Pacific Journal of Mathematics

DIVERGENCE FUNCTIONS OF HIGHER-DIMENSIONAL THOMPSON'S GROUPS

YUYA KODAMA

Volume 333 No. 2 December 2024

PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.309

DIVERGENCE FUNCTIONS OF HIGHER-DIMENSIONAL THOMPSON'S GROUPS

YUYA KODAMA

We prove that higher-dimensional Thompson's groups have linear divergence functions. By the work of Druţu, Mozes, and Sapir, this implies none of the asymptotic cones of nV has a cut-point.

1. Introduction

Thompson's groups F, T, and V are finitely presented infinite groups defined by Richard Thompson in the 1960s. They are all known to be mysterious groups. For example, T and V are the first examples of finitely presented, infinite, and simple, and it is known that the amenability of F is a difficult open problem. Because they have several unpredictable properties, by focusing on such properties, many "generalized" Thompson's groups were also defined. Higher-dimensional Thompson's groups, denoted by $2V, 3V, \ldots$, are some such groups defined by Brin [3]. The group V acts on the Cantor set $\mathfrak{C}$, and the group nV acts on the powers of the Cantor set $\mathfrak{C}^n$. It is known that nV is also finitely presented [4; 14] and simple [3; 5]. In addition, it was shown that for $n, m \in \mathbb{Z}_{>0}$, the group nV is isomorphic to mV if and only if $n = m$ holds [1]. In [16], it was proved that nV has Serre's property FA, and hence is one-ended.

In 2018, Golan and Sapir showed that the divergence functions of F, T, and V are linear [12]. This function was first mentioned by Gromov [13], and later, Gersten gave the formal definition [11] as a quasi-isometric invariant of geodesic metric spaces. Roughly speaking, the order of the function indicates whether the Cayley graphs of the group are "close" to the Euclidean or hyperbolic spaces. In fact, the orders of the functions of the direct powers of the infinite cyclic group $\mathbb{Z}^2, \mathbb{Z}^3, \ldots$ are linear, and it is known that the orders of the functions of hyperbolic groups are at least exponential [2]. In [12], they asked whether their proof could be extended to generalized Thompson's groups. In recent years, similar results have been obtained for some groups by extending the original arguments [17; 19; 18]. For recent results on functions of groups other than generalized Thompson's groups, see [15].

MSC2020: 20F65.

Keywords: divergence function, higher-dimensional Thompson's groups, asymptotic cones.

In this paper, we also extend the argument given by Golan and Sapir. That is, we show the following:

Theorem 1.1. *Higher-dimensional Thompson's groups have linear divergence functions.*

This paper is organized as follows. In Section 2, we summarize the definition of the higher-dimensional Thompson's group $2V$ and the notion of divergence functions. In Section 3, we first prepare estimates of the word length of elements in $2V$ to ensure that the condition required by the definition of the divergence function is satisfied. Subsequently, we construct a "good" path from any $g \in 2V$ to a specific element in $2V$, where the element is determined only from the word length of g. It should be noted that we assume $n = 2$ in most of this paper; however, all the proofs can be generalized to n with the appropriate modifications.

2. Preliminaries

2.1. ***Definition of the higher-dimensional Thompson's group 2V.*** In this section, we define only the higher-dimensional Thompson's group $2V$. Readers unfamiliar with the Thompson's groups are referred to [8].

2.1.1. *Homeomorphisms on the direct product of Cantor sets.* Let $\mathfrak{C}$ be the Cantor set $\{0, 1\} \times \{0, 1\} \times \cdots$. For a finite word w on $\{0, 1\}$ and a finite or infinite word ζ on $\{0, 1\}$, let $w\zeta$ denote their concatenation. Following [3], we will describe partitions of $\mathfrak{C}^2$ by using subdivisions of the unit square $[0, 1]^2$. Subsequently, we will define homeomorphisms from $\mathfrak{C}$ to itself that are obtained from such partitions.

Firstly, we call $[0, 1]^2$ itself the *trivial pattern*. Let us consider a rectangle $[a_1, a_2] \times [b_1, b_2] \subset [0, 1]^2$. By dividing this rectangle in half, we obtain two new rectangles. The way of obtaining rectangles $[a_1, (a_1 + a_2)/2] \times [b_1, b_2]$ and $[(a_1 + a_2)/2, a_2] \times [b_1, b_2]$ is called *vertical subdivision*, and the way of obtaining rectangles $[a_1, a_2] \times [b_1, (b_1 + b_2)/2]$ and $[a_1, a_2] \times [(b_1 + b_2)/2, b_2]$ is called *horizontal subdivision*. A *pattern* is defined as a finite set of rectangles in $[0, 1]^2$ obtained from the trivial pattern by applying finitely many vertical and horizontal subdivisions. See Figure 1. For a rectangle $[a_1, a_2] \times [b_1, b_2]$, the lengths $a_2 - a_1$ and $b_2 - b_1$ of its horizontal and vertical edges are called its *horizontal* and *vertical lengths*, respectively.

Next, by using patterns, we define partitions of $\mathfrak{C}^2$ inductively. The trivial pattern corresponds to $\mathfrak{C}^2$ itself. Assume that a rectangle in a pattern corresponds to a subset $\{w\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'\zeta_2 \mid \zeta_2 \in \mathfrak{C}\} \subset \mathfrak{C}^2$ where w and w' are finite words on $\{0, 1\}$. For the vertical subdivision, we define that the left rectangle corresponds to the subset $\{w0\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and the right rectangle corresponds to the subset $\{w1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$. Similarly, for the horizontal subdivision, we define that the bottom rectangle corresponds to the

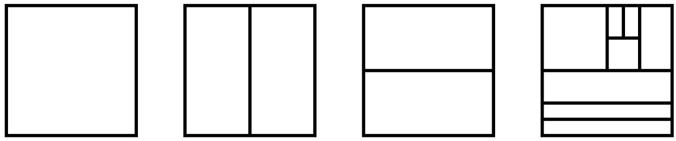


Figure 1. The trivial pattern, the pattern once divided vertically, the pattern once divided horizontally, and a pattern.

subset $\{w\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'0\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and the top rectangle corresponds to the subset $\{w\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'1\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$. Consequently, a set of rectangles of a pattern gives a partition of $\mathfrak{C}^2$. For example, the pattern once divided vertically illustrated in Figure 1 corresponds to $\{0\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \mathfrak{C} \cup \{1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \mathfrak{C}$. We often identify a partition of $\mathfrak{C}^2$ with the corresponding partition of $[0, 1]^2$.

For a pattern with m rectangles, we assign a number from 0 to $m-1$ to each rectangle. Such a pattern is called *numbered pattern*. Let P and P' be numbered patterns with m rectangles. Let R_i (resp. R'_i) be a rectangle of P (resp. P') numbered i. Then there exist two subsets $\{a_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and $\{a'_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b'_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ corresponding to R_i and R'_i, respectively. By mapping each element $(a_i\zeta_1, b_i\zeta_2) \in \{a_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ to $(a'_i\zeta_1, b'_i\zeta_2) \in \{a'_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b'_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, we obtain a homeomorphism on $\mathfrak{C}^2$.

Definition 2.1. *The higher-dimensional Thompson's group* $2V$ is the subgroup of $\mathrm{Homeo}(\mathfrak{C}^2)$ consisting of all homeomorphisms obtained from pairs of numbered patterns with the same number of rectangles.

Following a familiar convention, we write fg for $g \circ f$; namely, we always consider the right action of $2V$ on $\mathfrak{C}^2$.

The *domain* and *target pattern* of a pair of numbered patterns are defined as the patterns that determine the partition of the domain and range set of $\mathfrak{C}^2$, respectively.

Remark 2.2. The group nV is defined similarly as a subgroup of $\mathrm{Homeo}(\mathfrak{C}^n)$. Using the unit n-cube instead of the unit square, n-subdivisions are defined, which yield partitions of $\mathfrak{C}^n$.

Note that two distinct pairs of numbered patterns may give the same map. Let (P, P') be a pair of numbered patterns with the same number of rectangles (such a pair is just called a *pair of numbered patterns*). Let R_i and R'_i be rectangles of P and P' numbered i, respectively. Apply a vertical subdivision to R_i, and assign i_1 to the left rectangle and i_2 to the right rectangle. Do the same to R'_i and assign i_1 and i_2. Consequently, the obtained maps from the two pairs are the same. The same also holds for the case of horizontal subdivisions. A *reduced* pair of numbered patterns is a pair of numbered patterns where the inverse operations (called *vertical* and *horizontal reductions*) cannot be applied to any rectangles.

In order to multiply two pairs of numbered patterns (P_+, P_-) and (Q_+, Q_-), take a common refinement P of P_- and Q_+, and by using the operations, construct pairs (P'_+, P) and (P, Q'_-) such that they give the same maps as (P_+, P_-) and (Q_+, Q_-), respectively. Then the pair (P'_+, Q'_-) is the desired one.

Unfortunately, unlike the original Thompson's groups, there is no known way to define a unique reduced numbered pair for each element in $2V$. The notion of grid diagrams provides a solution to this issue. A detailed explanation will be given in Section 2.1.3.

2.1.2. *Pairs of colored binary trees.* Just as elements of the Thompson's groups are represented by pairs of binary trees, there exists a similar approach for $2V$. Like numbered patterns, the binary trees do not give unique representatives for elements of $2V$. However, because they can represent elements more simply than numbered patterns, we will frequently use colored binary trees in this paper.

We always assume that binary trees are rooted; namely, they have one specific vertex called the *root*. Vertices whose degree is one are called *leaves*. A graph with only one vertex (and no edges) is also regarded as a binary tree. We define a *caret* to be the binary tree consisting of three vertices, where the degree of the root is two and of the remaining vertices is one. See Figure 2.

Any binary tree can be constructed inductively by attaching carets to leaves. A *colored binary tree* is a binary tree where each caret is colored. We use two colors $\{a, b\}$ for $2V$ and use n colors for nV. We first explain the relationship between colored binary trees and patterns.

As previously stated, any pattern can be obtained inductively from the trivial pattern. Similarly, corresponding binary trees are also defined inductively. In the case of the trivial pattern, the corresponding binary tree is the one that consists of only the root. More precisely, consider the root as a leaf, the trivial pattern as a rectangle, and then this rectangle corresponds to this leaf. Since there exists no caret in this tree, we do not color it. Assume that for a pattern, there exists a colored binary tree with leaves such that there is a one-to-one correspondence between the set of rectangles and the set of leaves. Let R be a rectangle of this

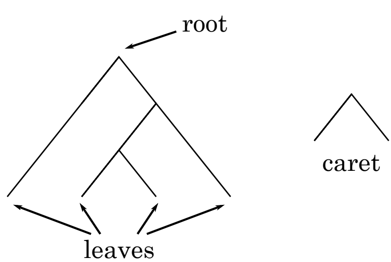


Figure 2. A binary tree and a caret.

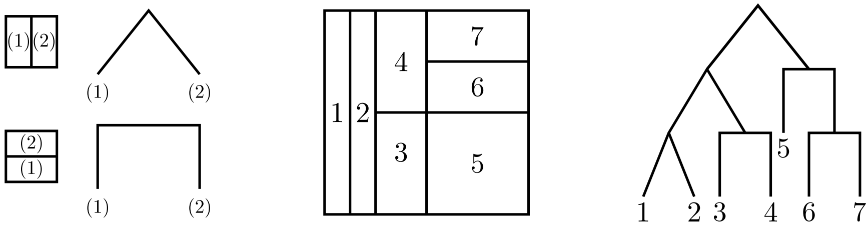


Figure 3. The two types of carets and a colored binary tree obtained from a pattern.

pattern, and assume that the corresponding leaf is i-th (from left to right). For the vertical subdivision of R, let R_1 be the left rectangle and R_2 be the right rectangle. Subsequently, we attach a caret colored by a to the i-th leaf and let R_1 correspond to the newly created left leaf and R_2 to the right leaf. Similarly, for the horizontal subdivision, we attach a caret colored by b to the i-th leaf and correspond the bottom rectangle to the left leaf and the top rectangle to the right leaf.

Following [6], in this paper, we represent carets colored by a by the "triangular carets" and carets colored by b by "square carets." See Figure 3.

The numbers in the rectangles and under the leaves represent the one-to-one correspondence between the set of leaves and rectangles. Observe that numbered patterns can also be represented by colored binary trees with numbers by writing numbers under the leaves in the same way. Consequently, each element in $2V$ can be represented by a pair of colored binary trees with numbers. A tree corresponding to a domain (resp. target) pattern is called a *domain* (resp. *target*) *tree*.

Note that, in general, more than one colored binary tree may give the same pattern. This is due to the fact that applying the following operations to a rectangle in a pattern are the same: the pattern obtained by subdividing vertically once and then subdividing two rectangles horizontally; subdividing horizontally once and then subdividing two rectangles vertically. See Figure 4.

As with pairs of numbered patterns, it is difficult to easily obtain a "good" pair of colored binary trees for each element of $2V$. However, we can use this to give an estimation of the word length, as we will see in Section 2.2. For a colored binary tree T, we define a *branch* of T to be a path from the root of T to a leaf of T. The

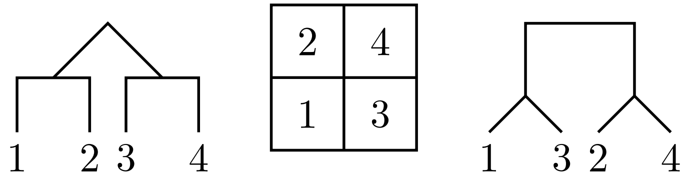


Figure 4. Two colored binary trees give the same pattern.

depth of T is then defined as the maximum length of the branches of T (with each edge having length one). It is clear that for any element g in $2V$, there exists a pair of colored binary trees giving g. Therefore, we may define the minimality of such pairs. First, we define a pair of colored binary trees to be *shallow* if it is a pair of colored binary trees whose target tree has the smallest depth among pairs that give the same map. A pair of colored binary trees is then defined as *minimal* if it is a shallow pair of binary trees with the smallest number of carets. It should be noted that such pairs may not be uniquely determined.

Remark 2.3. To make the relationship between grid diagrams and pairs of colored binary trees clearer, the definition of minimality is slightly modified from that in [6].

2.1.3. *Grid diagrams.* In this section, we explain how to represent each element in $2V$ using a "grid" based on [7]. However, unlike [7], the "grid" is constructed on the target patterns for the sake of our proof. Note that in [7, the second paragraph of Section 6], it was also pointed out that the same result holds in our setting. Indeed, we use only the fact that there exists a unique representative for each element in $2V$ (and nV). In this case, we only need to change the pattern we focus on from domain to target. Hence we omit most of the proofs in this section.

Definition 2.4. A *grid pattern* is defined as a pattern obtained by subdividing the unit square using only line segments of length one.

The pattern illustrated in Figure 3 is not a grid pattern, since the lengths of the horizontal line segment on $[1/4, 1] \times \{1/2\}$ and the horizontal line segment between rectangles numbered 6 and 7 are less than one. An example is illustrated in Figure 5.

Definition 2.5. A *grid diagram* is a pair of numbered patterns with the same number of rectangles, and the target pattern is a grid pattern.

Proposition 2.6 [7, Proposition 2.3]. *Each element in* $2V$ *admits a grid diagram as a representative.*

Sketch of proof. We can repeat horizontal and vertical subdivisions (so that the pairs of numbered patterns obtained always give the same map) until the target pattern becomes a grid pattern. □

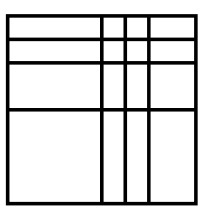


Figure 5. A grid pattern.

Moreover, it is known that there exists a unique representative for any element in $2V$. In order to give such a representative, we first recall the definition that a grid diagram is reduced.

Let (P, G) be a grid diagram where G is a grid pattern. In G, take a strip $I_i \times [0, 1] \subset [0, 1] \times [0, 1]$ where I_i are horizontal edges of the rectangles in G. In other words, take a rectangle of G where the rectangle is $I_i \times J_j \subset [0, 1] \times [0, 1]$, and then consider $I_i \times [0, 1]$. We apply vertical subdivisions to all rectangles in $I_i \times [0, 1]$, and then apply vertical subdivisions to the corresponding rectangles in P such that the obtained grid pattern and (P, G) give the same homeomorphism on $\mathfrak{C}^2$. We call this operation a *vertical global subdivision*. We say a grid diagram is *vertically reduced* if the inverse operation of the vertical global subdivision cannot be applied to any two adjacent strips of the target pattern. Similarly, we can define a *horizontal global subdivision* and a grid diagram to be *horizontally reduced*. Finally, a grid diagram is said to be *reduced* if it is vertically and horizontally reduced. Then we have the following:

Theorem 2.7 [7, Theorem 3.2]. *Any element in $2V$ has a unique reduced grid diagram.*

Note that since each reduced grid diagram is a pair of numbered patterns, there exist corresponding pairs of colored binary trees.

2.2. ***A generating set of $2V$ and estimations of word length.*** Consider a finite set

$$X_{2V} := \left\{x_0, x_1, x_2, y_i, B_i, C_i, \hat{x}_j, \hat{y}_1, \pi_i, \overline{\pi_i}, \alpha_i, \beta_i, \hat{B}_0, \gamma_0, {}_hx_j, {}_h\hat{x}_j \;\middle|\; i \in \{0, 1\}, j \in \{1, 2\}\right\}$$

defined in Figures 6, 7 and 8. The colored binary trees without numbers are all assigned $1, 2, \ldots$ to the leaves from left to right. Because this set contains well-known generating sets (cf. [3; 4; 6]), this also generates $2V$. Note that this set is an inefficient set specialized for calculating the divergence function of $2V$.

Following [6], we also express an element of $2V$ as "$P\varPi Q^{-1}$ form" and use a part of the form in the proof of the main theorem. We define $A_i := A_0^{-(i-1)} A_1 A_0^{(i-1)}$ for $i \geq 1$ and $B_i := B_0^{-(i-1)} B_1 B_0^{(i-1)}$ for $i \geq 1$. Then the following holds:

Theorem 2.8 [6, Theorem 2.2]. *For an element g of $2V$, we have its expression $P\varPi Q^{-1}$, where*

(1) *P and Q are represented by words of the form*

$$C_{m_1} \cdots C_{m_p} W_{i_1} \cdots W_{i_r},$$

where C_i are in Figure 7, W_i are words on $\{A_i, B_i\}$ without inverse elements, $m_1 < m_2 < \cdots < m_p$, and $i_1 < i_2 < \cdots < i_r$.

(2) *$\varPi$ is represented by a word on $\{\pi_0, \pi_1, \ldots\} \cup \{\overline{\pi_0}, \overline{\pi_1}, \ldots\}$.*

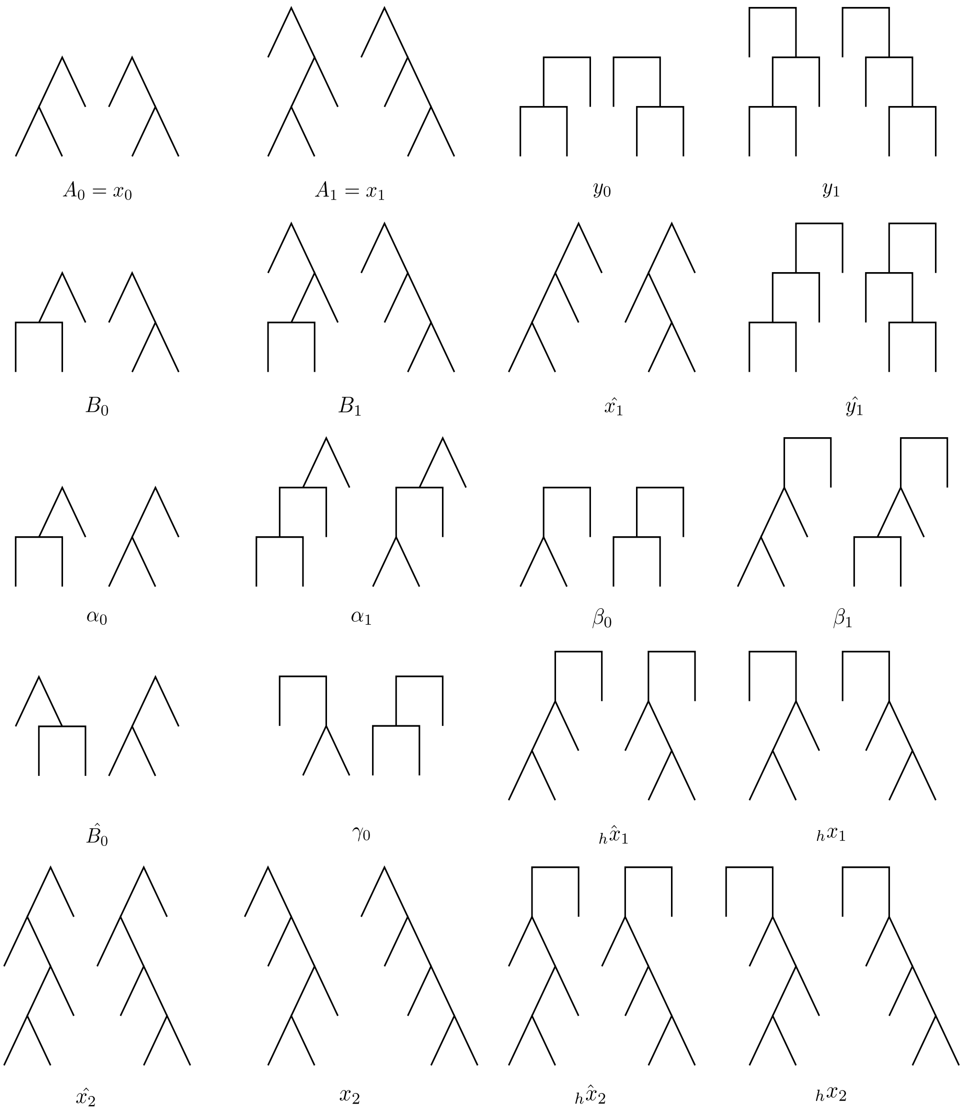


Figure 6. Generators of $2V$ except C_0, C_1, π_0, π_1, $\overline{\pi_0}$, and $\overline{\pi_1}$.

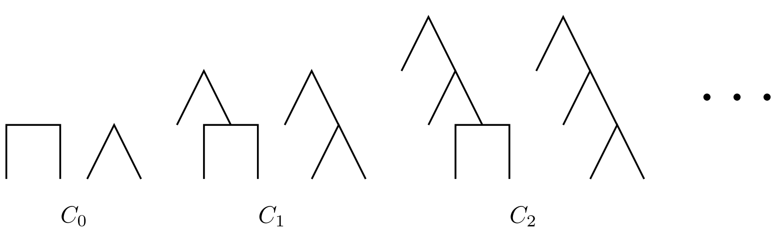


Figure 7. Generators $C_0, C_1, \ldots$.

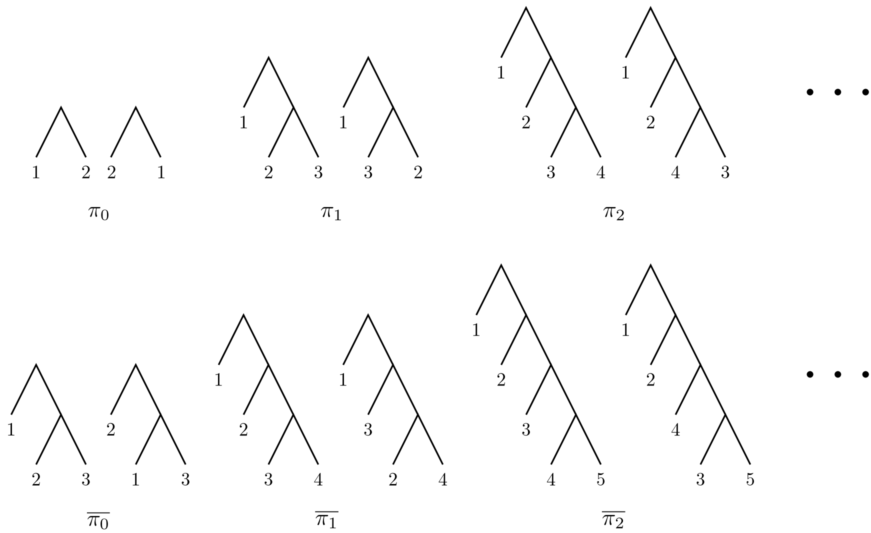


Figure 8. Generators $\pi_0, \pi_1, \ldots$ and $\overline{\pi_0}, \overline{\pi_1}, \ldots$.

In [6], Burillo and Cleary constructed this expression for each pair of colored binary trees. We also use this construction. See the proof of [6, Theorem 2.1] for details. Note that for element $g \in 2V$, if we take an expression $P \Pi Q^{-1}$ corresponding to a minimal pair of colored binary trees (T_+, T_-), then we have $P = (T_+, R_k)$, $\Pi = (R_k, \pi, R_k)$, and $Q = (T_-, R_k)$ where π is the corresponding permutation and R_k is the all-right tree with only vertical carets and k is the depth of g. Here, an *all-right tree* is a binary tree obtained by a finite number of attaching a caret only to the right leaf of a caret, and an all-right tree consisting of two types of carets is called *all-right colored binary tree*.

We will consider the word metric with respect to the above generating set. We first recall a known result of an estimation of the word length for pairs of binary trees.

Proposition 2.9 (cf. [6, Lemma 4.1]). *For an element of $2V$ which is represented by a minimal pair of colored binary trees with depth D, its word length with respect to X_{2V} is at least $D/4$.*

Remark 2.10. The difference between the denominators in [6, Lemma 4.1] and this lemma arises from the difference in the generating sets. The definition of minimality is also different, yet this lemma can still be shown similarly: if we take one of the shortest words with length n, we can say that the depth is at most $4n$ since each process of multiplying a generator increases the length of the branches by at most four.

As previously stated, a minimal pair of colored binary trees may not be uniquely determined for an element in $2V$. Hence, we use the notion of grid diagrams. We define the *size* of a rectangle $R = \{[a_i, a_{i+1}] \times [b_j, b_{j+1}]\}$ of a pattern as $-(\log_2(a_{i+1} - a_i) + \log_2(b_{j+1} - b_j))$ and write it as $\|R\|$. For a pattern P, the *fineness* of P is defined as the largest size of the rectangles in P and written as $\|P\|$. For an element $g \in 2V$, the *fineness* of g is defined as $\|G\|$, where G is the target numbered pattern of the reduced grid diagram of g. Note that the fineness of g is the same as the depth of the target tree of a pair of colored binary trees obtained from the grid diagram of g. Indeed, at the end of Section 2.1.2, we defined the depth of a pair of trees as the depth of the target tree. In the proof of the main theorem, we also use the following estimations:

Proposition 2.11. *Let g be in $2V$ with fineness k. Then the word length of g with respect to X_{2V} is at least $k/8$.*

Proof. Consider a minimal pair of colored binary trees of g and let D be its depth. Take a pair of numbered patterns (P_1, P_2) corresponding to this pair of colored binary trees. Then, we have $\|P_2\| = D$. We construct a grid diagram (P, G) of g from (P_1, P_2) without vertical and horizontal global subdivisions. Note that (P, G) may not be reduced. Since each rectangle in G has horizontal and vertical lengths of at least $1/2^D$, the fineness of G is at most $2D$. By considering the reduced grid diagram of g, the fineness of g is at most $\|G\|$, namely, $k \leq \|G\|$ holds. By Proposition 2.9, the word length of g is at least $D/4$. Hence, the desired result is obtained. □

By a similar argument, the following also holds:

Corollary 2.12. *Let (P_1, P_2) be a pair of numbered patterns corresponding to $g \in 2V$, and $R = \{w_1 x_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2 x_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ be a rectangle of P_2 where w_1, w_2 are words on $\{0, 1\}$ and x_1, x_2 are in $\{0, 1\}$. Assume that neither of the following conditions (1) nor (2) is satisfied:*

(1)
- $g^{-1}(R) = \{w'_1 x_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, *where w'_1 and w'_2 are some word on $\{0, 1\}$, and*
- $g^{-1}((w_1 \hat{x}_1 \zeta_1, w_2 x_2 \zeta_2)) = (w'_1 \hat{x}_1 \zeta_1, w'_2 \zeta_2)$ *for every* $(w_1 \hat{x}_1 \zeta_1, w_2 x_2 \zeta_2) \in \{w_1 \hat{x}_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2 x_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, *where $\hat{x}_1$ is in $\{0, 1\}$ with $\hat{x}_1 \neq x_1$.*

(2)
- $g^{-1}(R) = \{w'_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'_2 x_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, *where w'_1 and w'_2 are some word on $\{0, 1\}$, and*
- $g^{-1}((w_1 x_1 \zeta_1, w_2 \hat{x}_2 \zeta_2)) = (w'_1 \zeta_1, w'_2 \hat{x}_2 \zeta_2)$ *for every* $(w_1 x_1 \zeta_1, w_2 \hat{x}_2 \zeta_2) \in \{w_1 x_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2 \hat{x}_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, *where $\hat{x}_2$ is in $\{0, 1\}$ with $\hat{x}_2 \neq x_2$.*

Then, the word length of g is at least $\|R\|/8$.

Proof. From the assumptions, the fineness of g is at least $\|R\|$. By Proposition 2.11, we have the desired result. □

We say a rectangle R is an *essential* rectangle if it satisfies the assumptions of Corollary 2.12. We call *horizontal* and *vertical twin* the subsets $R^v := \{w_1\hat{x}_1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2x_2\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and $R^h := \{w_1x_1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2\hat{x}_2\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, respectively, which are defined from $R = \{w_1x_1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2x_2\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ in the assumptions of Corollary 2.12. Then the assumptions mean that we cannot apply vertical and horizontal reductions to R and "congruent rectangles" adjacent to R. In other words, the assumptions are that R^v is not mapped to the $(g^{-1}(R))^v$ with keeping the orientation and R^h is not mapped to $(g^{-1}(R))^h$ with keeping the orientation.

2.3. ***Divergence functions of finitely generated groups.*** For finitely generated groups, the property of having linear divergence functions is a quasi-isometric invariant. Since we see asymptotic properties of functions, we introduce an equivalence relation on functions from $\mathbb{R}_{>0}$ to itself as follows: let f and g be functions from $\mathbb{R}_{>0}$ to itself. We first define $f \preceq g$ if there exist $A, B, C, D, E \geq 0$ such that

$$f(x) \leq Ag(Bx + C) + Dx + E$$

holds for all $x \in \mathbb{R}_{>0}$. Then we define $f \approx g$ if $f \preceq g$ and $g \preceq f$ hold. This is an equivalence relation on the set of functions from $\mathbb{R}_{>0}$ to itself. Note that all linear functions and constant functions are equivalent.

Let G be a finitely generated group with a finite generating set X. Let Γ be the Cayley graph of G with respect to X. For $\delta \in (0, 1)$, we first define the δ-divergence function of Γ as follows: let $\Omega(g_1, g_2)$ be the set of all paths connecting $g_1, g_2 \in \Gamma$. Let $\|\omega\|$ denote the length of the path ω. Then for $x \in \mathbb{R}_{>0}$, define the function ϕ_δ by setting

$$\phi_\delta(x) := \max\left\{ \min\{\|\omega\| \mid \omega \in \Omega(g_1, g_2) \text{ avoiding } B(e, \delta x)\} \;\middle|\; |g_1| = |g_2| = x \right\},$$

where $B(e, \delta x)$ denotes the open ball of radius δx with a center at identity of G, and $|g_1|, |g_2|$ denote the lengths from identity of G to g_1, g_2 in Γ. If there does not exist such a path, take $\phi_\delta(x) = \infty$. For each $\delta \in (0, 1)$, the equivalence class of ϕ_δ is invariant under quasi-isometries [11]. Hence, the δ-divergence function of G is well-defined as an equivalence class of functions.

Definition 2.13. We say *a group G has a linear divergence function* if there exists $\delta \in (0, 1)$ such that the δ-divergence function of G is in the equivalence class of linear maps.

From this definition, it is clear that if the δ-divergence function of G is in the equivalence class of linear maps, then the δ'-divergence function of G is also in the equivalence class of linear functions for every $0 < \delta' \leq \delta$. In fact, Druţ, Mozes, and Sapir [9; 10] showed the following theorem, which establishes a relationship between divergence functions and a topological property of asymptotic cones.

Theorem 2.14 [9; 10]. *The following are equivalent*:

(1) *G has a linear divergence function.*

(2) *For every $\delta \in (0, 1/54)$, the function ϕ_δ is in the equivalence class of linear functions.*

(3) *None of the asymptotic cones of G has a cut point.*

3. Proof of Theorem 1.1

3.1. ***Size of the bottom left rectangle.*** For a given pattern P, let $R_0(P)$ be defined as the rectangle of P containing the point $(0, 0) \in [0, 1] \times [0, 1]$. We first study the change of the word length when we multiply some of the generators in X_{2V}.

Lemma 3.1. *Let (P_+, P_-) be a pair of numbered patterns representing $g \in 2V$. Assume that $R_0(P_-)$ is essential and a subset of* $[0, 1/4] \times [0, 1]$. *Then*:

(1) *There exists a pair of numbered patterns $(P_+(gx_0^{-1}), P_-(gx_0^{-1}))$ representing gx_0^{-1} such that $R_0(P_-(gx_0^{-1}))$ is essential, $R_0(P_-(gx_0^{-1})) \subset [0, 1/8] \times [0, 1]$ and $\|R_0(P_-(gx_0^{-1}))\| = \|R_0(P_-)\| + 1$.*

(2) *There exists a pair of numbered patterns $(P_+(g\hat{B}_0), P_-(g\hat{B}_0))$ representing $g\hat{B}_0$ such that $R_0(P_-(g\hat{B}_0))$ is essential, $R_0(P_-(g\hat{B}_0)) \subset [0, 1/8] \times [0, 1]$ and $\|R_0(P_-(g\hat{B}_0))\| = \|R_0(P_-)\| + 1$.*

(3) *If $R_0(P_-)$ is a subset of $[0, 1/4] \times [0, 1/2]$, then there exists a pair of numbered patterns $(P_+(gC_0), P_-(gC_0))$ representing gC_0 such that $R_0(P_-(gC_0))$ is essential, $R_0(P_-(gC_0)) \subset [0, 1/8] \times [0, 1]$ and $\|R_0(P_-(gC_0))\| = \|R_0(P_-)\|$.*

Proof. For part (1), we have $R_0(P_-) \cup (R_0(P_-))^v \cup (R_0(P_-))^h \subset [0, 1/2] \times [0, 1]$ if $(R_0(P_-))^h$ is defined. Consider the composition of pairs of patterns. The rectangle $R_0(P_-)$ is unchanged even if the common refinement is taken. Hence, $x_0^{-1}(R_0(P_-))$ is an essential rectangle of a pair of patterns representing gx_0^{-1}. The remaining claims are clear from the definition of x_0^{-1} and because if $(R_0(P_-))^h$ is not defined, neither is $(x_0^{-1}(R_0(P_-)))^h$. By the same argument, part (2) also follows.

For part (3), if $R_0(P_-)$ is a subset of $[0, 1/4] \times [0, 1/4]$, then it is also followed by a similar argument of the proof of part (1). If $R_0(P_-)$ is $[0, a] \times [0, 1/2]$ for some a with $a \leq 1/4$, then $C_0(R_0(P_-))^h$ is not defined. Hence, we also have the desired result. □

By a similar argument, we also have the following:

Lemma 3.2. *Let (P_+, P_-) be a pair of numbered patterns representing $g \in 2V$. Assume that $R_0(P_-)$ is essential and a subset of* $[0, 1] \times [0, 1/4]$. *Then*:

(1) *There exists a pair of numbered patterns* $(P_+(gy_0^{-1}), P_-(gy_0^{-1}))$ *representing* gy_0^{-1} *such that* $R_0(P_-(gy_0^{-1}))$ *is essential*, $R_0(P_-(gy_0^{-1})) \subset [0, 1] \times [0, 1/8]$ *and* $\|R_0(P_-(gy_0^{-1}))\| = \|R_0(P_-)\| + 1$.

(2) *There exists a pair of numbered patterns* $(P_+(g\gamma_0), P_-(\gamma_0))$ *representing* $g\gamma_0$ *such that* $R_0(P_-(g\gamma_0))$ *is essential*, $R_0(P_-(g\gamma_0)) \subset [0, 1] \times [0, 1/8]$ *and* $\|R_0(P_-(g\gamma_0))\| = \|R_0(P_-)\| + 1$.

(3) *If* $R_0(P_-)$ *is a subset of* $[0, 1/2] \times [0, 1/4]$, *there exists a pair of numbered patterns* $(P_+(gC_0^{-1}), P_-(gC_0^{-1}))$ *representing* gC_0^{-1} *such that* $R_0(P_-(gC_0^{-1}))$ *is essential*, $R_0(P_-(gC_0^{-1})) \subset [0,1]\times[0,1/8]$ *and* $\|R_0(P_-(gC_0^{-1}))\| = \|R_0(P_-)\|$.

3.2. *Construction of the path.* In the rest of this paper, we write $|\cdot|$ for the word length of $2V$ with respect to X_{2V}. For a word w and w', we write $w \equiv w'$ when w and w' are the same as words, and $\|w\|$ denotes the length of w. If we have $w \equiv w'w''$ for some words w, w' and w'' (the word w'' may be the empty word), then w' is said to be a *prefix* of w and denoted by $w' \leq w$. Theorem 1.1 immediately follows from the following proposition:

Proposition 3.3. *There exist constants* δ, D *and a positive integer* Q *such that the following holds*: *let* $g \in 2V$ *with* $|g| \geq 4$. *Then there exists a path of length at most* $D|g|$ *in the Cayley graph of* $2V$ *which avoids a* $\delta|g|$*-neighborhood of the identity and which has the initial vertex* g *and the terminal vertex* $\hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$.

In other words, *there exists a word* ω *on the generating set such that* $\|\omega\| < D|g|$; *for any prefix* ω' *of* ω, *we have* $g\omega' > \delta|g|$ *such that*

$$g\omega = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}.$$

Proof. Let $(P_+(g), P_-(g))$ be a reduced pair of numbered patterns representing g such that $R_0(P_-(g))$ is essential. Note that $R_0(P_-(g))$ can be made essential by changing the choice of rectangles to be reduced if necessary. We will define six subwords, denoted by $\omega_1, \ldots, \omega_6$, and then the concatenation of these subwords, $\omega_1 \cdots \omega_6$, will be the desired word, ω.

In subpath 1, we may assume that $R_0(P_-(g))$ is a subset of $[0, 1/2] \times [0, 1]$. Indeed, since g is not the identity map, if not, we have $R_0(P_-(g)) \subset [0, 1] \times [0, 1/2]$. Then by replacing the "vertical argument" with the "horizontal one", it is possible to join the argument from subpath 3 onward. More precisely, we replace all vertical and horizontal carets in generators in subpaths 1, 2 with horizontal and vertical carets, respectively, and replace all the set $I_1 \times I_2 \subset [0, 1] \times [0, 1]$ in the definition of subpath 1 with $I_2 \times I_1$, and construct the $P\Pi Q^{-1}$ expression of g_1 by using an all-right tree with only horizontal carets instead of vertical carets. There are no generators in Figure 6 which generate Π constructed above, but this is not a problem since all we need is Q and the required generators are defined in Figure 6.

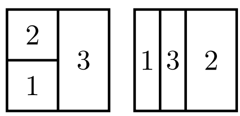
Subpath 1. We define ω_1 to be one of the following:

(a) If $R_0(P_-(g))$ is a subset of $[0, 1/4] \times [0, 1]$, then define ω_1 to be the empty word.

(b) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1]$, then define ω_1 to be $\hat{x}_1$.

(c) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2]$ and $\alpha_0(R_0(P_-(g)))$ is an essential rectangle of $g\alpha_0$, then define ω_1 to be α_0.

(d) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2]$ and $\alpha_0(R_0(P_-(g)))$ is not an essential rectangle of $g\alpha_0$, then define ω_1 to be $B_0\pi_1 x_0^{-1}$, as in the figure:



$B_0\pi_1 x_0^{-1}$

(e) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2^i]$ where $i \geq 2$ and $\alpha_0(R_0(P_-(g)))$ is an essential rectangle of $g\alpha_0$, then define ω_1 to be α_0.

(f) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2^i]$ where $i \geq 2$ and $\alpha_0(R_0(P_-(g)))$ is not an essential rectangle of $g\alpha_0$, then define ω_1 to be α_1.

Lemma 3.4. *In any case, g_1 has a reduced pair $(P_+(g_1), P_-(g_1))$ of numbered patterns such that $R_0(P_-(g_1))$ is essential and a subset of* $[0, 1/4] \times [0, 1]$. *Moreover, $\|\omega_1\| \leq 3$ holds and for any prefix $\omega' \leq \omega_1$, we have $|g\omega'| \geq |g|/4$.*

Proof. Since the latter statements are obvious, it is sufficient to show the first statement. But since cases (a), (c), (d), (e), and (f) of the remaining claim are also clear, we consider only case (b). Let $\{w_1\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$ be a rectangle corresponding to $g^{-1}(R_0(P_-(g))$. In the process of multiplying $\hat{x}_1$, the rectangle $g^{-1}(R_0(P_-(g))$ is subdivided into $\{w_1 00\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$, $\{w_1 01\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$, and $\{w_1 1\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$. Since the rectangles of a target pattern of $g\hat{x}_1$ corresponding to the first two are $\{00\zeta \mid \zeta \in \mathfrak{C}\} \times \{\zeta \mid \zeta \in \mathfrak{C}\}$ and $\{010\zeta \mid \zeta \in \mathfrak{C}\} \times \{\zeta \mid \zeta \in \mathfrak{C}\}$, respectively, the rectangle $\{00\zeta \mid \zeta \in \mathfrak{C}\} \times \{\zeta \mid \zeta \in \mathfrak{C}\}$ of $g\hat{x}_1$ is essential. □

The idea of the following subpath comes from [12; 18].

Subpath 2. We fix an integer $M \geq 100$. Consider the expression $P\Pi Q^{-1}$ of a minimal pair of colored binary trees of g_1 and let $C_{m_1} \cdots C_{m_p}$ be the maximal words on $\{C_i\}$ contained in Q, where $0 \le m_1 < \cdots < m_p$. When m_1 is not zero, we have

$$\begin{aligned} C_{m_1} \cdots C_{m_p} &= (x_0^{-(m_1-1)} C_1 x_0^{(m_1-1)} \cdots x_0^{-(m_p-1)} C_1 x_0^{(m_p-1)}) \\ &= x_0^{-(m_1-1)} C_1 x_0^{-(m_2-m_1)} \cdots C_1 x_0^{-(m_p-m_{p-1})} C_1 x_0^{(m_p-1)} \\ &= x_0^{-(m_1-1)} \hat{B}_0 x_0^{-(m_2-m_1-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 x_0^{m_p}, \end{aligned}$$

and define

$$\begin{aligned} \omega_2 \equiv\ & x_0^{-(m_1-1)} \hat{B}_0 x_0^{-(m_2-m_1-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 x_0^{-(M|g_1|-m_p)} \\ & \cdot x_1 \cdot x_0^{(M|g_1|-m_p)} \hat{B}_0^{-1} x_0^{(m_p-m_{p-1}-1)} \hat{B}_0^{-1} \cdots x_0^{(m_2-m_1-1)} \hat{B}_0^{-1} x_0^{(m_1-1)}. \end{aligned}$$

When m_1 is zero, define

$$\begin{aligned} \omega_2 \equiv\ & C_0 x_0^{-(m_2-1)} \hat{B}_0 x_0^{-(m_3-m_2-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 x_0^{-(M|g_1|-m_p)} \\ & \cdot x_1 \cdot x_0^{(M|g_1|-m_p)} \hat{B}_0^{-1} x_0^{(m_p-m_{p-1}-1)} \hat{B}_0^{-1} \cdots x_0^{(m_3-m_2-1)} \hat{B}_0^{-1} x_0^{(m_2-1)} C_0^{-1}. \end{aligned}$$

Let $g_2 = g_1\omega_2$.

See Figure 9 for example.

Lemma 3.5. (1) *We have* $\|\omega_2\| < 4M|g|$.

(2) *For every prefix* ω' *of* ω_2, *we have* $|g_1\omega'| > |g|/64$.

Proof. The proof is only provided when m_1 is not zero; however, it can be shown similarly when m_1 is zero. We first note that $m_p \le 4|g_1|$ holds by Proposition 2.9.

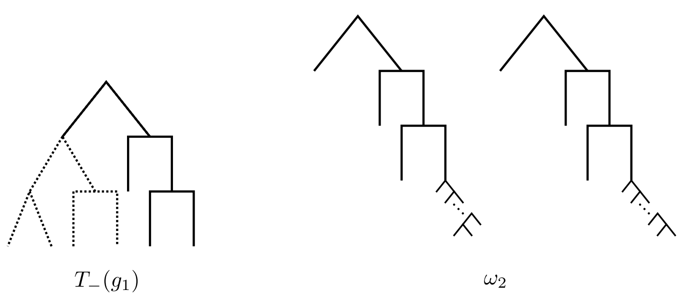


Figure 9. An example of the construction of ω_2. The dotted carets in $T_-(g_1)$ are not used when constructing ω_2. The all-right tree $T'_-(g_1)$ is obtained by removing the dotted carets from $T_-(g_1)$. Note that the tree $T'_-(g_1)$ is the domain tree of the product $C_{m_1} \cdots C_{m_p}$. By using $T'_-(g_1)$, all we needed to define ω_2 are the generators $\hat{B}_0$, x_0, and x_1. See also Figure 10.

A straightforward calculation yields the upper bound of $\|\omega_2\|$. Indeed, we have

$$\begin{aligned}&\big((m_1-1)+1+(m_2-m_1-1)+1+\cdots+(m_p-m_{p-1}-1)+1+(M|g_1|-m_p)\big)\times 2+1\\&=2M|g_1|+1\\&\le 2M(|g|+3)+1\\&<4M|g|.\end{aligned}$$

For part (2), we first consider $\omega' \le \omega_2$ which does not contain x_1. When $\|\omega'\| \le \lfloor |g_1|/2 \rfloor$, we have $|g_1\omega'| \ge |g_1|/2$. Indeed, if not, we have

$$|g_1| = |g_1\omega'(\omega')^{-1}| \le |g_1\omega'| + \|\omega'\| < \frac{|g_1|}{2} + \frac{|g_1|}{2}.$$

Hence we have $|g_1\omega'| \ge |g_1|/2 \ge (|g|-3)/2 \ge |g|/8$. Next, we assume that ω' does not contain x_1 and $\|\omega'\| > |g_1|/2$ holds. Then by Lemmas 3.4 and 3.1, there exists a pair of numbered patterns $(P_+(g_1\omega'), P_-(g_1\omega'))$ representing $g_1\omega'$ such that $R_0(P_-(g_1\omega'))$ is essential. Then we have

$$\|R_0(P_-(g_1\omega'))\| = \|R_0(P_-(g_1))\| + \|\omega'\| > \|\omega'\| > \frac{|g_1|}{2}. \tag{3-1}$$

By Corollary 2.12, we have

$$|g_1\omega'| \ge \frac{\|R_0(P_-(g_1\omega'))\|}{8} > \frac{|g_1|}{16} \ge \frac{|g|}{64}. \tag{3-2}$$

Next, we assume ω' contains x_1 and no $\hat{B}_0^{-1}$. Then there exists $i \ge 0$ such that

$$g_1\omega' = g_1 x_0^{-(m_1-1)}\hat{B}_0 x_0^{-(m_2-m_1-1)}\cdots\hat{B}_0 x_0^{-(m_p-m_{p-1}-1)}\hat{B}_0(x_0^{-(M|g_1|-m_p)}x_1 x_0^{(M|g_1|-m_p)})x_0^{-i}$$

as an element in $2V$. Since $x_0^{-(M|g_1|-m_p)}x_1x_0^{(M|g_1|-m_p)}$ is identity on $[0, 1/2]\times[0, 1]$, the essentiality of the rectangle is preserved when $i=0$. This means that when $i=0$, we have $\|R_0(P_-(g_1\omega'))\| > |g_1|/2$ by inequality (3-1). According to Lemma 3.1(1), multiplying x_0^{-1} from the right increases the size of the rectangle. Then for any $i \ge 0$, we have $\|R_0(P_-(g_1\omega'))\| > |g_1|/2$. This implies that we have $|g_1\omega'| > |g|/64$ for the same reason as in inequality (3-2).

Finally, consider the case where ω' contains x_1 and $\hat{B}_0^{-1}$. From the construction of ω_2, it can be observed that g_2 is obtained by attaching carets to the minimal pair of colored binary trees $(T_+(g_1), T_-(g_1))$. This implies that there exists a pair of numbered patterns of g_2 such that the rectangle of the target pattern that contains a point $(1, 1) \in [0, 1] \times [0, 1]$ is essential, and its size is at least $M|g_1|+3-4|g_1|$. Indeed, by Proposition 2.9, the length of the right most branch of $T_-(g_1)$ is at most $4|g_1|$. Hence the number of attached vertical carets to the

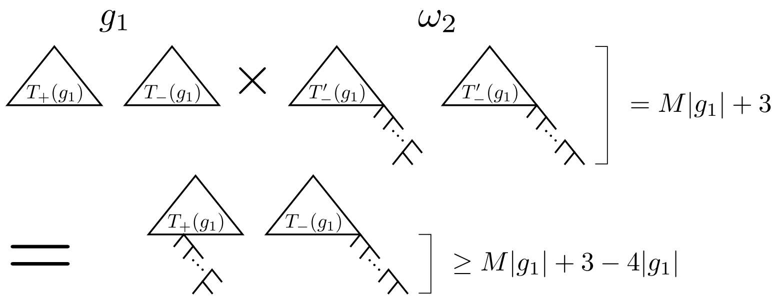


Figure 10. An illustration of the multiplication of g_1 and ω_2, where $(T_+(g_1), T_-(g_1))$ is a minimal pair of colored binary trees of g_1 and $T'_-(g_1)$ is the maximal all-right colored binary tree contained in $T_-(g_1)$.

rightmost branch is at least $M|g_1| + 3 - 4|g_1|$. Also, from the construction, vertical reduction cannot be applied. Therefore the size of the rectangle containing $(1, 1)$ is at least $M|g_1| + 3 - 4|g_1|$ by focusing only on the vertical subdivisions, since the size of a rectangle is defined as the sum of the number of vertical and horizontal subdivisions. See Figure 10 for an illustration of this argument.

We now consider $g_1\omega'$ as $g_2\omega''$ by a certain word ω''. Note that we have $\|\omega''\| \leq m_p$ by a straightforward calculation. This means that $\|\omega''\|$ is sufficiently smaller than $|g_2|$, so we estimate $|g_1\omega'|$ based on g_2. Since $m_p \leq 4|g_1|$ holds, by Corollary 2.12 and Lemma 3.4, we have

$$|g_1\omega'| = |g_2\omega''| \geq |g_2| - \|\omega''\| \geq \frac{M|g_1| + 3 - 4|g_1|}{8} - 4|g_1| \tag{3-3}$$
$$> \tfrac{1}{8}(M - 36)|g_1| \geq 8|g_1| \geq 2|g|.$$

Hence for any prefix $\omega' \leq \omega_2$, we have $|g_1\omega'| > |g|/64$.

□

Subpath 3. Let ω_3 be a minimal word on X_{2V} such that $\omega_3 = g_1^{-1}$ holds. Let $g_3 = g_2\omega_3$.

Lemma 3.6. (1) *We have* $\|\omega_3\| \leq 2|g|$.

(2) *For every prefix* ω' *of* ω_3, *we have* $|g_2\omega'| > 2|g|$.

Proof. The first claim is obvious by Lemma 3.4. For the second claim, we note that $|\omega'| \leq |g_1|$ holds since ω_3 is a minimal word. Then we have

$$|g_2\omega'| \geq |g_2| - |\omega'| \geq |g_2| - |g_1| > 2|g|,$$

as estimated in inequality (3-3).

□

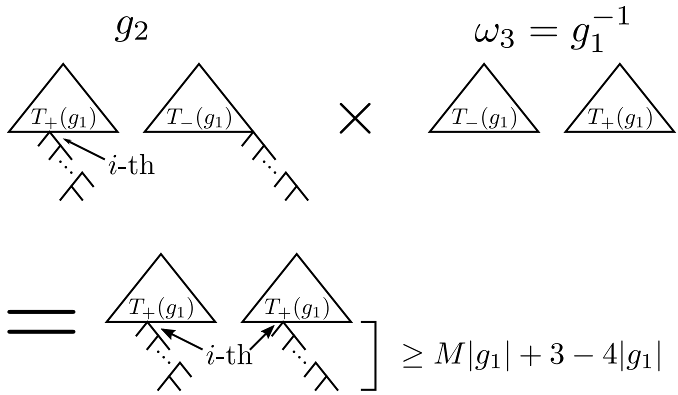


Figure 11. An illustration of g_3. Observe that g_3 is the identity map on the subset corresponding to all leaves except the i-th leaf.

Subpath 4. We fix an integer $Q \geq 48M$. We define ω_4 based on a subset $[0, 1] \times [0, 1]$ where g_3 is the identity map. From the construction, one of the following holds (see also Figure 11):

(a) g_3 is the identity map on $[0, 1] \times [0, 1/2]$;

(b) g_3 is the identity map on $[0, 1] \times [1/2, 1]$;

(c) g_3 is the identity map on $[0, 1/2] \times [0, 1]$; and

(d) g_3 is the identity map on $[1/2, 1] \times [0, 1]$.

Then we define ω_4 as the word determined from one of the following cases of capital letters corresponding to each small letter:

(A) let $\omega_4 \equiv {}_h\hat{x}_1{}^{-(Q|g|+1)}{}_h\hat{x}_{2h}\hat{x}_1{}^{Q|g|+1}$;

(B) let $\omega_4 \equiv {}_hx_1{}^{-(Q|g|+1)}{}_hx_{2h}x_1{}^{Q|g|+1}$;

(C) let $\omega_4 \equiv \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}$; and

(D) let $\omega_4 \equiv x_1{}^{-Q|g|}x_2x_1{}^{Q|g|}$.

Let $g_4 = g_3\omega_4$.

Lemma 3.7. (1) *We have* $\|\omega_4\| \leq 3Q|g|$.

(2) *For every prefix* ω' *of* ω_4, *we have* $|g_3\omega'| > 3|g|$.

(3) *As elements in* $2V$, g_3 *and* ω_4 *commute.*

Proof. Part (1) follows from a straightforward estimation, and part (3) is obvious since the supports of g_3 and ω_4 are disjoint. For part (2), in any case, we note that generators in ω_4 preserve the rectangle with its size at least $M|g_1| + 3 - 4|g_1|$. Since the process of obtaining the essential rectangle from this rectangle requires only at most one horizontal reduction, $g_3\omega_4$ is also represented by a pair of numbered patterns with an essential rectangle of size at least $M|g_1|+2-4|g_1|$. Indeed, when ω_4 is defined in the case of (A), since ${}_h\hat{x}_1, {}_h\hat{x}_2$ is the identity on $[0, 1] \times [0, 1/2]$, the vertical reduction cannot be applied in the process and the horizontal reduction may be possible. Also, since any generator in ω_4 vertically divides $[0, 1] \times [0, 1/2]$, the number of this horizontal reduction is at most one. For case (B), only at most one reduction is applied for the same reason as in case (A), and no such reduction is possible in cases (C) and (D). Hence the size is at least $M|g_1|+2-4|g_1|$ in any cases. By Corollary 2.12 and Lemma 3.4, we have

$$|g_3\omega'| \geq \frac{M|g_1|+2-4|g_1|}{8} > 12|g_1| \geq 3|g|,$$

which is the desired result.

□

Subpath 5. Let ω_5 be a minimal word on X_{2V} such that $\omega_5 = g_3^{-1}$ holds. Let $g_5 = g_4\omega_5$.

Lemma 3.8. (1) *We have* $\|\omega_5\| \leq 5M|g|$.

(2) *For every prefix* ω' *of* ω_5, *we have* $|g_4\omega'| > M|g|$.

Proof. For part (1), by Lemmas 3.4, 3.5 and 3.6, we have

$$\|\omega_5\| \leq |g| + \|\omega_1\| + \|\omega_2\| + \|\omega_3\| \leq |g| + 3 + 4M|g| + 2|g| < 5M|g|.$$

For part (2), we first note that ω_4 is represented by a pair of patterns with an essential rectangle of size at least $Q|g|+4$. In particular, by a similar argument to the proof of Lemma 3.7, the horizontal length of this rectangle is unchanged for $g_4 = g_3\omega_4$. Hence by Corollary 2.12, we have $|g_4| \geq (Q|g|+3)/8$. Therefore we have

$$|g_4\omega'| \geq |g_4| - \|\omega_5\| > 6M|g| - 5M|g| = M|g|.$$

This completes the proof.

□

We may now obtain $g\omega_1\omega_2\omega_3\omega_4\omega_5 = \omega_4$, which only depends on $|g|$.

The final step is to connect any of cases (A) to (D) defined in subpath 4 to $\hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$ by a final subpath. In order to define this subpath, we write the subpaths defined in cases (A) to (D) as $\omega_4(A)$, $\omega_4(B)$, $\omega_4(C)$ and $\omega_4(D)$, respectively.

Subpath 6. If the path ω_4 is $\omega_4(A)$, let $\omega_6 \equiv \omega_4(B)\omega_4(C)$. If the path ω_4 is $\omega_4(B)$, let $\omega_6 \equiv \omega_4(A)\omega_4(C)$. If the path ω_4 is $\omega_4(C)$, let $\omega_6 \equiv \omega_4(D)$. Finally, if the path ω_4 is $\omega_4(D)$, let $\omega_6 \equiv \omega_4(C)$.

Lemma 3.9. (1) *In any case, we have* $g_5\omega_6 = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$ *as an element in* $2V$.

(2) *We have* $\|\omega_6\| \le 6Q|g|$.

(3) *For every prefix* ω' *of* ω_6, *we have* $|g_5\omega'| > 6M|g|$.

Proof. For part (1), we note that $\omega_4(A)\omega_4(B) = \omega_4(D)$ holds as elements in $2V$. Observe that the supports of $\omega_4(A)$ and $\omega_4(B)$ are disjoint, and the same holds for $\omega_4(C)$ and $\omega_4(D)$. Since we have $\omega_4(C)\omega_4(D) = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$, we obtain the desired result. Part (2) follows from Lemma 3.7.

Finally, part (3) follows from the following observation: in the process of multiplying generators of ω_6, there always exists a pair of numbered patterns with an essential rectangle whose horizontal length is at least $Q|g|+2$. Hence by Corollary 2.12, we have $|g_5\omega'| \ge (Q|g|+2)/8 > 6M|g|$. □

Now, we define ω as $\omega_1 \cdots \omega_6$. From Lemmas 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, take $D = 10Q$ and $\delta = 1/64$. Then we have that $g\omega = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$, $\|\omega\| < D|g|$, and $|g\omega'| > \delta|g|$ for any prefix ω' of ω. This completes the proof of Proposition 3.3. □

Acknowledgements

The author would like to thank Professor Tomohiro Fukaya for his helpful comments. He also appreciates Professor Lucy Lifschitz for teaching him about the divergence function of the Lodha–Moore group. Last but not least, he expresses his gratitude to the referee for his or her kind comments. The author was supported by JSPS KAKENHI Grand number 24K22836.

References

[1] C. Bleak and D. Lanoue, "A family of non-isomorphism results", *Geom. Dedicata* **146** (2010), 21–26. MR Zbl

[2] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren Math. Wiss. **319**, Springer. Zbl

[3] M. G. Brin, "Higher dimensional Thompson groups", *Geom. Dedicata* **108** (2004), 163–192. MR Zbl

[4] M. G. Brin, "Presentations of higher dimensional Thompson groups", *J. Algebra* **284**:2 (2005), 520–558. MR Zbl

[5] M. G. Brin, "On the baker's map and the simplicity of the higher dimensional Thompson groups nV", *Publ. Mat.* **54**:2 (2010), 433–439. MR Zbl

[6] J. Burillo and S. Cleary, "Metric properties of higher-dimensional Thompson's groups", *Pacific J. Math.* **248**:1 (2010), 49–62. MR Zbl

[7] J. Burillo, S. Cleary, and B. Nucinkis, "Grid diagrams for higher-dimensional Thompson's groups", preprint, 2024. arXiv 2403.02562

[8] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Introductory notes on Richard Thompson's groups", *Enseign. Math.* (2) **42**:3-4 (1996), 215–256. MR Zbl

[9] C. Druțu, S. Mozes, and M. Sapir, "Divergence in lattices in semisimple Lie groups and graphs of groups", *Trans. Amer. Math. Soc.* **362**:5 (2010), 2451–2505. MR Zbl

[10] C. Druțu, S. Mozes, and M. Sapir, "Corrigendum to "Divergence in lattices in semisimple Lie groups and graphs of groups"", *Trans. Amer. Math. Soc.* **370**:1 (2018), 749–754. MR Zbl

[11] S. M. Gersten, "Quadratic divergence of geodesics in CAT(0) spaces", *Geom. Funct. Anal.* **4**:1 (1994), 37–51. MR Zbl

[12] G. Golan and M. Sapir, "Divergence functions of Thompson groups", *Geom. Dedicata* **201**, 227–242. MR Zbl

[13] M. Gromov, "Asymptotic invariants of infinite groups", pp. 1–295 in *Geometric group theory* (Sussex, 1991), London Math. Soc. Lecture Note Ser. **182**, Cambridge Univ. Press, 1993. MR Zbl

[14] J. Hennig and F. Matucci, "Presentations for the higher-dimensional Thompson groups nV", *Pacific J. Math.* **257**:1 (2012), 53–74. MR Zbl

[15] L. Issini, "On linear divergence in finitely generated groups", preprint, 2023. arXiv 2311.12938

[16] M. Kato, "Higher dimensional Thompson groups have Serre's property FA", preprint, 2015. Zbl arXiv 1504.06680

[17] Y. Kodama, "Divergence function of the braided Thompson group", *Kyoto J. Math.* **63**:2 (2023), 435–470. MR Zbl

[18] L. Lifschitz, "Divergence of the Lodha–Moore group", *Proc. Amer. Math. Soc.* (published online 2024).

[19] X. Sheng, "Divergence property of the Brown–Thompson groups and braided Thompson groups", *Transform. Groups* (published online January 2024).

Received June 7, 2024. Revised December 16, 2024.

YUYA KODAMA
GRADUATE SCHOOL OF SCIENCE AND ENGINEERING
KAGOSHIMA UNIVERSITY
KAGOSHIMA 890-0065
JAPAN
yuya@sci.kagoshima-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2024 is US $645/year for the electronic version, and $875/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 333 No. 2 December 2024