Pacific Journal of Mathematics

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Volume 333 No. 1 November 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 1, 2024
https://doi.org/10.2140/pjm.2024.333.59

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We study the Kähler–Einstein metrics and Calabi extremal metrics on some unbounded domains defined by

$$\Omega := \{(\zeta, \tilde{\zeta}, w) \in \mathbb{C}^s \times \mathbb{C}^n \times \mathbb{C}^m : e^{\|w\|^2}\|\zeta\|^2 + \|\tilde{\zeta}\|^2 < 1\},$$

which are type-III cohomogeneity-one manifolds. We introduce a Kähler metric g associated with the Kähler form

$$\omega := \sqrt{-1}(\partial\bar{\partial}F(X) + a\partial\bar{\partial}\|w\|^2 + b\partial\bar{\partial}\log\|\zeta\|^2)$$

on Ω. Using the method of cohomogeneity, we find many new Kähler–Einstein metrics and prove that any Calabi extremal metric with the scalar curvature as a linear function of the potential function of the circle action on the first vector ζ must be a metric with a constant scalar curvature. At last, we give one application of our main results. That is, in the case $s = m = 1$, we prove that the Kähler–Einstein metric is equivalent, but not equal, to the Bergman metric.

1. Introduction

Kähler–Einstein metrics are an interesting subject of complex geometry; see [17; 22] for examples. Both in the compact cases and the domain cases, people have used ordinary differential equations to find concrete or pseudoconcrete solutions. By this, we mean that the solutions could be expressed in an explicit way, as in [5; 3; 11; 13; 14; 15; 16], even though the functions themselves might be implicit, but not as in [9; 18; 19; 20; 21; 22]. One of the purposes of this paper is trying to connect domain cases as in [11; 13] to compact cases as in [5; 6; 7; 8; 16].

We believe that most of the published works in the domain cases are actually in the so-called type-III cases, such as those in [5; 6; 7; 8]. Therefore, by working out some new unbounded domains with Kähler–Einstein metrics or metrics with

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MSC2020: 32Q02, 32T27.

Keywords: Kähler–Einstein metric, extremal metric, twisted Fock–Bargmann–Hartogs domains, scalar curvature.

constant scalar curvature, we both explain our philosophy and provide an exposition of [5; 6; 7; 8].

Although the examples in [5; 6; 7; 8; 16] might not necessarily be cohomogeneity one, the cohomogeneity-one cases guarantee that one can reduce the Kähler–Einstein equation into an ordinary differential equation. However, just being cohomogeneity one does not give us an explicit solution at all. See [9], for example. What we need is a type-III cohomogeneity-one structure. A complex manifold is cohomogeneity one if there is a Lie group in its automorphism group such that it has a real hypersurface orbit. It is of type III if there is a holomorphic vector field E which commutes with this Lie group.

Motivated by the above reasons, we consider a class of unbounded nonhyperbolic Reinhardt domains defined by

$$\Omega := \{(\zeta, \tilde{\zeta}, w) \in \mathbb{C}^s \times \mathbb{C}^n \times \mathbb{C}^m : e^{\|w\|^2}\|\zeta\|^2 + \|\tilde{\zeta}\|^2 < 1\}, \tag{1-1}$$

which are called twisted Fock–Bargmann–Hartogs domains (see Kim and Yamamori's article [12]). In general, there is a $\mu \in \mathbb{R}^+$ in front of $\|w\|^2$ in (1-1). Because of the biholomorphic invariance of the Bergman metric, and the biholomorphic invariance of some Kähler metrics, we omit it. The domains in (1-1) are type-III cohomogeneity-one manifolds (see Lemma 2.4 below).

Let (Ω, g) be a Kähler manifold endowed with its Kähler metric g, in which g is a Kähler metric associated with the Kähler form

$$\omega := \sqrt{-1}(\partial\bar{\partial}F(X) + a\partial\bar{\partial}\|w\|^2 + b\partial\bar{\partial}\log\|\zeta\|^2) \tag{1-2}$$

on Ω, where $a, b \in \mathbb{R}$, $F(X)$ is a smooth function of X on $(0, 1)$, and

$$X = \frac{e^{\|w\|^2}\|\zeta\|^2}{1 - \|\tilde{\zeta}\|^2}, \quad (\zeta, \tilde{\zeta}, w) \in \Omega. \tag{1-3}$$

We notice that when $s = 1$, b is not needed. In particular, we also need the conditions

$$f > 0, \quad f + a > 0, \quad f' > 0, \quad 0 \le X < 1 \tag{1-4}$$

and if $s > 1$

$$f + b > 0, \quad 0 < X < 1, \quad \lim_{X \to 0^+}(f + b) = 0, \tag{1-5}$$

where $f := XF'$. In fact, the function f or $U := f - n - 2$ will be just a potential function of E with respect to the metric (see Lemma 3.3). After transforming the free variable into U, one gets a simple equation and can solve it explicitly.

Another new ingredient is that when we calculate the volume, instead of using the cohomogeneity-one property as we did earlier in [9], we calculate the volume directly.

The precise statements of our main results are as follows.

Theorem 1.1. *Let* (Ω, g) *be a Kähler manifold endowed with its Kähler metric* g, *where* g *is a Kähler metric associated with the Kähler form* (1-2). *Then* g *is a Kähler–Einstein metric if and only if* $a = -n-1$, $s = 1$, *that is*, *the dimension of the fiber of* Ω *is one*, *and* $F(X)$ *is the solution of the differential equation*

$$(Q\varphi)_f = f^n(f-n-1)^{m+1}$$

with the initial condition $f|_{X=0} = n+2$, *in which* $Q := (f-n-1)^m f^n$ *and* $\varphi := Xf'$.

After we deal with the Kähler–Einstein equation, we also deal with the Calabi extremal metric with the scalar curvature as a linear function of the potential function f. We call this the *special extremal metric*. By the completeness property, we prove that on these domains, the special extremal metrics must be metrics with constant scalar curvatures.

Theorem 1.2. *Let* (Ω, g) *be a Kähler manifold endowed with its Kähler metric* g, *where* g *is a Kähler metric associated with the Kähler form* (1-2). *Then* g *is a special extremal metric if and only if the scalar curvature of* g *is constant.*

The paper is organized as follows. In Section 2, we present some well-known results, such as the definition of the Calabi extremal metric and the holomorphic automorphism groups of the twisted Fock–Bargmann–Hartogs domains Ω defined in (1-1). We also prove that these domains are type-III cohomogeneity-one manifolds. Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively. At last, in Section 5, we give one application of our main results. That is, in the case $s = m = 1$, we prove that the Kähler–Einstein metric is equivalent to the Bergman metric.

2. Preliminaries

We first give the definition of the Calabi extremal metric. E. Calabi [1; 2] introduced extremal metrics on *compact* Kähler manifolds to solve a variational problem involving the integral of the scalar curvature. Given an n-dimensional Kähler manifold (M, g) with its Kähler form

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i,j} g_{i\bar{j}}\, dz_i \wedge d\overline{z_j},$$

the Calabi extremal metric is the critical metric for the Calabi functional

$$C_a(\omega) = \frac{1}{V(M)}\int_M S^2(\omega)\omega^n,$$

where $V(M) = \int_M \omega^n$, and $S(\omega)$ is the scalar curvature of ω. We refer the readers to Tian [20] for more information on this topic. Calabi proved the following result:

Proposition 2.1 [1; 2]. *On a compact Kähler manifold, a Kähler metric g is extremal if and only if the lifting of dS is a holomorphic vector field.*

Without the compactness condition, we still define:

Definition 2.2 (see [1; 2]). On a Kähler manifold, we call a Kähler metric g an extremal metric if the lifting of dS is a holomorphic vector field.

Next we also need the holomorphic automorphism group of Ω described by Kim and Yamamori [12].

Lemma 2.3 [12]. *The automorphism group of Ω is generated by the mappings*

$$\Phi_{U_1,U_2,U_3} : (\zeta, \tilde{\zeta}, w) \mapsto (U_1\zeta, U_2\tilde{\zeta}, U_3 w);$$

$$\Phi_{a,\upsilon} : (\zeta, \tilde{\zeta}, w) \mapsto (e^{-\langle w,\upsilon\rangle - \frac{1}{2}\|\upsilon\|^2} F_a(\tilde{\zeta})\zeta, h_a(\tilde{\zeta}), w + \upsilon),$$

where $U_1 \in U(s)$, $U_2 \in U(n)$, $U_3 \in U(m)$, $\upsilon \in \mathbb{C}^m$, $a \in \mathbb{B}^n$,

$$F_a(\tilde{\zeta}) = \frac{(1 - \|a\|^2)^{\frac{1}{2}}}{1 - \langle \tilde{\zeta}, a\rangle}$$

and $h_a(\cdot)$ is an automorphisms of $\mathbb{B}^n$ taking a to 0.

By Lemma 2.3, it is easy to check that Ω is a cohomogeneity-one manifold with noncompact automorphism group. In fact, on the one hand, for any $(\zeta_0, \tilde{\zeta}_0, w_0) \in \Omega$, by choosing a unitary matrix U_1 of order s such that $U_1\zeta = (\|\zeta\|, 0, \ldots, 0)$, consider the automorphism Φ_{U_1} and $\Phi_{a,\upsilon}$ of Ω defined as

$$\Phi_{U_1}(\zeta, \tilde{\zeta}, w) := (U_1\zeta, \tilde{\zeta}, w);$$

$$\Phi_{\tilde{\zeta}_0,-w_0}(\zeta, \tilde{\zeta}, w) := (e^{\langle w,w_0\rangle - \frac{1}{2}\|w_0\|^2} F_{\tilde{\zeta}_0}(\tilde{\zeta})\zeta, h_{\tilde{\zeta}_0}(\tilde{\zeta}), w - w_0).$$

Thus,

$$\Phi_{U_1} \circ \Phi_{\tilde{\zeta}_0,-w_0}(\zeta_0, \tilde{\zeta}_0, w_0) := \left(\frac{e^{\frac{1}{2}\|w_0\|^2}}{(1 - \|\tilde{\zeta}_0\|^2)^{\frac{1}{2}}} \|\zeta_0\|, 0, \ldots, 0, \mathbf{0}, \mathbf{0} \right),$$

which implies that Ω has a real hypersurface orbit under a Lie group action. On the other hand, since

$$X = \frac{e^{\|w\|^2} \|\zeta\|^2}{1 - \|\tilde{\zeta}\|^2}, \quad (\zeta, \tilde{\zeta}, w) \in \Omega,$$

it is obvious that $0 \le X < 1$, since, similar to the argument of Kim, Yamamori, and Zhang [13], we could prove that X is preserved by the automorphism group of Ω. Therefore, for fixed constant c, the noncompactness of the leaf $X = c$, in conjunction with the invariance of each leaf under holomorphic automorphisms, yields the fact that $\mathrm{Aut}(\Omega)$ is noncompact.

Lemma 2.4. *The twisted Fock–Bargmann–Hartogs domains Ω in (1-1) are type-III cohomogeneity-one manifolds.*

Proof. According to the above discussion, we only need to prove that there is a holomorphic vector field which commutes with $\mathrm{Aut}(\Omega)$. By Lemma 2.3, U_1 is a unitary matrix of order s and the center of U_1 is

$$\mathrm{Cent}(U_1) = \{\mathrm{diag}(e^{i\theta}, \dots, e^{i\theta}) : \theta \in \mathbb{R}\}.$$

Let $\Phi_{\theta,I_n,I_m} := \Phi_{e^{i\theta} I_s, I_n, I_m}$. We claim that

$$\Phi_{\theta,I_n,I_m} \circ \Phi_{a,\upsilon} = \Phi_{a,\upsilon} \circ \Phi_{\theta,I_n,I_m}.$$

In fact,

$$\begin{aligned}\Phi_{\theta,I_n,I_m} \circ \Phi_{a,\upsilon}(\zeta, \tilde{\zeta}, w) &= \Phi_{\theta,I_n,I_m}(e^{-\langle w,\upsilon\rangle - \frac{1}{2}\|\upsilon\|^2} F_a(\tilde{\zeta})\zeta, h_a(\tilde{\zeta}), w+\upsilon) \\ &= (e^{-\langle w,\upsilon\rangle - \frac{1}{2}\|\upsilon\|^2} F_a(\tilde{\zeta})(e^{i\theta}\zeta), h_a(\tilde{\zeta}), w+\upsilon) \\ &= \Phi_{a,\upsilon} \circ \Phi_{\theta,I_n,I_m}(\zeta, \tilde{\zeta}, w).\end{aligned}$$

Therefore, $\Phi_{\theta,I_n,I_m} \in \mathrm{Cent}(\mathrm{Aut}(\Omega))$. That is, the holomorphic vector field of Φ_{θ,I_n,I_m} is a holomorphic vector field which commutes with this Lie group. The proof is complete. □

We only consider metrics with a Kähler form

$$\omega := \sqrt{-1}(\partial\bar{\partial} F(X) + a\partial\bar{\partial}\|w\|^2 + b\partial\bar{\partial}\log\|\zeta\|^2)$$

defined in (1-2). The reason is that the Bergman metric, if it exists, is invariant under the automorphism group. Therefore, it should have a Kähler form in this format. That is, in this case, there are Kähler metrics in this format.

Remark 2.5 (see [4]). Although $F(X)$ might be singular at $X = 0$, the metric is smooth; consider, for example, the Bergman metric. Therefore, our metric is well defined on Ω, not just on $\Omega - \{X = 0\}$. This is consistent with earlier publications. In fact, Ω is a B^s bundle over $\mathbb{C}^m \times B^n$ by

$$\pi : (z_{(1)}, z_{(2)}, w) \to (w, z_{(2)})$$

with $\{X = 0\}$ being the zero section. One might ask why we do not do the same for $\Omega - \{X = 0\}$. The problem is that by the rigidity of the Kähler metric, we might not be able to get a complete Kähler–Einstein metric. See page 86 of [4] for an example. From (see [4])

$$\varphi(2U+m)^{n-1} = (2U+m)^n - \frac{l(2U+m)^{n+1}}{n+1} + C,$$

we get that if $U \to -\infty$, then $2U + m$ will be negative somewhere. So U must be finite when $r \to 0^+$; we denote this value by U_0. We have $\varphi(U_0) = 0 = C$. If the right-hand side only have single zero the integral of $\varphi^{-1/2}$ is finite near 0. Therefore, a Kähler–Einstein metric cannot be complete after deleting the zero section.

For convenience of the exposition, define $\zeta := z_{(1)} = (z_1, \dots, z_s) \in \mathbb{C}^s$, $\tilde{\zeta} := z_{(2)} = (z_{s+1}, \dots, z_{s+n}) \in \mathbb{C}^n$, and $w := z_{(3)} = (z_{s+n+1}, \dots, z_{s+n+m}) \in \mathbb{C}^m$.

3. Kähler–Einstein metric

Lemma 3.1. *The Ricci form of Ω associated with the Kähler metric g is*

$$\text{(3-6)}\quad \operatorname{Ric} = \sqrt{-1}\big(-\partial\bar\partial \log((f+b)^{s-1} f^n (f+a)^m X^{n+1}\varphi) + \partial\bar\partial \log \|z_{(1)}\|^{2s+2n+2} + (n+1)\partial\bar\partial\|z_{(3)}\|^2\big),$$

where $f := XF'$ and $\varphi := Xf'$. In particular, the Ricci form (3-6) has a similar expression as the Kähler form (1-2), that is,

$$\operatorname{Ric} = \sqrt{-1}(\partial\bar\partial F_{\text{Ric}} + a_{\text{Ric}}\partial\bar\partial\|z_{(3)}\|^2 + b_{\text{Ric}}\partial\bar\partial \log \|z_{(1)}\|^2),$$

where $F_{\text{Ric}} := -\log((f+b)^{s-1} f^n (f+a)^m X^{n+1}\varphi)$, $a_{\text{Ric}} := n+1$ and $b_{\text{Ric}} := s+n+1$.

Proof. By (1-2), we only need to calculate the determinant of the matrix corresponding to the Kähler form ω. Firstly, we have

$$\partial\bar\partial F(X) = \partial(F'\bar\partial X) = \partial(F'X\bar\partial \log X) = Xf'(\partial \log X \wedge \bar\partial \log X) + f\partial\bar\partial \log X.$$

Moreover, we have

$$\begin{aligned}\log X &= \log e^{\|z_{(3)}\|^2} + \log\|z_{(1)}\|^2 - \log(1-\|z_{(2)}\|^2)\\ &= \|z_{(3)}\|^2 + \log\|z_{(1)}\|^2 - \log(1-\|z_{(2)}\|^2);\end{aligned}$$

$$\begin{aligned}\partial \log X &= \sum_{i=1}^{s} \frac{\partial \log X}{\partial z_i} dz_i + \sum_{k=s+1}^{s+n} \frac{\partial \log X}{\partial z_k} dz_k + \sum_{l=s+n+1}^{s+n+m} \frac{\partial \log X}{\partial z_l} dz_l\\ &= \sum_{i=1}^{s} \frac{\overline{z_i}}{\|z_{(1)}\|^2} dz_i + \sum_{k=s+1}^{s+n} \frac{\overline{z_k}}{1-\|z_{(2)}\|^2} dz_k + \sum_{l=s+n+1}^{s+n+m} \overline{z_l}\, dz_l;\end{aligned}$$

$$\bar\partial \log X = \sum_{i=1}^{s} \frac{z_i}{\|z_{(1)}\|^2} d\overline{z_i} + \sum_{k=s+1}^{s+n} \frac{z_k}{1-\|z_{(2)}\|^2} d\overline{z_k} + \sum_{l=s+n+1}^{s+n+m} z_l\, d\overline{z_l};$$

$$\begin{aligned}\text{(3-7)}\quad \partial\bar\partial \log X &= \sum_{i,p=1}^{s} \frac{\delta_{ip}\|z_{(1)}\|^2 - z_i\overline{z_p}}{\|z_{(1)}\|^4} dz_p \wedge d\overline{z_i}\\ &\quad + \sum_{k,q=s+1}^{s+n} \frac{\delta_{kq}(1-\|z_{(2)}\|^2) + z_k\overline{z_q}}{(1-\|z_{(2)}\|^2)^2} dz_q \wedge d\overline{z_k} + \sum_{l,t=s+n+1}^{s+n+m} \delta_{lt}\, dz_t \wedge d\overline{z_l}.\end{aligned}$$

Combining the above results, we obtain

$$\begin{aligned}\partial\bar\partial F(X) = {}& Xf'\sum_{i,\vartheta=1}^{s}\frac{\overline{z_i}z_\vartheta}{\|z_{(1)}\|^4}dz_i\wedge d\overline{z_\vartheta} + f\sum_{j,p=1}^{s}\frac{\delta_{jp}\|z_{(1)}\|^2 - z_j\overline{z_p}}{\|z_{(1)}\|^4}dz_p\wedge d\overline{z_j}\\
&+Xf'\sum_{k,\lambda=s+1}^{s+n}\frac{\overline{z_k}z_\lambda}{(1-\|z_{(2)}\|^2)^2}dz_k\wedge d\overline{z_\lambda}\\
&+f\sum_{\tau,q=s+1}^{s+n}\frac{\delta_{\tau q}(1-\|z_{(2)}\|^2)+z_\tau\overline{z_q}}{(1-\|z_{(2)}\|^2)^2}dz_q\wedge d\overline{z_\tau}\\
&+Xf'\sum_{l,\mu=s+n+1}^{s+n+m}z_\mu\overline{z_l}dz_l\wedge d\overline{z_\mu}+f\sum_{\rho,t=s+n+1}^{s+n+m}\delta_{\rho t}dz_t\wedge d\overline{z_\rho}\\
&+Xf'\sum_{i=1}^{s}\sum_{\lambda=s+1}^{s+n}\frac{\overline{z_i}z_\lambda}{\|z_{(1)}\|^2(1-\|z_{(2)}\|^2)}dz_i\wedge d\overline{z_\lambda}\\
&+Xf'\sum_{i=1}^{s}\sum_{\mu=s+n+1}^{s+n+m}\frac{\overline{z_i}z_\mu}{\|z_{(1)}\|^2}dz_i\wedge d\overline{z_\mu}\\
&+Xf'\sum_{k=s+1}^{s+n}\sum_{\vartheta=1}^{s}\frac{\overline{z_k}z_\vartheta}{\|z_{(1)}\|^2(1-\|z_{(2)}\|^2)}dz_k\wedge d\overline{z_\vartheta}\\
&+Xf'\sum_{k=s+1}^{s+n}\sum_{\mu=s+n+1}^{s+n+m}\frac{\overline{z_k}z_\mu}{1-\|z_{(2)}\|^2}dz_k\wedge d\overline{z_\mu}\\
&+Xf'\sum_{l=s+n+1}^{s+n+m}\sum_{\vartheta=1}^{s}\frac{\overline{z_l}z_\vartheta}{\|z_{(1)}\|^2}dz_l\wedge d\overline{z_\vartheta}\\
&+Xf'\sum_{l=s+n+1}^{s+n+m}\sum_{\lambda=s+1}^{s+n}\frac{\overline{z_l}z_\lambda}{1-\|z_{(2)}\|^2}dz_l\wedge d\overline{z_\lambda}.\end{aligned}$$

Meanwhile, it is easy to calculate that

$$\begin{aligned}\partial\bar\partial\|z_{(3)}\|^2 &= \sum_{l,\mu=s+n+1}^{s+n+m}\delta_{l\mu}dz_l\wedge d\overline{z_\mu};\\
\partial\bar\partial\log\|z_{(1)}\|^2 &= \sum_{i,\vartheta=1}^{s}\frac{\delta_{i\vartheta}\|z_{(1)}\|^2 - z_i\overline{z_\vartheta}}{\|z_{(1)}\|^4}dz_\vartheta\wedge d\overline{z_i}.\end{aligned}\tag{3-8}$$

Hence the quadratic matrix associated with Kähler form ω is

$$H := \begin{pmatrix} a_{11}I_s+b_{11}\overline{z_{(1)}}^Tz_{(1)} & a_{12}\overline{z_{(1)}}^Tz_{(2)} & a_{13}\overline{z_{(1)}}^Tz_{(3)}\\ a_{21}\overline{z_{(2)}}^Tz_{(1)} & a_{22}I_n+b_{22}\overline{z_{(2)}}^Tz_{(2)} & a_{23}\overline{z_{(2)}}^Tz_{(3)}\\ a_{31}\overline{z_{(3)}}^Tz_{(1)} & a_{32}\overline{z_{(3)}}^Tz_{(2)} & a_{33}I_m+b_{33}\overline{z_{(3)}}^Tz_{(3)}\end{pmatrix},\tag{3-9}$$

where

$$a_{11} := \frac{f+b}{\|z_{(1)}\|^2}, \qquad b_{11} := \frac{Xf'-f-b}{\|z_{(1)}\|^4}, \qquad a_{12} := \frac{Xf'}{\|z_{(1)}\|^2(1-\|z_{(2)}\|^2)},$$

$$a_{13} := \frac{Xf'}{\|z_{(1)}\|^2}, \qquad a_{21} := \frac{Xf'}{\|z_{(1)}\|^2(1-\|z_{(2)}\|^2)}, \qquad a_{22} := \frac{f}{1-\|z_{(2)}\|^2},$$

$$b_{22} := \frac{Xf'+f}{(1-\|z_{(2)}\|^2)^2}, \qquad a_{23} := \frac{Xf'}{1-\|z_{(2)}\|^2}, \qquad a_{31} := \frac{Xf'}{\|z_{(1)}\|^2},$$

$$a_{32} := \frac{Xf'}{1-\|z_{(2)}\|^2}, \qquad a_{33} := f+a, \qquad b_{33} := Xf'.$$

Next we compute the determinant of H. Let $\mathbf{0}_j$ be the vector of 0's of length j. We choose unitary matrices U_1, U_2 and U_3 such that $U_1 z_{(1)} = (\|z_{(1)}\|, \mathbf{0}_{s-1})$, $U_2 z_{(2)} = (\|z_{(2)}\|, \mathbf{0}_{n-1})$ and $U_3 z_{(3)} = (\|z_{(3)}\|, \mathbf{0}_{m-1})$, that is, one can assume that $z_i = 0$ except $i = 1, s+1, n+s+1$. After multiplying H by $U = \operatorname{diag}(U_1, U_2, U_3)$ to $UH\bar{U}^T$, we obtain

$$\det H = \det(UH\bar{U}^T) := \det\begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{pmatrix}, \tag{3-10}$$

where M_j is diagonal for $j = 1, 5, 9$ and only has one nonzero element in the upper left corner for other i's, and

$$M_1 = \begin{pmatrix} \frac{Xf'}{\|z_{(1)}\|^2} & 0 & \cdots & 0 & 0 \\ 0 & \frac{f+b}{\|z_{(1)}\|^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{f+b}{\|z_{(1)}\|^2} & 0 \\ 0 & 0 & \cdots & 0 & \frac{f+b}{\|z_{(1)}\|^2} \end{pmatrix};$$

$$M_2 = M_4 = \begin{pmatrix} \frac{Xf'\|z_{(2)}\|}{\|z_{(1)}\|(1-\|z_{(2)}\|^2)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix};$$

$$M_3 = M_7 = \begin{pmatrix} \frac{Xf'}{\|z_{(1)}\|}\|z_{(3)}\| & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix};$$

$$M_5=\begin{pmatrix} \frac{f}{1-\|z_{(2)}\|^2}+\frac{(Xf)'\|z_{(2)}\|^2}{(1-\|z_{(2)}\|^2)^2} & 0 & \cdots & 0 & 0\\ 0 & \frac{f}{1-\|z_{(2)}\|^2} & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & \frac{f}{1-\|z_{(2)}\|^2} & 0\\ 0 & 0 & \cdots & 0 & \frac{f}{1-\|z_{(2)}\|^2} \end{pmatrix};$$

$$M_6=M_8=\begin{pmatrix} Xf'\frac{\|z_{(2)}\|\|z_{(3)}\|}{1-\|z_{(2)}\|^2} & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0 \end{pmatrix};$$

$$M_9=\begin{pmatrix} f+a+Xf'\|z_{(3)}\|^2 & 0 & 0 & \cdots & 0 & 0\\ 0 & f+a & 0 & \cdots & 0 & 0\\ 0 & 0 & f+a & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & f+a & 0\\ 0 & 0 & 0 & \cdots & 0 & f+a \end{pmatrix}.$$

From the simple form (3-10), it is easy to derive that

$$\begin{aligned}
\det H &= \left(\frac{f+b}{\|z_{(1)}\|^2}\right)^{s-1}\left(\frac{f}{1-\|z_{(2)}\|^2}\right)^{n-1}(f+a)^{m-1}\\
&\qquad\cdot\det\begin{pmatrix} \frac{Xf'}{\|z_{(1)}\|^2} & \frac{Xf'\|z_{(2)}\|}{\|z_{(1)}\|(1-\|z_{(2)}\|^2)} & \frac{Xf'}{\|z_{(1)}\|}\|z_{(3)}\|\\ \frac{Xf'\|z_{(2)}\|}{\|z_{(1)}\|(1-\|z_{(2)}\|^2)} & \frac{f}{1-\|z_{(2)}\|^2}+\frac{(Xf)'\|z_{(2)}\|^2}{(1-\|z_{(2)}\|^2)^2} & Xf'\frac{\|z_{(2)}\|\|z_{(3)}\|}{1-\|z_{(2)}\|^2}\\ \frac{Xf'}{\|z_{(1)}\|}\|z_{(3)}\| & Xf'\frac{\|z_{(2)}\|\|z_{(3)}\|}{1-\|z_{(2)}\|^2} & f+a+Xf'\|z_{(3)}\|^2 \end{pmatrix}\\
&= \left(\frac{f+b}{\|z_{(1)}\|^2}\right)^{s-1}\left(\frac{f}{1-\|z_{(2)}\|^2}\right)^{n-1}(f+a)^{m-1}\\
&\qquad\cdot\det\begin{pmatrix} \frac{Xf'}{\|z_{(1)}\|^2} & \frac{Xf'\|z_{(2)}\|}{\|z_{(1)}\|(1-\|z_{(2)}\|^2)} & \frac{Xf'}{\|z_{(1)}\|}\|z_{(3)}\|\\ 0 & \frac{f}{1-\|z_{(2)}\|^2} & 0\\ 0 & 0 & f+a \end{pmatrix}\\
&= \left(\frac{f+b}{\|z_{(1)}\|^2}\right)^{s-1}\left(\frac{f}{1-\|z_{(2)}\|^2}\right)^{n}(f+a)^{m}\frac{Xf'}{(1-\|z_{(2)}\|^2)\|z_{(1)}\|^2}\\
&= \frac{(f+b)^{s-1}f^n(f+a)^mXf'}{\|z_{(1)}\|^{2s}(1-\|z_{(2)}\|^2)^{n+1}}\\
&= \frac{(f+b)^{s-1}f^n(f+a)^mX^{n+2}f'}{\|z_{(1)}\|^{2s+2n+2}e^{(n+1)\|z_{(3)}\|^2}},
\end{aligned}$$

where, for the last line, we apply (1-3), that is, $1-\|z_{(2)}\|^2=e^{\|z_{(3)}\|^2}\|z_{(1)}\|^2X^{-1}$. In order to ensure $\det H>0$, when $0\le X<1$, we add the extra conditions (1-4) and (1-5).

The corresponding Ricci curvature is

$$\begin{aligned}\mathrm{Ric}&=-\partial\bar\partial\log\det H\\&=-\partial\bar\partial\log((f+b)^{s-1}f^n(f+a)^mX^{n+2}f')\\&\qquad\qquad+\partial\bar\partial\log\|z_{(1)}\|^{2s+2n+2}+(n+1)\partial\bar\partial\|z_{(3)}\|^2.\end{aligned}$$

The proof is finished. □

Remark 3.2. The above result implies that the Ricci curvature cannot be zero. Also,

$$\begin{aligned}F_{\mathrm{Ric}}&:=-\log((f+b)^{s-1}f^n(f+a)^mX^{n+1}\varphi),\\ f_{\mathrm{Ric}}&:=XF'_{\mathrm{Ric}}=-\frac{(s-1)\varphi}{f+b}-\frac{n\varphi}{f}-\frac{m\varphi}{f+a}-(n+1)-\varphi_f.\end{aligned}$$

Here we use the fact that

$$\frac{dG}{dX}=\frac{dG}{df}\frac{df}{dX}=G_ff'=G_f\frac{\varphi}{X}.$$

In particular, we have $f_{\mathrm{Ric}}(0)=-(n+1)-1=-n-2<0$. Therefore, we also cannot have Ric being positive.

Let E be the holomorphic vector field of Φ_{θ,I_n,I_m} in Lemma 2.4.

Lemma 3.3. *The function $f:=XF'$ or $U:=f-n-2$ is just a potential function of E with respect to the metric g.*

Proof. The holomorphic vector field corresponding to Φ_{θ,I_n,I_m} is

$$E=\sum_{i=1}^{s}\left(z_i\frac{\partial}{\partial z_i}+\overline{z_i}\frac{\partial}{\partial\overline{z_i}}\right)=\sum_{i=1}^{s}\left(x_i\frac{\partial}{\partial x_i}+y_i\frac{\partial}{\partial y_i}\right).$$

On the other hand, let $p_0:=(X^{1/2},0,\dots,0,\mathbf{0},\mathbf{0})$. We have

$$\begin{aligned}df|_{p_0}&=(Xf'(\partial\log X+\bar\partial\log X))|_{p_0}\\&=Xf'|_{p_0}\left(\sum_{i=1}^{s}\frac{\overline{z_i}dz_i+\zeta_id\overline{z_i}}{\|z_{(1)}\|^2}+\sum_{k=s+1}^{s+n}\frac{\overline{z_k}dz_k+z_kd\overline{z_k}}{1-\|z_{(2)}\|^2}+\sum_{l=s+n+1}^{s+n+m}(\overline{z_l}dz_l+z_ld\overline{z_l})\right)\Bigg|_{p_0}\\&=f'|_{p_0}(\overline{z_1}dz_1+z_1d\overline{z_1})|_{p_0}=2f'|_{p_0}(x_1dx_1+y_1dy_1).\end{aligned}$$

Then

$$df|_{p_0}(E|_{p_0})=2f'|_{p_0}(x_1dx_1+y_1dy_1)\left(x_1\frac{\partial}{\partial x_1}+y_1\frac{\partial}{\partial y_1}\right)=2g|_{p_0}(E|_{p_0},E|_{p_0}).$$

Let $\alpha = \sum_i \alpha_{\bar{i}} d\overline{z_i}$ be a (0, 1) form. We define $\uparrow\alpha := \sum_{i,j} g^{j\bar{i}} \alpha_{\bar{i}} \frac{\partial}{\partial z_j}$. Recall that

$$H|_{p_0} = \begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{pmatrix}\Bigg|_{p_0}$$

with $M_2|_{p_0} = M_4|_{p_0} = M_3|_{p_0} = M_7|_{p_0} = M_6|_{p_0} = M_8|_{p_0} = \mathbf{0}$. We also have

$$M_1|_{p_0} = \begin{pmatrix} f' & 0 & \cdots & 0 & 0 \\ 0 & \frac{f+b}{X} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{f+b}{X} & 0 \\ 0 & 0 & \cdots & 0 & \frac{f+b}{X} \end{pmatrix}.$$

Then

$$\uparrow\bar{\partial} f|_{p_0} = 2z_1 \frac{\partial}{\partial z_1} = 2E|_{p_0}.$$

But all these are invariant under the automorphism group. Therefore, we have

$$\uparrow\bar{\partial} f = 2E.$$

The proof is complete. □

Theorem 3.4. *Let* (Ω, g) *be a Kähler manifold endowed with its Kähler metric* g, *where* g *is a Kähler metric associated with the Kähler form* (1-2). *Then* g *is a Kähler–Einstein metric if and only if* $a = -n-1$, $s = 1$, *that is*, *the dimension of the fiber of* Ω *is one*, *and* $F(X)$ *is the solution of the differential equation*

$$(Q\varphi)_f = f^n(f-n-1)^{m+1}$$

with the initial condition $f|_{X=0} = n+2$, *in which* $Q := (f-n-1)^m f^n$ *and* $\varphi := Xf'$.

Proof. Let $\mathrm{Ric} = -\omega$. Then we obtain

$$-\partial\bar{\partial}F(X) - a\partial\bar{\partial}\|w\|^2 - b\partial\bar{\partial}\log\|\zeta\|^2 = -\partial\bar{\partial}\log((f+b)^{s-1} f^n (f+a)^m X^{n+2} f') \\ + (s+n+1)\partial\bar{\partial}\log\|z_{(1)}\|^2 + (n+1)\partial\bar{\partial}\|z_{(3)}\|^2,$$

which implies that

$$\begin{cases} a = -(n+1), b = -(s+n+1) & \text{if } s \neq 1; \\ a = -(n+1) & \text{if } s = 1. \end{cases} \tag{3-11}$$

The second case is due to the fact that, when $s=1$, $\partial\bar\partial \log|z_1|^2=0$. Moreover, we get

$$\begin{aligned}
&\partial\bar\partial \log((f-s-n-1)^{s-1} f^n (f-n-1)^m X^{n+2} f') = \partial\bar\partial F(X)\\
&\qquad\Longrightarrow e^{F(X)} = (f-s-n-1)^{s-1} f^n (f-n-1)^m X^{n+2} f'\\
&\qquad\Longleftrightarrow F(X) = \log((f-s-n-1)^{s-1} f^n (f-n-1)^m X^{n+2} f')\\
&\qquad\Longrightarrow f = XF' = X\big(\log((f-s-n-1)^{s-1} f^n (f-n-1)^m X^{n+2} f')\big)'.
\end{aligned}$$

Under (3-11), define $\varphi := Xf'$ and $Q = (f-s-n-1)^{s-1}(f-n-1)^m f^n$. For a function $G(X)$, let $G_f := \frac{dG}{df}$. Then

$$\begin{aligned}
f &= XF' = X(\log(Q\varphi X^{n+1}))' = X\frac{(Q\varphi X^{n+1})'}{Q\varphi X^{n+1}}\\
&= \frac{(Q\varphi)'X^{n+1} + (n+1)Q\varphi X^n}{Q\varphi X^n} = n+1+\frac{(Q\varphi)'X}{Q\varphi}\\
&= n+1+\frac{X}{Q\varphi}\frac{d(Q\varphi)}{df}\frac{df}{dX} = n+1+\frac{1}{Q}(Q\varphi)_f.
\end{aligned}\tag{3-12}$$

Now we look for the initial condition of the differential equation (3-12), that is, the value of $f|_{X=0}$. Since $\frac{d\varphi}{df} = \frac{(Xf')'}{f'} = 1+\frac{Xf''}{f'}$, we have

$$\begin{aligned}
f &= n+1+\frac{Q_f\varphi}{Q}+\varphi_f\\
&= n+2+\frac{Xf''}{f'}+\frac{Xf'((f-s-n-1)^{s-1}(f-n-1)^m f^n)'}{(f-s-n-1)^{s-1}(f-n-1)^m f^n},
\end{aligned}$$

which implies that $f|_{X=0}=n+2$. But, if $s\neq 1$, then $s>1$, $f>n+1+s>n+2$ for all $0\le X<1$, a contradiction. Thus we derive that $s=1$, and then $Q=(f-n-1)^m f^n$. Without confusion, we also use the same notation Q. Again from (3-12), we obtain the differential equation

$$Q(f-n-1) = (Q\varphi)_f \iff (Q\varphi)_f = f^n(f-n-1)^{m+1}.$$

We want to integrate both sides with respect to f. Before that we let $U := f-n-2$ and we get

$$\begin{aligned}
\int (Q\varphi)_f\,df &= \int f^n(f-n-1)^{m+1}\,df\\
&= \int (U+n+2)^n(U+1)^{m+1}\,dU\\
&= \int (U+1)^{m+1}\sum_{k=0}^{n} C_n^k (U+1)^k(n+1)^{n-k}\,dU\\
&= \sum_{k=0}^{n}(n+1)^{n-k}C_n^k\int (U+1)^{m+1+k}\,dU.
\end{aligned}$$

Thus

$$Q\varphi=\sum_{k=0}^{n}\frac{(n+1)^{n-k}C_n^k}{m+k+2}(U+1)^{m+k+2}+c_0. \tag{3-13}$$

According to the initial conditions $U|_{X=0}=0$ and $\varphi|_{X=0}=0$,

$$c_0=-\sum_{k=0}^{n}\frac{(n+1)^{n-k}C_n^k}{m+k+2}.$$

Define

$$P_1(U):=\sum_{k=0}^{n}\frac{(n+1)^{n-k}C_n^k}{m+k+2}(U+1)^{m+k+2}+c_0,\qquad Q(U):=Q.$$

Then

$$\frac{Q(U)}{P_1(U)}dU=\varphi^{-1}dU=\frac{1}{XU'}dU=\frac{1}{X}dX=d\log X.$$

From this equation we could derive an implicit expression of the Kähler–Einstein metric on Ω for the case when $s=1$.

Next we check the completeness. Since $X=|z_1|^2$, $df=dU$, $U|_{X=0}=0$ and $\lim_{X\to 1^+}U=+\infty$, first we have

$$\begin{aligned}\int_0^1\frac{\varphi^{\frac12}}{|z_1|}\,d|z_1|&=\int_0^1\varphi^{\frac12}\,d\log|z_1|=\frac12\int_0^1\varphi^{\frac12}\,d\log X\\&=\frac12\int_0^{+\infty}\varphi^{\frac12}\left(\frac{df}{d\log X}\right)^{-1}dU\\&=\frac12\int_0^{+\infty}\varphi^{-\frac12}\,dU=\frac12\int_0^{+\infty}\left(\frac{Q(U)}{P_1(U)}\right)^{\frac12}dU\\&=+\infty,\end{aligned}$$

where, for the last line, we apply the fact that the degree of polynomial $P_1(U)$ is $n+m+2$, while the degree of polynomial $Q(U)$ is $n+m$. Therefore, the metric is complete if one can prove that $\lim_{X\to 1}U=+\infty$. This is the same as proving that $\lim_{U\to+\infty}X(U)=1$. We could write $Q/P_1:=(1/U^2)(Q/\widetilde{P}_1)$, where $\widetilde{P}_1=P_1/U^2$ is a rational function of U with the same degree as that of Q. Then

$$\int\frac{Q}{P_1}\,dU=\int d\log X\implies\int_1^U\frac{Q}{P_1}\,dU=\log X+C$$

with a constant C. Then

$$\lim_{U\to 0}\int_1^U\frac{1}{U}\frac{Q}{\frac{P_1}{U}}\,dU=-\infty=\lim_{X\to 0}(\log X+C).$$

This is because P_1/Q has an order-one zero at 0, and

$$\lim_{U\to+\infty}\int_1^U \frac{1}{U^2}\frac{Q}{\widetilde{P}_1}\,dU = C.$$

By the left-hand side being finite, the constant C is uniquely determined. Therefore, we have $U(0)=0$ and $U(1)=+\infty$. Hence

$$\lim_{X\to 1} U = +\infty.$$

Here, we notice that $U' = f' > 0$. Thus the proof is finished. □

Example 3.5. Let $n=m=s=1$. By (3-12), we have

$$\begin{aligned}\int (Q\varphi)_f\,df &= \int f(f-2)^2\,df\\ &\implies Q\varphi = \tfrac{1}{4}f^4 - \tfrac{4}{3}f^3 + 2f^2 + C_0 = \tfrac{1}{12}(f-3)(3f^3-7f^2+3f+9),\end{aligned}$$

where $C_0 = -\frac{9}{4}$. Furthermore, the equation $3f^3-7f^2+3f+9=0$ has a real root

$$x_0 := \frac{-\sqrt[3]{594\sqrt{3}+1034}-\sqrt[3]{1034-594\sqrt{3}}+7}{9} \approx -0.82867.$$

Hence $3f^3-7f^2+3f+9 = (f-x_0)(3f^2+(3x_0-7)f-9/x_0)$ and we have the decomposition

$$\frac{12(f^2-2f)}{(f-3)(3f^3-7f^2+3f+9)} = \frac{1}{f-3} + \frac{B}{f-x_0} + \frac{Cf+D}{3f^2+\tilde{a}f+\tilde{b}},$$

where B, C and D are constants which could be computed explicitly, $\tilde{a} = 3x_0-7$, and $\tilde{b} = -\frac{9}{x_0}$. Letting f tend to infinity, we get $3B+C=-3$. Therefore, we have

$$\begin{aligned}\int \frac{12(f^2-2f)}{(f-3)(3f^3-7f^2+3f+9)}\,df &= \int d\log X\\ &\implies \log(f-3) + B\log(f-x_0) + \frac{C}{6}\log(3f^2+\tilde{a}f+\tilde{b})\\ &\qquad + \frac{2\big(D-\frac{C\tilde{a}}{6}\big)}{\sqrt{12\tilde{b}-\tilde{a}^2}}\arctan\bigg(\frac{6f+\tilde{a}}{\sqrt{12\tilde{b}-\tilde{a}^2}}\bigg) + d_0 = \log X,\end{aligned}$$

where

$$d_0 = -\frac{C}{6}\log 3 - \frac{\big(D-\frac{C\tilde{a}}{6}\big)\pi}{\sqrt{12\tilde{b}-\tilde{a}^2}}$$

could be computed by the boundary condition $f|_{X=1} = +\infty$ and $3B+C=-3$.

Example 3.6. Let $n=0$, $m=s=1$. We have

$$\begin{aligned}(f-1)Xf' &= \int (f-1)^2\,df = \tfrac{1}{3}f^3 - f^2 + f - \tfrac{2}{3}\\ &= \tfrac{1}{3}(f^3-3f^2+3f-2) = \tfrac{1}{3}(f-2)(f^2-f+1)\\ \Longrightarrow &\int \frac{f-1}{\frac{1}{3}f^3-f^2+f-\frac{2}{3}}\,df = \int d\log X\\ \Longrightarrow &-\tfrac{1}{2}\log(f^2-f+1)+\sqrt{3}\arctan\bigl(\tfrac{1}{3}\sqrt{3}(2f-1)\bigr)+\log(f-2)+d_1 = \log X,\end{aligned}$$

where

$$d_1 = -\tfrac{1}{2}\sqrt{3}\pi$$

could also be computed by the boundary condition $f|_{X=1}=+\infty$.

Remark 3.7. The above two examples have been considered by Jing [11] (locally) and Kim, Yamamori, and Zhang [13] by different methods from ours.

4. Extremal metric

Theorem 4.1. *Let (Ω, g) be a Kähler manifold endowed with its Kähler metric g, where g is a Kähler metric associated with the Kähler form (1-2). Then g is a special extremal metric if and only if the scalar curvature of g is constant.*

Proof. In the sense of automorphism of Ω, any point $(z_{(1)}, z_{(2)}, z_{(3)}) \in \Omega$ could be viewed as $p_0 := (X^{1/2}, 0, \dots, 0, \mathbf{0}, \mathbf{0})$. Therefore, we only need to consider the scalar curvature S of Ω at p_0. By (3-9), we have

$$H|_{p_0} = \begin{pmatrix} A_{11} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & fI_n & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & (f+a)I_m\end{pmatrix}, \tag{4-14}$$

where $A_{11} = \operatorname{diag}\bigl(f', \frac{f+b}{X}, \dots, \frac{f+b}{X}\bigr)$ is a diagonal matrix of order s. Let $c=f(0)$. Then the associated Kähler form is

$$\omega|_{p_0} = f'dz_1\wedge d\overline{z_1} + \frac{f+b}{X}\sum_{i=2}^{s} dz_i\wedge d\overline{z_i} + f\sum_{k=s+1}^{s+n} dz_k\wedge d\overline{z_k} + (f+a)\sum_{l=s+n+1}^{s+n+m} dz_l\wedge d\overline{z_l}.$$

On the other hand, we denote the Ricci curvature of ω at p_0 by $\mathrm{Ric}|_{p_0}$. By Lemma 3.1, the Ricci form (3-6) has a similar expression as the Kähler form (1-2). Hence, in the corresponding expression as in the format (4-14), we replace a, b, f in (4-14) by

$$a_{\mathrm{Ric}} := n+1, \quad b_{\mathrm{Ric}} := s+n+1$$

and

$$\tilde{f} := f_{\mathrm{Ric}} = -X(\log(Q\varphi X^{n+1}))' = -n-1-\frac{(Q\varphi)_f}{Q},$$

respectively (see Remark 3.2), where $Q := (f+b)^{s-1} f^n (f+a)^m$. Therefore, the corresponding matrix of $\mathrm{Ric}|_{p_0}$ could be obtained by

$$R|_{p_0} = \begin{pmatrix} B_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{f} I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\tilde{f}+n+1) I_m \end{pmatrix}, \tag{4-15}$$

where $B_{11} = \mathrm{diag}\big(\tilde{f}', \frac{\tilde{f}+s+n+1}{X}, \dots, \frac{\tilde{f}+s+n+1}{X}\big)$ is a diagonal matrix of order s. Furthermore,

$$\tilde{f}' = -\left(\frac{(Q\varphi)_f}{Q}\right)' = -\left(\frac{(Q\varphi)_f}{Q}\right)_f f' = -\left(\frac{(Q\varphi)_{ff}}{Q} - \frac{(Q\varphi)_f Q_f}{Q^2}\right) f'.$$

Combining (4-14) and (4-15), we obtain the scalar curvature S of Ω at p_0, that is,

$$\begin{aligned} S|_{p_0} &= \mathrm{tr}(H|_{p_0}^{-1} R|_{p_0}) \\ &= \frac{\tilde{f}'}{f'} + \frac{(s-1)(\tilde{f}+s+n+1)}{f+b} + \frac{n\tilde{f}}{f} + \frac{m(\tilde{f}+n+1)}{f+a} \\ &= -\frac{(Q\varphi)_{ff}}{Q} + \frac{(Q\varphi)_f Q_f}{Q^2} - \frac{s-1}{(f+b)Q}(Q\varphi)_f + \frac{s(s-1)}{f+b} \\ &\qquad - \frac{n}{fQ}(Q\varphi)_f - \frac{n(n+1)}{f} - \frac{m}{(f+a)Q}(Q\varphi)_f \\ &= -\frac{(Q\varphi)_{ff}}{Q} + \frac{s(s-1)}{f+b} - \frac{n(n+1)}{f}, \end{aligned} \tag{4-16}$$

where, for the last line, we apply

$$\begin{aligned} \frac{Q_f}{Q^2} &= \frac{1}{Q^2}\big((s-1)(f+b)^{s-2} f^n (f+a)^m + n(f+b)^{s-1} f^{n-1} (f+a)^m \\ &\qquad + m(f+b)^{s-1} f^n (f+a)^{m-1}\big) \\ &= \frac{s-1}{(f+b)Q} + \frac{n}{fQ} + \frac{m}{(f+a)Q}. \end{aligned} \tag{4-17}$$

By our assumption, g is special extremal metric. Then

$$-\frac{(Q\varphi)_{ff}}{Q} + \frac{s(s-1)}{f+b} - \frac{n(n+1)}{f} = c_1 f + c_2$$

for some $c_1, c_2 \in \mathbb{R}$. Therefore,

$$(Q\varphi)_{ff} = -(c_1 f + c_2)Q - \frac{n(n+1)}{f} Q + \frac{s(s-1)}{f+b} Q := P_2(f),$$

where $P_2(f)$ is a polynomial of f with degree $n+m+s$. Thus

$$\int (Q\varphi)_{ff}\,df = \int P_2(f)\,df.$$

By the initial condition $f|_{X=0} = c$, we can derive the expression of $(Q\varphi)_f$. Repeating the same procedure, if $c_1 \neq 0$, we get that the expression of $Q\varphi$ is a polynomial of f with degree $n+m+s+2$, again since Q is a polynomial of f with degree $n+m+s-1$. According to the completeness for the extremal metric, we have

$$\int_c^{+\infty} \varphi^{-\frac{1}{2}}\,df \approx \int_c^{+\infty} f^{-\frac{3}{2}}\,df = \frac{2}{\sqrt{c}} \neq \infty.$$

Then we derive that g is extremal if and only if $c_1 = 0$, that is, its scalar curvature is constant. □

5. Some applications

As an application of Theorem 1.1, we give the comparison between the Bergman metric and the Kähler–Einstein metric in the case $s = m = 1$. In this case we denote Ω in (1-1) by Ω_{n+2}. That is,

$$\Omega_{n+2} := \{(\zeta, \tilde{\zeta}, w) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C} : e^{|w|^2}|\zeta|^2 + \|\tilde{\zeta}\|^2 < 1\}.$$

Let g^{B} be the Bergman metric of Ω_{n+2}, and g be the Kähler–Einstein metric considered in Theorem 1.1. Then we obtain the following result.

Theorem 5.1. *The Bergman metric g^B and the Kähler–Einstein metric g on Ω_{n+2} are equivalent.*

Proof. We divide the proof into three steps.

Step 1. We first give the Bergman metric g^{B} on Ω_{n+2}.

In Huo [10], the Bergman kernel of Ω_{n+2} is given by

$$K(z,\bar{z}) = \frac{(n+1)!}{\pi^{n+2}} \frac{e^{|w|^2}(1 - \|\tilde{\zeta}\|^2 + (n+1)e^{|w|^2}|\zeta|^2)}{(1 - \|\tilde{\zeta}\|^2 - e^{|w|^2}|\zeta|^2)^{n+3}}. \tag{5-18}$$

Let $X = e^{|w|^2}|\zeta|^2/(1 - \|\tilde{\zeta}\|^2)$, $(\zeta, \tilde{\zeta}, w) \in \Omega_{n+2}$. Simple calculations show that

$$\text{(5-18)} = \frac{(n+1)!}{\pi^{n+2}} \frac{X^{n+2} e^{-(n+1)|w|^2}(1 + (n+1)X)}{(1-X)^{n+3}|\zeta|^{2(n+2)}}.$$

Let $a = -n-1$ in (1-2), and

$$F_B(X) := \log \frac{X^{n+2}(1 + (n+1)X)}{(1-X)^{n+3}}.$$

Then the Kähler form in (1-2) associated with Ω_{n+2} reduces to

$$\omega_B := \sqrt{-1}(\partial\bar{\partial} F_B(X) - (n+1)\partial\bar{\partial}|w|^2).$$

It is easy to conclude that $\omega_B = \sqrt{-1}\partial\bar{\partial}\log K(z, \bar{z})$. That is, the Kähler metric associated with ω_B is just the Bergman metric g^{B}. According to our previous notation in (4-14), we have

$$g^{\mathrm{B}}|_{p_0} = \operatorname{diag}(f_B', \underbrace{f_B, \dots, f_B}_{n}, f_B - n - 1), \tag{5-19}$$

where

$$f_B := XF_B' = -\frac{1}{1+(n+1)X} + \frac{n+3}{1-X}, \quad f_B' = \frac{n+1}{(1+(n+1)X)^2} + \frac{n+3}{(1-X)^2}.$$

We also set

$$\varphi_B := Xf_B' = \frac{(n+1)X}{(1+(n+1)X)^2} + \frac{(n+3)X}{(1-X)^2}. \tag{5-20}$$

Step 2. We analyze the Kähler–Einstein metric g on Ω_{n+2}.

By Theorem 1.1, the Kähler metric g associated with the Kähler form (1-2) is a Kähler–Einstein metric on Ω_{n+2} if and only if $a = -n-1$, and $F(X)$ is the solution of the differential equation

$$(Q\varphi)_f = f^n(f-n-1)^2 \tag{5-21}$$

with the initial condition $f|_{X=0} = n+2$, in which $Q := (f-n-1)f^n$ and $\varphi := Xf'$. Then

$$\begin{aligned} Q\varphi &= \int f^n(f-n-1)^2\, df \\ &= \int f^n(f^2 - 2(n+1)f + (n+1)^2)\, df \\ &= \frac{f^{n+3}-(n+2)^{n+3}}{n+3} - \frac{2(n+1)(f^{n+2}-(n+2)^{n+2})}{n+2} \\ &\qquad + (n+1)(f^{n+1}-(n+2)^{n+1}) := P_1(U). \end{aligned}$$

Moreover, by (4-14), the Kähler–Einstein metric g at p_0 associated with the Kähler form (1-2) has the formula

$$g|_{p_0} = \operatorname{diag}(f', \underbrace{f, \dots, f}_{n}, f-n-1). \tag{5-22}$$

Let $U := f-n-2$. Then $U' = f'$, $U|_{X=0} = 0$. By (3-13) and $\varphi|_{X=0} = 0$, we have

$$\varphi = XU' = \frac{P_1(U)}{Q(U)} = U\frac{\widehat{P_1}(U)}{Q(U)}, \tag{5-23}$$

where $\widehat{P_1}(U)$ is a positive polynomial of U on $\{U \geq 0\}$ with degree $n+2$, and

$$Q(U) = (U+1)(U+n+2)^n$$

is a polynomial of U with degree $n+1$. We could derive that

$$\frac{Q(U)}{U\widehat{P_1}(U)} = \varphi^{-1} = \frac{1}{U} + \frac{P(U)}{\widehat{P_1}(U)}.$$

The reason is that if we multiply the equation by U and let U tend to 0, the middle item is 1 and the second term on the right-hand side tends to 0. Here we notice that the degree of $P(U)$ is less than $n+1$. Thus

$$\frac{Q(U)}{U\widehat{P_1}(U)} dU = d\log X \implies X = UT(U), \tag{5-24}$$

where $T(U)$ is the exponential of the integral of a rational function of U on $\{U \geq 0\}$, with a positive polynomial $\widehat{P_1}(U)$ as the denominator.

Step 3. We compare the Bergman metric g^{B} and the Kähler–Einstein metric g on Ω_{n+2}.

From (5-19) and (5-22), we only need to consider the fractions

$$\frac{g^{\mathrm{B}}_{11}|_{p_0}}{g_{11}|_{p_0}} := \frac{f_B'}{f'}, \qquad \frac{g^{\mathrm{B}}_{ii}|_{p_0}}{g_{ii}|_{p_0}} := \frac{f_B}{f}, \quad i = 2, \dots, n, \qquad \frac{g^{\mathrm{B}}_{(n+1)(n+1)}|_{p_0}}{g_{(n+1)(n+1)}|_{p_0}} := \frac{f_B - n - 1}{f - n - 1}.$$

Since f_B (or f_B') and f (or f') are continuous functions for $X \in (0, 1)$, we consider only the limits $\lim_{X\to 0^+}$ and $\lim_{X\to 1^-}$.

From the above statement, we first have

$$\lim_{X\to 0^+} \frac{f_B}{f} = \frac{f_B(0)}{f(0)} = 1,$$

and

$$\lim_{X\to 0^+} \frac{f_B'}{f'} = \lim_{X\to 0^+} \frac{Xf_B'}{Xf'} = \lim_{X\to 0^+} \frac{\varphi_B}{\varphi} = \lim_{X\to 0^+} \frac{\varphi_B'}{\varphi'} = \lim_{X\to 0^+} \frac{\varphi_B'}{U'\varphi_U} < +\infty$$

exists but is not zero. For the last line, we applied (5-20), (5-23) and (5-24), and the fact that

$$\varphi_B' \to 2n+4, \quad \varphi_U \to \frac{\widehat{P_1}(0)}{Q(0)}, \quad U' \to T(0)^{-1}$$

when $X \to 0^+$.

On the other hand, we have

$$\lim_{X\to 1^-} \frac{f_B}{f} = \lim_{X\to 1^-} \frac{f_B'}{f'} = \lim_{X\to 1^-} \frac{\varphi_B}{\varphi} = \lim_{X\to 1^-} \frac{n+3}{(1-X)^2\varphi} = (n+3) \lim_{X\to 1^-} \left(\frac{\varphi^{-\frac{1}{2}}}{1-X}\right)^2.$$

Next, we prove that

$$\lim_{X\to 1^-}\left(\frac{\varphi^{-\frac{1}{2}}}{1-X}\right)^2=\frac{1}{n+3}.$$

Since

$$\lim_{X\to 1^-}\frac{\varphi^{-\frac{1}{2}}}{1-X}=\frac{1}{2}\lim_{X\to 1^-}\frac{\varphi'}{\varphi^{\frac{3}{2}}}=\frac{1}{2}\lim_{U\to+\infty}\frac{\varphi_U\varphi}{\varphi^{\frac{3}{2}}}=\frac{1}{2}\lim_{U\to+\infty}\frac{\varphi_U}{\varphi^{\frac{1}{2}}}.$$

By (5-21) and (5-23), we have

$$Q_U\varphi+Q\varphi_U=(U+n+2)^n(U+1)^2$$
$$\Longrightarrow \frac{\varphi_U}{\varphi^{\frac{1}{2}}}=\frac{Q(U+1)-Q_U\varphi}{Q\varphi^{\frac{1}{2}}}=\frac{Q^2(U+1)-Q_UP_1}{Q\sqrt{P_1Q}}.$$

Note that $\deg(Q)=n+1$, $\deg(P_1)=n+3$, the coefficient before the highest-order term of Q is 1, and the coefficient before the highest-order term of P_1 is $\frac{1}{n+3}$. Hence,

$$\frac{1}{2}\lim_{U\to+\infty}\frac{\varphi_U}{\varphi^{\frac{1}{2}}}=\frac{1}{2}\lim_{U\to+\infty}\frac{Q^2(U+1)-Q_UP_1}{Q\sqrt{P_1Q}}=\frac{1-\frac{n+1}{n+3}}{\frac{2}{\sqrt{n+3}}}=\frac{1}{\sqrt{n+3}}.$$

Therefore, we proved that

$$\lim_{X\to 1^-}\frac{f_B-n-1}{f-n-1}=\lim_{X\to 1^-}\frac{f_B}{f}=\lim_{X\to 1^-}\frac{f_B'}{f'}=1. \qquad \square$$

Acknowledgements

This project is supported by the National Natural Science Foundation of China (Grant No. 12171140, for which Guan is the PI and the second corresponding author) and Tang as the first corresponding author is also partially supported by the National Natural Science Foundation of China (Grant No. 12101185).

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Received March 1, 2024. Revised October 28, 2024.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 333 No. 1 November 2024
