\beta-1} A_F\big(p^{\delta-\beta}((n+r\alpha+m\alpha^2)p^{-\beta}, r+2m\alpha, mp^\beta)\big)\Bigg).\end{aligned}$$

First we compare the coefficients of both sides of (3). Let (n,r) be any pair of integers with $r^2 < 4mn$. By using (13), we have

$$\begin{aligned}(14)\quad & c_{F\,|\,T_{s,p^\delta}\,|\,\rho_m}(n,r)\\ &= A_{F\,|\,T_{s,p^\delta}}(n,r,m)\\ &= A_F(p^\delta(n,r,m))\\ &\quad+\sum_{\beta=1}^{\delta} p^{(k-2)\beta}\Bigg(\sum_{\substack{\alpha=0\\ p^\beta\,|\,(n+r\alpha+m\alpha^2)}}^{p^\beta-1} A_F\big((n+r\alpha+m\alpha^2)\,p^{\delta-2\beta},(r+2m\alpha)\,p^{\delta-\beta},mp^\delta\big)\Bigg).\end{aligned}$$

By using Proposition 1.1, we get

$$\begin{aligned}(15)\quad & c_{F\,|\,\rho_{mp^\delta}\,|\,V^*_{m,p^\delta}}(n,r)\\ &= c_{F\,|\,\rho_{mp^\delta}}(p^\delta n,\,p^\delta r)\\ &\quad+\sum_{\beta=1}^{\delta} p^{(k-2)\beta}\Bigg(\sum_{\substack{s=0\\ p^\beta\,|\,(n+rs+ms^2)}}^{p^\beta-1} c_{F\,|\,\rho_{mp^\delta}}\big((n+rs+ms^2)\,p^{\delta-2\beta},\,(r+2ms)\,p^{\delta-\beta}\big)\Bigg).\end{aligned}$$

Now by comparing (14) and (15) we get that $F\,|\,T_{s,p^\delta}\,|\,\rho_m = F\,|\,\rho_{mp^\delta}\,|\,V^*_{m,p^\delta}$.

Next we compare the coefficients of both sides of (4) and then of (5). By using (11), we have

$$\begin{aligned}(16)\quad & c_{F\,|\,T_{s,p}\,|\,\rho_p}(n,r)\\ &= A_{F|T_{s,p}}(n,r,p)\\ &= A_F(pn,\,pr,\,p^2)\\ &\qquad + p^{k-2}\Bigg(\sum_{\substack{\alpha=0\\ p|(n+r\alpha)}}^{p-1} A_F\left(\frac{n+r\alpha+p\alpha^2}{p},\,r+2p\alpha,\,p^2\right)+A_F(np,r,1)\Bigg)\\ &\qquad + p^{2k-3}\,\delta_p(n,r)\,A_F\left(\frac{n}{p},\,\frac{r}{p},\,1\right).\end{aligned}$$

By using Proposition 1.1, we have

$$(17)\quad c_{F\,|\,\rho_{p^2}\,|\,V^*_{p,p}}(n,r)=\sum_{d\,|\,p} d^{k-2}\sum_{\substack{s=0\\ d\,|\,(ps^2+rs+n)}}^{d-1} c_{F\,|\,\rho_{p^2}}\left(\frac{(ps^2+rs+n)\,p}{d^2},\,(r+2ps)\frac{p}{d}\right).$$

By using [3, Theorem 4.2, 7], we have

$$(18)\qquad c_{F\,|\,\rho_1|V_{1,p}}(n,r)=\sum_{d\,|\,(n,r,p)} d^{k-1}c_{F\,|\,\rho_1}\left(\frac{np}{d^2},\,\frac{r}{d}\right).$$

Comparing (16), (17) and (18), we get that

$$F\,|\,T_{s,p}\,|\,\rho_p = F\,|\,\rho_{p^2}\,|\,V^*_{p,p}+p^{k-2}\,F\,|\,\rho_1\,|\,V_{1,p}.$$

By using (12), we have

$$
\begin{aligned}
(19)\quad & c_{F|T_{s,p^2}|\rho_p}(n,r) \\
&= A_{F|T_{s,p^2}}(n,r,p) \\
&= A_F(p^2n,p^2r,p^3) + p^{2k-3}\delta_p(n,r)\,A_F(n,r,p) \\
&\quad + p^{k-2}\Bigg(\sum_{\substack{\alpha=0\\ p\mid(n+r\alpha)}}^{p-1} A_F\big(n+r\alpha+p\alpha^2,(r+2p\alpha)p,p^3\big) + A_F(np^2,rp,p)\Bigg) \\
&\quad + p^{2k-4}\Bigg(\sum_{\substack{\alpha=0\\ p^2\mid(n+r\alpha+p\alpha^2)}}^{p^2-1} A_F\Big(\frac{n+r\alpha+p\alpha^2}{p^2},r+2p\alpha,p^3\Big) \\
&\qquad\qquad + \sum_{\substack{\beta=0\\ p\mid(r\beta+1)}}^{p-1} A_F\Big(np^2,r+2np\beta,n\beta^2+\frac{r\beta+1}{p}\Big)\Bigg) \\
&\quad + p^{3k-5}\Bigg(\sum_{\substack{\alpha=0\\ p^2\mid(n+r\alpha+p\alpha^2),\,p\mid r}}^{p-1} A_F\Big(\frac{n+r\alpha+p\alpha^2}{p^2},\frac{r+2p\alpha}{p},p\Big)\Bigg).
\end{aligned}
$$

By using Proposition 1.1, we have

$$
\begin{aligned}
(20)\quad & c_{F|\rho_{p^3}|V^*_{p,p^2}}(n,r) \\
&= \sum_{d\mid p^2} d^{k-2} \sum_{\substack{s=0\\ d\mid(ps^2+rs+n)}}^{d-1} c_{F|\rho_{p^3}}\Big(\frac{(ps^2+rs+n)p^2}{d^2},(r+2ps)\frac{p^2}{d}\Big).
\end{aligned}
$$

By using [3, page 56, (24)], we have

$$
\begin{aligned}
(21)\quad & c_{F|\rho_p|T_p}(n,r) \\
&= \begin{cases}
c_{F|\rho_p}(p^2n,pr) & \text{if } p\nmid r,\\
c_{F|\rho_p}(p^2n,pr) - p^{k-2}c_{F|\rho_p}(n,r) & \text{if } p\mid r,\ p\nmid n,\\
c_{F|\rho_p}(p^2n,pr) + p^{k-2}(p-1)\,c_{F|\rho_p}(n,r) & \\
\quad + p^{2k-3}\sum_{\alpha=0,\,p^2\mid(n+r\alpha+p\alpha^2)}^{p-1} c_{F|\rho_p}\big(\frac{n+r\alpha+p\alpha^2}{p^2},\frac{r+2p\alpha}{p}\big) & \text{if } p\mid r,\ p\mid n.
\end{cases}
\end{aligned}
$$

Suppose $p\nmid r$. Then there exists unique $\beta\in\{0,1,\dots p-1\}$ such that $p\mid r\beta+1$. Suppose that $r\beta+1=lp$ for some $l\in\mathbb{Z}$. Then we have

$$
(22)\quad \begin{pmatrix}-p & r\\ -\beta & l\end{pmatrix}\begin{pmatrix}n & r/2\\ r/2 & m\end{pmatrix}\begin{pmatrix}-p & -\beta\\ r & l\end{pmatrix} = \begin{pmatrix}np^2 & (r+2np\beta)/2\\ (r+2np\beta)/2 & n\beta^2+(r\beta+1)/p\end{pmatrix}.
$$

By comparing the three equations (19), (20), (21) and also using (22), we get that

$$
F|T_{s,p^2}|\rho_p = F|\rho_{p^3}|V^*_{p,p^2} + p^{k-2}F|\rho_p|T_{J,p} + p^{2k-4}F|\rho_p.
$$

5. Proof of Theorem 1.6

Suppose $F\,|\,\rho_m = \varphi_m \in J_{k,m}^{\text{cusp, new}}$. By using the Eichler–Zagier isomorphism Z_m given in [8, Theorem 5.4], we get that $\varphi_m\,|\,Z_m$ is in the space

$$S_{k-1/2}^{+,m,\text{new}}(4m) := \big\{ f \in S_{k-1/2}^{+,\,\text{new}}(4m) : a_f(n) = 0 \text{ unless } (-1)^{k-1}n \equiv \square \pmod{4m} \big\},$$

where $S_{k-1/2}^{+,\,\text{new}}(4m)$ is the subspace of newforms inside the Kohnen's plus space $S_{k-1/2}^{+}(4m)$ studied in [4]. Moreover, we have

$$a_{\varphi_m\,|\,Z_m}(|D|) = c_{\varphi_m}\Big(\frac{r^2-D}{4m}, r\Big)$$

for any $0 > D$, $r \in \mathbb{Z}$ with $D \equiv r^2 \pmod{4m}$. Let p be an odd prime dividing m and let U_{p^2} be the level dividing operator $\sum_{n\geqslant 1} a(n)\,q^n \mapsto \sum_{n\geqslant 1} a(p^2 n)\,q^n$. Since $\varphi_m\,|\,Z_m \in S_{k-1/2}^{+,\,\text{new}}(4m)$, by using [4, Theorem 1] we get that

$$\varphi_m\,|\,Z_m\,|\,(U_{p^2} + p^{k-2} w_p) = 0,$$

where w_p is the involution operator $w_{p,k-1/2}^{m}$ defined in [4, Section 2, page 39]. From [8, Lemma 5.9] we know that w_p acts as the identity operator on $S_{k-1/2}^{+,m}(4m)$. Therefore we have $\varphi_m\,|\,Z_m\,|\,(U_{p^2} + p^{k-2}) = 0$. Hence for any $0 > D$, $r \in \mathbb{Z}$, $D \equiv r^2 \pmod{4m}$, we have

$$c_{\varphi_m}\Big(p^2\frac{r^2-D}{4m}, pr\Big) + p^{k-2} c_{\varphi_m}\Big(\frac{r^2-D}{4m}, r\Big) = 0.$$

For any $n \geqslant 1$, $r \in \mathbb{Z}$ with $r^2 < 4mn$, by taking $D = r^2 - 4mn$ we have

$$c_{\varphi_m}(p^2 n, pr) + p^{k-2} c_{\varphi_m}(n, r) = 0. \tag{23}$$

Suppose $F\,|\,T_{s,p} = G \in S_k(\Gamma_2)$. By using (11), we write down $\big(np, r, \frac{m}{p}\big)$-th coefficients of $F\,|\,T_{s,p}$ to get

$$p^{k-2} \sum_{\nu \pmod{p},\, p\,|\,\left(r\nu+\nu^2\frac{m}{p}\right)} A_F\Big(n + \frac{r\nu + \nu^2(m/p)}{p}, r + 2\nu\frac{m}{p}, m\Big) + A_F(p^2 n, pr, m) = A_G\Big(np, r, \frac{m}{p}\Big).$$

Suppose $p \nmid r$. Then there are exactly two choices for $\nu \pmod{p}$ in the left-hand side sum namely $\nu = 0$ and $\nu = -r(\overline{m/p})$, where $\overline{m/p}$ denotes the inverse of m/p modulo p. Assume that $(m/p)(\overline{m/p}) = 1 + lp$ for some $l \in \mathbb{Z}$. Then we have

$$p^{k-2}\big(c_{\varphi_m}(n, r) + c_{\varphi_m}(n + r^2 l(\overline{m/p}), -r - 2rlp)\big) + c_{\varphi_m}(p^2 n, pr) = c_{G\,|\,\rho_{m/p}}(np, r).$$

Since $(r + 2rlp)^2 - 4m(n + r^2 l(\overline{m/p})) = r^2 - 4mn$ and φ_m is in the space $J_{k,m}^{\text{cusp, new}}$, by using [8, Corollary 5.3] we get that $c_{\varphi_m}(n + r^2 l(\overline{m/p}), -r - 2rlp) = c_{\varphi_m}(n, r)$.

Hence we have

$$2p^{k-2}c_{\varphi_m}(n,r)+c_{\varphi_m}(p^2n,pr)=c_{G\,|\,\rho_{m/p}}(np,r). \tag{24}$$

From (23) and (24), we get that $c_{\varphi_m}(n,r)=c_{G\,|\,\rho_{m/p}}(np,r)$ for any $n\geqslant 1,\ r\in\mathbb{Z}$ with $p\nmid r,\ r^2<4mn$. But by using [3, Theorem 4.2, (7)] we get that

$$c_{G\,|\,\rho_{m/p}(np,r)}=c_{G\,|\,\rho_{m/p}\,|\,V_{m/p,p}}(n,r).$$

But $\phi_m\in J_{k,m}^{\text{cusp, new}}$, therefore by using [11, Lemma 3.1] we get that $\varphi_m=F\,|\,\rho_m=0$.

6. Proof of Theorem 1.7 and its corollaries

Proof of Theorem 1.7. We prove the theorem by induction on the number of prime factors of m.

Let p be any odd prime such that $\lambda_F(p)=0$. By using the decomposition given by (8), we have the following orthogonal decomposition $F\,|\,\rho_p=\phi_1\,|\,V_{1,p}+\phi_p$, where $\phi_1\in J_{k,1}^{\text{cusp, new}}$ and $\phi_p\in J_{k,p}^{\text{cusp, new}}$. Note that $J_{k,1}^{\text{cusp, new}}=J_{k,1}^{\text{cusp}}$. By using the identity (1) and the fact that $\lambda_F(p)=0$ we get that $F\,|\,\rho_p\,|\,V_{1,p}^*=0$. Then we have

$$\langle\phi_1\,|\,V_{1,p},\phi_1\,|\,V_{1,p}\rangle=\langle F\,|\,\rho_p,\phi_1\,|\,V_{1,p}\rangle=\langle F\,|\,\rho_p\,|\,V_{1,p}^*,\phi_1\rangle=0.$$

Therefore we have $F\,|\,\rho_p=\phi_p\in J_{k,p}^{\text{cusp, new}}$. By applying Theorem 1.6, we get that $F\,|\,\rho_p=0$.

Let m be any odd, squarefree, positive integer which is a multiple of at least 2 primes. Then again by using the decomposition (8), we have

$$F\,|\,\rho_m\in J_{k,m}^{\text{cusp}}=\bigoplus_{l\,|\,m,\,l\neq m}J_{k,l}^{\text{cusp, new}}\,|\,V_{l,(m/l)}\oplus J_{k,m}^{\text{cusp, new}}.$$

Note that all the direct sums in the above decomposition are orthogonal. We write

$$F\,|\,\rho_m=\sum_{l\,|\,m,\,l\neq m}\varphi_l\,|\,V_{l,(m/l)}+\varphi_m,$$

where $\varphi_l\in J_{k,l}^{\text{cusp, new}}$ and $\varphi_m\in J_{k,m}^{\text{cusp, new}}$. Suppose $\lambda_F(m)=0$. First, by using the identity (1) we deduce that $\varphi_1\,|\,V_{1,m}=0$. Next, by using the multiplicative property of $\lambda_F(m)$ we get that $\lambda_F(p)=0$ for some odd prime $p\,|\,m$. For any $l\,|\,m,\ l\neq m$ with $p\nmid l$, by using the fact that $V_{l,(m/l)}=V_{l,(m/lp)}V_{(m/p),p}$ we have

$$\begin{aligned}\langle\varphi_l\,|\,V_{l,(m/l)},\varphi_l\,|\,V_{l,(m/l)}\rangle&=\langle F\,|\,\rho_m,\varphi_l\,|\,V_{l,(m/l)}\rangle\\&=\langle F\,|\,\rho_m\,|\,V_{l,(m/l)}^*,\varphi_l\rangle=\langle F\,|\,\rho_m\,|\,V_{(m/p),p}^*V_{l,(m/lp)}^*,\varphi_l\rangle.\end{aligned}$$

By using the identity (3) for $\delta=1$, we have $F\,|\,\rho_m\,|\,V_{(m/p),p}^*=\lambda_F(p)\,F\,|\,\rho_{m/p}=0$. On the other hand, for any $l\,|\,m,\ l\neq m$ with $p\,|\,l$, let p' be any odd prime dividing m/l.

Again, by using the fact that $V_{l,(m/l)} = V_{l,(m/lp')} V_{(m/p'),p'}$ and the identity (3), we have

$$\begin{aligned}\langle \varphi_l \,|\, V_{l,(m/l)}, \varphi_l \,|\, V_{l,(m/l)} \rangle &= \langle F \,|\, \rho_m \,|\, V^*_{l,(m/l)}, \varphi_l \rangle \\ &= \langle F \,|\, \rho_m \,|\, V^*_{(m/p'),p'} V^*_{l,(m/lp')}, \varphi_l \rangle \\ &= \lambda_F(p') \langle F \,|\, \rho_{m/p'} \,|\, V^*_{l,(m/lp')}, \varphi_l \rangle .\end{aligned}$$

Since $\lambda_F(m/p') = \lambda_F(p)\,\lambda_F(m/pp') = 0$ and m/p' has fewer prime factors than m, by using the induction hypothesis we get that $F \,|\, \rho_{m/p'} = 0$. Hence we get that $F \,|\, \rho_m = \varphi_m \in J^{\text{cusp, new}}_{k,m}$. Now, by using Theorem 1.6 we get that $F \,|\, \rho_m = 0$. □

Proof of Corollary 1.10. Let $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right)$ and $m = ax_0^2 + bx_0y_0 + cy_0^2$ for some $x_0, y_0 \in \mathbb{Z}$. Since m is squarefree, we have $\gcd(x_0, y_0) = 1$. Let $A = \left(\begin{smallmatrix} x_1 & x_0 \\ y_1 & y_0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $S = {}^tATA$, where ${}^tA$ denotes the transpose of A. Then the lower right entry of S would be m. We have $a_F(S) = a_F({}^tATA) = a_F(T) \neq 0$ and hence $F \,|\, \rho_m \neq 0$. Using Theorem 1.7, we get the corollary. □

Proof of Corollary 1.11. Since F is a Siegel eigenform, we have the nonvanishing of the first Fourier–Jacobi coefficient of F [7], that is, $F \,|\, \rho_1 \neq 0$. Since $F \,|\, \rho_1 \in J^{\text{cusp}}_{k,1}$, by using (6) we have the following theta decomposition $F \,|\, \rho_1 = h_0\,\theta_{1,0} + h_1\theta_{1,1}$. Since $F \,|\, \rho_1 \neq 0$, by using (7) we get that $h_0 \neq 0$. Since $h_0 \in S_{k-1/2}(4)$ and $\dim S_{k-1/2}(4) = k/2 - 2$ for k even, there exists an n_0 with $1 \leqslant n_0 \leqslant k/2 - 2$ such that the n_0-th Fourier coefficient $a_{h_0}(n_0)$ of h_0 is nonzero. Then we have

$$a_{h_0}(n_0) = a_F \begin{pmatrix} n_0 & 0 \\ 0 & 1 \end{pmatrix} \neq 0.$$

Now by using Corollary 1.10, we conclude the proof. □

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