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**WEAK GROTHENDIECK COMPACTNESS PRINCIPLE
FOR SYMMETRIC SPACES**

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We strengthen and simplify the weak Grothendieck compactness principle for symmetric sequence spaces, and extend it to both symmetric function spaces and noncommutative symmetric spaces of measurable operators.

1. Introduction

In [25, p. 112], Grothendieck showed that for a Banach space X, any compact subset with respect to the norm can be enclosed within the (norm) closed convex hull of a norm null sequence. It is said there that this result is implicitly stated in an article of Dieudonné and Schwartz [7, Proof of Theorem 5]. Following [19] and despite Grothendieck's remark, we refer to this result as the Grothendieck compactness principle.

However, an analogous principle does not hold if the norm topology is replaced by the weak topology. For example, Lindenstrauss and Phelps [32, Corollary 1.2] proved that the closed unit ball in an infinite-dimensional reflexive Banach space cannot be contained in the closed convex hull of a weakly null sequence. In the context of specific spaces, it is noted in [38] that the closed unit ball of ℓ_2, viewed as a subset of c_0, is not contained in the closed convex hull of a weakly null sequence in c_0.

Nonetheless, in a recent study, an analogous principle for the weak topology was established in the context of the Schur property (i.e., when strong convergence and weak convergence coincide). The principle is as follows: every weakly compact subset of a Banach space X is contained in the closed convex hull of a weakly null sequence if and only if X has the Schur property; see [18].

Such a result when the norm topology is replaced by the weak topology is referred to as the *weak Grothendieck compactness principle* (wGCP) [18; 19]. We note that the class of such Banach spaces is rather scarce. For example, among symmetric sequence spaces there are no spaces with the Schur property but ℓ_1; see [29, Corollary 1]. The following version of the wGCP applicable for all symmetric sequence spaces different from ℓ_1 and with no subspace isomorphic to c_0 was obtained in [19, Theorem 1].

Theorem 1.1. *Let X be a Banach space with a symmetric basis. The space X is not isomorphic to ℓ_1 and does not contain a subspace isomorphic to c_0 if and only if the following are equivalent for all subsets $K \subset X$:*

(i) *K is relatively weakly compact.*

(ii) *K is contained in the weak closure of $\mathrm{sym}((x_n)_{n\geq 0})$ for some weakly null sequence $(x_n)_{n\geq 0}$ in X.*

Here, $\mathrm{sym}(Y)$ is a *symmetric hull* of $Y \subset X$ defined by

$$\mathrm{sym}(Y) := \{z \in \ell_\infty : \mu(z) = \mu(y) \text{ for some } y \in Y\},$$

and $\mu(z)$ is the classical decreasing rearrangement of the sequence $|z|$ (see definition in [35] and in Section 2.1 below).

We start with the study of the wGCP on symmetric sequence spaces in Section 4.1, where we provide an elementary proof of a stronger result than in Theorem 1.1 (see Theorem 4.1). The advantage of our approach is twofold:

(a) It eliminates the need for the condition that X does not contain a subspace isomorphic to c_0 in the implication (i) $\Rightarrow$ (ii) (it is necessary for the converse implication).

(b) It holds for the norm closure in the condition (ii) rather than the weak closure, which makes it closer to the original statement of the Grothendieck compactness principle.

The strength of our approach is evident in its applicability to the study of the wGCP in the noncommutative counterpart of a symmetric sequence space — Banach ideals in $B(\mathcal{H})$. In Section 4.4 we obtain the wGCP in the general setting of a noncommutative symmetric space $X(\mathcal{M}, \tau)$ associated with a von Neumann algebra $\mathcal{M}$; see Theorem 4.19. In fact, the majority of the results in this paper are its special cases. Nevertheless, in the interest of clarity and recognizing the subtle distinctions among these special cases, we opt for a step-by-step exposition of our results starting with (commutative) symmetric sequence and function spaces. Theorem 1.2 below is the most important consequence of Theorem 4.19.

Theorem 1.2. *Let $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ be a separable Banach ideal in $B(\mathcal{H})$ with the Fatou property. If $\mathcal{I} \neq \mathcal{L}_1$, then the following properties are equivalent:*

(i) $K \subset \mathcal{I}$ is relatively weakly compact.

(ii) *There exists a weakly null sequence* $(x_n)_{n \geq 0} \subset \mathcal{I}$ *such that*

$$K \subset \overline{\mathrm{sym}((x_n)_{n \geq 0})}^{\|\cdot\|_{\mathcal{I}}}.$$

Here, a symmetric hull $\mathrm{sym}(Y)$ of a set $Y \subset \mathcal{I}$ may be defined by

$$\mathrm{sym}(Y) = \{uxv : x \in Y \text{ and } u, v \in B(\mathcal{H}) \text{ are unitaries}\}.$$

In Section 5 we show that the condition $\mathcal{I} \neq \mathcal{L}_1$ in Theorem 1.2 is also necessary for the implication (ii) $\Rightarrow$ (i). We also provide a quick answer in the affirmative to a question posed in [19, Remark 1].

2. Preliminaries

2.1. Symmetric sequence spaces. If $x = (x(k))_{k=0}^{\infty}$ is a bounded sequence of real or complex numbers, then, in what follows, $\mu(x) = (\mu(k, x))_{k=0}^{\infty}$ denotes the decreasing rearrangement of the sequence $|x|$ defined by the formula

$$\mu(k, x) = \inf_{\substack{A \subset \mathbb{Z}_{\geq 0} \\ \mathrm{card}(A) = k}} \sup_{i \in A^c} |x(i)|, \quad k \in \mathbb{Z}_{\geq 0}.$$

A Banach sequence space X is called a *symmetric sequence space* if $X \subset \ell_{\infty}$ and, for every $x \in X$ and for every $y \in \ell_{\infty}$ such that $\mu(y) \leq \mu(x)$, we have $y \in X$ and $\|y\|_X \leq \|x\|_X$.

Note that if X is a symmetric sequence space, $x \in X$, and π is an arbitrary permutation of $\mathbb{Z}_{\geq 0}$, then the sequence $x \circ \pi$ belongs to X and $\|x \circ \pi\|_X = \|x\|_X$.

Among the most important examples of symmetric sequence spaces are the ℓ_p-spaces, $1 \leq p \leq \infty$. For details on symmetric spaces we refer the reader to the books [2; 31; 33; 35].

The Köthe dual $X^{\times}$ of a symmetric (Banach) sequence space X is defined as the space of all $y \in \ell_{\infty}$ such that $xy \in \ell_1$ for every $x \in X$. It is a Banach sequence space when equipped with the norm

$$\|y\|_{X^{\times}} = \sup_{\|x\|_X \leq 1} \left| \sum_{n \geq 0} x(n) y(n) \right|.$$

We define

$$\langle x, y \rangle = \sum_{n \geq 0} x(n) y(n), \quad x \in X, \ y \in X^{\times}.$$

Note that $X^{\times}$ is embedded isometrically into a Banach dual space X^*, and they coincide if and only if X is separable [30, Corollary 6.1.2].

Every symmetric sequence space X is continuously embedded into its second Köthe dual $X^{\times\times} = (X^{\times})^{\times}$, and $\|x\|_{X^{\times\times}} \leq \|x\|_X$ for $x \in X$. A symmetric sequence

space X has the *Fatou property* if for every $(x_n)_{n \geq 0} \subset X$ such that $\sup_{n \geq 0} \|x_n\|_X < \infty$ and such that $x_n \to x$ componentwise as $n \to \infty$, it follows that $x \in X$ and

$$\|x\|_X \leq \liminf_{n \to \infty} \|x_n\|_X.$$

A symmetric sequence space X has the Fatou property if and only if the natural inclusion of X into $X^{\times\times}$ is a surjective isometry [30, Theorem 6.1.7]. A separable symmetric sequence space has the Fatou property if and only if it does not have a subspace isomorphic to c_0 (see [10; 11; 16, Theorem 6.5]).

Definition 2.1. We say that a sequence $(a_n)_{n \geq 0}$ in a Banach space X is equivalent to the standard unit basis $(e_n)_{n \geq 0}$ in ℓ_1 if there exists $C > 1$ such that

$$C^{-1} \sum_{k \geq 0} |c_k| \leq \left\| \sum_{k \geq 0} c_k a_k \right\|_X \leq C \sum_{k \geq 0} |c_k|, \quad (c_n)_{n \geq 0} \in \ell_1.$$

2.2. Symmetric function spaces. Suppose that $\alpha \in (0, \infty]$, $I = (0, \alpha)$ and Σ is the σ-algebra of Lebesgue measurable subsets of I. By (I, m) we denote the measure space (I, Σ, m) equipped with the Lebesgue measure m. Let $L_0(I) = L_0(I, m)$ be the space of all equivalence classes of Lebesgue measurable real-valued functions on I. Let $S(I) = S(I, m)$ be its subspace consisting of all functions $f \in L_0(I)$ such that $m(\{|f| > s\}) < \infty$ for some $s > 0$. For $f \in S(I)$, we denote by $\mu(f)$ the decreasing rearrangement of the function $|f|$, that is,

$$\mu(t; f) = \inf\{s \geq 0 : m(\{|f| > s\}) \leq t\}, \quad t > 0.$$

Definition 2.2. We say that $(X(I), \|\cdot\|_X)$ is a symmetric Banach function space on I if the following hold:

  (i) $X(I)$ is a subspace of $S(I)$.

 (ii) $(X(I), \|\cdot\|_X)$ is a Banach space.

(iii) If $f \in X(I)$ and if $g \in S(I)$ are such that $|g| \leq |f|$, then $g \in X(I)$ and $\|g\|_X \leq \|f\|_X$.

(iv) If $f \in X(I)$ and if $g \in S(I)$ are such that $\mu(g) = \mu(f)$, then $g \in X(I)$ and $\|g\|_X = \|f\|_X$.

  For the general theory of symmetric function spaces, we refer to [31; 34].

2.3. Noncommutative symmetric spaces. Let $\mathcal{M}$ be a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ equipped with a faithful normal semifinite trace τ.

  A closed and densely defined operator x affiliated with $\mathcal{M}$ is called τ-measurable if $\tau(e^{|x|}(s, \infty)) < \infty$ for sufficiently large s, where $e^{|x|}$ stands for the spectral measure of $|x|$. We denote the set of all τ-measurable operators by $S(\mathcal{M}, \tau)$.

For every $x \in S(\mathcal{M}, \tau)$, we define its (generalized) singular value function $\mu(x)$ by setting [35, Section 2.3]

$$\mu(t, x) = \inf\{s \geq 0 : \tau(e^{|x|}(s, \infty)) \leq t\}, \quad t \geq 0.$$

For more details on generalized singular value functions, we refer to [10; 17; 22; 35].

A closed, densely defined operator x in $\mathcal{H}$, affiliated with the von Neumann algebra $\mathcal{M}$, is called τ-compact if $\tau(e^{|x|}(s, \infty)) < \infty$ for all $s > 0$, or, equivalently, $\mu(s, x) \to 0$ as $s \to \infty$.

Recall the following construction of a noncommutative symmetric space (or a symmetric Banach operator space) $X(\mathcal{M}, \tau)$ from [28] (see also its detailed exposition in [17; 35]). If X is a symmetric Banach function space on $(0, \tau(\mathbb{1}))$, then we set

$$X(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \mu(x) \in X(0, \tau(\mathbb{1}))\}.$$

Similarly, if X is a symmetric sequence space and $\mathcal{M}$ is atomic with all atoms having the same trace, then we may define

$$X(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : (\mu(n; x))_{n \geq 0} \in X\}.$$

There exists the natural norm on $X(\mathcal{M}, \tau)$ defined by

$$\|x\|_{X(\mathcal{M}, \tau)} := \|\mu(x)\|_{X(0, \tau(\mathbf{1}))}, \quad x \in X(\mathcal{M}, \tau).$$

We note that $X(\mathcal{M}, \tau)$ is a Banach space with respect to $\|\cdot\|_{X(\mathcal{M}, \tau)}$ and it is called the *noncommutative symmetric (operator) space* associated with $(\mathcal{M}, \tau)$ corresponding to a function space $(X, \|\cdot\|_{X_{(0, \tau(1))}})$ [17; 28; 35]. Equivalently:

Definition 2.3. Let $X(\mathcal{M}, \tau)$ be a linear subspace of $S(\mathcal{M}, \tau)$ equipped with a complete norm $\|\cdot\|_X = \|\cdot\|_{X(\mathcal{M}, \tau)}$. We say that $X(\mathcal{M}, \tau)$ is a noncommutative symmetric space if for $x \in X(\mathcal{M}, \tau)$ and for every $y \in S(\mathcal{M}, \tau)$ with $\mu(y) \leq \mu(x)$, we have $y \in X(\mathcal{M}, \tau)$ and $\|y\|_X \leq \|x\|_X$.

If $\mathcal{M} = B(\mathcal{H})$ (so that τ is the standard trace Tr), then the space $X(\mathcal{M}, \tau)$ is a Banach ideal in $B(\mathcal{H})$. Conversely, every Banach ideal in $B(\mathcal{H})$ is of the shape $X(\mathcal{M}, \tau)$ with $\mathcal{M} = B(\mathcal{H})$ and $\tau = \mathrm{Tr}$ (possibly after passing to an equivalent norm in the ideal).

For a noncommutative symmetric space $X(\mathcal{M}, \tau)$, we define its Köthe dual $X^\times(\mathcal{M}, \tau)$ by setting

$$X^\times(\mathcal{M}, \tau) = \{y \in S(\mathcal{M}, \tau) : \sup\{\tau(|xy|) : x \in X(\mathcal{M}, \tau), \|x\|_X \leq 1\} < \infty\},$$

and for $y \in X^\times(\mathcal{M}, \tau)$, the functional $\|y\|_{X^\times}$ is defined by setting

$$\|y\|_{X^\times} = \sup\{\tau(|xy|) : x \in X(\mathcal{M}, \tau), \|x\|_X \leq 1\}.$$

We note that $(X^\times(\mathcal{M}, \tau), \|\cdot\|_{X^\times})$ is a noncommutative symmetric space; see [13, Proposition 5.4]. It coincides with a Banach dual, that is, $X^*(\mathcal{M}, \tau) = X^\times(\mathcal{M}, \tau)$ and $X^*(0, \tau(\mathbb{1})) = X^\times(0, \tau(\mathbb{1}))$ whenever $X(0, \tau(\mathbb{1}))$ is separable (see, e.g., [10, Proposition 42; 13]).

3. Weak convergence of a shifted sequence

For a sequence $x = (x(k))_{k \geq 0}$ of real or complex numbers we use the notation (called a shifted sequence)

$$0^{\oplus n} \oplus x = (\underbrace{0, \ldots, 0}_{n \text{ times}}, x(0), x(1), \ldots), \quad n \in \mathbb{Z}_{\geq 0}.$$

The following main result of this section, which shows that a shifted sequence of a relatively weakly compact sequence is weakly null, is the key ingredient to prove the implication (i) $\Rightarrow$ (ii) in the wGCP for symmetric sequence spaces in Theorem 4.1 (without the Fatou property).

Theorem 3.1. *Let $X \neq \ell_1$ be a separable symmetric sequence space. For every relatively weakly compact sequence $(z_n)_{n \geq 0} \subset X$, the sequence*

$$(x_n)_{n \geq 0} := (0^{\oplus n} \oplus z_n)_{n \geq 0}$$

is weakly null.

The proof is postponed until the end of the section. We start by presenting the following two known facts. The first fact for separable spaces should be compared with [37, Proposition 2.5.15].

Fact 3.2. *Let X be a symmetric sequence space. A weakly null sequence in X cannot be equivalent to the unit basis $(e_n)_{n \geq 0}$ in ℓ_1.*

Fact 3.3. *Let X be a separable Banach space and let $K \subset X$. There exists a countable norm-dense subset of K.*

We also need several auxiliary lemmas.

Lemma 3.4 (see [37, Proposition 2.5.13]). *Let X be a separable symmetric sequence space and let $(x_n)_{n \geq 0} \subset X$ be a bounded sequence which consists of pairwise disjointly supported elements. One of the following (mutually exclusive) situations occurs:*

(i) *The sequence $(x_n)_{n \geq 0}$ is weakly null.*

(ii) *There is a subsequence $(x_{n_j})_{j \geq 0}$ of $(x_n)_{n \geq 0}$ equivalent to the unit basis in ℓ_1.*

In what follows, $\max(A)$ denotes the maximal element in a finite set $A \subset \mathbb{Z}_{\geq 0}$.

Lemma 3.5. *Let X be a separable symmetric sequence space. Suppose that the sequence $(a_j)_{j\geq 0} \subset X$ consists of finitely supported and pairwise disjointly supported elements. If the sequence $(a_j)_{j\geq 0}$ is weakly null, then the sequence $(0^{\oplus n_j} \oplus a_j)_{j\geq 0}$ is also weakly null for every strictly increasing sequence $(n_j)_{j\geq 0}$.*

Proof. Assume, on the contrary, that there is a strictly increasing sequence $(n_j)_{j\geq 0}$ such that the sequence $(A_j)_{j\geq 0} := (0^{\oplus n_j} \oplus a_j)_{j\geq 0}$ is not weakly null. Recall that $X^* = X^\times$ due to separability of X. Choose $y \in X^\times$ such that $\langle y, A_j \rangle \nrightarrow 0$ as $j \to \infty$. Since $(A_j)_{j\geq 0}$ is a bounded sequence in X, we may choose a strictly increasing sequence $(j_k)_{k\geq 0}$ such that

(1) $\langle y, A_{j_k} \rangle \to c \neq 0$ as $k \to \infty$;

(2) $n_{j_{k+1}} - n_{j_k} > \max(\mathrm{supp}(a_{j_k}))$ for all $k \geq 0$.

The second condition means that the sequence $(A_{j_k})_{k\geq 0}$ consists of pairwise disjointly supported elements. The first condition means that the sequence $(A_{j_k})_{k\geq 0}$ is not weakly null. By Lemma 3.4, there exists a subsequence $(A_{j_{k_l}})_{l\geq 0}$ of $(A_{j_k})_{k\geq 0}$ which is equivalent to the unit basis in ℓ_1.

Both sequences $(A_{j_k})_{k\geq 0}$ and $(a_{j_k})_{k\geq 0}$ consist of pairwise disjointly supported elements. So do their subsequences $(A_{j_{k_l}})_{l\geq 0}$ and $(a_{j_{k_l}})_{l\geq 0}$ and $\mu(A_{j_{k_l}}) = \mu(a_{j_{k_l}})$ for all $l \geq 0$. Therefore, for any sequence $(c_l)_{l\geq 0} \in \ell_1$ we have

$$\left\| \sum_{l\geq 0} c_l a_{j_{k_l}} \right\|_X = \left\| \sum_{l\geq 0} c_l A_{j_{k_l}} \right\|_X \approx \|(c_l)_{l\geq 0}\|_1.$$

However, $a_{j_{k_l}} \to 0$ weakly as $l \to \infty$, which contradicts Fact 3.2. $\qquad\square$

The following lemma, commonly referred to as the subsequence splitting lemma, is known for function spaces on $(0, 1)$; see, e.g., [40, Lemma 3.1]; for function spaces on $(0, \infty)$ and for ideals of compact operators in $B(\mathcal{H})$ but only with a splitting into three sequences, see, e.g., [1, Theorem 3.2; 15, Proposition 3.2], respectively. Similar results are also available with splitting into two sequences (see [42; 44]). It is a tool that allows us to pass to disjointly supported elements in sequence spaces.

Lemma 3.6. *Let X be a separable symmetric sequence space. For every weakly null sequence $(a_k)_{k\geq 0} \subset X$, there exists a strictly increasing sequence $(k_l)_{l\geq 0}$ of positive integers and sequences $(b_l)_{l\geq 0}, (c_l)_{l\geq 0}$ in X such that*

$$a_{k_l} = b_l + c_l, \quad l \geq 0,$$

where

(i) *the sequence $(c_l)_{l\geq 0}$ tends to 0 in the norm of X;*

(ii) *the sequence $(b_l)_{l\geq 0}$ consists of finitely supported elements;*

(iii) *the sequence $(b_l)_{l\geq 0}$ consists of pairwise disjointly supported elements.*

Proof. Firstly, note that weak convergence to zero implies coordinatewise convergence to zero.

Set $k_0 = 0$. Since X is separable, it follows that [2, Chapter I, Corollary 5.6]

$$\|a_{k_0}\chi_{[n,\infty)}\|_X \to 0, \quad n \to \infty.$$

Choose $n_0 \in \mathbb{N}$ such that

$$\|a_{k_0}\chi_{[n_0,\infty)}\|_X \le 2^{-0}.$$

Since $(a_k)_{k\ge 0}$ converges to zero coordinatewise, we have

$$\|a_k\chi_{[0,n_0)}\|_X \le \|a_k\chi_{[0,n_0)}\|_1 \to 0, \quad k \to \infty.$$

Choose $k_1 > k_0$ such that

$$\|a_{k_1}\chi_{[0,n_0)}\|_X \le 2^{-1}.$$

Since X is separable, it follows that

$$\|a_{k_1}\chi_{[n,\infty)}\|_X \to 0, \quad n \to \infty.$$

Choose $n_1 > n_0$ such that

$$\|a_{k_1}\chi_{[n_1,\infty)}\|_X \le 2^{-1}.$$

Choose $k_2 > k_1$ such that

$$\|a_{k_2}\chi_{[0,n_1)}\|_X \le 2^{-2}.$$

Proceed ad infinitum. We obtain strictly increasing sequences $(k_l)_{l\ge 0}$ and $(n_l)_{l\ge 0}$ such that

$$\|a_{k_l}\chi_{[n_l,\infty)}\|_X \le 2^{-l}, \quad l \ge 0,$$
$$\|a_{k_l}\chi_{[0,n_{l-1})}\|_X \le 2^{-l}, \quad l \ge 1.$$

Set $b_l = a_{k_l}\chi_{[n_{l-1},n_l)}$ for $l \ge 1$, and $b_0 = a_{k_0}\chi_{[0,n_0)}$. By construction, the sequence $(b_l)_{l\ge 0}$ consists of pairwise finitely and disjointly supported elements, that is, conditions (ii) and (iii) are satisfied. For $l \ge 1$ set

$$c_l = a_{k_l} - b_l = a_{k_l}\chi_{[0,n_{l-1})} + a_{k_l}\chi_{[n_l,\infty)} \quad \text{and} \quad c_0 = a_{k_0} - b_0 = a_{k_0}\chi_{[n_0,\infty)}.$$

Hence by the triangle inequality we have

$$\|c_l\|_X \le \|a_{k_l}\chi_{[0,n_{l-1})}\|_X + \|a_{k_l}\chi_{[n_l,\infty)}\|_X \le 2^{1-l}, \quad l \ge 1.$$

This yields the condition (i), which completes the proof. $\square$

Lemma 3.7. *Let X be a separable symmetric sequence space. If a sequence $(a_j)_{j\ge 0}$ is weakly null, then so is the sequence $(0^{\oplus n_j} \oplus a_j)_{j\ge 0}$ for every strictly increasing sequence $(n_j)_{j\ge 0}$.*

Proof. Assume, on the contrary, that there is a strictly increasing sequence $(n_j)_{j \geq 0}$ such that the sequence $(A_j)_{j \geq 0} := (0^{\oplus n_j} \oplus a_j)_{j \geq 0}$ is not weakly null. As in the proof of Lemma 3.5, there are $y \in X^\times$ and a strictly increasing sequence $(j_k)_{k \geq 0}$ such that $\langle y, A_{j_k} \rangle \to c \neq 0$ as $k \to \infty$.

Since the sequence $(a_{j_k})_{k \geq 0}$ is weakly null, we may choose a strictly increasing sequence $\{k_l\}_{l \geq 0}$ of positive integers by Lemma 3.6 such that $a_{j_{k_l}} = b_l + c_l$ for all $l \geq 0$, where $c_l \to 0$ in norm as $l \to \infty$ and $(b_l)_{l \geq 0}$ is finitely supported and pairwise disjointly supported sequence. Since the sequence $(a_{j_{k_l}})_{l \geq 0}$ is weakly null and since the sequence $(c_l)_{l \geq 0}$ converges to 0 in norm as $l \to \infty$, it follows that the sequence $(b_l)_{l \geq 0}$ is weakly null.

By Lemma 3.5, the sequence $(0^{\oplus n_{j_{k_l}}} \oplus b_l)_{l \geq 0}$ is also weakly null. Since $\|c_l\|_X = \|a_{j_{k_l}} - b_l\|_X \to 0$ as $l \to \infty$ and X is symmetric, it follows that the sequence $(A_{j_{k_l}} - 0^{\oplus n_{j_{k_l}}} \oplus b_l)_{l \geq 0}$ also converges to 0 in norm as $l \to \infty$, and hence is weakly null. In particular, $\langle y, A_{j_{k_l}} \rangle - \langle y, 0^{\oplus n_{j_{k_l}}} \oplus b_l \rangle \to 0$ as $l \to \infty$. Since $\langle y, A_{j_{k_l}} \rangle \to c$ as $l \to \infty$, it follows that $\langle y, 0^{\oplus n_{j_{k_l}}} \oplus b_l \rangle \to c \neq 0$, which is a contradiction. □

The following lemma is the only place in this section where we use the assumption $X \neq \ell_1$.

Lemma 3.8. *Let $X \neq \ell_1$ be a separable symmetric sequence space. For every $z \in X$ and for every strictly increasing sequence of positive integers $(n_j)_{j \geq 0}$, the sequence $(0^{\oplus n_j} \oplus z)_{j \geq 0}$ is weakly null.*

Proof. Suppose first that z is finitely supported.

Assume, on the contrary, that the sequence $(y_j)_{j \geq 0} := (0^{\oplus n_j} \oplus z)_{j \geq 0}$ is not weakly null for some strictly increasing sequence $(n_j)_{j \geq 0}$. Recall that $X^* = X^\times$ due to separability of X; see, for instance, [10, Proposition 42] or [13]. Choose $f \in X^\times$ such that $\langle f, y_j \rangle \not\to 0$ as $j \to \infty$. Since $\sup_{j \geq 0} \|y_j\|_X = \|z\|_X$, we may choose a strictly increasing sequence $(j_k)_{k \geq 0}$ such that

(1) $\langle f, y_{j_k} \rangle \to c \neq 0$ as $k \to \infty$;

(2) $n_{j_{k+1}} - n_{j_k} > \max(\text{supp}(z))$, $k \geq 0$.

It follows from Lemma 3.4 that there exists a subsequence $(y_{j_{k_l}})_{l \geq 0}$ equivalent to the unit basis in ℓ_1. Thus,

$$\left\| \bigoplus_{l \geq 0} c_l z \right\|_X = \left\| \sum_{l \geq 0} c_l y_{j_{k_l}} \right\|_X \approx \sum_{l \geq 0} |c_l|, \quad (c_l)_{l \geq 0} \in \ell_1.$$

Therefore, there is $C > 0$ such that for any $(c_l)_{l \geq 0} \in \ell_1$ we have

$$\|(c_l)_{l \geq 0}\|_1 = \sum_{l \geq 0} |c_l| \leq C \left\| \bigoplus_{l \geq 0} c_l z \right\|_X \leq C \|z\|_\infty \cdot \text{card}(\text{supp}(z)) \cdot \|(c_l)_{l \geq 0}\|_X,$$

and hence

$$\|(c_l)_{l \geq 0}\|_X \approx \|(c_l)_{l \geq 0}\|_1, \quad (c_l)_{l \geq 0} \in \ell_1.$$

The latter equality contradicts the condition $X \neq \ell_1$, which completes the proof for finitely supported z.

Let $z \in X$ be arbitrary, fix $y \in X^\times$ such that $\|y\|_{X^\times} = 1$ and $\varepsilon > 0$. Since X is separable, one may split $z = z_1 + z_2$, where $\|z_2\|_X \leq \varepsilon$ and z_1 is finitely supported. By the preceding step, we have $0^{\oplus n_j} \oplus z_1 \to 0$ weakly as $j \to \infty$ for every strictly increasing sequence $(n_j)_{j \geq 0}$. Fix $j(\varepsilon)$ such that

$$|\langle y, 0^{\oplus n_j} \oplus z_1 \rangle| < \varepsilon, \quad j \geq j(\varepsilon).$$

Therefore, since $|\langle y, 0^{\oplus n_j} \oplus z_2 \rangle| \leq \|y\|_{X^\times} \|0^{\oplus n_j} \oplus z_2\|_X$ (see [10, Remark 3]) and since $\|0^{\oplus n_j} \oplus z_2\|_X = \|z_2\|_X$ (due to the fact that X is symmetric), we have

$$|\langle y, 0^{\oplus n_j} \oplus z \rangle| \leq |\langle y, 0^{\oplus n_j} \oplus z_1 \rangle| + |\langle y, 0^{\oplus n_j} \oplus z_2 \rangle| \leq 2\varepsilon, \quad j \geq j(\varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\langle y, 0^{\oplus n_j} \oplus z \rangle \to 0, \quad j \to \infty.$$

Since $y \in X^\times$ is an arbitrary normalized element, the assertion follows. $\quad\square$

Proof of Theorem 3.1. Let $(z_n)_{n \geq 0} \subset X$ be a relatively weakly compact sequence and $(x_n)_{n \geq 0} := (0^{\oplus n} \oplus z_n)_{n \geq 0}$.

Assume, on the contrary, that the sequence $(x_n)_{n \geq 0}$ is not weakly null. There are $y \in X^\times$ and a subsequence $(x_{n_j})_{j \geq 0}$ (where $(n_j)_{j \geq 0}$ is strictly increasing) such that $\langle y, x_{n_j} \rangle \to c \neq 0$ as $j \to \infty$.

Passing to a further subsequence (if necessary), we may assume $\langle y, x_{n_j} \rangle \to c \neq 0$ and $z_{n_j} \to z$ weakly as $j \to \infty$ for some $z \in X$. Thus, $(z_{n_j} - z)_{j \geq 0}$ is weakly null, and the sequences $(0^{\oplus n_j} \oplus (z_{n_j} - z))_{j \geq 0}$ and $(0^{\oplus n_j} \oplus z)_{j \geq 0}$ are also weakly null by Lemmas 3.7 and 3.8, respectively. Hence, $(0^{\oplus n_j} \oplus z_{n_j})_{j \geq 0}$ is also weakly null. In particular, $\langle y, x_{n_j} \rangle = \langle y, 0^{\oplus n_j} \oplus z_{n_j} \rangle \to 0$ as $j \to \infty$. This contradicts $\langle y, x_{n_j} \rangle \to c \neq 0$ as $j \to \infty$ and completes the proof. $\quad\square$

4. Weak Grothendieck compactness principle

4.1. *Weak Grothendieck compactness principle for symmetric sequence spaces.*
Let X be a symmetric sequence space and let $Y \subset X$ be an arbitrary set. Recall that a symmetric hull of Y is defined by

$$\mathrm{sym}(Y) = \{z \in \ell_\infty : \mu(z) = \mu(y) \text{ for some } y \in Y\}.$$

The following theorem, which strengthens the wGCP for symmetric sequence spaces from [19, Theorem 1], is the main result of this subsection. In this theorem, we take the norm closure rather than the weak closure in condition (ii) below.

Theorem 4.1. *Let $X \neq \ell_1$ be a separable symmetric sequence space. For a subset $K \subset X$, consider the following properties*:

(i) *K is relatively weakly compact.*

(ii) *There exists a weakly null sequence $(x_n)_{n \geq 0}$ in X such that*

$$K \subset \overline{\mathrm{sym}((x_n)_{n \geq 0})}^{\|\cdot\|_X}.$$

It is always true that (i) *implies* (ii). *If, in addition, X possesses the Fatou property, then* (ii) *implies* (i).[1]

Although it is said in the remark following [19, Theorem 1] that the Fatou property is not necessary for the converse implication, the proof for this case was omitted. On the other hand, the Fatou property is necessary for the converse implication; see [19, Theorem 1] for an example of a subset K of X containing a subspace isomorphic to c_0 such that (ii) holds but (i) fails. We also provide an elementary proof of Theorem 4.1 using different techniques, in particular, the theory of Köthe duality, the so-called subsequence splitting property (Lemma 3.6), and the following fundamental result of Garling (see Theorem 16 in [23]).

Theorem 4.2 (Garling). *Let $X \neq \ell_1$ be a symmetric sequence space with the Fatou property. The set $K \subset X$ is relatively $\sigma(X, X^\times)$-compact if and only if so is the set $\mathrm{sym}(K)$.*

The following is an immediate corollary of Garling's result.

Lemma 4.3. *Let $X \neq \ell_1$ be a separable symmetric sequence space with the Fatou property. The set $K \subset X$ is relatively weakly compact if and only if so is the set $\mathrm{sym}(K)$.*

Proof. Since X is separable, it follows that $X^\times = X^*$; see [10, Proposition 42; 13]. Hence, $\sigma(X, X^\times)$-topology coincides with the weak topology on X. The assertion follows from Theorem 4.2. □

Proof of Theorem 4.1. The implication (ii) $\Rightarrow$ (i) follows immediately from Lemma 4.3.

Let us now prove the implication (i) $\Rightarrow$ (ii). Let K be a (relatively) weakly compact subset of X and let $(z_n)_{n \geq 0} \subset K$ be a countable norm-dense subset in K (which exists due to Fact 3.3). For every $n \geq 0$ set $x_n = 0^{\oplus n} \oplus z_n$. The fact that the sequence $(x_n)_{n \geq 0}$ is weakly null follows from Theorem 3.1.

Note that

$$z_m \in \mathrm{sym}((x_n)_{n \geq 0}), \quad m \geq 0.$$

---

[1]A separable symmetric sequence space X is called a KB-space (or a Kantorovich–Banach space) if it has the Fatou property. That is, for all KB-spaces different from ℓ_1 we have (i) $\Longleftrightarrow$ (ii).

Thus,

$$(z_m)_{m \geq 0} \subset \mathrm{sym}((x_n)_{n \geq 0}).$$

Taking the norm-closure, we obtain

$$K \subset \overline{(z_m)_{m \geq 0}}^{\| \cdot \|_X} \subset \overline{\mathrm{sym}((x_n)_{n \geq 0})}^{\| \cdot \|_X},$$

which completes the proof of the implication (i) $\Rightarrow$ (ii). $\square$

4.2. Weak Grothendieck compactness principle for symmetric function spaces over a positive semiaxis.

We extend our results (and hence, results of [19]) on the wGCP from symmetric sequence spaces to symmetric function spaces over $(0, \infty)$.

Define a space

$$(L_1 + L_\infty)(0, \infty) = \{f \in S(0, \infty) : f = g + h, \ g \in L_1(0, \infty), \ h \in L_\infty(0, \infty)\}$$

equipped with a natural norm

$$\|f\|_{L_1 + L_\infty} = \inf\{\|g\|_1 + \|h\|_\infty : f = g + h, \ g \in L_1, \ h \in L_\infty\}.$$

The space $((L_1 + L_\infty)(0, \infty), \| \cdot \|_{L_1 + L_\infty})$ is a symmetric function space on $(0, \infty)$; see [34, Proposition 2.a.2].

Recall that for any symmetric function space $X(0, \infty)$ we have

$$(L_1 \cap L_\infty)(0, \infty) \subset X(0, \infty) \subset (L_1 + L_\infty)(0, \infty)$$

as sets and the inclusion maps are of norm one with respect to the natural norms in these spaces [31, Theorem 4.1], i.e., if $f \in (L_1 \cap L_\infty)(0, \infty)$, then $\|f\|_X \leq \|f\|_{L_1 \cap L_\infty}$ and if $f \in X(0, \infty)$, then $\|f\|_{L_1 + L_\infty} \leq \|f\|_X$.

Let $X(0, \infty)$ (or $X(0, 1)$) be a symmetric function space, and let $K \subset X(0, \infty)$ be an arbitrary set. Define a symmetric hull of a set K (with obvious modification for $K \subset X(0, 1)$) by

$$\mathrm{sym}(K) = \{g \in (L_1 + L_\infty)(0, \infty) : \mu(g) = \mu(f) \text{ for some } f \in K\}.$$

The following key result is an immediate consequence of [14, Proposition 2.1].

Proposition 4.4. *Let $X(0, \infty)$ be a separable symmetric Banach function space with the Fatou property[2] such that $X(0, \infty) \not\subset L_1(0, \infty)$. A subset K in $X(0, \infty)$ is relatively weakly compact if and only if so is $\mu(K)$.*

For every $n \in \mathbb{Z}_{\geq 0}$ let $\tau_n : X(0, \infty) \to X(0, \infty)$ be a translation operator defined by

$$(4\text{-}1) \qquad \tau_n f(\cdot) = f(\cdot - n), \quad f \in X(0, \infty).$$

---

[2]Recall that $X(0, \alpha)$ is a KB-space if and only if $X(0, \alpha)$ is separable and has the Fatou property if and only if it is weakly sequentially complete if and only if it contains no copy of c_0; see [9, Proposition 4.8; 11, Proposition 3.2].

Lemma 4.5. *Let $X(0, \infty)$ be a separable symmetric Banach function space. Let $g \in X(0, \infty)$, let $(g_n)_{n \geq 0} \subset X(0, \infty)$ and let $(n_j)_{j \geq 0}$ be a strictly increasing sequence in $\mathbb{Z}_{\geq 0}$. If $\tau_{n_j} g_{n_j} \to g$ weakly as $j \to \infty$, then $g = 0$.*

Proof. For every $n \geq 0$ define

$$F_n = \overline{\text{sgn}(g)} \chi_{(n,n+1)} \in X^{\times}(0, \infty).$$

Hence, for every $n \geq 0$ we have

$$\langle F_n, \tau_{n_j} g_{n_j} \rangle \to \langle F_n, g \rangle = \int_n^{n+1} |g|, \quad j \to \infty.$$

On the other hand,

$$F_n \cdot \tau_{n_j} g_{n_j} = 0, \quad j > n+1.$$

Thus, for every $n \geq 0$

$$\langle F_n, \tau_{n_j} g_{n_j} \rangle = \int_n^{n+1} (F_n \cdot \tau_{n_j} g_{n_j}) \to 0, \quad j \to \infty.$$

By uniqueness of the limit, we have

$$\int_n^{n+1} |g| = 0.$$

Thus, $g = 0$ on $(n, n+1)$. Since $n \in \mathbb{Z}_{\geq 0}$ is arbitrary, the assertion follows. $\qquad\square$

Lemma 4.6. *Let $X(0, \infty)$ be a separable symmetric Banach function space with the Fatou property such that $X(0, \infty) \not\subset L_1(0, \infty)$. If $(g_n)_{n \geq 0} \subset X(0, \infty)$ is relatively weakly compact, then $(f_n)_{n \geq 0} := (\tau_n g_n)_{n \geq 0}$ is weakly null.*

Proof. Assume, on the contrary, $(\tau_n g_n)_{n \geq 0}$ is not weakly null. Hence, $\tau_n g_n \not\to 0$ as $n \to \infty$ in $\sigma(X, X^{\times})$-topology. Therefore, by the Eberlein–Šmulian theorem (see [30, VIII.2]) there exists $F \in X^{\times}$ and a strictly increasing sequence $(n_j)_{j \geq 0}$ such that

(4-2) $$\langle F, \tau_{n_j} g_{n_j} \rangle \to c \neq 0, \quad j \to \infty.$$

Set $K_0 = \{g_n : n \geq 0\}$ and $K_1 = \text{sym}(K_0)$. By assumption, K_0 is relatively weakly compact. Hence, by Proposition 4.4, so is $\mu(K_0)$. Since $\mu(K_0) = \mu(K_1)$, it follows that $\mu(K_1)$ is also relatively weakly compact. By Proposition 4.4, so is K_1. Since

$$(\tau_{n_j} g_{n_j})_{j \geq 0} \subset \text{sym}(K_0) = K_1,$$

it follows that the sequence $(\tau_{n_j} g_{n_j})_{j \geq 0}$ is relatively weakly compact. Hence, there exists a strictly increasing sequence $(j_k)_{k \geq 0}$ and $g \in X$ such that

$$\tau_{n_{j_k}} g_{n_{j_k}} \to g$$

weakly as $k \to \infty$. It follows from Lemma 4.5 that $g = 0$. Therefore,

$$\langle F, \tau_{n_{j_k}} g_{n_{j_k}} \rangle \to 0, \quad k \to \infty,$$

which contradicts (4-2). □

Theorem 4.7. *Let $X(0, \infty)$ be a separable symmetric Banach function space with the Fatou property such that $X(0, \infty) \not\subset L_1(0, \infty)$. For $K \subset X(0, \infty)$, the following conditions are equivalent:*

  (i) *K is relatively weakly compact in $X(0, \infty)$.*

  (ii) *There exists a weakly null sequence $(f_n)_{n \geq 0}$ in $X(0, \infty)$ such that*

$$K \subset \overline{\text{sym}((f_n)_{n \geq 0})}^{\| \cdot \|_{X(0,\infty)}}.$$

Proof. The implication (ii) $\Rightarrow$ (i) follows from Proposition 4.4.

(i) $\Rightarrow$ (ii). Let K be a relatively weakly compact subset of X and let $(g_n)_{n \geq 0} \subset K$ be a countable norm-dense subset in K (which exists due to Fact 3.3). Set $f_n = \tau_n g_n$ for all $n \geq 0$. It follows by Lemma 4.6 that the sequence $(f_n)_{n \geq 0}$ is weakly null.
    Finally,

$$K \subset \overline{(g_n)_{n \geq 0}}^{\| \cdot \|_X} \subset \overline{\text{sym}((f_n)_{n \geq 0})}^{\| \cdot \|_X}. \qquad \square$$

Remark 4.8. The main theorem in [18] is not applicable for symmetric function spaces $X(0, \infty)$ as there are no symmetric function spaces $X(0, \infty)$ (moreover, there is no noncommutative symmetric space affiliated with an atomless semifinite von Neumann algebra with a semifinite faithful normal trace) having the Schur property (see [27, Proposition 6.1]).

4.3. *Weak Grothendieck compactness principle for symmetric function spaces over a finite interval.* Recall that for any symmetric function space $X(0, 1)$,

$$L_\infty(0, 1) \subset X(0, 1) \subset L_1(0, 1)$$

as sets and the inclusion maps are of norm one; see [34, Definition 2.a.1].
    The following key result is an immediate consequence of [14, Proposition 2.1].

Proposition 4.9. *Let $X(0, 1)$ be a separable symmetric Banach function space with the Fatou property. A subset K of $X(0, 1)$ is relatively weakly compact if and only if so is $\text{sym}(K)$.*

Proof. Note that K is relatively weakly compact in $X(0, 1)$ if and only if so is $\mu(K)$ by [14, Proposition 2.1]. Since $\mu(\text{sym}(K)) = \mu(K)$, it follows from [14, Proposition 2.1] that K is relatively weakly compact in $X(0, 1)$ if and only if so is $\text{sym}(K)$. □

For function spaces on $(0, 1)$, we need a replacement for a translation operator τ_n, $n \geq 0$. For this purpose, equip the space $\bigtimes_{n=0}^{\infty} (0, 1)$ with a product measure and let $\gamma : (0, 1) \to \bigtimes_{n=0}^{\infty} (0, 1)$ be an isomorphism of measure spaces. For every $m \geq 0$ define the mappings $j_m : L_0(0, 1) \to L_0(\bigtimes_{n=0}^{\infty} (0, 1))$ by setting

$$(j_m(f))(t_0, t_1, \dots) = f(t_m), \quad f \in L_0(0, 1), \ (t_0, t_1, \dots) \in \bigtimes_{n=0}^{\infty} (0, 1).$$

Define the sequence $(i_m)_{m \geq 0}$ of independent copies (maps from $L_0(0, 1)$ to itself) by setting

$$i_m(f) = (j_m(f)) \circ \gamma, \quad f \in L_0(0, 1), \ m \geq 0.$$

Let $\mathfrak{B}_k$ be the Lebesgue σ-algebra on $\bigtimes_{n=0}^{k} (0, 1)$ and let m_k be the Lebesgue measure on $\mathfrak{B}_k$. For every $k \geq 0$, set

$$\mathcal{F}_k := \left\{ B \times \left(\bigtimes_{n=k+1}^{\infty} (0, 1) \right) : B \in \mathfrak{B}_k \right\}.$$

Set

$$m(A) = m_k(B), \quad A = B \times \left(\bigtimes_{n=k+1}^{\infty} (0, 1) \right), \ B \in \mathfrak{B}_k.$$

Let $\mathcal{F}$ be the Lebesgue completion of the algebra $\bigcup_{k \geq 0} \mathcal{F}_k$ with respect to the measure m. The sequence $(\mathcal{F}_k)_{k \geq 0}$ forms a filtration on an algebra $\mathcal{F}$. We denote the conditional expectation operator onto $\mathcal{F}_k$ by $\mathcal{E}_k$.

Lemma 4.10. *For every $h \in L_2(\bigtimes_{n=0}^{\infty} (0, 1))$, we have $\mathcal{E}_k h \to h$ in $L_2(\bigtimes_{n=0}^{\infty} (0, 1))$ as $k \to \infty$.*

Proof. Note that $(\mathcal{E}_l h)_{l \geq 0}$ is an $L_2(\bigtimes_{n=0}^{\infty} (0, 1))$-bounded martingale relative to $(\mathcal{F}_l)_{l \geq 0}$. By Doob's martingale convergence theorem there is $g \in L_2(\bigtimes_{n=0}^{\infty} (0, 1))$ such that $\mathcal{E}_l h \to g$ in $L_2(\bigtimes_{n=0}^{\infty} (0, 1))$ as $l \to \infty$. Thus,

$$\mathcal{E}_k(\mathcal{E}_l h) \to \mathcal{E}_k g, \quad l \to \infty.$$

Since

$$\mathcal{E}_k(\mathcal{E}_l h) = \mathcal{E}_k h, \quad l \geq k,$$

it follows that

$$\mathcal{E}_k h = \mathcal{E}_k g, \quad k \geq 0.$$

If $B \in \mathcal{F}_k$, then

$$\int_B (h - g) = \int (h - g)\chi_B = \int \mathcal{E}_k(h - g)\chi_B = 0.$$

Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\int_C |h - g| < \varepsilon \quad \text{for every } C \in \mathcal{F} \text{ with } m(C) < \delta.$$

Let now $A \in \mathcal{F}$. Choose $k \in \mathbb{N}$ and $B \in \mathcal{F}_k$ such that $m(A \triangle B) < \delta$. Thus,

$$\left| \int_A (h-g) \right| \leq \left| \int_B (h-g) \right| + \int_{A \triangle B} |h-g| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_A (h-g) = 0, \quad A \in \mathcal{F}.$$

Hence, $h - g = 0$ and, therefore, $\mathcal{E}_k h \to h$ in $L_2\left(\underset{n=0}{\overset{\infty}{\times}} (0,1) \right)$ as $k \to \infty$. $\qquad\square$

Recall that a subset K of a space $L_1(0,1)$ is called *equi-integrable* if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup \left\{ \int_E |f| \, dm : f \in K \right\} < \varepsilon \quad \text{whenever } E \subset (0,1) \text{ with } m(E) < \delta.$$

Equivalently, K is equi-integrable if and only if (see [3, Definitions 4.5.1 and 4.5.2]

$$\lim_{n \to \infty} \sup_{f \in K} \int_{\{|f|>n\}} |f| \, dm = 0.$$

A function f in $L_1(0,1)$ is said to be *mean zero* if $\int_0^1 f \, dm = 0$.

Lemma 4.11. *Let $(g_n)_{n \geq 0} \subset L_1(0,1)$ be an equi-integrable sequence of mean-zero elements and let $(n_j)_{j \geq 0}$ be a strictly increasing sequence in $\mathbb{Z}_{\geq 0}$. We have $i_{n_j}(g_{n_j}) \to 0$ weakly in $L_1(0,1)$ as $j \to \infty$.*

Proof. It is enough to show

$$\langle j_{n_j}(g_{n_j}), h \rangle \to 0, \quad j \to \infty,$$

for every $h \in L_\infty\left(\underset{n=0}{\overset{\infty}{\times}} (0,1) \right)$.

Suppose first the sequence $(g_n)_{n \geq 0}$ is bounded in $L_2(0,1)$. Fix $\varepsilon > 0$. By Lemma 4.10, there are $k \in \mathbb{Z}_{\geq 0}$ and $h' \in L_2\left(\underset{n=0}{\overset{\infty}{\times}} (0,1) \right)$ such that h' depends only on the first k coordinates and such that $\|h - h'\|_2 < \varepsilon$. We have

$$\langle j_{n_j}(g_{n_j}), h' \rangle = 0, \quad j > k.$$

Therefore,

$$\left| \langle j_{n_j}(g_{n_j}), h \rangle \right| \leq \left| \langle j_{n_j}(g_{n_j}), h' \rangle \right| + \left| \langle j_{n_j}(g_{n_j}), h' - h \rangle \right| = \left| \langle j_{n_j}(g_{n_j}), h' - h \rangle \right|$$
$$\leq \| j_{n_j}(g_{n_j}) \|_2 \| h' - h \|_2 \leq \varepsilon \sup_{n \geq 0} \|g_n\|_2, \quad j > k.$$

It follows that

$$\limsup_{j \to \infty} \left| \langle j_{n_j}(g_{n_j}), h \rangle \right| \leq \varepsilon \sup_{n \geq 0} \|g_n\|_2.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{j \to \infty} \left| \langle j_{n_j}(g_{n_j}), h \rangle \right| = 0.$$

Consider now the general case. Fix $\varepsilon > 0$ and, using equi-integrability of $(g_n)_{n \geq 0}$, choose $m \in \mathbb{Z}_{\geq 0}$ large enough so that

$$\sup_{n \geq 0} \| g_n \chi_{\{|g_n| > m\}} \|_1 \leq \varepsilon.$$

Set $g_n' := g_n \chi_{\{|g_n| \leq m\}}$ for every $n \geq 0$. Hence, a sequence $(g_n')_{n \geq 0}$ is bounded in $L_2(0, 1)$ such that $\| g_n - g_n' \|_1 \leq \varepsilon$ for $n \geq 0$. Therefore, we have

$$\left| \langle j_{n_j}(g_{n_j}), h \rangle \right| \leq \left| \langle j_{n_j}(g_{n_j}'), h \rangle \right| + \left| \langle j_{n_j}(g_{n_j} - g_{n_j}'), h \rangle \right|$$

$$\leq \left| \langle j_{n_j}(g_{n_j}'), h \rangle \right| + \| j_{n_j}(g_{n_j} - g_{n_j}') \|_1 \| h \|_\infty$$

$$\leq \left| \langle j_{n_j}(g_{n_j}'), h \rangle \right| + \varepsilon \| h \|_\infty.$$

Thus,

$$\limsup_{j \to \infty} \left| \langle j_{n_j}(g_{n_j}), h \rangle \right| \leq \limsup_{j \to \infty} \left| \langle j_{n_j}(g_{n_j}'), h \rangle \right| + \varepsilon \| h \|_\infty = \varepsilon \| h \|_\infty.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{j \to \infty} \left| \langle j_{n_j}(g_{n_j}), h \rangle \right| = 0,$$

thereby completing the proof. $\square$

The following is a well-known Dunford–Pettis criterion of weak compactness in $L_1(0, 1)$; see [6, Theorem 15, p. 76; 20; 36, Theorem 23, p. 20].

Proposition 4.12 (Dunford–Pettis). *For a subset $K \subset L_1(0, 1)$ the following conditions are equivalent*:

  (i) *K is relatively weakly compact.*

 (ii) *K is equi-integrable.*

Lemma 4.13. *Let $X(0, 1)$ be a separable symmetric Banach function space with the Fatou property. If $(g_n)_{n \geq 0} \subset X(0, 1)$ is a relatively weakly compact sequence of mean-zero elements, then $(f_n)_{n \geq 0} := (i_n(g_n))_{n \geq 0}$ is weakly null.*

Proof. Assume, on the contrary, $(i_n(g_n))_{n \geq 0}$ is not weakly null. Hence, $i_n(g_n) \nrightarrow 0$ in $\sigma(X, X^\times)$-topology (since $X^* = X^\times$ due to separability of X) as $n \to \infty$. Hence, there exists $F \in X^\times$ and a strictly increasing sequence $(n_j)_{j \geq 0}$ such that

(4-3) $$\langle F, i_{n_j}(g_{n_j}) \rangle \to c \neq 0, \quad j \to \infty.$$

Set $K_0 = \{g_n : n \geq 0\}$ and $K_1 = \text{sym}(K_0)$. By assumption, the set K_0 is relatively weakly compact. Hence, by Proposition 4.9, so is $\mu(K_0)$. Since $\mu(K_0) = \mu(K_1)$, it

follows that $\mu(K_1)$ is also relatively weakly compact. By Proposition 4.9, so is K_1. Since $\mu(i_{n_j}(g_{n_j})) = \mu(g_{n_j})$, it follows that

$$(i_{n_j}(g_{n_j}))_{j \geq 0} \subset \mathrm{sym}(K_0) = K_1.$$

It follows that the sequence $(i_{n_j}(g_{n_j}))_{j \geq 0}$ is relatively weakly compact. Hence, there exist a strictly increasing sequence $(j_k)_{k \geq 0}$ and $g \in X$ such that

$$i_{n_{j_k}}(g_{n_{j_k}}) \to g$$

weakly as $k \to \infty$. In particular,

$$i_{n_{j_k}}(g_{n_{j_k}}) \to g, \quad k \to \infty,$$

in $\sigma(L_1, L_\infty)$-topology. As the sequence $(g_n)_{n \geq 0}$ is relatively weakly compact in X, it is also relatively weakly compact in $L_1(0, 1)$. By the Dunford–Pettis criterion (Proposition 4.12), it is equi-integrable in $L_1(0, 1)$. It follows from Lemma 4.11 that $g = 0$. Therefore,

$$i_{n_{j_k}}(g_{n_{j_k}}) \to 0$$

weakly in X as $k \to \infty$, which contradicts (4-3). □

Theorem 4.14. *Let $X(0, 1)$ be a separable symmetric Banach function space with the Fatou property. For $K \subset X(0, 1)$, the following conditions are equivalent:*

(i) *K is relatively weakly compact.*

(ii) *There exists a weakly null sequence $(f_n)_{n \geq 0} \subset X(0, 1)$ such that*

$$K \subset \overline{\mathrm{sym}((f_n)_{n \geq 0})}^{\|\cdot\|_{X(0,1)}}.$$

Proof. The implication (ii) $\Rightarrow$ (i) follows immediately from Proposition 4.9.

Let us now prove the implication (i) $\Rightarrow$ (ii). Let K be a relatively weakly compact subset of X and let $(h_n)_{n \geq 0} \subset K$ be a countable norm-dense subset in K (which exists due to Fact 3.3). For every $n \geq 0$ set

$$g_n(t) = \begin{cases} h_n(2t), & t \in \left(0, \frac{1}{2}\right), \\ -h_n(2t - 1), & t \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Note that the sequence $(g_n)_{n \geq 0} \subset X$ consists of mean-zero elements. Set $f_n = i_n(g_n)$ for all $n \geq 0$. It follows by Lemma 4.13 that the sequence $(f_n)_{n \geq 0}$ is weakly null.

Finally,

$$K \subset \overline{(h_n)_{n \geq 0}}^{\|\cdot\|_X} \subset \overline{\mathrm{sym}((f_n)_{n \geq 0})}^{\|\cdot\|_X}.$$ □

4.4. *Weak Grothendieck compactness principle for noncommutative symmetric spaces associated with a semifinite von Neumann algebra.* Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. We say that $x \in S(\mathcal{M}, \tau)$ is submajorized by $y \in (L_1 + L_\infty)(\mathcal{M}, \tau)$ in the sense of Hardy–Littlewood–Pólya (written as $x \prec\prec y$) if

$$\int_0^t \mu(s, x)\, ds \leq \int_0^t \mu(s, y)\, ds, \quad 0 \leq t \leq \tau(\mathbb{1}).$$

A symmetric space $X(\mathcal{M}, \tau)$ is said to be fully symmetric if the norm is monotone with respect to the Hardy–Littlewood–Polya majorization.

The following key result is an immediate consequence of [14, Theorem 5.4]. For the case of compact operators in $B(\mathcal{H})$, see [24, Theorem 12].

Theorem 4.15. *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $X(0, \tau(\mathbb{1}))$ be a separable symmetric Banach function space with the Fatou property. If $\tau(\mathbb{1}) = \infty$, suppose, in addition, that $X(0, \tau(\mathbb{1})) \not\subset L_1(0, \tau(\mathbb{1}))$.*

If $K \subset X(\mathcal{M}, \tau)$, then the following conditions are equivalent:

(i) *K is relatively weakly compact in $X(\mathcal{M}, \tau)$.*

(ii) *$\mathrm{sym}(K)$ is relatively weakly compact in $X(\mathcal{M}, \tau)$.*

(iii) *$\mu(K)$ is relatively weakly compact in $X(0, \tau(\mathbb{1}))$.*

Proof. Set $E(0, \tau(\mathbb{1})) = X^\times(0, \tau(\mathbb{1}))$. It follows that $E(0, \tau(\mathbb{1}))$ has the Fatou property (see [31, Chapter II, Theorem 4.10]) and is, therefore, fully symmetric; see [10, Corollary 33]. The condition $X(0, \tau(\mathbb{1})) \not\subset L_1(0, \tau(\mathbb{1}))$ (for $\tau(\mathbb{1}) = \infty$) guarantees that $E(\mathcal{M}, \tau)$ consists of τ-compact operators. Hence, $E(0, \tau(\mathbb{1}))$ satisfies the conditions in [14, Theorem 5.4]. By that theorem, K is relatively $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$-compact if and only if $\mu(K)$ is relatively $\sigma(E^\times(0, \tau(\mathbb{1})), E(0, \tau(\mathbb{1})))$-compact. Since $X(0, \tau(\mathbb{1}))$ has the Fatou property, it follows that $X = X^{\times\times} = E^\times$ (see [30, Theorem 6.1.7]) in both commutative and noncommutative settings. Therefore, K is relatively $\sigma(X(\mathcal{M}, \tau), X^\times(\mathcal{M}, \tau))$-compact if and only if $\mu(K)$ is relatively $\sigma(X(0, \tau(\mathbb{1})), X^\times(0, \tau(\mathbb{1})))$-compact. Since $X(0, \tau(\mathbb{1}))$ is separable, it follows that $X^\times = X^*$ in both commutative and noncommutative settings. This completes the proof.   □

The following proposition shows existence of a conditional expectation in a noncommutative setting (see [43]).

Proposition 4.16 (Umegaki). *Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra which is proper in the sense that $\tau|_{\mathcal{N}}$ is semifinite. For each $x \in (L_1 + L_\infty)(\mathcal{M}, \tau)$, there exists a uniquely determined element $\mathcal{E}(x) = \mathcal{E}_{\mathcal{N}}(x) \in (L_1 + L_\infty)(\mathcal{N}, \tau)$, called the*

conditional expectation with respect to $\mathcal{N}$, such that

$$\tau(xy) = \tau(\mathcal{E}(x)y) \quad \textit{for all } y \in (L_1 \cap L_\infty)(\mathcal{N}, \tau).$$

The conditional expectation operator $\mathcal{E}$ has numerous useful properties; see, e.g., [8, Proposition 2.1; 43]. Among those we need

(4-4) $\mathcal{E}(x) \prec\prec x \quad$ for all $x \in (L_1 + L_\infty)(\mathcal{M}, \tau).$

The following lemma follows immediately from [21, Lemma 9].

Lemma 4.17. *Let $(\mathcal{M}, \tau)$ be a finite atomless von Neumann algebra equipped with a normal faithful finite trace τ. For every $0 \leq x \in S(\mathcal{M}, \tau)$, there exists a normal unital trace-preserving $*$-monomorphism $i : L_\infty(0, \tau(\mathbb{1})) \to \mathcal{M}$ (which extends by normality to a $*$-monomorphism $i : S(0, \tau(\mathbb{1})) \to S(\mathcal{M}, \tau)$) such that $i(\mu(x)) = x$.*

Proof. Let $0 \leq x \in \mathcal{M}$. By [21, Lemma 9] there is an increasing net of projections $\{p_s\}_{0 \leq s \leq \tau(\mathbb{1})}$ in $\mathcal{M}$ such that

$$x = \int_0^{\tau(\mathbb{1})} \mu(s, x) \, dp_s.$$

Define $i : L_\infty(0, \tau(\mathbb{1})) \to \mathcal{M}$ by

$$i(h) = \int_0^{\tau(\mathbb{1})} h(s) \, dp_s, \quad h \in L_\infty(0, \tau(\mathbb{1})).$$

It is a normal unital trace-preserving $*$-monomorphism such that $i(\mu(x)) = x$.

Now since i is normal, it follows that, given $0 \leq x \in S(\mathcal{M}, \tau)$, it may be extended to a trace-preserving $*$-monomorphism $i : S(0, \tau(\mathbb{1})) \to S(\mathcal{M}, \tau)$ such that $i(\mu(x)) = x$. □

We note that analogous results to the following lemma are available in [4, Lemma 4.1; 12, Theorem 3.5].

Lemma 4.18. *Let $(\mathcal{M}, \tau)$ be an atomless semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $0 \leq x \in S(\mathcal{M}, \tau)$ be τ-compact. There exists a (not necessarily unital) normal trace-preserving $*$-monomorphism*

$$i : L_\infty\big(0, \tau(s(x))\big) \to \mathcal{M},$$

(which extends by normality to a $*$-monomorphism $i : S\big(0, \tau(s(x)) \to S(\mathcal{M}, \tau)\big)$ such that

$$i(\mu(x)) = x, \quad i(1) = s(x).$$

In addition, $\tau|_{i(L_\infty(0, \tau(s(x))))}$ is semifinite.

Proof. Since x is τ-compact, it follows that, for each $k \geq 1$,

$$e_k := e^x([k, k+1)) \vee e^x\left(\left[\frac{1}{k+1}, \frac{1}{k}\right)\right)$$

are τ-finite projections. Set

$$s_k = \sum_{l=1}^{k} \tau(e_l), \quad k \geq 1, \quad \text{and} \quad s_0 = 0.$$

By Lemma 4.17, for every $k \geq 1$ there exists a trace-preserving normal unital $*$-isomorphism $\pi_k : L_\infty(s_{k-1}, s_k) \to e_k \mathcal{M} e_k$ such that

$$\pi_k(\mu(x)\chi_{(s_{k-1}, s_k)}) = x e_k, \quad \pi_k(\chi_{(s_{k-1}, s_k)}) = e_k.$$

Define

$$i(y) = \sum_{k \geq 1} \pi_k(y\chi_{(s_{k-1}, s_k)}), \quad y \in L_\infty\big(0, \tau(s(x))\big).$$

Hence, $i : L_\infty\big(0, \tau(s(x))\big) \to \mathcal{M}$ is a trace-preserving $*$-isomorphism.

Clearly, $\sum_{l=1}^{k} e_l \in i\big(L_\infty\big(0, \tau(s(x))\big)\big)$ for every $k \geq 1$. Since each e_l is τ-finite, it follows that $\sum_{l=1}^{k} e_l$ is τ-finite for every $k \geq 1$. Since $\sum_{l=1}^{k} e_l \uparrow s(x)$ as $k \to \infty$, it follows that $\tau|_{i(L_\infty(0, \tau(s(x))))}$ is semifinite. $\qquad\square$

Now we are ready to prove the main result of this paper.

Theorem 4.19. *Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ, which is either atomless or purely atomic* (*with all atoms having the same trace*). *Let $X(0, \tau(\mathbb{1}))$ be a separable symmetric Banach function* (*or sequence space if τ is purely atomic*) *space with the Fatou property. If $\tau(\mathbb{1}) = \infty$, suppose, in addition, that $X(0, \tau(\mathbb{1})) \not\subset L_1(0, \tau(\mathbb{1}))$.*

For a subset $K \subset X(\mathcal{M}, \tau)$, the following conditions are equivalent:

(i) *K is relatively weakly compact in $X(\mathcal{M}, \tau)$.*

(ii) *There exists a weakly null sequence $(x_n)_{n \geq 0}$ in $X(\mathcal{M}, \tau)$ such that*

$$K \subset \overline{\mathrm{sym}((x_n)_{n \geq 0})}^{\|\cdot\|_{X(\mathcal{M}, \tau)}}.$$

Proof. If $\mathcal{M}$ is purely atomic, then we assume, in addition, that the trace is infinite. Otherwise, we are in the finite-dimensional situation and the assertion is trivial.

(ii) $\Rightarrow$ (i). Follows from Theorem 4.15.

(i) $\Rightarrow$ (ii). Assume K is relatively weakly compact in $X(\mathcal{M}, \tau)$. By Theorem 4.15, so is $\mu(K)$ in $X(0, \tau(\mathbb{1}))$. By Theorem 4.7 (for atomless infinite trace) or by Theorem 4.14 (for atomless finite trace) or by Theorem 4.1 (for purely atomic infinite trace), there exists a weakly null sequence $(y_n)_{n \geq 0}$ in $X(0, \tau(\mathbb{1}))$ such that

$$\mu(K) \subset \overline{\mathrm{sym}_{X(0, \tau(\mathbb{1}))}((y_n)_{n \geq 0})}^{\|\cdot\|_{X(0, \tau(\mathbb{1}))}}.$$

Fix a trace-preserving unital $*$-monomorphism $i : L_\infty(0, \tau(\mathbb{1})) \to \mathcal{M}$ (respectively, $i : \ell_\infty \to \mathcal{M}$ for purely atomic trace). Existence of such i is guaranteed by Lemma 4.18 in the atomless case and is trivial in the purely atomic case. Since i is trace-preserving (and, therefore, normal), it follows that we can extend (see [39, Poposition 3.3]) $i : (L_1 + L_\infty)(0, \tau(\mathbb{1})) \to (L_1 + L_\infty)(\mathcal{M}, \tau)$ (the extension is not needed in the purely atomic case) preserving the trace. Hence, it follows that

$$\mu(i(y)) = \mu(y), \quad y \in (L_1 + L_\infty)(0, \tau(\mathbb{1})).$$

In particular, $i : X(0, \tau(\mathbb{1})) \to X(\mathcal{M}, \tau)$ is a trace-preserving $*$-monomorphism.

We claim that the sequence $(i(y_n))_{n \geq 0}$ is weakly null in $X(\mathcal{M}, \tau)$. Indeed, take $z \in X^\times(\mathcal{M}, \tau)$ and let $\mathcal{E}(z)$ be the conditional expectation of z onto the $*$-subalgebra $i(L_\infty(0, \tau(\mathbb{1})))$ (respectively, $i(\ell_\infty)$). Note here that $X(0, \tau(\mathbb{1}))$ has the Fatou property and is, therefore, fully symmetric. Since $X(0, \tau(\mathbb{1}))$ is fully symmetric, it follows from (4-4) that $X(\mathcal{M}, \tau)$ is closed with respect to taking conditional expectations. By Proposition 4.16, we have

$$\tau(i(y_n) \cdot z) = \tau(i(y_n) \cdot \mathcal{E}(z)) = \int_0^{\tau(\mathbb{1})} (y_n \cdot i^{-1}(\mathcal{E}(z))) \ \big(\text{or } \Sigma\big(y_n \cdot i^{-1}(\mathcal{E}(z))\big)\big) \to 0$$

as $n \to \infty$. Thus, $i(y_n) \to 0$ in $\sigma(X(\mathcal{M}, \tau), X^\times(\mathcal{M}, \tau))$-topology. Since X is separable, it follows that $X^*(\mathcal{M}, \tau) = X^\times(\mathcal{M}, \tau)$. Hence, $(i(y_n))_{n \geq 0}$ is weakly null in $X(\mathcal{M}, \tau)$.

Set $x_n = i(y_n)$ for every $n \geq 0$. It remains to show that

$$K \subset \overline{\text{sym}_{X(\mathcal{M}, \tau)}((x_n)_{n \geq 0})}^{\|\cdot\|_{X(\mathcal{M}, \tau)}}.$$

To see the latter inclusion, let $z \in K$ be arbitrary. Since $\mu(z) \in \mu(K)$, it follows that there is a sequence (dependent on z) $(z_k)_{k \geq 0} \subset \text{sym}_{X(0, \tau(\mathbb{1}))}((y_n)_{n \geq 0})$ such that $z_k \to \mu(z)$ as $k \to \infty$ in $X(0, \tau(\mathbb{1}))$.

Since $X(0, \tau(\mathbb{1}))$ is separable, it follows that $\mu(s, z) \to 0$ as $s \to \infty$, that is, $z \in X(\mathcal{M}, \tau)$ is τ-compact. Fix a trace-preserving $*$-monomorphism $i_z : L_\infty(0, \tau(\mathbb{1})) \to \mathcal{M}$ (respectively, $i_z : \ell_\infty \to \mathcal{M}$) such that

$$i_z(\mu(z)) = |z|, \quad i_z(1) = s(|z|).$$

Existence of i_z is guaranteed by Lemma 4.18 in the atomless case and by the Schmidt decomposition in the purely atomic case.

Let $z = u|z|$ be a polar decomposition of z. Set

$$w_k = u \cdot i_z(z_k), \quad k \geq 0.$$

Note that

$$w_k^* w_k = i_z(\overline{z_k}) \cdot u^* u \cdot i_z(z_k) = i_z(\overline{z_k}) \cdot i_z(1) \cdot i_z(z_k) = i_z(\overline{z_k} z_k), \quad k \geq 0.$$

Hence,

$$\mu(w_k) = \mu^{\frac{1}{2}}(w_k^* w_k) = \mu^{\frac{1}{2}}(i_z(\overline{z_k} z_k)) = \mu^{\frac{1}{2}}(\overline{z_k} z_k) = \mu(z_k), \quad k \geq 0.$$

Since $z_k \in \text{sym}_X((y_n)_{n\geq 0})$ for all $k \geq 0$, it follows that

$$w_k \in \text{sym}_{X(\mathcal{M},\tau)}((x_n)_{n\geq 0}), \quad k \geq 0.$$

On the other hand,

$$\|z - w_k\|_{X((\mathcal{M},\tau))} = \left\| u \cdot i_z(\mu(z)) - u \cdot i_z(z_k) \right\|_{X((\mathcal{M},\tau))}$$
$$\leq \left\| i_z(\mu(z)) - i_z(z_k) \right\|_{X((\mathcal{M},\tau))} = \|\mu(z) - z_k\|_{X(0,\tau(\mathbb{1}))} \to 0$$

as $k \to \infty$. That is,

$$z \in \overline{\text{sym}_{X(\mathcal{M},\tau)}((x_n)_{n\geq 0})}^{\|\cdot\|_{X(\mathcal{M},\tau)}}. \qquad \square$$

5. Concluding remarks

We discuss whether the condition $X(0, \infty) \not\subset L_1(0, \infty)$ in Theorem 4.19 (and in Theorem 4.7; and the condition $X \neq \ell_1$ in Theorem 4.1) is necessary (and sufficient) for implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii). First, we show that this condition on X is necessary for the implication (ii) $\Rightarrow$ (i) in Theorems 4.1 (provided X has the Fatou property), 4.7 and 4.19.

Remark 5.1. Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite infinite trace τ, which is either atomless or purely atomic (with all atoms having the same trace). Suppose $X(0, \infty)$ is a symmetric function space such that $X(0, \infty) \subset L_1(0, \infty)$. There exists a weakly null sequence $(x_n)_{n\geq 0} \subset X(\mathcal{M}, \tau)$ such that $\text{sym}((x_n)_{n\geq 0})$ is not relatively weakly compact in $X(\mathcal{M}, \tau)$.

Proof. Let $(p_n)_{n\geq 0}$ be a sequence of pairwise orthogonal projections in $\mathcal{M}$ such that $\sum_{n\geq 0} p_n = 1$ and such that $\tau(p_n) = 1$ for every $n \geq 0$. Let $\mathcal{N}$ be an abelian von Neumann subalgebra in $\mathcal{M}$ generated by $(p_n)_{n\geq 0}$.

We claim that the sequence $(p_n)_{n\geq 0}$ is not relatively weakly compact in $X(\mathcal{M}, \tau)$. Indeed, otherwise there exists a strictly increasing sequence $(n_k)_{k\geq 0}$ such that $p_{n_k} \to a \in X(\mathcal{M}, \tau)$ weakly as $k \to \infty$. Let $a = u|a|$ be a polar decomposition and let $q \in \mathcal{M}$ be a τ-finite projection. We have

$$\tau(p_{n_k} \cdot qu^*) \to \tau(a \cdot qu^*) = \tau(|a| \cdot q), \quad k \to \infty.$$

On the other hand, $\mathcal{E}_{\mathcal{N}}(qu^*) \in L_1(\mathcal{N}, \tau)$. Thus,

$$\tau(p_{n_k} \cdot qu^*) = \tau(p_{n_k} \cdot \mathcal{E}_{\mathcal{N}}(qu^*)) \to 0, \quad k \to \infty.$$

Combining these equalities, we obtain

$$\tau(|a| \cdot q) = 0$$

for every τ-finite projection $q \in \mathcal{M}$. Thus, $|a| = 0$. It follows that $p_{n_k} \to 0 \in X(\mathcal{M}, \tau)$ weakly as $k \to \infty$. However, $1 \in X^{\times}(\mathcal{M}, \tau)$ and, therefore,

$$1 = \tau(p_{n_k} \cdot 1) \to \tau(0 \cdot 1), \quad k \to \infty.$$

This contradiction shows that the sequence $(p_n)_{n \geq 0}$ is not relatively weakly compact.

Set

$$x_0 = p_0, \qquad x_n = 0, \quad n \geq 1.$$

Clearly, the sequence $(x_n)_{n \geq 0}$ is weakly null. Since

$$(p_n)_{n \geq 0} \subset \mathrm{sym}(p_0) \subset \mathrm{sym}((x_n)_{n \geq 0}),$$

it follows that $\mathrm{sym}((x_n)_{n \geq 0})$ is not relatively weakly compact. $\qquad \square$

Now we show that the condition $X \neq \ell_1$ is also necessary for the implication (i) $\Rightarrow$ (ii) in Theorem 4.1.

Remark 5.2. There exists a relatively weakly compact set $K \subset \ell_1$ such that for every weakly null sequence $(x_n)_{n \geq 0} \subset \ell_1$ we have $K \not\subset \overline{\mathrm{sym}((x_n)_{n \geq 0})}^{\|\cdot\|_1}$.

Proof. Set

$$K := \left(e_0 + \frac{1}{m} e_m \right)_{m \geq 1}.$$

Note that K is relatively weakly compact.

By the way of contradiction, let $(x_n)_{n \geq 0} \subset \ell_1$ be a weakly null sequence such that $K \subset \overline{\mathrm{sym}((x_n)_{n \geq 0})}^{\|\cdot\|_1}$. It follows by the Schur property of ℓ_1 that $x_n \to 0$ as $n \to \infty$ in ℓ_1. Fix $N \in \mathbb{N}$ such that $\|x_n\|_1 \leq 1$ for $n > N$.

For a given $m \in \mathbb{N}$, there exists a sequence $(n_k)_{k \geq 0} \subset \mathbb{Z}_+$ and a sequence $(y_k)_{k \geq 0} \subset \ell_1$ such that $\mu(y_k) = \mu(x_{n_k})$ for every $k \geq 0$ and such that $y_k \to e_0 + \frac{1}{m} e_m$ as $k \to \infty$ in ℓ_1.

Set $A = \{k \geq 0 : n_k > N\}$. If A is infinite, then

$$\lim_{\substack{k \to \infty \\ k \in A}} y_k = e_0 + \frac{1}{m} e_m.$$

Thus,

$$\lim_{\substack{k \to \infty \\ k \in A}} \|y_k\|_1 = \left\| e_0 + \frac{1}{m} e_m \right\|_1 = 1 + \frac{1}{m}.$$

On the other hand, $\|y_k\|_1 = \|x_{n_k}\|_1 \leq 1$ for all $k \in A$. Thus,

$$\limsup_{\substack{k \to \infty \\ k \in A}} \|y_k\|_1 \leq 1.$$

Since the last two equations are in contradiction, it follows that A is finite. Without loss of generality, assume that A is empty.

Hence, there is a sequence $(n_k)_{k \geq 0} \subset \{0, \ldots, N\}$ and a sequence $(y_k)_{k \geq 0} \subset \ell_1$ such that $\mu(y_k) = \mu(x_{n_k})$ for every $k \geq 0$ and such that $y_k \to e_0 + \frac{1}{m} e_m$ as $k \to \infty$ in ℓ_1.

Passing to a further subsequence, we may assume that there exists $n \in \{0, \ldots, N\}$ and a sequence $(y_k)_{k \geq 0} \subset \ell_1$ such that $\mu(y_k) = \mu(x_n)$ for every $k \geq 0$ and such that $y_k \to e_0 + \frac{1}{m} e_m$ as $k \to \infty$ in ℓ_1. Thus,

$$\mu(y_k) \to \mu\left(e_0 + \frac{1}{m} e_m\right) = e_0 + \frac{1}{m} e_1, \quad k \to \infty,$$

in ℓ_1. Since $\mu(y_k) = \mu(x_n)$ for every $k \geq 0$, it follows that $\mu(x_n) = e_0 + \frac{1}{m} e_1$.

Finally, for a given $m \in \mathbb{N}$, there exists $n \in \{0, \ldots, N\}$ such that $\mu(x_n) = e_0 + \frac{1}{m} e_1$. Hence, the set $\{e_0 + \frac{1}{m} e_1\}_{m \geq 1}$ is finite. The latter is obviously false. Hence, our assumption was incorrect and, therefore, $K \not\subset \overline{\text{sym}((x_n)_{n \geq 0})}^{\|\cdot\|_1}$ for every weakly null sequence $(x_n)_{n \geq 0} \subset \ell_1$. $\qquad \square$

Recall that a Banach space X is said to be a *Grothendieck space* [5] whenever weak* and weak convergence of sequences in X^* coincide. Ironically, the weak Grothendieck compactness principle holds for symmetric sequence spaces with the Fatou property if and only if X is not a Grothendieck space (see [27, Table 7]).

The following remark is standard in the commutative setting; for the proof of the noncommutative case, see, e.g., [26, Remark 4.4] and also [41, Lemma 8]. Recall that a space of τ-compact operators is defined by

$$S_0(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \mu(\infty, x) = 0\}.$$

Remark 5.3. Let $\mathcal{M}$ be an arbitrary von Neumann algebra equipped with an infinite semifinite faithful normal trace τ. Let $X(\mathcal{M}, \tau)$ be a noncommutative symmetric space. The inclusion $X(\mathcal{M}, \tau)^\times \subset S_0(\mathcal{M}, \tau)$ holds if and only if $X(0, \infty) \not\subset L_1(0, \infty)$.

Finally, we provide a quick answer in the affirmative to a question posed in [19, Remark 1].

Remark 5.4. A weak closure of a rearrangement invariant set A (i.e., $\text{sym}(A) = A$) in a symmetric sequence, function or operator space X is again rearrangement invariant.

Proof. Let A be a rearrangement invariant subset of a symmetric space X, that is, for all $y \in X$ such that $\mu(y) = \mu(x)$ for some $x \in A$ it follows that $y \in A$. Let x be in the weak closure of A. Hence, there is a net $(x_i)_{i \in I} \subset A$ weakly converging to x. For $y \in X$ with $\mu(x) = \mu(y)$, write $y = uxv$ for some partial isometries u and v. Set $y_i := u x_i v$ for all $i \in I$. Note that $y_i \in A$ for all $i \in I$. Therefore, $y_i = u x_i v \to_i uxv = y$ weakly, that is, y is in the weak closure of A. $\qquad \square$

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