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Fomin–Kirillov algebras are quadratic approximations of Nichols algebras associated with the conjugacy class of transpositions in a symmetric group and a (rack) 2-cocycle q^+ with values in $\{\pm 1\}$. Bazlov generalized their construction replacing the class of transpositions by the classes of reflections in an arbitrary finite Coxeter group. We prove that Bazolv's cocycle q^+ is twist equivalent to the constant cocycle $q^- \equiv -1$, generalizing a result of Vendramin. As a consequence, the Nichols algebras associated with the two different cocycles have the same Hilbert series and one is quadratic if and only if the other is quadratic. We further apply recent results of Heckenberger, Meir and Vendramin and Andruskiewitsch, Heckenberger and Vendramin to complete the classification of the finite-dimensional Nichols algebras of Yetter–Drinfeld modules over the dihedral groups.

1. Introduction

Racks and rack cocycles appear in the study of knot theory, as they naturally provide a solution of the braid equation. It became apparent that they are key basic objects in the study of Nichols (shuffle) algebras, and the classification of pointed Hopf algebras [2], since Nichols algebras are graded braided Hopf algebras that depend only on the datum of a braided vector space. Several tools have been developed in order to detect finiteness properties of Nichols algebras (such as finite dimension, finite GK-dimension, finite presentation) that depend only on the rack and not on the cocycle, and properties that are preserved under suitable transformations of the cocycle; see, among others, [2; 5; 6; 7; 8; 9; 25; 31].

Relevant racks in the study of Nichols algebras can be described as subsets X of a group G that are stable by conjugation [2, Definition 1.3]. For such racks, *twisting* a cocycle by a 2-cocycle of G with trivial coefficients corresponds, at the level of Nichols algebras, to performing a (dual) Drinfeld twist [7; 34]. As a consequence,

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Nichols algebras associated with the same rack and twist-equivalent cocycles have the same Hilbert series, same dimension and same GK-dimension.

We first focus on finitely generated Coxeter groups and a special family of racks and 2-cocycles, that is of interest due to its relation with the cohomology of the flag variety. Here the rack is the class (or union of classes) of reflections in a Coxeter group W, and the 2-cocycle is a cocycle with values in $\{\pm 1\}$. If W is a Weyl group, the quadratic approximation of the corresponding Nichols algebra is a noncommutative algebra containing the cohomology algebra of the flag variety of the corresponding algebraic group. For $W = \mathbb{S}_n$ this is the so-called Fomin–Kirillov algebra introduced in [21] for creating an algebraic and combinatorial framework to perform Schubert calculus. The construction was later generalized to the case of arbitrary Weyl groups in [12]. For $W = \mathbb{S}_3$, $\mathbb{S}_4$ and $\mathbb{S}_5$ the Nichols algebra in question is quadratic and finite dimensional [22; 36]. For $n > 5$ a proof of infinite dimensionality of the Fomin–Kirillov algebras appeared in [11], but it is not known whether the Nichols algebras are quadratic. This property and dimension are preserved by twists.

A novel approach to the analysis of the dimension and of the degree of a minimal set of generating relations may come through geometry, using the category equivalence in [33]. Through this equivalence, Nichols algebras correspond to intersection cohomology (IC) complexes on the infinite-dimensional space $\mathrm{Sym}(\mathbb{C})$ of monic polynomials, with a so-called factorization datum. One may expect that the Nichols algebras associated with a conjugacy class C of a finite group G and the constant cocycle equal to -1 might be easier to handle because the restriction of the corresponding IC complexes to the dense open subset $\mathrm{Sym}_{\neq}(\mathbb{C})$ of multiplicity-free polynomials is the push-forward through the covering map of the constant sheaf on the Hurwitz space with Galois group G and local monodromy C [19, §2.4, 3.3; 33, §3.3 A]. These spaces are widely studied. For this reason, it becomes relevant to know whether the cocycle associated with the rack of reflections in Coxeter groups as in [12; 21; 36] are twist equivalent to the -1 constant cocycle.

For $W = \mathbb{S}_n$ this problem was answered in the affirmative in [39], and for the dihedral group an attempt was done in [17]. We provide a unified approach to give a positive answer that applies to all finitely generated Coxeter groups as long as the entries of its Coxeter matrix $A(W)$ are all finite. We also show that the cocycles are cohomologous if and only if the entries in $A(W)$ are all odd. Hence, for dihedral groups of regular polygons with an odd number of edges, we recover the well-known fact that the Nichols algebras corresponding to these two cocycles are isomorphic [36, §5].

We next combine the information we gained in order to reorganize and complement what is known about the dimension of Nichols algebras associated with reflections in Coxeter groups, showing that the key case to be studied is type A_5. Finally, we address the case of Coxeter groups of rank 2, that is, the dihedral groups $I_2(n)$ of order $2n$. This is of particular interest because all finite groups generated by two involutions are dihedral. Finite-dimensional Nichols algebras over $I_2(n)$ for $4 \mid n$ were classified in [13; 20]. The case of $n = 3$ is to be found in [6] and $I_2(2)$ is not irreducible. Using the main results in [10; 31] and the work in [1] we address the remaining open case, i.e., when 4 does not divide n and $n \neq 2, 3$, completing the classification of finite-dimensional Nichols algebras over dihedral groups. In particular, we show that there is no finite-dimensional Nichols algebra over $I_2(n)$ when n is odd and different from 3. Hence, the only finite-dimensional complex pointed Hopf algebra with group of grouplikes isomorphic to $I_2(n)$ for such an n is the group algebra.

The paper is structured as follows: In Section 2 we introduce the basic notions on Coxeter groups, racks, cocycles and Nichols algebras; in Section 2.4 we define the cocycles q^+ and q^- on the rack of reflections in an arbitrary Coxeter group and translate twist equivalence of q^+ and q^- into existence of a section, generalizing the strategy in [39]. The core argument is in Section 3 where we apply results in [38] in order to show existence of the sought section (Theorem 2.8). In Section 4 we collect and complement to what is known on Nichols algebras associated with the rack of reflections in finite Coxeter groups, we apply Theorem 2.8 and the main results in [10; 31] in order to list the Coxeter groups for which a conjugacy class of reflections could possibly support a finite-dimensional Nichols algebra. Section 4.1 is devoted to the completion of the classification of finite-dimensional Nichols algebras over dihedral groups.

2. Basic notions

2.1. ***Coxeter groups and root systems.*** Let (W, S) be a Coxeter system, with preferred generating set $S = \{s_1, \dots, s_l\}$, Coxeter graph $\Gamma(W)$, length function $\ell \colon W \to \mathbb{N}$ and Coxeter matrix $A(W) = (m_{ij})_{i,j=1,\dots,l}$, where $|s_i s_j| = m_{ij}$. We further assume that $m_{ij} < \infty$ for any i, j. Following [14] we fix a basis $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of $\mathbb{R}^l$ and define the bilinear form $(\cdot\,,\cdot)$ on $\mathbb{R}^l$ by setting $(\alpha_i, \alpha_j) := -\cos \frac{\pi}{m_{ij}}$ for all $i, j \in \{1, \dots, l\}$. We identify W with the image of the geometric representation obtained mapping $s_i \in S$ to the reflection with respect to α_i. Then $\Phi = W\Delta$ is the set of roots and we denote by Φ^+ and Φ^- the set of positive and negative roots, respectively. We also set

$$T := \{s_\alpha \mid \alpha \in \Phi^+\} = \{w s_\alpha w^{-1} \mid \alpha \in \Delta,\ w \in W\}$$

for the set of reflections in W so $\Phi = \{\pm\alpha \mid s_\alpha \in T\}$. For $x \in T$ we indicate by α_x the unique root in Φ^+ satisfying $x = s_{\alpha_x}$.

We will be mainly interested in W-conjugacy classes in T: there are as many as the number of connected components of the graph obtained from $\Gamma(W)$ by removing the edges with even label [15, Proposition 3, Chapitre IV, §1, Théorème 1, Chapitre VI, §4].

We denote by $\Sigma = S^{(\mathbb{N})}$ the free group on S whose elements are words in the alphabet S and by $\pi : \Sigma \to W$ the projection. We denote by a^{op} the word *opposite* to a, that is, the word obtained from a by reversing the order of its letters. Clearly, $\pi(a^{\mathrm{op}}) = \pi(a)^{-1}$.

For $x \in W$ the subset $\mathcal{R}(x) \subset \Sigma$ stands for the set of reduced expressions of x. An element $x \in W$ lies in T if and only if it has a nontrivial palindromic reduced expression, that is, a nonempty reduced expression $r(x)$ of x such that $r(x) = r(x)^{\mathrm{op}}$; see [38, Proposition 2.4]. For $x \in T$ we denote by $\mathcal{P}(x)$ the set of *palindromic* reduced expressions of x.

For the reader's convenience we recollect some very basic facts on length of conjugate reflections that will be needed in the sequel.

Lemma 2.1. *Let $\beta \in \Phi^+$ and $\alpha \in \Delta$. Then*:

(i) $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) + 2 \iff s_\beta(\alpha) \in \Phi^+ \setminus \{\alpha\} \iff (\alpha, \beta) < 0$ *and* $\beta \neq \alpha$.

(ii) $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) - 2 \iff s_\beta(\alpha) \in \Phi^- \setminus \{-\alpha\} \iff (\alpha, \beta) > 0$ *and* $\beta \neq \alpha$.

(iii) $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) \iff s_\beta s_\alpha = s_\alpha s_\beta \iff (\alpha, \beta) = 0$ *or* $\beta = \alpha$.

(iv) *If $\beta \in \Phi^+ \setminus \Delta$, there is always $\alpha \in \Delta$ such that $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) - 2$.*

Proof. We prove (i). In virtue of [15, Exercise V.4.8], the condition $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) + 2$ is equivalent to $s_\beta(\alpha) \in \Phi^+$ and $s_\alpha s_\beta(\alpha) \in \Phi^+$, that is, equivalent to $s_\beta(\alpha) \in \Phi^+ \setminus \{\alpha\}$ by [14, Lemma 4.4.3]. Now, $s_\beta(\alpha) = \alpha - 2(\alpha, \beta)\beta \in \Phi^+ \setminus \{\alpha\}$ if and only if $(\alpha, \beta) < 0$ and $\beta \neq \alpha$. Then, (ii) follows similarly and (iii) by exclusion. To prove (iv) assume that (ii) never occurs for β. Then $(\beta, \alpha) \leqslant 0$ for any $\alpha \in \Delta$ and thus also for any $\alpha \in \Phi^+$, whence $(\beta, \beta) \leqslant 0$, a contradiction. □

2.2. *Racks, cocycles and Nichols algebras.*

Definition 2.2. A rack is a pair $(X, \triangleright)$ where X is a nonempty set and $\triangleright : X \times X \to X$ is a function, such that the map $x \mapsto i \triangleright x$ is bijective for all $i \in X$, and $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$.

We will be mainly interested in racks of the form $X \subset G$, where G is a group, X is stable by conjugation and $x \triangleright y = xyx^{-1}$ for $x, y \in X$.

Definition 2.3 [2, §2.2; 16, §2]. Let X be a rack and let A be an abelian group. A map $q : X \times X \to A$ is a 2-cocycle (or just a cocycle) if

$$q(x, y \triangleright z)q(y, z) = q(x \triangleright y, x \triangleright z)q(x, z) \tag{2-1}$$

for all $x, y, z \in X$. We call $Z^2_R(X, A)$ the set of rack cocycles. Two cocycles $q, q' \in Z^2_R(X, A)$ are said to be cohomologous if there exists $\gamma : X \to A$ such that

$$q(x, y) = \gamma(x \triangleright y)^{-1} q'(x, y)\gamma(y) \quad \text{for all } x, y \in X. \tag{2-2}$$

Our main motivation for studying racks and cocycles comes from Nichols algebras, whose construction we briefly recall. The base field here is $\mathbb{C}$.

Given a rack X and a cocycle q, the operator

$$c_q : \mathbb{C}X \otimes \mathbb{C}X \to \mathbb{C}X \otimes \mathbb{C}X, \quad x \otimes y \mapsto q(x, y) x \triangleright y \otimes x,$$

satisfies the braid equation on $(\mathbb{C}X)^{\otimes 3}$:

$$(c_q \otimes \mathrm{id})(\mathrm{id} \otimes c_q)(c_q \otimes \mathrm{id}) = (\mathrm{id} \otimes c_q)(c_q \otimes \mathrm{id})(\mathrm{id} \otimes c_q). \tag{2-3}$$

We call any pair (V, c), where V is a vector space and $c \in \mathrm{GL}(V^{\otimes 2})$ satisfies (2-3), a braided vector space, and c a braiding for V.

We recall that, for $n \geqslant 2$, the n-th braid group B_n is generated by σ_i for $i = 1, \ldots, n-1$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i \leqslant n-2$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i, j = 1, \ldots, n-1$ with $|i - j| > 1$. There is a unique section, called the Matsumoto section, $M : \mathbb{S}_n \to B_n$ satisfying $M(\sigma\tau) = M(\sigma)M(\tau)$ whenever $\ell(\sigma\tau) = \ell(\sigma)\ell(\tau)$.

Given a braided vector space (V, c), for any $n \geqslant 2$, there is a representation

$$\rho_n : B_n \to \mathrm{GL}(V^{\otimes n}), \quad \sigma_i \mapsto \mathrm{id}^{\otimes(i-1)} \otimes c \otimes \mathrm{id}^{\otimes(n-i-1)},$$

that combined with the Matsumoto section, gives rise to the n-th quantum symmetrizer $\Omega_n := \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)) \in \mathrm{End}(V^{\otimes n})$.

The Nichols algebra associated with (V, c) is the quotient of the tensor algebra $\mathcal{B}(V, c) := T(V)/\bigoplus_{n \geqslant 0} \mathrm{Ker}\, \Omega_n$. If (V, c) comes from a rack-cocycle pair (X, c) we will also denote the Nichols algebra by $\mathcal{B}(X, q)$. It is usually very difficult to detect for which pairs (X, q) the corresponding Nichols algebra is finite dimensional, or has finite GK-dimension, or whether it is finitely presented. Big progresses have been obtained in order to detect properties of $\mathcal{B}(X, q)$ from the knowledge of X only. However, there are cases on which dependence on the cocycle is apparent [31, Table 1] and some cocycles are easier to handle than others.

Remark 2.4. (1) If X is a rack and q, q' are cohomologous rack cocycles, then $\mathcal{B}(X, q) \simeq \mathcal{B}(X, q')$; see [2, Theorem 4.14].

(2) Let (V, c) be a braided vector space. If V' is a subspace of V such that $c(V' \otimes V') \subset V' \otimes V'$, then the inclusion $V' \subset V$ induces an injective algebra homomorphism $\mathcal{B}(V', c|_{V' \otimes V'}) \to \mathcal{B}(V, c)$ [4, Remark 1.1]. Hence, for any pair (X, q) and any inclusion $X' \to X$ such that $X' \triangleright X' \subset X'$ (we call such an X' a subrack of X), we have an algebra inclusion $\mathcal{B}(X', q|_{X' \times X'}) \subseteq \mathcal{B}(X, q)$.

(3) Let G be a group. A Yetter–Drinfeld module for G is the datum of a G-graded representation $V = \bigoplus_{g \in G} V_g$ of G such that $hV_g \subset V_{hgh^{-1}}$ for any $h, g \in G$. Setting $c_V(v \otimes w) := gw \otimes v$ for any $v \in V_g$ and $w \in V$ gives a braiding on V. Yetter–Drinfeld modules decompose as direct sums of irreducible ones. The latter are, as representations, of the form $V = \operatorname{Ind}_H^G U$, where H is the centralizer of some $g \in G$ and $\eta \colon H \to \mathrm{GL}(U)$ is an irreducible representation of H. The G-grading on V is determined by setting $V_g := U$. If η is 1-dimensional, then there is a rack cocycle q on the conjugacy class X of g, with values in $\mathbb{C}^*$ such that $(V, c_V) = (\mathbb{C}X, q)$. For general η, one needs the more general notion of cocycle with values in $\mathrm{GL}(U)$ [7, Section 2.4]. If V is a Yetter–Drinfeld module for G, we will write $\mathcal{B}(V)$ instead of $\mathcal{B}(V, c_V)$.

(4) If $\tau \in \operatorname{Aut}(G)$ and $V = \bigoplus_{g \in G} V_g$ is a Yetter–Drinfeld module for a group G, then twisting V by τ gives another Yetter–Drinfeld module structure V^τ by setting $g.v := \tau(g)v$ and $(V^\tau)_g := V_{\tau(g)}$ for any $g \in G$ and $v \in V$. The braidings c_V and c_{V^τ} coincide, so $\mathcal{B}(V) \simeq \mathcal{B}(V^\tau)$.

(5) Let X be a rack admitting a subrack decomposition $X = X_1 \coprod X_2$ such that $X_i \triangleright X_j = X_j$ for $i, j \in \{1, 2\}$, and let $q \in Z^2_R(X, \mathbb{C})$. Assume, in addition, that $c_q^2 = \mathrm{id}$ on $\mathbb{C}X_i \otimes \mathbb{C}X_j$, for $i \neq j$, and that either all Nichols algebras involved are finite dimensional or that $(X_i, q|_{X_i \times X_i})$, for $i = 1, 2$, are as in point (3). Then, there is an algebra isomorphism $\mathcal{B}(X_1, q|_{X_1 \times X_1}) \widetilde{\otimes} \mathcal{B}(X_2, q|_{X_2 \times X_2}) \simeq \mathcal{B}(X, q)$, where $\widetilde{\otimes}$ stands for the tensor product of algebras where the multiplication is twisted via c_q [24, Theorem 2.2; 27, Proposition 1.10.12].

2.3. ***Group 2-cocycles and sections.*** We recall that if G and A are groups, with A abelian, the group $Z^2(G, A)$ of 2-cocycles of G with values in A consists of those maps $\phi : G \times G \to A$ satisfying

$$\phi(xy, z)\phi(x, y) = \phi(x, yz)\phi(y, z) \quad \text{for all } x, y, z \in G.$$

If E is a central extension of G, with conjugation action $\blacktriangleright$, projection $\pi_G : E \to G$ and $\operatorname{Ker}(\pi_G) = A$, then for any section $\rho : G \to E$ of π_G, we will denote by ϕ_ρ the 2-cocycle defined by $\phi_\rho(x, y) := \rho(xy)\rho(y)^{-1}\rho(x)^{-1}$ for $x, y \in G$. A direct calculation shows that

$$\phi_\rho(x, y)(\rho(x) \blacktriangleright \rho(y)) = \phi_\rho(x \triangleright y, x)\rho(x \triangleright y) \quad \text{for all } x, y \in G. \tag{2-4}$$

Definition 2.5 [7, §3.4]. Let G be a group, let $X \subset G$ be stable by conjugation and let $q, q' \in Z^2_R(X, A)$. We say that q and q' are twist equivalent if there exists $\phi \in Z^2(G, A)$ such that

$$q(x, y) = \phi(x, y)\phi(x \triangleright y, x)^{-1} q'(x, y) \quad \text{for all } x, y \in X. \tag{2-5}$$

Remark 2.6. If X is a rack and q, q' are twist-equivalent rack cocycles, then $\mathcal{B}(X, q)$ and $\mathcal{B}(X, q')$ are cocycle twist-equivalent, in particular, they have the same Hilbert series [7, §3.4; 34, §2.7, 3.4].

Lemma 2.7. *Let G be a group and let X be a subset of G stable by conjugation. Let E be a central extension of G with kernel A and conjugation $\blacktriangleright$ and let $q_1, q_2 \in Z^2_R(X, A)$. If there is a section $\rho : G \to E$ satisfying*

$$\rho(x) \blacktriangleright \rho(y) = q_1(x, y)^{-1} q_2(x, y)\rho(x \triangleright y) \quad \textit{for all } x, y \in X, \tag{2-6}$$

then q_1 and q_2 are twist equivalent. If, in addition, $E \simeq A \times G$ is a trivial extension, then q_1 and q_2 are cohomologous.

Proof. The first statement follows from (2-4) and (2-5). If $E \simeq A \times G$ is trivial and π_A is the projection on A, then applying π_A to (2-6) gives (2-2) with $\gamma := \pi_A \circ \rho$. □

2.4. ***Twist equivalence for the racks of reflections.*** We now introduce two specific cocycles q^+ and $q^- \in Z^2_R(T, \mathbb{C}^*)$, as restriction to $T \times T$ of the following functions on $W \times T$. Let $w \in W$ and $y \in T$. Recall that α_y is the positive root associated with y. We set

$$q^+(w, y) = \begin{cases} 1 & \text{if } w(\alpha_y) \in \Phi^+, \\ -1 & \text{if } w(\alpha_y) \in \Phi^-, \end{cases} \quad \text{and} \quad q^-(w, y) = \det(w). \tag{2-7}$$

The function q^+ was introduced in [12] for finite W, generalizing the 2-cocycle for reflections in $\mathbb{S}_n$ [21; 36]. A direct calculation shows that q^+ and q^- satisfy

$$q^\pm(w_1 w_2, x) = q^\pm(w_1, w_2 \triangleright x) q^\pm(w_2, x) \quad \text{for all } w_1, w_2 \in W \text{ and } x \in T, \tag{2-8}$$

which implies (2-1); see [36, Section 5].

The main result in [39, Theorem 3.8] states that if $W = \mathbb{S}_n$, then q^+ and q^- are cohomologous for $n = 3$ and twist equivalent for $n \geqslant 4$. In the next section we will prove the following generalization to (possibly infinite) Coxeter groups of this result.

Theorem 2.8. *Assume that $A(W)$ has all its entries in $\mathbb{N}$. Then, the cocycles q^+ and q^- on T are twist equivalent.*

3. Proof of Theorem 2.8

3.1. ***Adapting Vendramin's construction.*** In [39] the group cocycle inducing the twist equivalence is obtained by constructing a suitable section from $\mathbb{S}_n$ to the Schur covering group of $\mathbb{S}_n$. This approach cannot be slavishly reproduced to the case of an arbitrary Coxeter group W because the Schur covering group of W may be trivial or too large for our purposes [32]. We replace the Schur covering group with a suitable, possibly trivial, central extension of W.

Let $\widetilde{W}$ be the group generated by $t_1, \dots, t_l, z$ with relations

$$z^2 = (t_i z)^2 = 1 \quad \text{and} \quad (t_i t_j)^{m_{ij}} = z^{m_{ij}+1} \qquad \text{for all } 1 \leq i, j \leq l. \tag{3-1}$$

By construction, z is central and the assignment $t_i \mapsto s_i$ for $i = 1, \dots, l$ and $z \mapsto 1$ defines a surjective homomorphism $\pi_W : \widetilde{W} \to W$. Making use of [32, page 264] one sees that $\ker(\pi_W) = \langle z \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

We define the cocycles $q_z^+, q_z^- \in Z_R^2(T, \langle z \rangle)$ by replacing the -1 by z, so that for the nontrivial group isomorphism $\psi : \langle z \rangle \to \{\pm 1\}$ we have $q^{\pm} = \psi \circ q_z^{\pm}$. It then suffices to show the twist equivalence of q_z^+ and q_z^- by means of a suitable section $W \to \widetilde{W}$.

Lemma 3.1. *Let $\rho : W \to \widetilde{W}$ be a section of π_W. Assume that ρ satisfies*

$$\rho(s) \blacktriangleright \rho(y) = \begin{cases} \rho(s \triangleright y) z & \text{if } s \neq y, \\ \rho(s \triangleright y) & \text{if } s = y \end{cases} \qquad \text{for all } s \in S \text{ and } y \in T. \tag{3-2}$$

Then

$$\rho(w) \blacktriangleright \rho(y) = q_z^+(w, y) q_z^-(w, y)^{-1} \rho(w \triangleright y) \quad \text{for all } w \in W \text{ and } y \in T. \tag{3-3}$$

Hence, q^+ and q^- are twist equivalent.

Proof. We prove (3-3) by induction on the length of $w \in W$, the case $\ell(w) = 1$ being (3-2) in virtue of [14, Lemma 4.4.3]. Assume $\ell(w) > 1$. Then $w = sw'$ for some $s \in S$ and $w' \in W$ such that $\ell(w') = \ell(w) - 1$. Applying in sequence the definition of ϕ_ρ from (2-4); centrality of z; the inductive hypothesis to w' and s, and (2-8) we obtain

$$\begin{aligned} \rho(w) \blacktriangleright \rho(y) &= (\phi_\rho(s, w') \rho(s) \rho(w')) \blacktriangleright \rho(y) \\ &= \rho(s) \blacktriangleright (\rho(w') \blacktriangleright \rho(y)) \\ &= \rho(s) \blacktriangleright (q_z^+(w', y) q_z^-(w', y)^{-1} \rho(w' \triangleright y)) \\ &= q_z^+(w', y) q_z^-(w', y)^{-1} \rho(s) \blacktriangleright \rho(w' \triangleright y) \\ &= q_z^+(w', y) q_z^-(w', y)^{-1} q_z^+(s, w' \triangleright y) q_z^-(s, w' \triangleright y)^{-1} \rho(s \triangleright (w' \triangleright y)) \\ &= q_z^+(sw', y) q_z^-(sw', y)^{-1} \rho(sw' \triangleright y). \end{aligned}$$

The last statement follows from Lemma 2.7. □

3.2. ***Towards the definition of a section.*** We build a section verifying (3-2) using a modified version of the conjugacy graph of (W, S) in [23, Section 3.2].

Definition 3.2. The *reflection conjugacy graph* $\widetilde{\Gamma}(W)$ of (W, S) is the labeled, directed graph having the elements of T as vertices, and labeled edges $x \xrightarrow{s} y$ whenever $x, y \in T$ satisfy $y = s \triangleright x$ with $s \in S$ and $\ell(x) = \ell(y) + 2$.

To simplify notation we write $x \xrightarrow{i} y$ instead of $x \xrightarrow{s_i} y$. More generally, for a sequence $s_{j_r}, \dots, s_{j_1}$ of elements in S, with $a = s_{j_r} \dots s_{j_1} \in \Sigma$, we write $x \xrightarrow{a} y$ if there is a path in $\widetilde{\Gamma}(W)$ from x to y, with labels $j_r, j_{r-1}, \dots, j_1$, or, equivalently, if $x = \pi(a) \triangleright y$ with $\ell(x) = \ell(y) + 2r$. Inductive application of Lemma 2.1(iv) guarantees that for any $x \in T$ there is always a path in $\widetilde{\Gamma}(W)$ starting in x and ending at some $s \in S$.

Remark 3.3. By [38, Proposition 2.4], the set of reduced expressions of $x \in T$ consists of all words of the form $a_L s a_R$ with $s \in S$ and $a_L, a_R \in \Sigma$, such that $x \xrightarrow{a_L} s$ and $x \xrightarrow{a_R^{\mathrm{op}}} s$. Hence, $\mathcal{P}(x)$ consists of all words of the form $a_L s a_L^{\mathrm{op}}$ where $s \in S$ and $x \xrightarrow{a_L} s$. In other terms, $\mathcal{P}(x)$ is in bijection with the paths in $\widetilde{\Gamma}(W)$ starting from x and ending in some $s \in S$. By construction all such paths have the same length.

We aim at defining a section of π_W inductively. To do this we need to compare different elements in $\mathcal{P}(x)$ for $x \in T$. First of all, we compare different elements in $\mathcal{R}(x)$. By [14, Theorem 3.3.1(ii)] it suffices to look at reduced expressions that differ by application of one braid relation, a situation that was considered by Stembridge.

Proposition 3.4 [38, Lemma 2.5]. *Let $x \in T$ and let $a = a_L s_i a_R$ and $b = b_L s_j b_R \in \mathcal{R}(x)$, with $s_i, s_j \in S$, and $a_L, a_R, b_L, b_R \in \Sigma$. Assume that a and b differ by one braid relation. Then, one of the following three alternatives holds*:

Case $a \rightsquigarrow_L b$: *$a_R = b_R$, $i = j$, $\pi(a_L) = \pi(b_L)$ and a_L and b_L differ by one braid relation.*

Case $a \rightsquigarrow_R b$: *$a_L = b_L$, $i = j$, $\pi(a_R) = \pi(b_R)$ and a_R and b_R differ by one braid relation.*

Case $a \rightsquigarrow_C b$: *m_{ij} is odd, $i \neq j$, and there exist $a'_L, a'_R \in \Sigma$ such that*

$$a = a'_L(\underbrace{\dots s_i s_j s_i s_j s_i \dots}_{m_{ij} \text{ terms}})a'_R \quad \text{and} \quad b = a'_L(\underbrace{\dots s_j s_i s_j s_i s_j \dots}_{m_{ij} \text{ terms}})a'_R.$$

We now focus on $\mathcal{P}(x)$. For any $x \in T$ and any $a = a_L s_i a_R \in \mathcal{R}(x)$ we consider the mirrored expression $\mu(a) = a_L s_i a_L^{\mathrm{op}} \in \Sigma$. By [18, (2.7)], the assignment μ gives a well-defined function $\mu \colon \mathcal{R}(x) \to \mathcal{P}(x)$, which restricts to the identity on $\mathcal{P}(x)$.

Proposition 3.5. *Let $x \in T$, and let $a = a_L s_i a_L^{\mathrm{op}}$ and $b = b_L s_j b_L^{\mathrm{op}} \in \mathcal{P}(x)$. Then there exists a sequence $a_d := a_{d,L} s_{i_d} a_{d,L}^{\mathrm{op}} \in \mathcal{P}(x)$, for $d = 0, \dots, r$, with $a_0 = a$ and $a_r = b$ and each $s_{i_d} \in S$ such that for every $d = 0, \dots, r-1$ either $a_{d+1,L}$ differs from $a_{d,L}$ by one braid move, or else $a_d \rightsquigarrow_C a_{d+1}$.*

Proof. By Proposition 3.4 there is a sequence $b_c := b_{c,L} s_{i_c} b_{c,R} \in \mathcal{R}(x)$, for $c = 0, \dots, q$, with $b_0 = a$ and $b_q = b$ and each $s_{i_c} \in S$ such that $b_c \rightsquigarrow_{D_c} b_{c+1}$ for some $D_c \in \{L, R, C\}$ and for all $c \in \{0, \dots, q-1\}$. Applying the mirroring function μ to all terms we obtain a sequence $\mu(b_0) = \mu(a) = a$, $\mu(b_1), \dots, \mu(b_q) = b \in \mathcal{P}(x)$. Now, if $D_c = R$, then $b_c = b_{c,L} s_{i_c} b_{c,R} \rightsquigarrow_R b_{c,L} s_{i_c} b_{c+1,R} = b_{c+1}$ and so $\mu(b_c) = \mu(b_{c+1})$. If $D_c = L$, then $b_c = b_{c,L} s_{i_c} b_{c,R} \rightsquigarrow_R b_{c+1,L} s_{i_c} b_{c,R} = b_{c+1}$ with $b_{c,L}$ and $b_{c+1,L}$ differing by one braid move. Hence, $\mu(b_c) = b_{c,L} s_{i_c} b_{c,L}^{\mathrm{op}}$ and $\mu(b_{c+1}) = b_{c+1,L} s_{i_c} b_{c+1,L}^{\mathrm{op}}$ with $b_{c,L}$ and $b_{c+1,L}$ differing by one braid move. Finally, if $D_c = C$, then

$$b_c = b'_{c,L}(\dots s_{i_c} s_{i_{c+1}} s_{i_c} s_{i_{c+1}} s_{i_c} \dots) b'_{c,R} \rightsquigarrow_C b'_{c,L}(\dots s_{i_{c+1}} s_{i_c} s_{i_{c+1}} s_{i_c} s_{i_{c+1}} \dots) b'_{c,R} = b_{c+1}$$

for some $b'_{c,L}, b'_{c,R} \in \Sigma$. Thus, $\mu(b_c) = b'_{c,L}(\dots s_{i_{c+1}} s_{i_c} s_{i_{c+1}} \dots) b_{c,L}^{\prime \mathrm{op}}$ and $\mu(b_{c+1}) = b'_{c,L}(\dots s_{i_c} s_{i_{c+1}} s_{i_c} \dots) b_{c,L}^{\prime \mathrm{op}}$, so $\mu(b_c) \rightsquigarrow_C \mu(b_{c+1})$. Removing redundancy, we obtain the desired sequence in $\mathcal{P}(x)$. □

We are now able to inductively define a section of π_W with good properties.

Lemma 3.6. *Let $\rho_0 : W \to \widetilde{W}$ be any set-theoretic section of π_W. The assignment*

$$\rho \colon W \to \widetilde{W}, \qquad x \mapsto \begin{cases} \rho_0(x) & \text{if } x \notin T, \\ t_i & \text{if } x = s_i \text{ for some } 1 \le i \le l, \\ t_i \blacktriangleright \rho(y) z & \text{if } x \xrightarrow{i} y \text{ for some } 1 \le i \le l, \end{cases} \tag{3-4}$$

determines a well-defined section of π_W.

Proof. For $x \in T$, let

$$x = x_0 \xrightarrow{i_1} x_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{r-1}} x_{r-1} \xrightarrow{i_r} s_i$$

be a path in $\widetilde{\Gamma}(W)$ ending at some $s_i \in S$. Applying (3-4), using that $z \in Z(\widetilde{W})$, gives

$$\rho(x) = t_{i_1} \blacktriangleright (t_{i_2} \blacktriangleright (\cdots \blacktriangleright t_{i_r} \blacktriangleright t_i) \cdots) z^r = t_{i_1} t_{i_2} \dots t_{i_r} t_i t_{i_r} \dots t_{i_2} t_{i_1} z^r, \tag{3-5}$$

and ρ is well defined if the term on the right-hand side is independent from the chosen path, or, equivalently, independent from the corresponding reduced expression $a = s_{i_1} \dots s_{i_r} s_i s_{i_r} \dots s_{i_1}$ in $\mathcal{P}(x)$. By Proposition 3.5, it is enough to verify that $\rho(x)$ does not change if a is modified by applying either one braid move to

$a_L = s_{i_1} \dots s_{i_r}$ (and consequently to a_L^{op}) or by applying the move $\rightsquigarrow_C$ to a. In the first scenario, a subword of the form

$$\underbrace{s_a s_b \dots}_{m_{ab}} \text{ in } a_L \quad \text{is replaced by} \quad \underbrace{s_b s_a \dots}_{m_{ab}}.$$

In this case, a subword of the form

$$\underbrace{t_a t_b \dots}_{m_{ab}} \text{ occurring before } t_i \text{ in (3-5)} \quad \text{is replaced by} \quad \underbrace{t_b t_a \dots}_{m_{ab}},$$

and symmetrically a subword of the form

$$\underbrace{t_b t_a \dots}_{m_{ab}} \text{ occurring after } t_i \text{ in (3-5)} \quad \text{is replaced by} \quad \underbrace{t_a t_b \dots}_{m_{ab}}.$$

Since

$$\underbrace{t_a t_b \dots}_{m_{ab}} = z^{m_{ab}+1} \underbrace{t_b t_a \dots}_{m_{ab}}$$

and z is central, the value of $\rho(x)$ does not change in this case. In the second scenario, the term

$$a = a_L s_i a_R = a'_L(\underbrace{\dots s_j s_i s_j \dots}_{m_{ij}})a'_R \quad \text{is replaced by} \quad a'_L(\underbrace{\dots s_i s_j s_i \dots}_{m_{ij}})a'_R$$

and m_{ij} is odd. In this case, the central term

$$\underbrace{\dots t_j t_i t_j \dots}_{m_{ij}} \text{ in (3-5)} \quad \text{is replaced by} \quad \underbrace{\dots t_i t_j t_i \dots}_{m_{ij}},$$

while the rest is unmodified. As

$$\underbrace{\dots t_j t_i t_j \dots}_{m_{ij}} = z^0 \underbrace{\dots t_i t_j t_i \dots}_{m_{ij}}$$

in $\widetilde{W}$, the value of $\rho(x)$ is unaltered, so ρ is well defined. Applying π_W to (3-5) shows that it is a section. □

3.3. *Good properties of the section.* The next step is to prove that the section ρ as in (3-4) satisfies Vendramin's condition. We fix some further notation.

Definition 3.7. The sequence $(U_n(X))_{n \geqslant 0}$ of Chebychev polynomials of the second kind is the sequence of polynomials in $\mathbb{Z}[X]$ defined recursively by

(3-6) $$U_0(X) = 1, \quad U_1(X) = 2X, \quad U_{n+1}(X) = 2XU_n(X) - U_{n-1}(X).$$

We recall the well-known formula

$$\sin(\theta)U_n(\cos\theta) = \sin((n+1)\theta). \tag{3-7}$$

The following lemma is key for proving inductively the good properties of ρ.

Lemma 3.8. *Let $\beta \in \Phi^+ \setminus \Delta$ and assume there are $\alpha_i, \alpha_j \in \Delta$ satisfying $\delta := (\alpha_j, \beta) > 0$ and $(\alpha_i, \beta) = 0$. For $p \geqslant 0$, let*

$$\alpha_{(p)} = \begin{cases} \alpha_i & \text{if } p \text{ is even,} \\ \alpha_j & \text{if } p \text{ is odd,} \end{cases} \qquad s_{(p)} := s_{\alpha_{(p)}}, \qquad \beta_0 := \beta, \qquad \beta_{p+1} := s_{(p+1)}\beta_{(p)}.$$

Then,

$$(\beta_p, \alpha_{(p+1)}) > 0 \quad \text{for } p = 0, \dots, m_{ij} - 2, \qquad (\beta_{m_{ij}-1}, \alpha_{(m_{ij})}) = 0 \tag{3-8}$$

and either

$$\ell(s_{\beta_{m_{ij}-1}}) = \ell(s_\beta) - 2m_{ij} + 2, \quad s_{(m_{ij})}s_{\beta_{m_{ij}-1}} = s_{\beta_{m_{ij}-1}}s_{(m_{ij})}, \quad \alpha_{(m_{ij})} \neq \beta_{m_{ij}-1} \tag{3-9}$$

or else

$$m_{ij} \text{ is even}, \quad \beta_{m_{ij}/2-1} = \alpha_{(m_{ij}/2)} \in \Delta, \quad \ell(s_\beta) = 2m_{ij} - 1. \tag{3-10}$$

Proof. For $p \geqslant 0$ we set

$$u_p := \frac{(\beta_p, \alpha_{(p+1)})}{\delta} = \frac{(\beta_p, \alpha_{(p-1)})}{\delta}, \qquad \gamma := -(\alpha_i, \alpha_j) = \cos\frac{\pi}{m_{ij}}.$$

Then $u_0 = 1$. For $p \geqslant 1$ we compute

$$\begin{aligned}
u_1 &= \frac{(s_j\beta, \alpha_i)}{\delta} = \frac{(\beta - 2(\beta, \alpha_j)\alpha_j, \alpha_i)}{\delta} = 0 + 2\gamma, \\
u_{p+1} &= \frac{(s_{(p+1)}\beta_p, \alpha_{(p)})}{\delta} = \frac{(\beta_p - 2(\beta_p, \alpha_{(p+1)})\alpha_{(p+1)}, \alpha_{(p)})}{\delta} \\
&= \frac{(\beta_p, \alpha_{(p)})}{\delta} - \frac{2(\beta_p, \alpha_{(p-1)})(\alpha_{(p+1)}, \alpha_{(p)})}{\delta} = \frac{(s_{(p)}\beta_{p-1}, \alpha_{(p)})}{\delta} + 2\gamma u_p \\
&= \frac{(\beta_{p-1}, s_{(p)}\alpha_{(p)})}{\delta} + 2\gamma u_p = -u_{p-1} + 2\gamma u_p.
\end{aligned}$$

Therefore $(\beta_p, \alpha_{(p+1)})/\delta = U_p(\gamma)$, the p-th Chebychev's polynomial evaluated at $\gamma = \cos(\pi/m_{ij})$. Then, (3-7) gives (3-8) and Lemma 2.1(i) and (ii) applied to $\beta_{(p)}$ and $\alpha_{(p+1)}$ imply that either

$$\ell(s_{\beta_{p+1}}) < \ell(s_{\beta_p}) \quad \text{for } p = 0, \dots, m_{ij} - 2, \tag{3-11}$$

giving (3-9), or else $\ell(s_{\beta_{q+1}}) = \ell(s_{\beta_q})$ for some $q \in \{0, \dots, m_{ij} - 2\}$. In this case, Lemma 2.1(iii) gives $\beta_q = \alpha_{(q+1)}$, whence

$$s_\beta = s_{(1)} s_{(2)} \dots s_{(q)} s_{(q+1)} s_{(q)} \dots s_{(1)}. \tag{3-12}$$

Applying the equality $s_i s_\beta s_i = s_\beta$ yields

$$s_i s_{(1)} s_{(2)} \dots s_{(q)} s_{(q+1)} s_{(q)} \dots s_{(1)} = s_{(1)} s_{(2)} \dots s_{(q)} s_{(q+1)} s_{(q)} \dots s_{(1)} s_i,$$

where both sides are products of $2q+2$ simple reflections. Bearing in mind that $s_{(1)} = s_j$, this gives $(s_i s_j)^{2q+2} = 1$ and so m_{ij} divides $2(q+1)$. By construction $m_{ij} > q+1$; hence $m_{ij} = 2(q+1)$ is even and $\beta_{m_{ij}/2-1} = \alpha_{(m_{ij}/2)}$. The claim on the length follows because $\ell(s_{\beta_{p+1}}) = \ell(s_{\beta_p}) - 2$ for $p \in \{0, \dots, q-1\}$. □

Lemma 3.9. *The section ρ defined by (3-4) satisfies (3-2).*

Proof. Let $y \in T$ and $s_i \in S$. Then we fall in one of the following situations: $y = s_i$, or $\ell(s_i \triangleright y) = \ell(y) \pm 2$, or $(\alpha_y, \alpha_i) = 0$, that we analyze separately.

- If $y = s_i$, then $\rho(s_i) \blacktriangleright \rho(y) = t_i \blacktriangleright t_i = t_i = \rho(y)$.

- If $\ell(s_i \triangleright y) = \ell(y)+2$, then by construction $\rho(s_i \triangleright y) = t_i \blacktriangleright \rho(y) z = \rho(s_i) \blacktriangleright \rho(y) z$. Multiplying both sides by z gives (3-2).

- If $\ell(s_i \triangleright y) = \ell(y) - 2$, then we invoke the previous case applied to $x = s_i \triangleright y$ and use that t_i and z are involutions and that z is central.

- If $(\alpha_y, \alpha_i) = 0$ then $y \neq s_i$ and $s_i \triangleright y = y$. We proceed by induction on the length of y. If $\ell(y) = 1$, then $y = s_j$ for some $j \in \{1, \dots, l\}$, with $m_{ij} = 2$ and $\rho(s_i \triangleright y) = \rho(y) = t_j = z t_i \blacktriangleright t_j = \rho(s_i) \blacktriangleright \rho(s_j) z$, confirming (3-2) in this case. Assume now that $\ell(y) > 1$. By Lemma 2.1, there always is a $j \in \{1, \dots, l\}$ satisfying $\ell(s_j y s_j) = \ell(y) - 2$. Setting $\beta = \alpha_y$, Lemma 3.8, from which we retain notation, gives either (3-9) or else (3-10).

If (3-10) holds, then $y = s_{(1)} \triangleright (s_{(2)} \triangleright \cdots \triangleright (s_{(m_{ij}/2-1)} \triangleright s_{(m_{ij}/2)}))$ and $\ell(y) = 2m_{ij} - 1$, so

$$\rho(s_i \triangleright y) = \rho(y) = (t_j t_i)^{m_{ij}/2-1} t_j z^{m_{ij}/2-1}.$$

As $(t_i t_j)^{m_{ij}} = z$, we have

$$(t_i t_j)^{m_{ij}/2} = z (t_j t_i)^{m_{ij}/2}.$$

Then,

$$\begin{aligned}\rho(s_i) \blacktriangleright \rho(y) z &= t_i (t_j t_i)^{m_{ij}/2-1} t_j t_i z^{m_{ij}/2} = t_i (t_j t_i)^{m_{ij}/2} z^{m_{ij}/2} \\ &= t_i (t_i t_j)^{m_{ij}/2} z^{m_{ij}/2+1} = (t_j t_i)^{m_{ij}/2-1} t_j z^{m_{ij}/2-1} = \rho(s_i \triangleright y).\end{aligned}$$

If instead, (3-9) holds, then $y = s_{(1)} \triangleright (s_{(2)} \triangleright \cdots \triangleright (s_{(m_{ij}-1)} \triangleright s_{\beta_{m_{ij}-1}}) \cdots)$ with $\ell(s_{\beta_{m_{ij}-1}}) = \ell(y) - 2m_{ij} + 2 < \ell(y)$ and $s_{(m_{ij})} \triangleright s_{\beta_{m_{ij}-1}} = s_{\beta_{m_{ij}-1}}$, so by induction $t_{(m_{ij})} \blacktriangleright \rho(s_{\beta_{m_{ij}-1}}) = \rho(s_{\beta_{m_{ij}-1}})z$. By definition of ρ we have

$$\rho(s_i \triangleright y) = \rho(y) = \underbrace{t_j t_i \ldots}_{m_{ij}-1 \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\ldots t_i t_j}_{m_{ij}-1 \text{ terms}} z^{m_{ij}-1}.$$

Therefore

$$\begin{aligned}\rho(s_i) \blacktriangleright \rho(y)z &= \underbrace{t_i t_j t_i \ldots}_{m_{ij} \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\ldots t_i t_j t_i}_{m_{ij} \text{ terms}} z^{m_{ij}} \\ &= \underbrace{t_j t_i t_j \ldots}_{m_{ij} \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\ldots t_j t_i t_j}_{m_{ij} \text{ terms}} z^{m_{ij}+2m_{ij}+2},\end{aligned}$$

where the last factor in $t_i t_j \ldots$ is now $\rho(s_{(m_{ij})-2}) = \rho(s_{(m_{ij})})$. By induction,

$$\rho(s_i) \blacktriangleright \rho(y)z = \underbrace{t_j t_i \ldots}_{m_{ij}-1 \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\ldots t_i t_j}_{m_{ij}-1 \text{ terms}} z^{m_{ij}+1} = \rho(s_i \triangleright y)$$

concluding the proof. □

We end this section characterizing when q^+ and q^- are cohomologous. For the case of $\mathbb{S}_n$, see [39, Remark 2.2].

Theorem 3.10. *Assume that $A(W)$ has all its entries in $\mathbb{N}$. Then, the following are equivalent*:

(1) *All the coefficients in $A(W)$ are odd.*

(2) *The group $\widetilde{W}$ is the trivial extension $\widetilde{W} = W \times \langle z \rangle$.*

(3) *The cocycles q^+ and q^- are cohomologous.*

Proof. The implication (1) ⇒ (2) is immediate from the definition of $\widetilde{W}$, and (2) ⇒ (3) follows from Lemmata 2.7, 3.1, and 3.9. We prove that (3) ⇒ (1). Suppose for a contradiction that $m := m_{ij}$ is even for some $i, j \in \{1, \ldots, l\}$, and that there exists $\gamma : W \to \mathbb{C}^*$ such that $q^-(x, y) = \gamma(x \triangleright y)^{-1} q^+(x, y)\gamma(y)$ for all $x, y \in T$. Set $s = s_i$, $s' = s_j$ and let $\sigma := (s's)^{m/2-1}s'$, so that $\sigma \in T$ and $s\sigma s = \sigma$. Then $\gamma(s \triangleright \sigma) = \gamma(\sigma)$. Thus $-1 = q^-(s, \sigma) = q^+(s, \sigma)$, contradicting (2-7). □

Remark 3.11. If all coefficients in $A(W)$ are odd, then the underlying graph of $\Gamma(W)$ is complete. Hence, the corresponding group W is finite if and only if $l \leqslant 2$. If $l = 1$, then $T = \{s_\alpha\}$ and $q^+ = q^-$ so the statement is trivial. If $l = 2$, then W is of type $I_2(2m + 1)$, for $m \geqslant 1$, i.e., it is the dihedral group of a regular $(2m+1)$-gon [15, Chapitre VI, §4, Théorème 1]. In this case, the cohomology of q^+ and q^- is proved in [36, Example 5.4(a)]. Notice that, even though the extension $\widetilde{W}$ is trivial, by [32], the Schur multiplier of W is elementary abelian of order $2^{(l-1)(l-2)/2}$, whence nontrivial whenever W is infinite.

4. Applications to Nichols algebras

Here, W is finite and the base field is $\mathbb{C}$. The Nichols algebra associated with (T, q^+) is of particular interest because it contains the coinvariant algebra of W [12; 21]. If W is crystallographic, the latter is isomorphic to the cohomology of the flag variety of the algebraic group with associated Weyl group W. It is in general an open question whether $\mathcal{B}(T, q^+)$ is finite dimensional, quadratic, or finitely presented [12; 21; 22; 34]. The main result in [11] states that the quadratic cover of $\mathcal{B}(T, q^+)$ is infinite dimensional for $W = \mathbb{S}_n$ and $n \geqslant 6$.

Combining Theorem 2.8 and Remark 2.6 readily gives the following.

Corollary 4.1. *If W is an arbitrary finite Coxeter group, then $\mathcal{B}(T, q^+)$ and $\mathcal{B}(T, q^-)$ have the same Hilbert series. This also holds for their quadratic approximations. In addition, $\mathcal{B}(T, q^+)$ is quadratic if and only if $\mathcal{B}(T, q^-)$ is quadratic.*

It also follows from Theorem 3.10 and Remark 2.4(1) that if W is of type $I_2(2m+1)$, for $m \geqslant 1$, that is, the dihedral group of a regular $(2m+1)$-gon, then $\mathcal{B}(T, q^+) \simeq \mathcal{B}(T, q^-)$, a result that was already present in [36, Example 5.4(b)]. By [36, Remark 5.2 part 2)] the braided vector spaces attached to $(T, q^\pm)$ are constructed as in Remark 2.4(3), so results in this setting apply. We summarize below some facts on the Nichols algebras $\mathcal{B}(T, q^\pm)$. Some of these results are known, we focus on giving a uniform point of view.

Remark 4.2. (1) If (W, S) is not irreducible, let $(W_1, S_1), \dots, (W_r, S_r)$ be its irreducible factors. Setting $T_i := W \triangleright S_i = W_i \triangleright S_i$ for $i = 1, \dots, r$ we have a rack decomposition $T = \coprod_{i=1}^r T_i$ where $t \triangleright t' = t'$ and $q^+(t, t') = q^+(t', t) = 1$ if $t \in T_i$ and $t' \in T_j$ with $i \neq j$. Hence $c_{q^+}^2$ and $c_{q^-}^2$ act as the identity on $\mathbb{C}T_i \otimes \mathbb{C}T_j$ whenever $i \neq j$. Remark 2.4(5) then gives $\mathcal{B}(T, q^+) \simeq \bigotimes_{i=1}^r \mathcal{B}(T_i, q_i^+)$ and $\mathcal{B}(T, q^-) \simeq \bigotimes_{i=1}^r \mathcal{B}(T_i, q_i^-)$, where $q_i^\pm$ stands for the restriction of $q^\pm$ to $T_i \times T_i$. Therefore, it is enough to study the case of irreducible Coxeter groups.

(2) Coxeter graph inclusions imply inclusions of the corresponding racks of reflections, hence a braided vector space inclusion for the cocycle q^-, and therefore a Nichols algebra inclusion by Remark 2.4(2).

(3) Let (W, S) be irreducible. If $\Gamma(W)$ has no even labeled edges, then the reflections form a single conjugacy class. Otherwise, W is of type B_l for $l \geqslant 3$, F_4, or $I_2(2m)$ for $m \geqslant 2$ and T is the union of two classes, represented by any s and $s' \in S$ that are joined in $\Gamma(W)$ by an even labeled edge. Setting $T_1 := W \triangleright s$ and

type of W	$\mathcal{B}(T_1, q^{\pm})$	$\mathcal{B}(T_2, q^{\pm})$	$\mathcal{B}(T, q^{\pm})$
$B_l,\ l \geqslant 3$	$\mathcal{B}(T_{D_l}, q^{\pm})$	$\left(\bigwedge k\right)^{\otimes l},\ \bigwedge \mathbb{C}^l$	infinite dimensional
F_4	$\mathcal{B}(T_{D_4}, q^{\pm})$	$\mathcal{B}(T_{D_4}, q^{\pm})$	infinite dimensional
$I_2(4)$	$\left(\bigwedge \mathbb{C}\right)^{\otimes 2}$	$\left(\bigwedge \mathbb{C}\right)^{\otimes 2}$	dimension 64
$I_2(2m),\ m > 2$	$\mathcal{B}(T_{I_2(m)}, q^{\pm})$	$\mathcal{B}(T_{I_2(m)}, q^{\pm})$	infinite dimensional

Table 1. Nichols algebras of (T, q^+) for W such that $\Gamma(W)$ has an even labeled edge.

$T_2 := W \triangleright s'$ we have a rack decomposition $T = T_1 \coprod T_2$ with $T_i \triangleright T_j = T_j$ for $i, j \in \{1, 2\}$. Observe that $c^2_{q^{\pm}}(s \otimes s') = \pm(s \triangleright s') \triangleright s \otimes s \triangleright s' \neq s \otimes s'$, so $c^2_{q^{\pm}} \neq \mathrm{id}$ on $\mathbb{C}T_1 \otimes \mathbb{C}T_2$. If $\min(|T_1|, |T_2|) > 2$ or $\max(|T_1|, |T_2|) > 4$, then $\dim \mathcal{B}(T, q) = \infty$ for *any* cocycle q, by [9, Theorem 2.9] applied to $Y = T$. By construction, for $i = 1, 2$, we have twist equivalence of the restrictions to T_i of q^+ and q^-.

(4) If W is of type B_l with $l \geqslant 3$, up to renumbering, T_1 is isomorphic to the rack T_{D_l} of reflections for W of type D_l, so $|T_1| = l^2 - l$ and T_2 is abelian, that is, it has trivial action, and $|T_2| = l$. By [9, Theorem 2.9], $\dim \mathcal{B}(T, q) = \infty$ for any cocycle q. On the other hand, $\mathcal{B}(T_2, q^-) = \bigwedge \mathbb{C}^l$ and $\mathcal{B}(T_2, q^+) = \left(\bigwedge \mathbb{C}\right)^{\otimes l}$, and hence $\dim \mathcal{B}(T_2, q^{\pm}) = 2^l$.

(5) If W is of type F_4 or $I_2(2m)$ for $m \geqslant 2$, then the classes T_1 and T_2 are interchanged by the automorphism of W coming from the symmetry of $\Gamma(W)$, so $T_1 \simeq T_2$ as racks. Using the descriptions of roots in [15, Planches IV, VIII], one sees that in type F_4 the racks T_1 and T_2 are isomorphic to T_{D_4}, so $|T_1| = |T_2| > 4$ whence $\dim \mathcal{B}(T, q) = \infty$ for *any* cocycle q.

(6) If W is of type $I_2(2m)$ for $m \geqslant 2$, the racks T_1 and T_2 are isomorphic to the rack $T_{I_2(m)}$ of reflections of type $I_2(m)$. In addition, it was shown in [5, Lemma 2.1] using [8, Theorem 3.6] that if $m > 2$, then $\dim \mathcal{B}(T_{I_2(2m)}, q) = \infty$ for *any* cocycle q. Several considerations concerning the rack T for dihedral groups are present in [36, Sections 5, 6]. In particular, [36, Example 6.5] shows that $\dim \mathcal{B}(T_{I_2(4)}, q^{\pm}) = 64$. The Nichols algebras and/or their quadratic covers for the rack of reflections in $I_2(2p)$ for p an odd prime has been studied in [2; 3, Example 3.3.5; 35].

In Table 1 we summarize what is currently known about the Nichols algebras of the pair (T, q^+) for W such that $\Gamma(W)$ has an even labeled edge, up to twist equivalence.

We now focus on the irreducible, finite Coxeter groups with one conjugacy class of reflections.

Proposition 4.3. *Let W be of type $I_2(2m+1)$ for $m > 1$. Then* $\dim \mathcal{B}(T, q^{\pm}) = \infty$.

Proof. Let $2m+1 = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorization and let $W = \langle s, s' \rangle$. Then

$$T = \{s(s's)^j \mid j = 0, \ldots, 2m+1\}$$

and for any n dividing $2m+1$, the subset $T_n = \{s(s's)^j \in T \mid n \text{ divides } j\}$ is a subrack of T because $s(s's)^j \triangleright s(s's)^l = s(s's)^{2j-l}$; see [1, Remark 3.3]. In particular, T contains a subrack of size p_i for any prime divisor of $2m+1$. The pair $(T_{p_i}, q^+|_{T_{p_i}})$ is precisely the pair corresponding to the rack of reflections of the dihedral group of size $2p_i$ and T_{p_i} is an indecomposable affine rack [2, Section 1.3.8]. If $p_i > 7$ for some i, then [31, Theorem 1.6] implies that $\dim \mathcal{B}(T_{p_i}, q^+|_{T_{p_i}}) = \infty$ and, a fortiori, $\dim \mathcal{B}(T, q^{\pm}) = \infty$. Assume now that $p_i = 5$ or 7 for some i. In the notation of [31], the rack T_{p_i} is $\mathrm{Aff}(p_i, p_i - 1)$. Therefore, if $p_i = 5, 7$, the rack T_{p_i} does not occur in [31, Table 1] and so $\dim \mathcal{B}(T, q^{\pm}) = \dim \mathcal{B}(T_{p_i}, q^{\pm}) = \infty$. We are left with the case $2m+1 = 3^b$ for some $b \geqslant 2$. In this case T is an indecomposable affine rack, and $\dim \mathcal{B}(T_{3^b}, q^{\pm}) = \infty$ by [10, Theorem 1.3]. □

Corollary 4.4. *If W is of type H_3 or H_4, then* $\dim \mathcal{B}(T, q^{\pm}) = \infty$.

Proof. The rack T contains a subrack isomorphic to $T_{I_2(5)}$, so the statement follows from Theorem 2.8, Remark 4.2(2) and Proposition 4.3. □

In the remaining cases W is in one of the crystallographic, simply laced families of groups A_n, $n \geqslant 1$, D_n for $n \geqslant 4$, and E_6, E_7, E_8. They afford a crystallographic root system $\widetilde{\Phi}$ as in [15, Chapitre VI.1.1], and all roots in $\widetilde{\Phi}$ have the same length. Here $T = \{s_\alpha \mid \alpha \in \widetilde{\Phi}^+\}$ and by [15, Chapitre VI.1.3] the subgroup generated by any pair of noncommuting reflections is isomorphic to $\mathbb{S}_3$. It is well known that $\dim \mathcal{B}(T_{A_n}, q^{\pm}) < \infty$ for $n \leqslant 4$ [22; 36], whilst infinite dimensionality for $n \geqslant 5$ is still open. Observe that none of the splitting criteria in [9, Sections 2.1, 2.2] and [8, Section 3.2] apply to the rack T: in the terminology of [9], the rack T is kthulhu.

Remark 4.5. It was kindly pointed out to us by I. Heckenberger that $\mathcal{B}(T_{D_4}, q^{\pm})$ is infinite dimensional as a consequence of [10, Theorem 6.14] because the Coxeter group of type D_4 is solvable, noncyclic, and generated by T, which has size > 7. Hence, $\dim \mathcal{B}(T_{D_n}, q^{\pm}) = \infty$ for any $n \geqslant 4$.

In all remaining cases $\Gamma(W)$ contains a graph of type A_5 (that is, $\mathbb{S}_6 \leqslant W$). The case of $\mathbb{S}_6$ has been addressed by several authors: by the main result in [11] it would be infinite dimensional provided $\mathcal{B}(T, -1)$ is quadratic, but the latter property has not been established despite several attempts. Summarizing we have:

Corollary 4.6. *Assume that* $\dim \mathcal{B}(T_{A_5}, q^{\pm}) = \infty$. *If* $\dim \mathcal{B}(T, q^{\pm}) < \infty$ *then W is either the dihedral group of order* 8, *the cyclic group of order* 2, $\mathbb{S}_3$, $\mathbb{S}_4$ *or* $\mathbb{S}_5$.

Through the equivalence of categories in [33], the Nichols algebra $\mathcal{B}(T, q^-)$ corresponds to a factorizable perverse sheaf whose underlying perverse sheaf is the intermediate extension of the local system on the ind-variety of configuration spaces in $\mathbb{C}$, corresponding to the collection of braid group representations $V^{\otimes n}$, for $n \geqslant 0$ associated with T and the trivial cocycle. The latter are obtained as the push-forward of the constant sheaf on the Hurwitz space with Galois group W and local monodromy T. One may hope to retrieve further information on these algebras (e.g., if they are quadratic) by using this geometric interpretation.

4.1. ***Nichols algebras over dihedral groups.*** Here, we restrict to the case in which W is of rank 2, that is, W is of type $I_2(n)$ for $n > 2$ and we write $S = \{s, s'\}$. The analysis of the finite-dimensional Nichols algebras over $I_2(n)$ when $4 \mid n$ and $n \geqslant 12$ was obtained in [20], the cases $n = 4, 8$ were then completed in [13, Chapter 2]. Thus in this section n is not divisible by 4 and we separate the analysis according to its parity.

Corollary 4.7. *Assume $n = 2m + 1$ for $m > 1$. Then, any Nichols algebra of a Yetter–Drinfeld module of W is infinite dimensional. Therefore, if H is a finite-dimensional complex, pointed Hopf algebra with group of grouplikes isomorphic to W, then $H = \mathbb{C}W$, the group algebra of W.*

Proof. By [6, Theorem 4.8], the only possible finite-dimensional Nichols algebra coming from a Yetter–Drinfeld module for W could come from the irreducible Yetter–Drinfeld module as in Remark 2.4(3) with $g \in S$ and η the nontrivial irreducible representation of $H = \langle g \rangle$. However, this corresponds precisely to the pair $(T_{I_2(2m+1)}, q^+)$, see [36, Section 5], which has an infinite-dimensional Nichols algebra by Proposition 4.3. The second statement follows from [2, §0.3]. □

We will now look at the case $n = 2r$, where r is odd. Let $\zeta \in \mathbb{C}^*$ be a primitive n-th root of 1 and let $C := \langle ss' \rangle$. The following irreducible Yetter–Drinfeld modules over W have a finite-dimensional associated Nichols algebra [1, Theorem 3.1]. The action can be extracted from [37, 5.3]. The grading of all of them is supported in C, thus to understand the braiding, it is enough to consider the C-action. This way, we can regard them as Yetter–Drinfeld modules over C. They are:

- $V_0 := \mathbb{C}v_0$, concentrated in degree $(ss')^r$ and with action $(ss')v_0 = -v_0$. The Nichols algebra is $\bigwedge V_0$;
- $V_{r,j} := \mathbb{C}v_{+r,j} \oplus \mathbb{C}v_{-r,j}$, for $j \in \{1, \dots, r-2\}$ and odd, with grading concentrated in degree $(ss')^r$ and action $(ss')v_{\pm r,j} = \zeta^{\pm j} v_{\pm r,j}$. The Nichols algebra is $\bigwedge V_{r,j}$;

- $V_{h,j} := \mathbb{C}v_{+h,j} \oplus \mathbb{C}v_{-h,j}$, for $j, h \in \{1, \dots, r-2\}$, both odd, and such that $r \mid jh$, with $v_{\pm h,j}$ in degree $(ss')^{\pm h}$, respectively, and action $(ss')v_{\pm h,j} = \zeta^{\pm j} v_{\pm h,j}$. The Nichols algebra is $\bigwedge V_{h,j}$.

Theorem 4.8. *Assume $n = 2r$ for $r > 3$ and odd. Let V be a Yetter–Drinfeld module of W. Then $\dim \mathcal{B}(V) < \infty$ if and only if as a Yetter–Drinfeld module over C,*

$$V \simeq V_0^{\oplus k} \oplus \bigoplus_{d=1}^{N} V_{h_d, j_d}^{\oplus k_d}$$

for some $k, k_d, N \geqslant 0$, and some distinct pairs (h_d, j_d) for $d \in \{1, \dots, N\}$ with $h_d \in \{1, \dots, r\}$ odd, $j_d \in \{1, \dots, r-2\}$ odd and such that $r \mid (h_d j_{d'} + h_{d'} j_d)$ for all $d, d' \in \{1, \dots, N\}$. In this case,

$$\mathcal{B}(V) \simeq \bigwedge V_0^{\oplus k} \mathbin{\widetilde{\otimes}} \Big(\widetilde{\bigotimes}_{d=1}^{N} \bigwedge V_{h_d, j_d}^{\oplus k_d} \Big).$$

where $\widetilde{\otimes}$ stands for the tensor product of algebras twisted by the braiding of V.

Proof. The analysis in [1, Theorem 3.1], combined with Proposition 4.3 and Remark 4.2(3), shows that the only irreducible Yetter–Drinfeld modules of W with finite-dimensional Nichols algebras are V_0 and $V_{h,j}$ with $h \in \{1, \dots, r\}$ odd, $j \in \{1, \dots, r-2\}$ odd and such that $r \mid jh$. Now we look at direct sums.

A direct calculation shows that the braiding c satisfies $c(x \otimes y) = -y \otimes x$ for $x \in V_0$, $y \in U$ or $x \in U$ and $y \in V_0$. Then we have the isomorphism $\mathcal{B}(V_0^{\oplus k} \oplus U) \simeq \bigwedge V_0^{\oplus k} \mathbin{\widetilde{\otimes}} \mathcal{B}(U)$ for any $k \geqslant 0$ after an iterated application of Remark 2.4(5).

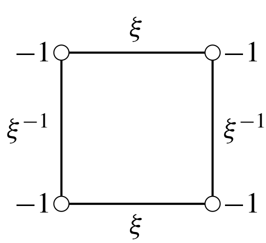
We now look at the braiding on $V_{h,j} \oplus V_{h',j'}$ where h, j, h', j' are odd; $1 \leqslant h, h' \leqslant r$, $1 \leqslant j, j' \leqslant r-2$, and $r \mid hj$ and $r \mid h'j'$. For $\epsilon, \epsilon' \in \{\pm 1\}$ we have

$$c(v_{\epsilon h,j} \otimes v_{\epsilon' h',j'}) = \zeta^{\epsilon\epsilon' hj'} v_{\epsilon' h',j'} \otimes v_{\epsilon h,j}. \tag{4-1}$$

Let $\xi := \zeta^{hj' + jh'}$. If $\xi = 1$, that is, if $r \mid (hj' + jh')$, then $c^2|_{V_{h,j} \oplus V_{h',j'}} = \mathrm{id}$, and so

$$\mathcal{B}(V_{h,j} \oplus V_{h',j'}) \simeq \mathcal{B}(V_{h,j}) \mathbin{\widetilde{\otimes}} \mathcal{B}(V_{h',j'}).$$

If, instead, $\xi \neq 1$, then we compute the generalized Dynkin diagram associated to the braiding on $V_{h,j} \oplus V_{h',j'}$ according to the recipe in [26, §2] obtaining



The above diagram does not occur in [26, Table 3], hence $\dim \mathcal{B}(V_{h,j} \oplus V_{h',j'}) = \infty$ by [25, Theorem 3, Corollary 5].

Now let $U = \bigoplus_{d=1}^{N} V_{h_d,j_d}^{\oplus k_d}$ for some $k_d, N \geqslant 0$, $h_d \in \{1,\dots,r\}$ odd, $j_d \in \{1,\dots,r-2\}$ odd and such that $r \mid h_d j_d$ for all d. If r does not divide $h_d j_{d'} + h_{d'} j_d$ for some d, d', then $\dim \mathcal{B}(U) = \infty$ by Remark 2.4(2). If, instead, $r \mid (h_d j_{d'} + h_{d'} j_d)$ for all d, d', then the square of the braiding between any irreducible component of U and the sum of the others is the identity. Thus, by Remark 2.4(5) and induction on the number of irreducible components, we obtain

$$\mathcal{B}(U) \simeq \widetilde{\bigotimes}_{d=1}^{N} \mathcal{B}(V_{h_d,j_d}^{\oplus k_d}) \simeq \widetilde{\bigotimes}_{d=1}^{N} \big(\bigwedge V_{h_d,j_d}\big)^{\widetilde{\otimes} k_d}.$$

Finally, observe that if $d = d'$ then $\zeta^{\epsilon\epsilon' h_d j_{d'}} = -1$, so $\big(\bigwedge V_{h_d,j_d}\big)^{\widetilde{\otimes} k_d} = \bigwedge V_{h_d,j_d}^{\oplus k_d}$. □

We now complement [1, Table 2] for $W = I_2(6)$ with the analysis of nonsimple Yetter–Drinfeld modules. Here we have to take into account also the simple Yetter–Drinfeld modules supported in $T_1 = W \triangleright s$ and $T_2 = W \triangleright s'$, whose corresponding Nichols algebra is the Fomin–Kirillov algebra FK_3 of dimension 12.

The Yetter–Drinfeld modules whose associated braided spaces correspond to $(T_1, q^{\pm})$ are the modules U_j for $j \in \{0, 1\}$ whose underlying vector space is $\mathbb{C}e_s^j \oplus \mathbb{C}e_{s'ss'}^j \oplus \mathbb{C}e_{ss'ss's}^j$ where e_g^j is in degree g for $g \in W$, and the action is given by

$$\begin{aligned} s \cdot e_s^j &= -e_s^j, & s \cdot e_{s'ss'}^j &= (-1)^{j+1} e_{ss'ss's}^j, & s \cdot e_{ss'ss's}^j &= -(-1)^j e_{s'ss'}^j,\\ (ss') \cdot e_s^j &= e_{ss'ss's}^j, & (ss') \cdot e_{s'ss'}^j &= (-1)^j e_s^j, & (ss') \cdot e_{ss'ss's}^j &= e_{s'ss'}^j. \end{aligned}$$

Similarly, the Yetter–Drinfeld modules whose associated braided spaces correspond to $(T_2, q^{\pm})$ are the modules U'_j for $j \in \{0, 1\}$ whose underlying vector space is $\mathbb{C}e_{s'}^j \oplus \mathbb{C}e_{ss's}^j \oplus \mathbb{C}e_{s'ss'ss'}^j$ where e_g is in degree g for $g \in W$, and the action is given by

$$\begin{aligned} s \cdot e_{s'}^j &= -e_{ss's}^j, & s \cdot e_{ss's}^j &= -e_{s'}^j, & s \cdot e_{s'ss'ss'}^j &= (-1)^{j+1} e_{s'ss'ss'}^j,\\ (ss') \cdot e_{s'}^j &= e_{ss's}^j, & (ss') \cdot e_{ss's}^j &= e_{s'ss'ss'}^j, & (ss') \cdot e_{ss'ss's}^j &= (-1)^j e_{s'}^j. \end{aligned}$$

In addition, to consider the braiding on sums of irreducible modules, we need to take into account the full action of W on the modules V_0 and $V_{3,1}$ following [1, Theorem 3.1]. According to [37, 5.3] we have $s \cdot v_{\pm 1,1} = v_{\mp 1,1}$ on $V_{3,1}$ and two possibilities for extending the action on V_0 from C to W: we denote the two extensions by $V_0^j := \mathbb{C}v_0^j$ for $j \in \{0, 1\}$ and set $s \cdot v_0^j = (-1)^j v_0^j$.

Twisting Yetter–Drinfeld modules as in Remark 2.4(4), by the automorphism τ of W that swaps s and s', preserves the isomorphism class of $V_{3,1}$ and interchanges V_0^0 and V_0^1. For $j = 0, 1$, it interchanges U_j and U'_j.

Proposition 4.9. *Let $W = I_2(6)$ and V be a Yetter–Drinfeld module for which $\dim \mathcal{B}(V) < \infty$. Then V is isomorphic to one of the modules*

- $V_{3,1}^{\oplus a} \oplus (V_0^0)^{\oplus b} \oplus (V_0^1)^{\oplus c}$ *for $a, b, c \geqslant 0$ and $a+b+c \geqslant 1$;*
- $U_j \oplus (V_0^j)^{\oplus a}$ *or* $U'_j \oplus (V_0^{1-j})^{\oplus a}$ *for $j \in \{1, 2\}$ and $a \geqslant 0$;*
- $U \oplus V_{3,1}$ *for $U \in \{U_0, U_1, U'_0, U'_1\}$;*

and $\mathcal{B}(V)$ is, respectively, $\left(\bigwedge V_{3,1}\right)^{\widetilde{\otimes} a} \widetilde{\otimes} \left(\bigwedge V_0^0\right)^{\widetilde{\otimes} b} \otimes \left(\bigwedge V_0^1\right)^{\widetilde{\otimes} c}$; $\mathrm{FK}_3 \widetilde{\otimes} \left(\bigwedge V_0^0\right)^{\widetilde{\otimes} a}$ *and the* 2304*-dimensional Nichols algebra in* [30, Theorem 8.2], *where $\widetilde{\otimes}$ denotes a twisted tensor product.*

Proof. By [1, Table 2], if V is irreducible then it is either $V_{3,1}$, V_0^j, U_j, or U'_j for $j \in \{0, 1\}$. Sums of copies of V_0^0, V_0^1 and of $V_{3,1}$ can be handled as in the proof of Theorem 4.8 because their grading is supported in C. Observe that the supports of U_0 and U_1 generate a group isomorphic to $\mathbb{S}_3$. Thus, by [6, Theorem 4.8] if $\dim \mathcal{B}(U_0^a \oplus U_1^b) < \infty$ for some $(a, b) \neq (0, 0)$, then $a+b=1$. The same argument applies to U'_0 and U'_1. For $\sigma \in T_1$ and $j, j' \in \{0, 1\}$ we have

$$c^2(e_\sigma^j \otimes v_0^{j'}) = (-1)^{j+j'} e_\sigma^j \otimes v_0^{j'}$$

so $\mathcal{B}((V_0^{j'})^{\oplus k} \oplus U_j)$ is a twisted tensor product of $\mathcal{B}((V_0^{j'})^{\oplus k})$ and $\mathcal{B}(U_j)$ if and only if $j = j'$. Remark 2.4(4) implies that the same property holds for the pair $V_0^{1-j} = (V_0^j)^\tau$ and $U'_j = U_j^\tau$.

For all other pairs (X, Y) of irreducible modules the support of $X \oplus Y$ generates W and $(c^2 - \mathrm{id})(X \oplus Y) \neq 0$. Under these conditions, [28, Corollary 7.2] gives a precise list of the possible supports for $X \oplus Y$ such that $\dim \mathcal{B}(X \oplus Y) < \infty$ and for each support, further conditions on W. In particular, if the size of the support is 6, then W must be a quotient of the group Γ_4 generated by a, b, ν, such that $\nu^4 = 1$, $\nu \mathrm{a} = \mathrm{a}\nu^{-1}$, $\nu \mathrm{b} = \mathrm{b}\nu^{-1}$, and $\mathrm{ba} = \nu \mathrm{ab}$. A direct calculation shows that this is not the case. Hence, $\dim \mathcal{B}(X \oplus Y) = \infty$ for $(X, Y) = (U_j, U'_{j'})$ for any $j, j' \in \{0, 1\}$. We are left with the pairs (U_0, V_0^1), (U_1, V_0^0), $(U_0, V_{3,1})$, and $(U_1, V_{3,1})$ and their twistings by τ. As a rack, their support is isomorphic to the one denoted by $Z_3^{3,1}$ in loc. cit. By [30, Theorem 2.1], the only pair of Yetter–Drinfeld modules X and Y of dimension, respectively, 3 and 1 and such that $X \oplus Y$ is supported by $Z_3^{3,1}$ is the one occurring in [30, Example 1.10]. There, it is required that, for z in the support of Y and g in the support of X, there holds $1 - \rho(z)\sigma(g) + (\rho(z)\sigma(g))^2 = 0$, where ρ denotes the action on the homogeneous component X_g and σ denotes the action on the homogeneous component Y_z. This cannot apply to the pairs (U_0, V_0^1) and (U_1, V_0^0) because the supports contain only involutions. The pair $(U_0, V_{3,1})$ occurs in [30, Example 1.11] for $g = s$, $z = (ss')^3$

and $\epsilon = (ss')^2$, hence $\mathcal{B}(U_0 \oplus V_{3,1})$ is the algebra described in [30, Theorem 8.4] whose dimension is 2304. Now, as (T_1, q^-) and (T_1, q^+) are twist equivalent by Theorem 2.8, the cocycles corresponding to U_0 and U_1 are twist equivalent, and therefore the cocycle corresponding to $U_0 \oplus V_{3,1}$ is twist equivalent to the cocycle corresponding to $U_1 \oplus V_{3,1}$, so $\mathcal{B}(U_0 \oplus V_{3,1})$ and $\mathcal{B}(U_1 \oplus V_{3,1})$ are twist equivalent. We finally need to consider the Yetter–Drinfeld modules of the form $U_0 \oplus V_{3,1}^{\oplus k}$ for $k > 1$. They are all braid indecomposable because $(\mathrm{id} - c^2)(U_0 \otimes V_{3,1}) \neq 0$, and $V_{3.1}$ is induced from a 2-dimensional representation of W, that is the centralizer of $(ss')^3$. Hence this case is ruled out by [29, Theorem 2.5]. □

Since the cases of $I_2(3) = \mathbb{S}_3$ and $I_2(4m)$ are to be found in [6; 13; 20], Corollary 4.7, Theorem 4.8 and Proposition 4.9 conclude the classification of finite-dimensional Nichols algebras over dihedral groups.

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