Pacific Journal of Mathematics

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Volume 333 No. 2 December 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.197

MULTIPLIER ALGEBRAS OF L^p-OPERATOR ALGEBRAS

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It is known that the multiplier algebra of an approximately unital and nondegenerate L^p-operator algebra is again an L^p-operator algebra. In this paper we investigate examples that drop both hypotheses. In particular, we show that the multiplier algebra of T_2^p, the algebra of strictly upper triangular 2×2 matrices acting on ℓ_2^p, is still an L^p-operator algebra for any p. To contrast this result, we first provide a thorough study of the augmentation ideal of $\ell^1(G)$ for a discrete group G. We use this ideal to define a family of nonapproximately unital degenerate L^p-operator algebras, $F_0^p(\mathbb{Z}/3\mathbb{Z})$, whose multiplier algebras cannot be represented on any L^q-space for any $q \in [1, \infty)$ as long as $p \in [1, p_0] \cup [p_0', \infty)$, where $p_0 = 1.606$ and p_0' is its Hölder conjugate.

1. Introduction

In 1971, C. S. Herz studied certain operators acting on L^p-spaces (see [17]). In fact, Herz constructed group algebras of operators acting on L^p-spaces, those were referred to as algebras of p-pseudofunctions (see [18] and also [9] for a modern approach). About 15 years ago, the study of these operators regained interest from the community thanks to the general research works of M. Daws, E.

MSC2020: primary 46H15, 46H35; secondary 47L10.
Keywords: multiplier algebra, augmentation ideal, representations on L^p-spaces, L^p-group algebras, p-operator norms.

Gardella, N. C. Phillips, N. Spronk, and H. Thiel, among others (see [5; 6; 12; 13; 14; 21], for instance). This area is now known as L^p-operator algebras. Given that L^p-operator algebras include the large and well studied class of C*-algebras, part of the research in this area focuses on understanding what C*-properties and constructions can and cannot be extended to the L^p-case. In particular it is worth mentioning that L^2-operator algebras include the class of C*-algebras, but also the larger class of nonselfadjoint operator algebras. Three big differences between C*-algebras and L^p-operator algebras, even when $p = 2$, which will be exploited here are: (1) L^p-operator norms are not unique, (2) some L^p-operator algebras cannot be nondegenerately represented on any Banach space, and (3) some L^p-operator algebras do not have contractive approximate identities (which will be henceforth abbreviated as cai's). A usually hard problem in L^p-operator algebras, arising in some sense due to the difficulty of computing L^p-operator norms, is to determine whether a given Banach algebra can be isometrically represented on an L^p-space. Different techniques have been employed to show that certain Banach algebras cannot be represented on L^p-spaces; see for instance Theorem 2.5 in [11] and Example 4.7 in [3], where it is shown that the class of L^p-operator algebras is not closed under taking quotients when $p \neq 2$. As a main result of this paper, we look at certain algebras with no contractive approximate identities to conclude that, for certain values of $p \neq 2$, the class of L^p-operator algebras is not closed under taking multiplier algebras.

This paper is organized as follows: We start with a brief review of the multiplier algebra construction (as double centralizers) for general Banach algebras. In Section 3, we present main results for the construction of the multiplier algebra, $M(A)$, of a general Banach algebra A. It is known that if a Banach algebra A has a cai and is nondegenerately representable on a Banach space E, then $M(A)$ can also be nondegenerately represented on E. In fact, in Corollary 3.6, we prove that for the known representation of $M(A)$ on E the action is given by two-sided multipliers, agreeing with the well known C*-result. Next, we investigate general properties of $M(A)$ when A has a nonunital identity which means that A an algebraic multiplicative identity with norm greater than 1 (see Definition 2.5). This condition automatically prevents A from having a cai and also from being nondegenerately represented on any Banach space, making these algebras suitable candidates to investigate whether their multiplier algebras can be nondegenerately represented on Banach spaces.

Starting in Section 4, we work only with multiplier algebras of L^p-operator algebras. As before, when A is a nondegenerate L^p-operator algebra with a cai, its multiplier algebra is also a nondegenerate L^p-operator algebra. We begin Section 4 by exploring whether these assumptions are necessary for the multiplier algebra of an L^p-operator algebra to be itself an L^p-operator algebra. Consequently, we produce two contrasting examples dealing with the multiplier algebra of degenerate L^p-operator algebras without cai's:

The first one, explored in Section 4B, deals with strictly upper triangular matrices, denoted T_d^p, acting on $\ell_d^p = \ell^p(\{1, \dots, d\})$. When $d = 2$, even though T_2^p is itself a degenerate L^p-operator algebra without a cai, we have the following main result:

Theorem 4.2. *For any $p \in [1, \infty)$, $M(T_2^p)$ is isometrically isomorphic to $\mathbb{C}^2$ with the supremum norm. In particular, for any $p, q \in [1, \infty)$, $M(T_2^p)$ is an L^q-operator algebra that is nondegenerately representable on ℓ_2^q.*

The second one is explored in Section 4C, where we focus on $\ell_0^1(G)$, the augmentation ideal of $\ell^1(G)$ for a discrete group G. As an algebra, $\ell_0^1(G)$ is an L^1-operator algebra that has an identity with norm strictly greater than 1 when G is finite and $\mathrm{card}(G) > 2$. The augmentation ideal $\ell_0^1(G)$ is defined as the kernel of the map $\ell^1(G) \to \mathbb{C}$ given by $a \mapsto \sum_{g \in G} a(g)$. We then study the main properties of this ideal and show that, when G is finite, $\ell_0^1(G)$ has no cai and cannot be nondegenerately represented on any Banach space (see Proposition 4.13(7)). For our main result, we construct a family of L^p-operator algebras whose multiplier algebras cannot be represented on any L^q-space for any $q \in [1, \infty)$ as long as $p \in [1, p_0] \cup [p_0', \infty)$, where $p_0 = 1.606$ and p_0' is its Hölder conjugate. This is done by looking at the image of the augmentation ideal $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ in the group algebra $F^p(\mathbb{Z}/3\mathbb{Z})$ under the left regular representation $\lambda_p : \ell^1(\mathbb{Z}/3\mathbb{Z}) \to \mathcal{L}(\ell^p(\mathbb{Z}/3\mathbb{Z}))$. That is, for each $p \in [1, \infty)$, we put

$$F_0^p(\mathbb{Z}/3\mathbb{Z}) = \lambda_p(\ell_0^1(\mathbb{Z}/3\mathbb{Z})),$$

and then denote its multiplier algebra as $M_0^p(\mathbb{Z}/3\mathbb{Z}) = M(F_0^p(\mathbb{Z}/3\mathbb{Z}))$. The Banach algebra $F_0^p(\mathbb{Z}/3\mathbb{Z})$ has an algebraic identity, denoted $\mathbb{1}_0$, whose norm is strict greater than 1. However, in Proposition 4.18 we show that $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is simply $F_0^p(\mathbb{Z}/3\mathbb{Z})$ equipped with a different norm in which $\mathbb{1}_0$ now has norm 1. Our main result in this subsection is then the following:

Theorem 4.21. *Let $p_0 = 1.606$ and let $p_0' = \frac{p_0}{p_0 - 1}$. Take any $p \in [1, p_0] \cup [p_0', \infty)$. Then $M_0^p(\mathbb{Z}/3\mathbb{Z})$, the multiplier algebra of $F_0^p(\mathbb{Z}/3\mathbb{Z})$, is not an L^q-operator algebra for any $q \in [1, \infty)$.*

The main ingredient in the proof of Theorem 4.21 follows a technique previously used by Blecher and Phillips in Section 4 of [3] to show that the class of L^p-operator algebras is not closed under taking quotients. To be more precise, we exhibit a bicontractive idempotent $e \in M_0^p(\mathbb{Z}/3\mathbb{Z})$ such that $\mathbb{1}_0 - 2e$ is not an invertible isometry when $p \in [1, p_0] \cup [p_0', \infty)$, contradicting a structural result of Bernau and Lacey, Theorem 2.9 below, regarding bicontractive idempotent acting on L^p-spaces. Unfortunately, the bicontractive idempotent argument might not work for groups other that $\mathbb{Z}/3\mathbb{Z}$. For instance, when $G = \mathbb{Z}/4\mathbb{Z}$ or $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the bicontractive idempotents in $M_0^p(G) = M(F_0^p(G))$ behave as expected for an L^p-operator algebra. The essential difficulty in deciding whether $M_0^p(G)$ is an L^p-operator algebra for a general finite group G is that its norm is generally hard to compute (see [16] for instance).

In Section 5, we restrict our attention to nontrivial finite abelian groups and show that when $n = \mathrm{card}(G) > 1$, the algebra $\ell_0^1(G)$ is algebraically isomorphic to $\mathbb{C}^{n-1}$ with pointwise multiplication via the Gelfand map. This yields a nice description of the idempotents of $M_0^p(\mathbb{Z}/n\mathbb{Z})$, see Proposition 5.8, that can be compared with Rudin's more general work for idempotents in $L^1(G)$ when G is a locally compact group (see W. Rudin's work [25] and [24]).

In the last section we present a brief discussion on different norms that make $\mathbb{C}^n$ an L^p-operator algebra, posing a final question regarding what properties a norm on $\mathbb{C}^n$ must satisfy to be an L^p-operator algebra. Answering this general question might provide a tool to decide when $M_0^p(G)$ is representable on an L^q-space.

We end our introduction with a brief overview of the general notation that will be used throughout the paper. When E is a Banach space, we denote by $\mathcal{L}(E)$ the space of all bounded linear maps $E \to E$. For $p \in [1, \infty)$, we sometimes write $L^p(\mu)$ for $L^p(\Omega, \mu)$, the L^p-space of a measure space (Ω, μ). In particular, when ν is the counting measure on Ω we simply write $\ell^p(\Omega)$ for $L^p(\Omega, \nu)$ and, for $d \in \mathbb{Z}_{\geq 2}$, we write ℓ_d^p for $\ell^p(\{1, \ldots, d\})$.

2. Preliminaries

We first establish standard definitions for representations of Banach algebras on Banach spaces. The definition of L^p-operator algebras will be given using this terminology. We then present basic facts about Banach algebras with either contractive approximate units or with nonunital identities.

Definition 2.1. Let A be a Banach algebra and let E be a Banach space. A *representation* of A on E is a continuous algebra homomorphism $\pi : A \to \mathcal{L}(E)$.

(1) We say that π is *contractive* if $\|\pi(a)\| \leq \|a\|$ for all $a \in A$.

(2) We say that π is *isometric* if $\|\pi(a)\| = \|a\|$ for all $a \in A$.

(3) We say that π is *nondegenerate* if

$$\pi(A)E = \mathrm{span}(\{\pi(a)\xi : a \in A \text{ and } \xi \in E\})$$

is dense in E, and that A is *nondegenerately representable* if it has a nondegenerate isometric representation.

Definition 2.2. Let $p \in [1, \infty)$. A Banach algebra A is an *L^p-operator algebra* if there is a measure space (Ω, μ) and an isometric representation of A on $L^p(\mu)$.

Definition 2.3. Let A be a Banach algebra. We say that A has a *contractive approximate identity* (*cai*) if there is a net $(e_\lambda)_{\lambda\in\Lambda}$ in A such that $\|e_\lambda\| \le 1$ for all $\lambda \in \Lambda$ and for all $a \in A$,

$$\lim_{\lambda\in\Lambda} \|ae_\lambda - a\| = \lim_{\lambda\in\Lambda} \|e_\lambda a - a\| = 0.$$

Lemma 2.4. *Let A be a Banach algebra with a cai. Then for any $a \in A$,*

$$\|a\| = \sup_{\|b\|=1} \|ab\| = \sup_{\|b\|=1} \|ba\|.$$

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a cai of A. Then for any $a \in A$,

$$\|a\| \ge \sup_{\|b\|=1} \|ba\| \ge \sup_{\lambda\in\Lambda} \frac{\|e_\lambda a\|}{\|e_\lambda\|} \ge \sup_{\lambda\in\Lambda} \|e_\lambda a\| \ge \|a\|.$$

The other equality is proved analogously. □

Definition 2.5. Let A be a Banach algebra. We say that A has an *identity element* if there is an element $1_A \in A$ such that, for all $a \in A$,

$$1_A \cdot a = a = a \cdot 1_A.$$

If $\|1_A\| = 1$, we call 1_A a *unit* of A; otherwise we say 1_A is a *nonunital identity* of A.

We will work below with a Banach algebra that has a nonunital identity, see Section 4C. The following two lemmas show that such Banach algebras cannot have cai's and cannot be nondegenerately represented on any Banach space.

Lemma 2.6. *Let A be a Banach algebra with a nonunital identity $1_A \in A$ (that is $\|1_A\| \ne 1$). Then, A cannot have a cai.*

Proof. It is clear that $\|1_A\| > 1$. Now suppose A has a cai, say $(e_\lambda)_{\lambda\in\Lambda}$. Then, for each $\lambda \in \Lambda$

$$0 \le |\|e_\lambda\| - \|1_A\|| \le \|e_\lambda - 1_A\| = \|e_\lambda 1_A - 1_A\|.$$

Therefore, by the squeeze theorem

$$\lim_\lambda |\|e_\lambda\| - \|1_A\|| = 0.$$

This implies that $\lim_\lambda \|e_\lambda\| = \|1_A\|$, whence $\|1_A\| \le 1$ which is a contradiction. □

Lemma 2.7. *Let A be a Banach algebra with a nonunital identity $1_A \in A$ (that is $\|1_A\| \ne 1$). Then A cannot be nondegenerately represented on any Banach space.*

Proof. Assume on the contrary that there is a Banach space E and an isometric nondegenerate representation $\pi : A \to \mathcal{L}(E)$. Then, for any $a \in A$ and any $\xi \in E$ we have

$$\pi(1_A)\pi(a)\xi = \pi(1_A a)\xi = \pi(a)\xi.$$

Nondegeneracy now gives that $\pi(1_A) = \mathrm{id}_E$, but this implies

$$1 = \|\mathrm{id}_E\| = \|\pi(1_A)\| = \|1_A\| > 1,$$

a contradiction. □

In Section 4, we will take the multiplier algebra of certain L^p-operator algebras in order to construct a family of unital Banach algebras that are not representable on any L^q-space for any $q \in [1, \infty)$. Our technique, which was also used by Blecher and Phillips in Section 4 of [3], will be to show that some bicontractive idempotents on the algebra do not behave as expected for L^q-operator algebras. We record precise definitions and known facts below.

Definition 2.8. Let A be a unital Banach algebra with unit 1_A.

(1) An idempotent $e \in A$ is *bicontractive* if $\|e\| \leq 1$ and $\|1_A - e\| \leq 1$.

(2) An element $s \in A$ is an *invertible isometry* if $\|s\| = 1$ and $\|s^{-1}\| = 1$.

The following is a combination of Theorem 2.1 in [1] and part (c) of Lemma 2.29 in [3].

Theorem 2.9. *Let $p \in [1, \infty)$ and let (Ω, μ) be a measure space. Then an idempotent $e \in \mathcal{L}(L^p(\mu))$ is bicontractive if and only if $1 - 2e$ is an invertible isometry, where $1 = \mathrm{id}_{L^p(\mu)}$.*

3. Multiplier algebras

Below we present general definitions and some results about multiplier algebras of Banach algebras.

Parts of following definition come from Section 2.5 in [4].

Definition 3.1. Let A be a Banach algebra.

(1) A *left multiplier of A* is a map $L \in \mathcal{L}(A)$ satisfying

$$L(ab) = L(a)b \quad \text{for all} \quad a, b \in A.$$

The set of left multipliers of A is denoted by $LM(A)$.

(2) A *right multiplier of A* is a map $R \in \mathcal{L}(A)$ satisfying

$$R(ab) = aR(b) \quad \text{for all} \quad a, b \in A.$$

The set of right multipliers of A is denoted by $RM(A)$.

(3) A *double centralizer of A* is a pair (L, R) with $L \in LM(A)$, $R \in RM(A)$, and

$$aL(b) = R(a)b \quad \text{for all} \quad a, b \in A.$$

We define $M(A)$, the *multiplier algebra of A*, to be the subset of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ (equipped with the supremum norm) consisting of double centralizers.

It is clear that $M(A)$ is a unital Banach subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$.

If A is a C*-algebra, $M(A)$ is equivalently defined as the set of two sided multipliers of A on any Hilbert space as long as A acts nondegenerately on it; see Definition 2.2.2 in [27] for instance. This will be also the case for a Banach algebra that has a cai and that can be nondegenerately represented on a Banach space. We start with the definition of a map from A to $M(A)$ that will be a canonical inclusion when A has a cai.

Definition 3.2. Let A be a Banach algebra. For each $a \in A$ we define maps $L_a : A \to A$ and $R_a : A \to A$ by left and right multiplication respectively, that is for any $b \in A$

$$L_a(b) = ab \quad \text{and} \quad R_a(b) = ba.$$

We get a map $\iota_A : A \to M(A)$ given by $\iota_A(a) = (L_a, R_a)$.

Lemma 3.3. *Let A be a Banach algebra with a cai, and let $(L, R) \in M(A)$. Then $\|L\| = \|R\|$. Furthermore, the map $\iota_A : A \to M(A)$ from Definition 3.2 is an isometric algebra homomorphism and $\iota_A(A)$ is a closed two-sided ideal in $M(A)$.*

Proof. Lemma 2.4 gives

$$\|L(a)\| = \sup_{\|b\|=1} \|bL(a)\| = \sup_{\|b\|=1} \|R(b)a\| \leq \|R\|\|a\|.$$

Thus, $\|L\| \leq \|R\|$. Similarly,

$$\|R(a)\| = \sup_{\|b\|=1} \|R(a)b\| = \sup_{\|b\|=1} \|aL(b)\| \leq \|a\|\|L\|,$$

whence $\|R\| \leq \|L\|$ and therefore $\|L\| = \|R\|$. Note that ι_A is clearly a linear map. To check it is multiplicative take $a, b \in A$ and compute

$$\iota_A(ab) = (L_{ab}, R_{ab}) = (L_a L_b, R_b R_a) = (L_a, R_a)(L_b, R_b) = \iota_A(a)\iota_A(b).$$

Once again, Lemma 2.4 gives $\|L_a\| = \|R_a\| = \|a\|$, showing that ι_A is isometric and that $\iota_A(A)$ is closed in $M(A)$. Finally, direct computations show that for any $a \in A$ and any $(L, R) \in M(A)$, $\iota_A(a)(L, R) = \iota_A(R(a))$ and $(L, R)\iota_A(a) = \iota_A(L(a))$, whence $\iota_A(A)$ is indeed a two-sided ideal in $M(A)$, finishing the proof. □

A well known and useful fact is that whenever A is a Banach algebra with a cai that is nondegenerately represented on a Banach space E, then $M(A)$ can be nondegenerately represented on E. For convenience, we state such result below and refer the reader to Theorem 4.1 and Remark 4.2 in [13] for a complete proof.

Theorem 3.4. *Let A be a Banach algebra with a cai and let $\pi : A \to \mathcal{L}(E)$ be a contractive nondegenerate representation of A on a Banach space E. Then π induces a unique nondegenerate, contractive, and unital representation $\widehat{\pi} : M(A) \to$*

$\mathcal{L}(E)$ such that, if ι_A is as in Definition 3.2, then $\pi = \widehat{\pi} \circ \iota_A$. Furthermore, $\widehat{\pi}$ is isometric when π is.

Remark 3.5. Theorem 3.4 can also be proved by either appropriately modifying the proof of Proposition 2.6.12 in [2] or by the methods described in Section 6 of [19]. We are grateful to both David Blecher and Hannes Thiel for pointing these references to us. An alternative way is to use Corollary 4.1.7 in [7]. This alternative proof uses that, just as in the C*-case, $M(A)$ is isometrically isomorphic to the two-sided multipliers of A on any Banach space E where A acts nondegenerately. This fact is actually equivalent to Theorem 3.4, as we see below in Corollary 3.6.

Corollary 3.6. *Let A be a Banach algebra with a cai and that is nondegenerately represented on a Banach space E via $\pi : A \to \mathcal{L}(E)$. Then the Banach algebra $M(A)$ is isometrically isomorphic to*

$$\{t \in \mathcal{L}(E) : t\pi(A) \subseteq \pi(A), \pi(A)t \subseteq \pi(A)\},$$

the algebra of two sided multipliers of $\pi(A)$, via the map $\widehat{\pi}$ from Theorem 3.4.

Proof. Theorem 3.4 gives that $\widehat{\pi} : M(A) \to \mathcal{L}(E)$ is an isometric and unital representation satisfying $\widehat{\pi} \circ \iota_A = \pi$. Set

$$B = \{t \in \mathcal{L}(E) : t\pi(A) \subseteq \pi(A), \pi(A)t \subseteq \pi(A)\}.$$

It only remains to show that the image of $\widehat{\pi}$ is indeed B. To do so, notice that for any $(L, R) \in M(A)$ and $a \in A$ we have $\widehat{\pi}(L, R)\pi(a) = \pi(L(a))$ and $\pi(a)\hat{\pi}(L, R) = \pi(R(a))$, whence $\widehat{\pi}(M(A)) \subseteq B$. To check the reverse inclusion, for any $t \in B$ we define $L_t, R_t \in \mathcal{L}(A)$ as follows:

$$L_t(a) = \pi^{-1}(t\pi(a)), \;\; R_t(a) = \pi^{-1}(\pi(a)t),$$

where $\pi^{-1} : \pi(A) \to A$ is understood as the inverse of the invertible isometry $\pi : A \to \pi(A)$. Since $t \in B$, we have $(L_t, R_t) \in M(A)$. Furthermore, for any $a \in A$, $\xi \in E$ we have

$$\widehat{\pi}(L_t, R_t)\pi(a)\xi = \pi(L_t(a))\xi = t\pi(a)\xi.$$

Hence, nondegeneracy implies that $\widehat{\pi}(L_t, R_t) = t$ and therefore we conclude that $B \subseteq \widehat{\pi}(M(A))$, finishing the proof. ☐

We will be mostly interested in $M(A)$ when A is a Banach algebra that has a nonunital identity as in Definition 2.5. This implies that A does not have a cai, as shown in Lemma 2.6, whence the previous two results cannot be used. However, the following proposition shows that in such cases $M(A)$ is actually the Banach algebra A equipped with a new norm.

Proposition 3.7. *Let A be a Banach algebra with identity $1_A \in A$, as in Definition 2.5, and let $\iota_A : A \to M(A)$ be as in Definition 3.2. Then the map $\varphi : M(A) \to A$ given by*

$$\varphi(L, R) = L(1_A)$$

is an isometric isomorphism between the Banach algebras $M(A)$ and $(A, |||-|||)$, where $|||a||| = \|\iota_A(a)\|$. In particular, $|||a||| = \|a\|$ for all $a \in A$ when 1_A is a unit.

Proof. It is routine to check that $\varphi : M(A) \to A$ is a linear map. Further, it is clear that $\varphi(\iota_A(a)) = a$, whence φ is surjective. If $\varphi(L, R) = 0$, then for any $a \in A$ we get $L(a) = L(1_A a) = L(1_A)a = \varphi(L, R)a = 0$ and $R(a) = R(a)1_A = aL(1_A) = a\varphi(L, R) = 0$, from where it follows that φ is injective. Next, to show that φ is multiplicative we take any $(L, R), (L', R') \in M(A)$, and compute

$$\begin{aligned}\varphi\big((L, R)(L', R')\big) &= \varphi(LL', R'R)\\ &= L(L'(1_A))\\ &= L(1_A L'(1_A))\\ &= L(1_A)L'(1_A) = \varphi(L, R)\varphi(L', R').\end{aligned}$$

Next, for any $(L, R) \in M(A)$ we obtain

$$|||\varphi(L, R)||| = \|\iota_A(L(1_A))\| = \|(L_{L(1_A)}, R_{L(1_A)})\| = \|(L, R)\|,$$

so that $\varphi : M(A) \to A$ is indeed isometric when A is equipped with $|||a||| = \|\iota_A(a)\|$. Finally, if 1_A is a unit, then A has a trivial cai and therefore Lemma 3.3 implies that ι_A is an isometry. □

Remark 3.8. Section 2 of [26] defines the multiplier algebra of a commutative Banach algebra A without absolute zero divisors other than 0 by

$$N(A) = \{L \in \mathcal{L}(A) : aL(b) = L(a)b \ \text{ for all } \ a, b \in A\}.$$

When A is commutative and $(L, R) \in M(A)$, we have $L = R$ and $(L, R) \mapsto L$ is an isometric algebra isomorphism from $M(A)$ to $N(A)$. Hence, both definitions are equivalent. If A is a commutative Banach algebra with identity $1_A \in A$ as in Definition 2.5, then A has no absolute zero divisors other than 0 and therefore Proposition 3.7 above implies that $M(A)$ is isometrically isomorphic, as a Banach algebra, to A equipped with the norm $a \mapsto \|L_a\|$.

4. Main results

In this section, we explore whether the conditions in Theorem 3.4 are necessary to guarantee that the multiplier algebra of an L^p-operator algebra is again an L^p-operator algebra. We do so via two contrasting examples presented in Sections 4B and 4C. On the one hand, we show that the multiplier algebra of 2×2 strictly upper triangular matrices acting on ℓ_2^p is an L^q-operator algebra for any $p, q \in [1, \infty)$. On the other hand, for each $p \in [1, \infty)$ we define a two dimensional L^p-operator algebra whose multiplier algebra is not an L^q-operator algebra for any $q \in [1, \infty)$ as long as $p \in [1, p_0] \cup [p_0', \infty)$ where $p_0 = 1.606$ and p_0' is its Hölder conjugate given by $p_0' = \frac{p_0}{p_0 - 1} = 2.65016501\ldots = 2.\overline{6501}$.

4A. ***Approximately unital and nondegenerate case.*** The following result is simply a consequence of Theorem 3.4 being a particular case of Corollary 3.6.

Theorem 4.1. *Let A be an L^p-operator algebra that has a cai and such that there is a nondegenerate representation π of A on $L^p(\mu)$. Then $M(A)$ is an L^p-operator algebra that is isometrically isomorphic to the algebra of two sided multipliers of $\pi(A)$, that is $M(A) \cong \{t \in L^p(\mu) : t\pi(a) \in \pi(A), \pi(a)t \in \pi(A)$ for all $a \in A\}$.*

Theorem 4.1 raises a natural question: Are the properties of having a cai and being nondegenerately represented necessary for $M(A)$ to be an L^p-operator algebra? Below, we present two examples of families of L^p-operator algebras that have neither property. In Section 4B, we investigate strictly upper triangular 2×2 matrices and show its multiplier algebra can be nondegenerately represented on an L^p-space. In Section 4C, we investigate the group L^p-operator algebras associated to the augmentation ideal $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ and show that for certain values of p, its multiplier algebra fails to be an L^q-operator algebra for any $q \in [1, \infty)$.

4B. ***Strictly upper triangular matrices.*** Let $p \in [1, \infty)$. We start by considering T_2^p, the algebra of strictly upper triangular 2×2 matrices acting on ℓ_2^p, that is,

$$T_2^p = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p(\mathbb{C}) = \mathcal{L}(\ell_2^p).$$

This is a degenerate L^p-operator algebra without identity, unit, nor cai. In fact, T_2^p is isometrically isomorphic to $\mathbb{C}$ with the trivial multiplication $zw = 0$ (see Example 3.6 in [9]). Notice that T_2^p is commutative, but all its elements are absolute zero divisors, whence Remark 3.8 does not apply. However, it is clear that $\mathcal{L}(T_2^p) = \mathcal{L}(\mathbb{C}) \cong \mathbb{C}$ and that $aL(b) = 0 = R(a)b$ for any $(L, R) \in M(T_2^p)$ and any $a, b \in T_2^p$. This, together with Definition 3.1, imply at once that

$$M(T_2^p) = LM(T_2^p) \times RM(T_2^p).$$

Furthermore, notice that $LM(T_2^p) = RM(T_2^p) = \mathcal{L}(T_2^p) \cong \mathbb{C}$. Therefore, $M(T_2^p)$ is isometrically isomorphic to $\mathbb{C}^2$ with the supremum norm, which is in turn isometrically identified with $C(\{1, 2\})$, the space of continuous functions on $\{1, 2\}$. Finally, for any $q \in [1, \infty)$, we have the map $\varphi_q : C(\{1, 2\}) \to \mathcal{L}(\ell_2^q)$ given by

$$(\varphi_q(\xi)x)(j) = \xi(j)x(j),$$

where $\xi \in C(\{1,2\})$, $x \in \ell_2^q$, and $j \in \{1,2\}$. This allows us to represent continuous functions on $\{1,2\}$ as multiplication operators on ℓ_2^q. It is well known and easy to see that φ_q is an isometric nondegenerate representation. Therefore, we have shown the following result:

Theorem 4.2. *For any $p \in [1,\infty)$, $M(T_2^p)$ is isometrically isomorphic to $\mathbb{C}^2$ with the supremum norm. In particular, for any $p, q \in [1,\infty)$, $M(T_2^p)$ is an L^q-operator algebra that is nondegenerately representable on ℓ_2^q.*

A more complicated example is T_3^p, the algebra of strictly upper triangular 3×3 matrices acting on ℓ_3^p. That is

$$T_3^p = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} : a_{12}, a_{13}, a_{23} \in \mathbb{C} \right\} \subset M_3^p(\mathbb{C}) = \mathcal{L}(\ell_3^p).$$

Notice that T_3^p is identified with $\mathbb{C}^3$ with multiplication given by

$$ab = (a_{12}, a_{13}, a_{23})(b_{12}, b_{13}, b_{23}) = (0, a_{12}b_{23}, 0).$$

If $p = 1$, then T_3^p is isometrically isomorphic to $\mathbb{C}^3$ with norm

$$\|a\| = \max\{|a_{12}|, |a_{13}| + |a_{23}|\}.$$

However, when $p \neq 1, 2, \infty$ it is known that computing the p-operator norm of a matrix is an NP-hard problem (see [16]). This complicates the study of $M(T_3^p)$ and at this moment we have not investigated whether $M(T_3^p)$ can be isometrically represented on an L^q-space.

4C. ***Augmentation ideals.*** We will now construct, for certain values of p, a class of L^p-operator algebras that have no cai, cannot be nondegenerately represented (on any Banach space), and whose multiplier algebra is not an L^q-operator algebra for any $q \in [1,\infty)$. We begin with the general construction of the augmentation ideal for an arbitrary discrete group G and eventually specialize to the finite case. In particular we will work with $G = \mathbb{Z}/3\mathbb{Z}$. The augmentation ideal of a group ring is a well known algebraic object (see [20, Section 11.2], [8, Example 7, p. 245] for instance) and is easily generalized to operator algebras arising from discrete groups. We include details here for completeness.

Let G be a discrete group with identity element denoted by $1_G \in G$. For any $g \in G$, the indicator function $\delta_g : G \to \mathbb{C}$ is given by

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

For each $g \in G$, we define the function $\Delta_g : G \to \mathbb{C}$ by

$$\Delta_g = \delta_g - \delta_{1_G}. \tag{4-1}$$

For each $p \in [1, \infty)$, recall that the Banach space $\ell^p(G)$ is the set of complex valued functions $a : G \to \mathbb{C}$ satisfying

$$\|a\|_p^p = \sum_{g \in G} |a(g)|^p < \infty.$$

Further, we define the convolution of any pair $(a, b) \in \ell^1(G) \times \ell^p(G)$ as the complex valued function $a * b : G \to \mathbb{C}$ defined at each $g \in G$ by

$$(a * b)(g) = \sum_{h \in G} a(h) b(h^{-1} g). \tag{4-2}$$

It is well known that $\|a * b\|_p \leq \|a\|_1 \|b\|_p$, whence $a * b \in \ell^p(G)$. Moreover, when $p = 1$, convolution gives rise to a multiplication on $\ell^1(G)$ that makes it a unital Banach algebra with unit δ_{1_G}.

Remark 4.3. Given the large presence of multiplication by convolution in this paper, when $a \in \ell^1(G)$ and $b \in \ell^p(G)$ for some $p \in [1, \infty)$ we will simply write ab instead of $a * b$ from (4-2). It will be understood by context when convolution is the multiplication that is being used.

It is also well known that $C_c(G)$, the subspace of compactly supported functions on G, lies densely in $\ell^1(G)$ with respect to the $\| - \|_1$ norm. Clearly, $\delta_g \in C_c(G)$ for each $g \in G$.

Lemma 4.4. *The map $\iota_0 : \ell^1(G) \to \mathbb{C}$ given by*

$$\iota_0(a) = \sum_{g \in G} a(g) \tag{4-3}$$

is a contractive algebra homomorphism.

Proof. It is clear that ι_0 is a well-defined linear map and that $|\iota_0(a)| \leq \|a\|_1$, whence $\|\iota_0\| \leq 1$. By Fubini's Theorem, we see that

$$\iota_0(ab) = \sum_{g \in G} (ab)(g) = \sum_{h \in G} a(h) \sum_{g \in G} b(h^{-1} g) = \iota_0(a) \iota_0(b),$$

finishing the proof. □

The main object of interest for this subsection is the augmentation ideal of $\ell^1(G)$, which is defined as the kernel of the map $\iota_0 : \ell^1(G) \to \mathbb{C}$ from (4-3). We record a precise definition below.

Definition 4.5. Let G be a discrete group. The *augmentation ideal* of $\ell^1(G)$ is defined as

$$\ell_0^1(G) = \ker(\iota_0) = \left\{ a \in \ell^1(G) : \iota_0(a) = 0 \right\}.$$

Similarly, we define $\mathbb{C}_0[G] \subseteq C_c(G)$ as

$$\mathbb{C}_0[G] = \{ a \in C_c(G) : \iota_0(a) = 0 \}.$$

Next we establish the main properties of $\ell^1_0(G)$, which we present as a series of Propositions.

Proposition 4.6. *Let G be a discrete group.*

(1) *The augmentation ideal $\ell^1_0(G)$ is a closed two-sided proper ideal of $\ell^1(G)$.*

(2) *The subspace $\mathbb{C}_0[G]$ is dense in $\ell^1_0(G)$ with respect to the $\| - \|_1$ norm.*

(3) *Let Δ_g be as in (4-1). Then* $\operatorname{span}\{\Delta_g : g \in G\}$ *is dense in $\ell^1_0(G)$. In fact,* $\operatorname{span}\{\Delta_g : g \in G\} = \mathbb{C}_0[G]$.

Proof. That $\ell^1_0(G)$ is a closed two-sided ideal follows from Lemma 4.4 above. Since $\delta_{1_G} \notin \ell^1_0(G)$, we also get that $\ell^1_0(G)$ is proper, so part (1) is done. For part (2), let $a \in \ell^1_0(G)$ and let $\varepsilon > 0$. Then, there is $b \in C_c(G)$ such that $\|a - b\|_1 < \frac{\varepsilon}{2}$ and observe that $b - \iota_0(b)\delta_{1_G} \in \mathbb{C}_0[G]$. Further, note that $|\iota_0(b)| = |\iota_0(a-b)| \leq \|a-b\|_1$. Therefore,

$$\|a - (b - \iota_0(b)\delta_{1_G})\|_1 \leq \|a - b\|_1 + |\iota_0(b)| \leq 2\|a - b\|_1 < \varepsilon.$$

This proves that $\mathbb{C}_0[G]$ is dense in $\ell^1_0(G)$, as wanted. Finally, for part (3), note first that $\operatorname{span}\{\Delta_g : g \in G\} \subseteq \mathbb{C}_0[G]$ because for each $g \in G$, it is clear that $\Delta_g \in C_c(G)$ and $\iota_0(\Delta_g) = \iota_0(\delta_g) - \iota_0(\delta_{1_G}) = 1 - 1 = 0$. For the reverse inclusion, take any $a \in \mathbb{C}_0[G]$, that is $\iota_0(a) = 0$ and $F = \operatorname{supp}(a)$ is a finite subset of G. Therefore,

$$a = \sum_{g \in F} a(g)\delta_g = \left(\sum_{g \in F} a(g)\delta_g\right) - \iota_0(a)\delta_{1_G} = \sum_{g \in F} a(g)\Delta_g.$$

This proves that $a \in \operatorname{span}\{\Delta_g : g \in G\}$ and we are done with part (3). □

The previous proposition shows that $\ell^1_0(G)$ is a Banach algebra in its own right with the 1-norm inherited from $\ell^1(G)$. When $p \in [1, \infty)$, we will mainly be interested in the image of $\ell^p_0(G)$ in $F^p(G)$, the full L^p operator algebra of G, which we define below in the discrete group case setting. We refer the interested reader to Definition 2.1 in [10] for a general definition of $F^p(G)$ for locally compact groups.

Definition 4.7. Let G be a discrete group and let $p \in [1, \infty)$.

(1) We define the *full group L^p-operator algebra of G*, denoted by $F^p(G)$, as the completion of $\ell^1(G)$ in the norm

$$a \mapsto \|a\|_{F^p(G)} = \sup\{\|\pi(a)\| : \pi \in \operatorname{Rep}_{L^p}(G)\},$$

where $\operatorname{Rep}_{L^p}(G)$ is the class of all nondegenerate contractive representations of $\ell^1(G)$ on L^p-spaces.

(2) We define *the augmentation ideal of $F^p(G)$*, denoted by $F^p_0(G)$, as the completion of $\ell^1_0(G)$ in the $\| - \|_{F^p(G)}$ norm.

The algebra $F^p(G)$ is in fact an L^p-operator algebra, this follows from the general crossed product case, see for instance Theorem 3.6 in [22]. We next show that $F_0^p(G)$ is actually a closed proper ideal in $F^p(G)$, making it also an L^p-operator algebra in its own right.

Proposition 4.8. *Let G be a discrete group and let $p \in [1, \infty)$. Then $F_0^p(G)$ is a nonzero closed two-sided proper ideal in $F^p(G)$.*

Proof. That $F_0^p(G)$ is a nonzero closed two-sided ideal follows at once from the fact that $\ell_0^1(G)$ is an ideal in $\ell^1(G)$. To show that it is proper, it suffices to prove that δ_{1_G}, the unit in $F^p(G)$, does not belong to $F_0^p(G)$. Assume on the contrary that there is a sequence $(a_n)_{n=1}^{\infty}$ in $\ell_0^1(G)$ such that a_n converges to δ_{1_G} in $F^p(G)$. Next, by Lemma 4.4, ι_0 is a contractive representation of $\ell^1(G)$ on the L^p-space $\ell_1^p = \mathbb{C}$, so it follows from definition of $\| - \|_{F^p(G)}$ that $|\iota_0(a)| \leq \|a\|_{F^p(G)}$ for all $a \in \ell_0^1(G)$. Therefore ι_0 extends continuously to a contraction $\iota_0 : F^p(G) \to \mathbb{C}$. In particular, since $\iota_0(a_n) = 0$ for all $n \in \mathbb{Z}_{\geq 1}$, it follows that

$$\begin{aligned} 1 = |\iota_0(\delta_{1_G})| &= |\iota_0(\delta_{1_G}) - \iota_0(a_n)| \\ &\leq \|\delta_{1_G} - a_n\|_{F^p(G)}. \end{aligned}$$

Taking the limit as $n \to \infty$ shows that $1 \leq 0$, a contradiction. Thus, $\delta_{1_G} \notin F_0^p(G)$, as wanted. □

For each $p \in [1, \infty)$, we will also let $\ell_0^1(G)$ act on $\ell^p(G)$ via left multiplication by convolution. That way the completion of the image of $\ell_0^1(G)$ in $\mathcal{L}(\ell^p(G))$ becomes an L^p-operator algebra. In particular, when $p = 1$, the algebra $\ell_0^1(G)$ is isometrically isomorphic to its image in $\mathcal{L}(\ell^1(G))$, making $\ell_0^1(G)$ an L^1-operator algebra. More precisely, this will be done by looking at the image of $\ell_0^1(G)$ in $F_{\mathrm{r}}^p(G)$, the reduced L^p-operator algebra of G. The algebra $F_{\mathrm{r}}^p(G)$ was first introduced by Herz in [18, Section 8], where it was called the algebra of p-pseudofunctions on G, and then reintroduced by Phillips in [22, Section 3] as a particular case of reduced L^p-operator crossed products. For convenience we state below the definition of $F_{\mathrm{r}}^p(G)$ for G discrete. We refer the interested reader to Definition 3.1 in [10] for the general definition of $F_{\mathrm{r}}^p(G)$ when G is locally compact.

Definition 4.9. Let G be a discrete group and let $p \in [1, \infty)$.

(1) Let $\lambda_p : \ell^1(G) \to \mathcal{L}(\ell^p(G))$ denote left multiplication by convolution; that is, for $a \in \ell^1(G)$, $b \in \ell^p(G)$, we put $\lambda_p(a)b = ab \in \ell^p(G)$ as in (4-2) (see Remark 4.3). We define the *reduced L^p-operator algebra of G*, denoted by $F_{\mathrm{r}}^p(G)$, as the closure of $\lambda_p(\ell^1(G))$ in $\mathcal{L}(\ell^p(G))$.

(2) We define *the augmentation ideal of $F_{\mathrm{r}}^p(G)$*, denoted by $F_{\mathrm{r},0}^p(G)$, as the closure of $\lambda_p(\ell_0^1(G))$ in $\mathcal{L}(\ell^p(G))$.

Since G is discrete, $\|a\|_p \leq \|a\|_1$ for all $a \in \ell^1(G)$, in fact the canonical inclusion $\ell^1(G) \hookrightarrow \ell^p(G)$ has norm 1. Furthermore, observe that for any $a \in \ell^1(G)$ we have

$$\|a\|_p \leq \|a\|_{F_r^p(G)} \leq \|a\|_{F^p(G)} \leq \|a\|_1. \tag{4-4}$$

Hence, when $p = 1$, the map $\lambda_1 : \ell^1(G) \to \mathcal{L}(\ell^1(G))$ is isometric and therefore we have isometric isomorphisms of Banach algebras

$$F^1(G) \cong F_{\mathrm{r}}^1(G) \cong \ell^1(G) \text{ and } F_0^1(G) \cong F_{\mathrm{r},0}^1(G) \cong \ell_0^1(G). \tag{4-5}$$

For general $p \in [1, \infty)$, the map $\lambda_p : \ell^1(G) \to \mathcal{L}(\ell^p(G))$ is an injective contractive nondegenerate representation of $\ell^1(G)$ on $\ell^p(G)$. Thus, $\lambda_p(\ell^1(G))$ is a dense subalgebra of $F_{\mathrm{r}}^p(G)$ that is algebraically isomorphic to $\ell^1(G)$. Moreover, by definition $F_{\mathrm{r}}^p(G)$ is a closed unital subalgebra of $\mathcal{L}(\ell^p(G))$, making it an L^p-operator algebra as in Definition 2.2.

Remark 4.10. As in Proposition 4.8, $F_{\mathrm{r},0}^p(G)$ is a nonzero closed two-sided ideal in $F_{\mathrm{r}}^p(G)$ and therefore an L^p-operator algebra. However, unlike Proposition 4.8, we no longer have that the ideal $F_{\mathrm{r},0}^p(G)$ is proper in general. This is explained due to the fact that in many instances $F_r^p(G)$ is a simple Banach algebra. Indeed, by Theorem 2.5 in [23], if $F_{\mathrm{r}}^2(G)$ is simple, then $F_r^p(G)$ is simple for any $p \in (1, \infty)$. In particular, for $p \in (1, \infty)$, $F_{\mathrm{r}}^p(G)$ will be simple when G has the Powers property (see also Theorem 3.7 in [15]). Thus, we can guarantee that for any $n \in \mathbb{Z}_{>1}$, $F_{r,0}^p(\mathbb{F}_n) = F_r^p(\mathbb{F}_n)$ where $\mathbb{F}_n$ is the free group in n generators.

However, Theorem 3.7 in [10] shows that, for any $p \in (1, \infty)$, if G is an amenable group, then $F^p(G)$ is isometrically isomorphic to $F_{\mathrm{r}}^p(G)$ as Banach algebras. Therefore, this fact combined with (4-5) give at once the following result:

Proposition 4.11. *Let G be a discrete group and let $p \in [1, \infty)$. If either $p = 1$ or G is amenable, then $F_{\mathrm{r},0}^p(G)$ is a proper ideal in $F_{\mathrm{r}}^p(G)$ that is degenerately represented on $\ell^p(G)$.*

In any case, it is more natural to work with the ideal $F_0^p(G)$, rather than $F_{\mathrm{r},0}^p(G)$, for $F_0^p(G)$ is guaranteed to be a proper ideal for any discrete group G. Moreover, for a finite nontrivial group G, we will show in Corollary 4.17 below that $F_0^p(G)$ cannot be nondegenerately represented on any Banach space, already voiding one of the assumptions in Theorem 4.1. Part of our work below is to show that, for a finite nontrivial group G, the assumption of having a cai is also not met for $F_0^p(G)$.

Remark 4.12. We mention that since finite groups are amenable, $F_0^p(G)$ and $F_{\mathrm{r},0}^p(G)$ are isometrically isomorphic as Banach algebras (see Theorem 3.7 in [10]). Therefore, if G is a nontrivial finite group, we will keep using the simpler notation $F_0^p(G)$, but for convenience we will use that $F_0^p(G)$ can be defined, in this case, as the closure of $\lambda_p(\ell_0^1(G))$ in $\mathcal{L}(\ell^p(G))$.

For the rest of the paper, we will concentrate on the properties of $F_0^p(G)$ when G is a finite nontrivial group.

Proposition 4.13. *Let G be a finite group with* $\mathrm{card}(G) \geq 2$ *and let* $p \in [1, \infty)$. *Then, for* $n = \mathrm{card}(G)$:

(1) *As vector spaces,*

$$F_0^p(G) = \ell_0^1(G) = \mathrm{span}\{\Delta_g : g \in G\} = \mathbb{C}_0[G] \cong \mathbb{C}^{n-1}.$$

(2) *For any* $a \in \ell_0^1(G)$,

$$a\Big(\sum_{g\in G} \delta_g\Big) = \Big(\sum_{g\in G} \delta_g\Big)a = 0.$$

(3) $F_0^p(G)$ *has an identity element* (see Definition 2.5) *given by*

$$\mathbb{1}_0 = -\frac{1}{n}\sum_{g\in G} \Delta_g.$$

(4) $\dfrac{((n-1)^p + n - 1)^{1/p}}{n} \leq \|\mathbb{1}_0\|_{F^p(G)} \leq 2 - \dfrac{2}{n}.$

(5) $\|\mathbb{1}_0\|_1 = 2 - \dfrac{2}{n}.$

(6) $\mathbb{1}_0$ *is not a unit in* $F_0^1(G)$ *when* $n > 2$ (see Definition 2.5).

(7) $F_0^1(G)$ *has no cai and cannot be nondegenerately represented on any Banach space when* $n > 2$.

Proof. In what follows we write $n = \mathrm{card}(G)$ and $G = \{g_1, \ldots, g_n\}$ where $g_1 = 1_G$. Since G is finite, $C_c(G) = \ell^1(G)$ and $\ell^1(G) \cong \mathbb{C}^n$ as vector spaces via the map $\delta_{g_j} \mapsto \mathbf{e}_j \in \mathbb{C}^n$ for $j = 1, \ldots, n$. Thus using Proposition 4.6 we now get $\mathbb{C}_0[G] = \ell_0^1(G)$ and $\ell_0^1(G) \cong \mathbb{C}^{n-1}$ as vector spaces via the map $\Delta_{g_j} \mapsto \mathbf{e}_{j-1} \in \mathbb{C}^{n-1}$ for $j = 2, \ldots, n$. Then $\ell_0^1(G)$ is a finite dimensional vector space and therefore already complete in the $\|-\|_{F^p(G)}$ norm. This gives that, for any $p \in [1, \infty)$, $F_0^p(G) = \ell_0^1(G)$ which is in turn identified with $\mathbb{C}^{n-1}$, equipped with a different norm for each p, proving part (1). Next, to show parts (2) and (3), we take any $a \in \ell_0^1(G)$ so that $\iota_0(a) = 0$. Then, since $(a\delta_g)(h) = a(hg^{-1})$ for any $g, h \in G$, it follows that for any $k \in \{1, \ldots, n\}$

$$\bigg(a\Big(\sum_{j=1}^n \delta_{g_j}\Big)\bigg)(g_k) = \sum_{j=1}^n a(g_k g_j^{-1}) = \iota_0(a) = 0,$$

proving that $a\big(\sum_{j=1}^n \delta_{g_j}\big) = 0$. A similar computation shows that $\big(\sum_{j=1}^n \delta_{g_j}\big)a = 0$, so part (2) follows. For part (3), notice that for any $j, k \in \{1, \ldots, n\}$,

$$(a\Delta_{g_j})(g_k) = (a\delta_{g_j})(g_k) - a(g_k) = a(g_k g_j^{-1}) - a(g_k).$$

Therefore, for any $k \in \{1, \dots, n\}$ we obtain

$$(a\mathbb{1}_0)(g_k) = \frac{1}{n}\sum_{j=1}^{n}\bigl(-a(g_k g_j^{-1}) + a(g_k)\bigr) = \frac{1}{n}(-\iota_0(a) + na(g_k)) = a(g_k),$$

from where it follows that $a\mathbb{1}_0 = a$. An analogous computation gives that $\mathbb{1}_0 a = a$ and therefore part (3) is done. Part (4) follows at once from (4-4) and the following direct computation for each $p \in [1, \infty)$:

$$\|\mathbb{1}_0\|_p = \frac{1}{n}\Bigl\|\sum_{j=2}^{n}(\delta_{g_j} - \delta_{g_1})\Bigr\|_p = \frac{1}{n}\bigl(|n-1|^p + (n-1)|-1|^p\bigr)^{1/p}.$$

Part (5) is simply (4) when $p = 1$. Next, notice that part (6) is implied by part (5) for $\|\mathbb{1}_0\|_1 > 1$ when $n > 2$. Finally, when $n > 2$, part (6) together with Lemma 2.6 shows that $F_0^1(G)$ does not have a cai. Combined with Lemma 2.7, this shows that $F_0^1(G)$ cannot be nondegenerately represented on any Banach space, so part (7) follows. □

Our final goal for this subsection is to investigate whether $M(F_0^p(G))$, the multiplier algebra of $F_0^p(G)$ (see Definition 4.9), is an L^p-operator algebra when G is finite and abelian. Proposition 4.13 (7) shows that, at least when $p = 1$, the hypotheses of Theorem 4.1 are not met for $A = F_0^p(G)$. We will show that for all $p \in [1, \infty) \setminus \{2\}$ and any finite group G with $\mathrm{card}(G) > 2$, $F_0^p(G)$ still does not meet the hypotheses in Theorem 4.1. In contrast to the algebra T_2^p (see Section 4B), we will see that for certain values of p, including 1, $M(F_0^p(\mathbb{Z}/3\mathbb{Z}))$ is not an L^q-operator algebra for any $q \in [1, \infty)$.

We start with a computational lemma that will be needed twice in the rest of the paper.

Lemma 4.14. *Let $p \in (1, \infty) \setminus \{2\}$ and let G be a finite group with $n = \mathrm{card}(G) > 2$. Define $a_0 : G \to \mathbb{C}$ by*

$$a_0(g) = \begin{cases} n-1 & \text{if } g = 1_G, \\ -1 & \text{if } g \neq 1_G. \end{cases}$$

For each $\varepsilon \in \mathbb{R}$, we define $b_\varepsilon : G \to \mathbb{C}$ by

$$b_\varepsilon = a_0 + \varepsilon \sum_{g \in G} \delta_g.$$

Then:

(1) *$a_0 \in \ell_0^1(G)$ and $b_\varepsilon \in \ell^p(G) \setminus \ell_0^1(G)$ for any $\varepsilon \neq 0$.*

(2) *For any $\varepsilon \in \mathbb{R}$, $\mathbb{1}_0 b_\varepsilon = a_0$ (when $\varepsilon \neq 0$ we think of b_ε as a perturbation of a_0 that leaves $\ell_0^1(G)$).*

(3) *There exists $\varepsilon_p \neq 0$ such that $\|a_0\|_p > \|b_{\varepsilon_p}\|_p$.*

(4) *In particular, when* $n = 3$, *we get*

$$\frac{\|a_0\|_p}{\|b_{\varepsilon_p}\|_p} = \frac{(2^p+2)(2^{\frac{1}{p-1}}+1)^p}{3^p(2^{\frac{p}{p-1}}+2)} > 1.$$

Proof. Part (1) is readily verified. Part (2) follows at once from parts (2) and (3) in Proposition 4.13. For part (3), note that asking for $\|a_0\|_p > \|b_\varepsilon\|_p$ is equivalent to

$$(n-1)^p + n - 1 > |n-1+\varepsilon|^p + (n-1)|\varepsilon - 1|^p. \tag{4-6}$$

Next, we define $f_p : (-1, 1) \to \mathbb{R}$ by $f_p(\varepsilon) = \|a_0\|_p - \|b_\varepsilon\|_p$. That is,

$$f_p(\varepsilon) = (n-1)^p + n - 1 - (n-1+\varepsilon)^p - (n-1)(1-\varepsilon)^p.$$

We need to show that exists $\varepsilon_p \neq 0$ such that $f_p(\varepsilon_p) > 0$. Clearly $f_p(0) = 0$. Furthermore, it is routine to verify that f_p is a differentiable concave function, that f_p has a maximum at

$$\varepsilon_p = \frac{(n-1)^{\frac{1}{p-1}} - (n-1)}{(n-1)^{\frac{1}{p-1}} + 1} \neq 0,$$

and that

$$f_p'(0) = \frac{1}{n-1}((n-1)^2 - (n-1)^p)p.$$

Thus, since $n > 2$, if $p \in (1, 2)$ we have $f_p'(0) > 0$ and $\varepsilon_p > 0$, which gives $f_p(\varepsilon_p) > 0$. Similarly, $p \in (2, \infty)$ implies $f_p'(0) < 0$ and $\varepsilon_p < 0$, from where it follows that $f_p(\varepsilon_p) > 0$. This now gives part (3). Finally, part (4) follows from plugging in $n = 3$ and ε_p in (4-6). □

We now obtain a generalization of part (6) in Proposition 4.13 for any $p \neq 2$. The proof for $p = 1$ was immediate, but since the right hand side of the inequality in part (4) of Proposition 4.13 is not always strictly greater than 1, a different argument is needed that works for any $p \in (1, \infty) \setminus \{2\}$ and any finite group G with $\mathrm{card}(G) > 2$.

Proposition 4.15. *Let* $p \in [1, \infty) \setminus \{2\}$ *and let* G *be a finite group with* $\mathrm{card}(G) > 2$. *Then* $\mathbb{1}_0$ *is a nonunital identity of* $F_0^p(G)$ (see Definition 2.5).

Proof. As before, we write $n = \mathrm{card}(G)$ and $G = \{g_1, \ldots, g_n\}$ where $g_1 = 1_G$. Proposition 4.13(6) gives the result for $p = 1$. Take $p \in (1, \infty) \setminus \{2\}$ and recall that

$$\|\mathbb{1}_0\|_{F^p(G)} = \|\lambda_p(\mathbb{1}_0)\| = \sup\left\{\frac{\|\mathbb{1}_0 b\|_p}{\|b\|_p} : b \in \ell^p(G), b \neq 0\right\}.$$

Now by Lemma 4.14 there is $a_0 \in \ell_0^1(G)$ and $b_{\varepsilon_p} \in \ell^p(G) \setminus \ell_0^1(G)$ such that $\mathbb{1}_0 b_{\varepsilon_p} = a_0$ and $\|a_0\|_p > \|b_{\varepsilon_p}\|_p$ for any $p \in (1, \infty) \setminus \{2\}$. Therefore,

$$\|\mathbb{1}_0\|_{F^p(G)} \geq \frac{\|\mathbb{1}_0 b_{\varepsilon_p}\|_p}{\|b_{\varepsilon_p}\|_p} = \frac{\|a_0\|_p}{\|b_{\varepsilon_p}\|_p} > 1,$$

finishing the proof. □

Remark 4.16. Notice that the argument given in the proof of Proposition 4.15 does not work when $p = 2$. Indeed when $p = 2$, with notation as in the proof of Lemma 4.14, we get $\varepsilon_2 = 0$ and $f_2(0) = 0$. In fact, f_2 is a parabola with vertex at the origin. The reason for this is that, when $p = 2$, the C*-equation gives that $\|\mathbb{1}_0\|_{F^2(G)} = 1$. Furthermore, it follows from Corollary 3.20 of [10] that

$$\|\mathbb{1}_0\|_{F^{p_2}(G)} \leq \|\mathbb{1}_0\|_{F^{p_1}(G)}$$

when either $1 \leq p_1 < p_2 \leq 2$ or $2 \leq p_2 < p_1 < \infty$. Proposition 4.15 seems to suggest that $\|\mathbb{1}_0\|_{F^p(G)}$ strictly decreases to 1 as p goes from 1 to 2 and that $\|\mathbb{1}_0\|_{F^p(G)}$ is strictly increasing as p goes from 2 to infinity. However, we have not investigated these potential strict inequalities for all p any further.

An immediate consequence of Proposition 4.15, whose proof is identical to that of part (7) in Proposition 4.13, is the following.

Corollary 4.17. *Let $p \in [1, \infty) \setminus \{2\}$ and let G be a finite group with* $\operatorname{card}(G) > 2$*. Then $F_0^p(G)$ has no cai and cannot be nondegenerately represented on any Banach space.*

Notice that the work done in Section 3 already gives us a clear description of what $M(F_0^p(G))$ is when G is finite and abelian:

Proposition 4.18. *Let G be a finite abelian group and let $p \in [1, \infty)$. Then, $M(F_0^p(G))$ is isometrically isomorphic to $(\ell_0^1(G), |||-|||_p)$ where*

$$|||a|||_p = \|L_a\|_{F_0^p(G) \to F_0^p(G)} = \sup\{\|ab\|_{F^p(G)} : b \in \ell_0^1(G), \|b\|_{F^p(G)} = 1\}.$$

Proof. Since G is abelian, $\ell_0^1(G)$ is commutative and, since G is finite, $\ell_0^1(G)$ has a multiplicative identity and therefore the only absolute zero divisor in $\ell_0^1(G)$ is 0. Thus, Remark 3.8 together with Proposition 4.13(1) give that $(L, R) \mapsto L$ is an isometric isomorphism from $M(F_0^p(G))$ to $\{L \in \mathcal{L}(F_0^p(G)) : aL(b) = L(a)b\}$. Proposition 3.7 now shows that $M(F_0^p(G))$ is indeed isometrically isomorphic to $(\ell_0^1(G), |||-|||_p)$ via $L_a \mapsto a$. □

The multiplier algebra of $F_0^p(G)$ will be our main object of study for the rest of the section. We introduce a compact notation that will be used henceforward:

$$M_0^p(G) = M(F_0^p(G)).$$

By Proposition 4.18, when G is finite abelian, then $M_0^p(G)$ is simply $\ell_0^1(G)$ equipped with the norm $|||-|||_p$. When $p=1$, the norm in $M_0^1(G)=M(F_0^1(G))=M(\ell_0^1(G))$ has a nicer description:

$$|||a|||_1 = \sup\{\|ab\|_1 : b \in \ell_0^1(G),\ \|b\|_1 = 1\}.$$

This norm is equivalent to the 1-norm in $\ell_0^1(G)$ via the following straightforward inequalities:

$$\frac{\|a\|_1}{\|\mathbb{1}_0\|_1} \leq |||a|||_1 \leq \|a\|_1 \tag{4-7}$$

where we clearly have equality if and only if $\mathrm{card}(G)=2$. In fact, these inequalities might be strict when $\mathrm{card}(G) \neq 2$:

Example 4.19. Let $G=\mathbb{Z}/3\mathbb{Z}=\{[0],[1],[2]\}$. Here, $\|\mathbb{1}_0\|_1=\frac{4}{3}$ and it is easy to check that $|||\Delta_{[j]}|||_1 = 2 = \|\Delta_{[j]}\|_1$ for $j \in \{1,2\}$. However, $|||\Delta_{[1]}+\Delta_{[2]}|||_1 = 3 \neq 4 = \|\Delta_{[1]}+\Delta_{[2]}\|_1$.

For general $p \in [1,\infty)$ we get, using (4-4), the following bounds for the norm in $M_0^p(G)$:

$$\frac{\|a\|_p}{\|\mathbb{1}_0\|_1} \leq \frac{\|a\|_{F^p(G)}}{\|\mathbb{1}_0\|_{F^p(G)}} \leq |||a|||_p \leq \|a\|_{F^p(G)} \leq \|a\|_1 \tag{4-8}$$

We end this section by explicitly showing, via a bicontractive idempotent argument, that for certain values of $p \in [1,\infty)\setminus\{2\}$, $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is not an L^q-operator algebra for any $q \in [1,\infty)$. Let $G=\mathbb{Z}/3\mathbb{Z}=\{[0],[1],[2]\}$ and recall that $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ is the span of elements $\Delta_{[1]}$ and $\Delta_{[2]}$. Furthermore, the multiplication in $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ obeys the following table:

	$\Delta_{[1]}$	$\Delta_{[2]}$
$\Delta_{[1]}$	$-2\Delta_{[1]}+\Delta_{[2]}$	$-\Delta_{[1]}-\Delta_{[2]}$
$\Delta_{[2]}$	$-\Delta_{[1]}-\Delta_{[2]}$	$-\Delta_{[1]}+2\Delta_{[2]}$

Recall that the element $\mathbb{1}_0 = -\frac{1}{3}\Delta_{[1]} - \frac{1}{3}\Delta_{[2]}$ is the identity in $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$. For any $a \in \ell_0^1(\mathbb{Z}/3\mathbb{Z})$, there are $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $a = \alpha_1\Delta_{[1]} + \alpha_2\Delta_{[2]}$. Therefore, for any $p \in [1,\infty)$ we have

$$\|a\|_p = \|\alpha_1\Delta_{[1]}+\alpha_2\Delta_{[2]}\|_p = (|-\alpha_1-\alpha_2|^p + |\alpha_1|^p + |\alpha_2|^p)^{1/p}.$$

In particular, $\|\mathbb{1}_0\|_1 = \frac{4}{3}$.

Proposition 4.18 shows that $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is isometrically isomorphic to $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ equipped with the norm $|||-|||_p$. For $a = \alpha_1\Delta_{[1]}+\alpha_2\Delta_{[2]} \in M_0^p(\mathbb{Z}/3\mathbb{Z})$, the inequalities in (4-8) become

$$|||a|||_p \leq \|a\|_1 = |-\alpha_1-\alpha_2| + |\alpha_1| + |\alpha_2| \tag{4-9}$$

and

$$\|\|a\|\|_p \geq \frac{\|a\|_p}{\|\mathbb{1}_0\|_1} = \frac{3}{4}(|-\alpha_1-\alpha_2|^p+|\alpha_1|^p+|\alpha_2|^p)^{1/p}. \tag{4-10}$$

We note that the norm on $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is an operator norm induced from a proper ideal of an L^1-space and is generally difficult to compute. Fortunately, there do exist certain elements a for which we can compute $|||a|||_p$ easily (see Example 4.19). In particular, we can compute it for the idempotents of $M_0^p(\mathbb{Z}/3\mathbb{Z})$.

Proposition 4.20. *Let $p \in [1,\infty)$ and let $\gamma = \frac{1}{3}\exp(2\pi i/3) \in \mathbb{C}$. We define $e = \gamma\Delta_{[1]} + \overline{\gamma}\Delta_{[2]}$. Then:*

(1) $|\gamma| = |\overline{\gamma}| = |\gamma+\overline{\gamma}| = \frac{1}{3}$, $|\gamma-\overline{\gamma}| = \frac{\sqrt{3}}{3}$.

(2) $e^2 = e$ *and* $\|e\|_1 = 1$.

(3) $\mathbb{1}_0 - e = \overline{\gamma}\Delta_{[1]} + \gamma\Delta_{[2]}$ *and* $\|\mathbb{1}_0 - e\|_1 = 1$.

(4) $|||e|||_p = 1$.

(5) $|||\mathbb{1}_0 - e|||_p = 1$.

(6) $|||\mathbb{1}_0 - 2e|||_1 = \frac{2\sqrt{3}}{3}$.

(7) $|||\mathbb{1}_0 - 2e|||_2 = 1$.

(8) $|||\mathbb{1}_0 - 2e|||_p \leq \|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})} \leq \left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}$ *if* $p \in [1,2]$.

(9) $|||\mathbb{1}_0 - 2e|||_p \leq \|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})} \leq \left(\frac{2\sqrt{3}}{3}\right)^{1-\frac{2}{p}}$ *if* $p \in [2,\infty)$.

Proof. Each of the parts (1), (2), and (3) are checked by direct computations. For part (4), observe that the bounds (4-9) and (4-10) give

$$0 < |||e|||_p \leq \|e\|_1 = 1.$$

Since $e^2 = e$, submultiplicativity gives

$$|||e|||_p^2 \geq |||e^2|||_p = |||e|||_p.$$

Dividing both sides by $|||e|||_p$, we obtain that $|||e|||_p \geq 1$. The proof of part (5) is analogous to the part (4). For part (6), first notice that

$$\|\mathbb{1}_0 - 2e\|_1 = \|(\overline{\gamma}-\gamma)\Delta_{[1]} + (\gamma-\overline{\gamma})\Delta_{[2]}\|_1 = |0| + 2|\overline{\gamma}-\gamma| = \frac{2\sqrt{3}}{3}.$$

Then, the upper bound from (4-9) gives us that

$$|||\mathbb{1}_0 - 2e|||_1 \leq \|\mathbb{1}_0 - 2e\|_1 = \frac{2\sqrt{3}}{3}.$$

On the other hand, since $\left\|\frac{1}{2}\Delta_{[1]}\right\|_1 = 1$, we compute

$$\begin{aligned}\big\|(\mathbb{1}_0-2e)\big(\tfrac{1}{2}\Delta_{[1]}\big)\big\|_1 &= \|((\overline{\gamma}-\gamma)\Delta_{[1]}+(\gamma-\overline{\gamma})\Delta_{[2]})(\tfrac{1}{2}\Delta_{[1]})\|_1\\ &= \tfrac{1}{2}\cdot|\overline{\gamma}-\gamma|\cdot\|(\Delta_{[1]}-\Delta_{[2]})\Delta_{[1]}\|_1\\ &= \tfrac{1}{2}\cdot\tfrac{\sqrt{3}}{3}\cdot\|-\Delta_{[1]}+2\Delta_{[2]}\|_1=\tfrac{2\sqrt{3}}{3}.\end{aligned}$$

Therefore, $|||\mathbb{1}_0-2e|||_1=\frac{2\sqrt{3}}{3}$. For part (7), first notice that

$$\mathbb{1}_0-2e=(\overline{\gamma}-\gamma)\Delta_{[1]}+(\gamma-\overline{\gamma})\Delta_{[2]}=(\overline{\gamma}-\gamma)(-\Delta_{[1]}+\Delta_{[2]})=\frac{-i\sqrt{3}}{3}(-\Delta_{[1]}+\Delta_{[2]}).$$

Then, by computing $(\mathbb{1}_0-2e)\delta_g$ for each $g\in\mathbb{Z}/3\mathbb{Z}$, we find that the standard matrix of $\mathbb{1}_0-2e$ is

$$\frac{-i\sqrt{3}}{3}\begin{pmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{pmatrix}. \tag{4-11}$$

Since the $\ell^2_d\to\ell^2_d$ operator norm is the maximum singular value of the matrix, (4-11) shows that $\|\mathbb{1}_0-2e\|_{F^2(\mathbb{Z}/3\mathbb{Z})}=1$. Combining this with the inner inequalities in (4-8) and the fact that $\|\mathbb{1}_0\|_{F^2(\mathbb{Z}/3\mathbb{Z})}=1$ (see Remark 4.16) we conclude that $|||\mathbb{1}_0-2e|||_2=1$, as wanted. Finally, for parts (8) and (9), the result is clear for $p=1$ and $p=2$. Next assume that $p\in(1,2)$ with $\frac{1}{p}=(1-\theta)+\frac{\theta}{2}$ for $\theta\in(0,1)$. Then, the Riesz–Thorin interpolation theorem shows that

$$\|\mathbb{1}_0-2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}\le\|\mathbb{1}_0-2e\|^{1-\theta}_{F^1(\mathbb{Z}/3\mathbb{Z})}\|\mathbb{1}_0-2e\|^{\theta}_{F^2(\mathbb{Z}/3\mathbb{Z})}=\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}.$$

Now, if $p'=\frac{p}{1-p}\in(2,\infty)$, Proposition 4.11 in [12] shows that the $F^{p'}(\mathbb{Z}/3\mathbb{Z})$ and $F^p(\mathbb{Z}/3\mathbb{Z})$ norms of $\mathbb{1}_0-2e$ agree and therefore

$$\|\mathbb{1}_0-2e\|_{F^{p'}(\mathbb{Z}/3\mathbb{Z})}\le\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}=\left(\frac{2\sqrt{3}}{3}\right)^{1-\frac{2}{p'}}.$$

The desired results now follow from the right inner bound in (4-8). □

The previous proposition gives all the necessary tools to show that for a certain $p_0\in(1,2)$, when $p\in[1,p_0)\cup(p_0',\infty)$, then the element $\mathbb{1}_0-2e\in M_0^p(\mathbb{Z}/3\mathbb{Z})$ is not an invertible isometry, which is essentially the proof of the following main result.

Theorem 4.21. *Let $p_0=1.606$ and let $p_0'=\frac{p_0}{p_0-1}$. Take $p\in[1,p_0]\cup[p_0',\infty)$. Then $M_0^p(\mathbb{Z}/3\mathbb{Z})$, the multiplier algebra of $F_0^p(\mathbb{Z}/3\mathbb{Z})$, is not an L^q-operator algebra for any $q\in[1,\infty)$.*

Proof. Suppose on the contrary that $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is an L^q-operator algebra for some $q\in[1,\infty)$. Then, since $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is a multiplier algebra it is unital (see Definition 3.1), whence Proposition 3.7 in [9] guarantees the existence of an isometric unital representation $\pi : M_0^p(\mathbb{Z}/3\mathbb{Z}) \to \mathcal{L}(L^q(\mu))$ for some measure space (Ω, μ).

Take $e = \gamma\Delta_{[1]} + \overline{\gamma}\Delta_{[2]}$ as in Proposition 4.20. Parts (2),(4), and (5) in Proposition 4.20 show that e is a bicontractive element in $M_0^p(\mathbb{Z}/3\mathbb{Z})$. We claim the following:

Claim 1. $|||\mathbb{1}_0 - 2e|||_p > 1$ when $p \in [1, p_0]$ or $p \in [p_0', \infty)$.

Once the claim above is proven, it will follow that $\mathbb{1}_0 - 2e$ is not an invertible isometry in $M_0^p(\mathbb{Z}/3\mathbb{Z})$. Thus, since π is a unital isometry, we would have found a bicontractive idempotent $\pi(e)$ in $\mathcal{L}(L^q(\mu))$ such that $1 - 2\pi(e)$ is not an invertible isometry when $p \in [1, p_0] \cup [p_0', \infty)$, contradicting Theorem 2.9.

Therefore it only remains to establish Claim 1, which already follows when $p = 1$ by Part (6) in Proposition 4.20. Otherwise, observe that when $p \in (1, \infty) \setminus \{2\}$ we have

$$|||\mathbb{1}_0 - 2e|||_p \geq \frac{\|(\mathbb{1}_0 - 2e)(\mathbb{1}_0 - 2e)\|_{F^p(\mathbb{Z}/3\mathbb{Z})}}{\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}} = \frac{\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})}}{\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}} \tag{4-12}$$

Next, let $a_0 \in \ell_0^1(\mathbb{Z}/3\mathbb{Z})$ and $b_{\varepsilon_p} \in \ell^p(\mathbb{Z}/3\mathbb{Z})$ be as in part (4) of Lemma 4.14. That is, for any $p \in (1, \infty) \setminus \{2\}$ we have

$$\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})} \geq \frac{\|a_0\|_p}{\|b_{\varepsilon_p}\|_p} = \frac{(2^p + 2)(2^{\frac{1}{p-1}} + 1)^p}{3^p(2^{\frac{p}{p-1}} + 2)} > 1. \tag{4-13}$$

Now take any $p \in (1, 2)$ and set $p' = \frac{p}{p-1} \in (2, \infty)$. Proposition 4.11 in [12] gives that the norms of $\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$ and $\|\mathbb{1}_0\|_{F^{p'}(\mathbb{Z}/3\mathbb{Z})}$ agree, so together with Part (8) in Proposition 4.20, combining the bounds from (4-12) and (4-13), we now get

$$|||\mathbb{1}_0 - 2e|||_p \geq \frac{\|\mathbb{1}_0\|_{F^{p'}(\mathbb{Z}/3\mathbb{Z})}}{(\frac{2\sqrt{3}}{3})^{\frac{2}{p}-1}} \geq \frac{\frac{\|a_0\|_{p'}}{\|b_{\varepsilon_{p'}}\|_{p'}}}{\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}} = \frac{(2^{\frac{p}{p-1}} + 2)(2^{p-1} + 1)^{\frac{p}{p-1}}}{3^{\frac{p}{p-1}}(2^p + 2)\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}}.$$

Next we define $h : (1, 2) \to \mathbb{R}$ using the right hand side in the inequality above, that is

$$h(p) = \frac{(2^{\frac{p}{p-1}} + 2)(2^{p-1} + 1)^{\frac{p}{p-1}}}{3^{\frac{p}{p-1}}(2^p + 2)\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}} = \frac{(2^{\frac{p}{p-1}} + 2)\left(2^{p-1} + 1\right)^{\frac{1}{p-1}}}{3^{\frac{1}{2} - \frac{1}{p} + \frac{p}{p-1}} \cdot 2^{\frac{2}{p}}}.$$

It can be checked that $p \mapsto h(p)$ is strictly decreasing on $(1, p_0]$, we omit the lengthy calculations. A direct computation now shows that $h(p_0) > 1.000098$, so it follows that $|||\mathbb{1}_0 - 2e|||_p > 1$ for any $p \in (1, p_0]$. Similarly, when $p \in (2, \infty)$ we get using Part (9) in Proposition 4.20

$$|||\mathbb{1}_0 - 2e|||_p \geq \frac{(2^p+2)(2^{\frac{1}{p-1}}+1)^p}{3^p(2^{\frac{p}{p-1}}+2)\left(\frac{2\sqrt{3}}{3}\right)^{1-\frac{2}{p}}} = h\Big(\frac{p}{p-1}\Big).$$

It follows from the $p \in (1, p_0]$ case that $|||\mathbb{1}_0 - 2e|||_p > 1$ for any $p \in [p_0', \infty)$, proving Claim 1, and therefore finishing the proof. □

Remark 4.22. It is likely that, for all $p \in [1,\infty)\setminus\{2\}$, the algebra $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is not an L^q-operator algebra for any $q \in [1,\infty)$. A proof of this fact could potentially be obtained by the same method employed in the proof of Theorem 4.21 above provided that we have either better bounds or exact values for the norms of $\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$ and $\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$, which are generally hard to compute (see [16]). For instance, we believe that the actual value for $\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$ when $p \in [1,2]$ is $\frac{\sqrt{3}}{3}(1+2^{p-1})^{\frac{1}{p}}$ which, if true, improves the value of p_0 to $p_0 = 1.675$.

Remark 4.23. Unfortunately, the bicontractive idempotent argument does not work when $G \in \{\mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\}$. Indeed, in this case if $e \in M_0(G)$ is any bicontractive idempotent, then it can be shown that $\mathbb{1}_0 - 2e \in M_0(G)$ is in fact an invertible isometry as expected for an L^p-operator algebra. Due to the difficulty of computing the norm of an element in $M_0^p(G)$, a different argument is likely needed to argue a version of Theorem 4.21 for higher order groups. At this time, for a general finite group G with $\mathrm{card}(G) > 3$ and $p \neq 2$, we do not know whether $M_0^p(G)$ can be isometrically represented on an L^q-space.

5. Idempotents and Gelfand transform for $\ell_0^1(G)$

In what follows we assume that G is a finite abelian group. In Proposition 4.18, for each $p \in [1,\infty)$, we obtained a unital Banach algebra $M_0^p(G) = M(F_0^p(G))$ whose underlying space is the augmentation ideal $\ell_0^1(G)$ normed with $|||-|||_p$. In addition, $M_0^p(G)$ is both a unital and a commutative Banach algebra when G is abelian. When $\mathrm{card}(G) > 2$ and $p \neq 2$, notice that $M_0^p(G)$ equipped with the inherited involution from $\ell^1(G)$ cannot be a C*-algebra. In fact, in such cases $M_0^p(G)$ is not isomorphically isometric to any closed subalgebra of $C(\Omega)$ where $\Omega = \mathrm{Max}(M_0^p(G))$ because $|||\Delta_g|||_p^2 \neq |||\Delta_g\Delta_g^*|||_p$ for all $g \in G \setminus \{1_G\}$, so the Gelfand map is not isometric. We will see, however, that the Gelfand map is an algebraic isomorphism.

Lemma 5.1. *For $n \in \mathbb{Z}_{>1}$, let*

$$G = \mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$$

and $(x_{[1]}, \ldots, x_{[n-1]}) \in \mathbb{C}^{n-1}$ with $x_{[0]} = 0$. Consider the system of equations

$$x_{[j]+[k]} = x_{[j]}x_{[k]} + x_{[j]} + x_{[k]}, \quad 1 \leq j, k \leq n-1. \tag{5-1}$$

Then the only nontrivial solutions for (5-1) are of the form

$$x_{[k]} = (x_{[1]}+1)^k - 1 \text{ for all } 1 \le k \le n-1 \text{ where } (x_{[1]}+1)^n = 1. \tag{5-2}$$

In particular, this means that (5-1) has exactly $n-1$ nontrivial solutions.

Proof. Given a solution for (5-1), we show that $x_{[k]} = (x_{[1]}+1)^k - 1$ for all $1 \le k \le n-1$. We do so by induction on k. When $k = 1$,

$$x_{[1]} = (x_{[1]}+1)^1 - 1 = x_{[1]}.$$

When $k = 2$,

$$x_{[2]} = x_{[1]+[1]} = x_{[1]}^2 + 2x_{[1]} = (x_{[1]}+1)^2 - 1$$

Suppose $x_{[j]} = (x_{[1]}+1)^j - 1$. Then

$$\begin{aligned} x_{[j]+[1]} &= x_{[j]}x_{[1]} + x_{[j]} + x_{[1]} \\ &= \big((x_{[1]}+1)^j - 1\big)\, x_{[1]} + \big((x_{[1]}+1)^j - 1\big) + x_{[1]} \\ &= x_{[1]}(x_{[1]}+1)^j - x_{[1]} + x_{[1]} + (x_{[1]}+1)^j - 1 \\ &= (x_{[1]}+1)(x_{[1]}+1)^j - 1 \\ &= (x_{[1]}+1)^{j+1} - 1 \end{aligned}$$

Now since $x_{[0]} = 0$, it follows that

$$0 = x_{[0]} = x_{[n-1]+[1]} = x_{[n-1]}x_{[1]} + x_{[n-1]} + x_{[1]} = (x_{[1]}+1)^n - 1,$$

so indeed any solution of the system (5-1) is of the form (5-2).

Next, we show that $x_{[k]} = (x_{[1]}+1)^k - 1$ for all $1 \le k \le n-1$ and $(x_{[1]}+1)^n = 1$ satisfies (5-1). Let $1 \le j, k \le n-1$. Then,

$$\begin{aligned} x_{[j]}x_{[k]} &= \big((x_{[1]}+1)^j - 1\big)\big((x_{[1]}+1)^k - 1\big) \\ &= (x_{[1]}+1)^{j+k} - (x_{[1]}+1)^k - (x_{[1]}+1)^j + 1, \end{aligned}$$

and since $x_{[j]} = (x_{[1]}+1)^j - 1$ for any $1 \le j \le n-1$, we conclude

$$x_{[j]}x_{[k]} + x_{[j]} + x_{[k]} = (x_{[1]}+1)^{j+k} - 1 = x_{[j+k]} = x_{[j]+[k]}.$$

Clearly, $(x_{[1]}+1)^n = 1$ satisfies $(x_{[1]}+1)^n - 1 = 0 = x_{[0]}$, so we are done. □

Maschke's and Wedderburn's theorems (see Chapter 16 of [8] for an elementary introduction) imply that the only nilpotent element of $C_c(G)$ for G finite abelian is $0 \in C_c(G)$. In particular, this means that $C_c(G)$ and all of its (two-sided) ideals have trivial nilradical. This leads to the following important observation.

Lemma 5.2. *Let I be any (two-sided) ideal in $\ell^1(G)$ for G finite abelian. Then the Gelfand transform $\Gamma_I : I \to C(\mathrm{Max}\,(I))$ is an algebraic isomorphism.*

We now begin with a technical lemma.

Lemma 5.3. *Let G be any discrete group and define for $m \in \mathbb{Z}_{>1}$*

$$\mathcal{B}^m = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \ : \ \varepsilon_j \in \{0,1\} \text{ for all } 1 \le j \le m\} \setminus \{(0,0,\ldots,0)\}.$$

Then for any collection of m elements $g_1, g_2, \ldots, g_m \in G \setminus \{1_G\}$,

$$\Delta_{g_1 g_2 \cdots g_m} = \sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m)\in\mathcal{B}^m} \Delta_{g_1}^{\varepsilon_1} \Delta_{g_2}^{\varepsilon_2} \cdots \Delta_{g_m}^{\varepsilon_m} \tag{5-3}$$

where we define $\Delta_{g_j}^0 = 1$ for all $1 \le j \le m$. In particular, this means

$$\Delta_{gg'} = \Delta_g \Delta_{g'} + \Delta_g + \Delta_{g'} \text{ for all } g, g' \in G \setminus \{1_G\}. \tag{5-4}$$

Proof. Fix $g, g' \in G \setminus \{1_G\}$. We argue by induction. A direct computation gives

$$\Delta_{gg'} = \Delta_g \Delta_{g'} + \Delta_{g'} + \Delta_g$$

Next, assume equation (5-3) holds for the product of $m-1$ nonidentity elements of G. Choose $g_1, g_2, \ldots, g_{m-1}, g_m \in G \setminus \{1_G\}$. Then, using the induction hypothesis at the last step,

$$\begin{aligned}\Delta_{g_1 g_2 \cdots g_{m-1} g_m} &= \Delta_{(g_1 g_2 \cdots g_{m-1}) g_m} = \Delta_{g_1 g_2 \cdots g_{m-1}} \Delta_{g_m} + \Delta_{g_1 g_2 \cdots g_{m-1}} + \Delta_{g_m} \\ &= \sum_{(\varepsilon_j)\in\mathcal{B}^m} \Delta_{g_1}^{\varepsilon_1} \Delta_{g_2}^{\varepsilon_2} \cdots \Delta_{g_{m-1}}^{\varepsilon_{m-1}} \Delta_{g_m}^{\varepsilon_m}. \qquad \square\end{aligned}$$

We are now ready to prove the cyclic group case.

Proposition 5.4. *Let $G = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}_{>1}$. Then $\ell_0^1(G)$ is algebraically isomorphic to $\mathbb{C}^{n-1}$ equipped with pointwise multiplication.*

Proof. We have already established that $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ is an ideal in $\ell^1(\mathbb{Z}/n\mathbb{Z})$ and so we immediately have an algebraic isomorphism between $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ and continuous functions on its maximal ideal space. Thus, it suffices to establish that the maximal ideal space of $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ consists of $n-1$ elements.

Let $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$. Thanks to (5-4), any multiplicative linear functional $\omega : \ell_0^1(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ must satisfy

$$\omega(\Delta_{[j]+[k]}) = \omega(\Delta_{[j]})\omega(\Delta_{[k]}) + \omega(\Delta_{[j]}) + \omega(\Delta_{[k]}) \tag{5-5}$$

for all $1 \le j, k \le n-1$. Letting $\omega(\Delta_{[j]}) = x_{[j]}$, we recover the system described in Lemma 5.1. We observe that each solution of the system (5-1) is completely determined by the choice of $x_{[1]}$, which must be chosen to satisfy $(x_{[1]}+1)^n = 1$. For each $1 \le j \le n-1$, let

$$\omega_j(\Delta_{[1]}) = \exp(2\pi i \cdot j/n) - 1 = x_{j,[1]}. \tag{5-6}$$

As discussed, this choice determines the remaining $\omega_j(\Delta_{[k]}) = x_{j,[k]}$ and so we have a one-to-one correspondence between the multiplicative linear functionals of the ideal $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ and the solutions of System (5-1). Of course, $\mathrm{Max}(\ell_0^1(\mathbb{Z}/n\mathbb{Z}))$ is in bijection with $\Omega = \{\omega_j : 1 \le j \le n-1\}$. Lemma 5.2 gives that

$$\Gamma : \ell_0^1(\mathbb{Z}/n\mathbb{Z}) \to C(\Omega)$$

is an algebraic isomorphism. Finally since the canonical map $C(\Omega) \to \mathbb{C}^{n-1}$ given by

$$f \mapsto (f(\omega_1), f(\omega_2), \dots, f(\omega_{n-1}))$$

is also an algebraic isomorphism, we have indeed shown that $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ and $\mathbb{C}^{n-1}$ are algebraically isomorphic, completing the proof. □

We now extend Proposition 5.4 to any finite abelian group. Consider

$$G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_m\mathbb{Z}),$$

where $n_1, \dots, n_m \in \mathbb{Z}_{>1}$. Then any $a \in \ell_0^1(G)$ may be written as

$$a = \sum_{(g_1,g_2,\dots,g_m)\in G} a(g_1, g_2, \dots, g_m)\Delta_{(g_1,g_2,\dots,g_m)}.$$

We note that $(g_1, g_2, \dots, g_m)(g_1', g_2', \dots, g_m') = (g_1 + g_1', g_2 + g_2', \dots, g_m + g_m')$ and we will abuse notation by writing g for $(0, \dots, g, \dots 0)$ and $g_1g_2\cdots g_m$ for $(g_1, g_2, \dots, g_m)$.

For each cyclic group $\mathbb{Z}/n_j\mathbb{Z}$, let $\{\omega_j^{(k)}\}_{0\le k\le n_j-1}$ be the multiplicative linear functionals associated with $\mathrm{Max}\,(\ell_0^1(\mathbb{Z}/n_j\mathbb{Z}))$ where $\omega_j^{(0)}$ represents the zero function on $\ell_0^1(\mathbb{Z}/n_j\mathbb{Z})$. We show that all multiplicative linear functionals of $\ell_0^1(G)$ are of the form

$$\omega(\Delta_{g_1g_2\cdots g_m}) = \sum_{(\varepsilon_j)\in\mathcal{B}^m} \omega_1^{(k_1)}(\Delta_{g_1}^{\varepsilon_1})\omega_2^{(k_2)}(\Delta_{g_2}^{\varepsilon_2})\cdots\omega_m^{(k_m)}(\Delta_{g_m}^{\varepsilon_m}) \tag{5-7}$$

for all $0 \le k_i \le n_i$, $1 \le i \le m$, where we interpret $\omega_{n_j}^{(k_j)}(\Delta_{g_k}^0)$ as equaling 1 when in a product with other nonzero terms.

Suppose $\omega : \ell_0^1(G) \to \mathbb{C}$ is a multiplicative linear functional. When ω is restricted to linear sums of Δ_{g_j} for $g_j \in \mathbb{Z}/n_j\mathbb{Z}$, which we will denote by ω_j, then ω_j is a multiplicative linear functional of $\ell_0^1(\mathbb{Z}/n_j\mathbb{Z})$. Thus, there must exist $k_j \in \{1, 2, \dots, n_j - 1\}$ such that $\omega_j = \omega_j^{(k_j)}$. Equation (5-7) then follows by previous comments and Lemma 5.3. Note that ω is the zero map exactly when all the $\omega_j^{(k_j)}$ are zero maps. Thus, we see that we have $n_1n_2\cdots n_m = \mathrm{card}(G)$ possible choices for ω, including the zero map.

This leads to the following theorem and corollary.

Theorem 5.5. *Let G be a finite abelian group with* $\mathrm{card}(G) > 1$. *Then the Gelfand transform from $\ell_0^1(G)$ to $C(\{1, 2, \dots, n-1\})$ is an algebraic isomorphism.*

Corollary 5.6. *Let G be a finite abelian group with* $\operatorname{card}(G) > 1$. *Then the algebras* $\ell^1_0(G)$, $\ell^1(\mathbb{Z}/(n-1)\mathbb{Z})$, $C(\{1, 2, \dots, n-1\})$, *and* $\mathbb{C}^{n-1}$ *are all algebraically isomorphic.*

Recall that we used a bicontractive idempotent as the primary tool to show that for certain values of $p \in [1, \infty) \setminus \{2\}$, the algebra $M^p_0(\mathbb{Z}/3\mathbb{Z})$ cannot be isometrically represented on any L^q-space. To this end, we present a description of the idempotents of $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$ and, in consequence, those of $M^p_0(\mathbb{Z}/n\mathbb{Z})$ since the algebraic structures are the same. We start with a short definition.

Definition 5.7. Let $a \in \ell^1_0(\mathbb{Z}/n\mathbb{Z})$ for $n \in \mathbb{Z}_{>1}$. Then, since a is uniquely determined by $[a] = (a([1]), \dots, a([n-1])) \in \mathbb{C}^{n-1}$, we define a canonical map $\ell^1_0(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}^{n-1}$ via $a \mapsto [a] \in \mathbb{C}^{n-1}$.

Let $n \in \mathbb{Z}_{>1}$. We define the *translation matrix of* $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$ by

$$X = \begin{pmatrix} x_{1,[1]} & x_{1,[2]} & \cdots & x_{1,[n-1]} \\ x_{2,[1]} & x_{2,[2]} & \cdots & x_{2,[n-1]} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,[1]} & x_{n-1,[2]} & \cdots & x_{n-1,[n-1]} \end{pmatrix}. \tag{5-8}$$

Proposition 5.4 implies that for any $v \in \mathbb{C}^{n-1}$ there exists a unique $a \in \ell^1_0(\mathbb{Z}/n\mathbb{Z})$ such that $X[a] = v$, where $[a]$ is as in Definition 5.7. In particular, if we let $\{e_k\}_{k=1}^{n-1}$ represent the standard basis of $\mathbb{C}^{n-1}$, then all nonzero idempotents of $\mathbb{C}^{n-1}$ are of the form

$$\varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_{n-1} e_{n-1} \quad \text{for} \quad (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}) \in \mathcal{B}^{n-1}$$

where $\mathcal{B}^{n-1}$ is as in Lemma 5.3. Then our discussion above yields the following Proposition.

Proposition 5.8. *Fix $n \in \mathbb{Z}_{>1}$ and let $a_k \in \ell^1_0(\mathbb{Z}/n\mathbb{Z})$ be the unique element such that $X[a_k] = e_k$. Then*:

(1) *Each a_k is an idempotent of norm at least one.*

(2) $a_k a_j = 0$ *for all* $k \neq j$.

(3) $a_1 + a_2 + \cdots + a_{n-1} = \mathbb{1}_0$.

(4) $\{a_k\}_{k=1}^{n-1}$ *forms a basis for* $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$.

(5) *Each nonzero idempotent of* $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$ *is of the form*

$$\varepsilon_1 a_1 + \varepsilon_2 a_2 + \cdots + \varepsilon_{n-1} a_{n-1} \quad \textit{for} \quad (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}) \in \mathcal{B}^{n-1}.$$

(6) $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$ *has precisely* 2^{n-1} *idempotents.*

6. L^p-operator norms on $\mathbb{C}^n$

We have shown that the conclusion of Theorem 4.1 can go either way when dropping the two hypotheses. As shown in Section 4B, for each $p \in [1, \infty)$, the algebra of strictly upper triangular matrices acting on ℓ_2^p, T_2^p, is a degenerate L^p-operator algebra without a cai and yet its multiplier algebra is nondegenerately representable on an L^q-space for any $q \in [1, \infty)$. In contrast, as shown in Section 4C, $F_0^p(\mathbb{Z}/3\mathbb{Z})$ is a degenerate L^p-operator algebra without a cai, but when $p \in [1, p_0] \cup [p_0', \infty)$, its multiplier algebra cannot be isometrically represented on an L^q-space for any $q \in [1, \infty)$. At this time, it is unclear what conditions are necessary to guarantee that the multiplier algebra of a degenerate L^p-operator algebra without a cai is again an L^p-operator algebra.

Given that the norms of L^p-operator algebras are not unique, a natural question is to characterize norms on $\mathbb{C}^n$ that make it into an L^p-operator algebra. This problem is already interesting for $p = 1$, for which we know of at least two norms on $\mathbb{C}^n$ which make it an L^1-operator algebra:

(1) $\mathbb{C}^n$ with the supremum norm is an L^1-operator algebra acting on ℓ_n^1 via multiplication operators.

(2) Let $\mathcal{F} : \ell^1(\mathbb{Z}/n\mathbb{Z}) \to C(\mathbb{Z}/n\mathbb{Z})$ the Fourier transform. Then we have algebra isomorphisms

$$\mathbb{C}^n \cong C(\mathbb{Z}/n\mathbb{Z}) \cong \mathcal{F}^{-1}(C(\mathbb{Z}/n\mathbb{Z})) = \ell^1(\mathbb{Z}/n\mathbb{Z})$$

which make $\mathbb{C}^n$ an L^1-operator algebra with norm coming from the identification with $\ell^1(\mathbb{Z}/n\mathbb{Z})$.

Question 6.1. Are these the only two norms that make $\mathbb{C}^n$ an L^1-operator algebra?

Theorem 4.21 exhibits a norm in $\mathbb{C}^2$ for which $\mathbb{C}^2$ fails to be an L^q-operator algebra for any $q \in [1, \infty)$, which is to say the identification of $M_0^1(\mathbb{Z}/3\mathbb{Z})$ with $\mathbb{C}^2$ carries neither of the norms listed above. This leads to a more general question.

Question 6.2. Let $n \in \mathbb{Z}_{>1}$ and consider $\mathbb{C}^n$ with pointwise multiplication. Is it possible to find all the norms on $\mathbb{C}^n$ that make it an L^1-operator algebra? An L^p-operator algebra for some $p \in [1, \infty)$?

Acknowledgments

The authors thank David Blecher and Eusebio Gardella for their encouragement and discussions, particularly regarding Question 6.2 above. We are also thankful to Hannes Thiel for pointing out several references and for encouraging us to generalize our main results to $p > 1$. Delfín also thanks N. Christopher Phillips, who read earlier versions of Section 3 and gave suggestions to improve the presentation.

Finally, we would like to thank the anonymous referee for a careful and thorough reading of the initial draft and for sending useful references and suggestions that greatly improved some content in this paper. In particular, we thank the referee for pointing out that it made more sense to look at the augmentation ideal of $F^p(G)$ instead of $F^p_{\mathrm{r}}(G)$ as we had initially considered.

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Received March 25, 2024. Revised November 12, 2024.

ANDREY BLINOV
andrew.blinov@gmail.com

ALONSO DELFÍN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
BOULDER, CO
UNITED STATES
alonso.delfin@colorado.edu

ELLEN WELD
DEPARTMENT OF MATHEMATICS AND STATISTICS
SAM HOUSTON STATE UNIVERSITY
HUNTSVILLE, TX
UNITED STATES
elw028@shsu.edu

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 333 No. 2 December 2024
