GRADING OF AFFINIZED WEYL SEMIGROUPS OF KAC–MOODY TYPE

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For any Kac–Moody root datum $\mathcal{D}$, D. Muthiah and D. Orr have defined a partial order on the semidirect product W_+^a of the integral Tits cone with the vectorial Weyl group of $\mathcal{D}$, and a compatible length function. We classify covers for this order and show that this length function defines a $\mathbb{Z}$-grading of W_+^a, generalizing the case of affine ADE root systems and giving a positive answer to a conjecture of Muthiah and Orr.

Introduction

Motivation.

Reductive groups over p-adic fields. Let G be a split reductive group scheme with the data of a Borel subgroup B containing a maximal torus T. Let $W = N_G(T)/T$

be its vectorial Weyl group and Y be its coweight lattice: $Y = \mathrm{Hom}(\mathbb{G}_m, \boldsymbol{T})$. The action of W on $\boldsymbol{T}$ induces an action of W on Y and allows us to form the semidirect product $W^a = Y \rtimes W$. This group, called the extended affinized Weyl group of $\boldsymbol{G}$, appears naturally in the geometry and the representation theory of $\boldsymbol{G}$ over discretely valued fields. A foundational work in this regard was done by N. Iwahori and H. Matsumoto [1965], when they exhibited a Bruhat decomposition of $\boldsymbol{G}(\mathbb{Q}_p)$ indexed by W^a.

Let $\mathcal{K}$ be a nonarchimedean local field with ring of integers $\mathcal{O}_{\mathcal{K}} \subset \mathcal{K}$, uniformizer $\pi \in \mathcal{O}_{\mathcal{K}}$ and residue field $\Bbbk_{\mathcal{K}} = \mathcal{O}_{\mathcal{K}}/\pi$. Let $G = \boldsymbol{G}(\mathcal{K})$, let $K = \boldsymbol{G}(\mathcal{O}_{\mathcal{K}})$ be its integral points and let I be its Iwahori subgroup, defined as

$$I = \{g \in K \mid g \in \boldsymbol{B}(\Bbbk_{\mathcal{K}}) \mod \pi\}.$$

The extended affinized Weyl group can be understood as $N_G(\boldsymbol{T}(\mathcal{K}))/\boldsymbol{T}(\mathcal{O}_{\mathcal{K}})$, so it admits a lift in G. Then, G admits a decomposition in I-double cosets indexed by W^a, the Iwahori–Matsumoto–Bruhat decomposition:

$$(0.1) \qquad G = \bigsqcup_{\pi^\lambda w \in W^a} I\pi^\lambda wI.$$

The group W^a is a finite extension of a Coxeter group and thus admits a Bruhat order which arises from the geometry of the homogeneous space G/I: for any $\pi^\lambda w \in W^a$, $I\pi^\lambda wI$ is a subvariety of pure dimension $\ell(\pi^\lambda w)$ in G/I, and its closure admits a disjoint decomposition in I orbits:

$$(0.2) \qquad \overline{I\pi^\lambda wI} = \bigsqcup_{\pi^\mu v \leq \pi^\lambda w} I\pi^\mu vI,$$

which extends the Iwahori–Matsumoto decomposition. The connection between the geometry of G/I and the combinatorial structure of W^a is deeper. In particular, R-Kazhdan–Lusztig polynomials introduced by Kazhdan and Lusztig [1980], defined as the number of points of certain intersections in G/I, are also given by a recursive formula based on the Bruhat order and the Bruhat length of W^a.

These polynomials appear in many topics around reductive groups over local fields, we aim to develop analogous polynomials when $\boldsymbol{G}$ is replaced by a general Kac–Moody group.

Extension to Kac–Moody groups. Replace $\boldsymbol{G}$ by a general split Kac–Moody group. Kac–Moody group functors are entirely defined by the underlying Kac–Moody root datum $\mathcal{D}$, as defined in [Rémy 2002, §2], and reductive groups correspond to root data of finite type. Then the Iwahori–Matsumoto decomposition no longer holds on $G = \boldsymbol{G}(\mathcal{K})$. However there is a partial Iwahori–Matsumoto decomposition: there

exists a subsemigroup G^+ of G such that

$$(0.3) \qquad G^+ = \bigsqcup_{\pi^\lambda w \in W^a_+} I\pi^\lambda wI.$$

The indexing set for this decomposition W^a_+ is a subsemigroup of $W^a = Y \rtimes W$, and it appears naturally in other related contexts, for example, when trying to construct an Iwahori–Hecke algebra for G [Braverman et al. 2016; Bardy-Panse et al. 2016]. Let us briefly explain how W^a_+ is defined.

Let Φ be the real root system of the root datum $\mathcal{D}$. It is an infinite set (unless $\mathcal{D}$ is reductive) of linear forms on Y coming with a subset of positive roots $\Phi_+ \subset \Phi$ such that $\Phi = \Phi_+ \sqcup -\Phi_+$. Let $Y^{++} = \{\lambda \in Y \mid \forall \alpha \in \Phi_+,\ \alpha(\lambda) \geq 0\}$ and $Y^+ = W \cdot Y^{++}$. Then W^a_+ is defined as $Y^+ \rtimes W$. In the reductive case, Y^+ coincides with Y and thus $W^a_+ = W^a$. However, W^a can no longer be conceived as a finite extension of a Coxeter system, hence there is a priori no Bruhat order on W^a_+, let alone on W^a. A well-behaved topology on G^+/I would allow us to define an order on W^a_+ through the analog of decomposition (0.2), but G^+/I does not seem to have a natural variety, nor even an ind-variety structure.

An order and two lengths on W^a_+. In Appendix B2 of their article on the construction of an Iwahori–Hecke algebra for G an affine Kac–Moody group over a p-adic field [Braverman et al. 2016], A. Braverman, D. Kazhdan and M. Patnaik propose the definition of a preorder on W^a_+ which would replace the Bruhat order of W^a and they conjecture that it is a partial order. In [2018], D. Muthiah extends the definition of this preorder to any Kac–Moody group G, defines a $\mathbb{Z} \oplus \varepsilon\mathbb{Z}$-valued length compatible with this preorder and hence shows that it is an order. In [2019], D. Muthiah and D. Orr then show that this length can be evaluated at $\varepsilon = 1$ to obtain a $\mathbb{Z}$-valued length strictly compatible with the order on W^a_+.

In order to build a Kazhdan–Lusztig theory of p-adic Kac–Moody groups, we want to understand how close this order is to the Bruhat order of an affine Coxeter group, which properties still hold and which do not. The definition of a $\mathbb{Z}$-length is already a significant step, but many important properties, which are known to hold for Bruhat orders, remain unknown in this context. Several were proved only for Kac–Moody root systems of affine simply laced type using the specific structure of an affinized Weyl group of W in this context.

Choice of vocabulary. The order on W^a_+ is often mentioned in the literature as "the double affine Bruhat order" and the associated length as "the double affine Bruhat length" because it is most studied in the case of G a Kac–Moody group of affine type (in which case W is an affine Weyl group). We refer to it as "the affinized

Bruhat order" and "the affinized Bruhat length", denoted by ℓ^a, because we do not suppose that W is an affine Weyl group. Note that, if W is finite, then the affinized Bruhat length and order on W_+^a are just the ones induced by its Coxeter group structure.

Main result. Our main result is a positive answer to [Muthiah and Orr 2019, Conjecture 1.5] in full generality.

For any partial order $\leq$ on a set X, we say that $\boldsymbol{y}$ covers $\boldsymbol{x}$ if $\boldsymbol{x} \neq \boldsymbol{y}$ and $\{\boldsymbol{z} \in X \mid \boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{y}\} = \{\boldsymbol{x}, \boldsymbol{y}\}$. A grading of X is a length function ℓ on X strictly compatible with $\leq$ and such that $\boldsymbol{y}$ covers $\boldsymbol{x}$ if and only if $\boldsymbol{x} \leq \boldsymbol{y}$ and $\ell(\boldsymbol{y}) - \ell(\boldsymbol{x}) = 1$. Gradings thus give an easy classification of covers and more generally of saturated chains in X. The Bruhat length for a Coxeter group equipped with the Bruhat order is the prototypical example of a grading.

Muthiah and Orr [2019] prove that if Φ is of affine ADE type, the affinized Bruhat length gives a $\mathbb{Z}$-grading of W_+^a for the affinized Bruhat order and conjecture this to be true in general. Our main result is a positive answer to this conjecture:

Theorem A. *Let $\mathcal{D}$ be any Kac–Moody root datum. Then the affinized length ℓ^a on W_+^a defines a $\mathbb{Z}$-grading of W_+^a strictly compatible with the affinized Bruhat order. Otherwise said, let $\boldsymbol{x}, \boldsymbol{y} \in W_+^a$ be such that $\boldsymbol{x} \leq \boldsymbol{y}$. Then*

$$(0.4) \qquad \boldsymbol{y} \text{ covers } \boldsymbol{x} \text{ if and only if } \ell^a(\boldsymbol{y}) - \ell^a(\boldsymbol{x}) = 1.$$

Along the way, we obtain several geometric properties of covers for the affinized Bruhat order which we expect to be insightful even if the root datum is reductive (so W is finite and W_+^a is an affine Weyl group) as they only rely on the Coxeter structure of W. In particular, we obtain in Proposition 3.20 a classification of covers which generalize results obtained using quantum Bruhat graphs, in the reductive setting by T. Lam and M. Shimozono [2010, Proposition 4.4] and F. Schremmer [2024, Proposition 4.5], and in the affine simply laced setting by A. Welch [2022, Theorem 2].

Further directions. In an upcoming joint work with A. Hébert, we prove that any element of W_+^a admits a finite number of covers for the affinized Bruhat order. We use this finiteness in the context of masures to define R-Kazhdan–Lusztig polynomials, following Muthiah's strategy exposed in [2019] and the work on twin masures of N. Bardy-Panse, A. Hébert and G. Rousseau [Bardy-Panse et al. 2022]. Our understanding of covers is useful to compute these R-polynomials and we intend to use R-polynomials to define P-Kazhdan–Lusztig polynomials.

Another interesting (but quite long reach) question is the following: W_+^a appears as the affinization of W, which may be taken as an affinized version of a finite

Coxeter group. Can we iterate the affinization process, e.g., to obtain a valid theory for reductive or Kac–Moody groups on valued fields of higher dimensions?

Lastly, little is known on the preorder defined on the whole semidirect product W^a; it could be insightful to study it and to connect it to the failure of the full Iwahori–Matsumoto decomposition of G.

Organization of the paper.

Proof strategy. The global strategy is to construct a nontrivial chain from x to y every time $y \geq x$ satisfies $\ell^a(y) - \ell^a(x) > 1$. Let proj^{Y^+} denote the projection $W_+^a = Y^+ \rtimes W \to Y^+$. We distinguish two cases which depend on the form of x and y: The first case is when $\mathrm{proj}^{Y^+}(y)$ lies in the orbit of $\mathrm{proj}^{Y^+}(x)$, we call such covers the vectorial covers. The other case is when $\mathrm{proj}^{Y^+}(y) \notin W \cdot \mathrm{proj}^{Y^+}(x)$, we call such covers the properly affine covers.

For vectorial covers we show that the affinized Bruhat order on the set $\{z \in W_+^a \mid x \leq z \leq y\}$ is, in some sense, a lift of several Bruhat-like orders on W. We are then able to construct chains between x and y from chains in W, and we deduce a classification of vectorial covers. The characterization of properly affine covers is, at first glance, more involved. Through a careful study of the relation between the vectorial chamber containing $\mathrm{proj}^{Y^+}(x)$ and the vectorial chamber containing $\mathrm{proj}^{Y^+}(y)$, we show that the length difference $\ell^a(y) - \ell^a(x)$ can be rewritten in a more workable form, making clear the conditions for which it is equal to one. Then the difficulty is to build, explicitly, a nontrivial chain every time one of these conditions is not satisfied.

Organization. Section 1 consists of preliminaries. In Section 1.1 we formally define everything we mentioned in this introduction. In particular we give the definition of the affinized Bruhat order and the two affinized Bruhat lengths as they are given in [Muthiah and Orr 2019]. To be more flexible, we chose to define the affinized Bruhat preorder on the whole affinized Weyl group $W^a = Y \rtimes W$, on which it may not be an order.

We show, amongst other preliminary results, that we indeed recover the affinized Bruhat order on W_+^a from this preorder in Section 1.3.

We also give, in Section 1.2, a geometric interpretation of W_+^a and its affinized Bruhat order, which is to be compared with the interpretation of the Bruhat order in the Coxeter complex of a Coxeter group. Even though it is not clearly mentioned in the rest of the paper, this geometric interpretation was very useful to construct chains and understand W_+^a.

In Section 2, we prove Theorem A for vectorial covers. We define relative versions of the Bruhat order on W in Section 2.1 and we connect these relative

Bruhat orders to the affinized Bruhat length in Section 2.2. This is enough to prove Theorem A when $\text{proj}^{Y^+}(y) = \text{proj}^{Y^+}(x)$ (see Theorem 2.13). Using finer results on parabolic quotients in Section 2.3, we extend it to vectorial covers such that $\text{proj}^{Y^+}(y) \in W \cdot \text{proj}^{Y^+}(x) \setminus \{\text{proj}^{Y^+}(x)\}$ (see Theorem 2.18).

In Section 3, we deal with properly affine covers. We first show in Section 3.1 that these covers are of a very specific form. Namely, if $x = \pi^{v(\lambda)}w$ with $v, w \in W$ and $\lambda \in Y^{++}$, then y needs to be of the form $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ or $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ for some $\beta \in \Phi_+$.

The strategy is then to get enough necessary conditions on v, w, λ, β for y to cover x, in order to obtain a simplified expression for $\ell^a(y) - \ell^a(x)$. Proposition 3.3 gives a first result in this direction. In Section 3.2 we fully exploit this strategy to obtain (3.14) for the length difference.

Finally, in Sections 3.3 and 3.4, we construct various chains from x to y to prove that the quantities appearing in (3.14) need to be minimal when y covers x, which allows us to conclude the argument in Section 3.5.

1. Preliminaries

1.1. *Definitions and notation.* Let $\mathcal{D} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ be a Kac–Moody root datum as defined in [Rémy 2002, §8]. It is a quintuplet such that:

(1) I is a finite indexing set and $A = (a_{ij})_{(i,j) \in I \times I}$ is a generalized Cartan matrix.

(2) X and Y are two dual free $\mathbb{Z}$-modules of finite rank, and we write $\langle \cdot, \cdot \rangle$ for the duality bracket.

(3) $(\alpha_i)_{i \in I}$ (resp. $(\alpha_i^\vee)_{i \in I}$) is a family of linearly independent elements of X (resp. Y): the simple roots (resp. simple coroots).

(4) For all $(i, j) \in I^2$ we have $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$.

1.1.1. *Vectorial Weyl group.* For every $i \in I$ set $s_i \in \text{Aut}_{\mathbb{Z}}(X) : x \mapsto x - \langle \alpha_i^\vee, x \rangle \alpha_i$. The generated group $W = \langle s_i \mid i \in I \rangle$ is the *vectorial Weyl group* of the Kac–Moody root datum.

The duality bracket $\langle \cdot, \cdot \rangle$ induces a contragredient action of W on Y, explicitly $s_i(y) = y - \langle y, \alpha_i \rangle \alpha_i^\vee$. The bracket is then W-invariant.

The vectorial Weyl group W is a Coxeter group with set of simple reflections $S = \{s_i \mid i \in I\}$; in particular it has a Bruhat order $<$ and a length function ℓ compatible with the Bruhat order. We refer to [Björner and Brenti 2005] for general definitions and properties of Coxeter groups. A *reflection* in a Coxeter group is any element conjugated to a simple reflection.

1.1.2. *Real roots.* Let $\Phi = W \cdot \{\alpha_i \mid i \in I\}$ be the set of real roots of $\mathcal{D}$. It is a root system in the classical sense, but possibly infinite. In particular let $\Phi_+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i \cap \Phi$ be the set of positive real roots. Then $\Phi = \Phi_+ \sqcup -\Phi_+$, and we write $\Phi_- = -\Phi_+$ for the set of negative roots.

The set $\Phi^\vee = W \cdot \{\alpha_i^\vee \mid i \in I\}$ is the set of *coroots*, and its subset $\Phi_+^\vee = \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee \cap \Phi^\vee$ is the set of *positive coroots*.

To each root β corresponds a unique coroot $\beta^\vee$: if $\beta = w(\alpha_i)$ then $\beta^\vee = w(\alpha_i^\vee)$. This map $\beta \mapsto \beta^\vee$ is well defined, bijective between Φ and $\Phi^\vee$ and sends positive roots to positive coroots. Note that $\langle \beta^\vee, \beta \rangle = 2$ for all $\beta \in \Phi$.

To each root β we associate a reflection $s_\beta \in W$: if $\beta = w(\pm\alpha_i)$ then $s_\beta := w s_i w^{-1}$. Explicitly it is the map $X \to X$ defined by $s_\beta(x) = x - \langle \beta^\vee, x \rangle \beta$. For any $\beta \in \Phi$ we have $s_\beta = s_{-\beta}$ and the map $\beta \mapsto s_\beta$ forms a bijection between the set of positive roots and the set $\{w s_i w^{-1} \mid (w, i) \in W \times I\}$ of reflections of W.

1.1.3. *Inversion sets.* For any $w \in W$, let

$$\text{Inv}(w) = \Phi_+ \cap w^{-1}(\Phi_-) = \{\alpha \in \Phi_+ \mid w(\alpha) \in \Phi_-\}.$$

These sets are strongly connected to the Bruhat order, as by [Kumar 2002, 1.3.13], for all $\alpha \in \Phi_+$

$$(1.1) \qquad \alpha \in \text{Inv}(w) \iff w s_\alpha < w \iff s_\alpha w^{-1} < w^{-1}.$$

They are related to the Bruhat length: $\ell(w) = |\text{Inv}(w)|$ [Kumar 2002, 1.3.14].

1.1.4. *Fundamental chamber and Tits cone.* We define the (closed) integral fundamental chamber by $Y^{++} = \{\lambda \in Y \mid \langle \lambda, \alpha_i \rangle \geq 0 \; \forall i \in I\}$. If $\lambda \in Y^{++}$, we say that it is a *dominant coweight*. Then, the integral Tits cone is $Y^+ := \bigcup_{w \in W} w(Y^{++})$. It is a convex cone of Y; in particular it is a semigroup for the group operation of Y, and it is equal to Y if and only if W is finite, if and only if Φ is finite, if and only if A is of finite type (see [Kumar 2002, 1.4.2]).

The integral fundamental chamber Y^{++} is a fundamental domain for the action of W on Y^+, and for any $\lambda \in Y^+$ we define λ^{++} to be the unique element of Y^{++} in its W-orbit.

There is a height function on Y^+, defined as follows:

Definition 1.1. Let $(\Lambda_i)_{i \in I}$ be a set of fundamental weights, that is to say $\langle \alpha_i^\vee, \Lambda_i \rangle = \delta_{ij}$ for any $i, j \in I$. We fix it once and for all. Let $\rho = \sum_{i \in I} \Lambda_i$. Then for any $\lambda \in Y$ define the *height* of λ as

$$(1.2) \qquad \text{ht}(\lambda) = \langle \lambda, \rho \rangle.$$

The height depends on the choice of fundamental weights, but its restriction to $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ do not:

$$\mathrm{ht}\Big(\sum_{i \in I} n_i \alpha_i^\vee \Big) = \sum_{i \in I} n_i.$$

Remark 1.2. The height function takes integral values on $Q^\vee$, but not necessarily on Y. In general, one can choose the fundamental weights such that $\mathrm{ht}(Y) \subset \frac{1}{N_{\mathrm{ht}}}\mathbb{Z}$ for some $N_{\mathrm{ht}} \in \mathbb{Z}_{>0}$. As noted by D. Muthiah and A. Puskás [2024, Remark 2.13], if $\mathcal{D}$ is of finite or affine type then the fundamental weights may be chosen such that $N_{\mathrm{ht}} \in \{1, 2\}$, but for more general Kac–Moody root systems the optimal choice for N_{ht} may be arbitrarily large.

1.1.5. *Parabolic subgroups, minimal coset representatives.* For $\lambda \in Y^+$, let Φ_λ denote the set $\{\alpha \in \Phi \mid \langle \lambda, \alpha \rangle = 0\}$ and $W_\lambda = \mathrm{Stab}_W(\lambda)$. We say that λ is *regular* if $\Phi_\lambda = 0$, or equivalently if $W_\lambda = 1_W$. More generally we say that λ is *spherical* if W_λ is finite.

Let $v \in W$ be such that $\lambda = v\lambda^{++}$. Then $W_\lambda v = v W_{\lambda^{++}}$ and, since λ^{++} is dominant, $W_{\lambda^{++}}$ is a standard parabolic subgroup, that is, a group of the form $W_J = \langle s \mid s \in J \rangle$ where $J \subset S$ is a set of simple reflections. More precisely, $J = \{s \in S \mid s(\lambda^{++}) = \lambda^{++}\}$.

By standard Coxeter group theory (see, for instance, [Björner and Brenti 2005, Section 2.2]), for any $u \in W$, the left coset $u W_{\lambda^{++}} = u W_J$ has a unique representative of minimal length which we denote by u^J, and one has a decomposition $u = u^J u_J$ with $u_J \in W_J$ such that

$$(1.3) \qquad\qquad \ell(u) = \ell(u^J) + \ell(u_J).$$

Notation 1.3. (1) For any $J \subset S$, we denote by W^J the set of minimal length representatives for W_J-cosets in W:

$$(1.4) \quad w \in W^J \iff \forall \widetilde{w} \in W_J, \, \ell(w\widetilde{w}) > \ell(w) \iff \forall s \in J, \, \ell(ws) > \ell(w).$$

If $\lambda \in Y^{++}$ is such that $W_\lambda = W_J$, then we may use W^λ as an alternative notation for W^J.

(2) For any $\lambda \in Y^+$ (not necessarily dominant), we denote by v^λ the minimal length element in W which satisfies $\lambda = v^\lambda \lambda^{++}$:

$$(1.5) \qquad\qquad v^\lambda = \min\{v \in W \mid \lambda = v\lambda^{++}\}.$$

In other words, for any $u \in W$ such that $\lambda = u\lambda^{++}$, we have $v^\lambda = u^J$, where J is the set of simple reflections such that $W_J = W_{\lambda^{++}}$.

1.1.6. *Affinized Weyl semigroup.* The action of W on Y allows us to form the semidirect product $Y \rtimes W$, which we denote by W^a. We denote its elements by $\pi^\lambda w$ with $\lambda \in Y$, $w \in W$.

By definition, $Y^+ \subset Y$ is stable by the action of W on Y; therefore we can form $W^a_+ = Y^+ \rtimes W$ which is a subsemigroup of W^a. This semigroup is called the affinized Weyl semigroup. Muthiah and Orr [2019] define a Bruhat order and an associated length function on W^a_+ which we aim to study in this article.

Denote by $\text{proj}^{Y^+} : W^a_+ \to Y^+$ the canonical projection, which sends $\pi^\lambda w$ onto λ. Denote by $\text{proj}^{Y^{++}} : W^a_+ \to Y^{++}$ the projection to Y^{++}: $\text{proj}^{Y^{++}}(\boldsymbol{x}) = (\text{proj}^{Y^+}(\boldsymbol{x}))^{++}$. Let us call $\text{proj}^{Y^+}(\boldsymbol{x})$ the *coweight* of $\boldsymbol{x}$, and $\text{proj}^{Y^{++}}(\boldsymbol{x})$ its *dominance class*.

1.1.7. *Affinized roots.* Let $\Phi^a = \Phi \times \mathbb{Z}$ be *the set of affinized roots* and denote by $\beta + n\pi$ the affinized root (β, n). The affinized root $\beta + n\pi$ is said to be positive if $n > 0$ or ($n = 0$ and $\beta \in \Phi_+$) and we write Φ^a_+ for the set of positive affinized roots. We have $\Phi^a = \Phi^a_+ \sqcup -\Phi^a_+$.

The semidirect product W^a acts on Φ^a by

$$(1.6) \qquad \pi^\lambda w(\beta + n\pi) = w(\beta) + \big(n + \langle \lambda, w(\beta) \rangle\big)\pi.$$

For any $n \in \mathbb{Z}$, its sign is denoted $\text{sgn}(n) \in \{-1, +1\}$, with the convention that $\text{sgn}(0) = +1$. Note that $|n| = \text{sgn}(n)n$. We also define the sign of an affinized root: $\text{sgn}(\beta + n\pi) \in \{-1, +1\}$ and $\text{sgn}(\beta + n\pi) = +1$ if and only if $\beta + n\pi \in \Phi^a_+$.

For $n \in \mathbb{Z}$ and $\beta \in \Phi_+$, set

$$(1.7) \qquad \beta[n] = \text{sgn}(n)\beta + |n|\pi \in \Phi^a_+,$$

$$(1.8) \qquad s_{\beta[n]} = \pi^{n\beta^\vee} s_\beta.$$

We also define $\beta[n] \in \Phi^a_+$ for $\beta \in \Phi_-$ by $\beta[n] = (-\beta)[-n]$, and $s_{\beta[n]} = s_{-\beta[-n]} = \pi^{n\beta^\vee} s_\beta$. The affinized root $\beta[n]$ is therefore the positive affinized root within the pair $\{\beta + n\pi, -(\beta + n\pi)\}$. Note that $s_{\beta[0]}$ is the vectorial reflection s_β.

1.1.8. *Bruhat order on W^a_+.* Recall Braverman, Kazhdan and Patnaik's definition of the Bruhat order $<$ introduced in [Braverman et al. 2016, Section B.2]: Let $\boldsymbol{x} \in W^a_+$ and let $\beta[n] \in \Phi^a_+$ be such that $\boldsymbol{x} s_{\beta[n]} \in W^a_+$. Then,

$$(1.9) \quad \boldsymbol{x} < \boldsymbol{x} s_{\beta[n]} \iff \text{sgn}(\beta + n\pi) = \text{sgn}(\boldsymbol{x}(\beta + n\pi)) \iff \boldsymbol{x}(\beta[n]) \in \Phi^a_+.$$

Explicitly, if $\boldsymbol{x} = \pi^\lambda w \in W^a_+$, the right-hand side condition can be written as

$$\text{sgn}(n)\big(n + \langle \lambda, w(\beta) \rangle\big) > 0 \qquad \text{or} \qquad n = -\langle \lambda, w(\beta) \rangle \quad \text{and} \quad \text{sgn}(n)w(\beta) > 0.$$

Then we extend this relation by transitivity, which makes it a preorder on W^a_+. Originally, Braverman, Kazhdan and Patnaik defined it only for affine vectorial

Weyl groups, but the definition extends to any vectorial Weyl group and Muthiah [2018] showed that it is an order on W_+^a in general.

1.1.9. *Extension to W^a.* As the whole semidirect product W^a acts on Φ_+^a, (1.9) makes sense for any $x \in W^a$, and we define $<$ on W^a as the closure by transitivity of the relation defined through (1.9) for $x \in W^a$. We show in the next section that if $x < y$ and $y \in W_+^a$, then $x \in W_+^a$. This ensures that the restriction of the W^a-preorder to W_+^a coincides with Braverman, Kazhdan, Patnaik's order on W_+^a. However $<$ may not be an order on W^a.

1.1.10. *Bruhat order through a right action.* We consider multiplication by reflections on the left. To switch between the right and left actions note that

$$(1.10) \quad s_{\beta[n]}\pi^\lambda w = \pi^{s_\beta \lambda + n\beta^\vee} s_\beta w = \pi^{\lambda + (n - \langle \lambda, \beta \rangle)\beta^\vee} w s_{w^{-1}(\beta)} = \pi^\lambda w s_{w^{-1}(\beta)[n - \langle \lambda, \beta \rangle]}.$$

In particular,

$$(1.11) \quad s_{\beta[0]}\pi^\lambda w = s_\beta \pi^\lambda w = \pi^{s_\beta \lambda} s_\beta w \quad \text{and} \quad s_{\beta[\langle \lambda, \beta \rangle]}\pi^\lambda w = \pi^\lambda s_\beta w = \pi^\lambda w s_{w^{-1}(\beta)}.$$

Using (1.10), the affinized Bruhat order can be recovered using a right action of W^a on Φ_+^a.

Proposition 1.4. *Let $\pi^\lambda w \in W^a$ and $(\beta, n) \in (\Phi \times \mathbb{Z}) \setminus (\Phi_- \times \{0\})$. Then*

$$(1.12) \quad s_{\beta[n]}\pi^\lambda w > \pi^\lambda w \iff \mathrm{sgn}(n)w^{-1}(\beta) + (|n| - \mathrm{sgn}(n)\langle \lambda, \beta \rangle)\pi \in \Phi_+^a.$$

Remark 1.5. The root appearing in the right-hand side of (1.12) is the affinized root $(\pi^\lambda w)^{-1}(\beta[n])$.

Proof. Let $\pi^\lambda w \in W^a$ and $\beta + n\pi \in \Phi^a$. Then by (1.9) and (1.10),

$$s_{\beta[n]}\pi^\lambda w > \pi^\lambda w \iff \mathrm{sgn}(\beta + n\pi) = \mathrm{sgn}(w^{-1}(\beta) + (n - \langle \lambda, \beta \rangle)\pi).$$

If $(\beta, n) \notin \Phi_- \times \{0\}$, then $\beta[n] = \mathrm{sgn}(n)(\beta + n\pi)$ so this is equivalent to

$$\mathrm{sgn}(n)\big(w^{-1}(\beta) + (n - \langle \lambda, \beta \rangle)\pi\big) \in \Phi_+^a,$$

which is (1.12). $\qquad\square$

Note that (1.12) is no longer correct if $\beta \in \Phi_-$ and $n = 0$, in which case it needs to be applied to $(-\beta)[0]$. Applying reflections on the left is better suited for the geometric interpretation we will give in Section 1.2.

1.1.11. *Terminology on partially ordered sets.* For $p \leq q \in \mathbb{Z}$, we denote by $[\![p, q]\!]$ the set $\{r \in \mathbb{Z} \mid p \leq r \leq q\}$. If $p > q$, then $[\![p, q]\!]$ is another notation for $[\![q, p]\!]$. We also write $]\!]p, q[\![$ for $[\![p, q]\!] \setminus \{p, q\}$.

Let $(\mathcal{P}, \leq)$ be a partially ordered set. For $x, y \in \mathcal{P}$, we say that x and y are comparable if $x \leq y$ or $y \leq x$. We say that y *covers* x, written as $x \lhd y$, if $x \neq y$ and

$\{z \mid x \leq z \leq y\} = \{x, y\}$. If $\mathcal{P} = W_a^+$, covers $x \lhd y$ such that $\text{proj}^{Y^{++}}(y) = \text{proj}^{Y^{++}}(x)$ are called *vectorial covers* and covers which are not vectorial covers are called *properly affine covers*.

A *chain* from x to y is a finite sequence $(x_0, \ldots, x_n)$ such that $x_0 = x$, $x_n = y$ and $x_k \leq x_{k+1}$ for all $k \in [\![0, n-1]\!]$. If $\mathcal{P} = W_+^a$ (resp. if $\mathcal{P}$ is a Coxeter group), we add the condition that $x_{k+1} x_k^{-1}$ is an affinized reflection (resp. a reflection). A chain is *saturated* if $x_k \lhd x_{k+1}$ for all $k \in [\![0, n-1]\!]$. We say that a subset $\mathcal{C}$ of $\mathcal{P}$ is *convex* if, for all $x, y \in \mathcal{C}$ and $z \in \mathcal{P}$,

(1.13) $$x \leq z \leq y \implies z \in \mathcal{C}.$$

Equivalently, $\mathcal{C}$ is convex if and only if any chain from one element of $\mathcal{C}$ to another is contained in $\mathcal{C}$.

Let $\ell : \mathcal{P} \to A$ be a function with values in a totally ordered set $(A, \leq_A)$. We say that it is *order-preserving* if, for all $x, y \in \mathcal{P}$,

(1.14) $$x \leq y \implies \ell(x) \leq_A \ell(y).$$

We say that ℓ is a *strictly compatible* (A-valued) length function if

(1.15) $$x < y \iff x, y \text{ are comparable and } \ell(x) <_A \ell(y).$$

We say that a strictly compatible $\mathbb{R}$-valued length function ℓ defines a $\mathbb{Z}$-*grading* of $\mathcal{P}$ if

(1.16) $$x \lhd y \iff x \leq y \text{ and } \ell(y) = \ell(x) + 1.$$

For instance, the Bruhat length on a Coxeter group W is strictly compatible with the Bruhat order, and defines a $\mathbb{N}$-valued grading of W. Muthiah and Orr associated length functions strictly compatible with the Bruhat order on W_+^a, generalizing the classical Bruhat length on Coxeter groups. We now formally introduce these lengths.

1.1.12. *Length functions on W_+^a.*

Definition 1.6. The *affinized length function* is the map $W_+^a \to \mathbb{R} \oplus \varepsilon \mathbb{Z}$ defined by

$$\ell_\varepsilon^a(\pi^\lambda w) = 2 \operatorname{ht}(\lambda^{++}) + \varepsilon\big(\big|\{\alpha \in \operatorname{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}\big| - \big|\{\alpha \in \operatorname{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}\big|\big).$$

The *affinized length with real values* is the affinized length function on which we set $\varepsilon = 1$:

$$\ell^a(\pi^\lambda w) = 2 \operatorname{ht}(\lambda^{++}) + \big(\big|\{\alpha \in \operatorname{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}\big| - \big|\{\alpha \in \operatorname{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}\big|\big).$$

Theorem 1.7 [Muthiah 2018, Theorem 4.24; Muthiah and Orr 2019, Theorem 3.6]. *The affinized length function and the affinized length function with real values are*

strictly compatible with the affinized Bruhat order on W_+^a. In other words, for any $x \in W_+^a$ and $\beta[n] \in \Phi_+^a$,

$$(1.17) \qquad x s_{\beta[n]} > x \iff \ell_\varepsilon^a(x s_{\beta[n]}) > \ell_\varepsilon^a(x) \iff \ell^a(x s_{\beta[n]}) > \ell^a(x).$$

In particular the affinized Bruhat order is a partial order.

Remark 1.8. The affinized length functions depend on the choice made for the height function. However since $\mathrm{proj}^{Y^{++}}(s_{\beta[n]}x) \in \mathrm{proj}^{Y^{++}}(x) + Q^\vee$ for any $x \in W_+^a$ and $\beta[n] \in \Phi_+^a$ such that $s_{\beta[n]}x \in W_+^a$ (this is a consequence of Corollary 1.12 below and of (1.10)), by Remark 1.2 the length difference between two comparable elements do not depend on the choice of height function. By the same remark if $\mathcal{D}$ is of finite or affine type, then the height function may be chosen such that ℓ^a takes integral values; therefore it was first introduced by Muthiah and Orr as "the affinized length with integral values". In general type, ℓ^a may take nonintegral values for every choice of height function but this could be artificially fixed: one could also define a strictly compatible length with integral values $\ell_{\mathbb{Z}}^a$ by setting $\ell_{\mathbb{Z}}^a(x) = \lfloor \ell^a(x) \rfloor$ (where $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ denotes any $\mathbb{Z}$-equivariant function).

In what follows we will mostly use ℓ^a and rarely mention ℓ_ε^a. We now refer to ℓ^a as the affinized Bruhat length.

1.2. *Geometric interpretation.* We introduced everything in a very algebraic way, but there is a strong geometric intuition behind root systems, vectorial Weyl groups and the vectorial Bruhat order, developed, for instance, in the context of buildings in [Ronan 1989]. There is also a geometrical interpretation of the Bruhat order on W_+^a which we develop in this paragraph; it takes place in the standard apartment of the masure associated to a Kac–Moody group with underlying Kac–Moody datum $\mathcal{D}$.

Let $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$. The lattice X embeds in its dual $V^\vee$ and the vectorial Weyl group W acts naturally on it. Inside V we have the (closed) fundamental chamber $C_f^v = \{v \in V \mid \langle v, \alpha_i \rangle \geq 0\}$ and the Tits cone $\mathcal{T} = W \cdot C_f^v$. A *vectorial chamber* is a set of the form $w \cdot C_f^v$ for $w \in W$. Since the interior of C_f^v has trivial stabilizer in W, the set of chambers is in natural bijection with W by $w \mapsto C_w^v := w \cdot C_f^v$.

To each root $\beta \in \Phi_+$ let $M_\beta = \{x \in V \mid \langle x, \beta \rangle = 0\}$; it is a hyperplane of V and, if $\beta = w(\alpha_i)$ with α_i a simple root, then $C_w^v \cap C_{ws_i}^v \subset M_\beta \cap \mathcal{T}$. The intersection $C_w^v \cap C_{ws_i}^v$ is called the panel of type s_i of w.

We can put a structure of simplicial complex on $\mathcal{T}$, for which chambers are the cells of maximal rank and panels are the cells of maximal rank within nonchambers. This simplicial complex is a realization of the *Coxeter complex* of (W, S). Each wall splits the Tits cone in two parts, and separate the set of vectorial chambers in two: say
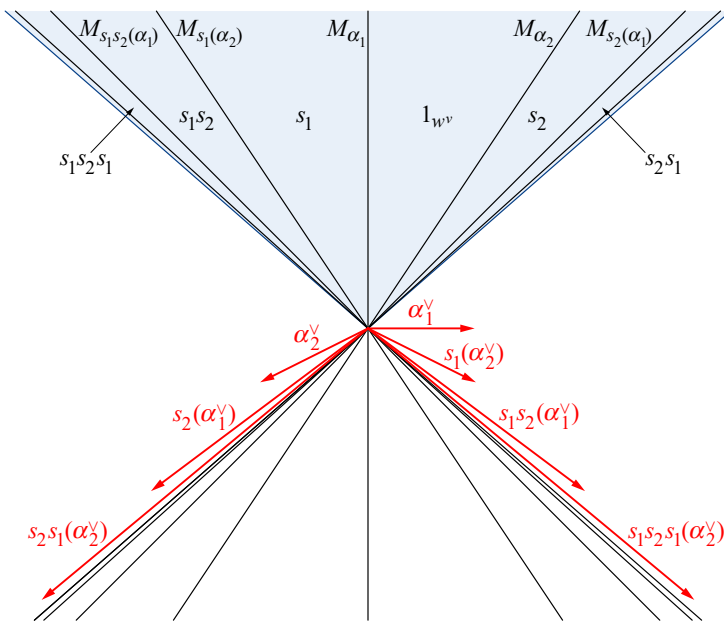
Figure 1. The Tits cone for a root system of Cartan Matrix $\left(\begin{smallmatrix} 2 & -3 \\ -2 & 2 \end{smallmatrix}\right)$.

that C_w^v is on the positive side of M_β if $w^{-1}(\beta) > 0$. In particular since β is a positive root, the positive side is always the one which contains the dominant chamber.

Then the vectorial Bruhat order can be interpreted by $s_\beta w > w$ if and only if, when we split $\mathcal{T}$ along M_β, the chambers C_w^v and C_f^v are in the same connected component of $\mathcal{T}$, that is to say C_w^v is on the positive side of M_β.

The inversion set of w^{-1}, $\mathrm{Inv}(w^{-1})$, can be interpreted as the set of walls separating the chamber $C_w^v = w \cdot C_f^v$ from the fundamental chamber C_f^v.

In Figure 1 we represent the Tits cone and its structure for a root system of rank 2 with Cartan matrix $\left(\begin{smallmatrix} 2 & -3 \\ -2 & 2 \end{smallmatrix}\right)$, which is of indefinite type. The Tits cone is colored in blue, and the vectorial chamber C_w^v is labeled by w. It is an approximation since W is infinite.

Let us now turn to the interpretation of the W_+^a-Bruhat order. Let $\mathbb{A}$ be a real affine space with direction V, we call $\mathbb{A}$ the (standard) affine apartment associated to $\mathcal{D}$. The tangent space of $\mathbb{A}$ is canonically isomorphic to $T\mathbb{A} = \mathbb{A} \times V$, with, for any $x \in \mathbb{A}$, $T_x\mathbb{A} = \{x\} \times V$.

The semigroup W_+^a has an affine action on $\mathbb{A}$, given by $\pi^\lambda w(x) = -\lambda + w(x)$, which induces an action on $T\mathbb{A}$ given by $\pi^\lambda w((x, v)) = (-\lambda + w(x), w(v))$. To any positive affinized root $\beta[n] \in \Phi_+^a$ corresponds an affine hyperplane

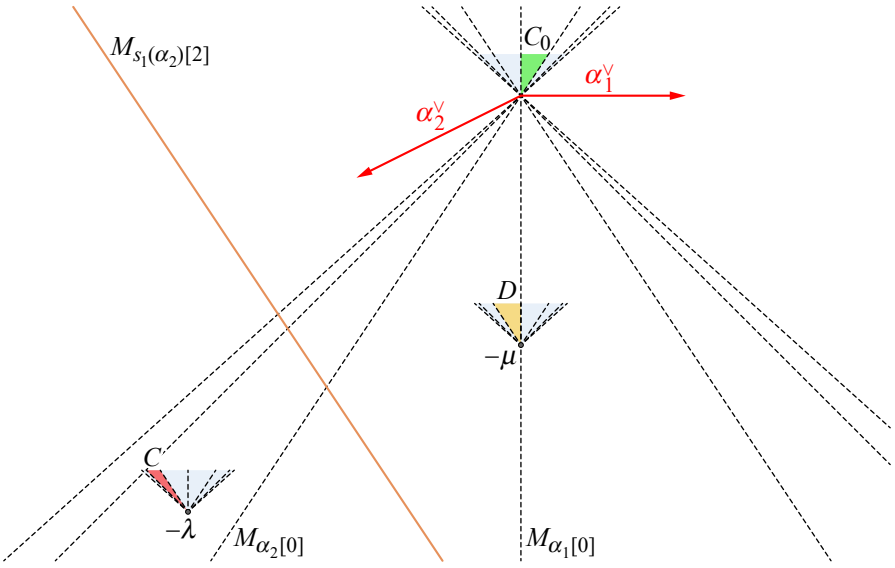(1.18) $$M_{\beta[n]} = \{x \in \mathbb{A} \mid \langle x, \beta \rangle + n = 0\},$$

Figure 2. The affine apartment for a root system of Cartan Matrix $\left(\begin{smallmatrix} 2 & -3 \\ -2 & 2 \end{smallmatrix}\right)$.

the *affine wall* associated to the affinized root $\beta[n]$. For any $x \in M_{\beta[n]}$ we have $T_x M_{\beta[n]} = \{x\} \times M_\beta \subset T_x \mathbb{A}$.

For any $\pi^\lambda w \in W_+^a$ let

$$(1.19) \qquad\qquad C_{\pi^\lambda w} = \{-\lambda\} \times C_w^v \subset T_{-\lambda}\mathbb{A} \subset T\mathbb{A},$$

we call it the *alcove of type $\pi^\lambda w$*. Mirroring the classical situation, $C_0 = \{0\} \times C_f^v$ is a fundamental domain for the action of W_+^a on $Y^+ \times \mathcal{T} \subset T\mathbb{A}$ and W_+^a acts on $\{C_x \mid x \in W_+^a\}$ simply transitively. Affine walls separate naturally the set of alcoves in two and we call the side containing C_0 the positive side.

Then the W_+^a-Bruhat order can be interpreted geometrically:

$$(1.20) \qquad s_{\beta[n]}\pi^\lambda w > \pi^\lambda w \iff C_{\pi^\lambda w} \text{ is on the positive side of } M_{\beta[n]}.$$

We give an illustration of the affine apartment in Figure 2.

In Figure 2 we represent the affine apartment for the same root datum as in Figure 1. The blue polygons represent the local Tits cones at three different points: the origin, $-\lambda \in -Y^+$ and $-\mu$, which is the image of $-\lambda$ by the reflection along the wall $M_{s_1(\alpha_2)[2]}$ (represented in yellow).

We have highlighted three alcoves: In green the alcove C_0; in red the alcove $C = C_{\pi^\lambda s_1 s_2}$ and in yellow $D = C_{\pi^\mu s_1}$ which is the image of C by $s_{s_1(\alpha_2)[2]}$. We see that D is on the same side of $M_{s_1(\alpha_2)[2]}$ as the fundamental alcove C_0; thus $\pi^\lambda s_1 s_2 = s_{s_1(\alpha_2)[2]}(\pi^\mu s_1) > \pi^\mu s_1$.

Note that $-\lambda$ lies in the negative vectorial chamber $-s_2 C_f^v$, that is to say that $s_2\lambda$ is dominant. Therefore $\pi^\lambda s_2$ is the minimal length element of $\pi^\lambda W$. We will make this more explicit in Section 2.2.

1.2.1. *Notation for segments.* For any two elements $x, y \in V = Y \otimes_{\mathbb{Z}} \mathbb{R}$, we define

$$[x, y] = \{tx + (1 - t)y \mid 0 \leq t \leq 1\} \quad \text{and} \quad]x, y[= \{tx + (1 - t)y \mid 0 < t < 1\}.$$

Note in particular that, if $x \in Y$ and $y = x + n\beta^\vee$ for $n \in \mathbb{Z}$ and $\beta \in \Phi$, then for any $m \in [\![0, n]\!]$ we have $x + m\beta^\vee \in [x, y] \cap Y$.

1.3. *Preliminary results.* Since the affinized Bruhat order is generated on W_+^a by the relations $s_{\beta[n]}x > x \iff \ell^a(s_{\beta[n]}x) > \ell^a(x)$ for affinized roots $\beta[n] \in \Phi_+^a$, covers are always of this form. In the rest of the paper, we always apply affinized reflections on the left.

Lemma 1.9. *Let $\pi^\lambda w \in W^a$ and $\beta[n] \in \Phi_+^a$. Write $\pi^\mu w'$ for $s_{\beta[n]}\pi^\lambda w$ and suppose that $(\pi^\lambda w)^{-1}(\beta[n]) \in \Phi_+^a$. Then $\lambda \in [\mu, s_\beta\mu]$. In particular*

$$(1.21) \qquad \mu \in Y^+ \implies \lambda \in Y^+.$$

Proof. Explicitly,

$$\pi^\mu w' = \pi^{n\beta^\vee} s_\beta . \pi^\lambda w = \pi^{s_\beta\lambda + n\beta^\vee} s_\beta w.$$

Thus

$$\mu = s_\beta\lambda + n\beta^\vee = \lambda + (n - \langle\lambda, \beta\rangle)\beta^\vee \quad \text{and} \quad s_\beta\mu = \lambda - n\beta^\vee.$$

Moreover, since $(\pi^\lambda w)^{-1}(\beta[n]) \in \Phi_+^a$, by (1.12),

$$|n| - \mathrm{sgn}(n)\langle\lambda, \beta\rangle = \mathrm{sgn}(n)(n - \langle\lambda, \beta\rangle) \geq 0.$$

Therefore, unless $n - \langle\lambda, \beta\rangle = 0$, n and $n - \langle\lambda, \beta\rangle$ have same sign, and thus $\lambda = s_\beta\mu + n\beta^\vee = \mu - (n - \langle\lambda, \beta\rangle)\beta^\vee$ lies in $[s_\beta\mu, \mu]$. If $n - \langle\lambda, \beta\rangle = 0$ then $\mu = \lambda$ and the result remains true.

The Tits cone $\mathcal{T}$ is convex [Kumar 2002, Proposition 1.4.2c)] and W-stable, so if $\mu \in \mathcal{T}$, then $[\mu, s_\beta\mu]$ is contained in $\mathcal{T}$ for any $\beta \in \Phi$. Therefore in the situation above, if $\mu \in Y^+ = \mathcal{T} \cap Y$, then $\lambda \in [\mu, s_\beta\mu] \cap Y \subset \mathcal{T} \cap Y = Y^+$, and thus $\mu \in Y^+ \implies \lambda \in Y^+$. $\qquad\square$

We directly obtain from Lemma 1.9 the following result.

Proposition 1.10. *The affinized Bruhat order defined on W_+^a coincides with the restriction of the preorder defined through (1.9) on the whole semidirect product W^a.*

1.3.1. *Properties of the height function.* We give here a few elementary results on the height function, which will be useful in our study of the affinized Bruhat length. They are also used in [Muthiah and Orr 2019, Section 3].

Proposition 1.11. *For any $w \in W$,*

$$(1.22) \qquad \rho - w^{-1}(\rho) = \sum_{\gamma \in \mathrm{Inv}(w)} \gamma.$$

Proof. This is [Kumar 2002, 1.3.22, Corollary 3], we prove it by induction on the length of w.

(1) If w is a simple reflection s_α then $\mathrm{Inv}(s_\alpha) = \{\alpha\}$ and $\rho - s_\alpha(\rho) = \langle \alpha^\vee, \rho \rangle \alpha = \alpha$ since $\langle \alpha^\vee, \rho \rangle = 1$ by definition of ρ.

(2) Suppose the result is true for elements of length n, and suppose that $\ell(w) = n+1$. Then write $w = w_1 s_\alpha$ for α a simple root and w_1 an element of length n. Then

$$\rho - w(\rho) = \rho - w_1(\rho) + w_1(\rho - s_\alpha(\rho)) = \sum_{\gamma \in \mathrm{Inv}(w_1^{-1})} \gamma + w_1(\alpha)$$

and since $\mathrm{Inv}(w^{-1}) = \mathrm{Inv}(w_1^{-1}) \sqcup \{w_1(\alpha)\}$ we get the result for w. $\qquad \square$

Corollary 1.12. *For any positive root $\beta \in \Phi_+$ we have*

$$(1.23) \qquad 2\,\mathrm{ht}(\beta^\vee) = \sum_{\gamma \in \mathrm{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle.$$

All the terms in the sum are positive.

Proof. Let $\beta \in \Phi_+$ be a positive root. Note that $-s_\beta(\beta^\vee) = \beta^\vee$ and thus $\langle \beta^\vee, \rho \rangle = \langle -s_\beta(\beta^\vee), \rho \rangle = \langle \beta^\vee, -s_\beta(\rho) \rangle$. Therefore by Proposition 1.11,

$$2\,\mathrm{ht}(\beta^\vee) = 2\langle \beta^\vee, \rho \rangle = \langle \beta^\vee, \rho - s_\beta(\rho) \rangle = \sum_{\gamma \in \mathrm{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle.$$

Also for any $\gamma \in \mathrm{Inv}(s_\beta)$, by definition $\gamma \in \Phi_+$ and $s_\beta(\gamma) = \gamma - \langle \beta^\vee, \gamma \rangle \beta^\vee \in \Phi_-$ so, since β is a positive root, the coefficient $\langle \beta^\vee, \gamma \rangle$ is necessarily positive. $\qquad \square$

Corollary 1.13. *Let $\mu \in Y^+$ and $u \in W$ be such that $\mu = u(\mu^{++})$. Then*

$$(1.24) \qquad \mathrm{ht}(\mu^{++}) = \mathrm{ht}(\mu) - \sum_{\tau \in \mathrm{Inv}(u^{-1})} \langle \mu, \tau \rangle.$$

The terms in this sum are nonpositive integers and

$$(1.25) \qquad \mathrm{ht}(\mu) \leq \mathrm{ht}(\mu^{++}).$$

The inequality is strict unless μ is dominant.

Proof. By definition $\mathrm{ht}(\mu^{++}) = \langle u^{-1}(\mu), \rho \rangle = \langle \mu, u(\rho) \rangle$, and, by Proposition 1.11,

$$\mathrm{ht}(\mu^{++}) = \langle \mu, u(\rho) \rangle = \left\langle \mu, \rho - \sum_{\tau \in \mathrm{Inv}(u^{-1})} \tau \right\rangle = \mathrm{ht}(\mu) - \sum_{\tau \in \mathrm{Inv}(u^{-1})} \langle \mu, \tau \rangle.$$

Moreover, for any $\tau \in \Phi$, we have $\langle \mu, \tau \rangle = \langle \mu^{++}, u^{-1}(\tau) \rangle$, so $\tau \in \mathrm{Inv}(u^{-1}) \implies \langle \mu, \tau \rangle \leq 0$ and the terms of the above sum are all nonpositive; we deduce (1.25). If μ is not dominant, then there exists $\tau \in \Phi_+$ such that $\langle \mu, \tau \rangle < 0$, and thus

$$\mathrm{ht}(\mu) < \mathrm{ht}(s_\tau \mu) \leq \mathrm{ht}(\mu^{++}). \qquad \square$$

Amongst other things, Corollary 1.13 directly implies the following result, which was first indicated to the author by Hébert and Muthiah.

Lemma 1.14. *Let $\lambda \in Y^+$ and $\beta \in \Phi_+$ such that $s_\beta \lambda \neq \lambda$. Suppose that $\mu \in]\lambda, s_\beta \lambda[$. Then $\mathrm{ht}(\mu^{++}) < \mathrm{ht}(\lambda^{++})$.*

Proof. Note that we do not suppose $\mu \in Y$. The height function is extended to $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$ linearly. Let $t \in]0, 1[$ be such that $\mu = t\lambda + (1-t)s_\beta \lambda$, and let $v \in W$ be such that $\mu^{++} = v\mu$. Then $\mathrm{ht}(\mu^{++}) = \mathrm{ht}(v\mu) = t\,\mathrm{ht}(v\lambda) + (1-t)\,\mathrm{ht}(vs_\beta \lambda)$. By Corollary 1.13, $\mathrm{ht}(v\lambda) \leq \mathrm{ht}(\lambda^{++})$ and $\mathrm{ht}(vs_\beta \lambda) \leq \mathrm{ht}(\lambda^{++})$ and, since $s_\beta \lambda \neq \lambda$, at least one of the two inequality is strict. We deduce $\mathrm{ht}(\mu^{++}) < \mathrm{ht}(\lambda^{++})$. $\qquad \square$

Proposition 1.15. *Let $x \in W_+^a$ and $\beta[n] \in \Phi_+^a$ such that $s_{\beta[n]} x \in W_+^a$. Then*

$$(1.26) \qquad \mathrm{proj}^{Y^{++}}(s_{\beta[n]} x) = \mathrm{proj}^{Y^{++}}(x) \iff n \in \{0, \langle \mathrm{proj}^{Y^+}(x), \beta \rangle\}.$$

Proof. To simplify notation, let $\lambda \in Y^+$ denote $\mathrm{proj}^{Y^+}(x)$. If $n \in \{0, \langle \lambda, \beta \rangle\}$ then by (1.11), $\mathrm{proj}^{Y^+}(s_{\beta[n]} x) \in \{s_\beta(\lambda), \lambda\}$ and therefore it has same dominance class.

Conversely, if $n \in]]0, \langle \lambda, \beta \rangle[[$ then

$$\mathrm{proj}^{Y^+}(s_{\beta[n]} x) = s_\beta(\lambda) + n\beta^\vee \in]\lambda, s_\beta(\lambda)[,$$

and if $n \notin [[0, \langle \lambda, \beta \rangle]]$ then

$$\lambda \in]s_\beta(\lambda) + n\beta^\vee, \lambda - n\beta^\vee[=]\mathrm{proj}^{Y^+}(s_{\beta[n]} x), s_\beta(\mathrm{proj}^{Y^+}(s_{\beta[n]} x))[.$$

Either way by Lemma 1.14, $\mathrm{ht}(\mathrm{proj}^{Y^{++}}(s_{\beta[n]} x)) \neq \mathrm{ht}(\mathrm{proj}^{Y^{++}}(x))$ and in particular $\mathrm{proj}^{Y^{++}}(s_{\beta[n]} x) \neq \mathrm{proj}^{Y^{++}}(x)$. $\qquad \square$

Remark 1.16. If $n = \langle \lambda, \beta \rangle$, then by (1.11), $s_{\beta[n]} \pi^\lambda w = \pi^\lambda w s_{w^{-1}(\beta)}$. Therefore Proposition 1.15 indicates that, if $y = s_{\beta[n]} x$, then $\mathrm{proj}^{Y^{++}}(y) = \mathrm{proj}^{Y^{++}}(x)$ if and only if y is obtained from x by applying a vectorial reflection either on the left-hand side (if $n = 0$) or on the right-hand side (if $n = \langle \lambda, \beta \rangle$). This justifies the terminology for vectorial covers and properly affine covers.

Proposition 1.17. *Let x, $y \in W_+^a$ and suppose that $x \le y$. Then*

$$(1.27) \qquad \mathrm{ht}(\mathrm{proj}^{Y^{++}}(x)) \le \mathrm{ht}(\mathrm{proj}^{Y^{++}}(y)),$$

with equality if and only if $\mathrm{proj}^{Y^{++}}(x) = \mathrm{proj}^{Y^{++}}(y)$.

In particular the function $\mathrm{ht} \circ \mathrm{proj}^{Y^{++}} : W_+^a \to \mathbb{R}$ is order-preserving.

Proof. It is enough to prove it for cover relations, if $y = s_{\beta[n]}x$ for some $\beta[n] \in \Phi_+^a$. In that case, by Lemma 1.9 we have $\mathrm{proj}^{Y^+}(x) \in \left[\mathrm{proj}^{Y^+}(y), s_\beta(\mathrm{proj}^{Y^+}(y))\right]$. If $\mathrm{proj}^{Y^+}(x) \in \{\mathrm{proj}^{Y^+}(y), s_\beta \mathrm{proj}^{Y^+}(y)\}$ then they have the same dominance class: $\mathrm{proj}^{Y^{++}}(x) = \mathrm{proj}^{Y^{++}}(y)$ and we obtain the equality case.

Otherwise, $\mathrm{proj}^{Y^+}(x) \in \left]\mathrm{proj}^{Y^+}(y), s_\beta(\mathrm{proj}^{Y^+}(y))\right[$, necessarily $s_\beta(\mathrm{proj}^{Y^+}(y)) \ne \mathrm{proj}^{Y^+}(y)$ and by Lemma 1.14 we deduce $\mathrm{ht}(\mathrm{proj}^{Y^{++}}(x)) < \mathrm{ht}(\mathrm{proj}^{Y^{++}}(y))$. $\square$

Corollary 1.18. *For any $\lambda^{++} \in Y^{++}$, the set $\{x \in W_+^a \mid \mathrm{proj}^{Y^{++}}(x) = \lambda^{++}\}$ is convex for the affinized Bruhat order.*

Proof. By Proposition 1.17 the function $W_+^a \to \mathbb{R} : x \mapsto \mathrm{ht} \circ \mathrm{proj}^{Y^{++}}$ is compatible with the affinized Bruhat order. Suppose that $x, y \in W_+^a$ satisfy $\mathrm{proj}^{Y^{++}}(x) = \mathrm{proj}^{Y^{++}}(y)$ and $x \le y$. Let $z \in W_+^a$ be such that $x \le z \le y$. Then by Proposition 1.17, $\mathrm{ht}(\mathrm{proj}^{Y^{++}}(x)) \le \mathrm{ht}(\mathrm{proj}^{Y^{++}}(z)) \le \mathrm{ht}(\mathrm{proj}^{Y^{++}}(y)) = \mathrm{ht}(\mathrm{proj}^{Y^{++}}(x))$. By the equality case in Proposition 1.17, we deduce $\mathrm{proj}^{Y^{++}}(z) = \mathrm{proj}^{Y^{++}}(x)$. $\square$

Remark 1.19. Note that, for $\lambda \in Y^+$, the set $\{x \in W_+^a \mid \mathrm{proj}^{Y^{++}}(x) = \lambda^{++}\}$ is the double W-orbit of π^λ:

$$(1.28) \qquad \{x \in W_+^a \mid \mathrm{proj}^{Y^{++}}(x) = \lambda^{++}\} = W\pi^\lambda W.$$

We show in Section 2 that the right W-orbits $\pi^\lambda W$ are also convex for the affinized Bruhat order.

We end this section with several metric properties of Coxeter groups, the results stated are proved in the context of Coxeter complexes and buildings in [Ronan 1989].

1.3.2. *Metric properties of Coxeter groups.* On any Coxeter group (W_0, S_0) we define a map $d : W_0 \times W_0 \to W_0$ by $d(v, w) = v^{-1}w$, called the *vectorial distance* of W_0. It is W_0-invariant: $d(uv, uw) = d(v, w)$ for any $u, v, w \in W_0$. We also define $d^\mathbb{N} = \ell \circ d$ where ℓ is the Bruhat length on (W_0, S_0) (note that ℓ and $d^\mathbb{N}$ depend on the set of simple reflections S_0, but the vectorial distance does not). These maps have properties analogous to the standard distance axioms, which justify the name (see [Ronan 1989, Chapter 3, §1]).

An unfolded gallery (resp. a gallery) in W_0 from w to v is a sequence $w = w_0, \dots, w_n = v$ such that $d^\mathbb{N}(w_i, w_{i+1}) = 1$ (resp. $d^\mathbb{N}(w_i, w_{i+1}) \in \{0, 1\}$) for all

$i \in [\![0, n-1]\!]$. A gallery is said to be minimal if its length n is equal to $d^{\mathbb{N}}(w_1, w_n)$, and a minimal gallery is necessarily unfolded. We refer to [Ronan 1989, Chapter 2] for properties of minimal galleries, but note that if $(w_0, \ldots, w_n)$ is a minimal gallery then $d^{\mathbb{N}}(w_0, w_i) = i$ and thus $(w_0, \ldots, w_i)$ is a minimal gallery from w_0 to w_i. Since the distance is W_0 invariant, $(v w_0, \ldots, v w_n)$ is also a minimal gallery for any $v \in W_0$. The next lemma is a reformulation of [Ronan 1989, Proposition 2.8].

Lemma 1.20. *Let (W_0, S_0) be a Coxeter system and let $v_1, v_2, w \in W_0$ be such that v_2 is not on a minimal gallery from v_1 to w. Then there is a reflection $r \in W_0$ such that $d(v_1, rw) > d(v_1, w)$ and $d(v_2, rw) < d(v_2, w)$.*

Proof. If v_2 is not on a minimal gallery from v_1 to w, by [Ronan 1989, Proposition 2.8] there is a root α — seen as a half-apartment: $\alpha = \{u \in W_0 \mid \ell(u) < \ell(s_\alpha u)\}$ — such that $v_1, w \in \alpha$ and $v_2 \notin \alpha$. Then consider the folding along α, defined by

$$\forall u \in W_0, \ \rho_\alpha(u) = \begin{cases} s_\alpha u & \text{if } u \notin \alpha, \\ u & \text{otherwise.} \end{cases}$$

It reduces the vectorial distance (see [Ronan 1989, §2]); hence

$$d(v_1, w) = d(\rho_\alpha(v_1), \rho_\alpha(s_\alpha w)) < d(v_1, s_\alpha w),$$
$$d(v_2, s_\alpha w) = d(s_\alpha v_2, w) = d(\rho_\alpha(v_2), \rho_\alpha(w)) < d(v_2, w). \qquad \square$$

Recall that for $J \subset S$, W_J is the subgroup generated by the set of simple reflections J. The Coxeter system (W_J, J) is an example of Coxeter system for which we will use Lemma 1.20. For any $w \in W$, the coset $w W_J$ is convex, in the sense that, if $w_1, w_2 \in w W_J$, then any minimal gallery from w_1 to w_2 lies in $w W_J$ (see [Ronan 1989, Lemma 2.10]).

Definition 1.21. For any $J \subset S$ and $v, w \in W$, the *projection of w on $v W_J$* is the unique element of $v W_J$ which reaches $\min_{\tilde{v} \in v W_J} d^{\mathbb{N}}(w, \tilde{v})$. It is denoted by $\text{proj}_{v W_J}(w)$. Any minimal gallery from v to an element of $w W_J$ goes through $\text{proj}_{v W_J}(w)$ (see [Ronan 1989, Theorem 2.10]).

2. Restriction to constant dominance classes

We study the affinized Bruhat order restricted to a dominance class, that is to say, for a given $\lambda^{++} \in Y^{++}$, we study the restriction of the affinized Bruhat order to the subset $(\text{proj}^{Y^{++}})^{-1}(\lambda^{++}) = W \pi^{\lambda^{++}} W$. By Corollary 1.18 these are convex subsets for the affinized Bruhat order. We start by showing that, for any $\lambda \in Y^+$, the subset $\pi^\lambda W = (\text{proj}^{Y^+})^{-1}(\lambda)$ of $(\text{proj}^{Y^{++}})^{-1}(\lambda^{++})$ is also convex for the affinized Bruhat order.

Lemma 2.1. *Let $\lambda \in Y^+$, recall Notation 1.3 for v^λ. Then $\mathrm{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda = \varnothing$. In particular for any $\beta \in \Phi_+$,*

$$(2.1) \qquad \mathrm{ht}(\lambda) < \mathrm{ht}(s_\beta \lambda) \iff \langle \lambda, \beta \rangle < 0 \iff s_\beta v^\lambda < v^\lambda.$$

Proof. Let $\lambda \in Y^+$ and $\alpha \in \mathrm{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda$. Then since $\alpha \in \Phi_\lambda$, s_α fixes λ, that is, $s_\alpha \in W_\lambda$. Moreover $(v^\lambda)^{-1}(\alpha) < 0$ so $s_\alpha v^\lambda < v^\lambda$, which contradicts the minimality of v^λ (note that, as $W_\lambda v^\lambda = v^\lambda W_{\lambda^{++}}$, v^λ is also the minimal representative for the right coset $W_\lambda v^\lambda$). Hence $\mathrm{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda = \varnothing$, and therefore any $\beta \in \mathrm{Inv}((v^\lambda)^{-1})$ satisfies $\langle \lambda, \beta \rangle \neq 0$.

For $\beta \in \Phi_+$, $\langle \lambda, \beta \rangle = \langle \lambda^{++}, (v^\lambda)^{-1}(\beta) \rangle$. Since λ^{++} is dominant, if this is negative then $\beta \in \mathrm{Inv}((v^\lambda)^{-1})$, and since $\mathrm{Inv}((v^\lambda)^{-1})$ and Φ_λ are disjoint, the converse is also true. Since $\beta \in \mathrm{Inv}((v^\lambda)^{-1}) \iff s_\beta v^\lambda < v^\lambda$ we deduce the second equivalence in (2.1). Moreover $\mathrm{ht}(s_\beta \lambda) = \mathrm{ht}(\lambda) - \langle \lambda, \beta \rangle \mathrm{ht}(\beta^\vee)$ by linearity of the height function, and since $\mathrm{ht}(\beta^\vee) > 0$, the first equivalence in (2.1) is clear. $\square$

Remark 2.2. The fact that $\mathrm{Inv}((v^\lambda)^{-1}) \cap \Phi_\lambda = \varnothing$ is visible geometrically in the Coxeter complex of W, in which Φ_λ is the set of walls containing λ and $\mathrm{Inv}(v^{-1})$ is the set of walls separating C_f^v and C_v^v. The chamber $C_{v^\lambda}^v$ is the closest chamber from the fundamental chamber amongst the chambers containing λ in their closure, in other words, $v^\lambda = \mathrm{proj}_{W_\lambda}(1_W)$.

Proposition 2.3. *Suppose that $\pi^\lambda w \in W_+^a$ and $r \in W$ is a reflection such that $r\lambda \neq \lambda$. Then*

$$(2.2) \qquad \pi^{r\lambda} r w > \pi^\lambda w \iff r v^\lambda < v^\lambda.$$

For any $\lambda^{++} \in Y^{++}$, the restriction of the function $\mathrm{ht} \circ \mathrm{proj}^{Y^+}$ to $(\mathrm{proj}^{Y^{++}})^{-1}(\lambda^{++})$ is order-preserving.

Proof. Suppose that $r \in W$ is a reflection which does not fix λ. By definition there exists a positive root $\beta \in \Phi_+$ such that $r = s_\beta$ and, since r does not fix λ, $\langle \lambda, \beta \rangle \neq 0$. Note that $\pi^{r\lambda} r w = s_{\beta[0]} \pi^\lambda w$ so, using (1.12), we have

$$\pi^{r\lambda} r w > \pi^\lambda w \iff -\langle \lambda, \beta \rangle > 0 \iff \langle \lambda, \beta \rangle < 0.$$

By Lemma 2.1 this is equivalent to $r v^\lambda < v^\lambda$, and to $\mathrm{ht}(\lambda) < \mathrm{ht}(r\lambda)$. This is enough to obtain (2.2). Moreover by convexity of $(\mathrm{proj}^{Y^{++}})^{-1}(\lambda^{++})$ (see Corollary 1.18) and by Proposition 1.15 it also implies that $\mathrm{ht} \circ \mathrm{proj}^{Y^+} : (\mathrm{proj}^{Y^{++}})^{-1}(\lambda^{++}) \to \mathbb{R}$ is order-preserving. $\square$

Note that the function $\mathrm{ht} \circ \mathrm{proj}^{Y^+}$ is not order-preserving on the whole semi-group W_+^a. For example suppose that $\lambda \in Y^{++}$ and $\beta \in \Phi_+$ are such that $\lambda + \beta^\vee$ is also

dominant. Then we can check that $\pi^{s_\beta(\lambda)} < \pi^{s_\beta(\lambda)-\beta^\vee}s_\beta$ whereas $\mathrm{ht}(s_\beta(\lambda)-\beta^\vee) < \mathrm{ht}(s_\beta(\lambda))$.

Proposition 2.3 implies the convexity of left W-cosets:

Corollary 2.4. *Let* $\lambda \in Y^+$. *Then the set* $\pi^\lambda W = \{\boldsymbol{x} \in W_+^a \mid \mathrm{proj}^{Y^+}(\boldsymbol{x}) = \lambda\}$ *is convex for the affinized Bruhat order.*

Proof. Let $\boldsymbol{x}, \boldsymbol{y} \in \pi^\lambda W$ such that $\boldsymbol{x} < \boldsymbol{y}$ and let $\boldsymbol{x} = \boldsymbol{x}_0 < \boldsymbol{x}_1 < \cdots < \boldsymbol{x}_n = \boldsymbol{y}$ be a chain from $\boldsymbol{x}$ to $\boldsymbol{y}$; in particular for all $k \in [\![0, n-1]\!]$, let $\beta_k[n_k] \in \Phi_+^a$ be such that $\boldsymbol{x}_{k+1} = s_{\beta_k[n_k]}\boldsymbol{x}_k$. For $k \in [\![0, n]\!]$, write $\boldsymbol{x}_k = \pi^{\lambda_k} w_k$ with $\lambda_k \in Y^+$, $w_k \in W$. By convexity of $(\mathrm{proj}^{Y^{++}})^{-1}(\lambda^{++})$, $\mathrm{proj}^{Y^{++}}$ is constant along the chain, therefore by Proposition 1.15, for all $k \in [\![0, n-1]\!]$ we have $\lambda_{k+1} \in \{\lambda_k, s_{\beta_k}(\lambda_k)\}$. From Proposition 2.3 we deduce that $v^{\lambda_{k+1}} \le v^{\lambda_k}$. Since $\lambda_0 = \lambda_n = \lambda$, $v^{\lambda_0} = v^{\lambda_n} = v^\lambda$, and thus $v^{\lambda_k} = v^\lambda$, so $\lambda_k = \lambda$ for all $k \in [\![0, n]\!]$. Hence $\pi^\lambda W$ is convex. $\square$

2.1. *Relative length on W.* We define a relative length and a relative Bruhat order on W, which naturally arises in the study of the affinized length ℓ^a on W_+^a. This connection was already observed by Muthiah and Orr [2018].

Definition 2.5. For any $v, w \in W$ let

$$(2.3) \qquad \ell_v(w) = |\mathrm{Inv}(w^{-1}) \setminus \mathrm{Inv}(v^{-1})| - |\mathrm{Inv}(w^{-1}) \cap \mathrm{Inv}(v^{-1})|.$$

This is a signed version of the Bruhat length, in particular $\ell_1 = \ell$.

We associate an order to ℓ_v by setting, for any element $w \in W$ and any reflection $r \in W$, $w <_v wr$ if and only if $\ell_v(w) < \ell_v(wr)$, and then let $<_v$ be the order generated by these relations. It is strictly compatible with ℓ_v. In particular $<_1$ is the classical Bruhat order.

As does the Bruhat length, the lengths ℓ_v have a geometric interpretation in the Coxeter complex associated to (W, S). For M a wall of the Coxeter complex and $w \in W$, let $\varepsilon_w(M) = -1$ if M separates C_f^v and C_w^v, and $\varepsilon_w(M) = +1$ otherwise. Then

$$(2.4) \qquad \qquad \ell_v(w) = \sum_{M \in \varepsilon_w^{-1}(-1)} \varepsilon_{v^{-1}}(M).$$

We will use this relative length to give an alternative definition of the affinized length. Let us first give an explicit formula for ℓ_v depending only on the classical length $\ell = \ell_1$.

Lemma 2.6. *If $sv > v$ with $v \in W$ and s a simple reflection then for any $w \in W$, $\ell_{sv}(w) = \ell_v(sw) - 1$.*

Proof. For any $w \in W$, the map $\gamma \mapsto s\gamma$ defines a bijection:

$$\mathrm{Inv}(w^{-1}) \setminus \{\alpha_s\} \cong \mathrm{Inv}(w^{-1}s) \setminus \{\alpha_s\}.$$

Moreover because $sv > v$, $\alpha_s \in \mathrm{Inv}(v^{-1}s)$ and $\alpha_s \notin \mathrm{Inv}(v^{-1})$.

Therefore

$$|\mathrm{Inv}(w^{-1}) \cap \mathrm{Inv}(v^{-1}s) \setminus \{\alpha_s\}| = |\mathrm{Inv}(w^{-1}s) \cap \mathrm{Inv}(v^{-1})|$$

and

$$|\mathrm{Inv}(w^{-1}) \setminus \mathrm{Inv}(v^{-1}s)| = \left|\mathrm{Inv}(w^{-1}s) \setminus (\mathrm{Inv}(v^{-1}) \cup \{\alpha_s\})\right|.$$

(1) If $\alpha_s \in \mathrm{Inv}(w^{-1})$ then $\alpha_s \notin \mathrm{Inv}(w^{-1}s)$ and

$$\ell_{sv}(w) = |\mathrm{Inv}(w^{-1}s) \setminus \mathrm{Inv}(v^{-1})| - \left(|\mathrm{Inv}(w^{-1}s) \cap \mathrm{Inv}(v^{-1})| + 1\right)$$
$$= \ell_v(sw) - 1.$$

(2) If $\alpha_s \notin \mathrm{Inv}(w^{-1})$ then $\alpha_s \in \mathrm{Inv}(w^{-1}s)$ and

$$\ell_{sv}(w) = \left(|\mathrm{Inv}(w^{-1}s) \setminus \mathrm{Inv}(v^{-1})| - 1\right) - |\mathrm{Inv}(w^{-1}s) \cap \mathrm{Inv}(v^{-1})|$$
$$= \ell_v(sw) - 1. \qquad \square$$

Proposition 2.7. *For all $v, w \in W$ the relative length $\ell_v(w)$ is given by*

$$(2.5) \qquad \qquad \ell_v(w) = \ell(v^{-1}w) - \ell(v).$$

Proof. Since $\ell = \ell_1$, we take a reduced expression for v and apply Lemma 2.6 recursively to get the result. $\qquad \square$

Corollary 2.8. *For any $v \in W$, the relative length ℓ_v is a grading of $(W, <_v)$.*

Proof. Let $v, w, w' \in W$. By Proposition 2.7, $\ell_v(w') - \ell_v(w) = \ell(v^{-1}w') - \ell(v^{-1}w)$ and w' covers w for $<_v$ if and only if $v^{-1}w'$ covers $v^{-1}w$ for the (standard) Bruhat order. Since the Bruhat length is a grading of $(W, <_1)$ (see [Björner and Brenti 2005, Theorem 2.2.6]), $v^{-1}w'$ covers $v^{-1}w$ if and only if $\ell(v^{-1}w') - \ell(v^{-1}w) = 1$ and $v^{-1}w' = v^{-1}wr$ for some reflection $r \in W$. Hence w' covers w for $<_v$ if and only if $\ell_v(w') - \ell_v(w) = 1$ and $w' = wr$ for some reflection $r \in W$: ℓ_v is a grading of $(W, <_v)$. $\qquad \square$

The order $<_v$ also has a geometric interpretation which will be important later on; it is given by the following corollary.

Corollary 2.9. *For any root $\alpha \in \Phi$ and elements $w, v \in W$, we have that $w <_v s_\alpha w$ if and only if, in the Coxeter complex of W, C_w^v and C_v^v are on the same side of the wall M_α.*

Proof. We have $\ell_v(s_\alpha w) - \ell_v(w) = \ell(v^{-1}s_\alpha w) - \ell(v^{-1}w) = d^{\mathbb{N}}(v, s_\alpha w) - d^{\mathbb{N}}(v, w)$. By the definition of the Coxeter complex this is positive if and only if C_v^v and C_w^v are on the same side of the wall M_α. $\square$

Therefore $<_v$ can be interpreted as a shift of the classical Bruhat order, corresponding geometrically to taking C_v^v as the fundamental chamber in the Coxeter complex.

2.2. *Relation with the affinized Bruhat length.* We relate the affinized Bruhat order and the relative order defined in Section 2.1. We start with an alternative expression for the affinized Bruhat length.

Proposition 2.10. *For any coweight $\lambda = v\lambda^{++} \in Y^+$, for any $w \in W$,*

$$(2.6) \quad \left|\{\alpha \in \mathrm{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}\right| - \left|\{\alpha \in \mathrm{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}\right| = \ell_{v^\lambda}(w).$$

Therefore

$$(2.7) \qquad \qquad \ell_\varepsilon^a(\pi^\lambda w) = 2\,\mathrm{ht}(\lambda^{++}) + \varepsilon \ell_{v^\lambda}(w),$$
$$(2.8) \qquad \qquad \ell^a(\pi^\lambda w) = 2\,\mathrm{ht}(\lambda^{++}) + \ell_{v^\lambda}(w).$$

Proof. For $\lambda \in Y^+$ and $v \in W$ such that $\lambda = v\lambda^{++}$, $\alpha \in \Phi_+$ satisfies $\langle \lambda, \alpha \rangle \geq 0$ if and only if $\alpha \in \Phi_\lambda \cup (\Phi_+ \setminus \mathrm{Inv}(v^{-1}))$. Hence,

$$\{\alpha \in \mathrm{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\} = (\mathrm{Inv}(w^{-1}) \setminus \mathrm{Inv}(v^{-1})) \bigsqcup (\mathrm{Inv}(w^{-1}) \cap \mathrm{Inv}(v^{-1}) \cap \Phi_\lambda)$$

and

$$\mathrm{Inv}(w^{-1}) \cap \mathrm{Inv}(v^{-1}) = \{\alpha \in \mathrm{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\} \bigsqcup (\mathrm{Inv}(w^{-1}) \cap \mathrm{Inv}(v^{-1}) \cap \Phi_\lambda).$$

Therefore

$$\left|\{\alpha \in \mathrm{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle \geq 0\}\right| - \left|\{\alpha \in \mathrm{Inv}(w^{-1}) \mid \langle \lambda, \alpha \rangle < 0\}\right|$$
$$= \ell_v(w) + 2|\mathrm{Inv}(w^{-1}) \cap \mathrm{Inv}(v^{-1}) \cap \Phi_\lambda|$$

By Lemma 2.1 we deduce (2.6). $\square$

Remark 2.11. By combining Corollary 1.13 with Proposition 2.10, we obtain the formulas already given by Muthiah and Orr [2019, Proposition 3.10].

Corollary 2.12. *Let $\lambda \in Y^+$ and $w \in W$. Suppose that $\pi^\mu w'' = s_{\beta[n]}\pi^\lambda w$ for some affinized root $\beta[n] \in \Phi^a$ such that $\mu^{++} = \lambda^{++}$. Then*

$$(2.9) \qquad \qquad \pi^\mu w'' > \pi^\lambda w \iff \ell_{v^\mu}(w'') > \ell_{v^\lambda}(w).$$

For any $\lambda \in Y^+$ and $w, w'' \in W$,

$$(2.10) \qquad \qquad \pi^\lambda w < \pi^\lambda w'' \iff w <_{v^\lambda} w''.$$

In particular, $\pi^\lambda v^\lambda$ is the minimal element of $\pi^\lambda W$.

Proof. Equivalence (2.9) is a direct consequence of (2.8) and strict compatibility of the affinized Bruhat length and the affinized Bruhat order. It implies by iteration that a chain for the relative order $<_{v^\lambda}$ from w to w'' lifts to a chain for the affinized Bruhat order from $\pi^\lambda w$ to $\pi^\lambda w''$. Conversely, by Corollary 2.4 proj^{Y^+} is constant along any chain from $\pi^\lambda w$ to $\pi^\lambda w''$, and therefore the projection on W of a chain from $\pi^\lambda w$ to $\pi^\lambda w''$ induces a chain for the relative Bruhat order $<_{v^\lambda}$ from w to w''. $\square$

We deduce a partial version of Theorem A, for vectorial covers with constant coweight.

Theorem 2.13. *Let* x, $y \in W_+^a$ *be such that* $\mathrm{proj}^{Y^+}(x) = \mathrm{proj}^{Y^+}(y)$ *and* $x \le y$. *Then*

$$(2.11) \qquad x \lhd y \iff \ell^a(y) = \ell^a(x) + 1.$$

More precisely, if $x = \pi^\lambda w$ *then* $y = \pi^\lambda r w$ *for some reflection* $r \in W$ *such that* rw *covers* w *for the relative Bruhat order* $<_{v^\lambda}$.

Proof. By (2.10), $\pi^\lambda w'$ covers $\pi^\lambda w$ if and only if w' covers w for the relative Bruhat order $<_{v^\lambda}$. By Corollary 2.8, this is equivalent to $\ell_{v^\lambda}(w') = \ell_{v^\lambda}(w) + 1$. Therefore by (2.8) we deduce that $x \lhd y \implies \ell^a(y) = \ell^a(x) + 1$. The converse is immediate by strict compatibility of the affinized Bruhat length (Theorem 1.7). $\square$

2.3. *Vectorial covers with nonconstant coweight.* Here, we prove Theorem A for vectorial covers with nonconstant coweight.

Beforehand, we need a few results on parabolic decomposition. The first lemma is an adaptation of a standard result on minimal coset representatives (see [Björner and Brenti 2005, Theorem 2.5.5]), and the second is proved by P-E. Chaput, L. Fresse and T. Gobet in [Chaput et al. 2021].

Lemma 2.14. *Let* J *be a subset of* S, *and recall* Notation 1.3 *for* W^J. *Let* v *be an element of* W^J *and* u *be any element of* W *such that* $u < v$. *Then, there is, for the Bruhat order, a saturated chain*

$$(2.12) \qquad u = u_0 \lhd u_1 \lhd \cdots \lhd u_N = v$$

such that, for any $i \in [\![1, N]\!]$, $u_{i-1}^{-1} u_i$ *does not belong to* W_J.

Proof. If v covers u, it is clear since $u < v$ is a saturated chain, and as v is a minimal coset representative, $u^{-1}v \notin W_J$. By induction it thus suffices, for a general pair (u, v), to construct $u_1 \in W$ such that u_1 covers u, $u^{-1}u_1 \notin W_J$ and $u_1 < v$; the rest of the chain is obtained by induction. Let $s_1 \ldots s_n$ be a reduced expression of v. Since $u < v$, there exists a reduced expression of u obtained from $s_1 \ldots s_n$ by deleting letters $s_{i_1}, \ldots, s_{i_N}$. Choose one such that i_N is minimal. Then

let $t \in W$ be the reflection defined by $t = s_n \ldots s_{i_N+1} s_{i_N} s_{i_N+1} \ldots s_n$. We show that $u_1 = ut$ satisfies the desired properties.

(1) By construction, an expression of ut is obtained from $s_1 \ldots s_n$ by deleting the $N-1$ letters $s_{i_1}, \ldots, s_{i_{N-1}}$. Therefore $ut < v$.

(2) Since an expression of vt is obtained from $s_1 \ldots s_n$ by deleting s_{i_N}, $vt < v$, and since v is the minimal coset representative of vW_J, t does not belong to W_J.

(3) It remains to show that ut covers u. By the first point, we have that $\ell(ut) \leq \ell(u) + 1$, so it suffices to show that $ut \not< u$. Suppose by contradiction that $ut < u$. Then, by the strong exchange property, an expression of ut is obtained from u by deleting one letter s_p of the reduced expression $s_1 \ldots \check{s}_{i_1} \ldots \check{s}_{i_N} \ldots s_n$ (where $\check{s}_i$ denotes a letter s_i taken away from the expression $s_1 \ldots s_n$).

   (a) Suppose that $p > i_N$. Then t can also be written as $s_n \ldots s_{p+1} s_p s_{p+1} \ldots s_n$, and $v = (vt)t = s_1 \ldots \check{s}_{i_N} \ldots \check{s}_p \ldots s_n$, which contradicts the hypothesis that $s_1 \ldots s_n$ is reduced.

   (b) Suppose now that there is $d \leq N-1$ such that $i_d < p < i_{d+1}$ (with the convention that $i_0 = -1$). Then $t = s_n \ldots \check{s}_{i_N} \ldots \check{s}_{i_{d+1}} \ldots s_p \ldots \check{s}_{i_{d+1}} \ldots \check{s}_{i_N} \ldots s_n$, and $u = (ut)t$ can be written from $s_1 \ldots s_n$ by deleting the terms of indices $i_1, \ldots, i_{N-1}$ and $p < i_N$, but not i_N. This contradicts the minimality of i_N. $\qquad\square$

Definition 2.15. For $v, w \in W$, we write

$$(2.13) \qquad\qquad v \leq_R w \iff \ell(w) = \ell(v) + \ell(wv^{-1}).$$

Remark 2.16. The relation $\leq_R$ is called the weak Bruhat order and it is related to minimal galleries: $v \leq_R w$ if and only if there is a minimal gallery from 1 to w^{-1} going through v^{-1}.

   Recall that for $J \subset S$ and $x \in W$, (x^J, x_J) denotes the unique pair of $W^J \times W_J$ such that $x = x^J . x_J$.

Lemma 2.17 [Chaput et al. 2021, Lemma 8.11]. *Let $J \subset S$ be a subset of simple reflections. Let u be an element of W and t be a reflection of $W \setminus W_J$ such that ut covers u. Then $(ut)_J \leq_R u_J$. In other words $((ut)_J)^{-1}$ lies on a minimal gallery from 1 to $(u_J)^{-1}$.*

Theorem 2.18. *Let $x, y \in W_+^a$ be such that $\operatorname{proj}^{Y^{++}}(y) = \operatorname{proj}^{Y^{++}}(x)$ and $x \leq y$. Then*

$$(2.14) \qquad\qquad x \lhd y \iff \ell^a(y) = \ell^a(x) + 1.$$

More precisely, write $\boldsymbol{x} = \pi^\lambda w$. Let J be the set of simple reflections stabilizing λ^{++} and let $v \in W^J$ be such that $\lambda = v\lambda^{++}$ (so $v = v^\lambda$ with Notation 1.3*). Then, if $\boldsymbol{x} \lhd \boldsymbol{y}$ and $\mathrm{proj}^{Y^+}(\boldsymbol{y}) \neq \mathrm{proj}^{Y^+}(\boldsymbol{x})$, there exists a unique reflection $r \in W$ such that:*

(1) *The reflection r does not stabilize λ and $\boldsymbol{y} = \pi^{r\lambda} r w$.*

(2) *For the Bruhat order on W, v covers rv.*

(3) *Set $u = rv$, so $u_J \in W_J$ denotes the element $(rv)_J$ and $rvu_J^{-1} \in W^J$. Then vu_J^{-1} is on a minimal gallery from v to w in W.*

Proof. If $\mathrm{proj}^{Y^+}(\boldsymbol{y}) = \mathrm{proj}^{Y^+}(\boldsymbol{x})$ then (2.14) is given by Theorem 2.13. Moreover if $\ell^a(\boldsymbol{y}) = \ell^a(\boldsymbol{x}) + 1$ and $\boldsymbol{y} \geq \boldsymbol{x}$ then, by strict compatibility of ℓ^a, $\boldsymbol{y}$ covers $\boldsymbol{x}$. We are thus reduced to prove that, if $\boldsymbol{y}$ covers $\boldsymbol{x}$ with $\mathrm{proj}^{Y^+}(\boldsymbol{y}) \neq \mathrm{proj}^{Y^+}(\boldsymbol{x})$ and $\mathrm{proj}^{Y^{++}}(\boldsymbol{y}) = \mathrm{proj}^{Y^{++}}(\boldsymbol{x})$, then $\ell^a(\boldsymbol{y}) = \ell^a(\boldsymbol{x}) + 1$.

Write $\boldsymbol{x} = \pi^\lambda w$ and $v = v^\lambda$. By definition of the affinized Bruhat order, if $\boldsymbol{x} \lhd \boldsymbol{y}$ then $\boldsymbol{y}$ is of the form $s_{\beta[n]}\boldsymbol{x}$ for some $\beta[n] \in \Phi_+^a$.

Let $\boldsymbol{y} = s_{\beta[n]}\boldsymbol{x} \in W_+^a$ with $\mathrm{proj}^{Y^{++}}(\boldsymbol{y}) = \lambda^{++}$ and $\mathrm{proj}^{Y^+}(\boldsymbol{y}) \neq \lambda$, in particular $n \neq \langle \lambda, \beta \rangle$. By Proposition 1.15, $n = 0$ so $\boldsymbol{y} = \pi^{r\lambda} r w$ for the reflection $r = s_\beta$ which does not stabilize λ. Let us write $u = rv$, and note that $u^J = v^{r\lambda}$. By (2.8),

$$(2.15) \qquad \ell^a(\boldsymbol{y}) - \ell^a(\boldsymbol{x}) = \ell_{u^J}(rw) - \ell_v(w).$$

By definition, $rv = u^J u_J$ with, by (1.3), $\ell(rv) = \ell(u_J) + \ell(u^J)$. We compute

$$
\begin{aligned}
(2.16) \quad \ell_{u^J}(rw) - \ell_v(w) &= \ell((rv(u_J)^{-1})^{-1}rw) - \ell(v^{-1}w) + \ell(v) - \ell(u^J) \\
&= \ell(u_J v^{-1} w) - \ell(v^{-1}w) + \ell(v) + \ell(u_J) - \ell(u) \\
&= (\ell(v) - \ell(u)) + \left(\ell(u_J) - (d^{\mathbb{N}}(v, w) - d^{\mathbb{N}}(vu_J^{-1}, w))\right).
\end{aligned}
$$

From (2.15) and (2.16), we deduce

$$(2.17) \quad \ell^a(\boldsymbol{y}) - \ell^a(\boldsymbol{x}) = (\ell(v) - \ell(u)) + \left(\ell(u_J) - (d^{\mathbb{N}}(v, w) - d^{\mathbb{N}}(vu_J^{-1}, w))\right).$$

In (2.17), by the triangle inequality and since $d^{\mathbb{N}}(v, vu_J^{-1}) = \ell(u_J)$, the second term $\ell(u_J) - (d^{\mathbb{N}}(v, w) - d^{\mathbb{N}}(vu_J^{-1}, w))$ is nonnegative, and it is equal to 0 if and only if $d^{\mathbb{N}}(v, w) = d^{\mathbb{N}}(v, vu_J^{-1}) + d^{\mathbb{N}}(vu_J^{-1}, w)$, so if and only if vu_J^{-1} is on a minimal gallery from v to w.

Recall Definition 1.21 of $\mathrm{proj}_{vW_J}(w)$. Since vu_J^{-1} lies in vW_J, a minimal gallery from vu_J^{-1} to w goes through $\mathrm{proj}_{vW_J}(w)$. Thus $\ell(u_J) - (d^{\mathbb{N}}(v, w) - d^{\mathbb{N}}(vu_J^{-1}, w))$ is equal to zero if and only if u_J^{-1} is on a minimal gallery from 1 to $v^{-1}\mathrm{proj}_{vW_J}(w)$ in W_J.

Let us first suppose that u_J^{-1} is not on a minimal gallery from 1 to $v^{-1}\mathrm{proj}_{vW_J}(w)$. We want to deduce that $\boldsymbol{y}$ does not cover $\boldsymbol{x}$. We thus want to produce a nontrivial

chain from $\pi^\lambda w$ to $\pi^{r\lambda} rw$. By Lemma 1.20, there is a reflection $t \in W_J$ such that

$$d^{W_J}(1, tv^{-1}\text{proj}_{vW_J}(w)) > d^{W_J}(1, v^{-1}\text{proj}_{vW_J}(w)),$$
$$d^{W_J}(u_J^{-1}, tv^{-1}\text{proj}_{vW_J}(w)) < d^{W_J}(u_J^{-1}, v^{-1}\text{proj}_{vW_J}(w)).$$

In W, this implies $d(vt, w) > d(v, w)$ and $d(vtu_J^{-1}, w) < d(vu_J^{-1}, w)$.

Let $\widetilde{w} = vtv^{-1}w$. We compute

$$\ell_v(\widetilde{w}) - \ell_v(w) = d^{\mathbb{N}}(vt, w) - d^{\mathbb{N}}(v, w) > 0,$$
$$\ell_{u^J}(rw) - \ell_{u^J}(r\widetilde{w}) = d^{\mathbb{N}}(vu_J^{-1}, w) - d^{\mathbb{N}}(vtu_J^{-1}, w) > 0,$$
$$\ell_{u^J}(r\widetilde{w}) - \ell_v(\widetilde{w}) = \ell(v) - \ell(u) + \ell(u_J) - (d^{\mathbb{N}}(v, \widetilde{w}) - d^{\mathbb{N}}(vu, \widetilde{w})) > 0.$$

Hence by Proposition 2.3 and Corollary 2.12,

(2.18) $$\pi^\lambda w < \pi^\lambda \widetilde{w} < \pi^{r\lambda} r\widetilde{w} < \pi^{r\lambda} rw.$$

Suppose now that vu_J^{-1} is on a minimal gallery from v to w. Then by (2.17), $\ell^a(\boldsymbol{y}) - \ell^a(\boldsymbol{x}) = \ell(v) - \ell(rv)$. Suppose that $\ell(v) - \ell(rv) = N > 1$. Let

(2.19) $$rv = u_0 \lhd u_1 \lhd \cdots \lhd u_N = v$$

be a saturated chain in W from rv to v given by Lemma 2.14 and, for $i \in [\![1, N]\!]$, let $\beta_i \in \Phi_+$ be such that $u_i = s_{\beta_i} u_{i-1}$, so $u_i = s_{\beta_i} \ldots s_{\beta_1} u \geq rv$. Note in particular that

(2.20) $$\ell(u_i) = \ell(rv) + i = \ell(v) - N + i.$$

Let us show that it induces a chain for the affinized Bruhat order

(2.21) $$\pi^\lambda w = s_{\beta_N[0]} \ldots s_{\beta_1[0]} \pi^{r\lambda} rw < s_{\beta_{N-1}[0]} \ldots s_{\beta_1[0]} \pi^{r\lambda} rw < \cdots < \pi^{r\lambda} rw.$$

Since $s_{\beta_i[0]} \ldots s_{\beta_1[0]} \pi^{r\lambda} rw = \pi^{u_i \lambda^{++}} s_{\beta_i} \ldots s_{\beta_1} rw$, by (2.9) it is enough to verify

(2.22) $$\forall i \in [\![0, n]\!], \ \ell_{u_i^J}(s_{\beta_i} \ldots s_{\beta_1} rw) = \ell_v(w) + N - i.$$

We compute

(2.23) $$\ell_{u_i^J}(s_{\beta_i} \ldots s_{\beta_1} rw) = \ell\big(((u_i(u_i)_J^{-1})^{-1} s_{\beta_i} \ldots s_{\beta_1} rw\big) - \ell(u_i^J)$$
$$= \ell((u_i)_J v^{-1} w) - \ell(u_i^J).$$

Since the saturated chain $u_0 < u_1 < \cdots < u_N$ is obtained from Lemma 2.14, u_i covers u_{i-1} such that the reflection $u_{i-1}^{-1} u_i$ does not belong to W_J, so by Lemma 2.17, $(u_i)_J \leq_R (u_{i-1})_J$ and by iteration we have $(u_i)_J \leq_R (u_0)_J = u_J$. Otherwise said, $(u_i)_J^{-1}$ is on a minimal gallery from 1 to u_J^{-1}. Therefore $v(u_i)_J^{-1}$ is on a minimal gallery from v to vu_J^{-1}, and hence on a minimal gallery from v to w. We deduce

(2.24) $$\ell((u_i)_J v^{-1} w) = \ell(v^{-1} w) - \ell((u_i)_J).$$

Combining (2.23) and (2.24) we obtain

$$(2.25) \qquad \ell_{u_i^J}(s_{\beta_i} \dots s_{\beta_1} r w) = \ell(v^{-1} w) - \ell((u_i)_J) - \ell(u_i^J).$$

Moreover,

$$(2.26) \qquad \ell((u_i)_J) + \ell(u_i^J) = \ell(u_i) = \ell(v) - (N - i)$$

by (1.3) and (2.20). Combining (2.25) and (2.26), we deduce (2.22). □

3. Properly affine covers

3.1. *A few properties of properly affine covers.* We now turn to the case of covers $\pi^\lambda w < \pi^\mu w'$ in W_+^a with $\mu^{++} \neq \lambda^{++}$. Such covers are called properly affine covers.

By (1.12), if $\pi^\mu s_\beta w = s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$ with $\beta[n] \in \Phi_+^a$, then $n \in \mathbb{Z} \setminus]]0, \langle \lambda, \beta \rangle [[$. Conversely, if $n \in \mathbb{Z} \setminus [[0, \langle \lambda, \beta \rangle]]$ then $s_{\beta[n]} \pi^\lambda w > \pi^\lambda w$, however $s_{\beta[n]} \pi^\lambda w$ may not be in W_+^a as $\lambda + \mathbb{Z}\beta^\vee \not\subset Y^+$. The limit cases $n \in \{0, \langle \lambda, \beta \rangle\}$ correspond to $\lambda^{++} = \mu^{++}$ dealt with in the previous section.

We first show that properly affine covers occur only for minimal n, in the following sense.

Proposition 3.1. *Let $\lambda \in Y^+$ and $w \in W$, and let $\beta \in \Phi$ and $n \in \mathbb{Z}$. Let us define $\sigma = \mathrm{sgn}(\langle \lambda, \beta \rangle) \in \{1, -1\}$. If $\pi^\mu w' = s_{\beta[n]} \pi^\lambda w \rhd \pi^\lambda w$ is a cover with $\lambda^{++} \neq \mu^{++}$, then $n \in \{-\sigma, \langle \lambda, \beta \rangle + \sigma\}$.*

Proof. For any $\nu \in Y^+$ if we identify the Coxeter complex of W with the positive Tits cone $\mathcal{T} \subset \mathbb{A}$, $C_{\nu^\nu}^\nu$ is the closest vectorial chamber, from the fundamental chamber, containing ν in its closure. All the elements of $\lambda + \sigma \mathbb{Z}_{>0} \beta^\vee$ are on the same side of M_β; hence by Corollary 2.9, for any two such ν, $\nu' \in \lambda + \sigma \mathbb{Z}_{>0} \beta^\vee$ and any $w \in W$,

$$(3.1) \qquad w <_{\nu^\nu} s_\beta w \iff w <_{\nu^{\nu'}} s_\beta w.$$

Suppose first that $n \in \langle \lambda, \beta \rangle + \sigma \mathbb{Z}_{>1}$ and let $\mu = \lambda + (n - \langle \lambda, \beta \rangle) \beta^\vee$. Then:

(1) If $w <_{\nu^\mu} s_\beta w$, we have the chain

$$(3.2) \qquad \pi^\lambda w < s_{\beta[\langle \lambda, \beta \rangle + \sigma]} \pi^\lambda w = \pi^{\lambda + \sigma \beta^\vee} s_\beta w < \pi^\mu w < \pi^\mu s_\beta w.$$

The second inequality comes from $\pi^\mu w = s_{\beta[n+\sigma]} \pi^{\lambda + \sigma \beta^\vee} s_\beta w$ and (1.12), and the third comes from (2.10).

(2) Else $s_\beta w <_{\nu^\mu} w$, so by (3.1), $s_\beta w <_{\nu^{\lambda + \sigma \beta^\vee}} w$ and we have the chain

$$(3.3) \qquad \pi^\lambda w < s_{\beta[\langle \lambda, \beta \rangle + \sigma]} \pi^\lambda w = \pi^{\lambda + \sigma \beta^\vee} s_\beta w < \pi^{\lambda + \sigma \beta^\vee} w < \pi^\mu s_\beta w.$$

Here the second inequality comes from (2.10). The third comes from $\pi^\mu s_\beta w = s_{\beta[n+\sigma]} \pi^{\lambda + \sigma \beta^\vee} w$ and (1.12).

Either way, for $n \in \langle \lambda, \beta \rangle + \sigma \mathbb{Z}_{>1}$, $s_{\beta[n]}\pi^\lambda w$ does not cover $\pi^\lambda w$. For $n \in -\sigma \mathbb{Z}_{>1}$ the argument is similar, because all the elements of $s_\beta \lambda - \sigma \mathbb{Z}_{>0}\beta^\vee$ are on the same side of M_β, in particular on the side of $\mu = s_\beta \lambda + n\beta^\vee$.

(1) If $w <_{v^\mu} s_\beta w$, we have a chain

(3.4) $$\pi^\lambda w < s_{\beta[-\sigma]}\pi^\lambda w = \pi^{s_\beta(\lambda+\sigma\beta^\vee)} s_\beta w < \pi^\mu w < \pi^\mu s_\beta w.$$

(2) Else $s_\beta w <_{v^\mu} w$ so $s_\beta w <_{v^{s_\beta(\lambda+\sigma\beta^\vee)}} w$ and we have a chain

(3.5) $$\pi^\lambda w < s_{\beta[-\sigma]}\pi^\lambda w = \pi^{s_\beta(\lambda+\sigma\beta^\vee)} s_\beta w < \pi^{s_\beta(\lambda+\sigma\beta^\vee)} w < \pi^\mu s_\beta w.$$

So the only possible covers (with varying coweights) are for $n \in \{-\sigma, \langle \lambda, \beta \rangle + \sigma\}$. $\square$

Remark 3.2. To follow up on Remark 1.16, by (1.10), we have $s_{\beta[\sigma+\langle \lambda, \beta \rangle]}\pi^\lambda w = \pi^\lambda w s_{w^{-1}(\beta)[\sigma]}$, where $\sigma = \text{sgn}(\langle \lambda, \beta \rangle)$. Therefore Proposition 3.1 tells us that, if y covers x in W_+^a, then y is obtained from x applying an affinized reflection $s_{\tilde\beta[n]}$ either on the left (for $s_{\beta[0]}$ and $s_{\beta[-\sigma]}$) or on the right (for $s_{\beta[\langle \lambda, \beta \rangle]}$ and $s_{\beta[\langle \lambda, \beta \rangle+\sigma]}$), with $n \in \{-1, 0, 1\}$.

This is still far from a sufficient condition and many cases of potential covers can still be eliminated. We give another necessary condition for $\pi^\mu s_\beta w = s_{\beta[n]}\pi^\lambda w > \pi^\lambda w$ to be a cover; this is a generalization of the chains produced in the proof of Theorem 2.18:

Proposition 3.3. Let $\pi^\mu s_\beta w = s_{\beta[n]}\pi^\lambda w > \pi^\lambda w$ with $\mu^{++} \neq \lambda^{++}$. Suppose that $s_\beta v^\mu$ is not on a minimal gallery from w to v^λ. Then $\pi^\mu s_\beta w > \pi^\lambda w$ is not a cover.

Proof. We express the difference of ε-length using (2.7):

(3.6) $$\ell_\varepsilon^a(\pi^\mu s_\beta w) - \ell_\varepsilon^a(\pi^\lambda w) = 2\,\text{ht}(\mu^{++} - \lambda^{++}) + \varepsilon(\ell_{v^\mu}(s_\beta w) - \ell_{v^\lambda}(w)).$$

If there exists a reflection $r \in W$ such that $\ell_{v^\lambda}(rw) > \ell_{v^\lambda}(w)$ and $\ell_{v^\mu}(s_\beta rw) < \ell_{v^\mu}(s_\beta w)$ then using (2.7) to compute the length ℓ_ε^a, we have a chain

(3.7) $$\pi^\lambda w < \pi^\lambda rw < \pi^\mu s_\beta rw < \pi^\mu s_\beta w.$$

Since $\ell_v(rw) - \ell_v(w) = \ell(v^{-1}rw) - \ell(v^{-1}w)$ for $v, r, w \in W$, Lemma 1.20 guaranties the existence of r, which proves the proposition. $\square$

In Figure 3 below, we give an example of a chain constructed this way in the A_1-affine case, with Cartan matrix $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix} \right)$.

In this example, α and β are the simple roots of an A_1-affinized root system, and we have chosen λ, w and $\beta[n]$ such that $v^\lambda = s_\alpha$, $v^\mu = s_\alpha s_\beta$ and $w = s_\beta$. $\pi^\lambda w$ corresponds to the alcove C_1 in light blue, and its image $\pi^\mu s_\beta w$ by $s_{\beta[6]}$ corresponds to C_4. Since $r = s_\beta s_\alpha s_\beta$ satisfies $d(v^\lambda, rw) = s_\alpha s_\beta s_\alpha > d(v^\lambda, w) = s_\alpha s_\beta$, and
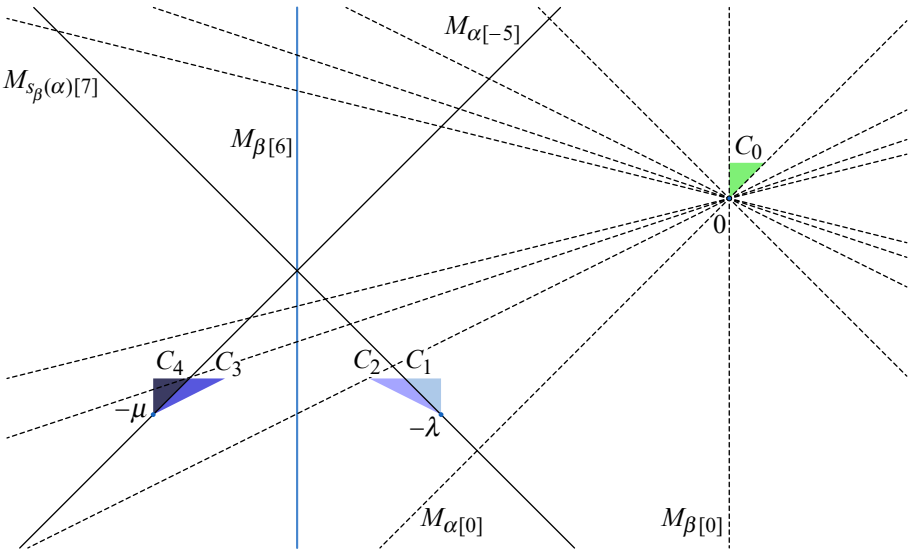
Figure 3. Example of a chain constructed as in Proposition 3.3.

$d(s_\beta v^\mu, rw) = s_\beta < s_\beta s_\alpha = d(s_\beta v^\mu, w)$, there is a chain $\pi^\lambda w < \pi^\lambda rw < \pi^\mu s_\beta rw < \pi^\mu s_\beta w$ which corresponds to the sequence of alcoves (C_1, C_2, C_3, C_4) in Figure 3.

Remark 3.4. Let $v_0, w_0 \in W$, $\mu_0 \in Y$ and $\alpha_0 \in \Phi$. To produce chains, note that (1.12) applied with the affinized reflection $s_{v_0(\alpha_0)[m+\langle\mu_0,\alpha_0\rangle]}$ to $\pi^{v_0(\mu_0)}w_0$ gives

$$(3.8) \qquad \forall m \in \mathbb{Z} \setminus [\![-\langle\mu_0, \alpha_0\rangle, 0]\!], \ \pi^{v_0(\mu_0+m\alpha_0^\vee)}s_{v_0(\alpha_0)}w_0 > \pi^{v_0(\mu_0)}w_0.$$

Applying the affinized reflection $s_{v_0(\alpha_0)[-m]}$ to $\pi^{v_0(\mu_0)}w_0$ instead we obtain

$$(3.9) \qquad \forall m \in \mathbb{Z} \setminus [\![-\langle\mu_0, \alpha_0\rangle, 0]\!], \ \pi^{v_0 s_{\alpha_0}(\mu_0+m\alpha_0^\vee)}s_{v_0(\alpha_0)}w_0 > \pi^{v_0(\mu_0)}w_0.$$

For $m \in \,]\!]-\langle\mu_0, \alpha_0\rangle, 0[\![$ the inequalities are reversed. The cases $m \in \{-\langle\mu_0, \alpha_0\rangle, 0\}$ need to be treated more carefully since they depend on the sign of the root $v_0(\alpha_0)$ (because (1.12) holds for the affinized reflection $s_{v_0(\alpha_0)[0]}$ only if $v_0(\alpha_0) \in \Phi_+$), on the sign of $\langle\mu_0, \alpha_0\rangle$ and on the vectorial element w_0.

3.2. *Another expression for the affinized length difference.* Outside of the case of vectorial covers dealt with in Theorems 2.13 and 2.18, if we write $x = \pi^{v\lambda}w$ with $\lambda \in Y^{++}$, $v, w \in W$ with v of minimal length in vW_λ, by Proposition 3.1 the only covers are of the form $y \in \{\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w, \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w\}$ for some $\beta \in \Phi_+$, so the rest of this paper is dedicated to covers of this sort.

Notation 3.5. From now on, unless stated otherwise, we use the following notation:

(1) $\lambda \in Y^{++}$ is a dominant coweight.

(2) $\beta \in \Phi_+$ is a positive root.

(3) $v \in W^\lambda$ is the minimal representative of a W_λ-coset.

(4) $w \in W$ is any element of W.

(5) $\boldsymbol{x} = \pi^{v(\lambda)}w$ and $\boldsymbol{y} \in \{\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w, \ \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w\}$ are elements of W^+.

The choice to denote by λ a dominant coweight is made in order to avoid the heavier notation λ^{++}. Recall from Notation 1.3 that W^λ is the set of minimal coset representatives of W/W_λ, where W_λ is the standard parabolic subgroup $\{w \in W \mid w(\lambda) = \lambda\}$.

In this subsection, we give another expression for $\ell^a(\boldsymbol{y}) - \ell^a(\boldsymbol{x})$.

The next two lemmas give information on the vectorial chamber of $v(\lambda + \beta^\vee)$.

Lemma 3.6. *Let $\lambda \in Y^{++}$ be a dominant coweight and let $\beta \in \Phi_+$ be a positive root such that $\lambda + \beta^\vee \in Y^+$. Let $u \in W$ be such that $\lambda + \beta^\vee$ belongs to the vectorial chamber C_u^v, that is to say $u^{-1}(\lambda + \beta^\vee) \in Y^{++}$. Then*

$$(3.10) \qquad \ell(s_\beta u) = \ell(s_\beta) + \ell(u).$$

Proof. Let $s_{\tau_1} \ldots s_{\tau_n}$ be a reduced expression of u, so that $\ell(u) = n$ and

$$\mathrm{Inv}(u^{-1}) = \{\tau_1, s_{\tau_1}(\tau_2), \ldots, s_{\tau_1} \ldots s_{\tau_{n-1}}(\tau_n)\}.$$

We show that $s_{\tau_{k+1}} \ldots s_{\tau_1} s_\beta > s_{\tau_k} \ldots s_{\tau_1} s_\beta$ for all $k \in [\![0, n-1]\!]$.

For any $\alpha \in \mathrm{Inv}(u^{-1})$ we have $\langle \lambda + \beta^\vee, \alpha \rangle \leq 0$ (because $\lambda + \beta^\vee \in C_u^v$). Since λ is dominant this implies $\langle \beta^\vee, \alpha \rangle \leq 0$.

Let $k \in [\![0, n-1]\!]$. Then $\langle \beta^\vee, s_{\tau_1} \ldots s_{\tau_k}(\tau_{k+1}) \rangle \leq 0$ so

$$s_\beta(s_{\tau_1} \ldots s_{\tau_k}(\tau_{k+1})) = s_{\tau_1} \ldots s_{\tau_k}(\tau_{k+1}) - \langle \beta^\vee, s_{\tau_1} \ldots s_{\tau_k}(\tau_{k+1}) \rangle \beta$$

is a positive root as a sum of positive roots. Thus $s_{\tau_{k+1}} \ldots s_{\tau_1} s_\beta > s_{\tau_k} \ldots s_{\tau_1} s_\beta$ for any $k \in [\![0, n-1]\!]$ and therefore $\ell(s_\beta u) = \ell(u^{-1} s_\beta) = n + \ell(s_\beta) = \ell(s_\beta) + \ell(u)$. $\square$

Lemma 3.7. *Let $\lambda \in Y^{++}$ be a dominant coweight and let $\beta \in \Phi_+$ be a positive root such that $\lambda + \beta^\vee \in Y^+$. Let $v \in W^\lambda$, $w \in W$ and let u denote the element $v^{\lambda+\beta^\vee}$.*

Then, if $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ (resp. $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$) covers $\boldsymbol{x} = \pi^{v(\lambda)}w$,

$$\ell(vu) = \ell(v) + \ell(u) \quad (\text{resp. } \ell(vs_\beta u) = \ell(v) + \ell(s_\beta u) \text{ and } \ell(vs_\beta) = \ell(v) + \ell(s_\beta)).$$

Proof. To simplify the notation, write W_J for $W_{(\lambda+\beta^\vee)^{++}}$. Note that, with the notation of Definition 1.21, $vu = \mathrm{proj}_{vuW_J}(v)$ since u is the element of minimal length in uW_J.

Suppose by contradiction that $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ covers $\boldsymbol{x}$ with $\ell(vu) < \ell(v) + \ell(u)$. Then $d^\mathbb{N}(1, vu) = \ell(vu) < d^\mathbb{N}(1, v) + d^\mathbb{N}(v, vu) = \ell(v) + \ell(u)$, so v is not on a

minimal gallery from 1 to vu. Therefore by Lemma 1.20, there is a reflection $r \in W$ such that $d(1, rvu) > d(1, vu)$ and $d(1, rv) < d(1, v)$, that is to say $rv < v$ and $rvu > vu$.

By minimality of u, r is not in $vuW_J(vu)^{-1}$: otherwise $rvu \in vuW_J$ satisfies $d(v, rvu) = d(rv, vu) < d(v, vu)$, because foldings reduce the vectorial distance and v, vu are on different sides of the wall M_r associated to r.

Since vu is the projection of v on vuW_J which is convex (see [Ronan 1989, Lemma 2.10]), and since the wall M_r separates v and vu, any element of vuW_J is on the same side of the wall M_r as vu, so $rvu\tilde{u} > vu\tilde{u}$ for any $\tilde{u} \in W_J$. In particular, let $\tilde{u} \in W_J$ be such that $rvu\tilde{u}$ is the minimal coset representative of $rvuW_J$. Then by Proposition 2.3, since $rvu\tilde{u} > vu\tilde{u}$, we have

$$(3.11) \qquad \pi^{rv(\lambda+\beta^\vee)} r s_{v(\beta)} w = \pi^{rvu\tilde{u}((\lambda+\beta^\vee)^{++})} r s_{v(\beta)} w$$
$$< \pi^{vu\tilde{u}((\lambda+\beta^\vee)^{++})} s_{v(\beta)} w = \pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w.$$

Therefore by Proposition 2.3 for the left- and right-hand side inequalities and (3.8) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda, \beta, rv, rw, 1)$ for the middle one, we have a chain

$$(3.12) \qquad \pi^{v(\lambda)} w < \pi^{rv(\lambda)} rw < \pi^{rv(\lambda+\beta^\vee)} s_{rv(\beta)} rw$$
$$= \pi^{rv(\lambda+\beta^\vee)} r s_{v(\beta)} w < \pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w.$$

Therefore if $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$ covers x then $\ell(vu) = \ell(v) + \ell(u)$.

Now assume by contradiction that $\pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w$ covers $\pi^\lambda w$ with $\ell(vs_\beta u) < \ell(v) + \ell(s_\beta u)$. Then, similarly there is a reflection $r \in W$ such that $rv < v$ and $rvs_\beta u\tilde{u} > vs_\beta u\tilde{u}$. By Proposition 2.3 for the left- and right-hand side inequalities and (3.9) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda, \beta, rv, rw, 1)$ for the middle one, we have a chain

$$(3.13) \qquad \pi^{v(\lambda)} w < \pi^{rv(\lambda)} rw < \pi^{rvs_\beta(\lambda+\beta^\vee)} s_{rv(\beta)} rw$$
$$= \pi^{rvs_\beta(\lambda+\beta^\vee)} r s_{v(\beta)} w < \pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w.$$

We deduce that if $\pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w$ covers x then

$$\ell(vs_\beta u) = \ell(v) + \ell(s_\beta u).$$

By Lemma 3.6 this is $\ell(v) + \ell(s_\beta) + \ell(u)$, by the triangle inequality we deduce that

$$\ell(v) + \ell(s_\beta) \geq \ell(vs_\beta) \geq \ell(vs_\beta u) - \ell(u) = \ell(v) + \ell(s_\beta)$$

and we obtain the second equality in this case. □

Proposition 3.8. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$, $w \in W$. Let $\beta \in \Phi_+$ be a positive root such that $\lambda + \beta^\vee \in Y^+$ and let u denote $v^{\lambda + \beta^\vee} \in W^{(\lambda + \beta^\vee)^{++}}$.*

Suppose that $y \in \{\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w, \pi^{v s_\beta (\lambda + \beta^\vee)} s_{v(\beta)} w\}$ covers $x = \pi^{v(\lambda)} w$. Then

$$(3.14) \quad \ell^a(y) - \ell^a(x) = (2\,\mathrm{ht}(\beta^\vee) - \ell(s_\beta)) - 2\Big(\ell(u) + \sum_{\tau \in \mathrm{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle\Big).$$

Proof. Let W_J denote the standard parabolic subgroup $W_{(\lambda + \beta^\vee)^{++}}$. Recall that $u = v^{\lambda + \beta^\vee}$ is the minimal element of W such that $u((\lambda + \beta^\vee)^{++}) = \lambda + \beta^\vee$, so it is the minimal representative of the coset uW_J. By Proposition 2.10 we have

$$(3.15) \quad \ell^a(\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w) - \ell^a(\pi^{v(\lambda)} w)$$
$$= 2\,\mathrm{ht}((\lambda + \beta^\vee)^{++}) - 2\,\mathrm{ht}(\lambda) + \ell_{v^{v(\lambda + \beta^\vee)}}(s_{v(\beta)} w) - \ell_v(w),$$

$$(3.16) \quad \ell^a(\pi^{v s_\beta (\lambda + \beta^\vee)} s_{v(\beta)} w) - \ell^a(\pi^{v(\lambda)} w)$$
$$= 2\,\mathrm{ht}((\lambda + \beta^\vee)^{++}) - 2\,\mathrm{ht}(\lambda) + \ell_{v^{v s_\beta (\lambda + \beta^\vee)}}(s_{v(\beta)} w) - \ell_v(w).$$

We unwrap these formulas with the help of previous results.

(1) In the case $y = \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$, let $\tilde{u} \in W_J$ be such that $vu\tilde{u} = (vu)^J = v^{v(\lambda + \beta^\vee)}$. The term $\ell_{v^{v(\lambda + \beta^\vee)}}(s_{v(\beta)} w) - \ell_v(w)$ rewrites as

$$\ell((u\tilde{u})^{-1} s_\beta v^{-1} w) - \ell(vu\tilde{u}) - \ell(v^{-1} w) + \ell(v).$$

Since $y > x$ is a covering, by Proposition 3.3, $v s_\beta u \tilde{u} = s_{v(\beta)} (vu)^J$ is on a minimal gallery from v to w, so $\ell(v^{-1} w) = \ell((v s_\beta u \tilde{u})^{-1} w) + \ell(s_\beta u \tilde{u})$. Moreover by Lemma 3.6, $\ell(s_\beta u \tilde{u}) = \ell(s_\beta) + \ell(u\tilde{u})$ and, by Lemma 3.7, $\ell(vu) = \ell(v) + \ell(u)$. Finally, by (1.3), since $u = u^J = v^{\lambda + \beta^\vee}$ and $vu\tilde{u} = (vu)^J = v^{v(\lambda + \beta^\vee)}$, we have $\ell(u\tilde{u}) = \ell(u) + \ell(\tilde{u})$ and $\ell(vu) = \ell(vu\tilde{u}) + \ell(\tilde{u})$. Thus

$$(3.17) \quad \ell_{v^{v(\lambda + \beta^\vee)}}(s_{v(\beta)} w) - \ell_v(w) = \ell((u\tilde{u})^{-1} s_\beta v^{-1} w) - \ell(v^{-1} w) - \ell(vu\tilde{u}) + \ell(v)$$
$$= -\ell(s_\beta u\tilde{u}) - \ell(vu) + \ell(\tilde{u}) + \ell(v)$$
$$= -\ell(s_\beta) - \ell(u\tilde{u}) - \ell(u) + \ell(\tilde{u})$$
$$= -\ell(s_\beta) - 2\ell(u).$$

(2) In the second case, let $\tilde{u} \in W_J$ be such that $v s_\beta u \tilde{u} = (v s_\beta u)^J = v^{v s_\beta (\lambda + \beta^\vee)}$. Then $\ell_{v^{v s_\beta (\lambda + \beta^\vee)}}(s_{v(\beta)} w) - \ell_v(w)$ rewrites as $\ell((u\tilde{u})^{-1} v^{-1} w) - \ell(v s_\beta u \tilde{u}) - \ell(v^{-1} w) + \ell(v)$. By Proposition 3.3, $\ell((u\tilde{u})^{-1} v^{-1} w) = \ell(v^{-1} w) - \ell(u\tilde{u})$. By (1.3),

$$\ell(u\tilde{u}) = \ell(u) + \ell(\tilde{u}) \quad \text{and} \quad \ell(v s_\beta u \tilde{u}) = \ell(v s_\beta u) - \ell(\tilde{u}).$$

By Lemmas 3.7 and 3.6,

$$\ell(v s_\beta u) = \ell(v) + \ell(s_\beta u) = \ell(v) + \ell(s_\beta) + \ell(u).$$

Thus, in this case,

$$(3.18) \quad \ell_{v^{vs_\beta(\lambda+\beta^\vee)}}(s_{v(\beta)}w) - \ell_v(w)$$
$$= \ell((u\tilde{u})^{-1}v^{-1}w) - \ell(vs_\beta u\tilde{u}) - \ell(v^{-1}w) + \ell(v)$$
$$= \ell(v^{-1}w) - \ell(u\tilde{u}) - (\ell(vs_\beta u) - \ell(\tilde{u})) - \ell(v^{-1}w) + \ell(v)$$
$$= -\ell(s_\beta) - 2\ell(u).$$

(3) By Lemma 1.13 we have

$$(3.19) \quad 2\,\mathrm{ht}((\lambda+\beta^\vee)^{++}) = 2\Big(\mathrm{ht}(\lambda+\beta^\vee) - \sum_{\tau\in\mathrm{Inv}(u^{-1})}\langle\lambda+\beta^\vee,\tau\rangle\Big)$$
$$= 2\Big(\mathrm{ht}(\lambda) + \mathrm{ht}(\beta^\vee) - \sum_{\tau\in\mathrm{Inv}(u^{-1})}\langle\lambda+\beta^\vee,\tau\rangle\Big).$$

By plugging (3.17), (3.19) into (3.15), and (3.18), (3.19) into (3.16) we obtain, either way,

$$\ell^a(\boldsymbol{y}) - \ell^a(\boldsymbol{x}) = 2\,\mathrm{ht}(\lambda) + 2\,\mathrm{ht}(\beta^\vee) - 2\sum_{\tau\in\mathrm{Inv}(u^{-1})}\langle\lambda+\beta^\vee,\tau\rangle - 2\,\mathrm{ht}(\lambda) - \ell(s_\beta) - 2\ell(u)$$
$$= (2\,\mathrm{ht}(\beta^\vee) - \ell(s_\beta)) - 2\Big(\ell(u) + \sum_{\tau\in\mathrm{Inv}(u^{-1})}\langle\lambda+\beta^\vee,\tau\rangle\Big). \qquad \square$$

Using Corollary 1.12, it is easy to see that $2\,\mathrm{ht}(\beta^\vee) - \ell(s_\beta)$ is always positive and that, on the contrary, $\ell(u) + \sum_{\tau\in\mathrm{Inv}(u^{-1})}\langle\lambda+\beta^\vee,\tau\rangle$ is always nonpositive. Therefore, the length difference is equal to 1 if and only if in the right-hand side of (3.14), the first term is equal to 1 and the second term cancels out. This motivates the following definitions.

Definition 3.9. A coweight $\mu \in Y^+$ is *almost dominant* if and only if

$$(3.20) \qquad\qquad \forall\tau \in \Phi_+, \ \langle\mu,\tau\rangle \geq -1.$$

A root $\beta \in \Phi_+$ is *a quantum root* if and only if

$$(3.21) \qquad\qquad \ell(s_\beta) = 2\,\mathrm{ht}(\beta^\vee) - 1.$$

The notion of quantum roots comes from the definition of quantum Bruhat graphs, (see [Lenart et al. 2015, §4.1]). With Notation 3.5, in Section 3.3 we prove that if $\boldsymbol{y}$ covers $\boldsymbol{x}$ then $\lambda + \beta^\vee$ is almost dominant and we prove in Section 3.4 that β needs to be a quantum root.

Remark 3.10. If $\lambda + \beta^\vee$ is dominant, then the second term in the right-hand side of (3.14) immediately cancels out, since in this case $u = 1_W$. In the reductive case, Φ is finite and therefore if λ is far enough in the fundamental chamber (meaning

that $\langle \lambda, \alpha_i \rangle$ is large for all $i \in I$, we say that λ is superregular), then $\lambda + \beta^\vee$ is always dominant. Accordingly, covers of $\pi^{v(\lambda)}w$ for λ superregular are easier to classify (see [Lam and Shimozono 2010, Proposition 4.4; Milićević 2021, Proposition 4.4]).

3.3. *Almost-dominance in properly affine covers.* We prove that the second term of the right-hand side of (3.14) need to be zero when y covers x (with Notation 3.5), through the following proposition:

Proposition 3.11. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$ and $w \in W$. Let $\beta \in \Phi_+$ be a positive root such that $\lambda + \beta^\vee \in Y^+$ and suppose that $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ or $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ covers $\pi^{v(\lambda)}w$. Then $\lambda + \beta^\vee$ is almost dominant, that is to say*

$$(3.22) \qquad\qquad \forall \tau \in \Phi_+, \ \langle \lambda + \beta^\vee, \tau \rangle \geq -1.$$

It is deduced from the following two technical lemmas; we give their proofs after the proof of Proposition 3.11.

Lemma 3.12. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$, $w \in W$, $\beta \in \Phi_+$. Suppose that there exists a pair $(\tau, n) \in \Phi_+ \times \mathbb{Z}$ such that*

(i) $n > 0$,

(ii) $\langle \lambda + n\tau^\vee, \beta \rangle \geq -1$,

(iii) $n < -\langle \lambda + \beta^\vee, \tau \rangle$.

Then, $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ and $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ do not cover $\pi^{v(\lambda)}w$.

Lemma 3.13. *Let $\lambda \in Y^{++}$ and $\beta \in \Phi_+$ be such that $\lambda + \beta^\vee$ lies in Y^+. Let $\tau \in \Phi_+$ be such that $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$ and suppose that $\langle \tau^\vee, \beta \rangle \leq -2$. Then*

$$\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1.$$

Proof of Proposition 3.11. We prove the contrapositive: Let $\tau \in \Phi_+$ be a positive root such that $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$. We will produce nontrivial chains from $\pi^{v(\lambda)}w$ to $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ and $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$. In particular since λ is dominant, $\langle \beta^\vee, \tau \rangle \leq -2$.

The numbers $\langle \tau^\vee, \beta \rangle$ and $\langle \beta^\vee, \tau \rangle$ have the same sign [Bardy 1996, Lemma 1.1.10], and therefore we have that $\langle \tau^\vee, \beta \rangle \leq -1$.

Suppose first that $\langle \tau^\vee, \beta \rangle \leq -2$. Then $\big(\tau, -(\langle \lambda + \beta^\vee, \tau \rangle + 1)\big)$ is a pair which satisfy the conditions of Lemma 3.12:

(i) This is true since $\langle \lambda + \beta^\vee, \tau \rangle \leq -2$, and $-(\langle \lambda + \beta^\vee, \tau \rangle + 1) \geq 1 > 0$.

(ii) By Lemma 3.13, $\langle \lambda + \beta^\vee, s_\tau(\beta)\rangle \geq -1$; thus

$$\begin{aligned}
\langle \lambda - ((\lambda + \beta^\vee, \tau) + 1)\tau^\vee, \beta\rangle &= \langle s_\tau(\lambda + \beta^\vee) - \beta^\vee - \tau^\vee, \beta\rangle \\
&= \langle \lambda + \beta^\vee, s_\tau(\beta)\rangle - 2 - \langle \tau^\vee, \beta\rangle \\
&\geq \langle \lambda + \beta^\vee, s_\tau(\beta)\rangle \geq -1.
\end{aligned}$$

(iii) Clearly $-(\langle \beta^\vee, \tau\rangle + 1) < -\langle \beta^\vee, \tau\rangle$.

Suppose now that $\langle \tau^\vee, \beta\rangle = -1$. We show that $(\tau, 1)$ is a pair satisfying the conditions of Lemma 3.12:

(i) The first point is trivially verified.

(ii) Since $\langle \tau^\vee, \beta\rangle = -1$ and λ is dominant, $\langle \lambda + \tau^\vee, \beta\rangle \geq -1$.

(iii) Since $\langle \lambda + \beta^\vee, \tau\rangle \leq -2$ we obtain $1 < -\langle \lambda + \beta^\vee, \tau\rangle$.

Hence, either way, if such a $\tau \in \Phi_+$ exists, then by Lemma 3.12 $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ and $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ do not cover $\pi^{v(\lambda)}w$. $\qquad\square$

Proof of Lemma 3.12. We use conditions (i), (ii), (iii) in the statement to produce chains from $\pi^{v(\lambda)}w$ to $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ and $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$.

Suppose first that (ii) is strict. Then we show that we have the chains

$$(3.23) \quad \pi^{v(\lambda)}w < \pi^{vs_\tau(\lambda+n\tau^\vee)}s_{v(\tau)}w < \pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w,$$

$$(3.24) \quad \pi^{v(\lambda)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee+n\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

(a) By (i), since λ is dominant and τ is a positive root, applying (3.8) with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda, \tau, v, w, n)$, we have

$$(3.25) \qquad\qquad \pi^{v(\lambda)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w.$$

Using (3.9) with the same parameters gives

$$(3.26) \qquad\qquad \pi^{v(\lambda)}w < \pi^{vs_\tau(\lambda+n\tau^\vee)}s_{v(\tau)}w.$$

(b) Since $\langle \tau^\vee, \tau\rangle = 2$, (iii) is equivalent to $-n < -\langle \lambda + \beta^\vee + n\tau^\vee, \tau\rangle$, so, using (3.9) for $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + \beta^\vee + n\tau^\vee, \tau, vs_\tau, s_{v(\tau)}s_{v(\beta)}w, -n)$, we get

$$(3.27) \qquad \pi^{vs_\tau(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\tau)}s_{v(\beta)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

Using (3.8) for $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + \beta^\vee + n\tau^\vee, \tau, vs_\beta, s_{v(\beta)}s_{v(\tau)}w, -n)$,

$$(3.28) \qquad \pi^{vs_\beta(\lambda+\beta^\vee+n\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

We now split the argument in two cases, according to whether (ii) is strict.

(1) Suppose that (ii) is strict, so $\langle \lambda + n\tau^\vee, \beta \rangle \geq 0$. Then by (3.8) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + n\tau^\vee, \beta, vs_\tau, s_{v(\tau)}w, 1)$, we get

$$(3.29) \qquad \pi^{vs_\tau(\lambda + n\tau^\vee)} s_{v(\tau)} w < \pi^{vs_\tau(\lambda + \beta^\vee + n\tau^\vee)} s_{v(\tau)} s_{v(\beta)} w.$$

Moreover by (3.9) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + n\tau^\vee, \beta, v, s_{v(\tau)}w, 1)$,

$$(3.30) \qquad \pi^{v(\lambda + n\tau^\vee)} s_{v(\tau)} w < \pi^{vs_\beta(\lambda + \beta^\vee + n\tau^\vee)} s_{v(\beta)} s_{v(\tau)} w.$$

Thus, if (ii) is strict, combining (3.25), (3.30) and (3.28) we obtain the chain (3.24). Moreover combining (3.26), (3.29) and (3.27) we obtain the chain (3.23). This proves Lemma 3.12 in this case.

(2) Suppose now that $\langle \lambda + n\tau^\vee, \beta \rangle = -1$. Then note that $\lambda + n\tau^\vee + \beta^\vee = s_\beta(\lambda + n\tau^\vee)$, and (3.8), (3.9) cannot be used for the middle inequalities of chains (3.24) and (3.23) anymore.

(a) Case of $\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$.

  (i) If $vs_\tau(\beta) \in \Phi_+$ then we can apply (1.12) to the element $\pi^{vs_\tau(\lambda + n\tau^\vee)} s_{v(\tau)} w$ and the positive affinized root $vs_\tau(\beta)[0]$, and since $\langle \lambda + n\tau^\vee, \beta \rangle = -1 < 0$, we still have

$$(3.31) \quad \pi^{vs_\tau(\lambda + n\tau^\vee)} s_{v(\tau)} w < s_{vs_\tau(\beta)[0]} \pi^{vs_\tau(\lambda + n\tau^\vee)} s_{v(\tau)} w = \pi^{vs_\tau(\lambda + \beta^\vee + n\tau^\vee)} s_{v(\tau)} s_{v(\beta)} w$$

and the chain (3.23) still holds by (3.26), (3.31) and (3.27).

  (ii) If $vs_\tau(\beta) \in \Phi_-$ note that, since $\langle \lambda + n\tau^\vee, \beta \rangle < 0$, $\langle \tau^\vee, \beta \rangle < 0$, so $s_\tau(\beta)$ is a positive root. Therefore $vs_\tau(\beta) \in \Phi_-$ is equivalent to $s_\tau(\beta) \in \mathrm{Inv}(v)$. Since $v \in W^\lambda$, by minimality of v we have $\langle \lambda, s_\tau(\beta) \rangle \neq 0$. Then, by Proposition 2.3,

$$(3.32) \qquad \pi^{v(\lambda)} w < \pi^{vs_{s_\tau(\beta)}(\lambda)} s_{vs_\tau(\beta)} w$$

and by (3.9) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda, \tau, vs_{s_\tau(\beta)}, s_{vs_\tau(\beta)}w, 1)$ we get

$$(3.33) \quad \pi^{vs_{s_\tau(\beta)}(\lambda)} s_{vs_\tau(\beta)} w < \pi^{vs_\tau s_\beta(\lambda + n\tau^\vee)} s_{v(\tau)} s_{v(\beta)} w = \pi^{vs_\tau(\lambda + \beta^\vee + n\tau^\vee)} s_{v(\tau)} s_{v(\beta)} w.$$

For the vectorial elements, we used the fact that $s_{vs_\tau(\beta)} = s_{v(\tau)} s_{v(\beta)} s_{v(\tau)} = vs_\tau s_\beta s_\tau v^{-1}$ and $s_{vs_{s_\tau(\beta)}(\tau)} = vs_\tau s_\beta s_\tau s_\beta s_\tau v^{-1}$, and hence $s_{vs_{s_\tau(\beta)}(\tau)} s_{vs_\tau(\beta)} = vs_\tau s_\beta v^{-1} = s_{v(\tau)} s_{v(\beta)}$.

Combining (3.32), (3.33) and (3.27) we obtain the chain

$$\pi^{v(\lambda)} w < \pi^{vs_\tau s_\beta s_\tau(\lambda)} s_{v(\tau)} s_{v(\beta)} s_{v(\tau)} w < \pi^{v(s_\tau(\lambda + \beta^\vee + n\tau^\vee))} s_{v(\tau)} s_{v(\beta)} w < \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$$

which replaces the chain (3.23).

(b) Case of $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$.

    (i) If $w^{-1}vs_\tau(\beta) \in \Phi_-$, by (1.12) applied to $\pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w$ and the affinized root $v(\beta)[\langle\lambda+n\tau^\vee, \beta\rangle]$, since $\langle\lambda+n\tau^\vee, \beta\rangle < 0$,

$$(3.34) \qquad \pi^{v(\lambda+n\tau^\vee)}s_{v(\tau)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w = \pi^{vs_\beta(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w.$$

Therefore the chain (3.24) still holds by (3.25), (3.34) and (3.28).

    (ii) If $w^{-1}vs_\tau(\beta) \in \Phi_+$, then using (1.12) with $\pi^{v(\lambda)}w$ and the affinized root $vs_\tau(\beta)\big[\langle\lambda, s_\tau(\beta)\rangle\big]$ (which is always possible because if $\langle\lambda, s_\tau(\beta)\rangle = 0$ then by minimality of v, $vs_\tau(\beta) \in \Phi_+$), we obtain

$$(3.35) \qquad\qquad \pi^{v(\lambda)}w < \pi^{v(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w.$$

Moreover, by (3.8) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda, \tau, v, s_{vs_\tau(\beta)}w, n)$, we get

$$(3.36) \qquad\qquad \pi^{v(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{v(\lambda+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w.$$

Hence combining (3.35), (3.36) and (3.28) we obtain a chain

$$\pi^{v(\lambda)}w < \pi^{v(\lambda)}s_{v(\tau)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee+n\tau^\vee)}s_{v(\beta)}s_{v(\tau)}w < \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

Therefore, in all cases, if such a pair (τ, n) exists, then $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ and $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$ do not cover $\pi^{v(\lambda)}w$. $\qquad\qquad\square$

Proof of Lemma 3.13. The proof relies on the assumption that $\lambda + \beta^\vee$ lies in the Tits cone, which is equivalent to saying that there is only a finite number of positive roots α such that $\langle\lambda+\beta^\vee, \alpha\rangle < 0$.

We will show that $\langle\lambda+\beta^\vee, (s_\tau s_\beta)^n(\tau)\rangle \geq 0$ for n large enough implies

$$(3.37) \qquad\qquad \langle\lambda+\beta^\vee, s_\tau(\beta)\rangle \geq -1,$$

which implies the lemma. To shorten the computation, let us write $a = -\langle\beta^\vee, \tau\rangle$ and $a^\vee = -\langle\tau^\vee, \beta\rangle$. So the assumptions $\langle\lambda+\beta^\vee, \tau\rangle \leq -2$ and $\langle\tau^\vee, \beta\rangle \leq -2$ imply that $a \geq 2 + \langle\lambda, \tau\rangle$ and $a^\vee \geq 2$. In the basis (β, τ) of $\mathbb{R}\beta \oplus \mathbb{R}\tau$, the matrix of $s_\tau s_\beta$ is $M = \left(\begin{smallmatrix} -1 & a \\ -a^\vee & aa^\vee-1 \end{smallmatrix}\right)$. We have $\chi_M = X^2 + (2-aa^\vee)X + 1$; thus, since $aa^\vee \geq 4$, $M^2 = (aa^\vee-2)M - I_2$. Write $M^n = \mu_n M + \nu_n I_2$ for $n \in \mathbb{N}$. Then an easy computation shows that $\nu_n = -\mu_{n-1}$ and $\mu_{n+1} = (aa^\vee-2)\mu_n - \mu_{n-1}$. In particular since $aa^\vee - 2 \geq 2$ and $\mu_0 = 0 < \mu_1$, by iteration $(\mu_n)_{n\in\mathbb{N}}$ is strictly increasing.

Let $x = \langle\lambda, \beta\rangle \geq 0$ and $y = \langle\lambda, \tau\rangle \in [\![0, a-2]\!]$. Then

$$(3.38) \qquad \langle\lambda+\beta^\vee, (s_\tau s_\beta)^n(\tau)\rangle = \big\langle\lambda+\beta^\vee, a\mu_n\beta + ((aa^\vee-1)\mu_n - \mu_{n-1})\tau\big\rangle$$
$$= (x+2)\mu_n a + ((aa^\vee-1)\mu_n - \mu_{n-1})(y-a).$$

Since $\lambda + \beta^\vee$ lies in the Tits cone, $\langle \lambda + \beta^\vee, (s_\tau s_\beta)^n(\tau) \rangle$ is nonnegative for n large enough. Moreover, $\mu_{n-1} < \mu_n$ for all $n \in \mathbb{N}$ and $a - y > 0$. Therefore we deduce from (3.38) that, for n large enough,

$$(x+2)\mu_n a \geq (a-y)((aa^\vee - 1)\mu_n - \mu_{n-1}) > (a-y)\mu_n(aa^\vee - 2).$$

Hence

$$(x+2) > (a-y)\left(a^\vee - \frac{2}{a}\right) = aa^\vee - a^\vee y - 2 + 2\frac{y}{a}.$$

Therefore $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle = x + 2 + a^\vee y - aa^\vee > -2 + 2\frac{y}{a}$ and, since it is an integer, we deduce $\langle \lambda + \beta^\vee, s_\tau(\beta) \rangle \geq -1 \geq 1 - a^\vee$, which proves the result. $\square$

Corollary 3.14. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$, $w \in W$. Let $\beta \in \Phi_+$ be a positive root such that $\lambda + \beta^\vee \in Y^+$. Suppose that $y \in \{\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w, \pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w\}$ covers $x = \pi^{v(\lambda)}w$.*

Then

$$(3.39) \qquad \ell^a(y) - \ell^a(x) = 2\,\mathrm{ht}(\beta^\vee) - \ell(s_\beta).$$

Proof. Let $u = v^{\lambda + \beta^\vee} \in W$. Then for any $\tau \in \mathrm{Inv}(u^{-1})$, by Lemma 2.1, $\langle \lambda + \beta^\vee, \tau \rangle$ is negative. By Proposition 3.11, $\langle \lambda + \beta^\vee, \tau \rangle = -1$ for any such τ. Therefore

$$(3.40) \qquad \sum_{\tau \in \mathrm{Inv}(u^{-1})} \langle \lambda + \beta^\vee, \tau \rangle = -|\mathrm{Inv}(u^{-1})| = -\ell(u).$$

We then directly obtain (3.39) from (3.14) and (3.40). $\square$

3.4. *Properly affine covers and quantum roots.*

We now prove in Proposition 3.19 that, with Notation 3.5, if β is not a quantum root, then y does not cover x. This is enough, together with Corollary 3.14, to conclude that $\ell^a(y) - \ell^a(x) = 1$. There is a subtlety if the root β lies in a subsystem of Φ of type G_2; we suppose that this is not the case in Lemmas 3.16 and 3.17, and we deal with the G_2 case in Lemma 3.18. Let us first give another characterization of quantum roots.

Lemma 3.15. *A root $\beta \in \Phi_+$ is a quantum root if and only if $\langle \beta^\vee, \gamma \rangle = 1$ for all $\gamma \in \mathrm{Inv}(s_\beta) \setminus \{\beta\}$.*

Proof. Recall that a quantum root is a root $\beta \in \Phi_+$ such that $2\,\mathrm{ht}(\beta^\vee) = \ell(s_\beta) + 1$. By Corollary 1.12, this is equivalent to

$$(3.41) \qquad \sum_{\gamma \in \mathrm{Inv}(s_\beta)} \langle \beta^\vee, \gamma \rangle = \ell(s_\beta) + 1.$$

For any $\gamma \in \mathrm{Inv}(s_\beta)$, γ is a positive root and $s_\beta(\gamma) = \gamma - \langle \beta^\vee, \gamma \rangle \beta$ is a negative root, and therefore $\langle \beta^\vee, \gamma \rangle \geq 1$. Moreover, $\langle \beta^\vee, \beta \rangle = 2$ and $|\mathrm{Inv}(s_\beta)| = \ell(s_\beta)$. Therefore (3.41) is satisfied if and only if $\langle \beta^\vee, \gamma \rangle$ is exactly one for all $\gamma \in \mathrm{Inv}(s_\beta) \setminus \{\beta\}$. $\square$

Lemma 3.16. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$, $w \in W$ and $\beta \in \Phi_+$. Let $\gamma \in \mathrm{Inv}(s_\beta) \setminus \{\beta\}$ be such that $\langle \beta^\vee, \gamma \rangle \geq 2$ and suppose that $\beta \notin \mathrm{Inv}(s_\gamma)$. Then $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$ and $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$ do not cover $\pi^{vs_\beta(\lambda)} w$.*

Proof. By the contrapositive of [Proposition 3.11](#), we can suppose that $\langle \lambda + \beta^\vee, \tau \rangle \geq -1$ for any $\tau \in \Phi_+$. Let γ be as in the statement and write $\alpha = s_\gamma(\beta) \in \Phi_+$. We will construct nontrivial chains in the same fashion as in the proof of [Lemma 3.12](#). Beforehand, we show by computation that $\langle \lambda + \gamma^\vee, \alpha \rangle \geq -1$. If $\langle \gamma^\vee, \beta \rangle = 1 = -\langle \gamma^\vee, \alpha \rangle$ this is clear since λ is dominant. If $\langle \gamma^\vee, \beta \rangle \geq 2$,

$$
\begin{aligned}
\langle \lambda + \gamma^\vee, \alpha \rangle &= \langle \lambda + \beta^\vee - \alpha^\vee + (1 - \langle \beta^\vee, \gamma \rangle)\gamma^\vee, \alpha \rangle \\
&= \langle \lambda + \beta^\vee, \alpha \rangle + (1 - \langle \beta^\vee, \gamma \rangle)\langle \gamma^\vee, \alpha \rangle - 2 \\
&= \langle \lambda + \beta^\vee, \alpha \rangle + (\langle \beta^\vee, \gamma \rangle - 1)\langle \gamma^\vee, \beta \rangle - 2.
\end{aligned}
$$

Since $\langle \beta^\vee, \gamma \rangle \geq 2$ and $\langle \gamma^\vee, \beta \rangle \geq 2$, $(\langle \beta^\vee, \gamma \rangle - 1)\langle \gamma^\vee, \beta \rangle \geq 2$, and by assumption $\langle \lambda + \beta^\vee, \alpha \rangle \geq -1$. Thus, $\langle \lambda + \gamma^\vee, \alpha \rangle \geq -1$ either way.

We construct chains which are slight modifications of the ones constructed in the proof of [Lemma 3.12](#). We prove that, except in a few particular cases, we have the chains

$$(3.42) \quad \pi^{v(\lambda)} w < \pi^{v(\lambda+\gamma^\vee)} s_{v(\gamma)} w < \pi^{v(\lambda+\gamma^\vee+\alpha^\vee)} s_{v(\alpha)} s_{v(\gamma)} w < \pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w,$$

$$(3.43) \quad \pi^{v(\lambda)} w < \pi^{vs_\gamma(\lambda+\gamma^\vee)} s_{v(\gamma)} w < \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)} s_{v(\gamma)} s_{v(\alpha)} w < \pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w.$$

Indeed:

(1) The coweight λ is dominant and $\gamma \in \Phi_+$, so $\langle \lambda, \gamma \rangle \geq 0$ and [(3.8)](#) (resp. [(3.9)](#)) applied with $(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda, \gamma, v, w, 1)$ proves the leftmost inequality in the chain [(3.42)](#) (resp. [(3.43)](#)).

(2) Note that $\lambda + \beta^\vee = (\lambda + \gamma^\vee + \alpha^\vee) + (\langle \beta^\vee, \gamma \rangle - 1)\gamma^\vee$. Moreover

$$(3.44) \qquad\qquad 0 < \langle \beta^\vee, \gamma \rangle - 1$$

and

$$(3.45) \qquad -\langle \lambda + \gamma^\vee + \alpha^\vee, \gamma \rangle = \langle \beta^\vee, \gamma \rangle - \langle \lambda, \gamma \rangle - 2 < \langle \beta^\vee, \gamma \rangle - 1.$$

Therefore by applying [(3.8)](#) (resp. [(3.9)](#)) to

$$(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + \gamma^\vee + \alpha^\vee, \gamma, v, s_{v(\alpha)} s_{v(\gamma)} w, \langle \beta^\vee, \gamma \rangle - 1)$$

$$(\text{resp. } (\mu_0, \alpha_0, v_0, w_0, m) = \lambda + \gamma^\vee + \alpha^\vee, \gamma, vs_\gamma s_\alpha, s_{v(\gamma)} s_{v(\alpha)} w, \langle \beta^\vee, \gamma \rangle - 1))$$

we obtain the rightmost inequality in the chain [(3.42)](#) (resp. [(3.43)](#)).

Let us now split cases in order to either prove the second inequality in chains (3.42) and (3.43) or provide alternative chains.

(1) Suppose first that $\langle \lambda + \gamma^\vee, \alpha \rangle \geq 0$. Then (3.8) (resp. (3.9)) applied with

(3.46) $$(\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + \gamma^\vee, \alpha, v, s_{v(\gamma)}w, 1)$$

(3.47) $$(\text{resp. } (\mu_0, \alpha_0, v_0, w_0, m) = (\lambda + \gamma^\vee, \alpha, vs_\gamma, s_{v(\gamma)}w, 1))$$

prove the middle inequality in the chain (3.42) (resp. (3.43)).

(2) Suppose that $\langle \lambda + \gamma^\vee, \alpha \rangle = -1$. Then $\lambda + \gamma^\vee + \alpha^\vee = s_\alpha(\lambda + \gamma^\vee)$ and the above chains do not always hold. We focus here on the case of $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$.

(a) If $v(\alpha) \in \Phi_+$, since $\langle \lambda + \gamma^\vee, \alpha \rangle < 0$, the inequality $\pi^{vs_\alpha(\lambda+\gamma^\vee)}s_{v(\alpha)}s_{v(\gamma)}w > \pi^{v(\lambda+\gamma^\vee)}s_{v(\gamma)}w$ still holds, by (1.12) applied with $s_{v(\alpha)[0]}$. Therefore the chain (3.42) still holds.

(b) If $v(\alpha) \in \Phi_-$, then $vs_\alpha < v$, and we have a chain

(3.48) $$\pi^{v(\lambda)}w < \pi^{vs_\alpha(\lambda)}s_{v(\alpha)}w < \pi^{vs_\alpha(\lambda+\gamma^\vee)}s_{v(\alpha)}s_{v(\gamma)}w$$
$$= \pi^{v(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\alpha)}s_{v(\gamma)}w < \pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w.$$

The reflection used for the first inequality is $s_{-v(\alpha)[0]}$, and it holds by (1.12) because $\langle v(\lambda), -v(\alpha) \rangle = -\langle \lambda, \alpha \rangle < 0$. Note that this is nonzero because v is the minimal representative of vW_λ and thus $vs_\alpha < v$ implies $s_\alpha \notin W_\lambda$ so $\langle \lambda, \alpha \rangle \neq 0$. By (3.44) and (3.45) we can use (3.8) with

(3.49) $$(\mu_0, \alpha_0, v_0, w_0, m) = \begin{cases} (\lambda, \gamma, vs_\alpha, s_{v(\alpha)}w, 1) \\ (\lambda + \alpha^\vee + \gamma^\vee, \gamma, v, s_{v(\alpha)}s_{v(\gamma)}w, \langle \beta^\vee, \gamma \rangle - 1) \end{cases}$$

in order to obtain the second and third inequalities of chain (3.48), respectively.

(3) We suppose that $\langle \lambda + \gamma^\vee, \alpha \rangle = -1$. We deal with the case of $\pi^{vs_{v(\beta)}(\lambda+\beta^\vee)}s_{v(\beta)}w$. Then

(3.50) $$\pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w = \pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{vs_\gamma(\alpha)}s_{v(\gamma)}w$$
$$= s_{vs_\gamma(\alpha)[\langle\lambda+\gamma^\vee,\alpha\rangle]}\pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{v(\gamma)}w.$$

Moreover $(s_{v(\gamma)}w)^{-1}(vs_\gamma(\alpha)) = w^{-1}v(\alpha)$. Thus, since $\langle \lambda + \gamma^\vee, \alpha \rangle < 0$:

(a) If $w^{-1}v(\alpha) \in \Phi_-$, by (1.12), $\pi^{vs_\gamma(\lambda+\gamma^\vee)}s_{v(\gamma)}w < \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)}s_{v(\gamma)}s_{v(\alpha)}w$ and the chain (3.43) still holds.

(b) If $w^{-1}v(\alpha) \in \Phi_+$, then, since $\langle \lambda, \alpha \rangle = \langle \gamma^\vee, \beta \rangle - 1 > 0$, by (1.12), $\pi^{v(\lambda)}w < s_{v(\alpha)[\langle\lambda,\alpha\rangle]}\pi^{v(\lambda)}w = \pi^{v(\lambda)}s_{v(\alpha)}w$. Then, by (3.9) with $(\mu_0, \alpha_0, v_0, w_0, m) =$

$(\lambda, \gamma, v, s_{v(\alpha)}w, 1)$, we have

$$\pi^{v(\lambda)} s_{v(\gamma)} s_{v(\beta)} s_{v(\gamma)} w < \pi^{vs_\gamma(\lambda+\gamma^\vee)} s_{v(\beta)} s_{v(\gamma)} w = \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)} s_{v(\gamma)} s_{v(\alpha)} w$$

and we have a chain

$$\pi^{v(\lambda)} w < \pi^{v(\lambda)} s_{v(\alpha)} w < \pi^{vs_\gamma s_\alpha(\lambda+\gamma^\vee+\alpha^\vee)} s_{v(\gamma)} s_{v(\alpha)} w < \pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w. \quad \square$$

Lemma 3.17. *Let $\beta \in \Phi_+$ be a positive root and suppose that there exists $\gamma \in$ Inv$(s_\beta) \setminus \{\beta\}$ such that $\langle \beta^\vee, \gamma \rangle \geq 2$ and $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \neq 3$. Then γ can be chosen such that $\beta \notin$ Inv(s_γ).*

Proof. Note that, by [Bardy 1996, Lemma 1.1.10], for any $\beta, \gamma \in \Phi$, $\langle \beta^\vee, \gamma \rangle$ and $\langle \gamma^\vee, \beta \rangle$ have the same sign, so if $\langle \beta^\vee, \gamma \rangle \geq 2$ and $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \neq 3$, either $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee \rangle \geq 4$, either $\langle \beta^\vee, \gamma \rangle = 2$ and $\langle \gamma^\vee, \beta \rangle = 1$. We treat separately these cases:

(1) Let us first suppose that there exists $\gamma \in$ Inv(s_β) such that $\langle \beta^\vee, \gamma \rangle = 2$ and $\langle \gamma^\vee, \beta \rangle = 1$. Suppose that $\beta \in$ Inv(s_γ), so $s_\gamma(\beta) = \beta - \gamma < 0$, and $s_\beta(\gamma) = \gamma - 2\beta < 0$. Then we show that $\beta \notin$ Inv$(s_{\tilde\gamma})$ for $\tilde\gamma = -s_\beta(\gamma)$:

$$s_{\tilde\gamma}(\beta) = s_\beta s_\gamma s_\beta(\beta) = -s_\beta(\beta - \gamma) = \gamma - \beta = -s_\gamma(\beta) > 0.$$

Moreover $s_\beta(\tilde\gamma) = -\gamma < 0$ and $\langle \beta^\vee, \tilde\gamma \rangle = \langle \beta^\vee, \gamma \rangle = 2$; therefore, γ can be chosen such that $\beta \notin$ Inv(s_γ).

(2) Let us now suppose that there exists $\gamma \in$ Inv(s_β) such that $\langle \beta^\vee, \gamma \rangle \geq 2$ and $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \geq 4$. Write $\beta = v_\beta(\beta_0) = s_{\alpha_1} \ldots s_{\alpha_n}(\beta_0)$ where the α_i and β_0 are simple roots, and suppose that n is of minimal length amongst possible expressions of β. Therefore $s_{\alpha_1} \ldots s_{\alpha_n} s_{\beta_0} s_{\alpha_n} \ldots s_{\alpha_1}$ is a reduced expression of s_β and

$$\text{Inv}(s_\beta) = \{s_{\alpha_1} \ldots s_{\alpha_{p-1}}(\alpha_p) \mid p \leq n\} \sqcup \{\beta\} \sqcup \{s_{\alpha_1} \ldots s_{\alpha_n} s_{\beta_0} s_{\alpha_n} \ldots s_{\alpha_{n+1-p}}(\alpha_{n-p}) \mid p \leq n\}.$$

Let k be the smallest such that $\gamma_k = s_{\alpha_1} \ldots s_{\alpha_{k-1}}(\alpha_k)$ satisfies $\langle \beta^\vee, \gamma_k \rangle \geq 2$ and $\langle \beta^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \beta \rangle \geq 4$.

The expression $s_{\alpha_1} \ldots s_{\alpha_{k-1}} s_{\alpha_k} s_{\alpha_{k-1}} \ldots s_{\alpha_1}$ is an expression of s_{γ_k}; thus

$$\text{Inv}(s_{\gamma_k}) \subset \{s_{\alpha_1} \ldots s_{\alpha_{p-1}}(\alpha_p) \mid p \leq k-1\} \sqcup \{\gamma_k\}$$
$$\sqcup \{s_{\alpha_1} \ldots s_{\alpha_k} s_{\alpha_{k-1}} \ldots s_{\alpha_{k+1-p}}(\alpha_{k-p}) \mid p \leq k-1\}.$$

Suppose by contradiction that $\beta \in$ Inv(s_{γ_k}). Since v_β is of minimal length, β is not in the first set; thus there is $p \in [\![1, k-1]\!]$ such that $\beta = s_{\alpha_1} \ldots s_{\alpha_k} s_{\alpha_{k-1}} \ldots s_{\alpha_{k+1-p}}(\alpha_{k-p})$.

We show that $\gamma_{k-p} = s_{\alpha_1} \dots s_{\alpha_{k-p-1}}(\alpha_{k-p}) \in \mathrm{Inv}(s_\beta)$ satisfies $\langle \beta^\vee, \gamma_{k-p} \rangle \geq 2$, which contradicts the minimality of k. Note that $\beta = -s_{\gamma_k}(\gamma_{k-p})$. We compute

$$
\begin{aligned}
\langle \beta^\vee, \gamma_{k-p} \rangle &= \langle -s_{\gamma_k}(\gamma_{k-p}^\vee), \gamma_{k-p} \rangle \\
&= -(2 - \langle \gamma_{k-p}^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \gamma_{k-p} \rangle) \\
&= \langle \beta^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \beta \rangle - 2.
\end{aligned}
$$

So since $\langle \beta^\vee, \gamma_k \rangle \langle \gamma_k^\vee, \beta \rangle \geq 4$, we get $\langle \beta^\vee, \gamma_{k-p} \rangle \geq 2$, and with a similar computation, we find that $\langle \gamma_{k-p}^\vee, \beta \rangle = \langle \beta^\vee, \gamma_{k-p} \rangle \geq 2$ as well, so $\langle \beta^\vee, \gamma_{k-p} \rangle \langle \gamma_{k-p}^\vee, \beta \rangle \geq 4$. This contradicts the minimality of k and thus $\beta \notin \mathrm{Inv}(s_{\gamma_k})$. $\square$

Lemma 3.18. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$ and $w \in W$. Let $\beta \in \Phi_+$ and let $\gamma \in \mathrm{Inv}(s_\beta)$ be such that $\beta \in \mathrm{Inv}(s_\gamma)$ and $\langle \beta^\vee, \gamma \rangle = 3$, $\langle \gamma^\vee, \beta \rangle = 1$.*
  Then $\pi^{v(\lambda+\beta^\vee)} s_{v(\beta)} w$ and $\pi^{vs_\beta(\lambda+\beta^\vee)} s_{v(\beta)} w$ do not cover $\pi^{v(\lambda)} w$.

Proof. We show that, with the assumptions of the statement, β and γ appear as positive roots of a root subsystem of Φ isomorphic to G_2, and we use this system to construct chains replacing the ones in the proof of Lemma 3.16.

First, note that $-s_\gamma(\beta)$ lies in $\mathrm{Inv}(s_\beta)$ (so $s_\beta s_\gamma(\beta)$ is positive). Indeed, we can write, as in the proof of Lemma 3.17, $\beta = s_{\alpha_1} \dots s_{\alpha_n}(\beta_0)$ for a minimal n, and $\gamma = s_{\alpha_1} \dots s_{\alpha_{k-1}}(\alpha_k)$ for some $k \leq n$. Then, since $\beta \in \mathrm{Inv}(s_\gamma)$, β is also of the form $s_{\alpha_1} \dots s_{\alpha_k} s_{\alpha_{k-1}} \dots s_{\alpha_{k-p+1}}(\alpha_{k-p})$ for some $p \leq k-1$, and thus

$$
-s_\gamma(\beta) = s_{\alpha_1} \dots s_{\alpha_{k-p-1}}(\alpha_{k-p}) \in \mathrm{Inv}(s_\beta).
$$

Therefore we have the following positive roots, and their associated coroots (the notation will become clear afterwards):

(1) $\theta_1 := -s_\gamma(\beta) = \gamma - \beta \in \Phi_+$, with associated coroot $\theta_1^\vee = -s_\gamma(\beta^\vee) = 3\gamma^\vee - \beta^\vee$.

(2) $\tilde{\beta} := -s_\beta(\gamma) = 3\beta - \gamma \in \Phi_+$, with associated coroot $\tilde{\beta}^\vee = -s_\beta(\gamma^\vee) = \beta^\vee - \gamma^\vee$.

(3) $\tilde{\gamma} := s_\beta s_\gamma(\beta) = 2\beta - \gamma \in \Phi_+$, with associated coroot $\tilde{\gamma}^\vee = s_\beta s_\gamma(\beta^\vee) = 2\beta^\vee - 3\gamma^\vee$.

Let us also define $\theta_2 = s_{\theta_1}(\gamma) = 3\beta - 2\gamma$, with associated coroot $\theta_2^\vee = \beta^\vee - 2\gamma^\vee$. Then one can check that $\{\theta_1, \theta_2\}$ form the positive simple roots of a G_2 root system (in the sense that $\langle \theta_1^\vee, \theta_2 \rangle = -3$ and $\langle \theta_2^\vee, \theta_1 \rangle = -1$), such that $\gamma = s_{\theta_1}(\theta_2)$, $\beta = s_{\theta_1} s_{\theta_2}(\theta_1)$, $\tilde{\gamma} = s_{\theta_2}(\theta_1)$ and $\tilde{\beta} = s_{\theta_2} s_{\theta_1}(\theta_2)$. However, θ_2 may not be a positive root in Φ, and we thus need to distinguish these two cases.

Let us first suppose that θ_2 lies in Φ_+. Notice that

$$
(3.51) \qquad \theta_1^\vee + \tilde{\beta}^\vee + \theta_2^\vee = (3\gamma^\vee - \beta^\vee) + (\beta^\vee - \gamma^\vee) + (\beta^\vee - 2\gamma^\vee) = \beta^\vee,
$$

and

$$(3.52) \qquad s_{\theta_1} s_{\tilde{\beta}} s_{\theta_2} = s_{\theta_1} (s_{\theta_2} s_{\theta_1} s_{\theta_2} s_{\theta_1} s_{\theta_2}) s_{\theta_2} = s_{\theta_1} s_{\theta_2} s_{\theta_1} s_{\theta_2} s_{\theta_1} = s_{\theta_2} s_{\tilde{\beta}} s_{\theta_1} = s_{\beta}.$$

Moreover, we have

$$(3.53) \qquad \langle \theta_2^\vee, \tilde{\beta} \rangle = \langle \beta^\vee - 2\gamma^\vee, 3\beta - \gamma \rangle = 1 > 0,$$

$$(3.54) \qquad \langle \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle = \langle 2\beta^\vee - 3\gamma^\vee, \gamma - \beta \rangle = -1.$$

(1) Suppose first that $\langle \lambda, \theta_1 \rangle > 0$. Since λ is dominant and by (3.53), (3.54),

$$-\langle \lambda + \theta_2^\vee, \tilde{\beta} \rangle < 0 \quad \text{and} \quad -\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle \le 0.$$

Using (3.8) (resp. (3.9)) with $(\mu_0, \alpha_0, m) = (\lambda, \theta_2, 1)$ for the first inequality, $(\mu_0, \alpha_0, m) = (\lambda + \theta_2^\vee, \tilde{\beta}, 1)$ for the second and $(\mu_0, \alpha_0, m) = (\lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1, 1)$ for the third (recall (3.51), (3.52)), we obtain the chain (3.55) (resp. (3.56))

$$(3.55) \quad \pi^{v(\lambda)} w < \pi^{v(\lambda + \theta_2^\vee)} s_{v(\theta_2)} w < \pi^{v(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\tilde{\beta})} s_{v(\theta_2)} w < \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w,$$

$$(3.56) \quad \pi^{v(\lambda)} w < \pi^{v s_{\theta_2}(\lambda + \theta_2^\vee)} s_{v(\theta_2)} w$$

$$< \pi^{v s_{\theta_2} s_{\tilde{\beta}} (\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\theta_2)} s_{v(\tilde{\beta})} w < \pi^{v s_\beta (\lambda + \beta^\vee)} s_{v(\beta)} w.$$

(2) If $\langle \lambda, \theta_1 \rangle = 0$, then $-\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle = 1$ so the last inequality in the chains (3.55) and (3.56) do not always hold, we have the following case distinction, which we already encountered in Lemmas 3.12 and 3.16:

(a) If $v(\theta_1) \in \Phi_+$, the chain (3.55) still holds, else $v s_{\theta_1} < v, \lambda + \beta^\vee = s_{\theta_1}(\lambda + \theta_2^\vee + \tilde{\beta})$ and we instead have the chain

$$\pi^{v(\lambda)} w < \pi^{v s_{\theta_1}(\lambda)} s_{v(\theta_1)} w < \pi^{v s_{\theta_1}(\lambda + \theta_2^\vee)} s_{v(\theta_1)} s_{v(\theta_2)} w < \pi^{v s_{\theta_1}(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\theta_1)} s_{v(\tilde{\beta})} s_{v(\theta_2)} w,$$

where the last term is actually equal to $\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$.

(b) If $w^{-1} v(\theta_1) \in \Phi_-$, then since $\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle < 0$, by (1.12) applied with the affinized root $v s_{\theta_2} s_{\tilde{\beta}}(\theta_1)[\langle \lambda + \theta_2^\vee + \tilde{\beta}^\vee, \theta_1 \rangle]$, the third inequality of chain (3.56) still holds, and thus the whole chain remains correct. Otherwise if $w^{-1} v(\theta_1) \in \Phi_+$ we instead have the chain

$$\pi^{v(\lambda)} w < \pi^{v(\lambda)} s_{v(\theta_1)} w < \pi^{v s_{\theta_2}(\lambda + \theta_2^\vee)} s_{v(\theta_2)} s_{v(\theta_1)} w < \pi^{v s_{\theta_2} s_{\tilde{\beta}}(\lambda + \theta_2^\vee + \tilde{\beta}^\vee)} s_{v(\theta_2)} s_{v(\tilde{\beta})} s_{v(\theta_1)} w,$$

where the last term is actually equal to $\pi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w$ since $\lambda + \theta_2^\vee + \tilde{\beta}^\vee = s_{\theta_1}(\lambda + \beta^\vee)$.

We now turn to the case of $\theta_2 \in \Phi_-$. Notice that $\beta^\vee = -\theta_2^\vee + \tilde{\gamma}^\vee + \gamma^\vee$ and $s_\beta = s_\gamma s_{\tilde{\gamma}} s_{\theta_2} = s_{\theta_2} s_{\tilde{\gamma}} s_\gamma$. Moreover, $\langle -\theta_2^\vee, \tilde{\gamma} \rangle = \langle 2\gamma^\vee - \beta^\vee, 2\beta - \gamma \rangle = -1$ and

$\langle -\theta_2^\vee + \tilde{\gamma}^\vee, \gamma \rangle = \langle \beta^\vee - \gamma^\vee, \gamma \rangle = 1$. Therefore, since λ is dominant and $-\theta_2$ is a positive root:

(1) If $\langle \lambda, \tilde{\gamma} \rangle > 0$, then using (3.8) (resp. (3.9)) with $(\mu_0, \alpha_0, m) = (\lambda, -\theta_2, 1)$ for the first inequality, $(\lambda - \theta_2^\vee, \tilde{\gamma}, 1)$ for the second and $(\lambda - \theta_2^\vee + \tilde{\gamma}^\vee, \gamma, 1)$ for the third, we obtain the chain (3.57) (resp. (3.58))

$$(3.57) \quad \pi^{v(\lambda)} w < \pi^{v(\lambda - \theta_2^\vee)} s_{v(\theta_2)} w < \pi^{v(\lambda - \theta_2^\vee + \tilde{\gamma}^\vee)} s_{v(\tilde{\gamma})} s_{v(\theta_2)} w < \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$$

$$(3.58) \quad \pi^{v(\lambda)} w < \pi^{v s_{\theta_2}(\lambda - \theta_2^\vee)} s_{v(\theta_2)} w$$
$$< \pi^{v s_{\theta_2} s_{\tilde{\gamma}}(\lambda - \theta_2^\vee + \tilde{\gamma}^\vee)} s_{v(\theta_2)} s_{v(\tilde{\gamma})} w < \pi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w.$$

(2) Suppose now that $\langle \lambda, \tilde{\gamma} \rangle = 0$, so $\lambda - \theta_2^\vee + \tilde{\gamma} = s_{\tilde{\gamma}}(\lambda - \theta_2^\vee)$. Then:

(a) If $v(\tilde{\gamma}) \in \Phi_+$, the chain (3.57) still holds. Else, $v s_{\tilde{\gamma}} < v$ and we instead have the chain

$$(3.59) \quad \pi^{v(\lambda)} w < \pi^{v s_{\tilde{\gamma}}(\lambda)} s_{v(\tilde{\gamma})} w < \pi^{v s_{\tilde{\gamma}}(\lambda - \theta_2^\vee)} s_{v(\tilde{\gamma})} s_{v(\theta_2)} w < \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w,$$

where the first inequality comes from Proposition 2.3 and the two others from (3.8).

(b) If $w^{-1} v(\tilde{\gamma}) \in \Phi_-$, then the chain (3.58) still holds. Else $w^{-1} v(\tilde{\gamma}) \in \Phi_+$ and we instead have the chain

$$(3.60) \quad \pi^{v(\lambda)} w < \pi^{v(\lambda)} s_{v(\tilde{\gamma})} w < \pi^{v s_{\theta_2}(\lambda - \theta_2^\vee)} s_{v(\theta_2)} s_{v(\tilde{\gamma})} w < \pi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w,$$

where the first inequality is deduced from (1.12) used with the affinized root $v(\tilde{\gamma})[\langle \lambda, \tilde{\gamma} \rangle]$, and the two others from (3.9) as for the chain (3.58).   □

Proposition 3.19. *Let $\lambda \in Y^{++}$, $v \in W^\lambda$ and $w \in W$. Let $\beta \in \Phi_+$ and suppose that $\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$ or $\pi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w$ cover $\pi^{v(\lambda)} w$. Then β is a quantum root.*

Proof. We prove the contrapositive. Suppose that β is not a quantum root. By Lemma 3.15, there is $\gamma \in \mathrm{Inv}(s_\beta) \setminus \{\beta\}$ such that $\langle \beta^\vee, \gamma \rangle \geq 2$. If $\beta \notin \mathrm{Inv}(s_\gamma)$ we apply Lemma 3.16. We can also apply it in case $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle \neq 3$ by Lemma 3.17. Finally if $\beta \in \mathrm{Inv}(s_\gamma)$ and $\langle \beta^\vee, \gamma \rangle \langle \gamma^\vee, \beta \rangle = 3$ we apply Lemma 3.18. Therefore if β is not a quantum root then $\pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$ and $\pi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w$ do not cover $\pi^{v(\lambda)} w$.   □

3.5. *Conclusion.* We now have everything to prove Theorem A:

Theorem A. *Suppose that $y, x \in W_+^a$ are such that $x \leq y$. Then*

$$(3.61) \qquad\qquad x \lhd y \iff \ell^a(y) = \ell^a(x) + 1.$$

Proof. If $x \leq y$ with $\ell^a(y) = \ell^a(x) + 1$, then by strict compatibility of ℓ^a (Theorem 1.7), y covers x. Conversely suppose that y covers x. If y and x have same dominance class then Theorem 2.18 implies that $\ell^a(y) = \ell^a(x) + 1$. Else, if $\mathrm{proj}^{Y^+}(y) \notin W \cdot \mathrm{proj}^{Y^+}(x)$, by Proposition 3.1, y is of the form $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w$ or $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w$, for $x = \pi^{v(\lambda)}w$ with $\lambda \in Y^{++}$, $v \in W^\lambda$, $w \in W$ and $\beta \in \Phi_+$. Then, by Corollary 3.14, we have $\ell^a(y) - \ell^a(x) = 2\,\mathrm{ht}(\beta^\vee) - \ell(s_\beta)$. Moreover, by Proposition 3.19, β is a quantum root and therefore, in this case as well,

$$\ell^a(y) - \ell^a(x) = 1. \qquad \square$$

Along the way, we have obtained a classification of covers, which we summarize in Proposition 3.20. This is to be compared with [Schremmer 2024, Proposition 4.5].

Proposition 3.20. *Let $x = \pi^{v(\lambda)}w \in W_+^a$ with $\lambda \in Y^{++}$, $v \in W^\lambda$ and $w \in W$. Let $J \subset S$ be the set of simple reflections such that $W_\lambda = W_J$, and recall Notation 1.3 and Definition 3.9. Then covers of x are the elements of the following form:*

(1) $\pi^{v(\lambda)}s_{v(\beta)}w = xs_{w^{-1}v(\beta)[0]}$ *for $\beta \in \Phi$ such that $\ell(s_\beta v^{-1}w) = \ell(v^{-1}w) + 1$.*

(2) $\pi^{vs_\beta(\lambda)}s_{v(\beta)}w = s_{v(\beta)[0]}x$ *for $\beta \in \Phi_+$ such that:*

  (a) $\langle \lambda, \beta \rangle \neq 0$.

  (b) $\ell(vs_\beta) = \ell(v) - 1$.

  (c) *If u denotes vs_β and u_J the maximal W_J-suffix of u, then vu_J^{-1} is on a minimal gallery from v to w.*

(3) $\pi^{v(\lambda+\beta^\vee)}s_{v(\beta)}w = s_{v(\beta)[\langle\lambda,\beta\rangle+1]}x = xs_{w^{-1}v(\beta)[1]}$ *for $\beta \in \Phi_+$ such that:*

  (a) *β is a quantum root.*

  (b) *$\lambda + \beta^\vee$ is an almost dominant coweight.*

  (c) *For $u = v^{\lambda+\beta^\vee}$, v is on a minimal gallery from 1 to vu, that is to say $\ell(vu) = \ell(v) + \ell(u)$.*

  (d) *For $\tilde{v} = v^{v(\lambda+\beta^\vee)}$, $s_{v(\beta)}\tilde{v}$ is on a minimal gallery from v to w.*

(4) $\pi^{vs_\beta(\lambda+\beta^\vee)}s_{v(\beta)}w = s_{v(\beta)[-1]}x$ *for $\beta \in \Phi_+$ such that:*

  (a) *β is a quantum root.*

  (b) *$\lambda + \beta^\vee$ is an almost dominant coweight.*

  (c) *For $u = v^{\lambda+\beta^\vee}$, v is on a minimal gallery from 1 to $vs_\beta u$.*

  (d) *For $\tilde{v} = v^{vs_\beta(\lambda+\beta^\vee)}$, $s_{v(\beta)}\tilde{v}$ is on a minimal gallery from v to w.*

In particular, suppose that $\lambda \in Y^{++}$ is regular and is such that $\lambda + \beta^\vee$ is also regular for any quantum root $\beta \in \Phi_+$. We then say that λ is superregular. Proposition 3.20 can be simplified for superregular coweights. This is to be compared with [Lam and Shimozono 2010, Proposition 4.4] and [Welch 2022, Theorem 2].

Proposition 3.21. *Let $x = \pi^{v(\lambda)}w \in W_+^a$ with $\lambda \in Y^{++}$ a superregular coweight and $v, w \in W$. Then covers of x are the elements of the following form*:

(1) $x s_{\beta[0]} = \pi^{v(\lambda)} w s_\beta$ *for* $\beta \in \Phi_+$ *such that* $\ell(v^{-1}w s_\beta) = \ell(v^{-1}w) + 1$.

(2) $s_{\beta[0]} x = \pi^{s_\beta v(\lambda)} s_\beta w$ *for* $\beta \in \Phi_+$ *such that* $\ell(s_\beta v) = \ell(v) - 1$.

(3) $x s_{w^{-1}v(\beta)[1]} = \pi^{v(\lambda + \beta^\vee)} s_{v(\beta)} w$ *for* $\beta \in \Phi_+$ *a quantum root such that* $\ell(v^{-1}w) = \ell(s_\beta) + \ell(s_\beta v^{-1}w)$ *(otherwise said $s_\beta v^{-1}w \leq_R v^{-1}w$)*.

(4) $s_{v(\beta)[-1]} x = \pi^{v s_\beta(\lambda + \beta^\vee)} s_{v(\beta)} w$ *for* $\beta \in \Phi_+$ *a quantum root such that* $\ell(v s_\beta) = \ell(v) + \ell(s_\beta)$ *(otherwise said $s_\beta \leq_R v s_\beta$)*.

For Kac–Moody root systems, the existence of superregular coweights is not clear a priori. However in an upcoming joint work with Hébert we prove that any Kac–Moody root system admits a finite number of quantum roots, which ensures the existence of superregular coweights. We also use this finiteness to deduce that any element of W_a^+ admits a finite number of covers; in particular intervals in W_a^+ are finite.

Acknowledgements

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