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CONTINUOUS SOBOLEV FUNCTIONS WITH SINGULARITY ON ARBITRARY REAL-ANALYTIC SETS

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Near every point of a real-analytic set in $\mathbb{R}^n$, we make use of Hironaka's resolution of singularities theorem to construct a family of continuous functions in $W^{1,1}_{\text{loc}}$ such that their weak derivatives have (removable) singularities precisely on that set.

1. Introduction

Given a domain U in $\mathbb{R}^n$, $n \geq 1$, denote by $W^{k,p}_{\text{loc}}(U)$ the Sobolev space consisting of functions on U whose k-th order weak derivatives exist and belong to $L^p_{\text{loc}}(U)$, $k \in \mathbb{Z}^+$, $p \geq 1$. We investigate a Sobolev property for the reciprocals of logarithms of the modulus of real-analytic functions near their zero sets. Namely, given a real-analytic nonconstant function f on U, consider

$$v := \frac{1}{\ln |f|} \quad \text{on } U. \tag{1-1}$$

As we are solely interested in the Sobolev behavior of v near $f = 0$, and additional singularities would be introduced near $|f| = 1$, we further assume, say, $|f| < \frac{1}{2}$ on U. Consequently v is continuous on U. Letting $f^{-1}(0)$ be the zero set of f in U, we have $v|_{f^{-1}(0)} = 0$, and v is differentiable on $U \setminus f^{-1}(0)$. Note that $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) \geq 1$ in general.

According to a classical result of Stein [1993, pp. 71], $\ln |f| \in \text{BMO}$ for any polynomial f. On the other hand, Shi and Zhang [2022] showed that for a real-analytic f on U, if $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) \geq 2$, then $\ln |f| \in W^{1,1}_{\text{loc}}(U)$. It is important to note that this codimension assumption is essential and cannot be dropped. In comparison to this result, although v in (1-1) exhibits slightly greater regularity than $\ln |f|$, our first main theorem shows that v belongs to $W^{1,1}_{\text{loc}}(U)$ regardless of the codimension of $f^{-1}(0)$.

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Theorem 1.1. *Let U be a domain in $\mathbb{R}^n$, $n \geq 1$. Let f be a real-analytic nonconstant function on U and $|f| < \frac{1}{2}$ on U. Then*:

(1) $\frac{1}{\ln|f|} \in W^{1,1}_{\text{loc}}(U)$.

(2) *If* $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) = 1$, *then* $\frac{1}{\ln|f|} \notin W^{1,p}_{\text{loc}}(U)$ *for any* $p > 1$.

The main idea of the proof is to use the coarea formula to transform the integrals under consideration into new ones along level sets of the function f. The L^1-integrability and the L^p-nonintegrability for $p > 1$ that we seek are thus consequences of certain quantitative properties of the level sets of f, which can be conveniently established by utilizing the powerful Hironaka's resolution of singularities theorem and the Łojasiewicz gradient inequality. A novelty of Theorem 1.1 is to provide ample $W^{1,1}_{\text{loc}}$ functions. For instance, $\frac{1}{\ln|P(x)|} \in W^{1,1}_{\text{loc}}$ for any polynomial P near its zeros. It is also interesting to point out that Theorem 1.1 indicates that Sobolev spaces in general do not satisfy an openness property, in the sense that there exists a class of functions in $W^{k,p}_{\text{loc}}(U)$ for some $p \geq 1$ but not in $W^{k,q}_{\text{loc}}(U)$ for any $q > p$.

Unfortunately our method cannot be applied directly in the smooth category, due to the absence of a Hironaka-type resolution property for smooth functions. It is natural to wonder if there is an easy way to verify the optimal Sobolev property of v, say, for any finitely vanishing smooth function f. For instance, consider the function $f(x, y) := y^2 - \sin(e^{1/x}\pi)e^{-1/x^2}$, which is smooth near $0 \subset \mathbb{R}^2$ and vanishes to second order at 0. It turns out, with a straightforward computation, that $\frac{1}{\ln|f|} \in W^{1,1}$ near 0.

As a consequence of Theorem 1.1, the weak derivative ∇v exists on U. Specifically, this implies that the singularity set $f^{-1}(0)$ of ∇v in the classical sense is actually a removable singularity in the weak sense. In other words, Theorem 1.1 allows us to construct, for any given real-analytic set, a continuous function in $W^{1,1}_{\text{loc}}$ such that its weak derivative has a removable singularity precisely on that set.

Corollary 1.2. *Let A be a real-analytic set in $\mathbb{R}^n$. For every $p \in A$, there exists an open neighborhood V of p and a continuous function $u \in W^{1,1}_{\text{loc}}(V)$, such that the set of removable singularities of ∇u is $A \cap V$.*

Finally, we study the Sobolev property of v in the special case when f is a holomorphic function on $U \subset \mathbb{C}^n$. Note that in this case $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) = 2$ unless $f \neq 0$ on U.

Theorem 1.3. *Let U be a domain in $\mathbb{C}^n$. Let f be a holomorphic nonconstant function on U and $|f| < \frac{1}{2}$ on U. Then*:

(1) $\frac{1}{\ln|f|} \in W^{1,2}_{\mathrm{loc}}(U)$.

(2) *If* $f^{-1}(0) \neq \varnothing$, *then* $\frac{1}{\ln|f|} \notin W^{1,p}_{\mathrm{loc}}(U)$ *for any* $p > 2$.

Corollary 1.4. *Let* A *be a complex analytic set in* $\mathbb{C}^n$. *For every* $p \in A$, *there exists an open neighborhood* V *of* p *and a continuous function* $u \in W^{1,2}_{\mathrm{loc}}(V)$, *such that the set of removable singularities of* ∇u *is* $A \cap V$.

In view of Theorems 1.1 and 1.3, it seems to have suggested a correlation between the codimension of the level sets and the Sobolev integrability index. Thus, one may ask whether $v \in W^{1,d}_{\mathrm{loc}}(U)$ if $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) = d$ for some $0 \le d \le n$. Unfortunately we do not have an answer to this question in general.

2. Proof of Theorem 1.1

Recall that the coarea formula states that, given $\phi \in L^1(U)$ and a real-valued Lipschitz function f on U,

$$\int_U \phi(x)\,|\nabla f(x)|\,dV_x = \int_{-\infty}^{\infty} \int_{f^{-1}(t)} \phi(x)\,dS_x\,dt. \tag{2-1}$$

Here given $t \in \mathbb{R}$, S_x is the $(n-1)$-dimensional Hausdorff measure of the level set $f^{-1}(t)$ of f defined by

$$f^{-1}(t) := \{x \in U : f(x) = t\}.$$

Towards the proof of the main theorems, we shall fix the real-analytic (or holomorphic) function f and use the following notation: two quantities A and B are said to satisfy $A \lesssim B$ if $A \le CB$ for some constant $C > 0$ which depends only on the f under consideration. We say $A \gtrsim B$ if and only if $B \lesssim A$, and $A \approx B$ if and only if $A \lesssim B$ and $B \lesssim A$ at the same time.

Given a set $A \subset \mathbb{R}^n$, denote by $m(A)$ the Hausdorff measure of A at its Hausdorff dimension. We first utilize Hironaka's resolution of singularities theorem to show the Hausdorff measure of level sets of real-analytic functions is bounded (from above). This will be essential in proving a Harvey–Polking type removable singularity lemma for the weak derivatives of v.

Theorem 2.1 [Atiyah 1970]. *Let* f *be a real-analytic nonconstant function defined near a neighborhood of* $0 \in \mathbb{R}^n$. *Then there exists an open set* $U \subset \mathbb{R}^n$ *near* 0, *a real-analytic manifold* $\widetilde{U}$ *of dimension n and a proper real-analytic map* $\phi : \widetilde{U} \to U$ *such that*:

(1) *The function* $\phi : \widetilde{U} \setminus \widetilde{f^{-1}(0)} \to U \setminus f^{-1}(0)$ *is an isomorphism, where* $\widetilde{f^{-1}(0)} := \{p \in \widetilde{U} : \phi(p) \in f^{-1}(0)\}$.

(2) *For each* $p \in \widetilde{U}$, *there exist local real-analytic coordinates* $(y_1, \dots, y_n)$ *centered at* p, *such that near* p *one has*

$$f \circ \phi(y) = u(y) \cdot \prod_{i=1}^{n} y_i^{k_i},$$

where u *is real-analytic and* $u \neq 0$, $k_i \in \mathbb{Z}^+ \cup \{0\}$.

Lemma 2.2. *Let* f *be a real-analytic nonconstant function on* U. *Then*

$$m(f^{-1}(t)) \lesssim 1 \quad \textit{for all } |t| \ll 1.$$

Proof. Without loss of generality, assume $0 \in U$ and $f(0) = 0$. Under the setup of Hironaka's resolution Theorem 2.1, for every $p \in \widetilde{f^{-1}(0)}$, let $(\widetilde{V}, \psi)$ be a coordinate chart near p in $\widetilde{U}$ such that, for $y \in \psi(\widetilde{V}) \subset \mathbb{R}^n$,

$$f \circ \Phi(y) := f \circ \phi \circ \psi^{-1}(y) = u(y) \cdot \prod_{i=1}^{n} y_i^{k_i}.$$

By properness of ϕ, $V := \phi(\widetilde{V})$ is an open subset of U near $\phi(p)$. Since ϕ is smooth on $\widetilde{U}$, by shrinking U if necessary, $\Phi : \psi(\widetilde{V}) \to V$ is smooth up to the boundary of $\psi(\widetilde{V})$. By change of coordinates formula,

$$\begin{aligned} m(f^{-1}(t) \cap V) &= \int_{\{f(x)=t\} \cap \phi(\widetilde{V})} dS_x \\ &= \int_{\{f \circ \Phi(y)=t\} \cap \psi(\widetilde{V})} \Phi^* \, dS_x \lesssim \int_{\{f \circ \Phi(y)=t\} \cap \psi(\widetilde{V})} dS_y. \end{aligned}$$

Thus, in view of this and the fact that $u \neq 0$ on $\widetilde{U}$, the proof boils down to showing that the $(n-1)$-dimensional Hausdorff measure satisfies

$$m(A^n(t)) \lesssim 1 \quad \text{for all } 0 < t \ll 1, \tag{2-2}$$

where

$$A^n(t) = \left\{ y \in \mathbb{R}^n : \prod_{i=1}^{n} y_i^{k_i} = t, 0 < y_i < 1, i = 1, \dots, n \right\}. \tag{2-3}$$

Here the constant multiple for "$\lesssim$" in (2-2) is only dependent on k_i, $i = 1, \dots, n$. Clearly, one only needs to prove the case when all $k_i > 0$. Let $k := \sum_{i=1}^{n} k_i$.

We shall employ the mathematical induction on the dimension n to prove (2-2) for all level sets in the form of (2-3). The $n = 1$ case is trivial. Assume the $n = l$ case holds. Namely, for every level set $A^l(t)$ in $\mathbb{R}^l$ defined by (2-3), $m(A^l(t)) \lesssim 1$

for $0 < t \ll 1$. When the dimension n equals $l+1$, one first has

$$A^{l+1}(t) \subset \bigcup_{j=1}^{l+1} A_j^{l+1}(t),$$

where, for each $j = 1, \dots, l+1$,

$$A_j^{l+1}(t) := \left\{ y \in \mathbb{R}^{l+1} : t^{1/k} \le y_j < 1, 0 < y_i < 1 \text{ if } i \ne j, \text{ and } \prod_{\substack{1 \le i \le l+1 \\ i \ne j}} y_i^{k_i} = t y_j^{-k_j} \right\}.$$

Since $A_j^{l+1}(t)$ is a finite union of smooth hypersurfaces in $\mathbb{R}^{l+1}$ away from a set of dimension $l-1$, by Fubini's theorem, the l-dimensional Hausdorff measure satisfies

$$m(A_j^{l+1}(t)) = \int_{t^{1/k}}^{1} \int_{\prod_{1 \le i \le l+1, i \ne j} y_i^{k_i} = t y_j^{-k_j}, 0 < y_i < 1, i \ne j} dS_{\hat{y}_j}\, dy_j,$$

and thus

$$m(A^{l+1}(t)) \le \sum_{j=1}^{l+1} \int_{t^{1/k}}^{1} \int_{\prod_{1 \le i \le l+1, i \ne j} y_i^{k_i} = t y_j^{-k_j}, 0 < y_i < 1, i \ne j} dS_{\hat{y}_j}\, dy_j. \tag{2-4}$$

Further denote $\hat{y}_j := (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_{l+1}) \in \mathbb{R}^l$,

$$t' := t y_j^{-k_j},$$

and

$$A_j^l(t') := \left\{ \hat{y}_j \in \mathbb{R}^l : 0 < y_i < 1, i \ne j, \text{ and } \prod_{\substack{1 \le i \le l+1^l \\ i \ne j}} y_i^{k_i} = t' \right\}.$$

Noting that $t' < t^{1-k_j/k}$ when $y_j > t^{1/k}$, we obtain from (2-4)

$$m(A^{l+1}(t)) \le (1 - t^{1/k}) \sum_{j=1}^{l+1} \sup_{0 < t' < t^{1-k_j/k}} m(A_j^l(t')).$$

On the other hand, since $k_j < k$, one has $t^{1-k_j/k} \ll 1$ when $t \ll 1$. By the induction assumption and the fact that $A_j^l(t')$ is in $\mathbb{R}^l$,

$$\sup_{0 < t' < t^{1-k_j/k}} m(A_j^l(t')) \lesssim 1 \quad \text{for all } 0 < t \ll 1.$$

This finally gives

$$m(A^{l+1}(t)) \lesssim 1 \quad \text{for all } 0 < t \ll 1.$$

The lemma is proved. □

Lemma 2.3. *Given a real-analytic nonconstant function f on U with $|f| < \frac{1}{2}$ on U, let v be defined in* (1-1), *and*

$$g := \frac{\nabla f}{f \cdot (\ln |f|)^2} \quad \textit{on } U. \tag{2-5}$$

Then $g \in L^1_{\mathrm{loc}}(U)$. One has

$$\nabla v = g \quad \textit{on } U$$

in the sense of distributions.

Proof. First, we show that $g \in L^1_{\mathrm{loc}}(U)$. Since f is real-analytic on U, shrinking U if necessary, one can assume f to be (globally) Lipschitz on U. Making use of the coarea formula (2-1), one gets

$$\begin{aligned}\int_U |g(x)|\, dV_x &= \int_U \frac{|\nabla f(x)|}{|f(x)|\,(\ln |f(x)|)^2}\, dV_x \\ &\le \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{f^{-1}(t)} \frac{1}{|f(x)|\,(\ln |f(x)|)^2}\, dS_x\, dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{m(f^{-1}(t))}{|t|\,(\ln |t|)^2}\, dt.\end{aligned}$$

Lemma 2.2 further allows us to infer

$$\int_U |g(x)|\, dV_x \lesssim \int_0^{\frac{1}{2}} \frac{1}{t(\ln t)^2}\, dt = \int_{\ln 2}^{\infty} \frac{1}{s^2}\, ds < \infty.$$

Next, we show that, given any testing function $\eta \in C_c^\infty(U)$,

$$-\int_U v \nabla \eta = \int_U \eta g. \tag{2-6}$$

Since v is differentiable away from $f^{-1}(0)$, a direct computation gives

$$\nabla v = g \quad \text{on } U \setminus f^{-1}(0). \tag{2-7}$$

In particular, (2-6) is trivially true if $K := f^{-1}(0) \cap \operatorname{supp} \eta = \varnothing$.

If $K \neq \varnothing$, given $\epsilon > 0$ let

$$K_\epsilon := \{x \in U : \operatorname{dist}(x, K) \le \epsilon\},$$

where $\operatorname{dist}(x, K)$ is the distance function from x to the set K. Let $\rho_\epsilon \in C^\infty(U)$ be such that $\rho_\epsilon = 0$ in K_ϵ, $\rho_\epsilon = 1$ in $U \setminus K_{3\epsilon}$ and $|\nabla \rho_\epsilon| \lesssim \frac{1}{\epsilon}$ on U. See, for instance, [Hörmander 2003, Theorem 1.2.1-2]. Then $\rho_\epsilon \eta \in C_c^\infty(U \setminus f^{-1}(0))$. Using (2-7) we immediately have

$$-\int_U v \nabla(\rho_\epsilon \eta) = \int_U \rho_\epsilon \eta g,$$

or equivalently,

$$-\int_U v\eta\nabla\rho_\epsilon - \int_U v\rho_\epsilon\nabla\eta = \int_U \rho_\epsilon\eta g. \tag{2-8}$$

We shall prove

$$\lim_{\epsilon\to 0}\int_U v\eta\nabla\rho_\epsilon = 0. \tag{2-9}$$

If so, then passing $\epsilon \to 0$ in (2-8), we obtain the desired equality (2-6) as a consequence of Lebesgue's dominated convergence theorem.

To prove (2-9), first by the assumption on ρ_ϵ,

$$\left|\int_U v\eta\nabla\rho_\epsilon\right| = \left|\int_{K_{3\epsilon}\setminus K_\epsilon} v\eta\nabla\rho_\epsilon\right| \lesssim \frac{C}{\epsilon}\int_{K_{3\epsilon}\setminus K_\epsilon} |v| \tag{2-10}$$

for some constant C dependent only on η. Since f is Lipschitz on U, for any $x_0 \in f^{-1}(0)$, $|f(x)| = |f(x) - f(x_0)| \lesssim |x - x_0|$. In particular,

$$|f(x)| \lesssim \operatorname{dist}(x, f^{-1}(0)).$$

Thus for all $x \in K_{3\epsilon} \setminus K_\epsilon$ (equivalently, $\epsilon < \operatorname{dist}(x, f^{-1}(0)) < 3\epsilon$), one has

$$|v(x)| = \frac{1}{\big|\ln|f(x)|\big|} \lesssim \frac{1}{\big|\ln \operatorname{dist}(x, f^{-1}(0))\big|} \approx \frac{1}{|\ln\epsilon|}$$

for all ϵ small enough. Hence by (2-10)

$$\left|\int_U v\eta\nabla\rho_\epsilon\right| \lesssim \frac{Cm(K_{3\epsilon})}{\epsilon\,|\ln\epsilon|}. \tag{2-11}$$

On the other hand, according to a nontrivial result of Loeser [1986, Theorem 1.1] and its consequent remarks,

$$m(K_{3\epsilon}) \lesssim \epsilon^{\operatorname{codim}_{\mathbb{R}} f^{-1}(0)} \lesssim \epsilon.$$

Here the last inequality has used the fact that $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) \geq 1$ due to the real-analyticity of f. The equality (2-9) follows by combining the above with (2-11). □

Proof of Theorem 1.1. Since $|f| < \frac{1}{2}$, we have $\big|\ln|f|\big| > \ln 2$ and so $|v| < \frac{1}{\ln 2} \in L^\infty(U)$. Part (1) follows from this and Lemma 2.3. For part (2), we only need to show that the function g defined in (2-5) does not belong to L^p_{loc} for any $p > 1$ near any neighborhood of $f^{-1}(0)$.

First, according to the Łojasiewicz inequality, by shrinking U if necessary, there exists some constant $\beta \in (0, 1)$ such that

$$|\nabla f(x)| \gtrsim |f(x)|^\beta, \quad x \in U. \tag{2-12}$$

As a consequence of this,

$$\begin{aligned}\int_U |\nabla v(x)|^p \, dV_x &= \int_U \frac{|\nabla f(x)|}{|\nabla f(x)|^{-(p-1)}|f(x)|^p \big|\ln|f(x)|\big|^{2p}} \, dV_x \\ &\gtrsim \int_U \frac{|\nabla f(x)|}{|f(x)|^{p-(p-1)\beta}\big|\ln|f(x)|\big|^{2p}} \, dV_x.\end{aligned}$$

Utilizing the coarea formula, we have, for some $\epsilon_0 > 0$,

$$\begin{aligned}\int_U |\nabla v(x)|^p \, dV_x &\gtrsim \int_{-\epsilon_0}^{\epsilon_0} \int_{f^{-1}(t)} \frac{1}{|f(x)|^{p-(p-1)\beta}\big|\ln|f(x)|\big|^{2p}} \, dS_x \, dt \\ &= \int_{-\epsilon_0}^{\epsilon_0} \frac{m(f^{-1}(t))}{|t|^{p-(p-1)\beta}\big|\ln|t|\big|^{2p}} \, dt.\end{aligned}$$

Since $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) = 1$, there is some $x_0 \in f^{-1}(0) \cap U$, such that $|\nabla f(x_0)| \neq 0$. Let V be a neighborhood of x_0 in U such that $|\nabla f| \gtrsim 1$ on V. Then for all t small enough, $m(f^{-1}(t) \cap V) \gtrsim 1$. Consequently, $m(f^{-1}(t)) \gtrsim 1$ for $0 < t \ll 1$. Thus

$$\int_U |\nabla v(x)|^p \, dV_x \gtrsim \int_0^{\epsilon_0} \frac{1}{t^{p-(p-1)\beta}|\ln t|^{2p}} \, dt.$$

Note that $p - (p-1)\beta > 1$ necessarily when $p > 1$ and $\beta < 1$. Hence the last term is unbounded. The proof is complete. □

Proof of Corollary 1.2. Since A is real-analytic, there exists an open neighborhood $V \subset \mathbb{R}^n$ of p and a real-analytic function f on V such that $A \cap V = \{x \in V : f(x) = 0\}$. Then $u = \frac{1}{\ln|f|}$ is the desired function satisfying the assumptions. □

For functions (such as $\ln|f|$) with singularities, its composition with another logarithm typically exhibits reduced singularities. The following theorem shows that composing extra logarithms does not improve Sobolev regularity in general.

Theorem 2.4. *Let U be a domain in $\mathbb{R}^n$, $n \geq 1$. Let f be a real-analytic nonconstant function on U and $|f| < \frac{1}{10}$ on U. Then*:

(1) $\frac{1}{\ln\ln|f|} \in W^{1,1}_{\mathrm{loc}}(U)$.

(2) *If* $\operatorname{codim}_{\mathbb{R}} f^{-1}(0) = 1$, *then* $\frac{1}{\ln\ln|f|} \notin W^{1,p}_{\mathrm{loc}}(U)$ *for any* $p > 1$.

Proof. Applying a similar approach as in the proof of Lemma 2.3, we first have

$$\nabla\left(\frac{1}{\ln\big|\ln|f|\big|}\right) = \frac{\nabla f}{f \cdot \ln|f| \cdot \big(\ln\big|\ln|f|\big|\big)^2} \quad \text{on } U$$

in the sense of distributions. Making use of the coarea formula and Lemma 2.2,

$$\begin{aligned}\int_U \left|\nabla\left(\frac{1}{\ln\left|\ln|f|\right|}\right)\right| &\le \int_{-\frac{1}{10}}^{\frac{1}{10}} \int_{f^{-1}(t)} \frac{1}{|f(x)|\,\left|\ln|f(x)|\right|\left(\ln\left|\ln|f(x)|\right|\right)^2}\,dS_x\,dt\\ &\lesssim \int_0^{\frac{1}{10}} \frac{1}{t|\ln t|(\ln|\ln t|)^2}\,dt\\ &= \int_{\ln 10}^{\infty} \frac{1}{t(|\ln t|)^2}\,dt = \int_{\ln\ln 10}^{\infty} \frac{1}{t^2}\,dt \lesssim 1.\end{aligned}$$

In the case when $p > 1$, there exists some $0 < \beta < 1$ by (2-12), and some small $\epsilon_0 > 0$ such that

$$\begin{aligned}\int_U \left|\nabla\left(\frac{1}{\ln\left|\ln|f|\right|}\right)\right|^p &\gtrsim \int_U \frac{|\nabla f(x)|}{|f(x)|^{p-(p-1)\beta}\left|\ln|f(x)|\right|^p\left(\ln\left|\ln|f(x)|\right|\right)^{2p}}\,dV_x\\ &\gtrsim \int_0^{\epsilon_0} \frac{1}{t^{p-(p-1)\beta}|\ln t|\,(\ln|\ln t|)^2}\,dt.\end{aligned}$$

Since $p-(p-1)\beta > 1$, the last term is divergent. This completes the proof of the theorem. ☐

3. Proof of Theorem 1.3

To prove Theorem 1.3 for holomorphic functions, we shall need the following well-known complex version Hironaka's resolution of singularities theorem. See, for instance, [Smith 2016].

Theorem 3.1. *Let f be a holomorphic function defined near a neighborhood of $0 \in \mathbb{C}^n$. Then there exists an open set $U \subset \mathbb{C}^n$ near 0, a complex manifold $\widetilde{U}$ of dimension n and a proper holomorphic map $\phi : \widetilde{U} \to U$ such that*:

(1) *The function $\phi : \widetilde{U}\setminus\widetilde{f^{-1}(0)} \to U\setminus f^{-1}(0)$ is a biholomorphism, where $\widetilde{f^{-1}(0)} := \{p \in \widetilde{U} : \phi(p) \in f^{-1}(0)\}$.*

(2) *For each $p \in \widetilde{U}$, there exist local holomorphic coordinates $(w_1, \dots, w_n)$ centered at p, such that near p one has*

$$f \circ \phi(w) = u(w) \cdot \prod_{i=1}^{n} w_i^{k_i},$$

where u is holomorphic and $u \neq 0$, $k_i \in \mathbb{Z}^+ \cup \{0\}$.

Proof of Theorem 1.3. (1) Since $\bar\partial f = 0$, and according to Lemma 2.3,

$$\partial v = \frac{\partial f}{2f \cdot (\ln|f|)^2} \in L^1_{\mathrm{loc}}(U)$$

in the sense of distributions, we only need to show that

$$\frac{\partial f}{f\cdot(\ln|f|)^2}\in L^2_{\mathrm{loc}}(U).$$

On the other hand, making use of Hironaka's resolution of singularities Theorem 3.1 for holomorphic functions, for every $p\in\widetilde{f^{-1}(0)}$, let $(\widetilde{V},\psi)$ be a coordinate chart near p in $\widetilde{U}$ such that, for $w\in\psi(\widetilde{V})\subset\{w\in\mathbb{C}^n:|w_j|<\frac{1}{2}\}$,

$$\tilde{f}(w):=f\circ\phi\circ\psi^{-1}(w)=u(w)\cdot\prod_{i=1}^{n}w_i^{k_i},$$

where $u\neq 0$ on $\psi(\widetilde{V})$ and $k_i\in\mathbb{Z}^+\cup\{0\}$. Let $V:=\phi(\widetilde{V})$, $\Phi:=\phi\circ\psi^{-1}$, and Jac_Φ be the complex Jacobian of the holomorphic map Φ. Note that the inverse matrix $(\mathrm{Jac}_\Phi)^{-1}$ is smooth on $\psi(\widetilde{V}\setminus\widetilde{f^{-1}(0)})$, and

$$|(\mathrm{Jac}_\Phi)^{-1}(w)\cdot\det(\mathrm{Jac}_\Phi)(w)|\lesssim 1\quad\text{for all }\ w\in\psi(\widetilde{V}\setminus\widetilde{f^{-1}(0)}).$$

By change of variables formula,

$$\begin{aligned}\int_V&\frac{|\partial_z f(z)|^2}{|f(z)|^2(\ln|f(z)|)^4}\,dV_z\\ &=\int_{\Phi^{-1}(V\setminus f^{-1}(0))}\Phi^*\biggl(\frac{|\partial_z f(z)|^2}{|f(z)|^2(\ln|f(z)|)^4}\,dV_z\biggr)\\ &\lesssim\int_{\psi(\widetilde{V}\setminus\widetilde{f^{-1}(0)})}\frac{|\partial_w\tilde{f}(w)|^2|(\mathrm{Jac}_\Phi)^{-1}(w)|^2}{|\tilde{f}(w)|^2(\ln|\tilde{f}(w)|)^4}\bigl|\det(\mathrm{Jac}_\Phi(w))\bigr|^2\,dV_w\\ &\lesssim\int_{\psi(\widetilde{V})}\frac{|\partial_w\tilde{f}(w)|^2}{|\tilde{f}(w)|^2(\ln|\tilde{f}(w)|)^4}\,dV_w.\end{aligned}$$

Thus, the proof boils down to showing that, for $j=1,\dots,n$,

$$\int_{\psi(\widetilde{V})}\frac{|\partial_{w_j}\tilde{f}(w)|^2}{|\tilde{f}(w)|^2(\ln|\tilde{f}(w)|)^4}\,dV_w\lesssim 1. \tag{3-1}$$

For simplicity, let $j=1$ in (3-1). If $k_1=0$, then

$$\partial_{w_1}\tilde{f}(w)=\partial_{w_1}u(w)\cdot\prod_{i=1}^{n}w_i^{k_i}.$$

Since $\frac{1}{(\ln|\tilde{f}(w)|)^4}\lesssim 1$ and $u\neq 0$, when w is near 0,

$$\frac{|\partial_{w_1}\tilde{f}(w)|^2}{|\tilde{f}(w)|^2(\ln|\tilde{f}(w)|)^4}=\frac{|\partial_{w_1}u(w)|^2}{|u(w)|^2(\ln|\tilde{f}(w)|)^4}\lesssim 1.$$

So (3-1) holds. If $k_1 > 0$, then

$$\partial_{w_1}\tilde{f}(w) = \partial_{w_1}u(w)\cdot\prod_{i=1}^{n} w_i^{k_i} + k_1 u(w)\cdot w_1^{k_1-1}\cdot\prod_{i=2}^{n} w_i^{k_i}.$$

Hence

$$\begin{aligned}\frac{|\partial_{w_1}\tilde{f}(w)|^2}{|\tilde{f}(w)|^2(\ln|\tilde{f}(w)|)^4} &\lesssim \frac{|\partial_{w_1}u(w)|^2}{|u(w)|^2(\ln|\tilde{f}(w)|)^4} + \frac{k_1^2}{|w_1|^2(\ln|\tilde{f}(w)|)^4}\\ &\lesssim 1 + \frac{1}{|w_1|^2(\ln|\tilde{f}(w)|)^4}.\end{aligned}$$

Note that when w is close to 0,

$$\big|\ln|\tilde{f}(w)|\big| = \left|\ln|u(w)| + \sum_{i=1}^{n} k_i \ln|w_i|\right| \gtrsim -\ln|w_1|. \tag{3-2}$$

This leads to

$$\begin{aligned}\int_{\psi(\tilde{V})} \frac{|\partial_{w_1}\tilde{f}(w)|^2}{|\tilde{f}(w)|^2(\ln|\tilde{f}(w)|)^4}\,dV_w &\lesssim 1 + \int_{\psi(\tilde{V})} \frac{1}{|w_1|^2\big|\ln|w_1|\big|^4}\,dV_w\\ &\lesssim 1 + \int_0^{\frac{1}{2}} \frac{1}{s(\ln s)^4}\,ds \lesssim 1.\end{aligned}$$

Equation (3-1) and thus part (1) are proved.

(2) Let U_1 be an open subset of U such that $f^{-1}(0)\cap U_1$ is regular. Then there exists a holomorphic coordinate change on U_1 such that under the new coordinates $(w_1,\ldots,w_n)$, one has $w_n = f(z)$. As a consequence of this,

$$\begin{aligned}\int_U \left|\frac{\partial_z f}{f\cdot(\ln|f|)^2}\right|^p dV_z &\ge \int_{U_1}\left|\frac{\partial_z f}{f\cdot(\ln|f|)^2}\right|^p dV_z\\ &\approx \int_{U_1}\frac{1}{|w_n|^p\big|\ln|w_n|\big|^{2p}}\,dV_w \gtrsim \int_0^{\epsilon_0}\frac{1}{s^{p-1}|\ln s|^{2p}}\,ds\end{aligned}$$

for some $\epsilon_0 > 0$. Since $p > 2$, the last term is unbounded. This proves part (2). □

Proof of Corollary 1.4. The proof is similar to that of Corollary 1.2, with Theorem 1.1 substituted by Theorem 1.3, and is omitted. □

An application of Theorem 1.3 is to provide ample data to the $\bar\partial$ problem in complex analysis, in particular, within the framework of Hörmander's classical L^2 theory for $\bar\partial$-closed forms with L^2_{loc} coefficients. Normally, generating smooth data is straightforward. In the following, we construct data with singularity on complex analytic varieties, where Hörmander's theory can still be applied.

Example 1. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. Let f be a nonconstant holomorphic function on Ω such that $f^{-1}(0) \neq \varnothing$. Choose a monotone increasing function $\chi \in C^\infty([0, \infty))$ such that $\chi(t) = t$ if $0 \leq t \leq \frac{1}{4}$, and $\chi(t) = \frac{1}{3}$ if $t \geq 1$. Then $g = \frac{1}{\ln \chi(|f|)} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ by Theorem 1.3. Furthermore, $u := \bar{\partial} g$ is a $\bar{\partial}$-closed $(0, 1)$ form with L^2_{loc} coefficients with singularities precisely on $f^{-1}(0)$.

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