*Pacific Journal of Mathematics*

# QUIVER BRASCAMP–LIEB INEQUALITIES

NICHOLAS HU

Volume 333 No. 1 November 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 1, 2024
https://doi.org/10.2140/pjm.2024.333.115

# QUIVER BRASCAMP–LIEB INEQUALITIES

NICHOLAS HU

**We formulate generalized Brascamp–Lieb inequalities for representations of bipartite quivers and establish necessary and sufficient conditions for such inequalities. Notably, we show contra Lieb that Gaussians do not saturate certain types of quiver Brascamp–Lieb inequalities.**

## 1. Introduction

The Brascamp–Lieb inequalities [7] are an important family of inequalities in analysis that subsumes several inequalities significant in their own right, including Hölder's inequality, the Loomis–Whitney inequality, and Young's inequality. Several variants and extensions of these inequalities have been developed, such as reverse Brascamp–Lieb inequalities [1], perturbed Brascamp–Lieb inequalities [6], and adjoint Brascamp–Lieb inequalities [3], some of which have proved to be very useful in Fourier restriction theory [12].

In addition, algorithms have been devised to compute optimal constants in Brascamp–Lieb inequalities [9; 11]. The algorithm of Garg et al. [9] does so by relating these constants to the so-called *capacity* of completely positive operators, and this notion of capacity was generalized by Chindris and Derksen [8] to algebraic objects known as *quiver representations*. In this paper, we come full circle by formulating and studying Brascamp–Lieb inequalities for quivers.

**1.1.** ***Brascamp–Lieb inequalities.*** Let us begin by briefly reviewing the theory of ordinary Brascamp–Lieb inequalities.

**Definition 1.1** (Brascamp–Lieb inequality). Let $H$ and $H^1, \dots, H^m$ be nontrivial finite-dimensional Hilbert spaces, and for each $1 \leq j \leq m$, let $B_j : H \to H^j$ be a surjective linear map and $p_j \in [1, \infty]$. A *Brascamp–Lieb inequality* is an inequality of the form

$$\int_H \prod_{j=1}^m f_j \circ B_j \, dx \lesssim \prod_{j=1}^m \| f_j \|_{L^{p_j}(H^j)} \tag{1-1}$$

that holds for all measurable functions $f_j : H^j \to [0, \infty]$, where the implicit constant (see Section 1.3) is allowed to be infinite.

*MSC2020:* 16G20, 26D15, 47B90.
*Keywords:* quiver, quivers, Brascamp, Lieb, inequality, inequalities.

Of course, a Brascamp–Lieb inequality is only useful when the implicit constant is finite. The conditions under which this occurs are well known [4; 5].

**Theorem 1.2.** *Let* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$ *denote the smallest constant for which inequality* (1-1) *holds, where* $(\boldsymbol{B}, \boldsymbol{p}) := ((B_1, \dots, B_m), (p_1, \dots, p_m))$; *it is known as the* ***Brascamp–Lieb constant*** *for the* ***Brascamp–Lieb datum*** $(\boldsymbol{B}, \boldsymbol{p})$. *Then* $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p})$ *is finite if and only if the* ***scaling condition***

$$\dim(H) = \sum_{j=1}^{m} p_j^{-1} \dim(H^j) \tag{1-2}$$

*and the* ***dimension condition***

$$\dim(V) \le \sum_{j=1}^{m} p_j^{-1} \dim(B_j V) \quad \textit{for all } V \le H \tag{1-3}$$

*hold. (Here* $V \le H$ *means that* $V$ *is a subspace of* $H$.*)*

Furthermore, Lieb [10] showed that the optimal constant is unchanged when the functions are restricted to be Gaussians of the form $f_j(x) := e^{-\pi \langle A_j x, x\rangle p_j^{-1}}$ for each $1 \le j \le m$, where $A_j \succ 0$ with respect to the Loewner order. Since

$$\int_{H^j} e^{-\pi \langle A_j x, x\rangle}\, dx = \det(A_j)^{-1/2},$$

this yields the following result and formula for Brascamp–Lieb constants.

**Theorem 1.3.** *Let* $\mathrm{BL}_{\mathrm{G}}(\boldsymbol{B}, \boldsymbol{p})$ *denote the smallest constant for which inequality* (1-1) *holds when* $f_j(x) := e^{-\pi \langle A_j x, x\rangle p_j^{-1}}$ *for some* $A_j \succ 0$ *(for each* $1 \le j \le m$*). Then*

$$\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \mathrm{BL}_{\mathrm{G}}(\boldsymbol{B}, \boldsymbol{p}) = \sup_{A_j \succ 0} \left[ \frac{\prod_{j=1}^{m} \det(A_j)^{p_j^{-1}}}{\det\bigl(\sum_{j=1}^{m} p_j^{-1} B_j^{\top} A_j B_j\bigr)} \right]^{1/2}. \tag{1-4}$$

*(Here the supremum is taken over* $A_1 \succ 0, \dots, A_m \succ 0$.*)*

**Example 1.4** (Young's convolution inequality). Let $H := \mathbb{R}^d \times \mathbb{R}^d$ and $H^j := \mathbb{R}^d$ for $1 \le j \le 3$ (with their standard inner products), and let $B_1(x, y) := x$, $B_2(x, y) := y$, and $B_3(x, y) := x - y$ for all $(x, y) \in H$. Then inequality (1-1) is *Young's convolution inequality*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_1(x) f_2(y) f_3(x - y)\, dx\, dy \lesssim \|f_1\|_{L^{p_1}(\mathbb{R}^d)} \|f_2\|_{L^{p_2}(\mathbb{R}^d)} \|f_3\|_{L^{p_3}(\mathbb{R}^d)}.$$

It is a result of Beckner [2] and Brascamp and Lieb [7] that

$$\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \left( \prod_{j=1}^{3} \frac{p_j^{1/p_j}}{p_j'^{\,1/p_j'}} \right)^{d/2} \quad \text{if } \sum_{j=1}^{3} \frac{1}{p_j} = 2,$$

where $1/p_j + 1/p_j' = 1$, and $\mathrm{BL}(\boldsymbol{B}, \boldsymbol{p}) = \infty$ for all other $\boldsymbol{p}$ (under the usual interpretations of $1/\infty = 0$ and $\infty^{1/\infty} = 1$).

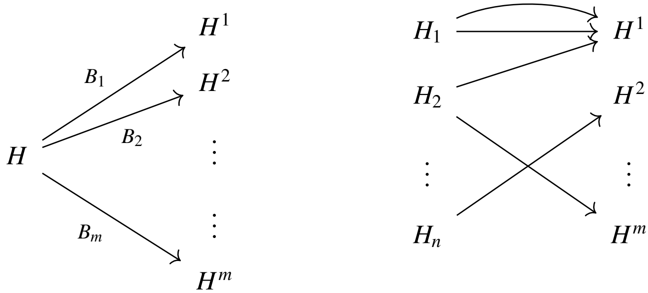


**Figure 1.** The $m$-subspace quiver (left) and a bipartite quiver (right).

**1.2.** ***Quiver Brascamp–Lieb inequalities.*** The relationships between the spaces and the linear maps in inequality (1-1) can be depicted by a type of graph known as a *quiver*.

**Definition 1.5** (quiver). A *quiver* (or *directed multigraph*) $\mathcal{Q}$ is a tuple $(\mathcal{V}, \mathcal{A}, s, t)$ consisting of a set $\mathcal{V}$ of *vertices*, a set $\mathcal{A}$ of *arrows* (or *edges*), a function $s : \mathcal{A} \to \mathcal{V}$ specifying the *source* (or *tail*) of each arrow, and a function $t : \mathcal{A} \to \mathcal{V}$ specifying the *target* (or *head*) of each arrow. If $\mathcal{V}$ is the disjoint union of two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ with $s(\mathcal{A}) \subseteq \mathcal{V}_1$ and $t(\mathcal{A}) \subseteq \mathcal{V}_2$, the quiver is said to be *bipartite*.

**Definition 1.6** (quiver representation). A *representation* of a quiver is an assignment of a vector space to each vertex and a linear map to each arrow.

In what follows, we will only consider quivers with finitely many vertices and arrows and will generally identify each quiver with a specific representation of it by nontrivial finite-dimensional Hilbert spaces and surjective linear maps (so, strictly speaking, the sets of vertices and arrows will be multisets). For instance, the quiver corresponding to inequality (1-1) is the bipartite *$m$-subspace quiver* defined by $\mathcal{V} = \{H\} \cup \{H^j\}_{j=1}^m$, $\mathcal{A} = \{B_1, \dots, B_m\}$, $s(B_j) = H$, and $t(B_j) = H^j$, as illustrated in Figure 1 (left).

Recently, Chindris and Derksen [8] considered a quantity analogous to that in formula (1-4) for a general bipartite quiver with $\mathcal{V} = \{H_i\}_{i=1}^n \cup \{H^j\}_{j=1}^m$, $s(\mathcal{A}) = \{H_i\}_{i=1}^n$, and $t(\mathcal{A}) = \{H^j\}_{j=1}^m$; see Figure 1 (right). Namely, if $\mathcal{A}_{ij}$ denotes the set of arrows from $H_i$ to $H^j$, $B_a$ denotes the surjective linear map representing an arrow $a \in \mathcal{A}$, and $p_j \in [1, \infty]$ for each $1 \le j \le m$, it was shown that the quantity

$$\sup_{A_j \succ 0} \left[ \frac{\prod_{j=1}^m \det(A_j)^{p_j^{-1}}}{\prod_{i=1}^n \det\bigl(\sum_{j=1}^m \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} B_a^\top A_j B_a\bigr)} \right]^{1/2} \tag{1-5}$$

is finite if and only if the scaling condition

$$\sum_{i=1}^n \dim(H_i) = \sum_{j=1}^m p_j^{-1} \dim(H^j) \tag{1-6}$$

and the dimension condition

$$\sum_{i=1}^{n}\dim(V_i) \le \sum_{j=1}^{m} p_j^{-1}\dim\Big(\sum_{i=1}^{n}\sum_{a\in\mathcal{A}_{ij}} B_a V_i\Big) \quad \text{for all } V_i \le H_i \tag{1-7}$$

hold. Clearly, conditions (1-6) and (1-7) and the supremum (1-5) reduce to conditions (1-2) and (1-3) and the supremum (1-4) in the case of the $m$-subspace quiver.

However, the natural question of whether the supremum (1-5) is the optimal constant in some generalization of the Brascamp–Lieb inequality was left uninvestigated. Indeed, a "quiver Brascamp–Lieb inequality" has yet to be formulated, so our first task is to reinterpret this algebraic result as an analytic one. Naïvely taking $f_j(x) := e^{-\pi\langle A_j x, x\rangle p_j^{-1}}$ as before allows one to write the supremum (1-5) as

$$\sup_{A_j\succ 0} \frac{\prod_{i=1}^{n}\int_{H_i}\prod_{j=1}^{m}\prod_{a\in\mathcal{A}_{ij}} f_j\circ B_a\,dx}{\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(H^j)}},$$

but this misleadingly suggests the untenable inequality

$$\prod_{i=1}^{n}\int_{H_i}\prod_{j=1}^{m}\prod_{a\in\mathcal{A}_{ij}} f_j\circ B_a\,dx \lesssim \prod_{j=1}^{m}\|f_j\|_{L^{p_j}(H^j)},$$

which in general has a different number of $f_j$ on each side. Aiming for the more viable inequality

$$\prod_{i=1}^{n}\int_{H_i}\prod_{j=1}^{m}\prod_{a\in\mathcal{A}_{ij}} f_j\circ B_a\,dx \lesssim \prod_{i=1}^{n}\prod_{j=1}^{m}\prod_{a\in\mathcal{A}_{ij}}\|f_j\|_{L^{p_j}(H^j)} = \prod_{j=1}^{m}\|f_j\|_{L^{p_j}(H^j)}^{\alpha_j}, \tag{1-8}$$

where $\alpha_j := \sum_{i=1}^{n}\#(\mathcal{A}_{ij})$ (and $\#(\mathcal{A}_{ij})$ denotes the cardinality of $\mathcal{A}_{ij}$), one can verify that scaling $p_j^{-1}$ by $\alpha_j$ in the supremum (1-5) yields

$$C_{\mathcal{Q},\boldsymbol{p}}^{-1/2}\cdot\sup_{A_j\succ 0}\frac{\prod_{i=1}^{n}\int_{H_i}\prod_{j=1}^{m}\prod_{a\in\mathcal{A}_{ij}} f_j\circ B_a\,dx}{\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(H^j)}^{\alpha_j}}$$

when $f_j(x) := e^{-\pi\langle A_j x, x\rangle \alpha_j p_j^{-1}}$, where

$$C_{\mathcal{Q},\boldsymbol{p}} := \prod_{j=1}^{m}\alpha_j^{\alpha_j p_j^{-1}\dim(H^j)}.$$

Thus, the result of Chindris and Derksen amounts to the theorem below.

**Theorem 1.7.** *Let* $\mathrm{BLCD_G}(\mathcal{Q},\boldsymbol{p})$ *denote the optimal constant in inequality* (1-8) *when* $f_j(x) := e^{-\pi\langle A_j x, x\rangle \alpha_j p_j^{-1}}$ *for some* $A_j\succ 0$, *so that*

$$\mathrm{BLCD_G}(\mathcal{Q},\boldsymbol{p}) = C_{\mathcal{Q},\boldsymbol{p}}^{1/2}\cdot\sup_{A_j\succ 0}\left[\frac{\prod_{j=1}^{m}\det(A_j)^{\alpha_j p_j^{-1}}}{\prod_{i=1}^{n}\det\big(\sum_{j=1}^{m}\sum_{a\in\mathcal{A}_{ij}}\alpha_j p_j^{-1}B_a^{\top}A_jB_a\big)}\right]^{1/2}.$$

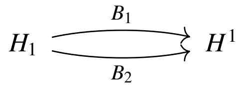


**Figure 2.** The quiver $\mathcal{Q}$.

*Then* $\mathrm{BLCD_G}(\mathcal{Q}, \boldsymbol{p})$ *is finite if and only if the* ***scaling condition***

$$\sum_{i=1}^{n} \dim(H_i) = \sum_{j=1}^{m} \alpha_j p_j^{-1} \dim(H^j) \tag{1-9}$$

*and the* ***dimension condition***

$$\sum_{i=1}^{n} \dim(V_i) \le \sum_{j=1}^{m} \alpha_j p_j^{-1} \dim\Big(\sum_{i=1}^{n} \sum_{a \in \mathcal{A}_{ij}} B_a V_i\Big) \quad \text{for all } V_i \le H_i \tag{1-10}$$

*hold.*

One might hope that a generalization of Theorem 1.3 holds so that the optimal constant $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ in inequality (1-8) (for general functions) coincides with $\mathrm{BLCD_G}(\mathcal{Q}, \boldsymbol{p})$, or at least that conditions (1-9) and (1-10) are also sufficient for the finiteness of $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$. Unfortunately, this turns out to be false — the finiteness of the general constant is in fact equivalent to the scaling condition (1-9) along with the stronger dimension condition (1-13) given below. We will prove this by considering ostensibly more general inequalities in which a function is associated with each *arrow* instead of each target space.

**Definition 1.8** (quiver Brascamp–Lieb inequalities). Let $\mathcal{Q}$ be a bipartite quiver represented by nontrivial finite-dimensional Hilbert spaces $H_1, \dots, H_n$ and $H^1, \dots, H^m$ and surjective linear maps from the former to the latter. In addition, let $\mathcal{A}_{ij}$ denote the set of arrows from $H_i$ to $H^j$, $B_a$ denote the map representing an arrow $a$, and $p_j \in [1, \infty]$ for each $1 \le j \le m$. A *quiver Brascamp–Lieb inequality* is an inequality of the form

$$\prod_{i=1}^{n} \int_{H_i} \prod_{j=1}^{m} \prod_{a \in \mathcal{A}_{ij}} f_a \circ B_a \, dx \lesssim \prod_{i=1}^{n} \prod_{j=1}^{m} \prod_{a \in \mathcal{A}_{ij}} \| f_a \|_{L^{p_j}(H^j)} \tag{1-11}$$

that holds for all measurable functions $f_a : H^j \to [0, \infty]$.

**Example 1.9.** Consider the quiver $\mathcal{Q}$ in Figure 2, where $H_1 := \mathbb{R}^3$, $H^1 := \mathbb{R}^2$, $B_1(x_1, x_2, x_3) := (x_1, x_2)$, and $B_2(x_1, x_2, x_3) := (x_2, x_3)$; and let $\boldsymbol{p} = (p_1) := \left(\frac{4}{3}\right)$.

Then inequality (1-8) reads

$$\int_{\mathbb{R}^3} f_1(x_1, x_2) f_1(x_2, x_3) \, dx \lesssim \| f_1 \|_{L^{4/3}(\mathbb{R}^2)} \| f_1 \|_{L^{4/3}(\mathbb{R}^2)} = \| f_1 \|^2_{L^{4/3}(\mathbb{R}^2)},$$

whereas inequality (1-11) reads

$$\int_{\mathbb{R}^3} f_1(x_1, x_2) f_2(x_2, x_3) \, dx \lesssim \| f_1 \|_{L^{4/3}(\mathbb{R}^2)} \| f_2 \|_{L^{4/3}(\mathbb{R}^2)}.$$

For quiver Brascamp–Lieb inequalities in the form of inequality (1-11), a generalization of Theorem 1.2 holds.

**Theorem 1.10.** *Let* $\mathrm{BL}(\mathcal{Q}, \boldsymbol{p})$ *denote the optimal constant in inequality* (1-11). *Then* $\mathrm{BL}(\mathcal{Q}, \boldsymbol{p})$ *is finite if and only if the* ***scaling condition***

$$\sum_{i=1}^{n} \dim(H_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \dim(H^j) \tag{1-12}$$

*and the* ***dimension condition***

$$\sum_{i=1}^{n} \dim(V_i) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \dim(B_a V_i) \quad \text{for all } V_i \leq H_i \tag{1-13}$$

*hold.*

In actuality, such inequalities are no more general than Chindris–Derksen-type inequalities, as the following result shows.

**Theorem 1.11.** *Let* $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ *denote the optimal constant in inequality* (1-8). *Then*

$$\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) \leq \mathrm{BL}(\mathcal{Q}, \boldsymbol{p}) \leq \prod_{j=1}^{m} \alpha_j^{\alpha_j} \cdot \mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}).$$

(*As above*, $\alpha_j = \sum_{i=1}^{n} \#(\mathcal{A}_{ij})$.)

**Corollary 1.12.** *Let* $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ *denote the optimal constant in inequality* (1-8). *Then* $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ *is finite if and only if the* ***scaling condition*** (1-12)

$$\sum_{i=1}^{n} \dim(H_i) = \sum_{j=1}^{m} \alpha_j p_j^{-1} \dim(H^j)$$

*and the* ***dimension condition*** (1-13)

$$\sum_{i=1}^{n} \dim(V_i) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \dim(B_a V_i) \quad \text{for all } V_i \leq H_i$$

*hold.*

As a result, we find that Gaussians do not saturate Chindris–Derksen-type inequalities in general.

**Corollary 1.13.** *There exists a quiver Brascamp–Lieb datum* $(\mathcal{Q}, \boldsymbol{p})$ *for which* $\mathrm{BLCD}_{\mathrm{G}}(\mathcal{Q}, \boldsymbol{p}) < \mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) = \infty$.

The sufficiency and necessity of the conditions in Theorem 1.10 will be separately established in Section 2. The proof of sufficiency involves splitting the bipartite quiver in question into multiple subspace quivers to which Theorem 1.2 applies; inequality (1-11) is then obtained as the product of inequalities of the form (1-1). The proof of necessity is an adaptation of the scaling argument used by Bennett et al. [5].

Following this, in Section 3, we will show that every instance of inequality (1-8) can be realized as an instance of inequality (1-11) and vice-versa, which will prove Theorem 1.11. Corollary 1.12 is merely a restatement of Theorem 1.10 for comparison with Theorem 1.7. Finally, we will verify that for the datum in Example 1.9, the conditions in Theorem 1.7 hold but those in Theorem 1.10 or Corollary 1.12 do not, which entails Corollary 1.13 — in essence, the failure of Theorem 1.3 for quivers.

**1.3. *Notation.*** We employ the standard notation $A \lesssim B$ to indicate that $A \le CB$ for some constant $C > 0$; if $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$. Occasionally, we adjoin subscripts to this notation to indicate dependence of the constant $C$ on other parameters; for instance, we write $A \lesssim_{\alpha,\beta} B$ when $A \le CB$ for some constant $C > 0$ depending on $\alpha$, $\beta$.

We also use $\#(\,\cdot\,)$ for the cardinality of a finite set, $|\cdot|$ for the measure of a set, and $\|\cdot\|$ for the norm of a vector.

## 2. Conditions for quiver Brascamp–Lieb inequalities

First, we prove that the scaling and dimension conditions in Theorem 1.10 are sufficient and necessary.

*Proof of sufficiency in Theorem 1.10.* By taking all but one subspace to be zero in condition (1-13), we find that

$$\dim(V_i) \le \sum_j \sum_{a\in\mathcal{A}_{ij}} p_j^{-1} \dim(B_a V_i) \quad \text{for all } V_i \le H_i \text{ and each } i.$$

In particular,

$$\dim(H_i) \le \sum_j \sum_{a\in\mathcal{A}_{ij}} p_j^{-1} \dim(H^j) \quad \text{for each } i,$$

so we also have

$$\dim(H_i) = \sum_j \sum_{a\in\mathcal{A}_{ij}} p_j^{-1} \dim(H^j) \quad \text{for each } i$$

on account of condition (1-12). Thus, the scaling condition (1-2) and the dimension condition (1-3) hold for each of the subspace quivers $\mathcal{Q}_i$ consisting of the source $H_i$, its incident arrows, and their targets *regarded as separate vertices* (see Figure 3 for an example), along with the corresponding weights $\boldsymbol{p}_i$. It follows from Theorem 1.2 that

$$\int_{H_i} \prod_j \prod_{a\in\mathcal{A}_{ij}} f_a \circ B_a \, dx \le \mathrm{BL}(\mathcal{Q}_i, \boldsymbol{p}_i) \prod_j \prod_{a\in\mathcal{A}_{ij}} \|f_a\|_{L^{p_j}(H^j)},$$

with $\mathrm{BL}(\mathcal{Q}_i, \boldsymbol{p}_i) < \infty$ for each $i$; taking the product of these inequalities over $i$ yields the conclusion. □

Our proof that the conditions are necessary adapts an argument of Bennett et al. [4].

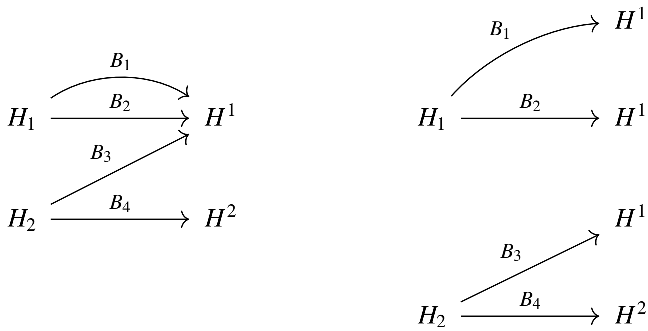


**Figure 3.** A bipartite quiver $\mathcal{Q}$ and its subspace quivers $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

*Proof of necessity in Theorem 1.10.* Let $c$ be a positive constant to be determined later and let $r$ and $R$ be arbitrary positive constants satisfying $r \le 1 \le R$. Given subspaces $V_i \le H_i$, define

$$S_i := \{(v_i, v_i') \in V_i \oplus V_i^\perp : \|v_i\| \le cR \text{ and } \|v_i'\| \le cr\} \subseteq H_i$$

for each $i$ and

$$S^a := \{(w_a, w_a') \in W^a \oplus (W^a)^\perp : \|w_a\| \le R \text{ and } \|w_a'\| \le r\} \subseteq H^j$$

for each $a \in \mathcal{A}_{ij}$ and each $i$, $j$, where $W^a := B_a V_i \le H^j$.

If $(v_i, v_i') \in S_i$ for some $i$ and $a \in \mathcal{A}_{ij}$, then $B_a v_i \in W^a$ and $\|B_a v_i\| \lesssim \|v_i\| \le cR$, so $B_a v_i \in S^a$ provided that $c$ is chosen sufficiently small. Similarly, if we write $B_a v_i' =: (w_a, w_a') \in W^a \oplus (W^a)^\perp$, then $\|w_a\| + \|w_a'\| \approx \|B_a v_i'\| \lesssim \|v_i'\| \le cr$, so $B_a v_i' \in S^a$ provided that $c$ is chosen sufficiently small because $r \le R$. Hence $B_a S_i \subseteq S^a + S^a \subseteq 2S^a$.

Now taking $f_a := 1_{2S^a}$, we find that

$$\begin{aligned}\prod_i \int_{H_i} \prod_j \prod_{a \in \mathcal{A}_{ij}} f_a \circ B_a \, dx &\ge \prod_i \int_{S_i} \prod_j \prod_{a \in \mathcal{A}_{ij}} f_a \circ B_a \, dx \\ &= \prod_i |S_i| \approx R^{\sum_i \dim(V_i)} r^{\sum_i \operatorname{codim}(V_i)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\prod_i \prod_j \prod_{a \in \mathcal{A}_{ij}} \|f_a\|_{L^{p_j}(H^j)} &\approx \prod_i \prod_j \prod_{a \in \mathcal{A}_{ij}} |S^a|^{p_j^{-1}} \\ &\approx R^{\sum_i \sum_j \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \dim(W^a)} r^{\sum_i \sum_j \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \operatorname{codim}(W^a)}.\end{aligned}$$

Sending $R \to \infty$, we deduce that $\sum_i \dim(V_i) \le \sum_i \sum_j \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \dim(W^a)$; sending $r \to 0$, we deduce that $\sum_i \operatorname{codim}(V_i) \ge \sum_i \sum_j \sum_{a \in \mathcal{A}_{ij}} p_j^{-1} \operatorname{codim}(W^a)$. Condition (1-13) is the first inequality, while condition (1-12) is the conjunction of the first inequality with $V_i := H_i$ and the second inequality with $V_i := \{0\}$ (recalling that the $B_a$ are assumed to be surjective, so $B_a H_i = H^j$). □

## 3. Examples of quiver Brascamp–Lieb inequalities

Next, we return to the question of when an inequality of the form (1-8) holds, which is answered by the work of Chindris and Derksen in the Gaussian case. Here we answer this question in general by proving Theorem 1.11, which implies that such an inequality is equivalent to an inequality of the form (1-11).

*Proof of Theorem 1.11.* Given inequality (1-11) and a function $f_j$ for each $j$, we can take $f_a := f_j$ for each $a \in \mathcal{A}_{ij}$ and each $i$ to obtain inequality (1-8) with constant $\mathrm{BL}(\mathcal{Q}, \boldsymbol{p})$, which shows that $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) \le \mathrm{BL}(\mathcal{Q}, \boldsymbol{p})$.

Conversely, given inequality (1-8) and a function $f_a$ for each arrow $a$, let us assume that $\|f_a\|_{L^{p_j}(H^j)} = 1$ for each $a$. We can then take $f_j := \sum_i \sum_{a \in \mathcal{A}_{ij}} f_a$ for each $j$ to obtain

$$\begin{aligned}\prod_i \int_{H_i} \prod_j \prod_{a \in \mathcal{A}_{ij}} f_a \circ B_a \, dx &\le \mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) \prod_i \prod_j \prod_{a \in \mathcal{A}_{ij}} \Big\| \sum_{i'=1}^{n} \sum_{a' \in \mathcal{A}_{i'j}} f_a \Big\|_{L^{p_j}(H^j)} \\ &\le \mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) \prod_i \prod_j \prod_{a \in \mathcal{A}_{ij}} \alpha_j = \mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) \prod_j \alpha_j^{\alpha_j},\end{aligned}$$

which is inequality (1-11) assuming that the $f_a$ are normalized. By homogeneity, the inequality holds in general with the same constant, which means that

$$\mathrm{BL}(\mathcal{Q}, \boldsymbol{p}) \le \prod_j \alpha_j^{\alpha_j} \cdot \mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}). \qquad \square$$

Now let us see how Example 1.9 gives an instance of condition (1-13) being strictly stronger than condition (1-10), thereby precluding a generalization of Lieb's result (Theorem 1.3) to the quiver setting.

*Proof of Corollary 1.13.* Consider the datum $(\mathcal{Q}, \boldsymbol{p})$ of Example 1.9. The scaling condition (1-9) or (1-12) is $\dim(\mathbb{R}^3) = \frac{3}{2}\dim(\mathbb{R}^2)$, which is obviously satisfied. However, the dimension condition (1-10) is

$$\dim(V_1) \le \tfrac{3}{2}\dim(B_1 V_1 + B_2 V_1) \quad \text{for all } V_1 \le \mathbb{R}^3,$$

whereas the dimension condition (1-13) is

$$\dim(V_1) \le \tfrac{3}{4}\dim(B_1 V_1) + \tfrac{3}{4}\dim(B_2 V_1) \quad \text{for all } V_1 \le \mathbb{R}^3.$$

Evidently, the latter does not hold — consider $V_1 = \mathrm{span}\{(1, 0, 0)\}$ — but in fact the former does.

To see this, first note that if $\dim(B_1 V_1 + B_2 V_1) = 0$, then $\dim(V_1) = 0$ since $V_1 \le \ker(B_1) \cap \ker(B_2) = \{0\}$, and if $\dim(B_1 V_1 + B_2 V_1) = 2$, the inequality is trivial. In the remaining case $\dim(B_1 V_1 + B_2 V_1) = 1$, we must have $B_1 V_1 \le \mathrm{span}\{w\}$ and $B_2 V_1 \le \mathrm{span}\{w\}$ for some $w = (w_1, w_2) \in \mathbb{R}^2$. From this, we find that $V_1 \le \mathrm{span}\{(w_1^2, w_1 w_2, w_2^2)\}$, so $\dim(V_1) \le 1$ and the inequality is satisfied.

As a result, $\mathrm{BLCD}_{\mathrm{G}}(\mathcal{Q}, \boldsymbol{p})$ is finite (by Theorem 1.7), yet $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ is not (by Theorem 1.10 or Corollary 1.12). $\square$

**Remark 3.1.** For this datum, we can also observe directly that $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ is infinite: if $S := ([0, N] \times [0, 1]) \cup ([0, 1] \times [0, N])$ and $f_1 := 1_S$, then the right-hand side of inequality (1-8) is $\|1_S\|^2_{L^{4/3}(\mathbb{R}^2)} = |S|^{3/2} \approx N^{3/2}$, while the left-hand side is

$$\int_{\mathbb{R}^3} 1_S(x_1, x_2)\, 1_S(x_2, x_3)\, dx \geq |[0, N] \times [0, 1] \times [0, N]| = N^2.$$

Even when $\mathrm{BLCD}_{\mathrm{G}}(\mathcal{Q}, \boldsymbol{p})$ and $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p})$ are both finite, they need not be equal — in Example 3.2, they are; in Example 3.3, they are not (in general).

**Example 3.2.** Consider the quiver $\mathcal{Q}$ in Figure 2 with $H_1 := \mathbb{R}^2$ and $H^1 := \mathbb{R}^1$, and let $\boldsymbol{p} = (p_1) := (1)$. In addition, suppose that $B_1(x) := b_1^\top x$ and $B_2(x) := b_2^\top x$, where $b_1, b_2 \in \mathbb{R}^2$ are such that $B := \begin{bmatrix} b_1 & b_2 \end{bmatrix}$ is invertible.

According to the formula in Theorem 1.7, we have

$$\begin{aligned}\mathrm{BLCD}_{\mathrm{G}}(\mathcal{Q}, \boldsymbol{p}) &= 2 \cdot \sup_{a_1 > 0} \left[ \frac{a_1^2}{\det(2 b_1 a_1 b_1^\top + 2 b_2 a_1 b_2^\top)} \right]^{1/2} \\ &= \frac{1}{\det(B B^\top)^{1/2}} = \frac{1}{|\det(B)|}.\end{aligned}$$

On the other hand, for any measurable function $f_1 : \mathbb{R}^1 \to [0, \infty]$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} f_1(b_1^\top x)\, f_1(b_2^\top x)\, dx &= \int_{\mathbb{R}^2} f_1(y_1)\, f_1(y_2)\, |\det((B^\top)^{-1})|\, dy \\ &= \frac{1}{|\det(B)|} \|f_1\|_{L^1(\mathbb{R}^1)} \|f_1\|_{L^1(\mathbb{R}^1)},\end{aligned}$$

which shows that

$$\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) = \frac{1}{|\det(B)|}$$

as well.

**Example 3.3.** Consider the quiver $\mathcal{Q}$ in Figure 2 with $H_1 := \mathbb{R}^1$ and $H^1 := \mathbb{R}^1$, and let $\boldsymbol{p} = (p_1) := (2)$. In addition, suppose that $B_1(x) := b_1 x$ and $B_2(x) := b_2 x$, where $b_1, b_2 \in \mathbb{R}^1 \setminus \{0\}$.

According to the formula in Theorem 1.7, we have

$$\mathrm{BLCD}_{\mathrm{G}}(\mathcal{Q}, \boldsymbol{p}) = 2^{1/2} \cdot \sup_{a_1 > 0} \left[ \frac{a_1}{b_1 a_1 b_1 + b_2 a_1 b_2} \right]^{1/2} = \left( \frac{2}{b_1^2 + b_2^2} \right)^{1/2}.$$

On the other hand, for any measurable function $f_1 : \mathbb{R}^1 \to [0, \infty]$, we have

$$\begin{aligned}\int_{\mathbb{R}} f_1(b_1 x)\, f_1(b_2 x)\, dx &\leq \left[ \int_{\mathbb{R}} f_1(b_1 x)^2\, dx \right]^{1/2} \left[ \int_{\mathbb{R}} f_1(b_2 x)^2\, dx \right]^{1/2} \\ &= \left[ \int_{\mathbb{R}} f_1(y)^2\, |b_1^{-1}|\, dy \right]^{1/2} \left[ \int_{\mathbb{R}} f_1(y)^2\, |b_2^{-1}|\, dy \right]^{1/2} \\ &= \frac{1}{|b_1 b_2|^{1/2}} \|f_1\|_{L^2(\mathbb{R}^1)} \|f_1\|_{L^2(\mathbb{R}^1)},\end{aligned}$$

which shows that

$$\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) \le \frac{1}{|b_1 b_2|^{1/2}}.$$

In fact, this is an equality. To see this, let $p > 1$ and

$$f_1(x) := \begin{cases} |x|^{-p/2} & \text{if } |x| \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

Assuming without loss of generality that $|b_1| \le |b_2|$, we compute that

$$\begin{aligned} \int_{\mathbb{R}} f_1(b_1 x)\, f_1(b_2 x)\, dx &= \int_{|x|\ge 1/|b_1|} \frac{1}{|b_1 x|^{p/2}} \cdot \frac{1}{|b_2 x|^{p/2}}\, dx \\ &= \frac{1}{|b_1 b_2|^{p/2}} \int_{|x|\ge 1/|b_1|} \Big(\frac{1}{|x|^{p/2}}\Big)^2 dx \\ &= \frac{1}{|b_1 b_2|^{p/2}} \cdot \frac{1}{|b_1|^{1-p}} \cdot \|f_1\|^2_{L^2(\mathbb{R}^1)}, \end{aligned}$$

whence the claim follows by taking $p \to 1$. As a result, we see that both the Gaussian constant and the general constant are finite for this datum, but that they are equal if and only if $|b_1| = |b_2|$.

## 4. Concluding remarks and questions

Inspecting the proof of Lieb's theorem [10], we find that it uses the *multilinearity* of inequality (1-1) in the $f_j$, which the quiver inequality (1-8) does not possess. Thus, one might have expected the optimal constant for Gaussians to differ from that for general functions in the latter inequality. Indeed, the function $f_1$ in Remark 3.1 can be thought of as the sum of two rough approximations to Gaussians, $g_1 := 1_{[0,N]\times[0,1]}$ and $h_1 := 1_{[0,1]\times[0,N]}$ (ignoring the overlap of their supports). The left-hand side of the inequality, roughly

$$\int_{\mathbb{R}^3} (g_1 + h_1)(x_1, x_2)(g_1 + h_1)(x_2, x_3)\, dx,$$

is incomparably larger than

$$\int_{\mathbb{R}^3} g_1(x_1, x_2) g_1(x_2, x_3)\, dx + \int_{\mathbb{R}^3} h_1(x_1, x_2) h_1(x_2, x_3)\, dx$$

for large $N$ because of the "cross terms" in the product, and consequently fails to be bounded by the right-hand side.

Although the theory of quiver Brascamp–Lieb inequalities does not appear to be as rich as that of ordinary Brascamp–Lieb inequalities, there are still some questions that could be investigated. For instance, can the inequalities in Theorem 1.11 be strict? We know that the second can be: consider the quiver $\mathcal{Q}$ in Figure 2 once more, with $H_1 := \mathbb{R}^1$, $H^1 := \mathbb{R}^1$, $B_1(x_1) := x_1$, and $B_2(x_1) := x_1$; and let $\boldsymbol{p} = (p_1) := (2)$. Then $\mathrm{BLCD}(\mathcal{Q}, \boldsymbol{p}) = \mathrm{BL}(\mathcal{Q}, \boldsymbol{p}) = 1$ by the Cauchy–Schwarz inequality and $\alpha_1 = 2$. We can also ask: if not Gaussians, what are the maximizers of quiver Brascamp–Lieb inequalities? Are there any interesting applications of such inequalities?

## Acknowledgments

We thank Terence Tao for his guidance and suggestions in the research for and writing of this paper, as well as Calin Chindriş for his thought-provoking discussions. We are obliged to the reviewer for helpful comments.

## References

[1] F. Barthe, "On a reverse form of the Brascamp–Lieb inequality", *Invent. Math.* **134**:2 (1998), 335–361. MR Zbl

[2] W. Beckner, "Inequalities in Fourier analysis", *Ann. of Math.* (2) **102**:1 (1975), 159–182. MR Zbl

[3] J. Bennett and T. Tao, "Adjoint Brascamp–Lieb inequalities", *Proc. Lond. Math. Soc.* (3) **129**:4 (2024), art. id. e12633. MR Zbl

[4] J. Bennett, A. Carbery, M. Christ, and T. Tao, "The Brascamp–Lieb inequalities: finiteness, structure and extremals", *Geom. Funct. Anal.* **17**:5 (2008), 1343–1415. MR Zbl

[5] J. Bennett, A. Carbery, M. Christ, and T. Tao, "Finite bounds for Hölder–Brascamp–Lieb multilinear inequalities", *Math. Res. Lett.* **17**:4 (2010), 647–666. MR Zbl

[6] J. Bennett, N. Bez, T. C. Flock, and S. Lee, "Stability of the Brascamp–Lieb constant and applications", *Amer. J. Math.* **140**:2 (2018), 543–569. MR Zbl

[7] H. J. Brascamp and E. H. Lieb, "Best constants in Young's inequality, its converse, and its generalization to more than three functions", *Advances in Math.* **20**:2 (1976), 151–173. MR Zbl

[8] C. Chindris and H. Derksen, "The capacity of quiver representations and Brascamp–Lieb constants", *Int. Math. Res. Not.* **2022**:24 (2022), 19399–19430. MR Zbl

[9] A. Garg, L. Gurvits, R. Oliveira, and A. Wigderson, "Algorithmic and optimization aspects of Brascamp–Lieb inequalities, via operator scaling", *Geom. Funct. Anal.* **28**:1 (2018), 100–145. MR Zbl

[10] E. H. Lieb, "Gaussian kernels have only Gaussian maximizers", *Invent. Math.* **102**:1 (1990), 179–208. MR Zbl

[11] M. Weber and S. Sra, "Computing Brascamp–Lieb constants through the lens of Thompson geometry", preprint, 2024. arXiv 2208.05013

[12] R. Zhang, "The Brascamp–Lieb inequality and its influence on Fourier analysis", pp. 585–628 in *The physics and mathematics of Elliott Lieb — the* 90*th anniversary*, *II*, edited by R. L. Frank et al., EMS Press, Berlin, 2022. MR Zbl

Received July 11, 2024. Revised September 30, 2024.

NICHOLAS HU
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
njhu@math.ucla.edu

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm



See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 333 No. 1 November 2024