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SL(2, $\mathbb{Z}$) MODULAR FORMS AND ANOMALY CANCELLATION FORMULAS FOR ALMOST COMPLEX MANIFOLDS

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Building on a kind of elliptic genus for almost complex manifolds introduced by Ping Li and its various properties established by him, we define a generalized elliptic genus where an extra complex bundle is involved. This generalized elliptic genus is a generalized Jacobi form. By this generalized Jacobi form, we can get some SL(2, $\mathbb{Z}$) modular forms. By these SL(2, $\mathbb{Z}$) modular forms, we get some interesting anomaly cancellation formulas for an almost complex manifold. As corollaries, we get some divisibility results of the holomorphic Euler characteristic number.

1. Introduction

For an arbitrary compact spin manifold one can define its elliptic genus. It is a modular form in one variable with respect to a congruence subgroup of level 2. For a compact complex manifold one can define its elliptic genus as a function in two complex variables. In the last case, the elliptic genus is the holomorphic Euler characteristic of a formal power series with vector bundle coefficients. If the first Chern class of the complex manifold is equal to zero, then the elliptic genus is a weak Jacobi form. In [9], Li extended the elliptic genus of an almost complex manifold to a twisted version where an extra complex vector bundle is involved. Under some conditions, Li proved this elliptic genus is a weak Jacobi form. We extend Li's elliptic genus and prove this generalized elliptic genus is not a weak Jacobi form and we call it the generalized Jacobi form. By this generalized Jacobi form, we can get some SL(2, $\mathbb{Z}$) modular forms as in [9].

In 1983, the physicists Alvarez-Gaumé and Witten [1] discovered the "miraculous cancellation" formula for gravitational anomaly which reveals a beautiful relation between the top components of the Hirzebruch $\widehat{L}$-form and $\widehat{A}$-form of a 12-dimensional

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smooth Riemannian manifold. Kefeng Liu [10] established higher-dimensional "miraculous cancellation" formulas for $(8k+4)$-dimensional Riemannian manifolds by developing modular invariance properties of characteristic forms. These formulas could be used to deduce some divisibility results. In [3; 5; 6], some more general cancellation formulas that involve a complex line bundle and their applications were established. In [7], Han, Liu and Zhang showed that both of the Green–Schwarz anomaly factorization formula for the gauge group $E_8 \times E_8$ and the Horava–Witten anomaly factorization formula for the gauge group E_8 could be derived through modular forms of weight 14. This answered a question of J. H. Schwarz. In [8], Han, Huang, Liu and Zhang introduced a modular form of weight 14 over $\mathrm{SL}(2, \mathbb{Z})$ and a modular form of weight 10 over $\mathrm{SL}(2, \mathbb{Z})$ and they got some interesting anomaly cancellation formulas on 12-dimensional manifolds. In [12], by some $\mathrm{SL}(2, \mathbb{Z})$ modular forms introduced in [4] and [11], we get some interesting anomaly cancellation formulas. As corollaries, we get some divisibility results of index of twisted Dirac operators. *Our motivation is to prove more anomaly cancellation formulas for almost complex manifolds by modular forms over* $\mathrm{SL}(2, \mathbb{Z})$ *induced by the above generalized Jacobi form.*

This paper is organized as follows. In Section 2, we introduce the generalized elliptic genus and prove it is a generalized Jacobi form. In Section 3, by this generalized Jacobi form, we can get some $\mathrm{SL}(2, \mathbb{Z})$ modular forms as in [9]. By these $\mathrm{SL}(2, \mathbb{Z})$ modular forms, we get some interesting anomaly cancellation formulas for an almost complex manifold. As corollaries, we get some divisibility results of the holomorphic Euler characteristic number.

2. Generalized elliptic genus for almost complex manifolds

Let (M, J) be a $2d$-dimensional almost complex manifold, let T be the holomorphic tangent bundle in the sense of J and let T^* be the dual of T. Let W denote a complex l-dimensional vector bundle on M. Denote the first Chern classes of T and W by $c_1(M)$ and $c_1(W)$, respectively. We denote by $2\pi\sqrt{-1}x_i$ $(1 \le i \le d)$ and $2\pi\sqrt{-1}w_j$ $(1 \le j \le l)$, respectively, the formal Chern roots of T and W. Then the Todd form of (M, J) is defined by

$$\mathrm{Td}(M) := \prod_{i=1}^{d} \frac{2\pi\sqrt{-1}x_i}{1 - e^{-2\pi\sqrt{-1}x_i}}.$$

Let $(\tau, z) \in \mathcal{H} \times \mathcal{C}$ where $\mathcal{H}$ is the upper half plane and $\mathcal{C}$ is the complex plane. Let $a_0 \ge 1$ be a positive integer. Let $y_r = e^{2\pi\sqrt{-1}m_r z}$ for $1 \le r \le a_0$ and a positive integer m_r. Let $q = e^{2\pi\sqrt{-1}\tau}$ and $c := \prod_{j=1}^{\infty}(1 - q^j)$.

For any complex number t, let

$$\wedge_t(E) = \boldsymbol{C}|_M + tE + t^2 \wedge^2 (E) + \cdots, \quad S_t(E) = \boldsymbol{C}|_M + tE + t^2 S^2(E) + \cdots$$

denote, respectively, the total exterior and symmetric powers of E, which live in $K(M)[[t]]$. The following relations between these operations hold:

$$S_t(E) = \frac{1}{\wedge_{-t}(E)}, \quad \wedge_t(E - F) = \frac{\wedge_t(E)}{\wedge_t(F)}.$$

If $\{\omega_i\}$, $\{\omega'_j\}$ are formal Chern roots for Hermitian vector bundles E, F, respectively,

$$\mathrm{ch}(\wedge_t(E)) = \prod_i (1 + e^{\omega_i} t), \quad \mathrm{ch}(S_t(E)) = \frac{1}{\prod_i (1 - e^{\omega_i} t)}.$$

Definition 2.1. The generalized elliptic genus of (M^{2d}, J) with respect to W, which we denote by $\mathrm{Ell}(M, W, \tau, z)$, is defined by

$$\mathrm{Ell}(M, W, \tau, z) := \left\{ \exp\left(\frac{a_0 c_1(W) - c_1(M)}{2} \right) Td(M)\, \mathrm{ch}(E(M, W, \tau, z)) \right\}^{(2d)},$$

where

$$\mathrm{E}(M, W, \tau, z) := c^{2(d - l a_0)} y_1^{-l/2} \cdots y_{a_0}^{-l/2} \bigotimes_{n=1}^{\infty} \left(\bigotimes_{r=1}^{a_0} \wedge_{-y_r q^{n-1}}(W^*) \wedge_{-y_r^{-1} q^n}(W) \right) \otimes \left(\bigotimes_{n=1}^{\infty} S_{q^n}(T^*) \otimes S_{q^n}(T) \right).$$

When $a_0 = 1$ and $m_1 = 1$, we get Li's elliptic genus. We know that our elliptic genus is not a special case of Li's elliptic genus since

$$\wedge_{t_1}(W) \otimes \wedge_{t_2}(W) \neq \wedge_{t_1 + t_2}(W) \quad \text{and} \quad \wedge_{t_1}(W) \otimes \wedge_{t_2}(W) \neq \wedge_{t_1 t_2}(W).$$

Using the same calculations as in Lemma 3.4 in [9], we obtain:

Lemma 2.2. *We have*

$$\mathrm{Ell}(M, W, \tau, z) = \left(\eta(\tau)^{3(d - l a_0)} \prod_{i=1}^{d} \frac{2\pi\sqrt{-1} x_i}{\theta(\tau, x_i)} \prod_{j=1}^{l} \prod_{r=1}^{a_0} \theta(\tau, w_j - m_r z) \right)^{(2d)},$$

where

$$\theta(\tau, z) = 2q^{1/8} \sin(\pi z) \prod_{j=1}^{\infty} ((1 - q^j)(1 - e^{2\pi\sqrt{-1} z} q^j)(1 - e^{-2\pi\sqrt{-1} z} q^j)),$$

$$\eta(\tau) := q^{1/24} \cdot c = q^{1/24} \prod_{j=1}^{\infty} (1 - q^j).$$

Theorem 2.3. *If $c_1(W)=0$ and the first Pontrjagin class $p_1(M)$ equals $a_0p_1(W)$, then the generalized elliptic genus* $\mathrm{Ell}(M, W, \tau, z)$ *satisfies*

(2-1) $$\mathrm{Ell}\Big(M, W, \frac{a\tau+b}{c\tau+d_0}, \frac{z}{c\tau+d_0}\Big) = (c\tau+d_0)^{d-la_0}\exp\Big(\pi\sqrt{-1}l\Big(\sum_{r=1}^{a_0} m_r^2\Big)\frac{cz^2}{c\tau+d_0}\Big)\,\mathrm{Ell}(M, W, \tau, z),$$

(2-2) $$\mathrm{Ell}(M, W, \tau, z+\lambda\tau+\mu) = (-1)^{\mu l(\sum_{r=1}^{a_0} m_r)+\lambda l a_0}\exp\Big(-\pi\sqrt{-1}l\Big(\sum_{r=1}^{a_0} m_r^2\Big)(2\lambda z+\lambda^2\tau)\Big)\cdot\mathrm{Ell}(M, W, \tau, z),$$

where $\left(\begin{smallmatrix} a & b \\ c & d_0\end{smallmatrix}\right)\in \mathrm{SL}(2,\mathbb{Z})$ *and* $\lambda, \mu\in\mathbb{Z}$. *We know that the generalized elliptic genus* $\mathrm{Ell}(M, W, \tau, z)$ *is not a Jacobi form and we call it the generalized Jacobi form.*

Proof. By the transformation laws

$$\eta^3\Big(-\frac{1}{\tau}\Big) = \Big(\frac{\tau}{\sqrt{-1}}\Big)^{3/2}\eta^3(\tau), \quad \eta^3(\tau+1)=e^{\pi\sqrt{-1}/4}\eta^3(\tau),$$
$$\theta(\tau, z+1) = -\theta(\tau, z), \quad \theta(\tau, z+\tau) = -q^{-1/2}\exp\big(-2\pi\sqrt{-1}z\big)\theta(\tau, z),$$
$$\theta(\tau+1, z) = e^{\pi\sqrt{-1}/4}\theta(\tau, z),$$
$$\theta\Big(-\frac{1}{\tau}, z\Big) = -\sqrt{-1}\Big(\frac{\tau}{\sqrt{-1}}\Big)^{1/2}\exp\big(\pi\sqrt{-1}\tau z^2\big)\theta(\tau, \tau z),$$

we get $\mathrm{Ell}(M, W, \tau, z)$ satisfies the transformation laws

$$\mathrm{Ell}(M, W, \tau+1, z) = \mathrm{Ell}(M, W, \tau, z),$$
$$\mathrm{Ell}(M, W, \tau, z+1) = (-1)^{l(\sum_{r=1}^{a_0} m_r)}\,\mathrm{Ell}(M, W, \tau, z),$$
$$\mathrm{Ell}(M, W, \tau, z+\tau) = (-1)^{la_0}\exp\Big(-\pi\sqrt{-1}l\Big(\sum_{r=1}^{a_0} m_r^2\Big)(\tau+2z)\Big)\,\mathrm{Ell}(M, W, \tau, z),$$
$$\mathrm{Ell}\Big(M, W, -\frac{1}{\tau}, \frac{z}{\tau}\Big) = \tau^{d-la_0}\exp\Big(\pi\sqrt{-1}l\Big(\sum_{r=1}^{a_0} m_r^2\Big)\frac{z^2}{\tau}\Big)\,\mathrm{Ell}(M, W, \tau, z).$$

By the above transformation laws, we can conclude (2-1) and (2-2). □

In the following, we introduce some generalized elliptic genus with extra complex bundle W and real vector bundle V. Let V be a $2b_0$-dimensional real Euclidean vector bundle with the Euclidean connection ∇^V and the curvature R^V. Let $\{\pm 2\pi\sqrt{-1}u_r\}$ $(1\le r\le b_0)$ be the formal Chern roots for $V\otimes C$. Let $\widetilde{V_C} = V_C - \dim V$.

Definition 2.4. The generalized elliptic genus of (M^{2d}, J) with respect to W and V, which we denote by $\mathrm{Ell}(M, W, V, \tau, z)$, $\widetilde{\mathrm{Ell}}(M, W, V, \tau, z)$, $\overline{\mathrm{Ell}}(M, W, V, \tau, z)$, are defined by

$$\begin{aligned}\mathrm{Ell}(M, W, V, \tau, z) &:= \Bigg(\exp\Big(\frac{a_0c_1(W)-c_1(M)}{2}\Big)Td(M)\,\mathrm{ch}(E(W,q,\tau))\det{}^{1/2}\cosh\Big(\frac{\sqrt{-1}}{4\pi}R^V\Big)\\ &\quad\cdot\mathrm{ch}\Big(\bigotimes_{m=1}^{\infty}\wedge_{q^m}(\widetilde{V_C})\otimes\bigotimes_{r=1}^{\infty}\wedge_{q^{r-1/2}}(\widetilde{V_C})\otimes\bigotimes_{s=1}^{\infty}\wedge_{-q^{s-1/2}}(\widetilde{V_C})\Big)\Bigg)^{(2d)},\end{aligned}$$

$$\begin{aligned}\widetilde{\mathrm{Ell}}(M, W, V, \tau, z) &:= \Bigg(\exp\Big(\frac{a_0c_1(W)-c_1(M)}{2}\Big)Td(M)\,\mathrm{ch}(E(W,q,\tau))\\ &\quad\cdot\Big(\det{}^{1/2}\cosh\Big(\frac{\sqrt{-1}}{4\pi}R^V\Big)\mathrm{ch}\Big(\bigotimes_{m=1}^{\infty}\wedge_{q^m}(\widetilde{V_C})\Big)\\ &\quad+\mathrm{ch}\Big(\bigotimes_{r=1}^{\infty}\wedge_{q^{r-1/2}}(\widetilde{V_C})\Big)+\mathrm{ch}\Big(\bigotimes_{s=1}^{\infty}\wedge_{-q^{s-1/2}}(\widetilde{V_C})\Big)\Big)\Bigg)^{(2d)},\end{aligned}$$

$$\begin{aligned}\overline{\mathrm{Ell}}(M, W, V, \tau, z) &:= \Bigg(\exp\Big(\frac{a_0c_1(W)-c_1(M)}{2}\Big)Td(M)\,\mathrm{ch}(E(W,q,\tau))\\ &\quad\cdot\det{}^{1/2}\Bigg(\frac{\sin\big(\frac{1}{4\pi^2}R^V\big)}{\frac{1}{4\pi^2}R^V}\Bigg)\mathrm{ch}\Big(\bigotimes_{m=1}^{\infty}\wedge_{-q^m}(\widetilde{V_C})\Big)\Bigg)^{(2d)}.\end{aligned}$$

Lemma 2.5. *We have*

$$\begin{aligned}\mathrm{Ell}(M, W, V, \tau, z) &= \Bigg(\eta(\tau)^{3(d-la_0)}\prod_{i=1}^{d}\frac{2\pi\sqrt{-1}x_i}{\theta(\tau,x_i)}\prod_{j=1}^{l}\prod_{r=1}^{a_0}\theta(\tau,w_j-m_rz)\\ &\quad\cdot\prod_{r=1}^{b_0}\Big(\frac{\theta_1(u_r,\tau)}{\theta_1(0,\tau)}\frac{\theta_2(u_r,\tau)}{\theta_2(0,\tau)}\frac{\theta_3(u_r,\tau)}{\theta_3(0,\tau)}\Big)\Bigg)^{(2d)},\end{aligned}$$

$$\begin{aligned}\widetilde{\mathrm{Ell}}(M, W, V, \tau, z) &= \Bigg(\eta(\tau)^{3(d-la_0)}\prod_{i=1}^{d}\frac{2\pi\sqrt{-1}x_i}{\theta(\tau,x_i)}\prod_{j=1}^{l}\prod_{r=1}^{a_0}\theta(\tau,w_j-m_rz)\\ &\quad\cdot\prod_{r=1}^{b_0}\Big(\frac{\theta_1(u_r,\tau)}{\theta_1(0,\tau)}+\frac{\theta_2(u_r,\tau)}{\theta_2(0,\tau)}+\frac{\theta_3(u_r,\tau)}{\theta_3(0,\tau)}\Big)\Bigg)^{(2d)},\end{aligned}$$

$$\begin{aligned}\overline{\mathrm{Ell}}(M, W, V, \tau, z) &= \Bigg(\eta(\tau)^{3(d-la_0)}\prod_{i=1}^{d}\frac{2\pi\sqrt{-1}x_i}{\theta(\tau,x_i)}\prod_{j=1}^{l}\prod_{r=1}^{a_0}\theta(\tau,w_j-m_rz)\\ &\quad\cdot\prod_{r=1}^{b_0}\Big(\frac{\theta(u_r,\tau)}{\theta_1(0,\tau)\theta_2(0,\tau)\theta_3(0,\tau)u_r}\Big)\Bigg)^{(2d)}.\end{aligned}$$

One has the following transformation laws of theta functions (see [2]):

$$\theta_1(v,\tau+1)=e^{\pi\sqrt{-1}/4}\theta_1(v,\tau),\quad \theta_1\left(v,-\frac{1}{\tau}\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_2(\tau v,\tau);$$

$$\theta_2(v,\tau+1)=\theta_3(v,\tau),\qquad \theta_2\left(v,-\frac{1}{\tau}\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_1(\tau v,\tau);$$

$$\theta_3(v,\tau+1)=\theta_2(v,\tau),\qquad \theta_3\left(v,-\frac{1}{\tau}\right)=\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2}e^{\pi\sqrt{-1}\tau v^2}\theta_3(\tau v,\tau).$$

By the above transformation laws, similar to Theorem 2.3, we have:

Theorem 2.6. *If $c_1(W)=0$ and the first Pontrjagin classes $p_1(M)$ and $p_1(E)$ equal $a_0p_1(W)$ and $p_1(E)=0$, respectively, then the generalized elliptic genus* $\mathrm{Ell}(M,W,V,\tau,z)$, $\widetilde{\mathrm{Ell}}(M,W,V,\tau,z)$, $\overline{\mathrm{Ell}}(M,W,V,\tau,z)$ *satisfies* (2-1) *and* (2-2).

3. Anomaly cancellation formulas for almost complex manifolds

We recall the definition of the Eisenstein series $G_{2k}(\tau)$:

$$G_{2k}(\tau):=-\frac{B_{2k}}{4k}+\sum_{n=1}^{\infty}\sigma_{2k-1}(n)\cdot q^n,$$

where $\sigma_k(n):=\sum_{m>0,\ m|n}m^k$ and B_{2k} are the Bernoulli numbers. It is well known that the whole grading ring of modular forms over $\mathrm{SL}(2,\mathbb{Z})$ are generated by $G_4(\tau)$ and $G_6(\tau)$. We recall Proposition 3.5 in [9].

Proposition 3.1 [9]. *Suppose a function $\varphi(\tau,z):\mathbb{H}\times\mathbb{C}\to\mathbb{C}$ satisfies*

$$\varphi\left(\frac{a\tau+b}{c\tau+d_0},\frac{z}{c\tau+d_0}\right)=(c\tau+d_0)^k\exp\left(\frac{2\pi\sqrt{-1}mcz^2}{c\tau+d_0}\right)\varphi(\tau,z);\quad \begin{pmatrix}a&b\\c&d_0\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}).$$

We define

$$\Phi(\tau,z):=\exp(-8\pi^2mG_2(\tau)z^2)\varphi(\tau,z):=\sum_{n\geq 0}a_n(\tau)\cdot z^n.$$

Then these $a_n(\tau)$ are modular forms of weight $k+n$ over $\mathrm{SL}(2,\mathbb{Z})$.

Proposition 3.2. *Let $c_1(W)=c_1(M)=0$ and the first Pontrjagin class $p_1(M)$ equal $a_0p_1(W)$. Then the series $a_n(M,W,\tau)$ determined by*

$$\exp\left(-4\pi^2l\left(\sum_{r=1}^{a_0}m_r\right)G_2(\tau)z^2\right)\mathrm{Ell}(M,W,\tau,z)=\sum_{n\geq 0}a_n(M,W,\tau)\cdot z^n$$

are modular forms of weight $d - la_0 + n$ *over* $\mathrm{SL}(2,\mathbb{Z})$. *The first five series of* $a_n(M, W, \tau)$ *are of the form*

$$\text{(3-1)}\quad a_0(M,W,\tau) = \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)} + q\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)} + q^2\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)} + O(q^3),$$

where

$$A_0 = T + T^* - 2(d - la_0) - W_0 - W_0^*, \quad W_0 = a_0 W,$$

and

$$A_1 = S^2T + T^*\otimes T + S^2T^* + \wedge^2 W_0^* + \wedge^2 W_0 + W_0^*\otimes W_0 + (2(d-la_0)-1)(W_0 + W_0^* - T - T^*) - (W_0 + W_0^*)\otimes(T+T^*) + (d-la_0)(2d-2la_0-3),$$

$$a_1(M,W,\tau) = \Bigg(2\pi\sqrt{-1}\,Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\Bigg(\sum_{r=1}^{a_0}m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0}m_r\Bigg)\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Bigg)\Bigg)^{(2d)} + q\big(Td(M)\,\mathrm{ch}(A_3)\big)^{(2d)} + O(q^2),$$

where

$$A_3 = 2\pi\sqrt{-1}(-2(d-la_0)+T+T^*-a_0(W+W^*))\cdot\sum_{r=1}^{a_0}m_r(1+\wedge_{-1}(W^*))^{a_0-1}\otimes(-W^*+2\wedge^2W^*+\cdots+(-1)^l l\wedge^l W^*) + (1+\wedge_{-1}(W^*))^{a_0}\Bigg(-2\pi\sqrt{-1}\sum_{r=1}^{a_0}m_r(W+W^*) - l\pi\sqrt{-1}\sum_{r=1}^{a_0}m_r\big(-2(d-la_0)+T+T^*-a_0(W+W^*)\big)\Bigg),$$

$$a_2(M,W,\tau) = \Bigg(-2\pi^2\,Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\Bigg(\sum_{r=1}^{a_0}m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0}m_r\Bigg)^2\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Bigg)\Bigg)^{(2d)}$$
$$+\frac{l}{6}\Bigg(\sum_{r=1}^{a_0}m_r^2\Bigg)\pi^2\Bigg(Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Bigg)\Bigg)^{(2d)} + O(q),$$

$$
\begin{aligned}
&a_3(M,W,\tau)\\
&=\Bigg(\tfrac{4}{3}\pi^3(\sqrt{-1})^3 Td(M)\,\mathrm{ch}\Big(\sum_{p_1,\ldots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\Big(\sum_{r=1}^{a_0}m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0}m_r\Big)\\
&\qquad\qquad \wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Big)\Bigg)^{(2d)}\\
&\quad+\frac{\sqrt{-1}l}{3}\Big(\sum_{r=1}^{a_0}m_r^2\Big)\pi^3\Bigg(Td(M)\,\mathrm{ch}\Big(\sum_{p_1,\ldots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\Big(\sum_{r=1}^{a_0}m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0}m_r\Big)\\
&\qquad\qquad \wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Big)\Bigg)^{(2d)}\\
&\quad+O(q),\\
&a_4(M,W,\tau)=\Bigg(\tfrac{2}{3}\pi^4 Td(M)\\
&\quad\cdot\mathrm{ch}\Big(\sum_{p_1,\ldots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\Big(\sum_{r=1}^{a_0}m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0}m_r\Big)^4\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Big)\Bigg)^{(2d)}\\
&\quad-\frac{l^2}{3}\Big(\sum_{r=1}^{a_0}m_r^2\Big)^2\pi^4\Bigg(Td(M)\,\mathrm{ch}\Big(\sum_{p_1,\ldots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\Big(\sum_{r=1}^{a_0}m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0}m_r\Big)^2\\
&\qquad\qquad \wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Big)\Bigg)^{(2d)}\\
&+\frac{l^2}{72}\Big(\sum_{r=1}^{a_0}m_r^2\Big)^2\pi^4\Bigg(Td(M)\,\mathrm{ch}\Big(\sum_{p_1,\ldots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Big)\Bigg)^{(2d)}\\
&\qquad+O(q),
\end{aligned}
$$

Proof. We know that $\mathrm{Ell}(M,W,\tau,0)=a_0(M,W,\tau)$ and

$$\mathrm{Ell}(M,W,\tau,0)=\big(Td(M)\,\mathrm{ch}(E(M,W,\tau,0))\big)^{(2d)},$$

where

$$
\begin{aligned}
\mathrm{E}(M,W,\tau,0)&:=c^{2(d-la_0)}\wedge_{-1}(W_0^*)\bigotimes_{n=1}^{\infty}\wedge_{-q^n}(W_0^*)\wedge_{-q^n}(W_0)\\
&\qquad\otimes\Big(\bigotimes_{n=1}^{\infty}S_{q^n}(T^*)\otimes S_{q^n}(T)\Big)\\
&=\wedge_{-1}(W^*)+q\wedge_{-1}(W^*)\otimes A_0+q^2\wedge_{-1}(W^*)\otimes A_1+O(q^3).
\end{aligned}
$$

So we get (3-1). If we set

$$\exp\left(-4\pi^2 l\left(\sum_{r=1}^{a_0} m_r\right)G_2(\tau)z^2\right) := C_0(Z)+C_1(z)q+O(q^2),$$

and

$$\mathrm{Ell}(M, W, \tau, z) := B_0(Z)+B_1(z)q+O(q^2),$$

we can get that

$$C_0(z)=1+\frac{l}{6}\left(\sum_{r=1}^{a_0} m_r^2\right)\pi^2 z^2+\frac{l^2}{72}\left(\sum_{r=1}^{a_0} m_r^2\right)^2\pi^4 z^4+O(z^6),$$

$$C_1(z)=-4l\left(\sum_{r=1}^{a_0} m_r^2\right)\pi^2 z^2-\frac{2l^2}{3}\left(\sum_{r=1}^{a_0} m_r^2\right)^2\pi^4 z^4+O(z^6),$$

$$\begin{aligned}
B_0(z)&=\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)}\\
&+\left(2\pi\sqrt{-1}Td(M)\,\mathrm{ch}\left(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\left(\sum_{r=1}^{a_0} m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0} m_r\right)\right.\right.\\
&\qquad\left.\left.\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\right)\right)^{(2d)} z\\
&+\left(-2\pi^2 Td(M)\,\mathrm{ch}\left(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\left(\sum_{r=1}^{a_0} m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^2\right.\right.\\
&\qquad\left.\left.\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\right)\right)^{(2d)} z^2\\
&+\left(\tfrac{4}{3}\pi^3(\sqrt{-1})^3 Td(M)\,\mathrm{ch}\left(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\left(\sum_{r=1}^{a_0} m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^3\right.\right.\\
&\qquad\left.\left.\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\right)\right)^{(2d)} z^3\\
&+\left(\tfrac{2}{3}\pi^4 Td(M)\,\mathrm{ch}\left(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\left(\sum_{r=1}^{a_0} m_r p_r-\frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^4\right.\right.\\
&\qquad\left.\left.\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\right)\right)^{(2d)} z^4+O(z^5),
\end{aligned}$$

$$\begin{aligned}B_1(y)=&\big(Td(M)\,\mathrm{ch}\big(\wedge_{-1}(W_0^*)\otimes(-2(d-la_0)+T+T^*-a_0(W+W^*))\big)\big)^{(2d)}\\&+\bigg(Td(M)\,\mathrm{ch}\big(2\pi\sqrt{-1}(-2(d-la_0)+T+T^*-a_0(W+W^*))\big)\\&\qquad\cdot\sum_{r=1}^{a_0}m_r(1+\wedge_{-1}(W^*))^{a_0-1}\otimes(-W^*+2\wedge^2W^*+\cdots+(-1)^l l\wedge^l W^*)\\&\qquad\qquad+(1+\wedge_{-1}(W^*))^{a_0}\bigg)\\&\quad\cdot\bigg(-2\pi\sqrt{-1}\sum_{r=1}^{a_0}m_r(W+W^*)-l\pi\sqrt{-1}\sum_{r=1}^{a_0}m_r(-2(d-la_0)+T+T^*\\&\qquad\qquad-a_0(W+W^*))\bigg)^{(2d)}z\\&+O(z^2).\end{aligned}$$

We know that

$$\sum_{n\geq 0}a_n(M,\,W,\,\tau)\cdot z^n=C_0(z)B_0(z)+(C_0(z)B_1(z)+C_1(z)B_0(z))q+\cdots. \tag{3-2}$$

Then we can get Proposition 3.2 by the penultimate display above and (3-2). □

Since there are no $\mathrm{SL}(2,\mathbb{Z})$ modular forms with the odd weight or nonzero weight ≤ 2, we have:

Proposition 3.3. *Let $c_1(W)=c_1(M)=0$ and the first Pontrjagin class $p_1(M)$ equal $a_0p_1(W)$.*

(1) *If either $d-la_0$ is odd or $d-la_0\leq 2$ but $d-la_0\neq 0$, then*

$$\begin{aligned}\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)}=\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)}&=0,\\ \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)}&=0,\end{aligned}$$

and for complex manifolds,

$$\chi(M,\,\wedge_{-1}(W_0^*))=\chi(M,\,\wedge_{-1}(W_0^*)\otimes(A_0))=\chi(M,\,\wedge_{-1}(W_0^*)\otimes(A_1))=0,$$

where $\chi(M,\,\wedge_{-1}(W_0^*))$ denotes the twisted holomorphic Euler characteristic number.

(2) *If either $d-la_0$ is even or $d-la_0\leq 1$ but $d-la_0\neq -1$, then*

$$\begin{aligned}\bigg(Td(M)\,\mathrm{ch}\bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\bigg(\sum_{r=1}^{a_0}m_rp_r-\frac{l}{2}\sum_{r=1}^{a_0}m_r\bigg)\\ \wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\bigg)\bigg)^{(2d)}&=0,\\ (Td(M)\,\mathrm{ch}(A_3))^{(2d)}&=0,\end{aligned}$$

and for complex manifolds,

$$\chi\left(M, \sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r}\left(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\right) \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right) = 0,$$

$$\chi(M, A_3) = 0.$$

(3) *If either $d - la_0$ is odd or $d - la_0 \leq 0$ but $d - la_0 \neq -2$, then*

$$\left(-2Td(M)\,\mathrm{ch}\left(\sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r}\left(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^2 \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right)\right)^{(2d)}$$
$$+\frac{l}{6}\left(\sum_{r=1}^{a_0} m_r^2\right)\left(Td(M)\,\mathrm{ch}\left(\sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r} \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right)\right)^{(2d)} = 0.$$

(4) *If either $d - la_0$ is even or $d - la_0 \leq -1$ but $d - la_0 \neq -3$, then*

$$\left(\tfrac{4}{3}\pi^3(\sqrt{-1})^3 Td(M)\cdot\mathrm{ch}\left(\sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r}\left(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^3 \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right)\right)^{(2d)}$$
$$+\frac{\sqrt{-1}\,l}{3}\left(\sum_{r=1}^{a_0} m_r^2\right)\pi^3\left(Td(M)\,\mathrm{ch}\left(\sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r}\left(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\right) \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right)\right)^{(2d)} = 0.$$

(5) *If either $d - la_0$ is odd or $d - la_0 \leq -2$ but $d - la_0 \neq -4$, then*

$$\left(\tfrac{2}{3}\pi^4 Td(M)\,\mathrm{ch}\left(\sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r}\left(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^4 \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right)\right)^{(2d)}$$
$$-\frac{l^2}{3}\left(\sum_{r=1}^{a_0} m_r^2\right)^2\pi^4\left(Td(M)\,\mathrm{ch}\left(\sum_{p_1,\ldots,p_{a_0}=0}^{l} (-1)^{\sum_{r=1}^{a_0} p_r}\left(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\right)^2 \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\right)\right)^{(2d)}$$

$$+\frac{l^2}{72}\Big(\sum_{r=1}^{a_0} m_r^2\Big)^2\pi^4\Big(Td(M)\,\mathrm{ch}\Big(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0}p_r}\wedge^{p_1}W^*\otimes\cdots\otimes\wedge^{p_{a_0}}W^*\Big)\Big)^{(2d)}$$
$$=0.$$

Similarly in cases (3)–(5), *we have expressions using the twisted holomorphic Euler characteristic number.*

Theorem 3.4. *Let $c_1(W)=c_1(M)=0$ and the first Pontrjagin class $p_1(M)$ equal $a_0p_1(W)$.*

(1) *If $d-la_0=4$, then*

$$\begin{aligned}(3\text{-}3)\qquad &\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)}=240\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)},\\ &\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)}=2160\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)},\end{aligned}$$

and for complex manifolds,

$$\begin{aligned}\chi(M,\wedge_{-1}(W_0^*)\otimes A_0)&=240\chi(M,\wedge_{-1}(W_0^*)),\\ \chi(M,\wedge_{-1}(W_0^*)\otimes A_1)&=2160\chi(M,\wedge_{-1}(W_0^*)),\end{aligned}$$

so $\chi(M,\wedge_{-1}(W_0^*)\otimes A_0)$ is an integer multiple of 240 *and $\chi(M,\wedge_{-1}(W_0^*)\otimes A_1)$ is an integer multiple of* 2160.

(2) *If $d-la_0=6$, then*

$$\begin{aligned}(3\text{-}4)\qquad &\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)}=-504\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)},\\ &\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)}=-16632\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)},\end{aligned}$$

and for complex manifolds,

$$\begin{aligned}\chi(M,\wedge_{-1}(W_0^*)\otimes A_0)&=-504\chi(M,\wedge_{-1}(W_0^*)),\\ \chi(M,\wedge_{-1}(W_0^*)\otimes A_1)&=-16632\chi(M,\wedge_{-1}(W_0^*)),\end{aligned}$$

so $\chi(M,\wedge_{-1}(W_0^*)\otimes A_0)$ is an integer multiple of 504 *and $\chi(M,\wedge_{-1}(W_0^*)\otimes A_1)$ is an integer multiple of* 16632.

(3) *If $d-la_0=8$, then*

$$\begin{aligned}(3\text{-}5)\qquad &\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)}=480\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)},\\ &\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)}=61920\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)},\end{aligned}$$

and for complex manifolds,

$$\begin{aligned}\chi(M,\wedge_{-1}(W_0^*)\otimes A_0)&=480\chi(M,\wedge_{-1}(W_0^*)),\\ \chi(M,\wedge_{-1}(W_0^*)\otimes A_1)&=61920\chi(M,\wedge_{-1}(W_0^*)),\end{aligned}$$

so $\chi(M, \wedge_{-1}(W_0^*) \otimes A_0)$ *is an integer multiple of* 480 *and* $\chi(M, \wedge_{-1}(W_0^*) \otimes A_1)$ *is an integer multiple of* 61920.

(4) *If* $d - la_0 = 10$, *then*

$$\text{(3-6)} \quad \begin{aligned} \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)} &= -264\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)}, \\ \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)} &= -135432\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)}, \end{aligned}$$

and for complex manifolds,

$$\begin{aligned} \chi(M, \wedge_{-1}(W_0^*) \otimes A_0) &= -264\chi(M, \wedge_{-1}(W_0^*)), \\ \chi(M, \wedge_{-1}(W_0^*) \otimes A_1) &= -135432\chi(M, \wedge_{-1}(W_0^*)), \end{aligned}$$

so $\chi(M, \wedge_{-1}(W_0^*) \otimes A_0)$ *is an integer multiple of* 264 *and* $\chi(M, \wedge_{-1}(W_0^*) \otimes A_1)$ *is an integer multiple of* 135432.

(5) *If* $d - la_0 = 12$, *then*

$$\text{(3-7)} \quad \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)} = 196560\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)} - 24\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)},$$

and for complex manifolds, $\chi(M, \wedge_{-1}(W_0^*) \otimes A_1)$ *is an integer multiple of* 24.

(6) *If* $d - la_0 = 14$, *then*

$$\text{(3-8)} \quad \begin{aligned} \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)} &= -24\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)}, \\ \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)} &= -196632\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)}, \end{aligned}$$

and for complex manifolds,

$$\begin{aligned} \chi(M, \wedge_{-1}(W_0^*) \otimes A_0) &= -24\chi(M, \wedge_{-1}(W_0^*)), \\ \chi(M, \wedge_{-1}(W_0^*) \otimes A_1) &= -196632\chi(M, \wedge_{-1}(W_0^*)), \end{aligned}$$

so $\chi(M, \wedge_{-1}(W_0^*) \otimes A_0)$ *is an integer multiple of* 24 *and* $\chi(M, \wedge_{-1}(W_0^*) \otimes A_1)$ *is an integer multiple of* 196632.

(7) *If* $d - la_0 = 16$, *then*

$$\text{(3-9)} \quad \big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_1)\big)^{(2d)} = 146880\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\big)^{(2d)} + 216\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_0)\big)^{(2d)},$$

and for complex manifolds, $\chi(M, \wedge_{-1}(W_0^*) \otimes A_1)$ *is an integer multiple of* 216.

Proof. $a_0(M, W, \tau)$ is a modular form of weight $d - la_0$ over $\mathrm{SL}(2, \mathbb{Z})$.

(1) If $d - la_0 = 4$, then $a_0(M, W, \tau)$ is proportional to

$$G_4(\tau) = 1 + 240q + 2160q^2 + 6720q^3 + \cdots,$$

so (3-3) holds.

(2) If $d-la_0=6$, then $a_0(M, W, \tau)$ is proportional to

$$G_6(\tau)=1-504q-16632q^2-122976q^3+\cdots,$$

so (3-4) holds.

(3) If $d-la_0=8$, then $a_0(M, W, \tau)$ is proportional to

$$G_4(\tau)^2=1+480q+61920q^2+\cdots,$$

so (3-5) holds.

(4) If $d-la_0=10$, then $a_0(M, W, \tau)$ is proportional to

$$G_4(\tau)G_6(\tau)=1-264q-135432q^2+\cdots,$$

so (3-6) holds.

(5) If $d-la_0=12$, then

$$a_0(M, W, \tau)=\lambda_1 G_4(\tau)^3+\lambda_2 G_6(\tau)^2, \tag{3-10}$$

where λ_1, λ_2 are degree-$2d$ forms. We have

$$G_4(\tau)^3=1+720q+179280q^2+\cdots,$$
$$G_6(\tau)^2=1-1008q+220752q^2+\cdots.$$

In (3-10), we compare the coefficients of 1, q, q^2, and we get three equations about λ_1, λ_2. Solving the three equations, we get (3-7).

(6) If $d-la_0=14$, then $a_0(M, W, \tau)$ is proportional to

$$G_4(\tau)^2G_6(\tau)=1-24q-196632q^2+\cdots,$$

so (3-8) holds.

(7) If $d-la_0=16$, then

$$a_0(M, W, \tau)=\lambda_1 G_4(\tau)^4+\lambda_2 G_4(\tau)G_6(\tau)^2, \tag{3-11}$$

where λ_1, λ_2 are degree-$2d$ forms. We have

$$G_4(\tau)^4=1+960q+354240q^2+\cdots, \tag{3-12}$$

$$G_4(\tau)G_6(\tau)^2=1-768q-19008q^2+\cdots. \tag{3-13}$$

By (3-11)–(3-13), we get (3-9). □

Similarly, by $a_1(M, W, \tau)$, we have:

Theorem 3.5. *Let $c_1(W)=c_1(M)=0$ and the first Pontrjagin class $p_1(M)$ equal $a_0p_1(W)$.*

(1) *If* $d - la_0 = 3$, *then*

$$\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_3)\big)^{(2d)} = 240\Bigg(2\pi\sqrt{-1}Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0} p_r}\Bigg(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\Bigg) \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\Bigg)\Bigg)^{(2d)}$$

and for complex manifolds, $\chi(M, \wedge_{-1}(W_0^*) \otimes A_3)$ *is an integer multiple of* 240.

(2) *If* $d - la_0 = 5$, *then*

$$\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_3)\big)^{(2d)} = -504\Bigg(2\pi\sqrt{-1}Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0} p_r}\Bigg(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\Bigg) \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\Bigg)\Bigg)^{(2d)}$$

and for complex manifolds, $\chi(M, \wedge_{-1}(W_0^*) \otimes A_3)$ *is an integer multiple of* 504.

(3) *If* $d - la_0 = 7$, *then*

$$\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_3)\big)^{(2d)} = 480\Bigg(2\pi\sqrt{-1}Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0} p_r}\Bigg(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\Bigg) \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\Bigg)\Bigg)^{(2d)}$$

and for complex manifolds, $\chi(M, \wedge_{-1}(W_0^*) \otimes A_3)$ *is an integer multiple of* 480.

(4) *If* $d - la_0 = 9$, *then*

$$\big(Td(M)\,\mathrm{ch}(\wedge_{-1}(W_0^*))\,\mathrm{ch}(A_3)\big)^{(2d)} = -264\Bigg(2\pi\sqrt{-1}Td(M)\,\mathrm{ch}\Bigg(\sum_{p_1,\dots,p_{a_0}=0}^{l}(-1)^{\sum_{r=1}^{a_0} p_r}\Bigg(\sum_{r=1}^{a_0} m_r p_r - \frac{l}{2}\sum_{r=1}^{a_0} m_r\Bigg) \wedge^{p_1} W^* \otimes \cdots \otimes \wedge^{p_{a_0}} W^*\Bigg)\Bigg)^{(2d)}$$

and for complex manifolds, $\chi(M, \wedge_{-1}(W_0^*) \otimes A_3)$ *is an integer multiple of* 264.

We remark that the ideas in the proofs of Theorems 3.4 and 3.5 appear in [9].

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