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We give an algorithm for computing the knot Floer homology of a (1, 1) knot from a particular presentation of its fundamental group.

1. Introduction

Heegaard Floer homology (introduced by Peter Ozsváth and Zoltán Szabó [30; 31]) provides various topological invariants for three- and four-manifolds. A null homologous knot in a three-manifold induces a filtration on the Heegaard Floer chain complex and its homology, called the knot Floer homology, which was introduced by Peter Ozsváth and Zoltán Szabó [29] and Jacob Rasmussen [37] independently.

The fundamental group is an important invariant for three-manifolds. The geometrization theorem (proposed by William Thurston [44] and proved by Grisha Perelman [33; 34; 35]) implies that the fundamental group completely determines a closed, orientable, irreducible, three-manifold up to orientation except for lens spaces (see [2] for details). Hence the (hat version) Heegaard Floer homology can be completely determined by the fundamental group of the three-manifold.

Heegaard Floer homology has a close relationship with the fundamental group. For an integer homology three-sphere Y, the Euler characteristic of $\mathrm{HF}^+_{\mathrm{red}}(Y)$ minus half its *correction term* equals Casson's invariant $\lambda(Y)$ under certain normalization [27], and Casson's invariant is the algebraic counting of $\mathrm{SU}(2)$ representations of $\pi_1(Y)$. On the homology level, the instanton Floer homology is a categorification of Casson's invariant [8] and its generators are the $\mathrm{SU}(2)$ representations of the fundamental group [8; 17]. Seiberg–Witten Floer homology is isomorphic to Heegaard Floer homology (by the work of Hutchings [14], Hutchings and Taubes [15; 16], Taubes [39; 40; 41; 42; 43], and Kutluhan, Lee, and Taubes [18; 19; 20; 21; 22] or Colin, Ghiggini, and Honda [4; 5; 6]), and Witten [45] conjectured a relation between the Seiberg–Witten and Donaldson invariants.

It is interesting to find a concrete connection between the fundamental group and the Heegaard Floer homology. In [28], Ozsváth and Szabó asked the following two closely related questions:

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Keywords: (1, 1) knot, fundamental group, knot Floer homology.

Question 1 [28, Question 7]. *Let K be a knot in S^3. Is there an explicit relationship between the fundamental group of the knot complement $S^3 \setminus K$ and the knot Floer homology $\widehat{\mathrm{HFK}}(S^3, K)$?*

Question 2 [28, Question 8]. *Is there an explicit relationship between the Heegaard Floer homology and the fundamental group of a three-manifold?*

We study Question 1 for (1, 1) knots. (1, 1) knots are those knots which can be placed in one-bridge position with respect to a genus one Heegaard splitting of the three-sphere. (1, 1) knots form a large family of knots: torus knots and two-bridge knots are all (1, 1) knots. Fujii [9] showed that the Alexander polynomial of any knot can be realized as the Alexander polynomial of some (1, 1) knot.

From the perspective of knot Floer homology, (1, 1) knots are particularly appealing. It was observed by Goda, Matsuda, and Morifuji in [10] that (1, 1) knots are exactly those knots which can be presented by a genus one doubly pointed Heegaard diagram and their knot Floer homology can be computed combinatorially (see also [29]). The diagrammatic characterization of the (1, 1) L-space knots and the (1, 1) almost L-space knots is given in [3; 11]. Nie [24] constructed infinitely many (1, 1) knots which are topologically slice, but not smoothly slice, and Himino [13] constructed infinitely many mutually nonconcordant hyperbolic (1, 1) knots whose Upsilon invariants are convex. Antonio, Celoria, and Stipsicz [1] proved that a family of the alternating torus knots are linearly independent in the concordance group and a family of twist knots are linearly independent in the concordance group.

The fundamental group of a (1, 1) knot has a presentation with two generators and one relator. Our main result is the following:

Theorem 1.1. *Let K be a $(1, 1)$ knot in S^3. Given a two-generator one-relator presentation $\pi_1(S^3 \setminus K) = \langle X, Y \mid R(X, Y)\rangle$ of its fundamental group coming from a genus one doubly pointed Heegaard diagram, $\widehat{\mathrm{HFK}}(S^3, K)$ can be computed directly from the relator $R(X, Y)$.*

The computation is provided by Algorithm 4.1. The proof of Theorem 1.1 relies on the fact that the presentation from a genus one Heegaard diagram contains enough information about the universal cover of the diagram, which was employed to compute $\widehat{\mathrm{HFK}}(S^3, K)$ for certain knots in [29, Section 6] and for (1,1) knots in [10]. Algorithm 4.1 can be generalized for (1, 1) knots in lens spaces (Section 5).

It would be interesting to generalize Theorem 1.1 to knots with Heegaard diagrams of higher genus, or remove that the presentation of the fundamental group arises from a genus one Heegaard diagram. Algorithm 4.1 actually applies to slightly more general group presentations of (1, 1) knots (Section 6), for which Algorithm 4.1 does compute the Alexander polynomial (Corollary 6.4) but may not compute $\widehat{\mathrm{HFK}}(S^3, K)$ (Example 6.6).

Algorithm 4.1 does not only apply to the fundamental group of (1, 1) knots, it also applies to any *pseudo-geometric* two-generator one-relator group presentations. Though we do not know whether the homology yields an invariant of the group, and what properties it captures.

This paper is organized as follows. In Section 2, we briefly introduce the knot Floer homology and (1, 1) knots. In Section 3, we give a method to find all (primitive) bigons and the basepoints contained in them from the presentation. In Section 4, we describe the algorithm from the special presentation to the knot Floer homology, and give some examples. In Section 5, we extend the algorithm for (1, 1) knots in lens spaces. In Section 6, we discuss our algorithm on general group presentations.

2. Preliminaries on Heegaard Floer homology and (1, 1) knots

We recall some facts about the knot Floer homology and (1, 1) knots. For details on knot Floer homology, see [29; 32; 37].

2.1. ***Knot Floer homology.*** A null-homologous knot K in a closed, oriented three-manifold Y can be represented by a *doubly pointed Heegaard diagram* $\mathcal{H} := (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, where:

- Σ is a closed, oriented surface of genus g in Y which splits Y into two handlebodies, denoted by V_α and V_β.
- $\boldsymbol{\alpha} := \{\alpha_1, \dots, \alpha_g\}$ (resp. $\boldsymbol{\beta} := \{\beta_1, \dots, \beta_g\}$) is a set of attaching circles for handlebody V_α (resp. V_β) that are homologically independent in $H_1(\Sigma)$. Denote by D_{α_i} and D_{β_i} their attaching disks.
- w and z are two basepoints in $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$.
- The knot K is specified by a proper arc joining z to w in $V_\alpha - \bigcup_i D_{\alpha_i}$, and a proper arc joining w to z in $V_\beta - \bigcup_i D_{\beta_i}$.

The g-th symmetric product $\mathrm{Sym}^g(\Sigma) := \Sigma^{\times g}/S_g$ of Σ, where S_g is the symmetric group on g elements, has a symplectic structure induced from a complex structure on Σ, so that the two g-dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ are Lagrangian submanifolds (by Perutz [36]). An intersection point $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a g-tuple $\{x_1, \dots, x_g\}$ such that each x_i belongs to $\alpha_i \cap \beta_{\sigma(i)}$ for some permutation $\sigma \in S_g$. Given two intersection points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, let $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ be the set of homotopy classes of *Whitney disks* connecting $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\{u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma) \mid u(-i) = \boldsymbol{x},\ u(i) = \boldsymbol{y},\ u(a) \subset \mathbb{T}_\alpha,\ u(b) \subset \mathbb{T}_\beta\},$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ whose boundary consists of two arcs $a = \{z \in \partial\mathbb{D} \mid \mathrm{Re}(z) \geq 0\}$ and $b = \{z \in \partial\mathbb{D} \mid \mathrm{Re}(z) \leq 0\}$. The *multiplicity* $n_w(\phi)$ of $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ at $w \in \Sigma$ is defined to be the algebraic intersection number of ϕ with $\{w\} \times \mathrm{Sym}^{g-1}(\Sigma)$.

The moduli space $\mathcal{M}(\phi)$ of pseudoholomorphic representatives of ϕ has a natural $\mathbb{R}$ action, and let $\widehat{\mathcal{M}}(\phi)$ be the unparametrized moduli space $\mathcal{M}(\phi)/\mathbb{R}$. The expected dimension of $\mathcal{M}(\phi)$ is determined by the *Maslov index* $\mu(\phi)$ of ϕ.

The chain complex $\widehat{\mathrm{CFK}}(\mathcal{H})$ is a free Abelian group generated by the intersection points $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with the differential defined by

$$\hat{\partial}_K \boldsymbol{x} = \sum_{\boldsymbol{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta} \sum_{\{\phi\in\pi_2(\boldsymbol{x},\boldsymbol{y}) \mid \mu(\phi)=1, n_w(\phi)=n_z(\phi)=0\}} \#\widehat{\mathcal{M}}(\phi)\cdot \boldsymbol{y}. \tag{1}$$

$(\widehat{\mathrm{CFK}}(\mathcal{H}), \hat{\partial}_K)$ is a chain complex whose homology $\widehat{\mathrm{HFK}}(Y, K)$ is an invariant of the knot K in Y, called the *knot Floer homology* of K [29; 37].

There are two gradings on $\widehat{\mathrm{CFK}}(S^3, K)$. The *Alexander grading* is the unique function $F : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ satisfying

$$F(\boldsymbol{x}) - F(\boldsymbol{y}) = n_z(\phi) - n_w(\phi) \quad \text{for all } \phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \tag{2}$$

and the additional symmetry

$$\#\{\boldsymbol{x} \mid F(\boldsymbol{x}) = i\} \equiv \#\{\boldsymbol{x} \mid F(\boldsymbol{x}) = -i\} \pmod 2 \quad \text{for all } i \in \mathbb{Z}. \tag{3}$$

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the *relative Maslov grading* or the *homological grading* satisfies

$$\mathrm{gr}(\boldsymbol{x}, \boldsymbol{y}) = \mu(\phi) - 2n_w(\phi) \quad \text{for all } \phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}).$$

A Heegaard diagram of Y can be obtained by removing the basepoint z from a doubly pointed Heegaard diagram of (Y, K), and $\widehat{\mathrm{HF}}(Y)$ can be obtained from $\widehat{\mathrm{CFK}}(Y, K)$ with additional differentials. When $Y = S^3$, we have $\widehat{\mathrm{HF}}(S^3) \cong \mathbb{Z}$, and by defining this homology to be supported in Maslov grading 0, we can define an *absolute Maslov grading* $M : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}$ with

$$\mathrm{gr}(\boldsymbol{x}, \boldsymbol{y}) = M(\boldsymbol{x}) - M(\boldsymbol{y}).$$

It is evident that the differential (1) preserves the Alexander grading and decreases the Maslov grading by one. Let $\widehat{\mathrm{CFK}}_m(S^3, K; s)$ be the subgroup of $\widehat{\mathrm{CFK}}(S^3, K)$ generated by those $\boldsymbol{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $F(\boldsymbol{x}) = s$ and $M(\boldsymbol{x}) = m$. Then $\widehat{\mathrm{HFK}}(S^3, K)$ can be decomposed as

$$\widehat{\mathrm{HFK}}(S^3, K) = \bigoplus_{m,s\in\mathbb{Z}} \widehat{\mathrm{HFK}}_m(S^3, K; s).$$

The knot Floer homology is a categorification of the Alexander polynomial.

Theorem 2.1 (Ozsváth–Szabó [29], Rasmussen [37]). *Let K be a knot in S^3 and $\Delta_K(T)$ its symmetrized Alexander polynomial. Then*

$$\sum_{m,s\in\mathbb{Z}} (-1)^m \cdot \mathrm{rank}\, \widehat{\mathrm{HFK}}_m(S^3, K; s) \cdot T^s = \Delta_K(T).$$

2.2. (1, 1) *knots and their fundamental groups.*

Definition 2.2 [7]. A proper embedded arc γ in a handlebody H is called *trivial* if there exists an embedded disk D in H such that $\gamma \subseteq \partial D$ and $\partial D \cap \partial H = \partial D \setminus \mathrm{Int}\,\gamma$. A link L in a three-manifold M is called a (g, n) *link* if there exists a genus g Heegaard splitting $M = H_1 \cup H_2$ such that $L \cap H_i$ $(i = 1, 2)$ is the union of n mutually disjoint properly embedded trivial arcs. The decomposition $(H_1, L \cap H_1) \cup (H_2, L \cap H_2)$ is called a (g, n) *decomposition* of (M, L).

We focus on (1, 1) knots in S^3. (1, 1) knots are precisely those knots which admit genus one doubly pointed Heegaard diagrams, or (1, 1) *Heegaard diagrams* for brevity. For more detailed discussions of Heegaard diagrams of (1, 1) knots, see [10; 25; 26].

A (1, 1) Heegaard diagram $\mathcal{H} = (T^2, \alpha, \beta, w, z)$ for a (1, 1) knot K gives a two-generator one-relator presentation of $\pi_1(S^3 \setminus K)$ as follows. Orient α and β so that their intersection number $[\alpha] \cdot [\beta] = +1$ and let t_α be an oriented arc connecting w to z in $T^2 \setminus \alpha$. Travel along β for a full round and record its intersection with α and t_α: write X (resp. X^{-1}) for a positive (resp. negative) intersection with α, and Y (resp. Y^{-1}) for a positive (resp. negative) intersection with t_α. Let $R(X, Y)$ be the resulting word. Then we have

$$\pi_1(S^3 \setminus K) \cong \langle X, Y \mid R(X, Y) \rangle. \tag{4}$$

To see that this is indeed a presentation of $\pi_1(S^3 \setminus K)$, we stabilize the Heegaard splitting as follows. Let V_α (resp. V_β) be the solid torus in the Heegaard splitting defined by $\mathcal{H}$ for which α (resp. β) bounds a disk. We push the interior of t_α into V_α and denote the new arc by $\tilde{t}_\alpha$, so that t_α and $\tilde{t}_\alpha$ cobound a disk D_1. Take a tubular neighborhood $N(\tilde{t}_\alpha)$ of $\tilde{t}_\alpha$ in V_α so that $N(\tilde{t}_\alpha) \cap \beta = \varnothing$ and $D_1 \setminus N(\tilde{t}_\alpha)$ is a disk, denoted by D_2. Let $V'_\alpha = V_\alpha \setminus N(\tilde{t}_\alpha)$ and V'_β be the closure of $V_\alpha \cup N(\tilde{t}_\alpha)$. Then we have a genus two Heegaard splitting

$$S^3 = V'_\alpha \bigcup_{\Sigma_2} V'_\beta, \tag{5}$$

with Heegaard surface $\Sigma_2 = \partial V'_\alpha = \partial V'_\beta$. Let $\alpha_2 = \partial D_2$. Let β_2 be a belt circle of $N(\tilde{t}_\alpha)$ which is a meridian of K. The Heegaard diagram $(\Sigma_2, \{\alpha, \alpha_2\}, \{\beta, \beta_2\})$ specifies the Heegaard splitting (5) and

$$\mathcal{H}' = (\Sigma_2, \{\alpha, \alpha_2\}, \{\beta\})$$

is a Heegaard diagram for the knot complement $S^3 \setminus K$, i.e., $S^3 \setminus K$ can be obtained from a genus two surface Σ_2 by first attaching two-handles $C_\alpha, C_{\alpha_2}, C_\beta$ so that the boundaries of the meridian disks of the two-handles are identified with α, α_2, β, respectively, and then attaching a three-ball to $\partial(\Sigma_2 \cup C_\alpha \cup C_{\alpha_2}) = S^2$. In this language, it is clear that the presentation (4) specifies $\pi_1(S^3 \setminus K)$: attaching C_α, C_{α_2},

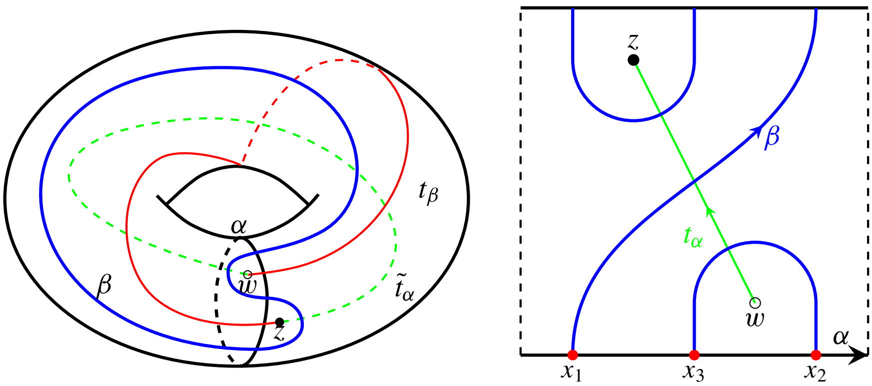

Figure 1. A Heegaard diagram for the right-handed trefoil $T_{2,3}$.

and the three-ball to Σ_2 yields a genus two handlebody with fundamental group free on two generators. Note that $\beta \cap \alpha_2 = \beta \cap t_\alpha$. According to Van Kampen's theorem, the relator obtained by attaching two-handle C_β is $R(X, Y)$ described as above.

Example 2.3. The right-handed trefoil $T_{2,3}$ has a Heegaard diagram $(T^2, \alpha, \beta, w, z)$ as illustrated in Figure 1. The $T_{2,3}$ is given by $\tilde{t}_\alpha \cup t_\beta$, where the arc $\tilde{t}_\alpha$ in the solid torus is described as above, and t_β is an arc in $T^2 \setminus \beta$ connecting z and w. For convenience, we represent the torus as a rectangle with opposite sides glued together, and push $\tilde{t}_\alpha$ to t_α in the torus, as shown in Figure 1 on the right. By choosing orientations on each curve, we obtain a presentation of $\pi_1(S^3 \setminus T_{2,3})$ (X-letters are labeled in order):

$$\pi_1(S^3 \setminus T_{2,3}) \cong \langle X, Y \mid R(X, Y) = X_1 \overline{Y} X_2 Y \overline{X}_3 Y \rangle.$$

Throughout the paper, we use $\overline{X}$ for X^{-1} and $\overline{Y}$ for Y^{-1} for brevity.

Assumption 2.4. Let K be a $(1, 1)$ knot in S^3 and $\langle X, Y \mid R(X, Y) \rangle$ be a presentation of $\pi_1(S^3 \setminus K)$ obtained as above. We make the following assumptions:

(1) The curves α and β are oriented so that $[\alpha] \cdot [\beta] = +1$, that is,

$$\#\{X \mid X \in R(X, Y)\} - \#\{\overline{X} \mid \overline{X} \in R(X, Y)\} = +1.$$

(2) $\#\{X \mid X \in R(X, Y)\} \geq 2$, or equivalently, K is knotted.

(3) The relator $R(X, Y)$ is *cyclically reduced*, that is, there are no subwords of the form $X\overline{X}$, $\overline{X}X$, $Y\overline{Y}$ or $\overline{Y}Y$ up to cyclic permutations of the relator.

Assumption 2.4 makes no actual restrictions. (1) is obtained by proper choice of orientations of α and β. (2) is obtained for nontrivial knots. Given a (1,1) Heegaard diagram, we can isotope the beta curve so that every bigon contains z and/or w (see [11]), and the resulting presentation will satisfy (3).

We write the relator as

$$R(X, Y) = R_1 R_2 \cdots R_i \cdots R_m,$$

where $R_i \in \{X, \overline{X}, Y, \overline{Y}\}$, and denote by R_i^j the proper subword $R_i R_{i+1} \cdots R_j$, with the convention that $R_{m+k} = R_k$. We are interested in the subword R_i^j with $\{R_i, R_j\} = \{X, \overline{X}\}$. We label the X-letters in R_i^j from 1 to n, and the i-th X-letter in R_i^j is either X_i or $\overline{X}_i$. Denote by W_a^b $(1 \leqslant a < b \leqslant n)$ the subword of R_i^j from X_a (resp. $\overline{X}_a$) to X_b (resp. $\overline{X}_b$) (hence $W_1^n = R_i^j$). We use the capital letters X and Y to denote the letters in the relator R and the lowercase letters x and y to denote the intersection points in the Heegaard diagram $\mathcal{H}$.

3. Bigons and disk words for (1, 1) knots

Let K be a (1, 1) knot in S^3 and $\langle X, Y \mid R(X, Y)\rangle$ be a presentation of $\pi_1(S^3 \setminus K)$ obtained from a (1, 1) Heegaard diagram $\mathcal{H} = (T^2, \alpha, \beta, z, w)$ of (S^3, K), satisfying Assumption 2.4. To compute $\widehat{\mathrm{HFK}}(S^3, K)$ from the presentation, we need to find all the information of the chain complex $\widehat{\mathrm{CFK}}(S^3, K)$. The generators are the intersection points of the curves α and β in the torus T^2, which correspond to X-letters (X or $\overline{X}$) in the relator R. On the other hand, the differential is determined by bigons in the Heegaard diagram $\mathcal{H}$. Since the relator R is cyclically reduced, all bigons contain at least one basepoint, and it follows that the differential is identically zero. Therefore it suffices to decide the Alexander and Maslov gradings of $\widehat{\mathrm{CFK}}(S^3, K)$ from the relator R. In order to do so, one may wish to find all bigons from R, and decide the number of basepoints contained in each bigon. However, there are difficulties to achieve that (see Example 3.5). Instead, we will work on a special class of bigons (Definition 3.1) that can be determined from R, and are sufficient to determine the gradings of $\widehat{\mathrm{CFK}}(S^3, K)$.

Definition 3.1. Let D be a Whitney disk in T^2 connecting x_1 and x_n. Let $\widetilde{x}_1$ and $\widetilde{x}_n$ be lifts of x_1 and x_n, respectively, in $\mathbb{C}$, and $\widetilde{D}$ be a lift of D connecting $\widetilde{x}_1$ and $\widetilde{x}_n$. $\partial\widetilde{D}$ consists of an α arc a and a β arc b. Suppose $\widetilde{\alpha}$ is the lift of α containing $\widetilde{x}_1$. The bigon $\widetilde{D}$ is called *primitive* if $\widetilde{\alpha} \cap b = \{\widetilde{x}_1, \widetilde{x}_n\}$, and a bigon D is called *primitive* if it has a primitive lift.

To describe our algorithm, we use the universal cover $\mathbb{C}$ of the torus T^2. We also use α, β, t_α, z and w to denote their lifts in $\mathbb{C}$ when there is no confusion. The key point is that bigons can lift to embedded bigons in the universal cover and it is possible to count the number of lifted basepoints inside primitive bigons from the relator R.

Primitive bigons are enough to determine the gradings of all generators. If D is a bigon connecting two points x_1 and x_n in $\mathbb{C}$, suppose $\alpha \cap b = \{x_1, x_2, \ldots, x_n\}$, labeled by the order of them on b. Then D is a combination of primitive bigons D_i

($i = 1, \dots, n-1$) that connecting two adjacent x_i and x_{i+1}, which can be used to compute the grading difference between x_1 and x_n.

Suppose D is a primitive bigon. Let $n_z(D)$ (resp. $n_w(D)$) be the number of basepoints z (resp. w) contained in D and write

$$P(D) = (n_z(D), n_w(D)). \tag{6}$$

We always consider the grading shift from point x_1 to x_n, so the numbers $n_z(D)$ and $n_w(D)$ may be negative. In fact, the sign of $n_z(D)$ and $n_w(D)$ depends on the orientation of D, as defined below:

Definition 3.2. Let D be a bigon in $\mathbb{C}$ whose boundary consists of an α arc a and a β arc b (b is oriented). The *orientation* of D is *positive* if D is on the right-hand side of the arc b, and *negative* if otherwise.

Definition 3.3. Let $\delta(W_1^n) = \#\{X \mid X \in W_1^n\} - \#\{\overline{X} \mid \overline{X} \in W_1^n\}$. A subword W_1^n of the relator $R(X, Y)$ is called a *disk word* if its two ends have opposite signs and $\delta(W_1^n) = 0$. A disk word W_1^n is called *primitive* if $\delta(W_1^k) \neq 0$ for all $1 < k < n$.

Definition 3.4. Let W_1^n be a disk word, let $\hat{X}_i$ denote X_i or $\overline{X}_i$. Define the *height* (relative to the endpoints) of $\hat{X}_i$ as

$$h(\hat{X}_i) = \delta(W_1^i) - \operatorname{sgn}(\hat{X}_i) \quad (1 \leq i \leq n),$$

where $\operatorname{sgn}(\hat{X}_i) = 1$ if $\hat{X}_i$ has the same sign as $\hat{X}_1$, and zero otherwise. If a subword W_i^j of W_1^n is primitive and $h(\hat{X}_i) = h(\hat{X}_j) = s$, we say that the primitive disk word W_i^j has *height* s in W_1^n. A primitive disk word W_1^n is *elementary* if $n = 2$, *upward* if $\hat{X}_1 = X_1$ and *downward* if $\hat{X}_1 = \overline{X}_1$.

For a primitive disk word W_1^n with $n > 2$, all $h(\hat{X}_i)$ $(1 < i < n)$ have the same sign. We have $h(\hat{X}_i) > 0$ $(1 < i < n)$ if W_1^n is upward, and $h(\hat{X}_i) < 0$ $(1 < i < n)$ if W_1^n is downward.

It is clear that the corresponding word of a bigon D in $\mathbb{C}$ is a disk word, since the intersection points of each lifting of α with the β part of ∂D occur in pairs of opposite signs. We call a disk word W_1^n a *real bigon* if it corresponds to a bigon in $\mathbb{C}$. An example of a disk word that does not correspond to a bigon is given in the following:

Example 3.5. Figure 2 shows a lifting of a Heegaard diagram compatible with the knot 5_2 in S^3. The curve β gives the relator

$$R(X, Y) = X_1 Y \overline{X}_2 Y X_3 \overline{Y}\,\overline{X}_4 \overline{Y} X_5 Y \overline{X}_6 Y X_7.$$

The two disk words $W_1^4 : X_1 Y \overline{X}_2 Y X_3 \overline{Y}\,\overline{X}_4$ and $W_3^6 : X_3 \overline{Y}\,\overline{X}_4 \overline{Y} X_5 Y \overline{X}_6$ are the same if we replace Y by $\overline{Y}$. W_1^4 corresponds to a real bigon, but W_3^6 does not.

Lemma 3.6. *A primitive disk word corresponds to a primitive bigon in $\mathbb{C}$ and vice versa.*

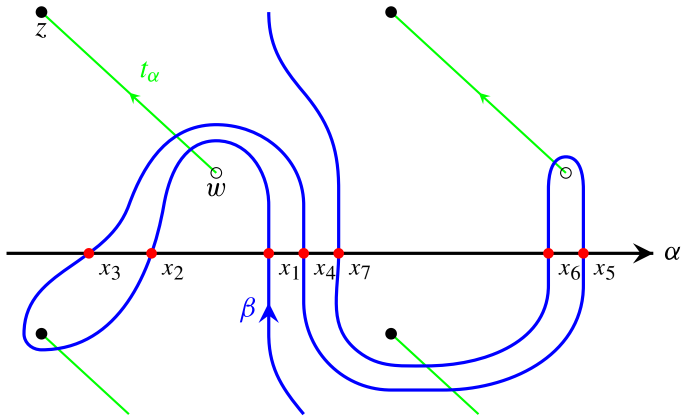

Figure 2. A Heegaard diagram of the knot 5_2 (lifting to $\mathbb{C}$).

Proof. Let W_1^n be a primitive disk word. Suppose it corresponds to the subarc b of β whose two endpoints x_1 and x_n are in the lifts α and α', respectively. Note that two sets $\{k \mid x_k \in \alpha \cap b,\ 1 < k\}$ and $\{k \mid x_k \in \alpha' \cap b,\ k < n\}$ cannot both be empty sets. Assuming without loss of generality that the former is not empty and that i is its minimum, then the subarc of b from x_1 to x_i and α bounds a primitive bigon, so that W_1^i is a disk word. Since W_1^n is assumed to be primitive, it follows that $i = n$ so that W_1^n actually corresponds to a primitive bigon in $\mathbb{C}$.

Suppose D is a primitive bigon. If its corresponding word is not primitive, then $\delta(W_1^k) = 0$ for some $1 < k < n$. Let $j = \min\{k \mid \delta(W_1^n) = 0,\ 1 < k < n\}$. Then the subword W_1^j is primitive by definition. As proved above, W_1^j corresponds to a primitive bigon, so that its endpoint x_j belongs to α, which cannot occur. □

Since the primitive disk word and the primitive bigon are in one-to-one correspondence, we will use these two terms alternatively by abuse of notation. All primitive bigons can be enumerated since the length of the relator R is finite. What we need to do is to find the number of basepoints contained in each of them directly from $R(X, Y)$.

3.1. *Elementary bigons.* In the beginning, we consider what the local diagram that corresponds to the simplest primitive disk word $XY\overline{X}$ should look like. As illustrated in Figure 3, it could have two choices.

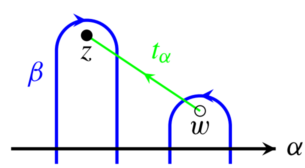

Figure 3. Two choices of an arc whose word is $XY\overline{X}$.

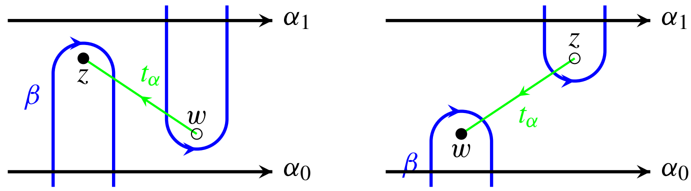

Figure 4. The first figure shows that $XY\overline{X}$ contains the basepoint z. The second figure is obtained by switching two basepoints z and w.

Note that the two cases cannot occur simultaneously in a diagram. If both are present, consider the word that follows the letter $\overline{X}$ to the next X-letter, it can only be $Y\overline{X}$ or $\overline{X}$ since the relator is reduced. Then continue the discussion with the next letter $\overline{X}$; it can be seen that none of the $\overline{X}$ is followed by X, which cannot happen. When we choose $XY\overline{X}$ to correspond to the one containing the basepoint z (resp. w), the word $\overline{X}YX$ must correspond to a bigon that contains the basepoint w (resp. z); see Figure 4.

Thus, after exchanging two basepoints z and w,[1] we can assume that the disk word $XY\overline{X}$ corresponds to the bigon containing the basepoint w so that the bigons corresponding to the four elementary disk words (or say *elementary bigons*) can be drawn as in Figure 5. The orientation of each elementary bigon, as well as the number of basepoints it contains, can be found, as shown in Table 1.

Remark 3.7. By Assumption 2.4, there are always two adjacent X-letters in R with opposite signs, and there exist bigons of the form $XY^k\overline{X}$ or $\overline{X}Y^kX$. We claim that k must be ± 1. If $|k| > 1$ on the contrary, then there exists a covering transformation Γ such that $\beta \cap \Gamma(\beta) \neq \varnothing$, as illustrated in Figure 6. The argument is similar to Lemma 3.9. It follows that there always exist elementary bigons containing basepoint z and w, respectively.

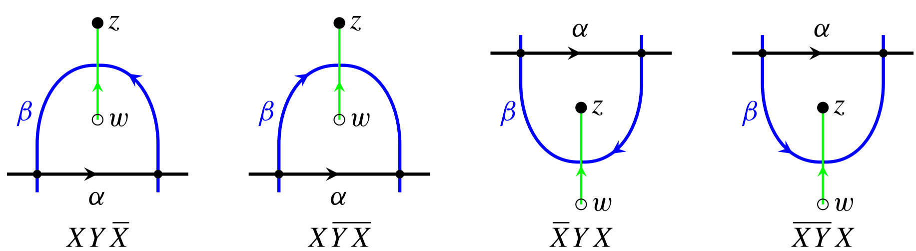

Figure 5. Four elementary disk words and their corresponding bigons.

[1]This procedure will reverse the orientation of the arc t_α. To obtain the same relator R, we can reverse the orientation of T^2 and α. The resulting diagram $(-T^2, -\alpha, \beta, w, z)$ is compatible with the mirror image K^* of the knot K. Although the knot Floer homology can distinguish chirality, the fundamental group cannot. Our algorithm yields the knot Floer homology of K or K^*.

elementary disk word	$XY\overline{X}$	$X\overline{Y}\overline{X}$	$\overline{X}YX$	$\overline{X}\overline{Y}X$
orientation (n_z, n_w)	negative $(0, -1)$	positive $(0, 1)$	positive $(1, 0)$	negative $(-1, 0)$

Table 1. Elementary bigons.

3.2. ***Bigons with height.*** Now we consider general primitive bigons. Firstly, the orientation of a primitive bigon can be found from its corresponding word.

Lemma 3.8. *If D is a real bigon, the number of positive and negative elementary words that it contains differs by one. Moreover, D is positive if and only if*

$$\#\{\textit{positive elementary bigons in } D\} - \#\{\textit{negative elementary bigons in } D\} = 1.$$

Proof. Suppose D is a real bigon whose boundary consists of an α arc a and a β arc b. Perturb β so that it is perpendicular to α at all intersection points. Then the total curvature of the arc b is π (resp. $-\pi$), counted counterclockwise, if the bigon is on the left (resp. right) when walking along b. That is, D is positive if and only if the total curvature is $-\pi$.

By assuming that β is perpendicular to α at all intersections, the arc b will have a nonzero contribution to the total curvature only when it passes near the basepoints. As can be seen from the picture of the elementary bigons (Figure 5), the arc b on a negative (resp. positive) elementary bigon will contribute π (resp. $-\pi$). Hence the total curvature is

$$\pi \cdot (\#\{\text{negative elementary bigons}\} - \#\{\text{positive elementary bigons}\}).$$

Thus, D is positive if and only if D contains one more positive elementary bigons than negative ones. □

In the rest of the section, let W_1^n be an upward positive disk word with $n > 2$. Let D be a corresponding bigon of W_1^n in $\mathbb{C}$. Suppose that $\partial D = a \cup b$ with $a \subseteq \widetilde{\alpha}$ and $b \subseteq \widetilde{\beta}$. It is clear that $h(x_2) = h(x_{n-1}) = 1$. Let b_1 be the subarc from x_1 to x_2, and b_2 be the subarc from x_{n-1} to x_n. The corresponding subwords for b_1 and b_2 are $W_1^2 = XY^lX$ and $W_{n-1}^n = \overline{X}\overline{Y}^{l'}\overline{X}$ for some integers l and l'. Write the two subwords as a pair (W_1^2, W_{n-1}^n) or $(XY^lX, \overline{X}\overline{Y}^{l'}\overline{X})$. Denote by S the square

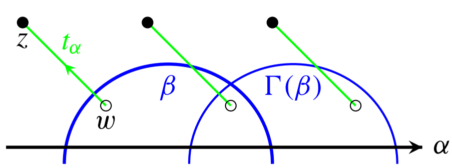

Figure 6. No subword of the form $XY^k\overline{X}$ with $|k| > 1$.

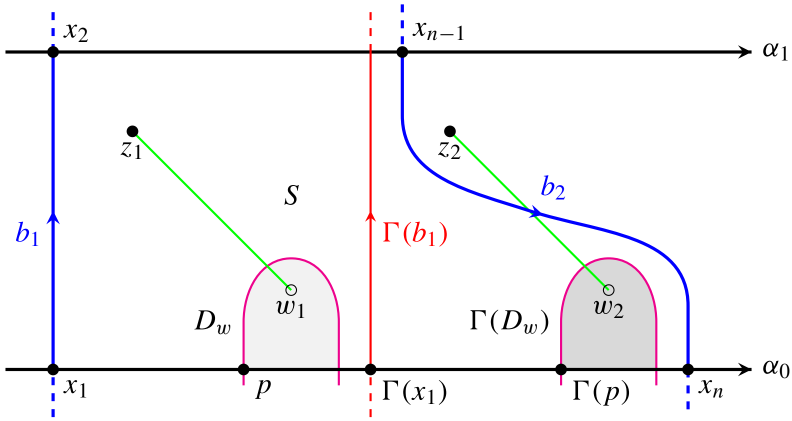

Figure 7. Domain S contains at least two lifts of the basepoint w.

domain bounded by the subarcs b_1, b_2 and two lifts of the α curve. The key to calculating $P(D)$ is to determine the number of basepoints contained in S.

Lemma 3.9. $\max\{|n_z(S)|, |n_w(S)|\} \le 1.$

Proof. Suppose the square domain S contains two lifts w_1, w_2 of the basepoint w. As in Figure 7, we assume that w_2 is on the right-hand side of w_1. Note that there exists an elementary bigon containing w_1, denoted by D_w. Then $D_w \subseteq S$. Let p be an endpoint of D_w. Consider the covering transformation Γ which maps w_1 to w_2. Then $w_2 \in \Gamma(D_w)$ and $\Gamma(D_w) \subset S$. The orientation of the lift curve $\widetilde{\alpha}$ gives an order

$$x_1 < p < \Gamma(p) < x_n.$$

Therefore $x_1 < \Gamma(x_1) < \Gamma(p) < x_n$. It implies that $\Gamma(b_1) \subset S$ and $\Gamma(D) \cap D \neq \varnothing$. It follows that $\widetilde{\beta} \cap \Gamma(\widetilde{\beta}) \neq \varnothing$ and we get a contradiction.

The case that S contains two lifts z_1, z_2 of z is similar: the image of x_2 under the covering transformation Γ' that maps z_1 to z_2 satisfies $x_2 < \Gamma'(x_2) < x_{n-1}$, so $\Gamma'(D)$ and D are overlapped. □

As a consequence, we have $|l - l'| \le 1$. Since D is upward and positive, we have $x_1 < x_n$, i.e., the subarc b_2 is on the right-hand side of b_1. When $|l - l'| = 1$, all possible pairs (W_1^2, W_{n-1}^n) are shown in Figure 8, and we have the following:

Corollary 3.10. *If the two subarcs b_1 and b_2 correspond to the words XY^lX and $\overline{X}\,\overline{Y}^{l'}\overline{X}$ with $|l - l'| = 1$, then the pair (W_1^2, W_{n-1}^n) has four cases and the basepoint contained in S is shown in Table 2.*

It is evident that $n_z(S) = n_w(S)$ if $l = l'$. We now consider the height-one primitive bigons contained in D. The simplest case is that there is exactly one such bigon, namely the one corresponding to W_2^{n-1}.

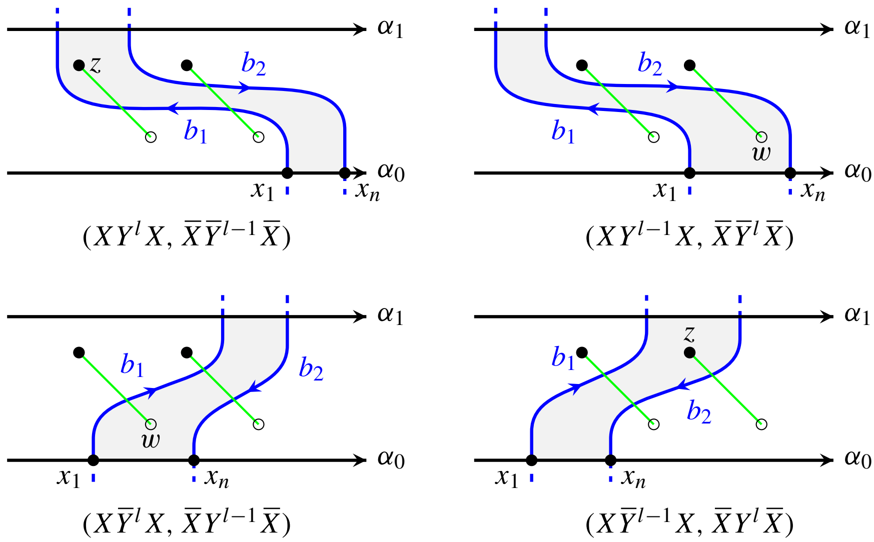

Figure 8. Four cases of the pair (W_1^2, W_{n-1}^n), and a basepoint contained in S.

Lemma 3.11. *Suppose the two subarcs b_1 and b_2 correspond to the words XY^lX and $\overline{X}\,\overline{Y}^l\overline{X}$. If W_2^{n-1} is a primitive disk word, then*

$$P(S) = (0, 0).$$

Proof. Without loss of generality, suppose that S contains a basepoint. Since $n_z(S) = n_w(S)$, S contains both z and w exactly once by Lemma 3.9. As illustrated in Figure 9, since the algebraic intersection number of α_0 and β is one, there will be other intersection points. Without loss of generality, suppose $x' \neq x_1$ is adjacent to x_n. Then the arcs bounded by x_n and x' on α_0 and β bound a primitive bigon D', which is on the other side of α_0 with D, i.e., D' is downward. Therefore, it must contain a subword of the form $\overline{X}Y^{\pm 1}X$, whose corresponding bigon contains the basepoint z. In fact, the first elementary bigon after the letter $\overline{X}_n$ is of this form, assumed to be $\overline{X}_N Y^{\pm 1} X_{N+1}$ $(N \geq n)$ and denoted by D''. By the assumption that S contains the basepoint z, there is a covering transformation Γ such that $\Gamma(D'')$ is contained in D. Furthermore, let s be the subarc of β from the point x_n to x_{N+1}. Then $\Gamma(s) \subseteq S$. In particular, the covering transformation Γ maps x_n to a point in

$(XY^lX,\ \overline{X}\,\overline{Y}^{l-1}\overline{X})$	$(XY^{l-1}X,\ \overline{X}\,\overline{Y}^{l}\overline{X})$	$(X\overline{Y}^lX,\ \overline{X}Y^{l-1}\overline{X})$	$(X\overline{Y}^{l-1}X,\ \overline{X}Y^{l}\overline{X})$
$(1, 0)$	$(0, 1)$	$(0, 1)$	$(1, 0)$

Table 2. Four cases of the pair (W_1^2, W_{n-1}^n) when $|l - l'| = 1$. The top row represents (W_1^2, W_{n-1}^n) and the bottom row represents $P(S)$.

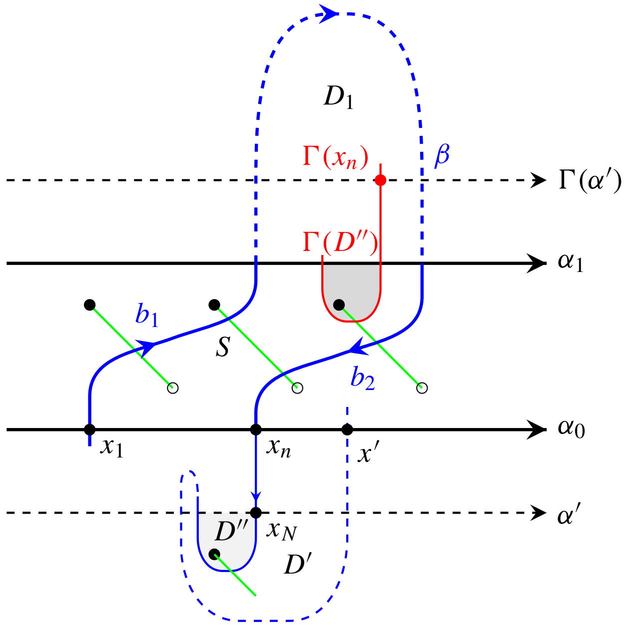

Figure 9. The case where the subword W_2^{n-1} is primitive. The red arc is a transformation of arc s.

the interior of D. Since $\beta \cap \Gamma(\beta) = \varnothing$, this implies that $\Gamma(D)$ is contained in D. We get a contradiction since Γ is nothing but a translation. □

Corollary 3.12. *$P(X^k Y \overline{X}^k) = P(XY\overline{X})$. This equation holds for the other three types as well. We call all of them elementary bigons.*

Corollary 3.13. *A lift of a real bigon does not contain a whole lift of t_α.*

Proof. Assume D is a lift of a real bigon that contains a whole lift $\tilde{t}_\alpha$ of t_α. Then $\tilde{t}_\alpha$ is contained in a primitive bigon whether or not D is primitive. However, it is obvious from the proof of Lemma 3.11 that this is impossible. □

Let $x_{k_1}, \dots, x_{k_{d+1}}$ ($k_1 = 2$, $k_{d+1} = n-1$) be all points of height one. Suppose D has more than one height-one primitive bigons ($d \geq 2$). Then the first and the last height-one points x_2 and x_{n-1} are connected by a series of primitive bigons D_i corresponding to the subword $W_{k_i}^{k_{i+1}}$ ($1 \leq i \leq d$). We call D_1 and D_d the first and the last height-one primitive bigons in W_1^n, respectively.

Lemma 3.14. *Suppose the two subarcs b_1 and b_2 correspond to the words XY^lX and $\overline{X}\overline{Y}^l\overline{X}$. Suppose D contains at least two primitive bigons of height one, say $D_1, \dots, D_d$ ($d \geq 2$). Then $P(S) = (1, 1)$ if and only if the first and last height-one primitive bigons D_1 and D_d have the same orientation as D; otherwise, $P(S) = (0, 0)$.*

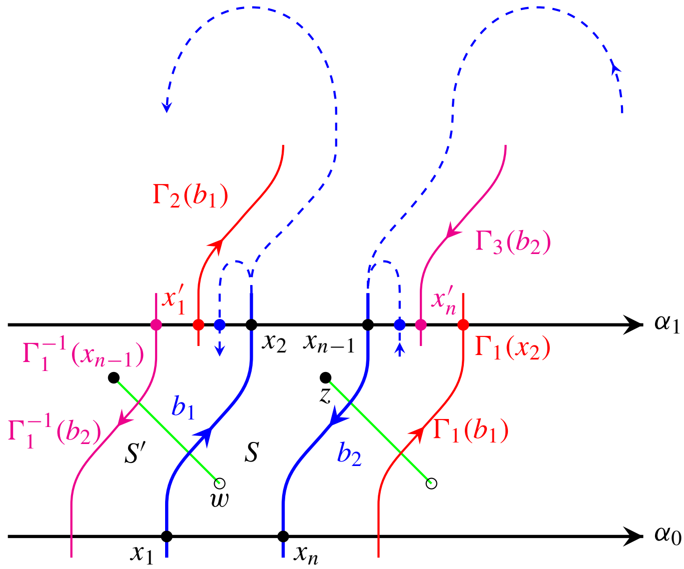

Figure 10. The case where the subword W_2^{n-1} is not primitive and S contains basepoints. The two red arcs are translations of the arc b_1, and magenta arcs are translations of b_2. The two dashed arcs on the left are two choices of D_1 with opposite orientation as D; the two dashed arcs on the right are two choices of D_d with opposite orientation as D, all of them cannot occur.

Proof. Note that $P(S) = (1, 1)$ or $(0, 0)$. Suppose $P(S) = (1, 1)$. As shown in Figure 10, consider the square domain S' bounded by the two lifts of the curve α, the arcs b_1 and $\Gamma_1^{-1}(b_2)$, where Γ_1 is a horizontal translation such that S' contains no basepoints. Let x_1' be a copy of x_1 at the height-one α-lifting and at the point on the left-closest to x_2. There is a unique transformation that maps x_1 to x_1', say Γ_2. One can find that x_1' is on the right-hand side of $\Gamma_1^{-1}(x_{n-1})$, i.e.,

$$\Gamma_1^{-1}(x_{n-1}) < x_1' < x_2,$$

because otherwise the bigons $\Gamma_2(D)$ and $\Gamma_1^{-1}(D)$ would overlap, which cannot happen.

For the first height-one primitive bigon D_1, consider the position of its other endpoint x_{k_2}. Since any two different lifts of the curve β are disjoint, if $x_{k_2} < x_1'$, then $\Gamma_2(D)$ is contained in D_1, which cannot occur; on the other hand, x_{k_2} cannot be a point between x_1' and x_2 since there is no basepoint contained in domain S'. Thus x_{k_2} must be on the right-hand side of x_2, i.e., D_1 has the same orientation as D. For the last height-one primitive bigon D_d, consider the copy of x_n at height-one α-lifting and right-closest to x_2, say x_n'. There is a transformation Γ_3 that maps

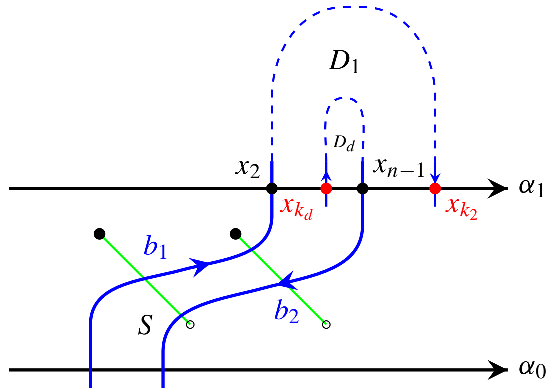

Figure 11. The case where D_1 and D_d have the same orientation, but S does not contain the basepoint. This figure shows that $x_{k_d} \in S$.

x_n to x'_n. By the same argument, one can find that $x_{k_{d-1}}$ is on the left-hand side of $x_{k_d} = x_{n-1}$, so that D_d has the same orientation as D.

Now suppose $P(S) = (0, 0)$, i.e., S does not contain any basepoints. Suppose D_1 and D_d have the same orientation as D. Then one of x_{k_2} and x_{k_d} lies in S. This is impossible. See Figure 11. □

Lemma 3.15.
$$P(D) = P(S) + \sum_{i=1}^{d} P(D_i).$$

Proof. Let a_1 be the subarc of the lift α_1 that connects the points x_2 and x_{n-1}. Suppose $a_1 \subset D$. As illustrated in Figure 12, cut D along a_1 into two domains: a square domain S and a real bigon (not necessarily primitive) connecting the points x_2 and x_{n-1}, say D', so that $D = S + D'$. Moreover, D' can be divided into a combination of primitive bigons $D_1, \ldots, D_d$. Thus

$$P(D) = P(S) + P(D') = P(S) + \sum_{i=1}^{d} P(D_i).$$

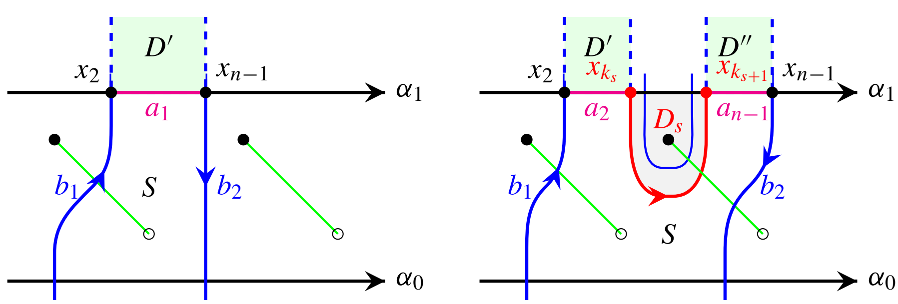

Figure 12. Cutting the primitive bigon D.

If D does not contain the entire subarc a_1, there must be a height-one primitive bigon of the form $\overline{X}Y^{\pm 1}X$ contained in S. Consider the outer-most one, and suppose it is the s-th height-one primitive bigon D_s, whose two endpoints we denote by x_{k_s} and $x_{k_{s+1}}$. Then D_s has the opposite orientation as S so that its corresponding word is $\overline{X}_{k_s}\overline{Y}X_{k_{s+1}}$. Let a_2 be the subarc of α_1 that is bounded by the points x_2 and x_{k_s}, and a_{n-1} be the subarc that is bounded by the points $x_{k_{s+1}}$ and x_{n-1}. Cutting the primitive bigon D along the line segments a_2 and a_{n-1}, we obtain three domains: two of them are real bigons (not necessarily primitive) whose corresponding words are $W_2^{k_s}$ and $W_{k_{s+1}}^{n-1}$, denoted by D' and D'', respectively, and the last one is a hexagon contained in S that does not contain the basepoint z, denoted by S_w — see Figure 12 — so that

$$D = D' + D'' + S_w, \quad S = S_w - D_s.$$

On the other hand, the two points x_2 and x_{k_s} are connected by a series of primitive bigons $D_1, \ldots, D_{s-1}$, and points $x_{k_{s+1}}$ and $x_{k_{d+1}}$ are connected by primitive bigons $D_{s+1}, \ldots, D_d$. Thus we obtain

$$\begin{aligned} P(D) = P(D') + P(D'') + P(S_w) &= \sum_{i=1}^{s-1} P(D_i) + \sum_{i=s+1}^{d} P(D_i) + P(S_w) \\ &= \sum_{i=1}^{d} P(D_i) - P(D_s) + P(S_w) = \sum_{i=1}^{d} P(D_i) + P(S). \end{aligned}$$ □

We have discussed, in detail, upward, positive primitive bigons. The result for the other three types of primitive bigons are similar. In summary, we have the following.

Theorem 3.16. *Let W_1^n be a primitive disk word and D be its corresponding bigon in $\mathbb{C}$. Then the number of basepoints in D can be computed from the word W_1^n.*

Proof. We prove by induction on the length of the word W_1^n. The conclusion holds for the four elementary disk words from Figure 5.

For the case of length n, each height-one primitive bigon D_i has length less than n, so that all of $P(D_i)$ can be read out from the subword of W_1^n by induction. On the other hand, Lemmas 3.10 and 3.14 imply that $P(S)$ can be computed from the word W_1^n. Theorem 3.16 then follows by Lemma 3.15. □

4. The fundamental group and $\widehat{\mathrm{HFK}}$

Now we describe the algorithm for computing $\widehat{\mathrm{HFK}}(S^3, K)$ from a presentation of $\pi_1(S^3 \setminus K)$ from a (1, 1) Heegaard diagram.

Algorithm 4.1. Let K be a (1, 1) knot and $\langle X, Y \mid R(X, Y)\rangle$ be a presentation of $\pi_1(S^3 \setminus K)$ from a (1, 1) Heegaard diagram. The chain complex $\widehat{\mathrm{CFK}}(S^3, K)$

has generators corresponding to the letters X and $\overline{X}$ in the relator R and trivial differential. The Alexander and Maslov gradings are determined as follows:

(I) Enumerate all the primitive disk words from the relator R and determine their orientation by Table 1 and Lemma 3.8.

(II) For each primitive disk word W_1^n, $P(W_1^n) = (n_z(W_1^n), n_w(W_1^n))$ is computed iteratively as follows:

(a) n_z and n_w for the elementary disk words are given in Table 1.

(b) Suppose W_1^n is upward and positive with $n > 2$. Let $x_{k_1}, \dots, x_{k_{d+1}}$ ($k_1 = 2$, $k_{d+1} = n - 1$) be all points of height one. Cut W_1^n into primitive disk words $W_{k_i}^{k_{i+1}}$, $i = 1, \dots, d$, and two subwords $W_1^2 = X_1 Y^l X_2$ and $W_{n-1}^n = \overline{X}_{n-1} \overline{Y}^{l'} \overline{X}_n$ ($l, l' \in \mathbb{Z}$, and $|l - l'| \le 1$ by Lemma 3.9). Then by Lemma 3.15

$$P(W_1^n) = P(S) + \sum_{i=1}^{d} P(W_{k_i}^{k_{i+1}}),$$

where $P(S)$ is determined by the following:

(i) If $|l - l'| = 1$, $P(S)$ are shown in Table 2 (Corollary 3.10).

(ii) If $l = l'$, then by Lemmas 3.11 and 3.14

$$P(S) = \begin{cases} (1, 1) & \text{if } d \ge 2 \text{ and } W_2^{k_2} \text{ and } W_{k_d}^{n-1} \text{ are positive;} \\ (0, 0) & \text{otherwise.} \end{cases}$$

(c) Suppose W_1^n is upward and negative. An upward positive disk word $\widetilde{W}_1^n$ can be obtained from W_1^n by replacing each Y-letter by its inverse. Then

$$P(W_1^n) = -P(\widetilde{W}_1^n).$$

(d) Suppose W_1^n is downward and negative. An upward positive disk word $\widetilde{W}_1^n$ can be obtained from W_1^n by replacing each X-letter by its inverse. Then

$$(n_z(W_1^n), n_w(W_1^n)) = -(n_w(\widetilde{W}_1^n), n_z(\widetilde{W}_1^n)).$$

(e) Suppose W_1^n is downward and positive. An upward positive disk word $\widetilde{W}_1^n$ can be obtained from W_1^n by replacing each X- and Y-letter by its inverse. Then

$$(n_z(W_1^n), n_w(W_1^n)) = (n_w(\widetilde{W}_1^n), n_z(\widetilde{W}_1^n)).$$

(III) For any two points x_1 and x_n that are connected by a primitive disk word W_1^n, the relative Alexander is determined by

$$F(x_1) - F(x_n) = n_z(W_1^n) - n_w(W_1^n), \tag{7}$$

and the relative Maslov grading can be computed as

$$M(x_1) - M(x_n) = \begin{cases} 1 - 2n_w(W_1^n) & \text{if } W_1^n \text{ is positive;} \\ -1 - 2n_w(W_1^n) & \text{if } W_1^n \text{ is negative.} \end{cases} \tag{8}$$

(IV) The absolute Alexander grading can be obtained by requiring

$$\#\{x \mid F(x) = i\} \equiv \#\{x \mid F(x) = -i\} \pmod 2 \quad \text{for all } i \in \mathbb{Z}. \tag{9}$$

To obtain the absolute Maslov grading, we forget the basepoint z, and we find a unique intersection point which is the generator of $\widehat{\mathrm{HF}}(T^2, \alpha, \beta, w) \cong \mathbb{Z}$. This process can only be done on the relator $R(X, Y)$, as follows:

(1) Erase primitive bigons whose n_w equals 0. (For example, all elementary bigons $\overline{X}^k Y^{\pm 1} X^k$ are removed.) The resulting relator is denoted by $R'(X, Y)$.

(2) The basepoint w also gives a relative grading by

$$w(x_1) - w(x_n) = n_w(W_1^n).$$

For the relator $R'(X, Y)$, delete those new primitive bigons whose relative w-grading of two endpoints is zero. (Here the relative grading is given by R, not R'.) The resulting relator is denoted by $R''(X, Y)$.

(3) There will be exactly one X-letter in $R''(X, Y)$; assume it is the k-th one. Thus we can define $M(x_k) = 0$ to obtain an absolute M-grading of all points.

In Step (IV), all primitive bigons deleted in the above two procedures do not contain the basepoint w, so it can be realized by isotopies of the β curve in the complement of w, and the resulting diagram $(T^2, \alpha, \beta'', w)$ specifies S^3, where β'' is the curve obtained from β by isotopies. We need to show that there is exactly one X-letter in $R''(X, Y)$. Specifically, we explain that no primitive bigon remains after these two steps. The proof is the same as the proof of Lemma 3.11. Suppose R'' contains a primitive bigon D_1 with $n_w(D_1) \neq 0$, and x_1 and x_n are two endpoints of D_1. Assume it is upward without loss of generality. Since the algebraic intersection number $[\alpha] \cdot [\beta''] = [\alpha] \cdot [\beta] = +1$, there will be another intersection point x'. Suppose x' is adjacent to x_n, and they form a primitive bigon D_2. Since D_2 remains after the second step, the w-grading implies that $n_w(D_2) \neq 0$, i.e., D_2 also contains the basepoint w. On the other hand, D_2 is downward. Therefore, there exists a translation Γ such that $\Gamma(D_2)$ and D_1 are overlapped, that is, $\Gamma(\beta'') \cap \beta'' \neq \varnothing$, which cannot happen.

To summarize, all the steps described above are relevant only to R. So we have the following theorem:

Theorem 4.2. *Let K be a $(1, 1)$ knot in S^3, $\langle X, Y \mid R(X, Y)\rangle$ be a cyclically reduced presentation of $\pi_1(S^3 \setminus K)$ that is coming from a $(1, 1)$ Heegaard diagram. The generators of the knot Floer homology of K are in one-to-one correspondence to the X-letters in the relator $R(X, Y)$; the Alexander and Maslov gradings of each generator can be determined by the above algorithm which involves only the*

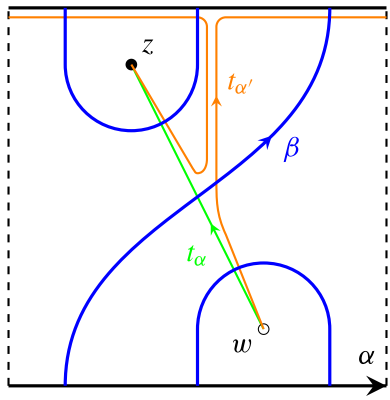

Figure 13. A Heegaard diagram compatible with the trefoil knot $T_{2,3}$. The relator R derived from t_α is $\overline{X}YX\overline{Y}XY$. The relator R' from $t_{\alpha'}$ is $\overline{X}\overline{Y}YYX\overline{Y}YXY$, which is the image of R under transformation l_1. It is further reduced to $\overline{X}YX^2Y$.

relator $R(X, Y)$. The knot Floer homology can be expressed as

$$\widehat{\mathrm{HFK}}_m(S^3, K; s) = \bigoplus_{\{x_i \mid F(x_i)=s, M(x_i)=m\}} \mathbb{Z} \cdot \langle x_i \rangle,$$

where x_i denotes the generator corresponding to the i-th X-letter in the relator $R(X, Y)$.

Rewrite the knot Floer homology as the Poincaré polynomial

$$P_R(t, q) = \sum_{s,m\in\mathbb{Z}} \operatorname{rank} \widehat{\mathrm{HFK}}_m(S^3, K; s) \cdot t^s q^m = \sum_i t^{F(x_i)} q^{M(x_i)}.$$

Proposition 4.3. *Let $\langle X, Y \mid R\rangle$ be a presentation coming from a $(1, 1)$ Heegaard diagram, and $P_R(t, m)$ be the Poincaré polynomial given by the algorithm.*

(1) *Let R' be the relator obtained from R by doing the transformation $l_k : X \mapsto Y^k X, Y \mapsto Y$ (resp. $r_k : X \mapsto XY^k, Y \mapsto Y$) for an integer k, and then reducing. Then $\langle X, Y \mid R'\rangle$ is also a presentation coming from a $(1, 1)$ Heegaard diagram, and $P_{R'}(t, q) = P_R(t, q)$.*

(2) *Let R'' be the relator obtained by transforming $\tau : X \mapsto X, Y \mapsto \overline{Y}$. Then $\langle X, Y \mid R'\rangle$ is also a presentation coming from a $(1, 1)$ Heegaard diagram, and $P_{R''}(t, q) = P_R(t^{-1}, q^{-1})$.*

Proof. The difference between two relators R and R' is the choice of the arc t_α on $T^2 - \alpha$. We consider only the case where $k = 1$ because $l_k = l_1^k$ (resp. $r_k = r_1^k$). Let $t_{\alpha'}$ be the arc obtained by handlesliding t_α across α (with orientation); see Figure 13 for an example on the trefoil knot. Then the intersection of t_α and β is one-to-one corresponding to the intersection of $t_{\alpha'}$ and β; and each intersection point of α and

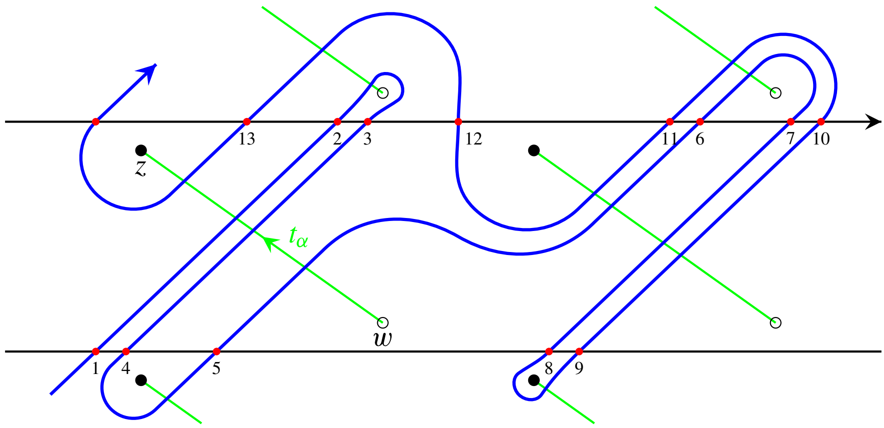

Figure 14. A Heegaard diagram of the knot 10_{161}. This diagram is taken from [10].

β corresponds to two intersection points of β with α and $t_{\alpha'}$, which are consecutive in β. Thus, R' is the image of R in l_1 or r_1. It is clear that R' also comes from the same Heegaard diagram as R, and t_α and $t_{\alpha'}$ are isotopic in the solid torus V_α. Therefore, R and R' represent the same knot.

Since we require that t_α is oriented from w to z, the second transformation can be achieved by switching the two basepoints z and w. Thus, the two relators represent the knot K and its mirror K^*, respectively. Therefore, $P_{R''}(t, q) = P_R(t^{-1}, q^{-1})$ is directly obtained. □

Example 4.4. A Heegaard diagram of the knot 10_{161} is illustrated in Figure 14 (as in [10]). The curve β gives the relator

$$R(X, Y) = X\overline{Y}X\overline{Y}\,\overline{X}Y\overline{X}\,\overline{Y}X\overline{Y}^2X\overline{Y}\,\overline{X}Y\overline{X}\,\overline{Y}X\overline{Y}XY\overline{X}YXY\overline{X}Y.$$

Label each X-letter in order. Consider the primitive bigon $D : X\overline{Y}^2X\overline{Y}\,\overline{X}Y\overline{X}$ from x_5 to x_8, for instance. It contains only one primitive bigon $D_1 : X_6\overline{Y}\,\overline{X}_7$ of height one, which is the second type in Table 1. The square domain S is the third type in Table 2. Thus

$$P(D) = P(S) + P(D_1) = (0, 1) + (0, 1) = (0, 2).$$

Removing primitive bigons with $n_w = 0$ yields

$$X_2\overline{Y}\,\overline{Y}YY\overline{Y}$$

so that we obtain $M(x_2) = 0$. The Alexander and Maslov gradings of the generators are listed in Table 3, and the Poincaré polynomial is

$$P_R(t, q) = t^{-3} + (1 + q)t^{-2} + 2qt^{-1} + 3q^2 + 2q^3t + (q^4 + q^5)t^2 + q^6t^3.$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}
F	-2	-3	-2	-1	0	0	1	2	3	2	1	0	-1
M	0	0	1	1	2	2	3	5	6	4	3	2	1

Table 3. The Alexander and Maslov gradings of the knot Floer homology of knot 10_{161}.

Example 4.5. Figure 15 shows a Heegaard diagram of the knot $D_+(T_{2,3}, 6)$ (the 6-twisted positive Whitehead double of the right-handed trefoil), as in [12]. The presentation is

$$\langle X, Y \mid X\bar{Y}\bar{X}^3 Y X^3 Y \bar{X}^3 \bar{Y} X^2 \bar{Y} \bar{X}^3 Y X^3 Y \bar{X}^3 \bar{Y} X^4 \rangle. \tag{10}$$

Label each X-letter in order. The corresponding word of the shaded domain D in Figure 15 is

$$\bar{X}^3 \bar{Y} X^2 \bar{Y} \bar{X}^3 Y X^3 Y \bar{X}^3 \bar{Y} X^4,$$

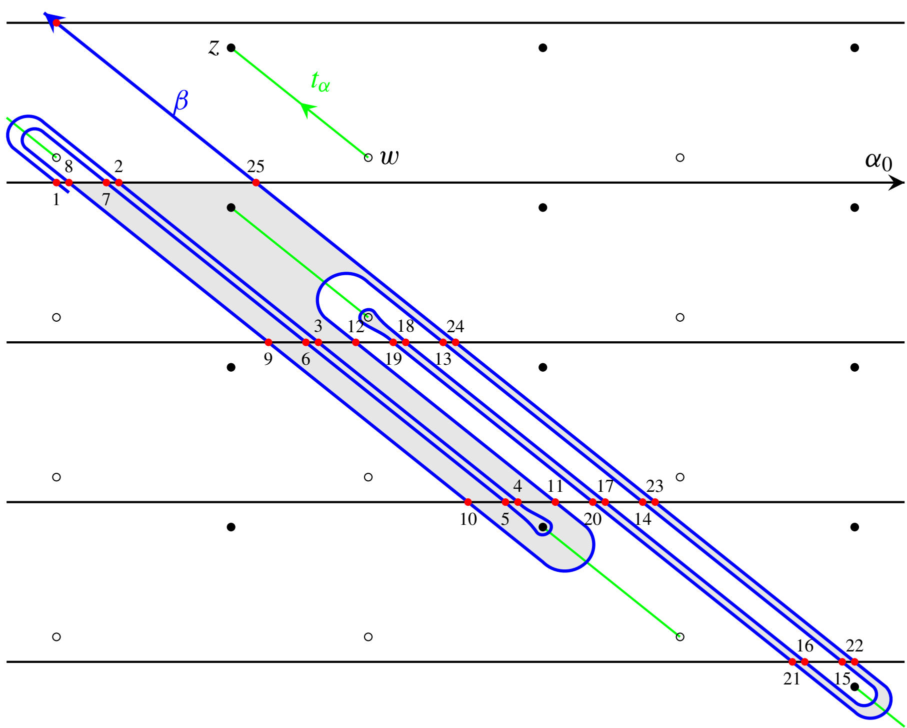

Figure 15. A Heegaard diagram of the knot $D_+(T_{2,3}, 6)$ (taken from [12]).

which is a primitive bigon from x_8 to x_{25}. Determine $P(D)$ directly from the word as follows: two height -1 points x_9 and x_{24} are connected by elementary bigons:

$$\begin{aligned} &D_1 : \overline{X}^2\overline{Y}X^2, && x_9 \to x_{12};\\ &D_2 : X\overline{Y}\overline{X}, && x_{12} \to x_{13};\\ &D_3 : \overline{X}^3YX^3, && x_{13} \to x_{18};\\ &D_4 : XY\overline{X}, && x_{18} \to x_{19};\\ &D_5 : \overline{X}^3\overline{Y}X^3, && x_{19} \to x_{24}. \end{aligned}$$

Note that D_1 and D_5 have the same orientation as D; thus

$$\begin{aligned} P(D) &= P(S) + \sum_{i=1}^{5} P(D_i)\\ &= (-1,-1) + (-1,0) + (0,1) + (1,0) + (0,-1) + (-1,0)\\ &= (-2,-1), \end{aligned}$$

which is consistent with the diagram shown in Figure 15.

The following steps find the unique X-letter that generates $\widehat{\mathrm{HF}}(S^3, w)$:

(1) Remove primitive bigons with $n_w = 0$. The resulting relator is

$$X_1\overline{Y}Y\overline{X}_8\overline{Y}YX_{25}.$$

(2) The points x_1 and x_8 are connected by three primitive bigons in the original relator R:

$$\begin{aligned} &E_1 : X\overline{Y}\overline{X}, && x_1 \to x_2;\\ &E_2 : \overline{X}^3YX^3, && x_2 \to x_7;\\ &E_3 : XY\overline{X}, && x_7 \to x_8. \end{aligned}$$

So, the w-grading shift is

$$w(x_1) - w(x_8) = w(x_1) - w(x_2) + w(x_2) - w(x_7) + w(x_7) - w(x_8) = 1 + 0 - 1 = 0.$$

On the other hand, $w(x_8) - w(x_{25}) = -1$, since the two points x_8 and x_{25} are connected by the bigon D in the above. Therefore, the subword $X_1\overline{Y}Y\overline{X}_8$ is removed in the second step, thus giving

$$\overline{Y}YX_{25}.$$

(3) We obtain $M(x_{25}) = 0$.

The Poincaré polynomial is

$$P_R(t,q) = (4q^{-1} + 2q)t^{-1} + (4q^2 + 9) + (2q^3 + 4q)t. \tag{11}$$

5. (1, 1) knots in lens spaces

We extend the algorithm for (1, 1) knots in lens spaces, which are automatically rationally null-homologous. See [32] for details.

Let K be a (1, 1) knot in the lens space $L(p, q)$ ($p > 0$) and $\langle X, Y \mid R(X, Y)\rangle$ be a presentation of $\pi_1(L(p, q) \setminus K)$ obtained from a (1, 1) Heegaard diagram $(T^2, \alpha, \beta, z, w)$ of $(L(p, q), K)$, satisfying:

(1) The curves α and β are oriented so that

$$\#\{X \mid X \in R(X, Y)\} - \#\{\overline{X} \mid \overline{X} \in R(X, Y)\} = p.$$

(2) The relator $R(X, Y)$ is cyclically reduced.

For a knot K in $L(p, q)$, the knot Floer homology $\widehat{\mathrm{HFK}}(L(p, q), K)$ can be decomposed as a direct sum:

$$\widehat{\mathrm{HFK}}(L(p, q), K) = \bigoplus_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,q))} \widehat{\mathrm{HFK}}(L(p.q), K; \mathfrak{s})$$

with respect to Spin^c structures on $L(p, q)$. The Alexander grading and the Maslov grading are defined for each direct summand $\widehat{\mathrm{HFK}}(L(p.q), K; \mathfrak{s})$. Note that there is an affine isomorphism between the space of Spin^c structures over a three-manifold and its second cohomology. In particular,

$$\mathrm{Spin}^c(L(p, q)) \cong H^2(L(p, q), \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

Let $\mathrm{Spin}^c(L(p, q)) = \{\mathfrak{s}_0, \ldots, \mathfrak{s}_{p-1}\}$. The set of relative Spin^c structures for $(L(p, q), K)$ is denoted by

$$\underline{\mathrm{Spin}}^c(L(p, q), K) \cong \mathrm{Spin}^c(L(p, q)) \times \mathbb{Z} = \{(\mathfrak{s}_i, s) \mid i = 0, \ldots, p-1,\ s \in \mathbb{Z}\}.$$

For each $\mathfrak{s}_i \in \mathrm{Spin}^c(L(p, q))$,

$$\begin{aligned}\widehat{\mathrm{HFK}}(L(p, q), K; \mathfrak{s}_i) &= \bigoplus_{\{\underline{\mathfrak{t}}\in \underline{\mathrm{Spin}}^c(L(p,q),K) \mid \underline{\mathfrak{t}} \text{ extends } \mathfrak{s}_i\}} \widehat{\mathrm{HFK}}(L(p, q), K; \underline{\mathfrak{t}}) \\ &= \bigoplus_{s\in\mathbb{Z}} \widehat{\mathrm{HFK}}(L(p, q), K; (\mathfrak{s}_i, s)).\end{aligned}$$

Note that $\widehat{\mathrm{HF}}(L(p, q); \mathfrak{s}_i) \cong \mathbb{Z}$. More generally, a rational homology sphere Y is called an *L-space* if $\widehat{\mathrm{HF}}(Y; \mathfrak{s}) \cong \mathbb{Z}$ for all $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Ignoring the basepoint z, the Heegaard diagram (T^2, α, β, w) specifies the lens space $L(p, q)$, so that there exist exactly p intersection points generating a $\widehat{\mathrm{HF}}(L(p, q); \mathfrak{s}_i)$, one for each $\mathfrak{s}_i$. Denote by $x_{\mathfrak{s}_i}$ the intersection point generating $\widehat{\mathrm{HF}}(L(p, q); \mathfrak{s}_i)$ and by $X_{\mathfrak{s}_i}$ the corresponding X-letter in the relator R. With w fixed, each intersection point x

of α and β specifies a Spin^c structure $\mathfrak{s}_w(x)$ of $L(p,q)$ [31]. Let $\mathfrak{S}$ be the set of intersection points of α and β (i.e., the set of X-letters in R), and

$$\mathfrak{S}_i := \{x \in \mathfrak{S} \mid \mathfrak{s}_w(x) = \mathfrak{s}_i\}, \quad i = 0, \dots, p-1.$$

For each $\mathfrak{s}_i$, $\widehat{\mathrm{CF}}(L(p,q); \mathfrak{s}_i)$ and $\widehat{\mathrm{CFK}}(L(p,q), K; \mathfrak{s}_i)$ are both generated by $\mathfrak{S}_i$.

The key of Algorithm 4.1 is to determine the number of basepoints in each primitive bigon, and the first three steps work for (1,1) knots in lens spaces as well. In Step (IV), we got the unique generator of $\widehat{\mathrm{HF}}(S^3)$ by reducing the relator for (1,1) knots in S^3. For (1, 1) knots in $L(p,q)$, the same process gives p generators $X_{\mathfrak{s}_0}, \dots, X_{\mathfrak{s}_{p-1}}$ in different Spin^c structures. Therefore, we can apply Step (IV) to each Spin^c structure separately. We only need to determine the Spin^c structure for each intersection point, that is, determine $\mathfrak{S}_i$ from the relator $R(X,Y)$.

Lemma 5.1. *If two X-letters are connected by a sequence of primitive bigons, then they correspond to the same* Spin^c *structure of $L(p,q)$.*

Proof. Suppose there is a sequence $X_{d_1}, \dots, X_{d_n}$ of X-letters such that any two adjacent X_{d_i} and $X_{d_{i+1}}$ are connected by a primitive bigon D_i, $i = 1, \dots, n-1$. The difference $\varepsilon(x_{d_i}, x_{d_{i+1}})$ equals 0 (see [31, Section 2.4]) since D_i is a primitive bigon. By [31, Lemma 2.19],

$$\mathfrak{s}_w(x_{d_{i+1}}) - \mathfrak{s}_w(x_{d_i}) = \mathrm{PD}[\varepsilon(x_{d_i}, x_{d_{i+1}})] = 0.$$

Therefore, all X_{d_i}, $i = 1, \dots, n$, correspond to the same Spin^c structure. □

Lemma 5.2. *Each X-letter in $R(X,Y)$ is connected to some $X_{\mathfrak{s}_i}$, $i = 0, \dots, p-1$, via a sequence of primitive bigons.*

Proof. Without loss of generality, we write the relator R so that the X-letter under consideration is the first letter X_1 in R. For each $X_{\mathfrak{s}_i} \in R$, denote its height (relative to X_1) by h_i. We claim that $h_i \not\equiv h_j \pmod p$ if $i \neq j$.

Suppose there exist i and j such that $h_i \equiv h_j \pmod p$, that is, $h_i - h_j = pu$ for some integer u (assume without loss of generality that $u \geq 0$). Pick the word R^{u+1}, the $u+1$ copies of R, which corresponds to a subarc of a lift of the β curve. Let $X_k^{(l)}$ denote the k-th X-letter in the l-th copy of R. By assumption, the height of $X_{\mathfrak{s}_j}^{(u+1)}$ is

$$h(X_{\mathfrak{s}_j}^{(u+1)}) = pu + h(X_{\mathfrak{s}_j}) = h_i = h(X_{\mathfrak{s}_i}).$$

Thus $X_{\mathfrak{s}_i}$ and $X_{\mathfrak{s}_j}^{(u+1)}$ lie in the same lift of the α curve in the universal cover $\mathbb{C}$. Denote by $X_{d_1}, X_{d_2}, \dots, X_{d_n}$ all the X-letters between $X_{\mathfrak{s}_i}$ and $X_{\mathfrak{s}_j}^{(u+1)}$ in R^{u+1} that have height h_i. Then they give a sequence of primitive bigons from $X_{\mathfrak{s}_i}$ to $X_{\mathfrak{s}_j}^{(u+1)}$. Thus $s_w(X_{\mathfrak{s}_i}) = s_w(X_{\mathfrak{s}_j})$ by Lemma 5.1, so $i = j$.

In particular, there is an $h_i \equiv 0 \pmod p$. Suppose $h_i = pv$ for some integer v. Note that $h(X_1) = 0$. The same argument as above gives a sequence of primitive bigons connecting X_1 and $X_{\mathfrak{s}_i}^{(v+1)}$. □

It follows that we can divide the set $\mathfrak{S}$ of generators into p disjoint subsets directly from the relator R:

Corollary 5.3. *$X_k \in \mathfrak{S}_i$ if and only if X_k and $X_{\mathfrak{s}_i}$ are connected by a sequence of primitive bigons, that is,*

$$\mathfrak{S}_i = \{X_k \in R(X, Y) \mid X_k \text{ and } X_{\mathfrak{s}_i} \text{ are connected by primitive bigons}\}. \tag{12}$$

Now we modify Algorithm 4.1 for (1, 1) knots in lens spaces.

Algorithm 5.4. Let K be a (1, 1) knot in the lens space $L(p, q)$ and $\langle X, Y \mid R(X, Y)\rangle$ be a presentation of $\pi_1(L(p, q) \setminus K)$ from a (1, 1) Heegaard diagram. Let $\mathfrak{S}$ be the set of letters X and $\overline{X}$ in the relator R. The chain complex $\widehat{\mathrm{CFK}}(L(p, q), K)$ has generators corresponding to elements in $\mathfrak{S}$ and vanishing differentials. The first two steps (Steps (I) and (II)) of determining Alexander and Maslov gradings are the same as Algorithm 4.1.

- (III') We apply the process of reducing the relator (Step (IV)) first. The resulting relator $R''(X, Y)$ contains exactly p X-letters $X_{\mathfrak{s}_0}, \dots, X_{\mathfrak{s}_{p-1}}$. We divide $\mathfrak{S}$ into p disjoint subsets $\mathfrak{S}_0, \dots, \mathfrak{S}_{p-1}$ by (12). $\widehat{\mathrm{HFK}}(L(p, q), K; \mathfrak{s}_i)$ is then generated by $\mathfrak{S}_i$.
- (IV') Apply Steps (III) and (IV) of Algorithm 4.1 for each $\mathfrak{S}_i$ to obtain the Alexander grading and the Maslov grading of the homology $\widehat{\mathrm{HFK}}(L(p, q), K; \mathfrak{s}_i)$, respectively.

6. On general presentation of the fundamental group

In previous sections, we always assume that the group presentation comes from a (1, 1) Heegaard diagram. This ensures that the curve β and its image in the covering transformation do not intersect, a fact that we use several times when determining the number of basepoints contained in each primitive bigon. However, Algorithm 4.1 can compute the knot Floer homology of a (1,1) knot K from a presentation of $\pi_1(S^3 \setminus K)$ which does not come from a (1,1) Heegaard diagram (Example 6.5). Indeed, Algorithm 4.1 itself does not a priori require the group to be the fundamental group of a knot complement.

Definition 6.1. Let $P = \langle X, Y \mid R(X, Y)\rangle$ be a presentation of a group G with $R(X, Y)$ cyclically reduced. P is *quasi-geometric* if

(1) $\#\{X \mid X \in R(X, Y)\} - \#\{\overline{X} \mid \overline{X} \in R(X, Y)\} = +1$;

(2) all subwords of the form $XY^k\overline{X}$ (or $\overline{X}Y^kX$) must have $|k| = 1$;

(3) there are no two subwords of the form XY^lX (or $\overline{X}\overline{Y}^l\overline{X}$) and $XY^{l'}X$ (or $\overline{X}\overline{Y}^{l'}\overline{X}$) that satisfy $|l - l'| > 1$, where $l, l' \in \mathbb{Z}$.

For a quasi-geometric presentation, Algorithm 4.1 (Steps (I)–(III)) can be applied: the primitive disk words can be enumerated; their orientations can be determined by counting the positive and negative elementary words that it contains; and the functions n_z and n_w on primitive disk words can be formally computed. Hence we get a relatively bigraded chain complex with trivial differential.

For a quasi-geometric presentation of the fundamental group of a (1, 1) knot K, the homology of the above chain complex may not be isomorphic to the knot Floer homology $\widehat{\mathrm{HFK}}(S^3, K)$. For example, the presentation

$$\langle X, Y \mid YX^3Y\overline{X}\overline{Y}\overline{X}\rangle \tag{13}$$

of the fundamental group of the trefoil knot $T_{2,3}$ is quasi-geometric. However, the homology of (13) from Algorithm 4.1 (Steps (I)–(III)) has rank five, while the rank of its knot Floer homology is three. However, we will show that its Euler characteristic agrees with the (unnormalized) Alexander polynomial.

The Alexander polynomial of a knot K can be computed from a presentation of $\pi_1(S^3 \setminus K)$ as follows [38]:

(1) For a knot K in S^3, the knot group $G \triangleq \pi_1(S^3 \setminus K)$ has a presentation of the form

$$\langle Y, g_1, \dots, g_p \mid r_1, \dots, r_p\rangle,$$

where $Y \mapsto 1$ and $g_i \mapsto 0$ under the abelianization $G \to G/[G:G] \cong \mathbb{Z}$.

(2) The commutator subgroup $C = [G:G]$ of G is generated by all words of the form

$$Y^{-k} g_i^{\pm 1} Y^k.$$

Each relator r_i is conjugated to a word r_i', which is a product of words of this form.

(3) Let $\widetilde{X}$ be the infinite cyclic cover of the knot complement $X = S^3 \setminus K$. Then the covering map $p : \widetilde{X} \to X$ gives a $\Lambda \triangleq \mathbb{Z}[t, t^{-1}]$-module isomorphism

$$p_* : H_1(\widetilde{X}; \mathbb{Z}) \to C/[C:C].$$

So we can obtain a Λ-module presentation of $C/[C:C]$ by taking the image of g_i under abelianization as the generator, say α_i, and replacing $Y^{-k} g_i^{\pm 1} Y^k$ by $\pm t^k \alpha_i$ in the relator r_i' (multiplication becomes addition). Therefore, the Alexander polynomial is given by

$$H_1(\widetilde{X}; \mathbb{Z}) \cong \Lambda/(\Delta_K(t)).$$

When $p = 1$, $H_1(\widetilde{X})$ is a cyclic Λ-module generated by a generator α. Rewrite the generator g of G by a and label each g-letter in the relator r' by a_i (signed) in sequence. An alternative description of the last step is as follows: Since the relator r' is a product of words form like $Y^{-k} g^{\pm 1} Y^k$, we define two gradings for each a_i as

$$A(a_i) = k, \quad (-1)^{S(a_i)} = \pm 1 \qquad \text{for } Y^{-k} a_i^{\pm 1} Y^k.$$

Then the (unnormalized) Alexander polynomial is

$$\sum_i (-1)^{S(a_i)} t^{A(a_i)}.$$

Proposition 6.2. *Let $\langle X, Y \mid R(X, Y)\rangle$ be a quasi-geometric presentation of a* (1, 1) *knot K in S^3. Algorithm 4.1* (*Steps* (I)–(III)) *gives a chain complex with two relative gradings F and M, whose Euler characteristic satisfies*

$$\sum_i (-1)^{M(x_i)} t^{F(x_i)} \overset{\circ}{=} \Delta_K(t), \tag{14}$$

where $f(t) \overset{\circ}{=} g(t)$ if $f(t) = \pm t^c g(t)$ for some integer c.

Proof. By assumption, the relator $R(X, Y)$ maps to $X + kY$ for some integer k under abelianization. Let $a = Y^k X$ and $R'(a, Y)$ be the reduced relator of $R(Y^{-k}a, Y)$. Consider the presentation

$$\langle Y, a \mid R'(a, Y)\rangle.$$

It is easy to see that $Y \mapsto 1$ and $a \mapsto 0$ under abelianization, so that we can do the above procedure. Label each a-letter and the origin X-letter in sequence, and require that a_i corresponds to the i-th X-letter. We claim that

$$F(x_i) - F(x_j) = A(a_i) - A(a_j), \quad M(x_i) - M(x_j) \equiv S(a_i) - S(a_j) \pmod 2. \tag{15}$$

In fact, it is sufficient to show that the two equations hold in the case where x_i, x_j are two endpoints of a primitive disk word.

Let W_1^n be a primitive disk word. By definition, $S(a_1) - S(a_n) \equiv 1 \pmod 2$. On the other hand,

$$M(x_1) - M(x_n) \equiv 1 - 2n_w(W_1^n) \equiv 1 \pmod 2.$$

So the second equation is automatically satisfied.

For the first equation in (15), we prove by induction on the length of the primitive disk word W_1^n. For the four elementary disk words, see $X_1 Y \overline{X}_2$ for example, we have $F(x_1) - F(x_2) = 0 - (-1) = 1$; on the other hand, the corresponding subword in $R'(a, Y)$ is $Y^{-k} a_1 Y a_2^{-1} Y^k$; thus $A(a_1) - A(a_2) = 1$ by definition. In general, we see an upward positive disk word $W_1^n = X_1 Y^l X_2 \cdots \overline{X}_{n-1} \overline{Y}^{l'} \overline{X}_n$, for example, where $|l - l'| \le 1$. By Step (II) and (7), we have

$$\begin{aligned} F(x_1) - F(x_n) &= n_z(S) + \sum_{i=1}^{d} n_z(W_{k_i}^{k_{i+1}}) - n_w(S) - \sum_{i=1}^{d} n_w(W_{k_i}^{k_{i+1}}) \\ &= n_z(S) - n_w(S) + \sum_{i=1}^{d} F(x_{k_i}) - F(x_{k_{i+1}}) \\ &= n_z(S) - n_w(S) + F(x_2) - F(x_{n-1}). \end{aligned}$$

On the other hand, the image of W_1^n in the relator $R'(a, Y)$ is

$$Y^{-k}a_1Y^{l-k}a_2\cdots a_{n-1}^{-1}Y^{k-l'}a_n^{-1}Y^k.$$

Thus

$$\begin{aligned}A(a_1)-A(a_n)&=A(a_1)-A(a_2)+A(a_{n-1})-A(a_n)+A(a_2)-A(a_{n-1})\\&=l-l'+A(a_2)-A(a_{n-1}).\end{aligned}$$

By the inductive hypothesis, we only need to show that

$$n_z(S)-n_w(S)=l-l'.$$

This follows directly from the definition of $P(S)$ in Step (b).

Therefore, from (15), we obtain

$$\sum_i(-1)^{M(x_i)}t^{F(x_i)}=\sum_i(-1)^{S(a_i)}t^{A(a_i)+c},$$

for some integer c. □

The presentation (13) does not come from a (1, 1) Heegaard diagram of the trefoil knot. Algorithm 4.1 fails on Step (IV) for (13): label the X-letters in order, and the resulting relator after removing primitive disk words with $n_w = 0$ is $X_2X_3Y\overline{X}_4\overline{Y}$, and then we cannot remove any letter since

$$w(x_4)-w(x_3)=w(x_4)-w(x_2)=1.$$

Definition 6.3. Let $P = \langle X, Y \mid R(X, Y)\rangle$ be a quasi-geometric presentation of a group G. P is *pseudo-geometric* if

(1) the relative Alexander grading can be normalized to satisfy (9);

(2) the relator obtained by the process of deleting subwords in Algorithm 4.1 Step (IV) has exactly one X-letter.

We remark that (9) automatically holds for knot groups by Proposition 6.2, but not for general groups. For example, the presentation $\langle X, Y \mid YX^2\overline{Y}XY\overline{X}^2\rangle$ is quasi-geometric, but (9) does not hold.

For a pseudo-geometric group presentation $G = \langle X, Y \mid R(X, Y)\rangle$, Algorithm 4.1 applies to give a chain complex with trivial differential and two absolute gradings F and M. Denote its homology by $H(R)$, and its Poincaré polynomial is

$$P_R(t,q)=\sum_i t^{F(x_i)}q^{M(x_i)}. \tag{16}$$

Corollary 6.4. *Let $\langle X, Y \mid R(X, Y)\rangle$ be a pseudo-geometric presentation of a* (1, 1) *knot K in S^3. Then its Poincaré polynomial* (16) *satisfies*

$$P_R(t,-1)=\Delta_K(t).$$

However, a pseudo-geometric presentation of $\pi_1(S^3 \setminus K)$ for a $(1, 1)$ knot K does not always come from a $(1, 1)$ Heegaard diagram. Algorithm 4.1 can compute the knot Floer homology of K from a pseudo-geometric presentation correctly in some cases (Example 6.5) and incorrectly in other cases (Example 6.6).

Example 6.5. The fundamental group of the torus knot $T_{p,q}$ has the presentation (by [23])

$$\langle W, Z \mid (W^p Z^t)^s Z\rangle,$$

where $q = ts + 1$. Consider the case that $p = 2$, $q = 7$, $s = 2$, $t = 3$. By setting $Z \mapsto \overline{X}$, $W \mapsto X^2 Y$, we get, for $\pi_1(S^3 \setminus T_{2,7})$, the presentation

$$\langle X, Y \mid Y X^2 Y \overline{X} Y X^2 Y \overline{X}^2\rangle. \tag{17}$$

Note that all primitive disk words are elementary, so this presentation is pseudo-geometric. Algorithm 4.1 applied to (17) is computed as $\widehat{\mathrm{HFK}}(S^3, T_{2,7})$. However, the presentation (17) does not come from a $(1, 1)$ Heegaard diagram.

Example 6.6. This is a pseudo-geometric presentation of $\pi_1(S^3 \setminus D_+(T_{2,3}, 6))$:

$$\langle X, Y \mid X^4 \overline{Y}\,\overline{X}^3 Y X^4 Y \overline{X}^3 \overline{Y} X \overline{Y}\,\overline{X}^3 Y X^4 Y \overline{X}^3 \overline{Y}\rangle, \tag{18}$$

which is obtained from the presentation (10) by performing the transformation $Y \mapsto XY$. Label each X-letter in order. To see that it is pseudo-geometric, consider the disk word D_1 from the fifth to the tenth X-letter, and the disk word D_2 from the twelfth to the fourth X-letter, both primitive. Applying the algorithm gives

$$P(D_1) = (1, 0), \quad P(D_2) = (-1, 0),$$

so that both are deleted in Step (IV) and the resulting relator is $\overline{Y} X_{11} Y$, which has a unique X-letter.

Algorithm 4.1 yields $F = M$ for all generators, and the resulting Poincaré polynomial is $P(t, q) = 6q^{-1}t^{-1} + 13 + 6qt$, which is different from (11). It follows that (18) does not come from a $(1, 1)$ Heegaard diagram of $D_+(T_{2,3}, 6)$.

Algorithm 4.1 does not a priori require the group to be the fundamental group of a $(1, 1)$ knot. We can apply Algorithm 4.1 to any pseudo-geometric two-generator one-relator group presentation. But we do not know whether it yields any invariants of the group, and what properties it captures.

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