Pacific Journal of Mathematics

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Volume 333 No. 1 November 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 1, 2024
https://doi.org/10.2140/pjm.2024.333.155

GEOMETRY AND DYNAMICS ON SUBLINEARLY MORSE BOUNDARIES OF CAT(0) GROUPS

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Given a sublinear function κ, κ-Morse boundaries $\partial_\kappa X$ of proper CAT(0) spaces are introduced by Qing, Rafi and Tiozzo (2024). It is a topological space that consists of a equivalence class of quasigeodesic rays and it is quasiisometrically invariant and metrizable. We study the sublinearly Morse boundaries with the assumption that there is a proper cocompact action of a group G on the CAT(0) space in question. We show that G acts minimally on $\partial_\kappa G$ and that contracting elements of G induces a weak north-south dynamic on $\partial_\kappa G$. Also, we show that a homeomorphism $f : \partial_\kappa G \to \partial_\kappa G'$ comes from a quasiisometry if and only if f is successively quasimöbius and stable. Lastly, we characterize exactly when the sublinearly Morse boundary of a CAT(0) space is compact.

1. Introduction

Much of the geometric group theory originates from the studying of hyperbolic groups and hyperbolic spaces. Hyperbolic groups have solvable word problem and their Gromov boundaries enjoy strong dynamical properties. One fundamental technique in the study of hyperbolic groups is to study the Gromov boundaries of these groups. Gromov took the collection of all infinite geodesic rays (up to fellow traveling) in the associated Cayley graph, equipped this set with cone topology, and defined the space to be the boundary ∂G of the hyperbolic group G. The boundary ∂G is independent of the choice of a generating set and has rich geometric, topological, and algebraic structures (see, for example, the survey in [Kapovich and Benakli 2002]).

If we view Gromov hyperbolic spaces as coarsely negatively curved, then the notion of CAT(0) include spaces with both local and global nonpositive curvature. Accordingly the extension of the boundary theory to CAT(0) spaces and groups has also been developing in recent decades. In this setting, the space of all geodesic rays together with the cone topology is called the visual boundary (denoted by $\partial_v X$). It is shown by Croke and Kleiner [2000] that the visual boundary of a CAT(0) space

MSC2020: 20F65.

Keywords: minimality, sublinearly Morse boundary, compact, quasimöbius.

is not in general a quasiisometry invariant. It is shown in [Qing 2016] that, in the Croke–Kleiner example, failure to obtain quasiisometry invariance can come from geodesic rays that spend linear amount of time (with respect to total time traveled) in each product region.

Hence, one can consider geodesic rays that spend a sublinear amount of time in each product region. Qing and Rafi [2022] introduce the sublinearly Morse boundary $\partial_\kappa X$ of a CAT(0) metric space X and show that $\partial_\kappa X$ is quasiisometry invariant and metrizable. Qing and Tiozzo [2022] show that, for a right-angled Artin group G, $\partial_\kappa G$ is a model for Poisson boundaries associated to a random walk (G, μ). Intuitively, a (quasi)geodesic ray is sublinearly Morse if it spends a sublinear amount of time in each maximal product region, with respect to total time traveled when it enters that product region. Moreover, it is shown in [Gekhtman et al. 2022] that for every CAT(0) group with a rank-one element, there exists a κ such that $\partial_\kappa G$ can be identified with the Poisson boundaries of the group. Gekhtman et al. [2022] also show that the sublinearly Morse directions in the visual boundary of a rank-one CAT(0) space with a geometric group action are generic with respect to Patterson–Sullivan measures. Most recently, it is shown that much like the Gromov boundaries, sublinearly Morse boundaries are sublinearly bi-Lipschitz equivalence invariant [Pallier and Qing 2024], providing a new way to tell when two groups are not sublinearly bi-Lipschitz equivalence. These are evidences that the sublinearly Morse directions behave similar to directions in hyperbolic spaces. In this paper, we continue to contribute to this comparison and focus on the dynamical property of the group action on κ-Morse boundaries. Much of the work in this paper is inspired by the methods in [Charney and Sultan 2015], [Murray 2019] and [Cashen and Mackay 2019]. In more general proper geodesic spaces, sublinearly Morse boundaries have been developed and studied, for instance, in [Durham and Zalloum 2022; Murray et al. 2022; Nguyen and Qing 2024; Pallier and Qing 2024; Qing et al. 2024; Qing and Yang 2024]. Parts of Theorems A and B have been recently studied in the setting of proper metric spaces in [Garcia et al. 2024].

Minimality of the group action. A group is said to act *minimally* on a topological space if every orbit is a dense subset of the space. We show that this property is enjoyed by the κ boundaries. In contrast with the identifications with Poisson boundaries in various settings, the minimality result evidence the fact that the boundary is not too large in excess of the orbit of a point under the group action.

Theorem A (Theorem 3.3). *Suppose G is a group that acts properly discontinuously, cocompactly and by isometries metrically on a* CAT(0) *space X. Then G acts minimally on $\partial_\kappa G$.*

Based on this result, we illustrate that for a subset of the group elements, their actions induces the following form of north-south dynamics on the boundary.

Theorem B (Theorem 3.6). *Suppose G is a group that acts properly discontinuously, cocompactly and by isometries metrically on a* CAT(0) *space X. Let $g \in G$ be a contracting element. For every open set V containing g^∞ and every compact set $C \in (\partial_\kappa G \setminus [g^\infty])$, there exists an N such that for all $n \geq N$, we have $g^n C \subset V$.*

Compact-type κ-boundaries. In the examples shown in [Qing and Rafi 2022], the boundaries are not compact. We show that when X is a proper hyperbolic space, κ-boundary is homeomorphic to the associated Gromov boundary. In fact, we show that this is exactly when a cocompact CAT(0) space X has compact sublinear boundaries. On the other hand, examples of CAT(0) space without a cocompact group action, whose sublinearly boundaries are compact can be constructed easily. However, it remains open to find a CAT(0) space X with noncompact sublinear boundary where $\partial_\kappa X$ is a perfect space. When X is a hyperbolic CAT(0) space, then the κ-boundary agrees with the Gromov boundary.

Theorem C (Theorem 4.5). *Suppose a group G acts properly discontinuously, cocompactly and by isometries metrically on a proper* CAT(0) *space X such that $\partial_\kappa X \neq \varnothing$, then the following are equivalent*:

(1) *Every geodesic ray in X is κ-contracting.*

(2) *Every geodesic ray in X is strongly contracting.*

(3) *$\partial_\kappa X$ is compact.*

(4) *The space X is hyperbolic.*

Corollary D (Corollary 4.4). *If X is a proper* CAT(0) *hyperbolic space then $\partial_\kappa X \simeq \partial X$.*

Rigidity. Paulin [1996] gives the following characterization: if $f : \partial X \to \partial Y$ is a homeomorphism between the boundaries of two proper, cocompact hyperbolic spaces, then the following are equivalent:

(1) f is induced by a quasiisometry $h : X \to Y$ and (2) f is quasimöbius.

Quasimöbius maps are maps such that changes in the cross ratio are controlled by a continuous function. It was shown in [Charney et al. 2019] that a homeomorphism on the Morse boundary with inverse limit topology is induced by a quasiisometry of the space if the homeomorphism is successively quasimöbius stable. Here we give a similar characterization for sublinearly Morse boundaries. We use the notion of *successively quasimöbius* discussed in [Qing and Rafi 2022], which is a 1-parameter family of quasimöbius maps on $\partial_\kappa X$.

Theorem E (Theorem 5.3). *Let X, Y be proper cocompact* CAT(0) *spaces with at least* 3 *points in their sublinear boundaries. A homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ is induced by a quasiisometry $h : X \to Y$ if and only if f is stable and successively quasimöbius.*

2. Preliminaries

Quasiisometry and quasiisometric embeddings.

Definition 2.1 (quasiisometric embedding). Let (X, d_X) and (Y, d_Y) be metric spaces. For constants $\mathsf{k} \geq 1$ and $\mathsf{K} \geq 0$, we say a map $f : X \to Y$ is a (k, K)-*quasiisometric embedding* if, for all points $x_1, x_2 \in X$

$$\frac{1}{\mathsf{k}} d_X(x_1, x_2) - \mathsf{K} \leq d_Y(f(x_1), f(x_2)) \leq \mathsf{k}\, d_X(x_1, x_2) + \mathsf{K}.$$

If, in addition, every point in Y lies in the K-neighborhood of the image of f, then f is called a (k, K)-quasiisometry. When such a map exists, X and Y are said to be *quasiisometric*.

A quasiisometric embedding $f^{-1} : Y \to X$ is called a *quasiinverse* of f if for every $x \in X$, $d_X(x, f^{-1}f(x))$ is uniformly bounded above. In fact, after replacing k and K with larger constants, we assume that f^{-1} is also a (k, K)-quasiisometric embedding:

$$\forall x \in X \quad d_X(x, f^{-1}f(x)) \leq \mathsf{K} \qquad \text{and} \qquad \forall y \in Y \quad d_Y(y, f f^{-1}(x)) \leq \mathsf{K}.$$

A *geodesic ray* in X is an isometric embedding $\beta : [0, \infty) \to X$. We fix a base-point $\mathfrak{o} \in X$ and always assume that $\beta(0) = \mathfrak{o}$, that is, a geodesic ray is always assumed to start from this fixed base-point.

Definition 2.2 (quasigeodesics). In this paper, a *quasigeodesic ray* is a continuous quasiisometric embedding $\beta : [0, \infty) \to X$ starting from the basepoint $\mathfrak{o}$.

The additional assumption that quasigeodesics are continuous is not necessary for the results in this paper to hold, but it is added for convenience and to make the exposition simpler.

If $\beta : [0, \infty) \to X$ is a (q, Q)-quasiisometric embedding, and $f : X \to Y$ is a (k, K)-quasiisometry then the composition $f \circ \beta : [t_1, t_2] \to Y$ is a quasiisometric embedding, but it may not be continuous. However, one can adjust the map slightly to make it continuous (see Definition 2.2 in [Qing and Rafi 2022]) such that $f \circ \beta$ is a $(\mathsf{kq}, 2(\mathsf{kq} + \mathsf{kQ} + \mathsf{K}))$-quasigeodesic ray.

Similar to above, a *geodesic segment* is an isometric embedding $\beta : [t_1, t_2] \to X$ and a *quasigeodesic segment* is a continuous quasiisometric embedding

$$\beta : [t_1, t_2] \to X.$$

Notation. Throughout the paper we will use $\alpha, \beta \ldots$ to denote quasigeodesic rays. If the quasigeodesic constants are $(1, 0)$, we use $\alpha_0, \beta_0, \ldots$ to signify that they are in fact geodesic rays. Meanwhile, we use $[\alpha], [\beta], \ldots$ to denote the κ-equivalence classes of quasigeodesic rays (see Definition 2.11), and we also use $\boldsymbol{a}, \boldsymbol{b}, \ldots$ to denote κ-equivalence classes without referring an element in each class.

In contrast, we use $\alpha(\infty)$ to denote equivalence classes of α in the visual boundary (see Definition 2.5). Furthermore, let α be a (quasi)geodesic ray $\alpha : [0, \infty) \to X$, if x_1, x_2 are points on α, then the segment of α between x_1 and x_2 is denoted $[x_1, x_2]_\alpha$. If a segment is presented without subscript, for example $[y_1, y_2]$, then it is a geodesic segment between the two points. Let β be a quasigeodesic ray. Define

$$\|x\| := d(\mathfrak{o}, x).$$

For $\mathsf{r} > 0$, let t_r be the first time where $\|\beta(t)\| = \mathsf{r}$ and define

$$\beta_\mathsf{r} := \beta(t_\mathsf{r}) \quad \text{and} \quad \beta|_\mathsf{r} := \beta[0, t_\mathsf{r}] = [\beta(0), \beta_\mathsf{r}]_\beta, \tag{1}$$

which are points and segments in X, respectively.

Properties of* CAT(0) *spaces. A geodesic metric space (X, d_X) is CAT(0) if geodesic triangles in X are at least as thin as triangles in Euclidean space with the same triple of side-lengths. To be precise, for any given geodesic triangle $\triangle pqr$, consider the unique triangle $\triangle \bar{p}\bar{q}\bar{r}$ in the Euclidean plane with the same triple of side-lengths. The triangle $\triangle pqr$ is at least as thin as $\triangle \bar{p}\bar{q}\bar{r}$ in the following sense: For any pair of points x, y on the triangle $\triangle pqr$, without loss of generality let x, y be on edges $[p, q]$ and $[p, r]$, if we choose points $\bar{x}$ and $\bar{y}$ on edges $[\bar{p}, \bar{q}]$ and $[\bar{p}, \bar{r}]$ of the triangle $\triangle \bar{p}\bar{q}\bar{r}$ so that $d_X(p, x) = d_\diamond(\bar{p}, \bar{x})$ and $d_X(p, y) = d_\diamond(\bar{p}, \bar{y})$, then

$$d_X(x, y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y}).$$

A metric space X is *proper* if closed metric balls are compact. For the remainder of the paper, we assume X is a proper CAT(0) space; a proper CAT(0) space has the following basic properties that are needed in this paper:

Lemma 2.3. *A proper* CAT(0) *space X has the following properties*:

(1) *For any two points x, y in X, there exists exactly one geodesic connecting them. Consequently, X is contractible via geodesic retraction to a base-point in the space.*

(2) *The nearest point projection from a point x to a geodesic line β_0 is a unique point denoted $\pi_{\beta_0}(x)$, or simply x_{β_0}. In fact, the closest point projection map to a geodesic*

$$\pi_{\beta_0} : X \to \beta_0$$

is Lipschitz with respect to distances. The nearest point projection from a point x to a quasigeodesic line β exists and is not necessarily unique. We denote the whole projection set $\pi_\beta(x)$.

(3) *For any $x \in X$, the distance function $d_X(x, \cdot)$ is convex. In other words, for any given any geodesic $[x_0, x_1]$ and $t \in [0, 1]$, if x_t satisfies $d_X(x_0, x_t) = t d(x_0, x_1)$ then we must have*

$$d_X(x, x_t) \le (1 - t)\, d_X(x, x_0) + t d_X(x, x_1).$$

We also need the following redirecting surgery for all proper metric spaces.

Lemma 2.4 [Qing and Rafi 2022, Lemma 4.3]. *Let X be a proper, complete metric space. Let b be a geodesic ray and γ be a* (q, Q)*-quasigeodesic ray. For* $\mathsf{r} > 0$, *assume that $d_X(b_{\mathsf{r}}, \gamma) \leq \frac{\mathsf{r}}{2}$. Then, there exists a* $(9\mathsf{q}, \mathsf{Q})$*-quasigeodesic γ' so that*

$$\gamma' \in [b] \quad \textit{and} \quad \gamma|_{\mathsf{r}/2} = \gamma'|_{\mathsf{r}/2}.$$

Boundaries of* CAT(0) *space.

The visual boundary. In this section we review a couple of topological boundaries of CAT(0) spaces that are important to the study of this paper.

Definition 2.5 (visual boundary). Let X be a CAT(0) space. The *visual boundary* of X, denoted $\partial_v X$, is the collection of equivalence classes of infinite geodesic rays, where α and β are in the same equivalence class, if and only if there exists some $C \geq 0$ such that $d(\alpha(t), \beta(t)) \leq C$ for all $t \in [0, \infty)$. The equivalence class of α in $\partial_v X$ we denote $\alpha(\infty)$.

Notice that by Proposition I.8.2 in [Bridson and Haefliger 1999], for each α representing an element of $\partial_v X$, and for each $x' \in X$, there is a unique geodesic ray α' starting at x' with $\alpha(\infty) = \alpha'(\infty)$.

We describe the topology of the visual boundary by a neighborhood basis: fix a base-point $\mathfrak{o}$ and let α be a geodesic ray starting at $\mathfrak{o}$. A neighborhood basis for α (see Figure 1) is given by sets of the form

$$\mathcal{U}_v(\alpha(\infty), r, \epsilon) := \big\{\beta(\infty) \in \partial_v X \mid \beta(0) = \mathfrak{o} \text{ and } d(\alpha(t), \beta(t)) < \epsilon \text{ for all } t < r\big\}.$$

In other words, two equivalence classes of points are close in the visual boundary if they have geodesic representatives that start at the same point and stay close (are at most ϵ apart) for a long time (at least r). Notice that the above definition of the topology on $\partial_v X$ references a base-point $\mathfrak{o}$. Nonetheless, Proposition I.8.8 in [Bridson and Haefliger 1999] proves that the topology of the visual boundary is base-point invariant.

Definition 2.6 (visibility). A *geodesic line* is an isometric embedding of the infinite interval $(-\infty, \infty)$. A point ζ in the visual boundary is said to be a *visibility point* if any other point $\zeta' \in \partial_v X$, there exists a geodesic line l with $l(\infty) = \zeta$ and $l(-\infty) = \zeta'$. A subset $Y \subseteq \partial X$ is said to be a *visibility space* if for any $\zeta, \zeta' \in Y$ with $\zeta \neq \zeta'$, there is a geodesic line l with $l(\infty) = \zeta$ and $l(-\infty) = \zeta'$.

Related to the above, it's shown in [Zalloum 2022] that each point of the sublinearly Morse boundary $\partial_\kappa X$ is a visibility point of $\partial_v X$.

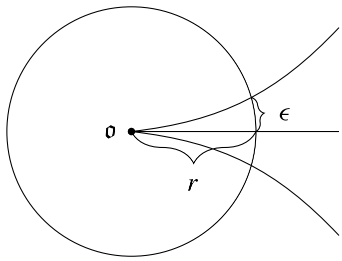

Figure 1. A basis for open sets.

Sublinearly Morse boundaries. Let $\kappa : [0, \infty) \to [1, \infty)$ be a sublinear function which is monotone increasing and concave, that is,

$$\lim_{t\to\infty} \frac{\kappa(t)}{t} = 0.$$

The assumption that κ is increasing and concave makes certain arguments cleaner, otherwise they are not really needed. One can always replace any sublinear function κ, with another sublinear function $\bar{\kappa}$ so that

$$\kappa(t) \le \bar{\kappa}(t) \le \mathsf{C}\,\kappa(t)$$

for some constant C and $\bar{\kappa}$ is monotone increasing and concave. For example, define

$$\bar{\kappa}(t) = \sup\bigl\{\lambda\kappa(u) + (1-\lambda)\,\kappa(v) \mid 0 \le \lambda \le 1,\ u, v > 0 \text{ and } \lambda u + (1-\lambda)\,v = t\bigr\}.$$

The requirement $\kappa(t) \ge 1$ is there to remove additive errors in the definition of κ-contracting geodesics (see Definition 2.10).

κ-Morse geodesic rays. The boundary of interest in this paper consists of points in $\partial_v X$ that are in the "hyperbolic-like". In proper CAT(0) spaces, they can be characterized in two equivalence ways.

Definition 2.7 (κ-neighborhood). For a closed set Z and a constant n define the (κ, n)-neighborhood of Z (see Figure 2) to be

$$\mathcal{N}_\kappa(Z, \mathsf{n}) = \{x \in X \mid d_X(x, Z) \le \mathsf{n}\cdot\kappa(x)\}.$$

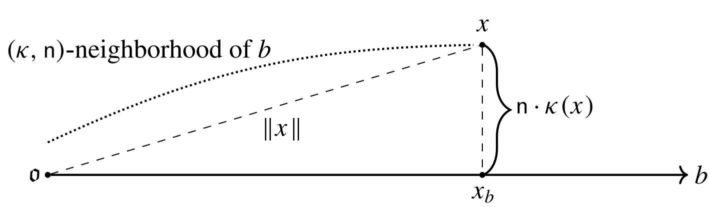

Figure 2. A κ-neighborhood of a geodesic ray b with multiplicative constant n.

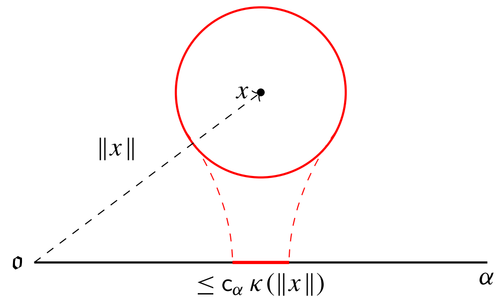

Figure 3. A κ-contracting geodesic ray.

Definition 2.8 (κ-Morse I, κ-Morse II). Let $Z \subseteq X$ be a closed set, and let κ be a concave sublinear function. We say that Z is *κ-Morse* if one of the following equivalent (see Proposition 3.10 in [Qing et al. 2024]) condition holds:

(I) There exists a proper function $\mathsf{m}_Z : \mathbb{R}^2 \to \mathbb{R}$ such that for any sublinear function κ' and for any $r > 0$, there exists R such that for any (q, Q)-quasigeodesic ray β with $\mathsf{m}_Z(\mathsf{q}, \mathsf{Q})$ small compared to r, if

$$d_X(\beta_R, Z) \leq \kappa'(R), \quad \text{then } \beta|_r \subset \mathcal{N}_\kappa(Z, \mathsf{m}_Z(\mathsf{q}, \mathsf{Q})).$$

(II) There is a function $\mathsf{m}'_Z : \mathbb{R}^2_+ \to \mathbb{R}_+$ so that if $\beta : [s, t] \to X$ is a (q, Q)-quasigeodesic with end points on Z then

$$[s, t]_\beta \subset \mathcal{N}_\kappa(Z, \mathsf{m}'_Z(\mathsf{q}, \mathsf{Q})).$$

Remark 2.9. By taking the maximum function of m_Z, m'_Z, we may and will always assume that both conditions hold for the same m_Z, which we refer to as the κ-Morse gauge. Further,

$$\mathsf{m}_Z(\mathsf{q}, \mathsf{Q}) \geq \max(\mathsf{q}, \mathsf{Q}). \tag{2}$$

Definition 2.10 (κ-contracting sets). For $x \in X$, define $\|x\| = d_X(\mathfrak{o}, x)$. For a closed subspace Z of X, we say Z is *κ-contracting* (see Figure 3) if there is a constant c_Z so that, for every $x, y \in X$

$$d_X(x, y) \leq d_X(x, Z) \quad \Rightarrow \quad \operatorname{diam}_X(x_Z \cup y_Z) \leq \mathsf{c}_Z \cdot \kappa(\|x\|).$$

In fact, to simplify notation, we drop $\|\cdot\|$ when it appears in the κ function and write $\kappa(x)$ instead of $\kappa(\|x\|)$.

In CAT(0) spaces, a geodesic is κ-contracting if and only if it is κ-Morse [Qing and Rafi 2022]. Furthermore, κ-contracting can be related to properties introduced in [Charney and Sultan 2015]. Recall from [Charney and Sultan 2015] that a geodesic α is such that if there exists a D such that for every $x, y \in X$

$$d_X(x, y) \leq d_X(x, \alpha) \quad \Rightarrow \quad \operatorname{diam}_X(x_\alpha \cup y_\alpha) \leq \mathsf{D}.$$

Then we say the geodesic is D-contracting. The geodesic is *strongly contracting* if it is D-contracting for some D, it follows that A geodesic ray is strongly contracting if and only if it is κcontracting for $\kappa = 1$.

κ-Morse quasigeodesic rays in X space are grouped into equivalence classes to form $\partial_\kappa X$.

Definition 2.11 (κ-equivalence classes in $\partial_\kappa X$). Let β and γ be two quasigeodesic rays in X. If β is in some κ-neighborhood of γ and γ is in some κ-neighborhood of β, we say that β and γ *κ-fellow travel* each other. This defines an equivalence relation on the set of quasigeodesic rays in X (to obtain transitivity, one needs to change n of the associated (κ, n)-neighborhood).

We denote the equivalence class that contains β by $[\beta]$.

Definition 2.12 (sublinearly Morse boundary). Let κ be a sublinear function as specified in Section 2 and let X be a CAT(0) space:

$$\partial_\kappa X := \{\text{all } \kappa\text{-Morse quasigeodesics}\}/\kappa\text{-fellow traveling}.$$

We define the topology of $\partial_\kappa X$ below.

We also use $\boldsymbol{a}, \boldsymbol{b}$ to denote κ-equivalence classes in $\partial_\kappa X$. We need the following fact that since X is CAT(0), there is a unique geodesic ray in each equivalence class:

Lemma 2.13 [Qing and Rafi 2022, Lemma 3.5]. *Let X be a* CAT(0) *space. Let $b : [0, \infty) \to X$ be a geodesic ray in X. Then b is the unique geodesic ray in any (κ, n)-neighborhood of b for any n. That is to say, there is an* 1-1 *embedding of the set of points in $\partial_\kappa X$ into the points of $\partial_v X$.*

Proof. For each element $\boldsymbol{a} \in \partial_\kappa X$, consider its unique geodesic ray α. The associated $\alpha(\infty)$ is an element of $\partial_v X$. By Lemma 3.5 in [Qing and Rafi 2022], each equivalence class contains a unique geodesic ray. Meanwhile, if two elements $\boldsymbol{a}, \boldsymbol{b} \in \partial_\kappa X$ contain the same geodesic ray, they are in fact the same set of quasigeodesics, therefore this map is well defined. Therefore we have an embedding of the set of points in $\partial_\kappa X$ into the points of $\partial_v X$. □

Coarse cone topology on $\partial_\kappa X$. We equip $\partial_\kappa X$ with a topology which is a coarse version of the visual topology. In visual topology, if two geodesic rays fellow travel for a long time, then they are "close". In this coarse version, if two geodesic rays and all the quasigeodesic rays in their respect equivalence classes remain close for a long time, then they are close. Now we define it formally. First, we say a quantity D is *small* compared to a radius $\mathsf{r} > 0$ if

$$\mathsf{D} \leq \frac{\mathsf{r}}{2\kappa(\mathsf{r})}. \tag{3}$$

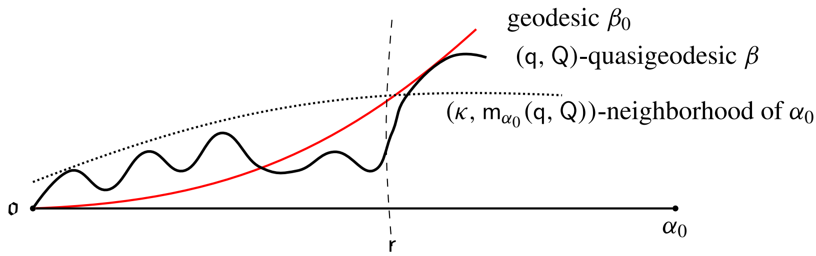

Figure 4. $\boldsymbol{b} \in \mathcal{U}_\kappa(\boldsymbol{a}, \mathsf{r})$ because the quasigeodesics of $\boldsymbol{b}$ such as β, β_0 stay inside the associated $(\kappa, \mathsf{m}_{\alpha_0}(\mathsf{q}, \mathsf{Q}))$-neighborhood of α_0 (as in Definition 2.7), up to distance r.

Recall that given a κ-Morse quasigeodesic ray β, we denote its associated κ-Morse gauge $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$. These are multiplicative constants that give the heights of the κ-neighborhoods.

Definition 2.14 (topology on $\partial_\kappa X$). Let $\boldsymbol{a} \in \partial_\kappa X$ and $\alpha_0 \in \boldsymbol{a}$ be the unique geodesic in the class a. Define $\mathcal{U}_\kappa(\boldsymbol{a}, \mathsf{r})$ to be the set of points $\boldsymbol{b}$ (see Figure 4) such that for any (q, Q)-quasigeodesic of $\boldsymbol{b}$, denoted β, such that $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$ is small compared to r, satisfies

$$\beta|_{\mathsf{r}} \subset \mathcal{N}_\kappa(\alpha_0, \mathsf{m}_{\alpha_0}(\mathsf{q}, \mathsf{Q})).$$

Let the topology of $\partial_\kappa X$ be the topology induced by this neighborhood system. The following fact shows that a κ-boundary is well defined with respect to the associated group.

Theorem 2.15 [Qing et al. 2024]. *Let X, Y be a proper metric space and let κ be a sublinear function. The κ-boundaries of X, Y are denoted $\partial_\kappa X, \partial_\kappa Y$. Any quasiisometry from X to Y induces a homeomorphism between $\partial_\kappa X$ and $\partial_\kappa Y$.*

3. Dense subsets and minimality of G-action

In this section we prove two results concerning dense subsets of $\partial_\kappa G$. First we show that the set of all Morse directions, $\partial_1 G$ is dense in $\partial_\kappa G$, secondly and more generally, the action of G is minimal on $\partial_\kappa G$ and as a consequence, a Morse element in G acts with north-south dynamics on the boundary. To begin with, in this section, let G acts properly discontinuously, cocompactly and by isometries metrically on a CAT(0) space X.

Let A be a geodesic ray or a geodesic segment. Since G acts cocompactly on X there exists a compact set F such that $X \subseteq \bigcup_{g \in G} g \cdot F$. In particular, there exists a set $\{g_i\}$ such that $A \subseteq \bigcup_{g_i} g_i \cdot F$. We can organize the g_i such that $\bigcup_{g_i} g_i \cdot F$ covers longer and longer initial segment of A as $i \to \infty$.

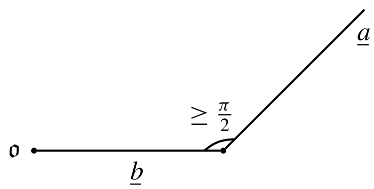

Figure 5. The local angle in a CAT(0) space.

Definition 3.1 (angles in CAT(0) spaces [Bridson and Haefliger 1999, II.3.1]). Let X be a CAT(0) space and let $\ell : [0, a] \to X$ and $\ell' : [0, a'] \to X$ be two geodesic paths issuing from the same point $\ell(0) = c'(0)$. Then the comparison angle $\angle_{\mathbb{E}}(c(t), c'(t'))$ is a nondecreasing function of both $t, t' \geq 0$, and the *Alexandrov angle* $\angle(c, c')$ is equal to

$$\lim_{t,t' \to 0} \angle_{c(0)}(c(t), c'(t')) = \lim_{t \to 0} \angle(c(t), c'(t)).$$

Hence, we define

$$\angle(c, c') = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(c(t), c'(t)).$$

We also refer to the Alexandrov angle as the local angle (see Figure 5).

Lemma 3.2. *Let c_0 be a concatenation of a geodesic segment and an infinite geodesic ray as follows*:

- *The geodesic segment is the initial segment of a κ-Morse geodesic ray labeled $\underline{b}$.*
- *The infinite geodesic ray is a κ-Morse geodesic ray labeled $\underline{a}$.*
- *Suppose in addition that the Alexandrov angle are the point of concatenation is bounded below by the right angle.*

Then c_0 is κ-Morse, and its Morse gauge is bounded above by

$$m_{c_0}(\mathsf{q}, \mathsf{Q}) \leq m'_{\underline{b}}(3\mathsf{q}, \mathsf{Q}) + m_{\underline{a}}(3\mathsf{q}, \mathsf{Q}).$$

Proof. We will show that c_0 satisfies the κ-Morse II condition. Consider a quasi-geodesic ray c' that sublinearly tracks c_0. Let p be the point of concatenation on c_0 and project p to c' and label the projection as $p_{c'}$ (see Figure 6). By the surgery lemma [Qing and Rafi 2022, Lemma 2.5], we have

$$[\mathfrak{o}, p_{c'}]_{c'} \cup [p, p_{c'}]$$

is a $(3\mathsf{q}, \mathsf{Q})$ quasigeodesic segment whose endpoints are on $\underline{b}$, thus by κ-Morse II it is in the $m'_{\underline{b}}(3\mathsf{q}, \mathsf{Q})$ neighborhood of $\underline{b}$. Define c'' to be the quasigeodesic ray with $c'' \subseteq c'$ and $c''(0) = p_{c'}$. Again, by [Qing and Rafi 2022, Lemma 2.5],

$$[p, p_{c'}] \cup c''$$

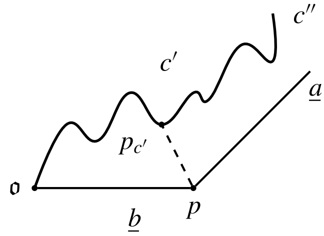

Figure 6. The quasigeodesic ray c' is covered by the union of a quasigeodesic segment $[\mathfrak{o}, p_{c'}]_{c'} \cup [p, p_{c'}]$, and a quasigeodesic ray $[p, p_{c'}] \cup c''$.

is (3q, Q) quasigeodesic ray that sublinearly tracks $\underline{a}$. Likewise by κ-Morse I it is in the $m_{\underline{a}}(3\mathsf{q}, \mathsf{Q})$ neighborhood of $\underline{a}$. Thus c' is in a $m'_{\underline{b}}(3\mathsf{q}, \mathsf{Q}) + m_{\underline{a}}(3\mathsf{q}, \mathsf{Q})$-neighborhood of c_0. □

Theorem 3.3. *Suppose G acts properly discontinuously, cocompactly and by isometries metrically on a proper* CAT(0) *space X and $|\partial_\kappa X| \geq 3$. For each $\mathsf{a} \in \partial_\kappa X$, such that $G \cdot \mathsf{a}$ is dense in $\partial_\kappa X$.*

Proof. By Theorem 1.1 in [Hamenstädt 2009], the action of G on its visual boundary ∂X is minimal. In particular, no element in the visual boundary is a global fixed point of G. Now, since elements of the κ-boundary $\partial_\kappa X$ are also elements in the visual boundary, we conclude that $G.\mathsf{a} \neq \mathsf{a}$ for any $\boldsymbol{a} \in \partial_\kappa X$.

Fix a point $\boldsymbol{a} \in \partial_\kappa X$, by the above, there exists a group element $g \in G$ such that $g\boldsymbol{a} \neq \boldsymbol{a}$. Since $\partial_\kappa X$ is a visibility space [Zalloum 2022], we can choose a biinfinite geodesic line, denoted l, that connects $\boldsymbol{a}$ and $g \cdot \boldsymbol{a}$ $\partial_\kappa G$, we will write $l(\infty)$ for $\boldsymbol{a}$ and $l(-\infty)$ for $g \cdot \boldsymbol{a}$. Given any point $\mathsf{b} \in \partial_\kappa X$ (not necessarily different from $l(\infty)$ or $l(-\infty)$), and let $\underline{b}$ be a geodesic ray representing b with $\underline{b}(0) = \mathfrak{o}$. It suffices to show that a subset of the points in $G \cdot \mathsf{a}$ converges to $\underline{b}$. If $\mathsf{b} = l(\infty)$ or $\mathsf{b} = l(-\infty)$ then we are done. Otherwise fix a point $p \in \ell$.

Since G acts on X cocompactly, there exists a constant C and elements $\{g_i \in G\}$ for $i = 1, 2, 3, \ldots$ such that $d(g_i \cdot p, \underline{b}(i)) \leq C$ for all $i \in \mathbb{N}$.

Denote $l_i := g_i \cdot \ell$. Also let $p_i := g_i \cdot p$ and consider the quasigeodesics

$$\alpha_i := [\mathfrak{o}, p_i] \cup [p_i, l_i^+), \quad \beta_i := [\mathfrak{o}, p_i] \cup [p_i, l_i^-).$$

Since l_i is a line in a CAT(0) space, at least one of the local angles $\angle(\mathfrak{o}, p_i, l_i(\infty))$, $\angle(\mathfrak{o}, p_i, l_i(-\infty))$ is greater than or equal $\frac{\pi}{2}$. Consequently, at least one of α_i or β_i is a $(3, 0)$-quasigeodesic ray (see Figure 7). For each i, let $\gamma_i \in \{\alpha_i, \beta_i\}$ be such a $(3, 0)$-quasigeodesic ray. Each γ_i is κ-Morse as its tail is κ-Morse. By Lemma 3.2, for all i and for all q, Q we have

$$m_{\gamma_i}(\mathsf{q}, \mathsf{Q}) < m'_b(3\mathsf{q}, \mathsf{Q}) + m_l(3\mathsf{q}, \mathsf{Q}) + C',$$

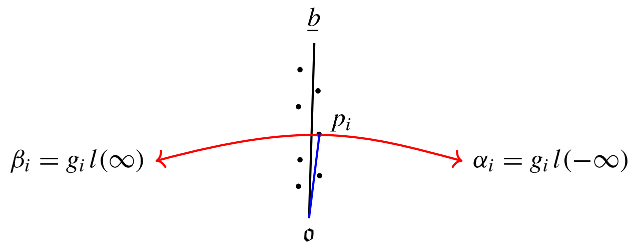

Figure 7. Translates of ℓ by $\{g_i\}$ along $\underline{b}$.

where C' is a constant depending only on C. Since m_{γ_i} does not depend on i we write it as m_γ. Let q, Q be small compared to r. Let

$$\kappa' = 3m_\gamma C\kappa + C.$$

Since $\underline{b}$ is κ-Morse I, we have that for each $r > 0$, and for each pair of (q, Q) small compared to r, there exists $R(\mathsf{q}, \mathsf{Q}, r, \kappa') \geq 1$ such that the conclusion of the κ-Morse I notion holds. Furthermore, by the proof of Theorem 3.14 in [Qing and Rafi 2022], since κ' is concave, if $R = R(\mathsf{q}, \mathsf{Q}, r, \kappa')$ satisfies the definition of κ-Morse I, then all $R > R(\mathsf{q}, \mathsf{Q}, r, \kappa')$ also satisfies the definition of κ-Morse I.

Recall that $\gamma_i \in \{\alpha_i, \beta_i\}$. Let $i = \lceil R \rceil$. By construction we have $d(\underline{b}_R, \gamma_{\lceil R \rceil}) \leq C$. Thus, there exists a point $s_{\lceil R \rceil} \in [0, \infty)$ with $d(\underline{b}_R, \gamma_{\lceil R \rceil}(s_{\lceil R \rceil})) \leq C$. In particular, since $\gamma_{\lceil R \rceil}$ is a $(3, 0)$-quasigeodesic, we have

$$R - C \leq d(\gamma(s_{\lceil R \rceil}), \mathfrak{o}) \leq R + C \quad \Rightarrow \quad s_{\lceil R \rceil} < 3(R + C).$$

Now it remains to show that $\gamma_{\lceil R \rceil} \in \mathcal{U}(\mathsf{b}, r)$. Let ζ be a (q, Q)-quasigeodesic in the class of $\gamma_{\lceil R \rceil}$, then by [Qing et al. 2024, Lemma 3.4] , we get that

$$\begin{aligned} d(\gamma_{\lceil R \rceil}(s_{\lceil R \rceil}), \zeta) &\leq m_\gamma \kappa(s_i) && \text{as } \zeta \text{ and } \gamma_{\lceil R \rceil} \text{ are both } \kappa\text{-Morse,} \\ &\leq m_\gamma \kappa(3(R + C)) && \text{as } \kappa \text{ is monotone nondecreasing,} \\ &\leq m_\gamma \kappa(3CR) \leq 3m_\gamma C\kappa(R) && \text{as } \kappa \text{ is convex.} \end{aligned}$$

This provides a point $x \in \zeta$ with $d(x, \gamma_i(s_i)) \leq m_\gamma C\kappa(R)$. Hence, by the triangle inequality, we get that $d(x, b(R)) \leq 3m_\gamma C\kappa(R) + C$. Now, recall that $\kappa' = 3m_\gamma C\kappa + C$, thus $R(\mathsf{q}, \mathsf{Q}, r, \kappa')$ is precisely that

$$d(x, b(R)) \leq 3m_\gamma C\kappa(R) + C \quad \Rightarrow \quad \zeta_r \subseteq \mathcal{N}_\kappa(\underline{b}, r).$$

This holds for every (q, Q)-quasigeodesic representative of γ_i and thus we have

$$\gamma_{\lceil R \rceil} \in \mathcal{U}(\mathsf{b}, r). \tag{4}$$

That is to say, for larger and larger r we can find an associated sequence of γ_i that is in $\mathcal{U}(\mathsf{b}, r)$, Thus up to a subsequence $\gamma_i = g_i \cdot \mathsf{a}$ limits to b. $\square$

As a consequence of the proof we can establish the following corollary.

Corollary 3.4. *Let l be a biinfinite axis of a rank-one element $g \in G$ and let $\boldsymbol{a}$ be another element in $\partial_\kappa X$ (not necessarily different from the ends of l). Then for each r, there exists a large enough n such that $g^k \cdot \boldsymbol{a} \in \mathcal{U}([l(\infty)], r)$ for all $k \geq n$.*

Proof. As established in the proof of Theorem 3.3, for a sequence of larger and larger r one can construct an associated sequence of γ_i that is in $\mathcal{U}([l(\infty)], r)$. By cocompactness, each γ_i has in its class a $(3, 0)$-quasigeodesic ray with an initial geodesic segment that is $[\mathfrak{o}, g_i \cdot p]$, where the sequence

$$\{g_1, g_2, g_3, \dots\}$$

tracks l. Since l is rank-one, the tracking sequence for $l(\infty)$ becomes

$$\{g, g^2, g^3, g^4 \dots\},$$

thus γ_i consists of a initial segment $[\mathfrak{o}, g^i \cdot p]$. Thus for each r there exists $\lceil R(r) \rceil$ such that if $i = \lceil R(r) \rceil$ then

$$[\gamma_i] \in \mathcal{U}([l(\infty)], r).$$

Furthermore, by the proof of Theorem 3.3, all $R > R(r)$ works for the definition of κ-Morse I, thus for all $i \geq \lceil R(r) \rceil$, we also have that

$$[\gamma_i] \in \mathcal{U}([l(\infty)], r).$$

By combining the two we then have that there exists a large enough n such that $g^k \cdot \boldsymbol{a} \in \mathcal{U}([l(\infty)], r)$ for all $k \geq n$. $\square$

Remark 3.5. Consider a point $\mathsf{b} \in \partial_\kappa X$ represented by a geodesic ray b. Let $p \in X$, $g_i \in G$ and $C > 0$ be such that every point $b(t)$ is within C of $g_i \cdot p$ for some i. The above argument shows that for any pair of points $\boldsymbol{a}, \boldsymbol{a}' \in \partial_\kappa X$, the orbit $g_i\{\boldsymbol{a}, \boldsymbol{a}'\}$ has a subsequence converging to b. Suppose further that $\boldsymbol{a} \neq \mathsf{b}$. If we are provided that $g_i \boldsymbol{a} = \boldsymbol{a}$ for all g_i, then $g_i \boldsymbol{a}'$ has a subsequence converging to b. This observation will be used in the proof of Theorem 3.6.

Now we prove a weak version of north-south dynamics for the action of a group on its κ-boundaries. By [Hamenstädt 2009, Proposition 4.3] an element in a CAT(0) group is contracting if and only if it is a rank-one element.

Theorem 3.6. *Let $g \in G$ be a rank-one element. Let g^∞ denotes the endpoint of the axis associated to g. For any open set U containing g^∞ and a compact set $\mathcal{C} \in (\partial_\kappa G \setminus [g^{-\infty}])$, there exists an N such that for all $n \geq N$, we have $g^n \cdot \mathcal{C} \subset U$.*

Proof. Consider any open set U that contains $g^{+\infty}$. By construction U contains a neighborhood specified in Definition 2.14 thus let $\mathcal{U}(g^{+\infty}, r)$ be the neighborhood in U. Let A be a line connecting g^∞ and $g^{-\infty}$. That is, A is an axis of g and $\{g^i\}$

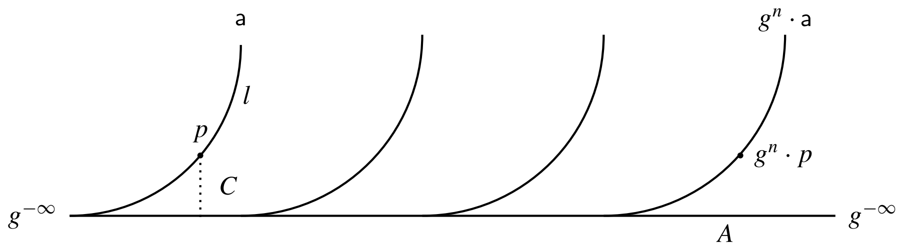

Figure 8. The quasigeodesic ray c' is covered by the union of a quasigeodesic segment $[\mathfrak{o}, p_{c'}]_{c'} \cup [p, p_{c'}]$, and a quasigeodesic ray $[p, p_{c'}] \cup c''$.

tracks A. For a point $\mathsf{a} \in \mathcal{C} \subset \partial_\kappa X$, we let l be a geodesic line connecting $[g^{-\infty}]$ to a and fix a point p on l. let this point p be a new basepoint. We can do this because $\partial_\kappa G$ is basepoint invariant there exists a natural homeomorphism of the boundaries when changing basepoint. Define $C := d(p, A)$, in particular, since g preserves the line A, we have $d(g^n p, gA) = d(g^n p, A) = C$. Hence, for each n the quasigeodesic ray

$$\gamma_n := [\mathfrak{o}, g^n \cdot p] \cup [g^n \cdot p, g^n \cdot \mathsf{a})$$

is a (3, 0)-quasigeodesic (see Figure 8).

By Corollary 3.4, we have that for each R, there exists a large enough n with $g^k \cdot \mathsf{a} \in \mathcal{U}([g^{+\infty}], R)$ for all $k \geq n$. This holds for each point in $\mathcal{C}$, i.e., for each point $\mathsf{a} \in \mathcal{C}$, there exists a large enough power n_{a} with $g^{n_{\mathsf{a}}} \cdot \mathsf{a} \in \mathcal{U}([g^{+\infty}], R) \subseteq U$. Now, notice that

$$g^{n_{\mathsf{a}}} \cdot \mathsf{a} \in \mathcal{U}([g^{+\infty}], R) \quad \Rightarrow \quad \mathsf{a} \in (g^{n_{\mathsf{a}}})^{-1} \mathcal{U}([g^{+\infty}], R).$$

Since g is rank-one, $(g^{n_{\mathsf{a}}})^{-1} = g^{-n_{\mathsf{a}}}$. We denote the open sets as

$$U(n, \mathsf{a}) := g^{-n_{\mathsf{a}}} \mathcal{U}(g^{\infty}, R).$$

Hence, the collection $\{U(n, \mathsf{a}) | \mathsf{a} \in \mathcal{C}\}$ forms a cover for $\mathcal{C}$ yielding a finite subcover

$$\{U(n_i, \mathsf{a}_i)\}^{i=1,2,3\ldots m}.$$

Now, choose $N := \max\{n_1, \cdots n_m\}$, and let n be any natural number such that $n \geq N$. We get

$$\begin{aligned} g^n \mathcal{C} &\subseteq g^n \Big(\bigcup_i U(n_i, \mathsf{a}_i) \Big) && \text{since } \textstyle\bigcup_i U(n_i, \mathsf{a}_i) \text{ is a cover,} \\ &= g^n \Big(\bigcup_i g^{-n_{\mathsf{a}_i}} \, \mathcal{U}([g^{+\infty}], R) \Big) && \text{definition of } U(n_i, \mathsf{a}_i), \\ &\subseteq \bigcup_i g^{m_i} \, \mathcal{U}(g^{\infty}, R) && \text{since } n \geq N \geq n_i, \ m_i = n - n_i, \\ &\subseteq \mathcal{U}(g^{\infty}, R) \subseteq U. \end{aligned}$$

□

4. Compact-type sublinearly Morse boundaries for CAT(0) spaces

It is shown in [Qing and Rafi 2022] that if $G = \mathbb{Z}^2 \star \mathbb{Z}$, then $\partial_\kappa G$ is not compact. In this section we show that the κ-boundary is compact if and only if the underlying group is hyperbolic.

Contracting geodesic rays and the visual boundary. We remind the reader of a few terminologies from [Charney and Murray 2017] and [Charney et al. 2019]. Recall that a geodesic γ is *strongly contracting* if it is in the sublinearly Morse boundary whose associated sublinear function $\kappa = 1$. This implies the existence of a constant D such that all disjoint balls project onto γ to a set of diameter at most D, in which case we say γ is D-*strongly contracting*. Consider the set of all D-strongly contracting geodesic rays emanating from $\mathfrak{o}$. We can think of this set as a subspace of the various boundaries we study in this paper: we use $\partial_v^{\mathsf{D}} X$ to denote the set of all D-contracting geodesic rays emanating from $\mathfrak{o}$ when equipped with the subspace topology of the visual boundary, and use $\partial_\kappa^{\mathsf{D}} X$ when equipped with the subspace topology of the κ-boundary. We note that the subspace $\partial_v^{\mathsf{D}} X$ is compact and metrizable.

Denote by $\partial_\kappa^{(n,\mathsf{D})} X$ the collection of all n-tuples $(a_1, a_2, \ldots, a_n)$ of distinct points $a_i \in \partial_\kappa X$ such that every biinfinite geodesic connecting a_i to a_j is D-strongly contracting.

Let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a nested sequence of topological spaces. The *direct limit* of $\{X_i\}$, denoted by $\varinjlim X_i$, is the space consisting of the union of all X_i given the following topology: A subset U is open in $\varinjlim X_i$ if $U \cap X_i$ is open in X_i for each i.

The following is a standard way to establish continuous maps between two nested sequences and the proof is left as an exercise for interested readers.

Lemma 4.1. *Let $\{X_i\}$ and $\{Y_j\}$ be two sequences of nested topological spaces. Let*

$$X = \varinjlim X_i \quad \textit{and} \quad Y = \varinjlim Y_i$$

be the direct limit of $\{X_i\}$ and $\{Y_i\}$ respectively. If $f : X \to Y$ is a map such that

- *for each i there exists some j with $f(X_i) \subseteq Y_j$,*
- *$f|_{X_i} : X_i \to Y_j$ is continuous,*

then f is continuous.

Consider the topological spaces $\partial_v^{\mathsf{D}} X$. The *Morse boundary* $\partial_\star X$ is the direct limit of the topological spaces $\partial_v^{\mathsf{D}} X$ where $\mathsf{D} \in \mathbb{N}$. In other words

$$\partial_\star X = \varinjlim \partial_v^{\mathsf{D}} X$$

Hence, a set U is open in $\partial_\star X$ if and only if $U \cap \partial_v^{\mathsf{D}} X$ is open for each D.

The following proposition states that equipping this subset with the subspace topology of the visual boundary or the subspace topology of the κ-boundary yields homeomorphic spaces. The intuitive reason for this is the following: since quasi-geodesics stay uniformly close to D-strongly contracting geodesics, the topology of fellow traveling of geodesics (the visual topology) and the topology of fellow traveling of quasigeodesics (the topology of the κ-boundary) coincide.

Proposition 4.2. *The inclusion map $id : \partial_v^{\mathsf{D}} X \to \partial_\kappa^{\mathsf{D}} X$ is a homeomorphism onto its image.*

Proof. We need to show that the map $id : \partial_v^{\mathsf{D}} X \to \partial_\kappa^{\mathsf{D}} X$ is a homeomorphism. Since $\partial_v^{\mathsf{D}} X$ is closed [Charney and Sultan 2015, Lemma 3.2] and X is proper, $\partial_v^{\mathsf{D}} X$ is compact. Also, the space $\partial_\kappa^{\mathsf{D}} X$ is metrizable by Theorem D in [Qing and Rafi 2022]. Hence, it suffices to show that the map id is a continuous map. Notice that since every geodesic ray in $\partial_v^{\mathsf{D}} X$ is D-strongly contracting for the same D, applying Definition 2.7, we get an associated Morse function such that every geodesic ray β_0 is $\mathsf{m}(\mathsf{D})$-Morse, where $\mathsf{m}(\mathsf{D})$ depends only on D and satisfies the following: For every constants $\mathsf{r} > 0$, $\mathsf{n} > 0$ and every sublinear function κ', there is an $\mathsf{R} = \mathsf{R}(\beta_0, \mathsf{r}, \mathsf{n}, \kappa') > 0$ where the following holds: Let $\eta : [0, \infty) \to X$ be a (q, Q)-quasigeodesic ray so that $\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$ is small compared to r, let t_{r} be the first time $\|\eta(t_{\mathsf{r}})\| = \mathsf{r}$ and let t_{R} be the first time $\|\eta(t_{\mathsf{R}})\| = \mathsf{R}$. Then

$$d_X(\eta(t_{\mathsf{R}}), \beta_0) \le \mathsf{n} \cdot \kappa'(\mathsf{R})$$

implies that

$$\eta[0, t_{\mathsf{r}}] \subset \mathcal{N}_1(\beta_0, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})) \subset \mathcal{N}_\kappa(\beta_0, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})).$$

This leads us to our first claim:

Claim. *Given $\boldsymbol{b} \in \partial_\kappa^{\mathsf{D}} X$, each neighborhood of $\boldsymbol{b}$, denoted $\mathcal{U}_\kappa\big(\boldsymbol{b}, \mathsf{r}\big)$, must contain a visual neighborhood basis of β_0, the unique geodesic ray in the class of $\boldsymbol{b}$.*

Proof of the Claim. To see this, let $\beta_0 \in \boldsymbol{b}$ be the unique geodesic ray starting at $\mathfrak{o}$. We wish to show that for any $\mathsf{r} > 0$, there exists r' and ϵ such that $\mathcal{U}_v\big(\beta_0, \mathsf{r}', \epsilon\big) \subseteq \mathcal{U}_\kappa(\boldsymbol{b}, \mathsf{r})$.

In other words, we want to show that for any $\mathsf{r} > 0$, there exists r' and ϵ if a geodesic ray $\alpha_0 \in \boldsymbol{a}$ with $\alpha_0(0) = \mathfrak{o}$ satisfies $d(\alpha_0(t), \beta_0(t)) < \epsilon$ for $t \le \mathsf{r}'$, then, any (q, Q)-quasigeodesic representative α of a with $\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$ small compared to r, we have

$$\alpha|_{\mathsf{r}} \subset \mathcal{N}_\kappa(\zeta, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})).$$

Remember that $\alpha|_{\mathsf{r}} = \alpha([0, t_{\mathsf{r}}])$ where t_{r} is the first time where $\|\alpha(t)\| = \mathsf{r}$.

Let r be given and let

$$n = \max\{\mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}) + 1 \mid \mathsf{q}, \mathsf{Q} \le \mathsf{r}\}.$$

By Definition 2.7, with $Z = \beta_0$, there exists an $R = R(\mathsf{r}, n)$ such that any (q, Q)-quasigeodesic representative β of a with $\mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})$ small compared to r, we have

$$d(\beta(t_R), b) < n \quad \Rightarrow \quad \beta_{\mathsf{r}} \subset \mathcal{N}_1(\beta_0, \mathsf{m}_\beta(\mathsf{q}, \mathsf{Q})).$$

Choose $\mathsf{r}' = \mathsf{r} + R$ and $\epsilon = 1$. Hence, we want to show that if $d(a(t), b(t)) < 1$ for $t \leq \mathsf{r} + R$, then $\beta_{\mathsf{r}} \subset \mathcal{N}_\kappa(b, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}))$ for β defined above. Since a is 1-Morse with gauge m, the Hausdorff distance between a and β is at most $\mathsf{m}(\mathsf{q}, \mathsf{Q})$. This implies that for any $0 < t \leq \mathsf{r} + R$, we have

$$d(a(t)), \beta(i_t)) < \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}) \quad \text{for some } i_t.$$

Therefore, if t_R is the first time with $\|\beta(t_R)\| = R$, we must have

$$d(a, \beta(t_R)) < \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}).$$

Now, since $d(a(t), b(t)) < 1$ for all $t < \mathsf{r} + R$ and as $d(a, \beta(t_R)) < \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})$, the triangle inequality gives

$$d(b, \beta(t_R)) \leq d(b, a) + d(a, \beta(t_R)) \leq 1 + \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q}),$$

which we can rewrite as

$$\beta_r \subset \mathcal{N}_1(b, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})) \subset \mathcal{N}_\kappa(b, \mathsf{m}_{\beta_0}(\mathsf{q}, \mathsf{Q})),$$

which proves the claim. □

Now we are left to show that the map id is continuous. Since both $\partial_v^{\mathsf{D}} X$ and $\partial_\kappa^{\mathsf{D}} X$ are metrizable spaces, it suffices to establish sequential continuity. Let $\{\mathsf{c}_n\}$, $\mathsf{c} \in \partial_v^{\mathsf{D}} X$ with $\mathsf{c}_n \to \mathsf{c}$. Assume that $\mathsf{c}_n \to \mathsf{c}$ in $\partial_v^{\mathsf{D}} X$, we want to show that $\mathsf{c}_n \to \mathsf{c}$ in $\partial_\kappa^{\mathsf{D}} X$. Using the above claim, since each neighborhood of c in $\partial_\kappa^{\mathsf{D}} X$ contains an open neighborhood of $\partial_v^{\mathsf{D}} X$, the statement is immediate. □

Corollary 4.3. *For a* CAT(0) *space X, the natural map $i : \partial_\star X \hookrightarrow \partial_\kappa X$ is continuous.*

Proof. By Proposition 4.2, $i_{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_\kappa X$ is continuous for each D. Since

$$\partial_\star X = \varinjlim \partial_v^{\mathsf{D}} X,$$

by definition $i : \partial_\star X \hookrightarrow \partial_\kappa X$ is continuous. □

Corollary 4.4. *Let X be proper* CAT(0) *hyperbolic space. Let κ be a sublinear function. The space $\partial_\kappa X$ is compact, and the κ-Morse boundary is homeomorphic to the Gromov boundary.*

Proof. Since X is a hyperbolic space, there exists a uniform constant D such that every geodesic ray is D-strongly contracting. This implies that the subspace $\partial_v^{\mathsf{D}} X$ defined above is the entire visual boundary, in other words, we have $\partial_v^{\mathsf{D}} X = \partial_v X$.

Also, since every geodesic ray is D-strongly contracting, the subspace $\partial_\kappa^{\mathsf{D}} X$ defined above is the full κ-boundary as a set. That is to say, $\partial_\kappa^{\mathsf{D}} X = \partial_\kappa X$ as sets. Proposition 4.2 then yields a homeomorphism between the visual boundary of X, $\partial_v X$ and the κ-boundary of X, $\partial_\kappa X$. By [Bridson and Haefliger 1999, III.H.3], the Gromov boundary is homeomorphic to the visual boundary of X. Thus we have

$$\partial X \cong \partial_v X \cong \partial_\kappa X.$$

As X is proper, ∂X is compact, thus the κ-boundary $\partial_\kappa X$ must also be compact. □

Theorem 4.5. *Suppose a group G acts properly discontinuously, cocompactly and by isometries metrically on a* CAT(0) *space X such that $\partial_\kappa X \neq \varnothing$, then the following are equivalent*:

(1) *Every geodesic ray in X is κ-contracting.*

(2) *Every geodesic ray in X is strongly contracting.*

(3) *$\partial_\kappa X$ is compact.*

(4) *The space X is hyperbolic.*

Proof. We will prove the equivalences by showing (3) ⇒ (1) ⇒ (4) ⇒ (3) and also (4) ⇔ (2). We start by showing (3) implies (1). The statement is vacuously true if $\partial_\kappa X$ is empty. If $\partial_\kappa X$ is nonempty, then by [Zalloum 2022], there exists a rank-one isometry g. This yields the existence of a strongly contracting geodesic line l_g that is an axis for g. Let $\mathfrak{o}$ be a point on l_g and let β be an arbitrary geodesic ray emanating from $\mathfrak{o}$. We show now that β is κ-contracting. Since the action of G on X is cocompact, there exists a $C \geq 0$ and a sequence of group elements $\{g_i\} \subseteq G$ such that $d(\beta(i), g_i \cdot \mathfrak{o}) \leq C$ for each $i \in \mathbb{N}$ (the dots in Figure 9). Now, consider the sets given by $g_i l_g$. Since g_i acts by isometry, these are biinfinite geodesic lines passing the points $g_i \mathfrak{o}$. Recall $[\,\cdot\,,\cdot\,]$ denote a geodesic segment between two points. By CAT(0) geometry, the concatenation of two geodesic segments at angle bounded below by $\frac{\pi}{2}$ forms a (3,0)-quasigeodesic segment. Lastly, we denote one end of $g_i l_g$ by $g_i l_g(\infty)$ and the other end by $g_i l_g(-\infty)$.

For each i, consider the concatenation

$$[\mathfrak{o}, g_i \mathfrak{o}] \cup [g_i \mathfrak{o}, g_i l_g(\infty)] \quad \text{and} \quad [\mathfrak{o}, g_i \mathfrak{o}] \cup [g_i \mathfrak{o}, l_g(-\infty)].$$

By CAT(0) geometry, one of these two concatenations consists of geodesic segments intersecting at angles bounded below by $\frac{\pi}{2}$. Thus one of the two concatenations is a (3,0)-quasigeodesic ray starting at $\mathfrak{o}$. Relabel the sequence of (3,0)-quasigeodesic rays defined by concatenating $[\mathfrak{o}, g_i \mathfrak{o}]$ with either $[g_i \mathfrak{o}, l_g(\infty)]$ or $[g_i \mathfrak{o}, l_g(-\infty)]$ to form a sequence of (3,0)-quasigeodesic rays, by γ_i. Since $\partial_\kappa X$ is compact, up to passing to a subsequence, $[\gamma_i]$ converges to an element $\mathsf{b} \in \partial_\kappa X$. Let b be the

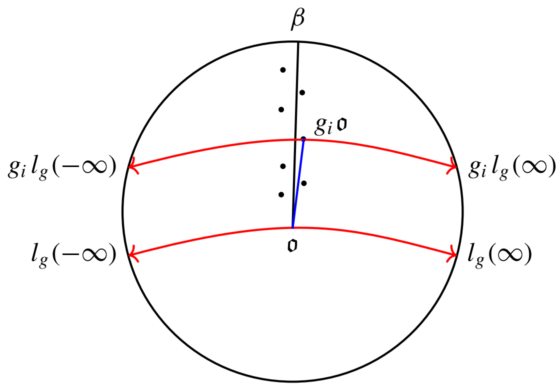

Figure 9. Translates of l_g by $\{g_i\}$ along α_0.

geodesic representative of b. The convergence implies that for each $\mathsf{r} > 0$, there exists k such that if $i \geq k$, the sequence γ_i satisfies

$$\gamma_i|_{\mathsf{r}} \subset \mathcal{N}_\kappa(b, \mathsf{m}_b(3, 0)).$$

Since the subsegment of γ_i given by $[\mathfrak{o}, g_i\mathfrak{o}]$ is in a C-neighborhood of β, we have for each r, $\beta|_r$ is in $\mathcal{N}_\kappa(b, C + \mathsf{m}_b(3, 0))$, and hence

$$\beta \in \mathcal{N}_\kappa(b, C + \mathsf{m}_b(3, 0)).$$

Lemma 2.13 then implies that $\beta = b$. Thus we have

$$[\beta] = [b] = \mathsf{b} \in \partial_\kappa X,$$

which finishes the proof.

Next we show that (1) implies (4). If every geodesic ray is κ-contracting, then X does not contain an isometric copy of $\mathbb{E}^2$, and hence, by the flat plane theorem [Bridson and Haefliger 1999, III.Γ.3 Theorem 3.1], the space X must be hyperbolic. The implication (4) $\Rightarrow$ (3) is Corollary 4.4.

Lastly, we prove the equivalence between (2) and (4). Since every geodesic ray is N-Morse for the same N in a δ-hyperbolic space, we have (4) $\Rightarrow$ (2). On the other hand, by way of contradiction, suppose X is not a hyperbolic space, then it must contain an isometrically embedded $\mathbb{E}^2$ by the flat plane theorem [Bridson and Haefliger 1999, III.Γ.3 Theorem 3.1]. Let $\mathfrak{o} \in \mathbb{E}^2$ and the geodesic rays that stays entirely in the is not D-strongly contracting for any D. Therefore (2) $\Rightarrow$ (4). $\square$

Remark 4.6. As we can see in [Behrstock 2019], one can have part of the sublinearly Morse boundary being compact while the rest of the sublinearly Morse boundary is not compact. In this case the theorem can be applied to a part of the space X whose sublinear boundary is the compact portion.

5. Successively quasimöbius homeomorphisms on the sublinearly Morse boundaries

Paulin [1996] characterizes homeomorphisms between boundaries of cocompact hyperbolic spaces that are induced by quasiisometries. They characterize such homeomorphisms as the ones that are *quasimöbius*.

quasimöbius maps.

Definition 5.1. Let X, Y be proper geodesic CAT(0) space.

- A map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be 1-*stable* if for every D, there exists D' such that $f(\partial_\kappa^{\mathsf{D}} X) \subseteq \partial_\kappa^{\mathsf{D}'} Y$.
- A map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be 2-*stable* if for every D, there exists D' such that
$$f(\partial_\kappa^{(2,\mathsf{D})} X) \subseteq \partial_\kappa^{(2,\mathsf{D}')} Y.$$

Notice that it follows from the above definition that a 2-stable map f maps $\partial_\kappa^{(n,\mathsf{D})} X$ to $\partial_\kappa^{(n,\mathsf{D}')} X$ for all $n \geq 2$. Hence, it makes sense to make the next definition.

A map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be *stable* if it is both 1- and 2-stable.

Definition 5.2. The cross-ratio of a four-tuple $(a, b, c, d) \in \partial_\kappa^{(4,\mathsf{D})} X$ is defined to be $[a, b, c, d] = \pm \sup_{\alpha \in (a,c)} d(\pi_\alpha(b), \pi_\alpha(d))$, where the sign is positive if the orientation of the geodesic $(\pi_\alpha(b), \pi_\alpha(d))$ agrees with that of (a, c) and is negative otherwise.

A stable map $f : \partial_\kappa X \to \partial_\kappa Y$ is said to be *successively quasimöbius* if for every D, there exists a continuous map $\psi_{\mathsf{D}} : [0, \infty) \to [0, \infty)$ such that for all 4-tuples $(a, b, c, d) \in \partial_\kappa^{(4,\mathsf{D})} X$, we have $[f(a), f(b), f(c), f(d)] \leq \psi_{\mathsf{D}}(|[a, b, c, d]|)$.

In this section, as an application of visibility [Zalloum 2022] and using the works [Charney and Murray 2017] and [Charney et al. 2019], we prove a weaker version of this characterization.

Theorem 5.3. *Let X, Y be proper cocompact* CAT(0) *spaces with at least* 3 *points in their sublinear boundaries. A homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ is induced by a quasiisometry $h : X \to Y$ if and only if f is stable and successively quasimöbius.*

The following is an immediate consequence.

Corollary 5.4. *Let G and H be* CAT(0) *groups. Then G is quasiisometric to H if and only if there exists a homeomorphism $f : \partial_\kappa G \to \partial_\kappa H$ which is successively quasimöbius and stable.*

We will often make use of the following theorem.

Theorem 5.5 [Charney et al. 2019]. *Let X, Y be proper cocompact* CAT(0) *spaces with at least* 3 *points in their Morse boundaries. A homeomorphism $f : \partial_\star X \to \partial_\star Y$*

is induced by a quasiisometry $h : X \to Y$ if and only if f is 2-stable and successively quasimöbius.

Lemma 5.6. *A quasiisometry $h : X \to Y$ induces a stable homeomorphism, i.e., $\partial_\kappa h : \partial_\kappa X \to \partial_\kappa Y$.*

Proof. Fix $\mathfrak{o} \in X$ and let $\mathfrak{o}' = h(\mathfrak{o})$ where h is a (k, K)-quasiisometry. Qing and Rafi [2022] show that a quasiisometry h induces a homeomorphism ∂h on their respective κ-boundaries. If γ is a D-strongly contracting geodesic ray, then by [Charney and Sultan 2015], the unique geodesic ray starting at $h(\mathfrak{o})$ and representing $[f(\gamma)]$ must be D'-strongly contracting where D' depends on D, k and K. This implies that $\partial_\kappa h$ is 1-stable. Now, Theorem 5.5 gives us that the map induced by h on the Morse boundary is 2-stable. Hence, we deduce that $\partial_\kappa h$ is stable. □

Lemma 5.7. *Any homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ such that f, f^{-1} are 1-stable induces a homeomorphism $g : \partial_\star X \to \partial_\star Y$ on their Morse boundaries, with $g(x) = f(x)$ for all $x \in \partial_\star X$.*

Proof. Let $f : \partial_\kappa X \to \partial_\kappa Y$ be a homeomorphism such that f and f^{-1} are 1-stable. Notice that by Theorem E in [Qing and Rafi 2022], we have that if $\kappa' < \kappa$, then the inclusion map

$$i : \partial_{\kappa'} X \to \partial_\kappa X$$

is continuous. Taking $\kappa' = 1$ yields that $i : \partial_1 X \to \partial_\kappa X$ is continuous. Hence, since both f and f^{-1} are 1-stable, the restriction of f to $\partial_1 X$ induces a homeomorphism $\bar{f} : \partial_1 X \to \partial_1 Y$, with $\bar{f} = f|_{\partial_1 X}$ where $\partial_1 X$ and $\partial_1 Y$ are given the subspace topology of the κ-boundary. Meanwhile,

$$\partial_1 X = \bigcup_{\mathsf{D}=1}^{\infty} \partial_1^{\mathsf{D}} X \quad \text{and} \quad \partial_1 Y = \bigcup_{\mathsf{D}=1}^{\infty} \partial_1^{\mathsf{D}} Y.$$

Since $\partial_1^{\mathsf{D}} X$ is equipped with the subspace topology of $\partial_1 X$, the inclusion map

$$i^{\mathsf{D}} : \partial_1^{\mathsf{D}} X \hookrightarrow \partial_1 X$$

is continuous. Using Proposition 4.2, we get that

$$i^{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_1 X$$

is continuous for every D, where $\partial_v^{\mathsf{D}} X$ is given the subspace topology of the visual boundary. Furthermore, since f is 1-stable, we have $\bar{f} \circ i^{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_1^{\mathsf{D}'} Y$ for some D' where $\bar{f} \circ i^{\mathsf{D}}$ is continuous. Using Proposition 4.2, we obtain a continuous map

$$\bar{f} \circ i^{\mathsf{D}} : \partial_v^{\mathsf{D}} X \hookrightarrow \partial_v^{\mathsf{D}'} Y \quad \text{for each } \mathsf{D}.$$

Hence, by Lemma 4.1, we get a continuous map $g : \partial_\star X \to \partial_\star Y$. Applying the same argument above to f^{-1} yields a continuous map $g' : \partial_\star Y \to \partial_\star X$ with

$$g \circ g' = id_{\partial_\star X} \quad \text{and} \quad g' \circ g = id_{\partial_\star Y}. \qquad \square$$

Proof of Theorem 5.3. ($\Rightarrow$) If h is a quasiisometry, then $f := \partial h$ is stable by Lemma 5.6 . Also, f is successively quasimöbius by Theorem 5.5.

($\Leftarrow$) Using Lemma 5.7 any stable homeomorphism $f : \partial_\kappa X \to \partial_\kappa Y$ induces a homeomorphism $g : \partial_\star X \to \partial_\star Y$ on their Morse boundaries, with $g(x) = f(x)$ for all $x \in \partial_\star X$. Since f is successively quasimöbius and $g(x) = f(x)$ for $x \in \partial_\star X$, Theorem 5.5 implies the existence of a quasiisometry $h : X \to Y$ such that we have $\partial h = g : \partial_\star X \to \partial_\star Y$. We wish to show that the induced map

$$\partial_\kappa h : \partial_\kappa X \to \partial_\kappa Y$$

agrees with f. Notice that as a set $\partial_\star X = \partial_1 X$, where $\partial_1 X$ is the subset of $\partial_\kappa X$ consisting of equivalence classes having a strongly contracting representative. Hence, we have $\partial_\kappa h(x) = \partial h(x)$ for all $x \in \partial_1 X \subseteq \partial_\kappa X$. Now, since $\partial h = g$, and $g(x) = f(x)$ on $\partial_1 X$, we get that

$$\partial_\kappa h(x) = \partial h(x) = g(x) = f(x)$$

for all $x \in \partial_1 X$. Therefore, $\partial_\kappa h(x) = f(x)$ for all $x \in \partial_1 X \subseteq \partial_\kappa X$. It remains to show that $\partial_\kappa h(x') = f(x')$ for all $x' \in \partial_\kappa X$. Let $x' \in \partial_\kappa X$, by Corollary 3.4, there exists a sequence $x_n \in \partial_1 X$ that converges to x':

$$x_n \to x'$$

in $\partial_\kappa X$. Since f is continuous on ∂X_κ, we have convergence

$$f(x_n) = \partial_\kappa h(x_n) \to f(x').$$

Also, since $\partial_\kappa h$ is continuous on $\partial_\kappa X$, we get that

$$\partial_\kappa h(x_n) \to \partial_\kappa h(x').$$

As $\partial_\kappa Y$ is Hausdorff, we obtain $\partial_\kappa h(x') = f(x')$. □

This result is far from satisfying, since successively quasimöbius requires one to check the quasimöbius condition for every D, it is a much stronger condition than quasimöbius. Currently there is no results directly characterizing quasimöbius maps on the κ-boundaries.

Acknowledgement

We thank Ruth Charney, Matthew Cordes, Curt Kent and Kasra Rafi for helpful conversations.

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Received January 7, 2023. Revised July 24, 2024.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 333 No. 1 November 2024