**GELFAND–CETLIN ABELIANIZATIONS
OF SYMPLECTIC QUOTIENTS**

PETER CROOKS AND JONATHAN WEITSMAN

GELFAND–CETLIN ABELIANIZATIONS
OF SYMPLECTIC QUOTIENTS

PETER CROOKS AND JONATHAN WEITSMAN

We show that generic symplectic quotients of a Hamiltonian *G*-space *M* by the action of a compact connected Lie group *G* are also symplectic quotients of the same manifold *M* by a compact torus. The torus action in question arises from certain integrable systems on g*, the dual of the Lie algebra of *G*. Examples of such integrable systems include the Gelfand–Cetlin systems of Guillemin and Sternberg (1980; 1983) in the case of unitary and special orthogonal groups, and certain integrable systems constructed for all compact connected Lie groups by Hoffman and Lane (2023). Our abelianization result holds for smooth quotients, and more generally for quotients which are stratified symplectic spaces in the sense of Sjamaar and Lerman (1991).

1. Introduction

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that M is a Hamiltonian G-space, i.e., a symplectic manifold equipped with a symplectic action of G and equivariant moment map $\mu : M \to \mathfrak{g}^*$. The symplectic or Marsden–Weinstein [9] quotient of M by G at level $\xi \in \mathfrak{g}^*$ is the topological space

$$M /\!/_\xi G := \mu^{-1}(\xi)/G_\xi,$$

where $G_\xi \subset G$ is the G-stabilizer of ξ. If G acts freely on $\mu^{-1}(\xi)$, then $M /\!/_\xi G$ is a smooth symplectic manifold. In the absence of this freeness assumption, $M /\!/_\xi G$ is a stratified symplectic space in the sense of Sjamaar and Lerman [10].

The purpose of this paper is to show that certain integrable systems on $\mathfrak{g}^*$ allow us to express generic symplectic quotients of a Hamiltonian G-space M as symplectic quotients of the same manifold M by the action of a compact torus. Such integrable systems include the Gelfand–Cetlin systems constructed by Guillemin and Sternberg [3; 4] for unitary and special orthogonal groups, as well as the more recent generalizations of Gelfand–Cetlin systems by Hoffman and Lane [8] to arbitrary Lie type.

An example of our main result arises in classical mechanics [2]. Suppose that we are given a Hamiltonian SO(3)-space M with an invariant Hamiltonian function $H : M \to \mathbb{R}$. The SO(3)-action gives rise to two Poisson-commuting conserved quantities: the total angular momentum, and the angular momentum in some fixed direction in the Lie algebra of SO(3) corresponding to a choice of maximal torus. These quantities give the components of a moment map for a densely defined 2-torus action on M, coming from the Gelfand–Cetlin system of Guillemin and Sternberg for the case of SO(3).[1] Our main result shows that for a nonzero value ξ of the angular momentum, the symplectic quotient $M/\!/_\xi SO(3)$ coincides with an appropriate symplectic quotient of M under the densely defined torus action. See [2] for more examples of these techniques.

1.1. *Main result.* We introduce the notion of a Gelfand–Cetlin datum $(\lambda_{\mathrm{big}}, \mathfrak{g}^*_{\mathrm{s\text{-}reg}})$ in Definition 1. This amounts to λ_{big} being a continuous map on $\mathfrak{g}^*$ that restricts to a Poisson moment map for a Hamiltonian action of a compact torus $\mathbb{T}_{\mathrm{big}}$ on an open dense subset $\mathfrak{g}^*_{\mathrm{s\text{-}reg}} \subset \mathfrak{g}^*$, along with some extra conditions that capture salient properties of the classical Gelfand–Cetlin systems. One of these conditions is that the open, symplectic submanifold $M_{\mathrm{s\text{-}reg}} := \mu^{-1}(\mathfrak{g}^*_{\mathrm{s\text{-}reg}}) \subset M$ be a Hamiltonian $\mathbb{T}_{\mathrm{big}}$-space with moment map $\lambda_M := (\lambda_{\mathrm{big}} \circ \mu)|_{M_{\mathrm{s\text{-}reg}}}$ for any Hamiltonian G-space M with moment map $\mu : M \to \mathfrak{g}^*$. The results of Guillemin and Sternberg [3; 4] imply that Gelfand–Cetlin data exist for all unitary and special orthogonal groups, while more recent results of Hoffman and Lane [8] imply that such data exist in all Lie types.

The following is the main result of our paper.

Theorem. *Let G be a compact connected Lie group, and M a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $(\lambda_{\mathrm{big}}, \mathfrak{g}^*_{\mathrm{s\text{-}reg}})$ is a Gelfand–Cetlin datum, and consider a point $\xi \in \mathfrak{g}^*_{\mathrm{s\text{-}reg}}$.*

(i) *The torus $\mathbb{T}_{\mathrm{big}}$ acts freely on $\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))$ if and only if G_ξ acts freely on $\mu^{-1}(\xi)$. In this case, there is a canonical symplectomorphism $M/\!/_\xi G \cong M_{\mathrm{s\text{-}reg}}/\!/_{\lambda_{\mathrm{big}}(\xi)} \mathbb{T}_{\mathrm{big}}$.*

(ii) *There is a canonical isomorphism $M/\!/_\xi G \cong M_{\mathrm{s\text{-}reg}} /\!/_{\lambda_{\mathrm{big}}(\xi)} \mathbb{T}_{\mathrm{big}}$ of stratified symplectic spaces.*

Part (ii) is strictly more general than (i). Part (i) is included for the sake of exposition and accessibility. One may regard this theorem as an approach to abelianizing the generic symplectic quotients of a Hamiltonian G-space M, i.e., to presenting such quotients as symplectic quotients by a compact torus.

---

[1] The square of the total angular momentum is a smooth function, but the orbits of its Hamiltonian flow do not have constant period, and so it does not generate a circle action. Taking the square root gives a function whose Hamiltonian flow generates a circle action, but which is only continuous, not differentiable, at zero. It therefore does not define a Hamiltonian flow at zero.

1.2. *Comparison to symplectic implosion.* By means of *symplectic implosion*, Guillemin, Jeffrey, and Sjamaar [5] offer a different approach to abelianizing symplectic quotients. Their construction replaces G and a Hamiltonian G-space M with a maximal torus $T \subset G$ and the *imploded cross-section* M_{impl}, respectively; the latter is a stratified symplectic Hamiltonian T-space whose dimension is generally less than that of M. Symplectic quotients of M by G are then shown to be realizable as symplectic quotients of M_{impl} by T. Our construction retains the same manifold M and replaces G with a torus $\mathbb{T}_{\mathrm{big}}$.

The imploded cross-section $(T^*G)_{\mathrm{impl}}$ features prominently in [5]. This stratified symplectic space also features prominently in [8], where Hoffman and Lane show Gelfand–Cetlin data to exist in arbitrary Lie type. It is natural to suspect that this coincidence is a shadow of some deeper connection between our work and symplectic implosion.

1.3. *Organization.* Section 2 briefly establishes some of our conventions concerning Lie theory and Hamiltonian geometry. Section 3 subsequently motivates and contextualizes the notion of a Gelfand–Cetlin datum. Our main result is then proved in Section 4 for smooth quotients. A generalization to stratified symplectic spaces is formulated and proved in Section 5.

2. Background and conventions

This section establishes some of our notation and conventions regarding Lie theory and Hamiltonian geometry.

2.1. *Tori.* The Lie algebra of the unitary group $\mathrm{U}(1)$ is the real vector space $i\mathbb{R} \subset \mathbb{C}$ of purely imaginary numbers. We will identify this vector space with $\mathbb{R}$ in the obvious way. It follows that $\mathbb{R}^k$ is the Lie algebra of $\mathrm{U}(1)^k$ for all nonnegative integers k, and that

$$\mathbb{R}^k \to \mathrm{U}(1)^k, \quad (x_1, \ldots, x_k) \mapsto (e^{ix_1}, \ldots, e^{ix_k})$$

is the exponential map for $\mathrm{U}(1)^k$. In certain contexts, we will implicitly use the dot product to regard $\mathbb{R}^k$ as the dual of the Lie algebra of $\mathrm{U}(1)^k$.

2.2. *General compact connected Lie groups.* Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and rank ℓ. One has the adjoint representations $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, as well as the coadjoint representations $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ and $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$. The G-representations induce G-actions on $\mathfrak{g}$ and $\mathfrak{g}^*$, and thereby give rise to stabilizer subgroups

$$G_x := \{g \in G : \mathrm{Ad}_g(x) = x\} \quad \text{and} \quad G_\xi := \{g \in G : \mathrm{Ad}_g^*(\xi) = \xi\}$$

of G for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. On the other hand, the $\mathfrak{g}$-representations allow us to define centralizers

$$\mathfrak{g}_x := \{y \in \mathfrak{g} : \mathrm{ad}_y(x) = 0\} \quad \text{and} \quad \mathfrak{g}_\xi := \{y \in \mathfrak{g} : \mathrm{ad}_y^*(\xi) = 0\}$$

for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. It follows that $\mathfrak{g}_x$ (resp. $\mathfrak{g}_\xi$) is the Lie algebra of G_x (resp. G_ξ). Let us also note that $\dim \mathfrak{g}_x \geq \ell$ and $\dim \mathfrak{g}_\xi \geq \ell$ for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. The regular loci

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} : \dim \mathfrak{g}_x = \ell\} \quad \text{and} \quad \mathfrak{g}_{\mathrm{reg}}^* := \{\xi \in \mathfrak{g}^* : \dim \mathfrak{g}_\xi = \ell\}$$

are open, dense, G-invariant subsets of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. An element $x \in \mathfrak{g}$ (resp. $\xi \in \mathfrak{g}^*$) then belongs to $\mathfrak{g}_{\mathrm{reg}}$ (resp. $\mathfrak{g}_{\mathrm{reg}}^*$) if and only if $\mathfrak{g}_x$ (resp. $\mathfrak{g}_\xi$) is a Cartan subalgebra of $\mathfrak{g}$. This is equivalent to G_x (resp. G_ξ) being a maximal torus of G.

A few remarks on maximal tori and Cartan subalgebras are warranted. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra, and write $T \subset G$ for the maximal torus with Lie algebra $\mathfrak{t}$. The exponential map $\exp : \mathfrak{g} \to G$ then restricts to a surjective homomorphism $\exp|_\mathfrak{t} : \mathfrak{t} \to T$ of abelian groups. The kernel of the latter is a free $\mathbb{Z}$-submodule of $\mathfrak{t}$ with rank equal to ℓ. It follows that the same is true of

$$\Lambda_\mathfrak{t} := \frac{1}{2\pi} \ker(\exp|_\mathfrak{t}) \subset \mathfrak{t}.$$

2.3. Hamiltonian geometry. Suppose that (M, σ) is a Poisson manifold, that is, $\sigma \in H^0(M, \Lambda^2 TM)$ is a Poisson bivector field on the manifold M. Note that σ may be regarded as a skew-symmetric bilinear map from two copies of T^*M to the trivial rank-1 vector bundle over M. Contracting σ with cotangent vectors in the first argument then determines a vector bundle morphism $\sigma^\vee : T^*M \to TM$. One calls (M, σ) *nondegenerate* if $\sigma^\vee$ is an isomorphism. In this case, $(\sigma^\vee)^{-1} = \omega^\vee$ for a unique symplectic form $\omega \in H^0(M, \Lambda^2 T^*M)$, where $\omega^\vee : TM \to T^*M$ is the vector bundle morphism obtained by contracting ω with tangent vectors in the first argument. This process gives rise to a bijective correspondence between symplectic structures on M and nondegenerate Poisson structures on M. We will thereby make no distinction between symplectic manifolds and nondegenerate Poisson manifolds.

If (M, σ) is a Poisson manifold, then σ can be recovered from the Poisson bracket $\{\cdot, \cdot\}$ that it induces. This bracket associates to smooth functions $f_1, f_2 : M \to \mathbb{R}$ the smooth function

$$\{f_1, f_2\} := \sigma(\mathrm{d}f_1 \wedge \mathrm{d}f_2) : M \to \mathbb{R}.$$

At the same time, one defines the Hamiltonian vector field of a smooth function $f : M \to \mathbb{R}$ by $X_f := -\sigma^\vee(\mathrm{d}f) \in H^0(M, TM)$. It follows that

$$\{f_1, f_2\} = -X_{f_1}(f_2) = X_{f_2}(f_1)$$

for all smooth functions $f_1, f_2 : M \to \mathbb{R}$.

Now let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. If G acts smoothly on a manifold M, then each $\eta \in \mathfrak{g}$ determines a generating vector field $\eta_M \in H^0(M, TM)$ by

$$(\eta_M)_m := \left.\frac{d}{dt}\right|_{t=0} \exp(-t\eta) \cdot m$$

for all $m \in M$. A Poisson manifold (M, σ) with a smooth G-action will be called a *Poisson Hamiltonian G-space* if σ is G-invariant and M comes equipped with a *moment map*. This last term refers to G-equivariant smooth map $\mu : M \to \mathfrak{g}^*$ satisfying $X_{\mu^\eta} = \eta_M$ for all $\eta \in \mathfrak{g}$, where $\mu^\eta : M \to \mathbb{R}$ is the result of pairing μ with η pointwise. We will reserve the term *Hamiltonian G-space* for a Poisson Hamiltonian G-space whose underlying Poisson structure is symplectic.

It will be advantageous to recall the Poisson Hamiltonian G-space structure on $\mathfrak{g}^*$. The Poisson bracket on $\mathfrak{g}^*$ is given by

$$\{f_1, f_2\}(\xi) = \xi([d_\xi f_1, d_\xi f_2])$$

for all smooth functions $f_1, f_2 : \mathfrak{g}^* \to \mathbb{R}$ and points $\xi \in \mathfrak{g}^*$, where

$$d_\xi f_1, d_\xi f_2 \in (\mathfrak{g}^*)^* = \mathfrak{g}$$

denote the differentials of f_1, f_2 at ξ, respectively. One finds that $\mathfrak{g}^*$ is a Poisson Hamiltonian G-space with respect to the coadjoint action, and with the identity $\mathfrak{g}^* \to \mathfrak{g}^*$ serving as the Poisson moment map.

3. Gelfand–Cetlin data

In this section, we define Gelfand–Cetlin data and introduce their main properties. This begins with the definition itself in Section 3.1. The existence of Gelfand–Cetlin data is addressed in Section 3.2, while concrete techniques for constructing such data are discussed in Sections 3.3 and 3.4. In Section 3.5, we describe concrete Gelfand–Cetlin data for unitary groups.

3.1. *Definition and relation to integrable systems.* Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and rank ℓ. Consider the quantities

$$\mathsf{u} := \tfrac{1}{2}(\dim \mathfrak{g} - \ell) \quad \text{and} \quad \mathsf{b} := \tfrac{1}{2}(\dim \mathfrak{g} + \ell),$$

and introduce the following tori of *small*, *intermediate*, and *big* ranks:

$$\mathbb{T}_{\text{small}} := U(1)^\ell, \quad \mathbb{T}_{\text{int}} := U(1)^{\mathsf{u}}, \quad \text{and} \quad \mathbb{T}_{\text{big}} := \mathbb{T}_{\text{small}} \times \mathbb{T}_{\text{int}} \cong U(1)^{\mathsf{b}}.$$

The respective Lie algebras of these tori are

$$\mathbb{R}_{\text{small}} := \mathbb{R}^\ell, \quad \mathbb{R}_{\text{int}} := \mathbb{R}^{\mathsf{u}}, \quad \text{and} \quad \mathbb{R}_{\text{big}} := \mathbb{R}_{\text{small}} \times \mathbb{R}_{\text{int}} \cong \mathbb{R}^{\mathsf{b}}.$$

Definition 1. A *Gelfand–Cetlin datum* is a pair $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$, consisting of a continuous map $\lambda_{\text{big}} = (\lambda_1, \ldots, \lambda_b) : \mathfrak{g}^* \to \mathbb{R}_{\text{big}}$ and open dense subset $\mathfrak{g}^*_{\text{s-reg}} \subset \mathfrak{g}^*$ that satisfy the following conditions:

(i) $\lambda_1, \ldots, \lambda_\ell$ are G-invariant on $\mathfrak{g}^*$ and smooth on $\mathfrak{g}^*_{\text{reg}}$.

(ii) $\{d_\xi \lambda_1, \ldots, d_\xi \lambda_\ell\}$ is a $\mathbb{Z}$-basis of the lattice $\Lambda_{\mathfrak{g}_\xi} \subset \mathfrak{g}_\xi$ for all $\xi \in \mathfrak{g}^*_{\text{reg}}$.

(iii) $\mathfrak{g}^*_{\text{s-reg}} \subset \mathfrak{g}^*_{\text{reg}}$.

(iv) $\lambda_{\text{big}}|_{\mathfrak{g}^*_{\text{s-reg}}} : \mathfrak{g}^*_{\text{s-reg}} \to \mathbb{R}_{\text{big}}$ is a smooth submersion and moment map for a Poisson Hamiltonian $\mathbb{T}_{\text{big}}$-space structure on $\mathfrak{g}^*_{\text{s-reg}}$.

(v) $\lambda_{\text{big}}|_{\mathfrak{g}^*_{\text{s-reg}}} : \mathfrak{g}^*_{\text{s-reg}} \to \lambda_{\text{big}}(\mathfrak{g}^*_{\text{s-reg}})$ is a principal $\mathbb{T}_{\text{int}}$-bundle.

(vi) If M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$, then

$$(\lambda_{\text{big}} \circ \mu)|_{\mu^{-1}(\mathfrak{g}^*_{\text{s-reg}})} : \mu^{-1}(\mathfrak{g}^*_{\text{s-reg}}) \to \mathbb{R}_{\text{big}}$$

   is a moment map for a Hamiltonian $\mathbb{T}_{\text{big}}$-space structure on $\mu^{-1}(\mathfrak{g}_{\text{s-reg}})$.

In this case, we adopt the notation

$$\lambda_{\text{small}} := (\lambda_1, \ldots, \lambda_\ell) : \mathfrak{g}^* \to \mathbb{R}_{\text{small}}, \quad \lambda_{\text{int}} := (\lambda_{\ell+1}, \ldots, \lambda_b) : \mathfrak{g}^* \to \mathbb{R}_{\text{int}},$$
$$M_{\text{s-reg}} := \mu^{-1}(\mathfrak{g}^*_{\text{s-reg}}), \quad \text{and} \quad \lambda_M := (\lambda_{\text{big}} \circ \mu)|_{M_{\text{s-reg}}} : M_{\text{s-reg}} \to \mathbb{R}_{\text{big}}.$$

We also refer to the elements of $\mathfrak{g}^*_{\text{s-reg}}$ as the *strongly regular* elements of $\mathfrak{g}^*$.

Remark 2. Condition (v) in Definition 1 is only slightly weaker than the existence of global action-angle coordinates on $\mathfrak{g}^*_{\text{s-reg}}$. This existence question features prominently in [1; 8].

It is instructive to consider this definition in relation to the theory of completely integrable systems. One is thereby led to the following result.

Proposition 3. *Let* $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ *be a Gelfand–Cetlin datum. If* $\mathcal{O} \subset \mathfrak{g}^*$ *is a coadjoint orbit and* $\mathcal{O}_{\text{s-reg}} := \mathcal{O} \cap \mathfrak{g}^*_{\text{s-reg}}$, *then*

$$\lambda_{\text{int}}|_{\mathcal{O}_{\text{s-reg}}} : \mathcal{O}_{\text{s-reg}} \to \lambda_{\text{int}}(\mathcal{O}_{\text{s-reg}}) \subset \mathbb{R}_{\text{int}}$$

is a completely integrable system, principal $\mathbb{T}_{\text{int}}$-*bundle, and moment map for a Hamiltonian action of* $\mathbb{T}_{\text{int}}$ *on* $\mathcal{O}_{\text{s-reg}}$.

Proof. Note that the Hamiltonian vector field of any smooth function $\mathfrak{g}^* \to \mathbb{R}$ is tangent to $\mathcal{O}$. It follows that $\mathcal{O}_{\text{s-reg}}$ is stable under the action of $\mathbb{T}_{\text{big}}$ on $\mathfrak{g}^*_{\text{s-reg}}$. Definition 1(iv) now implies that $\mathcal{O}_{\text{s-reg}}$ is a Hamiltonian $\mathbb{T}_{\text{big}}$-space with moment map $\lambda_{\text{big}}|_{\mathcal{O}_{\text{s-reg}}}$. We conclude that $\lambda_{\text{int}}|_{\mathcal{O}_{\text{s-reg}}}$ is a moment map for the Hamiltonian action of $\mathbb{T}_{\text{int}} \subset \mathbb{T}_{\text{big}}$ on $\mathcal{O}_{\text{s-reg}}$.

Since λ_{small} is constant-valued on $\mathcal{O}$, Definition 1(v) tells us that

$$\lambda_{\text{int}}|_{\mathcal{O}_{\text{s-reg}}} : \mathcal{O}_{\text{s-reg}} \to \lambda_{\text{int}}(\mathcal{O}_{\text{s-reg}})$$

is a principal $\mathbb{T}_{\mathrm{int}}$-bundle. It therefore remains only to prove that

$$\dim \mathbb{T}_{\mathrm{int}} = \tfrac{1}{2}\dim \mathcal{O}_{\text{s-reg}}.$$

This follows immediately from the fact that $\dim \mathbb{T}_{\mathrm{int}} = \mathfrak{u} = \tfrac{1}{2}(\dim \mathfrak{g} - \ell)$. $\square$

3.2. *Existence of Gelfand–Cetlin data.*

It is natural to wonder about the generality in which Gelfand–Cetlin data exist. The earliest constructions are due to Guillemin and Sternberg [3; 4], and apply to all unitary groups $\mathrm{U}(n)$ and special orthogonal groups $\mathrm{SO}(n)$. The underlying techniques are based on Thimm's method, as described in [4]. Further details are outlined in Sections 3.3–3.5. The case of symplectic groups is considerably more subtle and addressed in Harada's paper [6].

Some recent work of Hoffman and Lane [8] implies the existence of Gelfand–Cetlin data for an arbitrary compact connected Lie group G. In more detail, one may consider completely integrable systems on an arbitrary Poisson manifold. A completely integrable system on $\mathfrak{g}^*$ is thereby a continuous map $\phi : \mathfrak{g}^* \to \mathbb{R}_{\mathrm{big}}$ with the following property: there exists an open dense subset $U \subset \mathfrak{g}^*_{\mathrm{reg}}$ such that $\phi|_U$ is a smooth submersion with pairwise Poisson-commuting component functions. A Gelfand–Cetlin datum is thereby a completely integrable system on $\mathfrak{g}^*$. On the other hand, suppose that M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. One might hope for $\phi \circ \mu : M \to \mathbb{R}_{\mathrm{big}}$ to be a completely integrable system on an open subset of M; such a system would be called *collective*. Another aspiration would be for $\phi \circ \mu : M \to \mathbb{R}_{\mathrm{big}}$ to be the moment map of a Hamiltonian torus action on an open subset of M. This central theme of [8] is reflected in Definition 1: Gelfand–Cetlin data are defined in such a way as to induce Hamiltonian torus actions on open subsets of Hamiltonian G-spaces. The construction of Gelfand–Cetlin data is done in Sections 6.2 and 6.3 of [8]. Hoffman and Lane [8] begin by associating completely integrable systems on $\mathfrak{g}^*$ to *good valuations* on the *base affine space*; see [8, Data 5.2] and [8, Theorem 6.2]. Propositions 6.4 and 6.5 of [8] then show such systems to have several properties; these imply the existence of pairs $(\lambda_{\mathrm{big}}, \mathfrak{g}^*_{\text{s-reg}})$ satisfying parts (i)–(iv) of Definition 1. To address part (v) of Definition 1, Hoffman and Lane introduce the notion of *toric contraction* for Hamiltonian G-spaces [8, Definition 6.7]. The existence of pairs $(\lambda_{\mathrm{big}}, \mathfrak{g}^*_{\text{s-reg}})$ satisfying parts (i)–(v) then follows from Propositions 6.8 and 6.9 of [8].

The Hoffman–Lane paper is part of a broader program aimed at generalizing the results of Harada and Kaveh [7].

3.3. *Construction of Gelfand–Cetlin data: integrality.*

We now discuss the construction of functions $\lambda_1, \ldots, \lambda_\ell : \mathfrak{g}^* \to \mathbb{R}$ satisfying conditions (i) and (ii) in Definition 1. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and rank ℓ. Choose a G-invariant inner product on $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, and a closed, fundamental Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$. The chamber $\mathfrak{t}_+$ is known to be a

fundamental domain for the adjoint action of G on $\mathfrak{g}$. Our inner product thereby identifies $\mathfrak{t}_+$ with a fundamental domain $\mathfrak{t}_+^* \subset \mathfrak{g}^*$ for the G-action on $\mathfrak{g}^*$. We may therefore define a continuous surjection $\pi : \mathfrak{g}^* \to \mathfrak{t}_+^*$ by the property that

$$(G \cdot \xi) \cap \mathfrak{t}_+^* = \{\pi(\xi)\} \quad \text{for all } \xi \in \mathfrak{g}^*,$$

where $G \cdot \xi \subset \mathfrak{g}^*$ is the coadjoint orbit of ξ. One sometimes calls π the *sweeping map* on $\mathfrak{g}^*$ with respect to $\mathfrak{t}$; its fibers are exactly the coadjoint orbits of G, and $\pi(\mathfrak{g}_{\mathrm{reg}}^*)$ is the interior $(\mathfrak{t}_+^*)^\circ$ of $\mathfrak{t}_+^*$. One also finds that the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g}_{\mathrm{reg}}^* & \lhook\joinrel\longrightarrow & \mathfrak{g}^* \\
{\scriptstyle \pi|_{\mathfrak{g}_{\mathrm{reg}}^*}} \downarrow & & \downarrow {\scriptstyle \pi} \\
(\mathfrak{t}_+^*)^\circ & \lhook\joinrel\longrightarrow & \mathfrak{t}_+^*
\end{array}
$$

is Cartesian. The left vertical map

$$\pi|_{\mathfrak{g}_{\mathrm{reg}}^*} : \mathfrak{g}_{\mathrm{reg}}^* \to (\mathfrak{t}_+^*)^\circ$$

is easily seen to be a smooth, surjective submersion.

Now choose a $\mathbb{Z}$-basis $\{\phi_1, \ldots, \phi_\ell\}$ of the $\mathbb{Z}$-submodule $\Lambda_{\mathfrak{t}} \subset \mathfrak{t}$. Note that the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ allows one to regard $\phi_1, \ldots, \phi_\ell$ as functions on $\mathfrak{t}_+^*$. With this in mind, each $k \in \{1, \ldots, \ell\}$ determines a function

$$\lambda_k := \phi_k \circ \pi : \mathfrak{g}^* \to \mathbb{R}.$$

The previous paragraph implies that $\lambda_1, \ldots, \lambda_\ell$ are smooth on $\mathfrak{g}_{\mathrm{reg}}^*$, while being G-invariant and continuous as functions on $\mathfrak{g}^*$. Given any $\xi \in \mathfrak{g}_{\mathrm{reg}}^*$, the differentials $\mathrm{d}_\xi \lambda_1, \ldots, \mathrm{d}_\xi \lambda_\ell \in (\mathfrak{g}^*)^* = \mathfrak{g}$ may be described as follows.

Proposition 4. *If $\xi \in \mathfrak{g}_{\mathrm{reg}}^*$, then $\{\mathrm{d}_\xi \lambda_1, \ldots, \mathrm{d}_\xi \lambda_\ell\}$ is a $\mathbb{Z}$-basis of the lattice $\Lambda_{\mathfrak{g}_\xi} \subset \mathfrak{g}_\xi$.*

Proof. Choose $g \in G$ for which $\mathrm{Ad}_g^*(\xi) \in (\mathfrak{t}_+^*)^\circ$. Since each function λ_k is G-invariant, one has

$$\mathrm{d}_\xi \lambda_k = \mathrm{d}_{\mathrm{Ad}_g^*(\xi)} \lambda_k \circ \mathrm{Ad}_g^* = \mathrm{Ad}_{g^{-1}}(\mathrm{d}_{\mathrm{Ad}_g^*(\xi)} \lambda_k)$$

for all $k \in \{1, \ldots, \ell\}$. We also observe that $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ restricts to a $\mathbb{Z}$-module isomorphism $\lambda_{\mathfrak{t}} \xrightarrow{\ \cong\ } \lambda_{\mathfrak{g}_\xi}$. It therefore suffices to take $\xi \in (\mathfrak{t}_+^*)^\circ$ and prove that $\mathrm{d}_\xi \lambda_k = \phi_k$ for all $k \in \{1, \ldots, \ell\}$.

Assume that $\xi \in (\mathfrak{t}_+^*)^\circ$, and fix $k \in \{1, \ldots, \ell\}$. Our invariant inner product allows us to regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$. We then note that $\mathfrak{g}^* = \mathfrak{t}^* \oplus T_\xi(G \cdot \xi)$, and that $T_\xi(G \cdot \xi)$ is contained in the kernel of $\mathrm{d}_\xi \lambda_k$. Let us also note that $T_\xi(G \cdot \xi)$ is the annihilator of $\mathfrak{t}$ in $\mathfrak{g}^*$, and as such is contained in the kernel of ϕ_k. These last two sentences reduce us to proving that $\mathrm{d}_\xi \lambda_k(\eta) = \phi_k(\eta)$ for all $\eta \in \mathfrak{t}^*$. On the other hand, we have $\mathrm{d}_\xi \lambda_k = \phi_k \circ \mathrm{d}_\xi \pi$. It therefore suffices to prove that $\mathrm{d}_\xi \pi(\eta) = \eta$ for

all $\eta \in \mathfrak{t}^*$. But this is an immediate consequence of the following two observations: $(\mathfrak{t}_+^*)^\circ$ is an open subset of $\mathfrak{t}^*$, and $\pi(\eta) = \eta$ for all $\eta \in (\mathfrak{t}_+^*)^\circ$.   □

3.4. *Construction of Gelfand–Cetlin data: Thimm's method.* Retain the objects and notation discussed in Section 3.3. Let $G = G_0 \supset G_1 \supset \cdots \supset G_m$ be a descending filtration of G by connected closed subgroups with respective Lie algebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m$. Let us also choose a Cartan subalgebra $\mathfrak{t}_j \subset \mathfrak{g}_j$ and closed, fundamental Weyl chamber $(\mathfrak{t}_j)_+ \subset \mathfrak{t}_j$ for each $j \in \{0, \ldots, m\}$. Our G-invariant inner product on $\mathfrak{g}$ gives rise to a G_j-module isomorphism $\mathfrak{g}_j \cong \mathfrak{g}_j^*$, by means of which $\mathfrak{t}_j$ and $(\mathfrak{t}_j)_+$ correspond to subsets $\mathfrak{t}_j^*$ and $(\mathfrak{t}_j^*)_+$ of $\mathfrak{g}_j^*$. As in Section 3.3, one may define a continuous surjection $\pi_j : \mathfrak{g}_j^* \to (\mathfrak{t}_j^*)_+$ by the condition that $(G_j \cdot \xi) \cap (\mathfrak{t}_j^*)_+ = \{\pi_j(\xi)\}$ for all $\xi \in \mathfrak{g}_j^*$.

Let $\ell = \ell_0 \geq \ell_1 \geq \cdots \geq \ell_m$ be the ranks of $G = G_0 \supset G_1 \supset \cdots \supset G_m$, respectively. Let us also choose a $\mathbb{Z}$-basis $\{\phi_{j1}, \ldots, \phi_{j\ell_j}\}$ of the lattice $\Lambda_{\mathfrak{t}_j} \subset \mathfrak{t}_j$ for each $j \in \{0, \ldots, m\}$. As in Section 3.3, we define the functions

$$\nu_{jk} := \phi_{jk} \circ \pi_j : \mathfrak{g}_j^* \to \mathbb{R}$$

for $k \in \{1, \ldots, \ell_j\}$. The same section implies that $\nu_{j1}, \ldots, \nu_{j\ell_j}$ are G_j-invariant and continuous on $\mathfrak{g}_j^*$, as well as smooth on $(\mathfrak{g}_j^*)_{\mathrm{reg}}$. We also have the following equivalent version of Proposition 4.

Proposition 5. *If $j \in \{0, \ldots, m\}$ and $\xi \in (\mathfrak{g}_j^*)_{\mathrm{reg}}$, then $\{d_\xi \nu_{j1}, \ldots, d_\xi \nu_{j\ell_j}\}$ is a $\mathbb{Z}$-basis of the lattice $\Lambda_{(\mathfrak{g}_j)_\xi} \subset (\mathfrak{g}_j)_\xi$.*

Let $\sigma_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ be the transpose of the inclusion $\mathfrak{g}_j \hookrightarrow \mathfrak{g}$ for each $j \in \{0, \ldots, m\}$. Consider the functions on $\mathfrak{g}^*$ defined by

$$\lambda_{jk} := \nu_{jk} \circ \sigma_j : \mathfrak{g}^* \to \mathbb{R} \quad \text{for } j \in \{0, \ldots, m\} \text{ and } k \in \{1, \ldots, \ell_j\}.$$

It will be convenient to enumerate these functions as

$$(3\text{-}1) \qquad \lambda_{\mathrm{big}} := (\lambda_1, \ldots, \lambda_c)$$
$$:= (\lambda_{01}, \ldots, \lambda_{0\ell}, \lambda_{11}, \ldots, \lambda_{1\ell_1} \ldots, \lambda_{m1}, \ldots, \lambda_{m\ell_m}) : \mathfrak{g}^* \to \mathbb{R}^c,$$

where $c := \ell_0 + \cdots + \ell_m$.

The discussion preceding Proposition 5 implies that λ_{big} is smooth on the open subset

$$\mathcal{U} := \bigcap_{j=0}^m \sigma_j^{-1}((\mathfrak{g}_j^*)_{\mathrm{reg}}) \subset \mathfrak{g}^*.$$

Let us also define

$$\mathfrak{g}_{\mathrm{s\text{-}reg}}^* := \{\xi \in \mathcal{U} : d_\xi \lambda_{\mathrm{big}} \text{ is surjective}\}.$$

These last few sentences give context for the following consequence of [3, Theorem 3.4].

Proposition 6. *Let λ_{big} and $\mathfrak{g}^*_{\mathrm{s\text{-}reg}}$ be as defined above.*

(i) *The restriction $\lambda_{\mathrm{big}}|_{\mathfrak{g}^*_{\mathrm{s\text{-}reg}}} : \mathfrak{g}^*_{\mathrm{s\text{-}reg}} \to \mathbb{R}^c$ is a moment map for a Poisson Hamiltonian $\mathrm{U}(1)^c$-space structure on $\mathfrak{g}^*_{\mathrm{s\text{-}reg}}$.*

(ii) *If M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$, then*

$$(\lambda_{\mathrm{big}} \circ \mu)|_{\mu^{-1}(\mathfrak{g}^*_{\mathrm{s\text{-}reg}})} : \mu^{-1}(\mathfrak{g}^*_{\mathrm{s\text{-}reg}}) \to \mathbb{R}^c$$

is a moment map for a Hamiltonian $\mathrm{U}(1)^c$-space structure on $\mu^{-1}(\mathfrak{g}^*_{\mathrm{s\text{-}reg}})$.

This result has the following immediate connection to Definition 1: the pair $(\lambda_{\mathrm{big}}, \mathfrak{g}^*_{\mathrm{s\text{-}reg}})$ is a Gelfand–Cetlin datum if and only if $c = b$, $\mathfrak{g}^*_{\mathrm{s\text{-}reg}}$ is dense in $\mathfrak{g}^*$, and $\lambda_{\mathrm{big}}|_{\mathfrak{g}^*_{\mathrm{s\text{-}reg}}} : \mathfrak{g}^*_{\mathrm{s\text{-}reg}} \to \lambda_{\mathrm{big}}(\mathfrak{g}^*_{\mathrm{s\text{-}reg}})$ is a principal $\mathbb{T}_{\mathrm{int}}$-bundle. Guillemin and Sternberg [3; 4] explicitly show these conditions to be achievable for $G = \mathrm{U}(n)$ and $G = \mathrm{SO}(n)$. Our next section outlines the details of the Guillemin–Sternberg construction for $G = \mathrm{U}(n)$.

3.5. *Example of Gelfand–Cetlin datum: the Gelfand–Cetlin system on $\mathfrak{u}(n)^*$.* Fix a positive integer n. Consider the Lie group $G := \mathrm{U}(n)$ of unitary $n \times n$ matrices, and its Lie algebra $\mathfrak{g} := \mathfrak{u}(n)$ of skew-Hermitian $n \times n$ matrices. Let us also consider the real $\mathrm{U}(n)$-module $\mathcal{H}(n)$ of Hermitian $n \times n$ matrices. In what follows, we will freely identify $\mathfrak{u}(n)^*$ with $\mathcal{H}(n)$ by means of the nondegenerate, G-invariant bilinear form

$$(3\text{-}2) \qquad \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathcal{H}(n) \to \mathbb{R}, \quad \eta \otimes \xi \mapsto -i\mathrm{tr}(\eta\xi).$$

Given an integer $j \in \{0, \ldots, n-1\}$, define the subgroup

$$G_j := \left\{ \left[\begin{array}{c|c} I_j & 0 \\ \hline 0 & A \end{array} \right] : A \in \mathrm{U}(n-j) \right\} \subset G = \mathrm{U}(n).$$

The descending chain $\mathrm{U}(n) = G = G_0 \supset G_1 \supset \cdots \supset G_{n-1}$ then induces such a chain $\mathfrak{u}(n) = \mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_{n-1}$ on the level of Lie algebras. We have

$$\mathfrak{g}_j = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & x \end{array} \right] : x \in \mathfrak{u}(n-j) \right\} \subset \mathfrak{u}(n) \quad \text{and} \quad \mathfrak{g}_j^* = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \xi \end{array} \right] : \xi \in \mathcal{H}(n-j) \right\} \subset \mathcal{H}(n)$$

for all $j \in \{0, \ldots, n-1\}$, where the second equation implicitly uses (3-2). The transpose $\sigma_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ of $\mathfrak{g}_j \subset \mathfrak{g}$ then sends $\xi \in \mathfrak{g}^* = \mathcal{H}(n)$ to the $(n-j) \times (n-j)$ submatrix in the bottom right-hand corner of ξ.

Now note that

$$\mathfrak{t}_j := \left\{ \left[\begin{array}{c|ccc} 0 & & 0 & \\ \hline & ia_1 & & \\ 0 & & \ddots & \\ & & & ia_{n-j} \end{array} \right] : a_1, \ldots, a_{n-j} \in \mathbb{R} \right\} \subset \mathfrak{g}_j$$

is a Cartan subalgebra for each $j \in \{0, \ldots, n-1\}$. Let $\phi_{jk} \in \mathfrak{t}_j$ be the result of setting $a_k = 1$ and $a_p = 0$ for $p \neq k$, noting that $\{\phi_{j1}, \ldots, \phi_{j(n-j)}\}$ is a $\mathbb{Z}$-basis of $\Lambda_{\mathfrak{t}_j} \subset \mathfrak{t}_j$. At the same time, consider the fundamental Weyl chamber

$$(\mathfrak{t}_j)_+ := \left\{ \begin{bmatrix} 0 & 0 \\ \hline & ia_1 & \\ 0 & & \ddots & \\ & & & ia_{n-j} \end{bmatrix} : a_1, \ldots, a_{n-j} \in \mathbb{R}, \ a_1 \geq \cdots \geq a_{n-j} \right\} \subset \mathfrak{t}_j$$

for each $j \in \{0, \ldots, n-1\}$. Under the pairing (3-2), $(\mathfrak{t}_j)_+$ corresponds to the cone

$$(\mathfrak{t}_j^*)_+ := \left\{ \begin{bmatrix} 0 & 0 \\ \hline & a_1 & \\ 0 & & \ddots & \\ & & & a_{n-j} \end{bmatrix} : a_1, \ldots, a_{n-j} \in \mathbb{R}, \ a_1 \geq \cdots \geq a_{n-j} \right\} \subset \mathcal{H}(n).$$

We then have sweeping maps $\pi_j : \mathfrak{g}_j^* \to (\mathfrak{t}_j^*)_+$ and compositions

$$\nu_{jk} := \phi_{jk} \circ \pi_j : \mathfrak{g}_j^* \to \mathbb{R}$$

for $j \in \{0, \ldots, n-1\}$ and $k \in \{1, \ldots, n-j\}$, as in Section 3.4. The functions

$$\lambda_{jk} := \nu_{jk} \circ \sigma_j : \mathcal{H}(n) \to \mathbb{R}$$

from Section 3.4 are therefore given by the following condition: if $\xi \in \mathcal{H}(n)$ and $j \in \{0, \ldots, n-1\}$, then $\lambda_{j1}(\xi) \geq \lambda_{j2}(\xi) \geq \cdots \geq \lambda_{j(n-j)}(\xi)$ are the eigenvalues of the $(n-j) \times (n-j)$ submatrix in the bottom right-hand corner of ξ.

Observe that the number of maps λ_{jk} is

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2} = \tfrac{1}{2}(\dim \mathfrak{u}(n) + n).$$

Our enumeration (3-1) therefore takes the form

$$\lambda_{\text{big}} := (\lambda_1, \ldots, \lambda_{\frac{n(n+1)}{2}})$$
$$:= (\lambda_{01}, \ldots, \lambda_{0n}, \lambda_{11}, \ldots, \lambda_{1(n-1)}, \ldots, \lambda_{(n-2)1}, \lambda_{(n-2)2}, \lambda_{(n-1)1}) : \mathcal{H}(n) \to \mathbb{R}^{\frac{n(n+1)}{2}}.$$

Let us also consider the open dense subset

$$\mathcal{H}(n)_{\text{s-reg}} := \left\{ \xi \in \mathcal{H}(n) : \begin{array}{l} \lambda_{j1}(\xi) > \cdots > \lambda_{j(n-j)}(\xi) \text{ for all } j \in \{0, \ldots, n-1\} \\ \lambda_{0k}(\xi) > \cdots > \lambda_{(n-k)k}(\xi) \text{ for all } k \in \{1, \ldots, n\} \end{array} \right\}$$

of $\mathfrak{g}^* = \mathcal{H}(n)$. By the paragraph following Proposition 6, $(\lambda_{\text{big}}, \mathcal{H}(n)_{\text{s-reg}})$ is a Gelfand–Cetlin datum if and only if $\lambda_{\text{big}}|_{\mathcal{H}(n)_{\text{s-reg}}} : \mathcal{H}(n)_{\text{s-reg}} \to \lambda_{\text{big}}(\mathcal{H}(n)_{\text{s-reg}})$ is a principal bundle for $\mathbb{T}_{\text{int}} = U(1)^{\frac{n(n-1)}{2}}$; this latter condition is verified in [3, Section 5].

4. The abelianization theorem

This section is devoted to the proof of our abelianization theorem for smooth quotients. Some preliminary results are established in Sections 4.1 and 4.2, while the main proof appears in Section 4.3.

4.1. *The universal maximal torus.* Adopt the notation and conventions given in Section 3.1, and let $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ be a Gelfand–Cetlin datum. Given any $\xi \in \mathfrak{g}^*_{\text{reg}}$, Definition 1(ii) tells us that $\{d_\xi \lambda_1, \ldots, d_\xi \lambda_\ell\}$ is a basis of $\mathfrak{g}_\xi$. This basis determines a vector space isomorphism

$$\kappa_\xi : \mathfrak{g}_\xi \xrightarrow{\cong} \mathbb{R}^\ell, \quad x_1 d_\xi \lambda_1 + \cdots + x_\ell d_\xi \lambda_\ell \mapsto (x_1, \ldots, x_\ell).$$

The torus $\mathbb{T}_{\text{small}}$ is then a universal maximal torus in the following sense.

Proposition 7. *If $\xi \in \mathfrak{g}^*_{\text{reg}}$, then κ_ξ integrates to a Lie group isomorphism*

$$\tau_\xi : G_\xi \xrightarrow{\cong} \mathbb{T}_{\text{small}}.$$

Proof. It suffices to prove that κ_ξ restricts to a $\mathbb{Z}$-module isomorphism from the kernel of $\exp|_{\mathfrak{g}_\xi} : \mathfrak{g}_\xi \to G_\xi$ to $(2\pi\mathbb{Z})^\ell \subset \mathbb{R}^\ell$. This is an immediate consequence of Proposition 4. □

Proposition 8. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$. Then $g \cdot m = \tau_\xi(g) \cdot m$ for all $g \in G_\xi$ and $m \in \mu^{-1}(\xi)$, where the left- and right-hand sides denote the actions of $G_\xi \subset G$ on M and $\mathbb{T}_{\text{small}} \subset \mathbb{T}_{\text{big}}$ on $M_{\text{s-reg}}$, respectively.*

Proof. Let X_ζ be the generating vector field on $M_{\text{s-reg}}$ determined by $\zeta \in \mathbb{R}_{\text{small}}$ via the action of $\mathbb{T}_{\text{small}}$ on $M_{\text{s-reg}}$. Write Y_η for the generating vector field on M determined by $\eta \in \mathfrak{g}_\xi$ through the action of $G_\xi \subset G$ on M. It suffices to prove that $(X_{\kappa_\xi(\eta)})_m = (Y_\eta)_m$ for all $\eta \in \mathfrak{g}_\xi$ and $m \in \mu^{-1}(\xi)$. Setting $\gamma_j := d_\xi \lambda_j$ and letting $e_j \in \mathbb{R}^\ell = \mathbb{R}_{\text{small}}$ denote the j-th standard basis vector, this is equivalent to establishing that $(X_{e_j})_m = (Y_{\gamma_j})_m$ for all $j \in \{1, \ldots, \ell\}$ and $m \in \mu^{-1}(\xi)$. On the other hand, Y_{γ_j} (resp. X_{e_j}) is the Hamiltonian vector field on M (resp. $M_{\text{s-reg}}$) associated to μ^{γ_j} (resp. the j-th component $\mu^*\lambda_j$ of $\lambda \circ \mu$). This further reduces us to proving that $d_m \mu^{\gamma_j} = d_m \mu^*\lambda_j$. But it is clear that

$$d_m \mu^*\lambda_j = d_\xi \lambda_j \circ d_m \mu = d_m \mu^{\gamma_j}$$

for all $m \in \mu^{-1}(\xi)$ and $j \in \{1, \ldots, \ell\}$. □

4.2. *Some supplementary results.* We now prove two supplementary facts needed to establish the main result of this paper. We continue with the notation and conventions of Sections 3.1 and 4.1.

Proposition 9. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *If $m \in \mu^{-1}(\xi)$ and $t \in \mathbb{T}_{\text{int}}$ satisfy $t \cdot m \in \mu^{-1}(\xi)$, then $t = e$.*

(ii) *The saturation of $\mu^{-1}(\xi)$ under the action of $\mathbb{T}_{\text{int}}$ on $M_{\text{s-reg}}$ is $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$.*

Proof. To verify (i), let $m \in \mu^{-1}(\xi)$ and $t \in \mathbb{T}_{\text{int}}$ be such that $t \cdot m \in \mu^{-1}(\xi)$. Let us also observe that μ is $\mathbb{T}_{\text{int}}$-equivariant when restricted to a map $M_{\text{s-reg}} \to \mathfrak{g}^*_{\text{s-reg}}$. These last two sentences imply that $\xi = t \cdot \xi$. Since $\mathbb{T}_{\text{int}}$ acts freely on $\mathfrak{g}^*_{\text{s-reg}}$, we must have $t = e$.

We now verify (ii). To this end, note that $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ is a $\mathbb{T}_{\text{int}}$-invariant subset of $M_{\text{s-reg}}$ that contains $\mu^{-1}(\xi)$. This implies that the saturation of $\mu^{-1}(\xi)$ is contained in $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$. For the opposite inclusion, suppose that $m \in \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$. Definition 1(v) tells us that $t \cdot \mu(m) = \xi$ for some $t \in \mathbb{T}_{\text{int}}$. By the equivariance property of μ mentioned in the previous paragraph, we must have $t \cdot m \in \mu^{-1}(\xi)$. This completes the proof of (ii). $\square$

Fix $\xi \in \mathfrak{g}^*_{\text{s-reg}}$. In light of the previous proposition, we may define the map

$$\delta_\xi : \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}} \to \lambda_M^{-1}(\lambda_{\text{big}}(\xi)), \quad (m, t) \mapsto t \cdot m.$$

The following result is immediate consequence of the previous proposition.

Corollary 10. *If $\xi \in \mathfrak{g}^*_{\text{s-reg}}$, then δ_ξ is a homeomorphism.*

4.3. *Proof of the abelianization theorem.* Let us continue with the notation and conventions set in Sections 3.1, 4.1, and 4.2.

Theorem 11. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *The stabilizer G_ξ acts freely on $\mu^{-1}(\xi)$ if and only if $\mathbb{T}_{\text{big}}$ acts freely on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$.*

(ii) *In the case of* (i), *there is a canonical symplectomorphism*

$$M /\!\!/_\xi G \cong M_{\text{s-reg}} /\!\!/_{\lambda_{big}(\xi)} \mathbb{T}_{\text{big}}.$$

Proof. We begin by verifying (i). In light of Proposition 7, the multiplication map

$$\rho_\xi : G_\xi \times \mathbb{T}_{\text{int}} \to \mathbb{T}_{\text{big}}, \quad (g, t) \mapsto \tau_\xi(g)\, t$$

is a Lie group isomorphism. We also note that the action of G_ξ on $\mu^{-1}(\xi)$ and multiplication action of $\mathbb{T}_{\text{int}}$ on itself define an action $G_\xi \times \mathbb{T}_{\text{int}}$ on $\mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}$. By Proposition 8, the homeomorphism δ_ξ is equivariant in the sense that

$$\delta_\xi((g, t) \cdot x) = \rho_\xi(g, t) \cdot \delta_\xi(x)$$

for all $(g, t) \in G_\xi \times \mathbb{T}_{\text{int}}$ and $x \in \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}$. It follows that G_ξ acts freely on $\mu^{-1}(\xi)$ if and only if $\mathbb{T}_{\text{big}}$ acts freely on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$.

We now prove (ii). By Corollary 10, the inclusion $\mu^{-1}(\xi) \hookrightarrow \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ descends to a diffeomorphism

$$\text{f} : \mu^{-1}(\xi) \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}}.$$

We also note that the $\mathbb{T}_{\text{big}}$-action on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ induces a residual action of the subtorus $\mathbb{T}_{\text{small}}$ on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}}$. Proposition 8 then tells us that

$$\text{f}(g \cdot m) = \tau_\xi(g) \cdot \text{f}(m)$$

for all $g \in G_\xi$ and $m \in \mu^{-1}(\xi)$. The map f therefore descends to a diffeomorphism

$$\varphi : M /\!/_\xi G \xrightarrow{\cong} M_{\text{s-reg}} /\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}.$$

It therefore suffices to prove that φ pulls the symplectic form β on $M_{\text{s-reg}} /\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}$ back to the symplectic form α on $M /\!/_\xi G$.

We have a commutative diagram

$$
\begin{array}{ccc}
\mu^{-1}(\xi) & \xrightarrow{\;\;\text{j}\;\;} & \lambda_M^{-1}(\lambda_{\text{big}}(\xi)) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \theta} \\
M /\!/_\xi G & \xrightarrow[\;\;\varphi\;\;]{} & M_{\text{s-reg}} /\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}
\end{array}
$$

where

$$\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi = M /\!/_\xi G$$

and

$$\theta : \lambda_M^{-1}(\lambda_{\text{big}}(\xi)) \to \lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{big}} = M_{\text{s-reg}} /\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}$$

are the canonical quotient maps and $\text{j} : \mu^{-1}(\xi) \hookrightarrow \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ is the inclusion. We also have inclusion maps $\text{k} : \mu^{-1}(\xi) \hookrightarrow M$ and $\text{l} : \lambda_M^{-1}(\lambda_{\text{big}}(\xi)) \hookrightarrow M$. Another consideration is that α (resp. β) is the unique 2-form on $M /\!/_\xi G$ (resp. $M_{\text{s-reg}} /\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}$) for which $\pi^* \alpha = \text{k}^* \omega$ (resp. $\theta^* \beta = \text{l}^* \omega$), where ω is the symplectic form on M. It therefore suffices to prove that $\pi^*(\varphi^* \beta) = \text{k}^* \omega$. On the other hand, our commutative diagram implies that

$$\pi^*(\varphi^* \beta) = \text{j}^*(\theta^* \beta) = \text{j}^*(\text{l}^* \omega) = \text{k}^* \omega. \qquad \square$$

5. Generalization to stratified symplectic spaces

We now provide a generalization of Theorem 11 in the realm of stratified symplectic spaces [10]. In Section 5.1, we recall the immediately pertinent parts of Sjamaar and Lerman's more general theory of stratified symplectic spaces. The generalization of Theorem 11 to stratified symplectic spaces appears in Section 5.2.

5.1. *Stratified symplectic spaces.* Let X be a topological space on which a compact torus T acts continuously. Given a closed subgroup $H \subset T$, let

$$X_H := \{x \in X : T_x = H\}$$

be the locus of points with T-stabilizer T_x equal to H. Denote by $\mathrm{Stab}(T, X)$ the set of all closed subgroups $H \subset T$ for which $X_H \neq \varnothing$.

Now let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. As discussed in the introduction to this paper, $M /\!/_\xi G$ is a stratified symplectic space [10] for all $\xi \in \mathfrak{g}^*$. This means that $M /\!/_\xi G$ is naturally partitioned into symplectic manifolds satisfying certain compatibility conditions. While we refer the reader to [10, Definition 1.12] for a precise definition and description of stratified symplectic spaces, the following exposition will be sufficient for our purposes.

Fix a point $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$, and recall that $G_\xi \subset G$ is a maximal torus. Adopt the more parsimonious notation

$$\mathrm{Stab}(G, \xi) := \mathrm{Stab}(G_\xi, \mu^{-1}(\xi)),$$

and note that $\mu^{-1}(\xi)$ is the disjoint union

$$\mu^{-1}(\xi) = \bigsqcup_{H \in \mathrm{Stab}(G,\xi)} \mu^{-1}(\xi)_H.$$

The arguments in the proof of [10, Theorem 2.1] imply that each subset $\mu^{-1}(\xi)_H$ is a locally closed, G_ξ-invariant submanifold of M. These arguments also imply that the topological quotient $(\mu^{-1}(\xi)_H)/G_\xi$ carries a unique manifold structure for which the canonical map $\pi : \mu^{-1}(\xi)_H \to (\mu^{-1}(\xi)_H)/G_\xi$ is a surjective submersion. One further consequence of [10, Theorem 2.1] is the existence of a symplectic form $\bar{\omega}$ on $(\mu^{-1}(\xi)_H)/G_\xi$ such that $\pi^*\bar{\omega}$ is the pullback of ω along the inclusion $\mu^{-1}(\xi)_H \hookrightarrow M$. It follows that $M /\!/_\xi G = \mu^{-1}(\xi)/G_\xi$ is a disjoint union

$$(5\text{-}1) \qquad M /\!/_\xi G = \bigsqcup_{H \in \mathrm{Stab}(G,\xi)} (\mu^{-1}(\xi)_H)/G_\xi$$

of symplectic manifolds, called the *symplectic strata* of $M /\!/_\xi G$.

Remark 12. The quotients $(\mu^{-1}(\xi)_H)/G_\xi$ need not be manifolds in the traditional sense of the term; each may have connected components of different dimensions. To obtain a stratification into genuine symplectic manifolds, one must refine (5-1) and declare the symplectic strata to be the connected components of the quotients $(\mu^{-1}(\xi)_H)/G_\xi$. The distinction between (5-1) and this refined stratification will not materially affect any argument in this paper.

Definition 13. Let G and K be compact connected Lie groups with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. Suppose that M (resp. N) is a Hamiltonian G-space (resp. Hamiltonian K-space) with moment map $\mu : M \to \mathfrak{g}^*$ (resp. $\nu : N \to \mathfrak{k}^*$). Take $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$ and

$\eta \in \mathfrak{k}^*_{\text{reg}}$. A pair of maps $\varphi : M /\!/_\xi G \to N /\!/_\eta K$ and $\phi : \text{Stab}(G, \xi) \to \text{Stab}(K, \eta)$ will be called an *isomorphism of stratified symplectic spaces* if the following conditions are satisfied:

(i) φ is a homeomorphism.

(ii) ϕ is a bijection.

(iii) φ restricts to a symplectomorphism $(\mu^{-1}(\xi)_H)/G_\xi \to (\nu^{-1}(\eta)_{\phi(H)})/K_\eta$ for each $H \in \text{Stab}(G, \xi)$.

Remark 14. Assume that this definition is satisfied. Equip $M /\!/_\xi G$ and $N /\!/_\eta K$ with the refined stratifications discussed in Remark 12. By (ii) and (iii), the association $S \mapsto \varphi(S)$ defines a bijection from the set of symplectic strata $S \subset M /\!/_\xi G$ to the set of symplectic strata in $N /\!/_\eta K$. Property (i) implies that this bijection is an isomorphism of partially ordered sets, i.e., any symplectic strata $S, T \subset M /\!/_\xi G$ satisfying $S \subset \overline{T}$ must also satisfy $\varphi(S) \subset \overline{\varphi(T)}$. We also know that φ restricts to a symplectomorphism $S \to \varphi(S)$ for all symplectic strata $S \subset M /\!/_\xi G$, as follows from (iii). In other words, an isomorphism in the sense of Definition 13 gives rise to an isomorphism between the refined symplectic stratifications on $M /\!/_\xi G$ and $N /\!/_\eta K$.

5.2. *A more general abelianization theorem.* Let us continue with the notation and conventions set in Section 4, as well as those in Section 5.1 concerning stratified symplectic spaces.

In preparation for our next proposition, we encourage the reader to recall Proposition 7 and Corollary 10.

Proposition 15. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *The association $H \mapsto \tau_\xi(H)$ defines a bijection*

$$\text{Stab}(G, \xi) \xrightarrow{\cong} \text{Stab}(\mathbb{T}_{\text{big}}, \lambda_{\text{big}}(\xi)).$$

(ii) *If $H \subset G_\xi$ is a closed subgroup, then δ_ξ restricts to a diffeomorphism*

$$\mu^{-1}(\xi)_H \times \mathbb{T}_{\text{int}} \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\tau_\xi(H)}.$$

Proof. As in the proof of Theorem 11(i), we have

$$\delta_\xi((g, t) \cdot x) = \rho_\xi(g, t) \cdot \delta_\xi(x)$$

for all $(g, t) \in G_\xi \times \mathbb{T}_{\text{int}}$ and $x \in \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}$. It follows that $K \mapsto \rho_\xi(K)$ defines a bijection

$$\text{Stab}(G_\xi \times \mathbb{T}_{\text{int}}, \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}) \xrightarrow{\cong} \text{Stab}(\mathbb{T}_{\text{big}}, \lambda_{\text{big}}(\xi)),$$

and that δ_ξ restricts to a homeomorphism

$$(\mu^{-1}(\xi) \times \mathbb{T}_{\text{int}})_K \xrightarrow{\;\cong\;} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\rho_\xi(K)}$$

for all closed subgroups $K \subset G_\xi \times \mathbb{T}_{\text{int}}$. On the other hand, we clearly have a bijection

$$\text{Stab}(G, \xi) \xrightarrow{\;\cong\;} \text{Stab}(G_\xi \times \mathbb{T}_{\text{int}}, \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}), \quad H \mapsto H \times \{e\} \subset G_\xi \times \mathbb{T}_{\text{int}}.$$

We also note that

$$(\mu^{-1}(\xi) \times \mathbb{T}_{\text{int}})_K = \mu^{-1}(\xi)_H \times \mathbb{T}_{\text{int}} \quad \text{and} \quad \rho_\xi(K) = \tau_\xi(H)$$

for $K = H \times \{e\}$. These last three sentences combine to imply the desired results. $\square$

The following is our generalization of Theorem 11 to stratified symplectic spaces.

Theorem 16. *If $\xi \in \mathfrak{g}_{\text{s-reg}}^*$, then there is a canonical isomorphism*

$$M /\!\!/_\xi G \cong M_{\text{s-reg}} /\!\!/_{\lambda_{big}(\xi)} \mathbb{T}_{\text{big}}$$

of stratified symplectic spaces.

Proof. By Corollary 10 and Proposition 15, the inclusion $\mu^{-1}(\xi) \hookrightarrow \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ descends to a homeomorphism

$$f : \mu^{-1}(\xi) \xrightarrow{\;\cong\;} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}},$$

whose restriction to $\mu^{-1}(\xi)_H$ is a diffeomorphism

$$\mu^{-1}(\xi)_H \xrightarrow{\;\cong\;} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\tau_\xi(H)}/\mathbb{T}_{\text{int}}$$

for all $H \in \text{Stab}(G, \xi)$. We also note that the $\mathbb{T}_{\text{big}}$-action on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ induces a residual action of the subtorus $\mathbb{T}_{\text{small}}$ on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}}$. Proposition 8 then tells us that

$$f(g \cdot m) = \tau_\xi(g) \cdot f(m)$$

for all $g \in G_\xi$ and $m \in \mu^{-1}(\xi)$. The map f therefore descends to a homeomorphism

$$\varphi : M /\!\!/_\xi G \xrightarrow{\;\cong\;} M_{\text{s-reg}} /\!\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}},$$

whose restriction to $(\mu^{-1}(\xi)_H)/G_\xi$ is a diffeomorphism

$$\varphi_H : (\mu^{-1}(\xi)_H)/G_\xi \xrightarrow{\;\cong\;} (\lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\text{big}}$$

for all $H \in \text{Stab}(G, \xi)$.

Now consider the bijection

$$\phi : \text{Stab}(G, \xi) \xrightarrow{\;\cong\;} \text{Stab}(\mathbb{T}_{\text{big}}, \lambda_{\text{big}}(\xi)), \quad H \mapsto \tau_\xi(H)$$

from Proposition 15(i). We claim that φ and ϕ define an isomorphism of stratified symplectic spaces, in the sense of Definition 13. In light of the previous

paragraph, it suffices to prove the following for all $H \in \mathrm{Stab}(G, \xi)$: φ_H pulls the symplectic form β on $(\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}$ back to the symplectic form α on $(\mu^{-1}(\xi)_H)/G_\xi$.

Proposition 15(ii) implies that $\mu^{-1}(\xi)_H \subset \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))$. This leads to the commutative diagram

$$
\begin{array}{ccc}
\mu^{-1}(\xi)_H & \xrightarrow{\ \ \mathrm{j}\ \ } & \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \theta} \\
(\mu^{-1}(\xi)_H)/G_\xi & \xrightarrow[\ \varphi_H\ ]{} & (\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}
\end{array}
$$

where

$$
\pi : \mu^{-1}(\xi)_H \to (\mu^{-1}(\xi)_H)/G_\xi
$$

and

$$
\theta : \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)} \to (\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}
$$

are the canonical quotient maps and $\mathrm{j} : \mu^{-1}(\xi)_H \hookrightarrow \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))$ is the inclusion. We also have inclusion maps $\mathrm{k} : \mu^{-1}(\xi) \hookrightarrow M$ and $\mathrm{l} : \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi)) \hookrightarrow M$. Another consideration is that α (resp. β) is the unique 2-form on $(\mu^{-1}(\xi)_H)/G_\xi$ (resp. $(\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}$) for which $\pi^*\alpha = \mathrm{k}^*\omega$ (resp. $\theta^*\beta = \mathrm{l}^*\omega$). It therefore suffices to prove that $\pi^*(\varphi_H^*\beta) = \mathrm{k}^*\omega$. On the other hand, our commutative diagram implies that

$$
\pi^*(\varphi_H^*\beta) = \mathrm{j}^*(\theta^*\beta) = \mathrm{j}^*(\mathrm{l}^*\omega) = \mathrm{k}^*\omega. \qquad \square
$$

Acknowledgements

References

[1] J. J. Duistermaat, "On global action-angle coordinates", *Comm. Pure Appl. Math.* **33**:6 (1980), 687–706. MR Zbl

[2] V. Guillemin and S. Sternberg, "The moment map and collective motion", *Ann. Physics* **127**:1 (1980), 220–253. MR Zbl

[3] V. Guillemin and S. Sternberg, "The Gelfand–Cetlin system and quantization of the complex flag manifolds", *J. Funct. Anal.* **52**:1 (1983), 106–128. MR Zbl

[4] V. Guillemin and S. Sternberg, "On collective complete integrability according to the method of Thimm", *Ergodic Theory Dynam. Systems* **3**:2 (1983), 219–230. MR Zbl

[5] V. Guillemin, L. Jeffrey, and R. Sjamaar, "Symplectic implosion", *Transform. Groups* **7**:2 (2002), 155–184. MR Zbl

[6] M. Harada, "The symplectic geometry of the Gelfand–Cetlin–Molev basis for representations of Sp($2n$, $\mathbb{C}$)", *J. Symplectic Geom.* **4**:1 (2006), 1–41. MR Zbl

[7] M. Harada and K. Kaveh, "Integrable systems, toric degenerations and Okounkov bodies", *Invent. Math.* **202**:3 (2015), 927–985. MR Zbl

[8] B. Hoffman and J. Lane, "Stratified gradient Hamiltonian vector fields and collective integrable systems", preprint, 2023. arXiv 2008.13656

[9] J. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetry", *Rep. Mathematical Phys.* **5**:1 (1974), 121–130. MR Zbl

[10] R. Sjamaar and E. Lerman, "Stratified symplectic spaces and reduction", *Ann. of Math.* (2) **134**:2 (1991), 375–422. MR Zbl

PETER CROOKS
DEPARTMENT OF MATHEMATICS AND STATISTICS
UTAH STATE UNIVERSITY
LOGAN, UT
UNITED STATES

peter.crooks@usu.edu

JONATHAN WEITSMAN
DEPARTMENT OF MATHEMATICS
NORTHEASTERN UNIVERSITY
BOSTON, MA
UNITED STATES

j.weitsman@northeastern.edu

PACIFIC JOURNAL OF MATHEMATICS

PACIFIC JOURNAL OF MATHEMATICS

Volume 333   No. 2   December 2024