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We show that on a generic curve, a bundle obtained by generic successive extensions is stable, so long as the slopes satisfy natural conditions. We compute the dimension of the set of such extensions. We use degeneration methods specializing the curve to a chain of elliptic components. This extends our previous work (1998, 2000).

Take a generic (compact, nonsingular) curve C of genus g defined over the complex numbers. A vector bundle E on C of rank r and degree d is said to be stable (resp. semistable) if for every vector subbundle E_1 of E of rank r_1 and degree d_1, the inequality $\frac{d_1}{r_1} < \frac{d}{r}$ (resp. $\leq$) is satisfied. The moduli space $\mathcal{U}(r, d)(C)$ parametrizes isomorphism classes of stable vector bundles of rank r and degree d on C. It is a nonsingular variety of dimension $r^2(g-1)+1$ which can be compactified by equivalence classes of semistable vector bundles.

Fix $E \in \mathcal{U}(r, d)(C)$ and an integer $r_1 < r$. Define $s_{r_1}(E) = r_1 d - r \max\{\deg E_1\}$ where E_1 moves in the set of subbundles of E of rank r_1. As E is stable, $s_{r_1}(E) > 0$ for all r_1. On the other hand, for a generic E, $s_{r_1}(E)$ is the smallest integer greater than or equal to $r_1(r-r_1)(g-1)$ and congruent with $r_1 d$ modulo r [Lange 1983, Satz, p. 448; Lange and Narasimhan 1983]. Fix then s such that $0 < s \leq r_1(r-r_1)(g-1)$. The (proper) subset of the moduli space of vector bundles given as

$$\mathcal{U}^s(r, d)(C) = \{E \in \mathcal{U}(r, d)(C) \text{ such that } s_{r_1}(E) = s\}$$

generically coincides with the space of stable bundles $E_{r,d}$ that fit in an exact sequence

$$0 \to E_{r_1,d_1} \to E_{r,d} \to \bar{E}_{r-r_1,d-d_1} \to 0.$$

Lange conjectured that a generic choice of the two bundles E_{r_1,d_1}, $\bar{E}_{r-r_1,d-d_1}$ together with a generic choice of the extension would give rise to a stable $E_{r,d}$ and therefore $\mathcal{U}^s(r, d)(C)$ would be nonempty and of the expected dimension. Lange's conjecture was proved in full generality in [Russo and Teixidor 1999].

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Our goal here is to extend this result to the case of several successive extensions. We show the following:

Theorem 0.1. *Let C be a generic curve of genus g. Fix a rank r and degree d. Choose a collection of integers $r_1 < r_2 < \cdots < r_k = r$ and degrees $d_1, \ldots, d_k = d$ such that*

$$\frac{d_1}{r_1} < \frac{d_2}{r_2} < \cdots < \frac{d_k}{r_k}$$

and

$$\begin{gathered} r_1 d_2 - r_2 d_1 \leq r_1(r_2 - r_1)(g-1), \\ r_2 d_3 - r_3 d_2 \leq r_2(r_3 - r_2)(g-1), \\ \ldots, \\ r_{k-1} d_k - r_k d_{k-1} \leq r_{k-1}(r_k - r_{k-1})(g-1). \end{gathered}$$

Define $\mathcal{U}(r_1, \ldots, r_k; d_1, \ldots, d_k)(C) \subseteq \mathcal{U}(r, d)(C)$ as the set of stable $E_{r,d}$ obtained after a sequence of extensions

$$\begin{gathered} 0 \to E_{r_1,d_1} \to E_{r_2,d_2} \to \bar{E}_{r_2-r_1,d_2-d_1} \to 0, \\ 0 \to E_{r_2,d_2} \to E_{r_3,d_3} \to \bar{E}_{r_3-r_2,d_3-d_2} \to 0, \\ \ldots, \\ 0 \to E_{r_{k-1},d_{k-1}} \to E_{r,d} \to \bar{E}_{r-r_{k-1},d-d_{k-1}} \to 0. \end{gathered}$$

Then, $\mathcal{U}(r_1, \ldots, r_k; d_1, \ldots, d_k)(C)$ is nonempty, irreducible and of codimension (in $\mathcal{U}(r, d)(C)$)

$$\begin{aligned} (r_1(r_2 - r_1) + r_2(r_3 - r_2) + \cdots + r_k(r_k - r_{k-1}))(g-1) \\ - (r_1 d_2 - r_2 d_1 + r_2 d_3 - r_3 d_2 + \cdots + r_{k-1} d_k - r_k d_{k-1}) \end{aligned}$$

Following the ideas in [Teixidor 1998; 2000], we will use a degeneration argument. We first prove the result for a particular reducible nodal curve and then we extend it to the generic curve. The condition on the curve being generic is a by-product of our method of proof. It is unlikely to be necessary. On the other hand, the genericity of the extensions seems essential. This was already the case when we considered a single extension. Then, we were able to give geometric conditions identifying the nonstable extensions (see [Russo and Teixidor 1999]).

We became interested in this question while studying families of rational curves in the moduli space of vector bundles on a fixed curve. The stability of a successive extension is crucial for the construction of families of rational curves in the moduli space (see [Mustopa and Teixidor 2024]).

The result in the case of a single extension has found applications to a variety of other topics, in particular to the study of Brill–Noether theory for vector bundles (see, for instance, [Casalaina-Martin and Teixidor 2011; Hitching et al. 2021; Kopper and Mandal 2023]) and to the existence of Ulrich bundles on ruled surfaces (see [Aprodu et al. 2020]). We expect that this more general result will find similar applications.

1. The problem on the reducible curve

Tensoring with line bundles results in isomorphisms between $\mathcal{U}(r,d)(C)$ and $\mathcal{U}(r,d+tr)(C)$, so there are only, up to isomorphism, at most r nonisomorphic moduli spaces of vector bundles on a curve (in fact, about half of that if you consider also dualization, but this is irrelevant to us now). Without loss of generality, we will assume that $0 \le d < r$.

We will be using bundles on reducible, nodal curves as limits of vector bundles on nonsingular curves. More specifically, we will consider chains of elliptic curves defined as follows: Let $C_1, \dots, C_g$ be g elliptic curves, $P_i \neq Q_i$ arbitrary points on C_i. Glue Q_i to P_{i+1}, $i = 1, \dots, g-1$, to form a nodal curve C of genus g that we call a chain of elliptic curves.

On a reducible curve, stability for a bundle depends on a choice of polarization. A polarization is usually defined as the choice of a line bundle on the variety. For our goal of defining stability of a vector bundle, what matters is the relative degree of the restriction of this line bundle to each component, that is, the numbers

$$a_i = \frac{\deg L_{|C_i}}{\deg L}, \quad i = 1, \dots, g,\ 0 < a_i < 1,\ \sum a_i = 1.$$

Then, a vector bundle E on C of constant rank r is said to be a_i-stable if for every subsheaf F of E

$$\frac{\chi(F)}{\sum a_i \operatorname{rank}(F_{|C_i})} < \frac{\chi(E)}{r}.$$

Fixing a polarization, there is a moduli space of (a_i)-stable vector bundles on the chain of elliptic curves (see [Seshadri 1982]). Stability forces the degree of the restriction of the vector bundle to each component of the curve to vary in certain intervals that depend on the (a_j). The moduli space of (a_j)-stable vector bundles on the chain is reducible, each component $M_{(d_j)}$ corresponding to a choice of allowable degrees d_i on the component C_i. Let us look now at the particular case in which the components of C are elliptic

From results of Atiyah [1957], a vector bundle on an elliptic curve is stable if and only if it is indecomposable (not a direct sum of other bundles) of coprime rank and degree. The indecomposable vector bundles that are not of coprime rank and degree are semistably equivalent to a direct sum of stable (and therefore indecomposable) vector bundles all of the same rank and degree. This is similar to what happens for vector bundle on a rational curve, where each vector bundle is a direct sum of line bundles. On rational curves, vector bundles appearing in families tend to be balanced, that is, most of the time they have degrees that differ in just one unit. An analogous result holds for families of vector bundles on reducible curves whose components are elliptic; we describe these known results below.

Fix a collection (d_j) in the range of degrees allowed by stability and denote by $M_{(d_j)}$ the component of the moduli space of vector bundles on C associated to this choice. Write h_i for the greatest common divisor of d_i, r, $d_i' = \frac{d_i}{h_i}$ and rank $r_i' = \frac{r}{h_i}$. Then, a vector bundle corresponding to a generic point in $M_{(d_j)}$ restricts to C_i to a direct sum of h_i indecomposable bundles of degree d_i' and rank r_i'. That is, a vector bundle corresponding to a generic point in M_{d_i} restricts to C_i to a direct sum of stable vector bundles all of the same rank and degree (see [Teixidor 1991, Theorem; 1995, Theorem 3.2]).

Our goal is to use results from the chain of elliptic curves to deduce similar conditions for nonsingular curves. When dealing with a family of curves in which the general member is nonsingular and the special member is the chain of elliptic curves, we can modify a vector bundle in the family tensoring with a line bundle with support on the special fiber. This action leaves the vector bundle on the general fiber unchanged but moves the degree on the various components of the special fiber by multiples of the rank. This allows us to choose the distribution of degrees among the components up to multiples of r and ignore the actual distribution of degrees among components of the curve imposed by the polarization, focusing instead on the remaining conditions needed for stability.

Proposition 1.1. *Let C be a chain of elliptic curves of genus g. Fix a rank r and degree d, $0 \leq d < r$, and a collection of integers $r_1 < r_2 < \cdots < r_k = r$. Choose degrees $d_1, \ldots, d_k = d$ with d_{k-1} the largest degree such that $\frac{d_{k-1}}{r_{k-1}} < \frac{d_k}{r_k}$, d_{k-2} the largest degree such that $\frac{d_{k-2}}{r_{k-2}} < \frac{d_{k-1}}{r_{k-1}}, \ldots, d_1$ the largest degree such that $\frac{d_1}{r_1} < \frac{d_2}{r_2}$. Then, there exists a stable bundle E that contains a chain of subbundles*

$$E_{r_1,d_1} \subseteq E_{r_2,d_2} \subseteq \cdots \subseteq E_{r_k,d_k} = E$$

with E_{r_i,d_i} stable of degree d_i and rank r_i. This E contains at most a finite number of such chains.

Proof. On the moduli space of vector bundles on the chain of elliptic curves, we focus on the component whose restriction have degree $d < r$ on C_1 and degree zero on the remaining components. Write h for the greatest common divisor of d, r. Define d', r' by $d = hd', r = hr'$. As explained above, on the chosen component of the moduli space of vector bundles, the generic vector bundle restricts to a direct sum of h indecomposable bundles of degree d' and rank r' on C_1 and as a direct sum of line bundles of degree zero on $C_2, \dots, C_g$.

More generally and in keeping with the discussion above, for any r_i, d_i, we will say that E_{r_i,d_i} is a generic vector bundle of degree d_i and rank r_i if it is a direct sum of h_i indecomposable vector bundles of coprime rank and degree:

$$E_{r_i,d_i} = \bigoplus_{j=1}^{h_i} F_i^j, \quad h_i = \gcd(r_i, d_i),\ r_i = h_i r_i',\ d_i = h_i d_i',\ \deg F_i = r_i',\ \operatorname{rank} F_i = r_i'.$$

On C_1, choose a generic vector bundle $E^1_{r_1,d_1}$ of degree d_1 and rank r_1, a generic vector bundle $E^1_{r_2,d_2}$ of degree d_2 and rank $r_2, \dots$, a generic vector bundle $E^1_{r_k,d_k}$ of degree d_k and rank r_k.

The conditions $\frac{d_1}{r_1} < \frac{d_2}{r_2} < \cdots < \frac{d_k}{r_k}$ guarantee (see [Teixidor 2000, Lemma 2.5]) that there exist inclusions

$$E^1_{r_1,d_1} \subseteq E^1_{r_2,d_2} \subseteq \cdots \subseteq E^1_{r_k,d_k}.$$

In fact, as $E^1_{r_i,d_i} = \bigoplus_{j=1}^{h_i} F_i^j$, $E^1_{r,d} = E_{r_k,d_k} = \bigoplus_{j=1}^{h} F_k^j$, $\operatorname{Hom}(E^1_{r_i,d_i}, E^1_{r,d}) = \bigoplus_{j,j'} \operatorname{Hom}(F_i^j, F_k^{j'})$. Then, from

$$\frac{d_i'}{r_i'} = \frac{d_i}{r_i} < \frac{d}{r} = \frac{d'}{r'},$$

the space of morphisms of F_i^j to $F_k^{j'}$ has dimension $r_i'd' - r'd_i'$. Therefore, the space of morphisms of $E^1_{r_i,d_i}$ to $E^1_{r,d}$ has dimension $hh_i(r_i'd' - r'd_i') = r_i d - r d_i \geq 0$, in particular, it is nonempty. We can choose the inclusions from $E^1_{r_i,d_i}$ into E_{r_k,d_k} so that the image does not coincide with any of the finite number of destabilizing subsheaves of E_{r_k,d_k} (it is enough to make sure that none of the morphisms F_i^j to $F_k^{j'}$ is zero).

We now describe a vector bundle on the chain by giving a vector bundle on each component and the gluing at the nodes:

On the curve C^1 take the vector bundle $E^1_{r_k,d_k} = E^1_{r,d}$ we just described. On the curves $C_2, \dots, C_g$, choose a direct sum of r line bundles of degree zero. On each of $C_2, \dots, C_g$, select a first set of r_1 among the r line bundles in the direct sum. Select then a second set of r_2 among the r line bundles containing the initial subset of r_1 already chosen, Select a third set of r_3 line bundles containing the subset of r_2 chosen in the previous step and so on. Form now a bundle on the chain of elliptic curves by gluing the bundles on each component so that when identifying Q_i with P_{i+1}, $i = 2, \dots, g-1$, each of the sets of r_j line bundles $j = 1, \dots, k$ on C_i glues with the set of r_j line bundles on C_{i+1}, $j = 1, \dots, k$ (but the gluings are otherwise generic). At Q_1, glue each set of r_j line subbundles on C_2 with the fiber of the image of the $E^1_{r_j,d_j}$ (but the gluings are otherwise generic). In this way, we obtain bundles of ranks $r_1 < r_2 < \cdots < r_k$ and degrees $d_1, \dots, d_k$ on the whole curve C each contained in the next.

On a reducible nodal curve, gluing vector bundles that are semistable on each of the components and of the degrees allowed by the polarization, one obtains a semistable bundle on the whole curve. Moreover, if one of the bundles we are gluing is stable or if none of the subbundles that contradict stability glue with each other, the whole vector bundle on the reducible curve is stable (see [Teixidor 1991; 1995]).

By construction, the vector bundles on each C_i are semistable. On C_1, the only subbundles of the bundle $E^1_{r_k,d_k} = \bigoplus_{j=1}^h E^j_{r',d'} = \bigoplus_{j=1}^h F^j_k$ that contradict stability are the h subsheaves F^j_k of degree d' and rank r' and their direct sums. Our choice of the inclusions of the subbundles in the bundle on C_1 and the gluings at the nodes guarantee that we have a stable overall bundle.

Note also that our choice of d_i means that $r_i d_{i+1} - r_{i+1}(d_i + 1) \le -1$ or equivalently $r_i d_{i+1} - r_{i+1} d_i \le r_{i+1} - 1$. In the interval $1 \le r_i \le r_{i+1} - 1$, this implies that $r_i d_{i+1} - r_{i+1} d_i \le r_i(r_{i+1} - r_i)$. Therefore, given a subspace of dimension r_i of the fiber of $E^1_{r_{i+1},d_{i+1}}$ at Q_1, there is at most a finite number of subbundles $E^1_{r_i,d_i}$ whose immersion in $E^1_{r_{i+1},d_{i+1}}$ glue with that fixed subspace (see Proposition 2.8 of [Teixidor 2000]). Therefore, the number of chains for a fixed $E_{r,d}$ on the reducible curve is finite. □

2. Extending the result to the nonsingular curve

We start by using the results on the reducible curve to extend it to a generic, nonsingular curve.

Proposition 2.1. *Let C be a generic curve of genus g. Fix a rank r and degree d, $0 \le d < r$, and a collection of integers $r_1 < r_2 < \cdots < r_k = r$. Choose degrees $d_1, \dots, d_k = d$ with d_{k-1} the largest degree such that $\frac{d_{k-1}}{r_{k-1}} < \frac{d_k}{r_k}$, d_{k-2} is the largest degree such that $\frac{d_{k-2}}{r_{k-2}} < \frac{d_{k-1}}{r_{k-1}}, \dots, d_1$ the largest degree such that $\frac{d_1}{r_1} < \frac{d_2}{r_2}$. Then, there exists a stable bundle E that contain a chain of subbundles*

$$E_{r_1,d_1} \subseteq E_{r_2,d_2} \subseteq \cdots \subseteq E_{r_k,d_k} = E$$

with E_{r_i,d_i} stable of degree d_i and rank r_i.

Proof. Take a family of curves where the special fiber is a chain of elliptic curves and the generic curve is nonsingular. Then, the result follows from Proposition 1.1 using the openness of the stability condition. □

We proved stability of the various steps of a chain of extensions under the harder conditions on slopes. This implies the similar result when the slopes are not as close:

Proposition 2.2. *Let C be a generic curve of genus g. Fix a rank r and degree d, $0 \leq d < r$, and two collections of integers $r_1 < r_2 < \cdots < r_k = r$, $d_1, \ldots, d_k = d$ such that*

$$\frac{d_1}{r_1} < \frac{d_2}{r_2} < \cdots < \frac{d_k}{r_k}.$$

Then, there exists a stable bundle E that contain a chain of subbundles

$$E_{r_1,d_1} \subseteq E_{r_2,d_2} \subseteq \cdots \subseteq E_{r_k,d_k} = E$$

with E_{r_i,d_i} stable of degree d_i and rank r_i.

Proof. Fix integers r, d, r_1, d_1 with $\frac{d_1}{r_1} < \frac{d}{r}$. The set of vector bundles of rank r and degree d which contain a subbundle of rank r_1 and degree $d_1 - 1$ is contained in the closure of those vector bundles that contain a subbundle of rank r_1 and degree d_1 (see [Russo and Teixidor 1999] Corollary 1.12) Therefore, the result follows from Proposition 2.1. □

Let us now look at dimension and irreducibility:

Proposition 2.3. *Fix integers d_1, d_2, r_1, r_2 such that $\frac{d_1}{r_1} < \frac{d_2}{r_2}$. Let $\mathcal{U}_1$ be an irreducible family of stable vector bundles of rank r_1 and degree d_1. Let $\overline{\mathcal{U}_2}$ be an irreducible family of stable vector bundles of rank $r_2 - r_1$ and degree $d_2 - d_1$. Then, the family of extensions*

$$0 \to E_1 \to E \to \overline{E_2} \to 0, \ E_1 \in \mathcal{U}_1, \overline{E_2} \in \overline{\mathcal{U}_2}$$

is also irreducible of dimension

$$\dim(\mathcal{U}_1) + \dim(\overline{\mathcal{U}_2}) + r_1(r_2 - r_1)(g-1) + r_1 d_2 - r_2 d_1 - 1.$$

Proof. For fixed $E_1, \overline{E_2}$, the space of extensions as above is parameterized by $H^1(\overline{E_2}^* \otimes E_1)$. We claim that $H^0(\overline{E_2}^* \otimes E_1) = 0$. If this were not the case then there would be a nonzero morphism of $\overline{E_2} \to E_1$. Its image I would be both a quotient of $\overline{E_2}$ and a subsheaf of E_1. The stability of the two bundles implies that

$$\frac{d_1}{r_1} = \mu(E_1) < \mu(I) < \mu(\overline{E_2}) = \frac{d_2 - d_1}{r_2 - r_1}.$$

This contradicts the assumption of our initial choice of ranks and degrees. It follows that $H^0(\overline{E_2}^* \otimes E_1) = 0$ and therefore $H^1(\overline{E_2}^* \otimes E_1)$ has dimension equal to $r_1 d_2 - r_2 d_1 + r_1(r_2 - r_1)(g-1)$, irrespectively of the choice of E_1, $\overline{E_2}$. Then the statement about the dimension follows. □

Proof of Theorem 0.1. Denote by $\mathcal{U}_1$ the space of all vector bundles of degree d_1 and rank r_1, $\overline{\mathcal{U}_2}$ the space of all vector bundles of degree $d_2 - d_1$ and rank $r_2 - r_1, \ldots, \overline{\mathcal{U}_k}$ the space of all vector bundles of degree $d_k - d_{k-1}$ and rank $r_k - r_{k-1}$. From Proposition 2.1, the set of bundles E that can be obtained by successive extensions is nonempty. Proposition 2.3 allows us to compute successively the dimensions of the space of extensions, starting with

$$\begin{aligned} \dim(\mathcal{U}_1) &= r_1^2(g-1)+1, \\ \dim(\overline{\mathcal{U}_2}) &= (r_2 - r_1)^2(g-1)+1, \\ &\ldots, \\ \dim(\overline{\mathcal{U}_k}) &= (r_k - r_{k-1})^2(g-1)+1. \end{aligned}$$

The last claim in Proposition 1.1 ensures that each vector bundle appears only a finite number of times as an extension of the given form. □

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