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VOGAN'S CONJECTURE ON LOCAL ARTHUR PACKETS OF p-ADIC GL_n AND A COMBINATORIAL LEMMA

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For GL_n over a p-adic field, Cunningham and Ray proved Vogan's conjecture, that is, local Arthur packets are the same as ABV packets. They used endoscopic theory to reduce the general case to a combinatorial lemma for irreducible local Arthur parameters, and their proof implies that one can also prove Vogan's conjecture for p-adic GL_n by proving a generalized version of this combinatorial lemma. Riddlesden recently proved this generalized lemma. We give a new proof of it, which has its own interest.

1. Introduction

Let F be a non-Archimedean field of characteristic zero and let W_F denote the Weil group of F. Let G be a connected reductive group defined over F. We define $G := \mathrm{G}(F)$ and $\Pi(G)$ the isomorphism classes of smooth irreducible representations of G. A local Arthur parameter ψ is a continuous homomorphism

$$\psi : W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to {}^L G,$$

such that

(1) the restriction of ψ to W_F has bounded image;

(2) the restrictions of ψ to both $\mathrm{SL}_2^D(\mathbb{C})$ and $\mathrm{SL}_2^A(\mathbb{C})$ are analytic;

(3) ψ commutes with the projections $W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to W_F$ and ${}^L G \to W_F$.

Here $\mathrm{SL}_2^D(\mathbb{C})$ is called the Deligne-SL_2 and $\mathrm{SL}_2^A(\mathbb{C})$ is called the Arthur-SL_2.

In his fundamental work [2], Arthur attached a local Arthur packet Π_ψ for each local Arthur parameter ψ of quasisplit classical groups. This is a finite (multi)set of smooth irreducible representations, satisfying certain regular and twisted endoscopic character identities [2, Theorem 2.2.1]. Assuming the Ramanujan conjecture, Arthur showed that the union of these local Arthur packets contains the local components of all discrete square-integrable automorphic representations.

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An interesting question is to construct each local Arthur packet Π_ψ over p-adic fields besides the abstract definition. In a series of work [16; 17; 18; 19; 20], Mœglin explicitly constructed each local Arthur packet and showed that it is multiplicity-free. However, there are difficulties in her construction when trying to compute the representations in the local Arthur packets in terms of the Langlands classification. To remedy this, for symplectic and split odd special orthogonal groups, Atobe gave a reformulation of Mœglin's construction [3] using extended multisegments, and gave an explicit algorithm to compute the Langlands classification data for the representations in a local Arthur packet. The main tools in these results are (partial) Aubert–Zelevinsky involution and partial Jacquet module (called derivatives in [4]), which are representation theoretic.

On the other hand, in [8], Cunningham, Fiori, Moussaoui, Mracek and Xu aimed to construct local Arthur packets over p-adic fields using a geometric approach. They extended [1] to p-adic reductive groups: For any L-parameter ϕ of a p-adic quasisplit connected reductive group G^*, they defined the ABV-packet Π_ϕ^{ABV}, a finite set of irreducible representations of pure inner forms of G^*, using microlocal vanishing cycle functors. For a fixed pure inner form G of G^*, we shall write $\Pi_\phi^{\mathrm{ABV}}(G) := \Pi_\phi^{\mathrm{ABV}} \cap \Pi(G)$. It is expected that the ABV-packets recover the local Arthur packets in the following sense. For each local Arthur parameter ψ of G, we associate an L-parameter ϕ_ψ by

$$\phi_\psi(w,x) := \psi\Big(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{\frac{-1}{2}} \end{pmatrix}\Big).$$

We say ϕ is of Arthur type if $\phi = \phi_\psi$ for some local Arthur parameter ψ. *Vogan's conjecture* is stated as follows.

Conjecture 1.1 [8, Conjecture 8.3.1(a)]. *For any local Arthur parameter ψ of* $\mathrm{G}(F)$,

$$\Pi_\psi = \Pi_{\phi_\psi}^{\mathrm{ABV}}(\mathrm{G}(F))$$

There are more precise statements matching the distributions associated with local Arthur packets and ABV-packets; we refer to [8, Conjecture 8.3.1] for details.

Conjecture 1.1 remains widely open. The only known case is $\mathrm{GL}_n(F)$, proved by Cunningham and Ray [5; 6]. To introduce our main result, we sketch their main idea in the following subsection.

1.1. ***Proof of Vogan's conjecture of $\mathrm{GL}_n(F)$ and the combinatorial lemma.*** First, we introduce a crucial ingredient of their proof, the *Pyasetskii involution* on L-parameters, which we denote by

$$\phi \mapsto \widehat{\phi}.$$

This involution is defined via the geometric structure on Vogan varieties. See [8, §6.4] or [23, §4.3] for a precise definition. When $G = \mathrm{GL}_n(F)$, there are bijections among irreducible representations of G, L-parameters of G, and a collection of multisegments (see Section 2.1 for details). In [21], Mœglin and Waldspurger showed that under these bijections, Pyasetskii involution on L-parameters matches the *Zelevinsky involution* on irreducible representations defined in [23, §4.1]. They gave a combinatorial algorithm on multisegments to realize these involutions (see Section 2.3). Later in [14], Knight and Zelevinsky gave a closed formula for the involution on multisegments, which is proved using the theory of flows in network.

For $\mathrm{GL}_n(F)$, the structure of L-packets and local Arthur packets are simple: they are all singletons. Therefore, the inclusion $\Pi_\psi = \Pi_{\phi_\psi} \subseteq \Pi^{\mathrm{ABV}}_{\phi_\psi}$ automatically holds. With this observation and the geometric structure of ABV-packets of $\mathrm{GL}_n(F)$ [7, Proposition 3.2.1], Cunningham and Ray demonstrated in the proof of [6, Theorem 5.3] that for an irreducible local Arthur parameter ψ of $\mathrm{GL}_n(F)$,

$$\Pi_\psi = \Pi^{\mathrm{ABV}}_{\phi_\psi}(\mathrm{GL}_n(F))$$

holds if the following lemma holds for ψ.

Lemma 1.2 [6, Lemma 4.8]. *Let ψ be an irreducible local Arthur parameter of $\mathrm{GL}_n(F)$. If ϕ is an L-parameter of $\mathrm{GL}_n(F)$ satisfying $\phi \geq \phi_\psi$ and $\widehat{\phi} \geq \widehat{\phi_\psi}$, then*

$$\phi = \phi_\psi.$$

Here the inequality is the closure ordering of L-parameters defined via the geometric structure on Vogan varieties, which is equivalent to the partial ordering on multisegments considered in [23] (see Section 2.2 for details). Cunningham and Ray proved the above lemma, and hence established Vogan's conjecture in this case. Later in [5], they used endoscopic lifting to reduce the general case to the case of irreducible parameters. This proved Vogan's conjecture for $\mathrm{GL}_n(F)$ completely.

On the other hand, the proof in [6, Theorem 5.3] implies that for an *arbitrary* local Arthur parameter ψ of $\mathrm{GL}_n(F)$, not necessarily irreducible, the equality

$$\Pi_\psi = \Pi^{\mathrm{ABV}}_{\phi_\psi}(\mathrm{GL}_n(F))$$

holds if the following generalized lemma holds for ψ.

Lemma 1.3. *Let ψ be an arbitrary local Arthur parameter of $\mathrm{GL}_n(F)$. If ϕ is an L-parameter of $\mathrm{GL}_n(F)$ satisfying $\phi \geq \phi_\psi$ and $\widehat{\phi} \geq \widehat{\phi_\psi}$, then*

$$\phi = \phi_\psi.$$

Riddlesden [22] proved Lemma 1.3, mainly using the network description of the Zelevinsky involution as in [14]. Therefore, combining with the proof of [6, Theorem 5.3], this provides a combinatorial approach to Vogan's conjecture for $\mathrm{GL}_n(F)$.

1.2. ***Main result.*** We present a new proof of Lemma 1.3, thereby offering an alternative approach to Vogan's conjecture for $\mathrm{GL}_n(F)$. Compared to [22], our proof is elementary and only involves the Mœglin–Waldspurger algorithm. This reflects an important technique in the study of general representations of Arthur type, and hence has its own interest and is expected to have applications, as outlined below.

The main idea of the proof is to verify that the ABV-packets of $\mathrm{GL}_n(F)$ satisfy an analogue of the following property for local Arthur packets of $G_n = \mathrm{Sp}_{2n}(F)$ or split $\mathrm{SO}_{2n+1}(F)$ proved in [11]. Suppose π is a representation of G_n of Arthur type. Then the L-parameter ϕ_π shares a specific common summand with the L-parameter $\phi_{\psi^{\max}(\pi)}$, where

$$\psi^{\max}(\pi) \in \Psi(\pi) := \{\psi \mid \pi \in \Pi_\psi\}$$

is "the" local Arthur parameter of π considered in [10]. If π is not tempered, one may define a representation π^- and a local Arthur parameter $(\psi^{\max}(\pi))^-$ of G_{n^-} with $n^- < n$ by removing this common summand from the L-parameters. Then $\pi^- \in \Pi_{(\psi^{\max}(\pi))^-}$. Conversely, repeating this process (with slight modifications) gives an algorithm to determine whether π is of Arthur type or not. See [12, §6] for a precise statement.

For $\mathrm{GL}_n(F)$, motivated by the phenomenon above, we show that the dual inequalities

$$\phi \geq \phi_\psi \quad \text{and} \quad \widehat{\phi} \geq \widehat{\phi_\psi}$$

imply that ϕ and ϕ_ψ must share certain common summands in Proposition 4.3. Define ϕ^- and ψ^- by removing these common summands. The dual inequalities are preserved, i.e.,

$$\phi^- \geq \phi_{\psi^-} \quad \text{and} \quad \widehat{\phi^-} \geq \widehat{\phi_{\psi^-}}.$$

Repeating the above procedure, we conclude that $\phi = \phi_\psi$, which proves the main result. Note that $\Psi(\pi)$ is always a singleton for $\mathrm{GL}_n(F)$, and hence $\psi^{\max}(\pi)$ does not play an essential role here.

1.3. ***Applications of the combinatorial lemma.*** Finally, we discuss some applications of the combinatorial lemma. We say that an L-parameter ϕ is *atomic* if

$$\{\phi' \mid \phi' \geq \phi,\ \widehat{\phi'} \geq \widehat{\phi}\} = \{\phi\}.$$

In upcoming work [9; 15], we show that this atomic property enables us to explicitly study the associated ABV-packet, as we will now explain.

First, suppose ϕ is an atomic L-parameter of $\mathrm{GL}_n(F)$ and let π be the corresponding irreducible representation. Note that the atomic property implies that $\Pi_\phi^{\mathrm{ABV}} = \{\pi\}$. In [15], we give an algorithm to express π as a linear combination of standard modules.

Next, we consider L-parameters of classical groups $\mathrm{Sp}_{2n}(F)$ or split $\mathrm{SO}_{2n+1}(F)$. On one hand, we can associate a "stable standard" distribution η_ϕ^{std} by summing over all standard modules in the L-packet Π_ϕ, which is stable. On the other hand, there is another distribution η_ϕ^{ABV} defined via the ABV-packet Π_ϕ^{ABV}, which is conjectured to be stable [8, Conjecture 8.4.2]. Assuming this conjecture, for each atomic L-parameter ϕ, we give an algorithm to express η_ϕ^{ABV} as a linear combination of stable standard distributions in [15].

Finally, in a joint work currently in progress [9], we are going to establish the conjecture that η_ϕ^{ABV} is always stable for classical groups under certain assumptions. As a nontrivial corollary, we show that if ϕ is atomic, then the ABV-packet Π_ϕ^{ABV} is atomically stable, meaning that no proper subset supports any stable distribution. This justifies the use of the term "atomic".

With the above results in mind, it is an interesting and important question to characterize/classify the set of atomic L-parameters. The combinatorial lemma (Lemma 1.3) implies that L-parameters of Arthur type of $\mathrm{GL}_n(F)$ are always atomic. The same holds for classical groups $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ and $\mathrm{O}_{2n}(F)$ by the explicit computation of Pyasetskii involution for these groups in [13]. Therefore, the above results can be applied to these ABV-packets of Arthur type. We remark that not all atomic L-parameters are of Arthur type. For example, ϕ_π is atomic but not of Arthur type for any generic nontempered representation π of $\mathrm{GL}_n(F)$. We expect that the careful study of the combinatorial aspect of Mœglin–Waldspurger algorithm in this paper will play important roles toward the classification of atomic L-parameters not necessarily of Arthur type.

Here is the structure of this paper. In Section 2, we recall the necessary notation and preliminaries. We recall the notion of multisegments in Section 2.1, the partial ordering on multisegments in Section 2.2, and the Mœglin–Waldspurger algorithm in Section 2.3. In Section 3, we rephrase the Mœglin–Waldspurger algorithm and develop certain notation and lemmas for the proofs in Section 4. Then we prove Lemma 1.3 in Section 4. We rephrase Lemma 1.3 in terms of multisegments in Section 4.1. We prove the key of reduction, Proposition 4.3, in Section 4.2. Finally, we prove Lemma 1.3 in Section 4.3.

2. Preliminaries

Let F be a non-Archimedean field of characteristic zero, and let W_F denote the Weil group of F. Let $|\cdot|$ represent the normalized absolute value on F, which we also regard as a character of $\mathrm{GL}_n(F)$ via composition with the determinant.

We denote by $\Pi(\mathrm{GL}_n(F))$ the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$, and by $\Phi(\mathrm{GL}_n(F))$ the set of equivalence classes of L-parameters for $\mathrm{GL}_n(F)$. We define

$$\Pi(\mathrm{GL}(F)) := \bigsqcup_{n\geq 1} \Pi(\mathrm{GL}_n(F)), \quad \Phi(\mathrm{GL}(F)) := \bigsqcup_{n\geq 1} \Phi(\mathrm{GL}_n(F)).$$

We denote by $+$ the sum of multisets (disjoint union), and by $\setminus$ the difference of multisets.

2.1. *Langlands classification for* $\mathbf{GL}_n(F)$. We recall the Langlands classification for $\mathrm{GL}_n(F)$, and the bijection among $\Pi(\mathrm{GL}_n(F))$, $\Phi(\mathrm{GL}_n(F))$ and multisegments of correct rank.

Let $\mathcal{C}(\mathrm{GL}_n(F))$ denote the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}_n(F)$. By local Langlands correspondence for $\mathrm{GL}_n(F)$, we may identify $\mathcal{C}(\mathrm{GL}_n(F))$ as the set of isomorphism classes of n-dimensional irreducible representations of W_F. We let

$$\mathcal{C} := \bigsqcup_{n\geq 1} \mathcal{C}(\mathrm{GL}_n(F)),$$

and denote by $\mathcal{C}_{\mathrm{unit}}$ the subset of $\mathcal{C}$ that consists of unitary supercuspidal representations.

Let P be a standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi subgroup $L \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_s}(F)$. An irreducible representation σ of L can be identified with

$$\sigma = \sigma_1 \otimes \cdots \otimes \sigma_s,$$

where $\sigma_i \in \Pi(\mathrm{GL}_{n_i}(F))$. We set

$$\sigma_1 \times \cdots \times \sigma_s := \mathrm{Ind}_P^{\mathrm{GL}_n(F)} \sigma,$$

which is the normalized parabolic induction.

A segment Δ is a set of the form

$$\{\rho|\cdot|^b, \rho|\cdot|^{b+1}, \ldots, \rho|\cdot|^e\},$$

where $\rho \in \mathcal{C}_{\mathrm{unit}}$, $b, e \in \mathbb{R}$ such that $e - b \in \mathbb{Z}_{\geq 0}$. We shall write $\Delta = [b, e]_\rho$ for short and call b the base value of Δ, e the end value of Δ, and $e - b + 1$ the length of Δ.

We also write

$$b(\Delta) := b, \quad e(\Delta) := e, \quad l(\Delta) := e - b + 1.$$

A multisegment, which we usually denote by α, β, γ or δ, is a finite multiset of segments. We denote the collection of segments by $\underline{\mathrm{Seg}}$ and the collection of multisegments by $\underline{\mathrm{Mseg}}$. For each $\rho \in \mathcal{C}_{\mathrm{unit}}$, let $\underline{\mathrm{Seg}}_\rho$ denote the subset of $\underline{\mathrm{Seg}}$ consisting of segments of the form $[b, e]_\rho$, and let $\underline{\mathrm{Mseg}}_\rho$ denote the subset of $\underline{\mathrm{Mseg}}$ consisting of multisets of segments in $\underline{\mathrm{Seg}}_\rho$. For each $\alpha \in \underline{\mathrm{Mseg}}$, there is a unique decomposition

$$\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho,$$

where $\alpha_\rho \in \underline{\mathrm{Mseg}}_\rho$ and $\alpha_\rho = \varnothing$ except for a finite number of $\rho \in \mathcal{C}_{\mathrm{unit}}$.

For each segment $[y, x]_\rho$, let $\Delta_\rho[x, y]$ be the unique irreducible subrepresentation of the parabolic induction

$$\rho|\cdot|^x \times \rho|\cdot|^{x-1} \times \cdots \times \rho|\cdot|^y.$$

We recall the Langlands classification for $\mathrm{GL}_n(F)$ now. Any irreducible representation $\pi \in \Pi(\mathrm{GL}_n(F))$ can be realized as the unique irreducible subrepresentation of a parabolic induction of the form

$$\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_f}[x_f, y_f],$$

where

- $n = \sum_{i=1}^f \dim(\rho_i)(x_i - y_i + 1)$,
- $\rho_i \in \mathcal{C}_{\mathrm{unit}}$, and
- $x_1 + y_1 \leq \cdots \leq x_f + y_f$.

Here $\dim(\rho_i)$ is the dimension of ρ_i as an irreducible representation of W_F. The multiset

$$\{\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_f}[x_f, y_f]\}$$

with the above requirement is unique. With this notation, we may write down the L-parameter of π as

$$\phi_\pi = \rho_1|\cdot|^{\frac{x_1+y_1}{2}} \otimes \mathrm{Sym}^{x_1-y_1} \oplus \cdots \oplus \rho_f|\cdot|^{\frac{x_f+y_f}{2}} \otimes \mathrm{Sym}^{x_f-y_f},$$

where Sym^{b+1} is the unique b-dimensional irreducible analytic representation of $\mathrm{SL}_2(\mathbb{C})$. Also, we associate, to π, the multisegment

$$\delta_\pi := \{[y_1, x_1]_{\rho_1}, \ldots, [y_f, x_f]_{\rho_f}\}.$$

Thus, with the above notation, we obtain the correspondence

$$\begin{array}{ccccc} \Pi(\mathrm{GL}(F)) & \longleftrightarrow & \Phi(\mathrm{GL}(F)) & \longleftrightarrow & \underline{\mathrm{Mseg}}, \\ \pi & \longmapsto & \phi_\pi & \longmapsto & \delta_\pi. \end{array}$$

Suppose ϕ is an L-parameter of $\mathrm{GL}_n(F)$. We shall simply write $\delta_\phi := \delta_\pi$ where π is the unique representation of $\Pi(\mathrm{GL}_n(F))$ such that $\phi_\pi = \phi$.

2.2. ***A partial ordering on multisegments.*** We recall the partial ordering on $\underline{\mathrm{Mseg}}$ introduced in [23].

Suppose $\Delta_1 = [b_1, e_1]_{\rho_1}$ and $\Delta_2 = [b_2, e_2]_{\rho_2}$ are two segments. We say Δ_1 and Δ_2 are *linked* if the union $\Delta_1 \cup \Delta_2$ (as a set) is also a segment, and $\Delta_1 \not\supseteq \Delta_2$, $\Delta_2 \not\supseteq \Delta_1$. In particular, Δ_1 and Δ_2 are linked only if $\rho_1 \cong \rho_2$ (recall that we require ρ_1, ρ_2 to be unitary).

Now let α, β be two multisegments. We say β is obtained from α by performing a single *elementary operation* if we can form β from α by replacing a submultiset $\{\Delta_1, \Delta_2\}$ of α by

$$\begin{cases} \{\Delta_1, \Delta_2\} & \text{if } \Delta_1, \Delta_2 \text{ are not linked,} \\ \{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\} & \text{if } \Delta_1, \Delta_2 \text{ are linked and } \Delta_1 \cap \Delta_2 \neq \varnothing, \\ \{\Delta_1 \cup \Delta_2\} & \text{if } \Delta_1, \Delta_2 \text{ are linked and } \Delta_1 \cap \Delta_2 = \varnothing. \end{cases}$$

Definition 2.1. Let α, β be two multisegments. We define $\alpha \geq \beta$ if α can be obtained from β by performing a sequence of elementary operations. This gives a partial ordering on $\underline{\mathrm{Mseg}}$.

For any multisegment $\delta = \{\Delta_1, \ldots, \Delta_r\}$, we define

$$\mathrm{supp}(\delta) := \sum_{i=1}^{r} \Delta_i,$$

which is a multiset of supercuspidal representations. Then, it follows from the definition that $\alpha \geq \beta$ only if $\mathrm{supp}(\alpha) = \mathrm{supp}(\beta)$. Also, if $\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho$ and $\beta = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \beta_\rho$, then $\alpha \geq \beta$ if and only if $\alpha_\rho \geq \beta_\rho$ for every ρ.

It is proved in [23, §2] that the partial ordering in Definition 2.1 exactly corresponds to the closure ordering on the orbits of Vogan varieties.

Theorem 2.2 [23, Theorem 2.2]. *Let ϕ_1, ϕ_2 be two L-parameters of* $\mathrm{GL}_n(F)$. *The following are equivalent*:

(a) $\delta_{\phi_1} \geq \delta_{\phi_2}$.

(b) $\phi_1 \geq \phi_2$.

Here the $\geq$ in part (b) is the closure ordering on the associated Vogan variety.

2.3. ***Mœglin–Waldspurger algorithm.*** We recall the statement of the Mœglin–Waldspurger algorithm and introduce the related notation. For simplicity, we shall assume any multisegment in this section is in $\underline{\mathrm{Mseg}}_\rho$ for a fixed $\rho \in \mathcal{C}_{\mathrm{unit}}$, and omit ρ in the notation.

Suppose $\Delta_1 = [b_1, e_1]$ and $\Delta_2 = [b_2, e_2]$ are linked. We say Δ_1 *precedes* Δ_2 if $b_1 < b_2$ and $e_1 < e_2$. For $\Delta = [b, e]$, define

$$\Delta^- := \begin{cases} [b, e-1] & \text{if } b \neq e, \\ \varnothing & \text{otherwise.} \end{cases}$$

With these definitions, we are ready to state the Mœglin–Waldspurger algorithm.

Algorithm 2.3 (Mœglin–Waldspurger algorithm). Suppose α is a multisegment. We associate a segment $M(\alpha)$ as follows.

(1) Set e to be the largest end value of segments in α. Set $m := e$.

(2) Consider all segments in α with end value m. Among these, choose a segment with the largest base value and call it Δ_m.

(3) Consider the set of all segments in α that precede Δ_m with end value $m-1$. If this is empty, go to step (5). Otherwise, choose a segment from this set with the largest base value and call it Δ_{m-1}.

(4) Set $m := m-1$ and go to step (3).

(5) Return $M(\alpha) = [m, e]$.

Next, we inherit the following notation from the above procedure. Define $\alpha \setminus M(\alpha)$ to be the multisegment obtained from α by replacing Δ_i with Δ_i^- for all $m \leq i \leq e$ and removing empty sets.

Finally, we set

$$\widetilde{\alpha} := \{M(\alpha)\} + (\widetilde{\alpha \setminus M(\alpha)})$$

if $\alpha \neq \varnothing$ and $\widetilde{\alpha} := \varnothing$ otherwise.

For a general multisegment $\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho$, we define

$$\widetilde{\alpha} := \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \widetilde{\alpha_\rho}.$$

The main result of [21] is that Algorithm 2.3 computes the Zelevinsky involution on multisegments and also the Pyasetskii involution on L-parameters of $\mathrm{GL}_n(F)$.

Theorem 2.4 [21, Theorem II.13]. *For any L-parameter ϕ of $\mathrm{GL}_n(F)$, we have*

$$\delta_{\widehat{\phi}} = \widetilde{\delta_\phi}.$$

3. Rephrasing Mœglin–Waldspurger algorithm

We introduce certain notation and a useful observation (Lemma 3.3 below) for the Mœglin–Waldspurger algorithm. Then, we rephrase Algorithm 2.3 in Corollary 3.4.

We inherit the notation in Algorithm 2.3 in the following discussion. For any multisegment β, set $\beta^0 := \beta$ and $\beta^i := \beta^{i-1} \setminus M(\beta^{i-1})$, so that for any i, we have

$$\widetilde{\beta} = \{M(\beta^l)\}_{l=0}^{i-1} + \widetilde{\beta^i}.$$

Write $\beta = \{\Delta_j\}_{j \in J}$, and $\beta^i = \{\Delta_j^i\}_{j \in J^i}$. For $i > 0$, we fix an injection $J^i \hookrightarrow J^{i-1}$, which identifies J^i as a subset of J^{i-1}, with the conditions

- $\Delta_j^i \subseteq \Delta_j^{i-1}$, and
- if $\Delta_j^{i-1} \neq \Delta_j^i$, then $\Delta_j^i = (\Delta_j^{i-1})^-$.

In this way, we identify each J^i as a subset of $J = J^0$. Define

$$K^i := \{j \in J^i \mid \Delta_j^i \neq \Delta_j^{i+1}\}.$$

Then,

$$M(\beta^i) = \{e(\Delta_j^i) \mid j \in K^i\}.$$

Write $M(\beta^i) = [m^i, e^i]$. For $m^i \leq l \leq e^i$, let k_l^i be the unique index in K^i such that $e(\Delta_{k_l^i}^i) = l$. Now we rephrase Algorithm 2.3.

Lemma 3.1. *With the above notation, the following properties uniquely characterize $M(\beta^i) = [m^i, e^i]$ and $\{\Delta_{k_l^i}^i\}_{m^i \leq l \leq e^i}$:*

(1) $e^i = \max\{e(\Delta_j^i) \mid j \in J^i\}$.

(2) $b(\Delta_{k_{e^i}^i}^i) = \max\{b(\Delta_j^i) \mid j \in J^i,\ e(\Delta_j^i) = e^i\}$.

(3) $b(\Delta_{k_l^i}^i) = \max\{b(\Delta_j^i) \mid j \in J^i,\ e(\Delta_j^i) = l,\ b(\Delta_j^i) < b(\Delta_{k_{l+1}^i}^i)\}$

(4) $\{j \in J^i \mid e(\Delta_j^i) = m^i - 1,\ b(\Delta_j^i) < b(\Delta_{k_{m^i}^i}^i)\} = \varnothing$.

Finally, let $e := e^0 = \max_{j \in J}\{e(\Delta_j)\}$ and set

$$t := \max\{i \in \mathbb{Z}_{\geq 0} \mid e^i = e(M(\beta^i)) = e\} + 1.$$

Note that $t = \#\{j \in J \mid e(\Delta_j) = e\}$. When there are more than one multisegment involved in the argument, we write $J^i = J^i(\beta)$, $K^i = K^i(\beta)$, $e = e(\beta)$ and $t = t(\beta)$ to specify the multisegment β.

We give an example to demonstrate this notation.

Example 3.2. Consider $\beta = \{\Delta_j\}_{j\in J}$, where $J = \{1, 2, \dots, 8\}$, and

$$\Delta_1 = [2, 2],\quad \Delta_2 = [0, 1],\quad \Delta_3 = [-2, 0],\quad \Delta_4 = [-3, -1],$$
$$\Delta_5 = [1, 2],\quad \Delta_6 = [-1, 1],\quad \Delta_7 = [-2, 0],\quad \Delta_8 = [-2, 1].$$

We have $e(\beta) = 2$ and $t(\beta) = 2$. Applying Algorithm 2.3 once, we obtain $M(\beta) = [m^0, e^0] = [-1, 2]$, and $\beta^1 = \beta \setminus M(\beta)$ is obtained from β by replacing $\{\Delta_j\}_{j=1}^4$ by $\{\Delta_j^-\}_{j=1}^4$. Thus we can choose $K^0 = \{1, 2, 3, 4\}$ and $(k_2^0, k_1^0, k_0^0, k_{-1}^0) = (1, 2, 3, 4)$. Note that one can also choose $K^0 = \{1, 2, 7, 4\}$ since $\Delta_3 = \Delta_7$.

With the choice $K^0 = \{1, 2, 3, 4\}$, we obtain that $J^1 = \{2, 3, \dots, 8\} \subseteq J$ and

$$\beta^1 = \{\Delta_j^1\}_{j=2}^8 = \{\Delta_j^-\}_{j=2}^4 \sqcup \{\Delta_j\}_{j=5}^8.$$

Apply Algorithm 2.3 again, we obtain $M(\beta^1) = [m^1, e^1] = [0, 2]$ and the only choice of K^1 is $\{5, 6, 7\}$ with $(k_2^1, k_1^1, k_0^1) = (5, 6, 7)$. We obtain $J^2 = \{2, 3, \dots, 8\} \subseteq J^1 \subseteq J$ and

$$\beta^2 = \{\Delta_j^2\}_{j=2}^8 = \{\Delta_j^-\}_{j=2}^4 \sqcup \{\Delta_j^-\}_{j=5}^7 \sqcup \{\Delta_8\}.$$

Observe that in the above example, $K^0 \cap K^1 = \varnothing$, or in other words, the index sets $\{K^i(\beta)\}_{i=0}^{t(\beta)-1}$ are mutually disjoint. One can also check this on the multisegment in Example 4.6, which is slightly more complicated. We prove this interesting phenomenon for all multisegments along with other properties in the following lemma.

Lemma 3.3. *With the notation developed in this section, the following holds for any multisegment* β:

(a) $m^0 \le m^1 \le \cdots \le m^{t-1}$.

(b) *For any* $0 \le i \le t-1$ *and* $m^i \le l \le e$, *we have containment*

$$\Delta^0_{k_l^0} \subseteq \Delta^1_{k_l^1} \subseteq \cdots \subseteq \Delta^i_{k_l^i}.$$

(c) *The sets* $\{K^i\}_{i=0}^{t-1}$ *are mutually disjoint.*

(d) *For* $0 \le i \le t-1$ *and* $m^i \le l \le e$, *we have*

$$\Delta_{k_l^i} = \Delta^0_{k_l^i} = \Delta^1_{k_l^i} = \cdots = \Delta^i_{k_l^i} \supsetneq (\Delta^i_{k_l^i})^- = \Delta^{i+1}_{k_l^i} = \cdots = \Delta^{t-1}_{k_l^i}.$$

Proof. First, observe that part (a) implies that

$$m^0 \le m^1 \le \cdots \le m^i \le l \le e,$$

and hence the indices $k_l^0, \dots, k_l^{i-1}$ in part (b) are well defined. Also, part (d) is a direct consequence of part (c). To prove parts (a), (b), (c), we apply induction on $t = t(\beta)$. If $t = 1$, the conclusions trivially hold. We assume $t > 1$ from now on.

We claim that for any $l \in \{e, e-1, \ldots, m^{t-1}\}$, the following hold for any $0 \leq i < t-1$:

(i) $m^i \leq l$.

(ii) $\Delta^i_{k^i_l} \subseteq \Delta^{t-1}_{k^{t-1}_l}$.

(iii) k^{t-1}_l is not in K^i.

We demonstrate that the claims imply the desired conclusion before we verify the claims. Set $J' := J \setminus K^{t-1}$ and $\beta' := \{\Delta_j\}_{j \in J'}$. Then,

$$t(\beta') = \#\{j \in J' \mid e(\Delta_j) = e\} = t-1.$$

Also, for any $0 \leq i < t-1$, claim (iii) implies $K^{t-1} \cap K^i = \varnothing$, and hence claim (ii) implies that for $m^i \leq l \leq e^i$,

$$\Delta^i_{k^i_l} \subseteq \Delta^{t-1}_{k^{t-1}_l} = \Delta^i_{k^{t-1}_l}.$$

This gives $b(\Delta^i_{k^i_l}) \geq b(\Delta^i_{k^{t-1}_l})$. From this inequality, for $0 \leq i < t-1$ and $m^i \leq l \leq e^i$, we inductively check that $[m^i, e^i]$ and $\Delta^i_{k^i_l} \in (\beta')^i$ satisfy the properties in Lemma 3.1 for β'. Thus, we can choose $K^i(\beta') = K^i(\beta)$. The induction hypothesis for β' then implies that:

(a') $m^0 \leq \cdots \leq m^{t-2}$.

(b') For any $0 \leq i \leq t-2$ and $m^i \leq l \leq e$, we have containment

$$\Delta^0_{k^0_l} \subseteq \Delta^1_{k^1_l} \subseteq \cdots \subseteq \Delta^i_{k^i_l}.$$

(c') The sets $\{K^i\}_{i=0}^{t-2}$ are mutually disjoint.

Thus, together with the claims, this verifies conditions (a), (b) and (c) for β.

Now we prove the claims by applying induction on l. When $l = e$, claim (i) trivially holds. Claim (ii) follows from Lemma 3.1(2). For claim (iii), write $\Delta^0_{k^{t-1}_e} = [x, y]$. Then, $\Delta^t_{k^{t-1}_e} = [x, y-s]$ where $s = \#\{0 \leq i < t-1 \mid k^{t-1}_e \in K^i\}$. Since $e(\Delta^{t-1}_{k^{t-1}_e}) = e$ and

$$y = e(\Delta^0_{k^{t-1}_e}) \leq \max_{\Delta^0 \in \beta}\{e(\Delta^0)\} = e$$

by definition, we obtain that

$$e = y - s \leq y \leq e,$$

which implies that $s = 0$. This completes the verification of the claims for $l = e$.

Suppose $e > r \geq m^{t-1}$ and the claims are already verified for $l = r+1$. We are going to verify the claims for $l = r$.

First, for any $0 \leq i < t-1$, claim (ii) for $l = r+1$ gives $\Delta^i_{k^i_{r+1}} \subseteq \Delta^{t-1}_{k^{t-1}_{r+1}}$. Combining with Lemma 3.1(3), we obtain that

$$b(\Delta^{t-1}_{k^{t-1}_r}) < b(\Delta^{t-1}_{k^{t-1}_{r+1}}) \leq b(\Delta^i_{k^i_{r+1}}). \tag{3-1}$$

On the other hand, claim (iii) for $l = r+1$ implies that $k^{t-1}_{r+1} \neq k^j_r$ for any $0 \leq j < t-1$, and hence

$$\Delta^0_{k^{t-1}_r} = \Delta^1_{k^{t-1}_r} = \cdots = \Delta^i_{k^{t-1}_r} = \cdots = \Delta^{t-1}_{k^{t-1}_r}. \tag{3-2}$$

In particular, their base values are all the same. As a consequence, the set

$$\{j \in J^i \mid e(\Delta^i_j) = r,\ b(\Delta^i_j) < b(\Delta^i_{k^i_{r+1}})\}$$

is nonempty since it contains k^{t-1}_r. We conclude that $m^i \leq r$ by Lemma 3.1(4). This proves claim (i) for $l = r$.

Next, for any $0 \leq i \leq t-1$, (3-1) and (3-2) give

$$e(\Delta^i_{k^{t-1}_r}) = e(\Delta^{t-1}_{k^{t-1}_r}) = r, \quad b(\Delta^i_{k^{t-1}_r}) = b(\Delta^{t-1}_{k^{t-1}_r}) < b(\Delta^i_{k^i_{r+1}}).$$

Therefore, k^{t-1}_r is in the set

$$\{j \in J^i \mid e(\Delta^i_j) = r,\ b(\Delta^i_j) < b(\Delta^i_{k^i_{r+1}})\}.$$

We conclude that $b(\Delta^i_{k^{t-1}_r}) \leq b(\Delta^i_{k^i_r})$ by Lemma 3.1(3). This verifies claim (ii) for $l = r$.

Finally for claim (iii), suppose the contrary that $k^{t-1}_r \in K^i$ for some $0 \leq i < t$ and take maximal such i. We must have

$$\Delta^{t-1}_{k^{t-1}_r} = \Delta^{t-2}_{k^{t-1}_r} = \cdots = \Delta^{i+1}_{k^{t-1}_r} = (\Delta^i_{k^{t-1}_r})^-,$$

and hence $k^{t-1}_r = k^i_{r+1}$. However, claim (ii) for $l = r+1$ gives

$$\Delta^{t-1}_{k^{t-1}_r} = \Delta^{i+1}_{k^i_{r+1}} = (\Delta^i_{k^i_{r+1}})^- \subsetneq \Delta^i_{k^i_{r+1}} \subseteq \Delta^{t-1}_{k^{t-1}_{r+1}}.$$

In particular, $b(\Delta^{t-1}_{k^{t-1}_r}) \geq b(\Delta^{t-1}_{k^{t-1}_{r+1}})$, contradicting Lemma 3.1(3). This completes the verification of claim (iii) for $l = r$ and the proof of the lemma. □

As a corollary, we can improve the statement of Lemma 3.1 when $0 \leq i \leq t-1$ as follows.

Corollary 3.4. *Let $\{K^i\}_{i=0}^{t(\beta)-1}$ be a collection of mutually disjoint subsets of $J(\beta)$. The following are equivalent*:

(a) *K^i can be labeled as $K^i = \{k^i_l\}_{m^i \le l \le e(\beta)}$ such that the following holds*:

(1) $e(\Delta_{k^i_l}) = l$.

(2) $b(\Delta_{k^i_e}) = \max\{b(\Delta_j) \mid j \in J \setminus \big(\bigsqcup_{0\le r<i} K^r\big),\ e(\Delta_j) = e\}$.

(3) $b(\Delta_{k^i_l}) = \max\{b(\Delta_j) \mid j \in J \setminus \big(\bigsqcup_{0\le r<i} K^r\big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k^i_{l+1}})\}$

(4) $\{j \in J \setminus \big(\bigsqcup_{0\le r<i} K^r\big) \mid e(\Delta_j) = m^i - 1,\ b(\Delta_j) < b(\Delta_{k^i_{m^i}})\} = \varnothing$.

(b) *The multisegment β^i can be obtained from β^{i-1} by replacing $\{\Delta_{k^i}\}_{k^i \in K^i}$ with $\{\Delta^-_{k^i}\}_{k^i \in K^i}$.*

Proof. The descriptions of parts (a) and (b) both determine the collection of segments $\{\Delta_{k^i_l}\}_{k^i \in K^i}$ uniquely. Thus, it suffices to examine the index sets $\{K^i\}_{i=0}^{t-1}$ chosen in the beginning of this section, where part (b) holds, satisfying all of the conditions in part (a).

Condition (1) holds by definition. Condition (2) holds since $e(\Delta^i_j) = e$ if and only if $\Delta^i_j = \Delta_j$ and $j \notin \bigsqcup_{0\le r<i} K^r$. For condition (3), observe that $\Delta^i_j = \Delta_j$ for $j \in J \setminus \bigsqcup_{0\le r<i} K^r$, and hence Lemma 3.1(3) implies that

$$b(\Delta_{k^i_l}) \ge \max\Big\{b(\Delta_j) \;\Big|\; j \in J \Big\backslash \Big(\bigsqcup_{0\le r<i} K^r\Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k^i_{l+1}})\Big\}.$$

On the other hand, Lemma 3.3(c) shows that

$$k^i_l \in J \Big\backslash \Big(\bigsqcup_{0\le r<i} K^r\Big),$$

so the equality holds. By the same observation, Lemma 3.1(4) implies condition (4). This completes the proof of the corollary. □

4. Proof of Lemma 1.3

4.1. ***Rephrasing Lemma 1.3 by multisegments.*** We rephrase Lemma 1.3 using the notation of multisegments recalled in Section 2. First, we define multisegments of *Arthur type*.

Definition 4.1. We say a multisegment α is of Arthur type if $\alpha = \delta_\phi$ for some L-parameter ϕ of Arthur type. The following is an equivalent but more explicit description.

1. For an irreducible local Arthur parameter $\rho \otimes \mathrm{Sym}^d \otimes \mathrm{Sym}^a$, we associate a multisegment

$$\delta_{\rho,d,a} := \Big\{\Big[\frac{a-d}{2}, \frac{a+d}{2}\Big]_\rho, \Big[\frac{a-d}{2} - 1, \frac{a+d}{2} - 1\Big]_\rho, \dots, \Big[\frac{-a-d}{2}, \frac{-a+d}{2}\Big]_\rho\Big\}.$$

2. For a local Arthur parameter of the form

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}, \tag{4-1}$$

we associate a multisegment

$$\delta_\psi := \sum_{\rho} \sum_{i \in I_\rho} \delta_{\rho, d_i, a_i}.$$

3. We say a multisegment α is of *Arthur type* if $\alpha = \delta_\psi$ for some ψ of the form (4-1).

With this definition, we may rephrase Lemma 1.3 as follows.

Lemma 4.2. *Let α be a multisegment of Arthur type. If β is a multisegment such that $\beta \geq \alpha$ and $\widetilde{\beta} \geq \widetilde{\alpha}$, then $\beta = \alpha$.*

4.2. *Preparation for reduction.* In this subsection we prove Proposition 4.3 below, which is the key to the reduction argument in the proof of Lemma 1.3. The motivation for the formulation of Proposition 4.3 is discussed in the introduction. Note that some of the statements in this subsection are similar to those in [22], but the proofs are different.

Throughout this subsection, we fix $\rho \in \mathcal{C}_{\mathrm{unit}}$ and write $\delta_{d,a} := \delta_{\rho,d,a}$, $[b, e] := [b, e]_\rho$ to simplify the notation.

Proposition 4.3. *Suppose α is of Arthur type with $\alpha = \delta_\psi$, where*

$$\psi = \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}.$$

Set

$$a + d := \max\{a_i + d_i \mid i \in I_\rho\},$$
$$d := \min\{d_i \mid i \in I_\rho,\ a_i + d_i = a + d\}.$$

Then, for any multisegment β such that $\beta \geq \alpha$ and $\widetilde{\beta} \geq \widetilde{\alpha}$, the following holds:

(i) *The multisegment β must contain a copy of $\delta_{d,a}$.*

(ii) *If we define α^- and β^- by removing a copy of $\delta_{d,a}$ from α and β, respectively, then $\alpha^- = \delta_{\psi^-}$ where*

$$\psi^- := \psi - \rho \otimes \mathrm{Sym}^{d} \otimes \mathrm{Sym}^{a},$$

and $\beta^- \geq \alpha^-$ and $\widetilde{\beta^-} \geq \widetilde{\alpha^-}$.

We first show that the choice of $a + d$ and d in Proposition 4.3 gives the following bounds. The definition of β^i and other notation in the proof can be found in Section 3.

Lemma 4.4. *Under the setting of Proposition 4.3, for any $i \geq 0$ and $[x, y] \in \beta^i$, the following holds*:

(a) $x \geq \frac{-a-d}{2}$.

(b) $y \leq \frac{a+d}{2}$. *If equality holds, then* $[x, y] \in \beta^l$ *for any* $0 \leq l \leq i$ *and* $x \leq \frac{a-d}{2}$.

Proof. For arbitrary multisegments α, β, we have the following observations:

- If $\beta \geq \alpha$, then $\mathrm{supp}(\beta) = \mathrm{supp}(\alpha)$.
- For $i \geq 0$, $\mathrm{supp}(\beta^i) = \mathrm{supp}(M(\beta^i)) + \mathrm{supp}(\beta^{i+1})$. In particular, we have $\mathrm{supp}(\beta) \supseteq \mathrm{supp}(\beta^i)$.

Now we return to the setting of Proposition 4.3. The choice of $a+d$ implies that

$$\begin{aligned} \tfrac{1}{2}(a+d) &= \max\{x \in \mathbb{R} \mid \rho|\cdot|^x \in \mathrm{supp}(\alpha) = \mathrm{supp}(\beta)\}, \\ \tfrac{1}{2}(-a-d) &= \min\{x \in \mathbb{R} \mid \rho|\cdot|^x \in \mathrm{supp}(\alpha) = \mathrm{supp}(\beta)\}, \end{aligned} \tag{4-2}$$

which shows part (a) and the first part of part (b) by the observations above.

Next, we show the second part of part (b) with the notation developed in Section 3 for β. Take a $j \in J^i$ such that $\Delta^i_j = [x, y]$. We have

$$\Delta^0_j \supseteq \Delta^1_j \supseteq \cdots \supseteq \Delta^i_j = [x, y].$$

Moreover, if any of the inclusions is strict, then $e(\Delta^0_j) > y$. If $y = \frac{a+d}{2}$, this cannot happen by (4-2). Therefore, we conclude that $[x, y] = \Delta^l_j \in \beta^l$ for any $0 \leq l \leq i$.

Finally, we verify the property that if $\left[x, \frac{a+d}{2}\right] \in \beta$ then $x \leq \frac{a-d}{2}$. If $\beta = \alpha$, then

$$\left[x, \tfrac{1}{2}(a+d)\right] \in \left\{\left[\tfrac{1}{2}(a+d) - d_i, \tfrac{1}{2}(a+d)\right] \mid i \in I_\rho,\ a_i + d_i = a+d\right\},$$

and hence $x \leq \frac{a-d}{2}$ by the definition of d. In general, β is obtained from α by performing a sequence of elementary operations. By (4-2), the desired property is preserved under each elementary operation, and hence β also has this property. This completes the proof of the lemma. □

For part (ii) in Proposition 4.3, we need the following computation, which works for a general multisegment β. We obtain the same conclusion as [22, Lemma 3.18] but with slightly weaker conditions.

Lemma 4.5. *Suppose a multisegment β contains a copy of*

$$\delta_{b,e,s} := \{[b, e], [b-1, e-1], \ldots, [b-s, e-s]\},$$

and any segment $[x, y] \in \beta$ satisfies

(i) $x \geq b-s$,

(ii) $y \leq e$.

Define β^- *by removing a copy of* $\delta_{b,e,s}$ *from* β. *Then we have*

$$\widetilde{\beta} = \widetilde{\beta^-} + \widetilde{\delta}_{b,e,s},$$

where

$$\widetilde{\delta}_{b,e,s} = \{[e-s, e], [e-s-1, e-1], \ldots, [b-s, b]\}.$$

We give an example to illustrate the reduction process in the proof.

Example 4.6. Let $\beta = \sum_{i=0}^{4} \{\Delta_{k^i}\}_{k^i \in K^i}$, where

$$\begin{aligned}
&\{\Delta_{k^0}\}_{k^0 \in K^0} = \{[2,2],[1,1],[0,0],[-1,-1],[-2,-2]\},\\
&\{\Delta_{k^1}\}_{k^1 \in K^1} = \{[1,2],[-1,1],[-\mathbf{2},\mathbf{0}],[-\mathbf{3},-\mathbf{1}]\},\\
&\{\Delta_{k^2}\}_{k^2 \in K^2} = \{[0,2],[-\mathbf{1},\mathbf{1}],[-3,0]\},\\
&\{\Delta_{k^3}\}_{k^3 \in K^3} = \{[\mathbf{0},\mathbf{2}],[-3,1]\},\\
&\{\Delta_{k^4}\}_{k^4 \in K^4} = \{[-2,2]\}.
\end{aligned} \tag{4-3}$$

Note that β contains $\delta_{0,2,3} = \{[0,2],[-1,1],[-2,0],[-3,-1]\}$, and the pair $(\beta, \delta_{0,2,3})$ satisfies assumptions (i), (ii) in the above lemma. Also, the subsets $\{K^i\}_{i=0}^4$ of $J(\beta) = \bigsqcup_{i=0}^4 K^i$ satisfy all of the conditions in Corollary 3.4(a), where recall that we let k_l^i denote the index in K^i such that $e(\Delta_{k_l^i}) = l$. Therefore, $\beta^4 = \sum_{i=0}^4 \{\Delta_{k^i}^-\}_{k^i \in K^i}$, and

$$\widetilde{\beta} = \{M(\beta^i)\}_{i=0}^4 + \widetilde{\beta^4} = \{[-2,2],[-1,2],[0,2],[1,2],[2,2]\} + \widetilde{\beta^4}.$$

For $-1 = e-s \le l \le e = 2$, we let $r_l := \max\{0 \le r \le 4 \mid \Delta_{k_l^r} = [e-s-l, e-l]\}$. Thus, $(r_2, r_1, r_0, r_{-1}) = (3,2,1,1)$. The corresponding segments $\Delta_{k_l^{r_l}}$ are displayed in bold text in (4-3).

We may construct the segment β^- from β by removing $\{\Delta_{k_l^{r_l}}\}_{-1 \le l \le 2}$. Let $J^- := J \setminus \{k_l^{r_l}\}_{-1 \le l \le 2}$, which we identify with $J(\beta^-)$. We define the mutually disjoint subsets $\{(K^i)^-\}_{i=0}^3$ of J^- by removing $\{\Delta_{k_l^{r_l}}\}_{-1 \le l \le 2}$ in (4-3) and then "push" the segments below them upward accordingly. That is, we define

$$\begin{aligned}
&\{\Delta_{(k^0)^-}\}_{(k^0)^- \in (K^0)^-} = \{[2,2],[1,1],[0,0],[-1,-1],[-2,-2]\},\\
&\{\Delta_{(k^1)^-}\}_{(k^1)^- \in (K^1)^-} = \{[1,2],[-1,1],[-3,0]\},\\
&\{\Delta_{(k^2)^-}\}_{(k^2)^- \in (K^2)^-} = \{[0,2],[-3,1]\},\\
&\{\Delta_{(k^3)^-}\}_{(k^3)^- \in (K^3)^-} = \{[-2,2]\}.
\end{aligned}$$

It follows that the mutually disjoint subsets $\{(K^i)^-\}_{i=0}^3$ of J^- satisfy all of the conditions in Corollary 3.4(a). Therefore, $(\beta^-)^3 = \sum_{i=0}^3 \{\Delta_{(k^i)^-}^-\}_{(k^i)^- \in (K^i)^-}$, and

$$\widetilde{\beta^-} = \{M((\beta^-)^i)\}_{i=0}^3 + \widetilde{(\beta^-)^3} = \{[-2,2],[0,2],[1,2],[2,2]\} + \widetilde{(\beta^-)^3}.$$

Now observe that β^4 contains a copy of $\{\Delta_{k_l^{r_l}}^-\}_{-1\le l\le 3} = \delta_{0,1,3}$ and the pair $(\beta^4, \delta_{0,1,3})$ satisfies assumptions (i), (ii) in the lemma. Moreover, $(\beta^4)^-$, which is obtained from β^4 by removing a copy of $\delta_{0,1,3}$, is exactly $(\beta^-)^3$, and $\{M(\beta^i)\}_{i=0}^4$ equals $\{M((\beta^-)^i)\}_{i=0}^3 \sqcup \{[2,-1]\}$. We conclude that the lemma holds for the pair $(\beta, \delta_{0,2,3})$ if it holds for the pair $(\beta^4, \delta_{0,1,3})$.

Now we apply the reduction process in the above specific example to prove the lemma in general cases.

Proof of Lemma 4.5. We use the notation developed in Section 3 for β, which we recall briefly as follows:

- $\beta = \{\Delta_j\}_{j\in J(\beta)}$.
- $e(\beta) := \max\{e(\Delta_j)\}_{j\in J(\beta)}$.
- $t(\beta) := \#\{j \in J(\beta) \mid e(\Delta_j) = e(\beta)\}$.
- $\{K^i(\beta)\}_{i=0}^{t-1}$ is a collection of (mutually disjoint) subsets of $J(\beta)$ satisfying all conditions in part (a) of Corollary 3.4.

Note that assumption (ii) implies that the number e matches $e(\beta)$. We shall simply write $J = J(\beta)$, $t = t(\beta)$, $K^i = K^i(\beta)$ in the following discussion.

We apply induction on $\ell := e-b+1$ to prove the lemma. When $\ell = 1$, it follows directly from Algorithm 2.3 that $M(\beta) = \widetilde{\delta}_{b,e,s}$, $\beta^1 = \beta^-$, and hence

$$\widetilde{\beta} = M(\beta) + \widetilde{\beta^1} = \widetilde{\beta^-} + \widetilde{\delta}_{b,e,s}.$$

From now on, we assume $\ell > 1$ and the conclusions are already established for the case that $e-b+1 = \ell-1$.

First, we give the following observations:

- $\Delta_{k_e^i} = [b,e]$ for some $0 \le i \le t-1$.
- If $\Delta_{k_{e-r}^i} = [b-r, e-r]$, and β^i contains a $[b-r-1, e-r-1]$ for some $0 \le r < s$, then $\Delta_{k_{e-r-1}^i} = [b-r-1, e-r-1]$.

As a consequence, $\bigsqcup_{i=0}^{t-1}\{\Delta_{k^i}\}_{k^i\in K^i}$ must contain a copy of $\delta_{b,e,s}$. Thus, for each $e-s \le l \le e$, the set

$$\{0 \le r \le t-1 \mid \Delta_{k_l^r} = [l-e+b, l]\}$$

is nonempty, and we define r_l to be the maximum of this set.

Next, we show that

$$r_{e-s} \le r_{e-s+1} \le \cdots \le r_e. \tag{4-4}$$

Indeed, for $e-s<l\le e$, parts (b) and (d) of Lemma 3.3 and the definition of r_l imply that for any $r_l<i\le t-1$,

$$[l-e+b,l]=\Delta_{k_l^{r_l}}\subsetneq\Delta_{k_l^i}.$$

Thus, Corollary 3.4(a)(3) shows that if $l-1\ge m^i$, then

$$b(\Delta_{k_{l-1}^i})<b(\Delta_{k_l^i})\le l-e+b-1,$$

and hence $\Delta_{k_{l-1}^i}\ne[(l-1)-e+b,l-1]$. This shows that $r_{l-1}\le r_l$, and hence (4-4) holds.

Now set $J^-:=J\setminus\{k_l^{r_l}\}_{l=e-s}^e$. We have $\beta^-=\{\Delta_j\}_{j\in J^-}$. For $0\le i\le t-2$, define mutually disjoint subsets $(K^i)^-$ of J^- as follows.

$$(K^i)^-:=\begin{cases}K^i & \text{if } i<r_{e-s},\\ \{k_e^i,\dots,k_{l(i)+1}^i\}\sqcup\{k_{l(i)}^{i+1},\dots,k_{m^{i+1}}^{i+1}\} & \text{if } r_{l(i)}\le i<r_{l(i)+1},\\ K^{i+1} & \text{if } i\ge r_e.\end{cases}$$

Note that if i is not in the first or the third case above, i.e., $r_e>i\ge r_{e-s}$, then the index $l(i)=\max\{e>l\ge e-s\mid i\ge r_{l(i)}\}$ is the unique index such that $r_{l(i)}\le i<r_{l(i)+1}$ holds.

We claim that the collection of index sets $\{(K^i)^-\}_{i=0}^{t-2}$ satisfies all of the conditions in Corollary 3.4(a) for β^-, and hence we can compute $\widetilde{\beta^-}$ using these index sets as described in part (b) of the same corollary.

Before verifying the claim, we demonstrate that the claim implies the desired conclusion. Define

$$K^-:=\bigsqcup_{i=0}^{t-2}(K^i)^-,\quad K:=\bigsqcup_{i=0}^{t-1}K^i.$$

Observe that $b(\Delta_{k_{e-s}^{r_{e-s}}})=b-s$, and hence by assumption (i), $M(\beta^{r_{e-s}})=[e-s,e]$. This implies that

$$K^-\sqcup\{k_l^{r_l}\}_{l=e-s}^e=K. \tag{4-5}$$

As a consequence of the above equality, we have

$$\beta^t=(\beta^-)^{t-1}\sqcup\delta_{b,e-1,s},$$

and hence β^t contains a copy of $\delta_{b,e-1,s}$. Also, β^t satisfies assumptions (i), (ii) with e replaced by $e-1$ according to its construction. Thus, the induction hypothesis implies that

$$\widetilde{\beta}^t=\widetilde{(\beta^-)^{t-1}}\sqcup\widetilde{\delta}_{b,e-1,s}. \tag{4-6}$$

Another consequence of (4-5) is that

$$\{M((\beta^{-})^{i})\}_{i=0}^{t-2}\sqcup\{[e-s,e]\}=\{M(\beta^{i})\}_{i=0}^{t-1}. \tag{4-7}$$

Indeed, observe that as multisets over $\mathcal{C}$,

$$\sum_{i=0}^{t-1}M(\beta^{i})=\sum_{k\in K}\{e(\Delta_k)\},$$

and the left-hand side of the above equality uniquely determines the collection $\{M(\beta^{i})\}_{i=0}^{t-1}$. The same argument works for β^{-}. Therefore, (4-5) implies (4-7).

Combining (4-6) and (4-7), we obtain that

$$\begin{aligned}\widetilde{\beta}&=\{M(\beta^{i})\}_{i=0}^{t-1}+\widetilde{\beta^{t}}=\{M((\beta^{-})^{i})\}_{i=0}^{t-2}\sqcup\{[e-s,e]\}+\widetilde{(\beta^{-})^{t-1}}\sqcup\widetilde{\delta}_{b,e-1,s}\\&=\widetilde{\beta^{-}}+\widetilde{\delta}_{e,b-1,s}.\end{aligned}$$

This gives the desired conclusion of the lemma.

Now we prove the claim. Write $(K^{i})^{-}=\{(k_l^{i})^{-}\}_{(m^{i})^{-}\le l\le e}$, where $e(\Delta_{(k_l^{i})^{-}})=l$. In other words,

$$(m^{i})^{-}=\begin{cases}m^{i} & \text{if } i<r_{e-s},\\ m^{i+1} & \text{if } i\ge r_{e-s},\end{cases}\qquad (k_l^{i})^{-}=\begin{cases}k_l^{i} & \text{if } i<r_l,\\ k_l^{i+1} & \text{if } l\ge (m^{i})^{-} \text{ and } i\ge r_l.\end{cases}$$

Conditions (1) and (2) of Corollary 3.4(a) for $\{(K^{i})^{-}\}_{i=0}^{t-2}$ hold by the construction.

Now we verify condition (3) for each $\Delta_{(k_l^{i})^{-}}$, where $0\le i\le t-2$ and $e>l\ge(m^{i})^{-}$. We separate into three cases: $r_{l+1}\ge r_l>i$, $r_{l+1}>i\ge r_l$, and $i\ge r_{l+1}\ge r_l$. The key observation is

$$\Bigl\{j\in J\setminus\Bigl(\bigsqcup_{0\le r<i}K^{r}\Bigr)\;\Big|\;e(\Delta_j)=l\Bigr\}=\Bigl\{j\in J^{-}\setminus\Bigl(\bigsqcup_{0\le r<i}(K^{r})^{-}\Bigr)\;\Big|\;e(\Delta_j)=l\Bigr\}\sqcup\{k_l^{\max(i,r_l)}\}. \tag{4-8}$$

Case 1. Suppose that $r_{l+1}\ge r_l>i$. Then, $(k_l^{i})^{-}=k_l^{i}$, $(k_{l+1}^{i})^{-}=k_{l+1}^{i}$, and $b(\Delta_{k_l^{i}})\ge b(\Delta_{k_l^{r_l}})$ by Lemma 3.3(b) and (d). Thus Corollary 3.4(a)(3) for $\Delta_{k_l^{i}}$ and (4-8) give

$$\begin{aligned}&b(\Delta_{(k_l^{i})^{-}})\\&=b(\Delta_{k_l^{i}})=\max\Bigl\{b(\Delta_j)\;\Big|\;j\in J\setminus\Bigl(\bigsqcup_{0\le r<i}K^{r}\Bigr),\ e(\Delta_j)=l,\ b(\Delta_j)<b(\Delta_{k_{l+1}^{i}})\Bigr\}\\&=\max\Bigl(\Bigl\{b(\Delta_j)\;\Big|\;j\in J^{-}\setminus\Bigl(\bigsqcup_{0\le r<i}(K^{r})^{-}\Bigr),\ e(\Delta_j)=l,\ b(\Delta_j)<b(\Delta_{k_{l+1}^{i}})\Bigr\}\\&\qquad\qquad\sqcup\{b(\Delta_{k_l^{r_l}})\}\Bigr)\\&=\max\Bigl\{b(\Delta_j)\;\Big|\;j\in J^{-}\setminus\Bigl(\bigsqcup_{0\le r<i}(K^{r})^{-}\Bigr),\ e(\Delta_j)=l,\ b(\Delta_j)<b(\Delta_{(k_{l+1}^{i})^{-}})\Bigr\}.\end{aligned}$$

This verifies condition (3) in this case.

Case 2. Suppose that $r_{l+1} > i \geq r_l$. Then,

$$(k_l^i)^- = k_l^{i+1} \quad \text{and} \quad (k_{l+1}^i)^- = k_{l+1}^i.$$

We first check that for any $j^- \in J^- \setminus \big(\bigsqcup_{0\leq r\leq i-1}(K^r)^-\big)$ such that $e(\Delta_{j^-}) = l$, $b(\Delta_{j^-}) < b(\Delta_{k_{l+1}^i})$ if and only if $b(\Delta_{j^-}) < b(\Delta_{k_{l+1}^{i+1}})$. Indeed, Lemma 3.3(b) and (d) imply that

$$b(\Delta_{k_{l+1}^i}) = b(\Delta^i_{k_{l+1}^i}) \geq b(\Delta^{i+1}_{k_{l+1}^{i+1}}) = b(\Delta_{k_{l+1}^{i+1}}),$$

which shows one direction. For the other direction, Corollary 3.4(a)(3) for $\Delta_{k_l^i}$ and (4-8) imply that

$$\begin{aligned} b(\Delta_{k_l^i}) &= \max\Big\{b(\Delta_j) \;\Big|\; j \in J \setminus \Big(\bigsqcup_{0\leq r\leq i-1} K^r\Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^i})\Big\} \\ &\geq b(\Delta_{j^-}). \end{aligned}$$

Then, since $r_{l+1} \geq i+1 > i \geq r_l$, Lemma 3.3(b) and (d) imply that

$$b(\Delta_{j^-}) \leq b(\Delta_{k_l^i}) \leq b(\Delta_{k_l^{r_l}}) = l - e + b < l + 1 - e + b = b(\Delta_{k_{l+1}^{r_{l+1}}}) \leq b(\Delta_{k_{l+1}^{i+1}}).$$

In particular,

$$b(\Delta_{j^-}) < b(\Delta_{k_{l+1}^{i+1}}).$$

As a consequence, we obtain that

$$\begin{aligned} &b(\Delta_{(k_l^i)^-}) \\ &= b(\Delta_{k_l^{i+1}}) = \max\Big\{b(\Delta_j) \;\Big|\; j \in J \setminus \Big(\bigsqcup_{0\leq r< i+1} K^r\Big), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\Big\} \\ &= \max\Big(\Big\{b(\Delta_j) \;\Big|\; j \in J \setminus \Big(\bigsqcup_{0\leq r< i} K^r\Big), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\Big\} \setminus \{b(\Delta_{k_l^i})\}\Big) \\ &= \max\Big\{b(\Delta_j) \;\Big|\; j \in J^- \setminus \Big(\bigsqcup_{0\leq r< i} (K^r)^-\Big), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\Big\} \\ &= \max\Big\{b(\Delta_j) \;\Big|\; j \in J^- \setminus \Big(\bigsqcup_{0\leq r< i} (K^r)^-\Big), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i}})\Big\} \\ &= \max\Big\{b(\Delta_j) \;\Big|\; j \in J^- \setminus \Big(\bigsqcup_{0\leq r< i} (K^r)^-\Big), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{(k_{l+1}^{i})^-})\Big\}. \end{aligned}$$

This verifies condition (3) in this case.

Case 3. Suppose that $i \geq r_{l+1} \geq r_l$. Then, $(k_l^i)^- = k_l^{i+1}$ and $(k_{l+1}^i)^- = k_{l+1}^{i+1}$. Corollary 3.4(a)(3) for $\Delta_{k_l^{i+1}}$ and (4-8) imply that

$$\begin{aligned} &b(\Delta_{(k_l^i)^-}) \\ &= b(\Delta_{k_l^{i+1}}) = \max\Big\{ b(\Delta_j) \;\Big|\; j \in J \setminus \Big(\bigsqcup_{0 \leq r < i+1} K^r \Big),\, e(\Delta_j) = l,\, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}}) \Big\} \\ &= \max\Big(\Big\{ b(\Delta_j) \;\Big|\; j \in J^- \setminus \Big(\bigsqcup_{0 \leq r < i+1} (K^r)^- \Big),\, e(\Delta_j) = l,\, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}}) \Big\} \\ &\qquad\qquad \sqcup \{ b(\Delta_{k_l^{i+1}}) \} \Big) \\ &= \max\Big\{ b(\Delta_j) \;\Big|\; j \in J^- \setminus \Big(\bigsqcup_{0 \leq r < i} (K^r)^- \Big),\, e(\Delta_j) = l,\, b(\Delta_j) < b(\Delta_{(k_{l+1}^i)^-}) \Big\}. \end{aligned}$$

This verifies condition (3) in this case.

Finally, we verify condition (4). We separate into three cases: $i < r_{e-s}$, $r_{l(i)+1} > i \geq r_{l(i)}$ for some $e - s \leq l(i) < e$, and $e \geq r_e$.

Case 1. Suppose that $i < r_{e-s}$. Then, $(m^i)^- = m^i$ and $(k_{(m^i)^-}^i)^- = k_{m^i}^i$. Therefore,

$$\begin{aligned} &\Big\{ j \in J^- \setminus \Big(\bigsqcup_{0 \leq r < i} (K^r)^- \Big) \;\Big|\; e(\Delta_j) = (m^i)^- - 1,\, b(\Delta_j) < b(\Delta_{(k_{(m^i)^-}^i)^-}) \Big\} \\ &\qquad = \Big\{ j \in J^- \setminus \Big(\bigsqcup_{0 \leq r < i} (K^r)^- \Big) \;\Big|\; e(\Delta_j) = m^i - 1,\, b(\Delta_j) < b(\Delta_{k_{m^i}^i}) \Big\} \\ &\qquad \subseteq \Big\{ j \in J \setminus \Big(\bigsqcup_{0 \leq r < i} K^r \Big) \;\Big|\; e(\Delta_j) = m^i - 1,\, b(\Delta_j) < b(\Delta_{k_{m^i}^i}) \Big\}. \end{aligned}$$

Since the last set is empty by Corollary 3.4(a)(4) for $K^i(\beta)$, this verifies condition (4) in this case.

Case 2. Suppose that $r_{l(i)+1} > i \geq r_{l(i)}$ for some $e - s \leq l(i) < e$. We write $l = l(i)$ for simplicity in the following discussion. We have $(m^i)^- = m^{i+1}$ and $(k_{(m^i)^-}^i)^- = k_{m^{i+1}}^{i+1}$. Observe that $\bigsqcup_{0 \leq r < i} (K^r)^- \subseteq \bigsqcup_{0 \leq r < i+1} K^r$. Moreover, the difference set can be written down explicitly:

$$\Big(\bigsqcup_{0 \leq r < i+1} K^r \Big) \setminus \Big(\bigsqcup_{0 \leq r < i} (K^r)^- \Big) = \{ k_{e-s}^{r_{e-s}}, \ldots, k_l^{r_l} \} \sqcup \{ k_{l+1}^i, \ldots, k_e^i \}.$$

As a consequence, if $j \in J^- \setminus \big(\bigsqcup_{0 \leq r < i} (K^r)^- \big)$ but $j \notin J \setminus \big(\bigsqcup_{0 \leq r < i+1} K^r \big)$, then $j \in \{ k_{l+1}^i, \ldots, k_e^i \}$. In particular, since $i + 1 \leq r_{l+1}$, Lemma 3.3(a) implies

$$e(\Delta_j) \geq l + 1 \geq m^{r_{l+1}} \geq m^{i+1} > m^{i+1} - 1.$$

Therefore,

$$\begin{aligned}\Big\{j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) \;\Big|\; e(\Delta_j) = (m^i)^- - 1,\ b(\Delta_j) < b(\Delta_{(k^i_{(m^i)^-})^-})\Big\} \\ = \Big\{j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) \;\Big|\; e(\Delta_j) = m^{i+1} - 1,\ b(\Delta_j) < b(\Delta_{k^{i+1}_{m^{i+1}}})\Big\} \\ = \Big\{j \in J \setminus \Big(\bigsqcup_{0\le r<i+1} K^r\Big) \;\Big|\; e(\Delta_j) = m^{i+1} - 1,\ b(\Delta_j) < b(\Delta_{k^{i+1}_{m^{i+1}}})\Big\},\end{aligned}$$

which is empty by Corollary 3.4(a)(4) for $K^{i+1}(\beta)$, this verifies condition (4) in this case.

Case 3. Suppose $i \ge r_e$. Then $(m^i)^- = m^{i+1}$ and $(k^i_{(m^i)^-})^- = k^{i+1}_{m^{i+1}}$. Similar to the previous case, we have

$$\Big(\bigsqcup_{0\le r<i+1} K^r\Big) \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) = \{k^{r_{e-s}}_{e-s}, \dots, k^{r_e}_e\},$$

and hence

$$J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) = J \setminus \Big(\bigsqcup_{0\le r<i+1} K^r\Big).$$

Thus a similar argument as in the previous case verifies condition (4) in this case. This completes the verification of the claim and the proof of the lemma. □

We remark that we have

$$\delta_{d,a} = \delta_{\frac{a-d}{2},\frac{a+d}{2},a}, \qquad \delta_{a,d} = \widetilde{\delta}_{\frac{a-d}{2},\frac{a+d}{2},a}.$$

As a corollary, this gives an alternate proof of the following fact.

Lemma 4.7. *Consider a local Arthur parameter of the form*

$$\psi = \bigoplus_{i\in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i},$$

and let $\alpha = \delta_\psi$. *Then* $\widetilde{\alpha} = \delta_{\widehat{\psi}}$, *where*

$$\widehat{\psi} = \bigoplus_{i\in I_\rho} \rho \otimes \mathrm{Sym}^{a_i} \otimes \mathrm{Sym}^{d_i}.$$

Proof. We apply induction on $|I_\rho|$. Define $a+d, d$ as in Proposition 4.3. We apply Lemma 4.5 on α, which contains a copy of $\delta_{d,a} = \delta_{\frac{a-d}{2},\frac{a+d}{2},a}$. Note that the assumptions are verified by Lemma 4.4 with $\beta = \alpha$ and $i = 0$. We have

$$\widetilde{\alpha} = \widetilde{\alpha^-} + \widetilde{\delta}_{\frac{a-d}{2},\frac{a+d}{2},a} = \widetilde{\alpha^-} + \delta_{a,d},$$

where $\alpha^- = \alpha - \delta_{d,a} = \delta_{\psi^-}$ and

$$\psi^- = \psi - \rho \otimes \mathrm{Sym}^d \otimes \mathrm{Sym}^a.$$

If $|I_\rho| = 1$, then α^- is empty, and hence $\widetilde{\alpha} = \delta_{a,d} = \delta_{\widehat{\psi}}$. If $|I_\rho| > 1$, then the induction hypothesis shows that $\widetilde{\alpha^-} = \delta_{\widehat{\psi^-}}$, which implies that $\widetilde{\alpha} = \delta_{\widehat{\psi}}$. This completes the proof of the lemma. □

Proof of Proposition 4.3. We use the notation $e = e(\beta), t = t(\beta), K^i = K^i(\beta)$ defined in Section 3. We have $e = \frac{a+d}{2}$ by Lemma 4.4(b). Let

$$r := \min\{0 \le i \le t-1 \mid M(\beta^i) \supseteq [(-a+d)/2, (a+d)/2]\}.$$

Note that the right-hand side is nonempty since

$$\widetilde{\beta} \ge \widetilde{\alpha} = \delta_{\widehat{\psi}} \supseteq \delta_{a,d} \ni [(-a+d)/2, (a+d)/2].$$

Recall our notation that $M(\beta^r) = [m^r, e]$ and $K^r = \{k_e^r, \dots, k_{m^r}^r\}$. Corollary 3.4(a)(3) implies

$$b(\Delta_{k^r_{m^r}}) < b(\Delta_{k^r_{m^r+1}}) < \cdots < b(\Delta_{k^r_e}),$$

and hence

$$b(\Delta_{k^r_{m^r}}) \le b(\Delta_{k^r_e}) - (e - m^r) \le b(\Delta_{k^r_e}) - \frac{a+d}{2} + \frac{-a+d}{2} = b(\Delta_{k^r_e}) - a, \tag{4-9}$$

where we use $m^r \le (-a+d)/2$ given by the containment $[m^r, e] \supseteq [(-a+d)/2, e]$ in the second inequality. On the other hand, Lemma 4.4(b) gives

$$b(\Delta_{k^r_e}) \le \frac{a-d}{2}. \tag{4-10}$$

Therefore,

$$b(\Delta_{k^r_{m^r}}) \le \frac{a-d}{2} - a = \frac{-a-d}{2}.$$

By Lemma 4.4(a), the equality must hold, and hence all the inequalities in (4-9) and (4-10) are indeed equalities. In particular, for $m^r \le l \le e$, we have $\Delta_{k^r_l} = [l-d, l]$, We conclude that

$$\beta = \{\Delta_j\}_{j \in J} \supseteq \{\Delta_{k^r}\}_{k^r \in K^r} = \{[l-d, l]\}_{\frac{-a-d}{2} \le l \le \frac{a+d}{2}} = \delta_{d,a}.$$

This proves part (i).

For part (ii), it is clear that $\alpha^- = \delta_{\psi^-}$ and $\beta^- \ge \alpha^-$. It remains to show that $\widetilde{\beta^-} \ge \widetilde{\alpha^-}$. We apply Lemma 4.5 on β, which contains a copy of $\delta_{d,a} = \delta_{\frac{a-d}{2}, \frac{a+d}{2}, a}$. Note that the assumptions in Lemma 4.5 are verified by Lemma 4.4 with $i = 0$. Thus, we obtain that

$$\widetilde{\alpha} = \widetilde{\alpha^-} + \delta_{a,d}, \quad \widetilde{\beta} = \widetilde{\beta^-} + \delta_{a,d},$$

which implies that $\widetilde{\beta^-} \ge \widetilde{\alpha^-}$ since $\widetilde{\beta} \ge \widetilde{\alpha}$. This completes the proof of part (ii) and the proposition. □

4.3. ***Proof of Lemma 1.3.*** It is equivalent to prove Lemma 4.2. Write $\alpha = \delta_\psi$ where

$$\psi = \bigoplus_{\rho \in \mathcal{C}_{\mathrm{unit}}} \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}.$$

Let

$$\psi_\rho := \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}.$$

We have the decomposition $\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho$, where $\alpha_\rho = \delta_{\psi_\rho}$ and $\beta = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \beta_\rho$. The pair of inequalities $\beta \geq \alpha$ and $\widetilde{\beta} \geq \widetilde{\alpha}$ is equivalent to the pairs of inequalities $\beta_\rho \geq \alpha_\rho$, $\widetilde{\beta}_\rho \geq \widetilde{\alpha}_\rho$ for every ρ. Therefore, we may assume that

$$\psi = \psi_\rho = \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}$$

for some $\rho \in \mathcal{C}_{\mathrm{unit}}$, and adopt the notation in Proposition 4.3.

Apply induction on $k := |I_\rho|$. When $k = 1$, Proposition 4.3(i) implies that $\alpha = \delta_{d,a} = \beta$. Suppose that $k > 1$. We construct α^-, β^- as in Proposition 4.3(ii). Then, the induction hypothesis implies that $\alpha^- = \beta^-$, and hence

$$\alpha = \alpha^- + \delta_{d,a} = \beta^- + \delta_{d,a} = \beta.$$

This completes the proof of the lemma. □

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