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# THREE INVARIANTS OF GEOMETRICALLY VERTEX DECOMPOSABLE IDEALS

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**We study three invariants of geometrically vertex decomposable ideals: the Castelnuovo–Mumford regularity, the multiplicity, and the $a$-invariant. We show that these invariants can be computed recursively using the ideals that appear in the geometric vertex decomposition process.**

**As an application, we prove that the $a$-invariant of a geometrically vertex decomposable ideal is nonpositive. We also recover some previously known results in the literature including a formula for the regularity of the Stanley–Reisner ideal of a pure vertex decomposable simplicial complex, and proofs that some well-known families of ideals are Hilbertian. Finally, we apply our recursions to the study of toric ideals of bipartite graphs. Included among our results on this topic is a new proof for a known bound on the $a$-invariant of a toric ideal of a bipartite graph.**

## 1. Introduction

Vertex decomposable simplicial complexes and their associated Stanley–Reisner ideals have been extensively studied in the fields of combinatorial algebraic topology and combinatorial commutative algebra; for example, see [Dochtermann and Engström 2009; Hà and Woodroofe 2014; Knutson et al. 2009; Provan and Billera 1980; Woodroofe 2009]. These complexes are known to have many nice combinatorial properties. Such complexes are defined recursively via vertex decompositions into subcomplexes, and this suggests that one can study their structure and invariants by means of those recursions. A generalization of this concept, *geometric vertex decomposition*, was introduced by A. Knutson, E. Miller and A. Yong in [Knutson et al. 2009]. They used this computational-algebraic technique to study Schubert determinantal ideals associated to vexillary permutations.

Building upon this work, P. Klein and the second author in [Klein and Rajchgot 2021] introduced the notion of a *geometrically vertex decomposable ideal*, which is a generalization of the Stanley–Reisner ideal of a vertex decomposable simplicial

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complex. In fact, a geometrically vertex decomposable squarefree monomial ideal is precisely the Stanley–Reisner ideal of a vertex decomposable simplicial complex, with the geometric vertex decomposition given by the vertex decomposition of the complex. Other well-known families of geometrically vertex decomposable ideals include Schubert determinantal ideals [Klein and Rajchgot 2021] and toric ideals of bipartite graphs [Cummings et al. 2023]. The technique of geometric vertex decomposition has been increasingly useful in various algebro-geometric contexts including applications in liaison theory [Klein and Rajchgot 2021], Gröbner geometry of matrix Schubert varieties [Klein 2023; Klein and Weigandt 2022; Knutson et al. 2009], and the study of Hessenberg varieties [Cummings et al. 2024; Da-Silva and Harada 2023]. It was also shown in [Cummings et al. 2023; Klein and Rajchgot 2021] that geometrically vertex decomposable ideals have many algebraic properties in common with Stanley–Reisner ideals of vertex decomposable simplicial complexes.

The purpose of this paper is to study algebraic invariants of geometrically vertex decomposable ideals. We shall exploit their inherently recursive structure to derive recursive formulae for their invariants. Our recursions reduce the study of our original geometrically vertex decomposable ideal to the study of two related geometrically vertex decomposable ideals, each of which is in one less variable. The following theorem summarizes our main results about these invariants. In the statement below, the ideals $C_{y,I}$ and $N_{y,I}$ refer to ideals formed from the decomposition of $I$; complete definitions are found in Section 2. Furthermore, $\operatorname{reg}(R/I)$, $e(R/I)$, and $a(R/I)$ refer to the Castelnuovo–Mumford regularity, multiplicity, and $a$-invariant of $R/I$, respectively.

**Theorem 1.1.** *Suppose that $I \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ is a homogeneous, geometrically vertex decomposable ideal. Then there exists a variable $y$ such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is a geometric vertex decomposition, and $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable. If the geometric vertex decomposition is nondegenerate, then*

(1) (Corollary 3.3) $\operatorname{reg}(R/I) = \max\{\operatorname{reg}(R/N_{y,I}), \operatorname{reg}(R/C_{y,I}) + 1\}$,

(2) (Corollary 4.3) $e(R/I) = e(R/N_{y,I}) + e(R/C_{y,I})$, *and*

(3) (Corollary 5.7) $a(R/I) = \max\{a(R/N_{y,I}) + 1, a(R/C_{y,I}) + 1\}$.

*When the geometric vertex decomposition is degenerate, we have*

$$\operatorname{reg}(R/I) = \operatorname{reg}(R/N_{y,I}), \quad e(R/I) = e(R/N_{y,I}), \quad \textit{and} \quad a(R/I) = a(R/N_{y,I}).$$

A key observation of Knutson, Miller, and Yong [Knutson et al. 2009] is that the Hilbert series of an ideal $I$ that has a geometric vertex decomposition is related to the Hilbert series of the smaller ideals (see Theorem 2.3). We use this result to show that, in the case that $I$ is also geometrically vertex decomposable, there is a relation among the associated $h$-polynomials (see Theorem 2.4). The proof of Theorem 1.1 then relies heavily on this relation among the $h$-polynomials.

Applying these recursions to various classes of ideals that are previously known to be geometrically vertex decomposable (see [Cummings et al. 2023; Klein and Rajchgot 2021]), we are able to offer a new approach to recover several known results. As an example, the recursive formula for the regularity of Stanley–Reisner ideals of pure vertex decomposable simplicial complexes, independently found by H.T. Hà and R. Woodroofe [Hà and Woodroofe 2014] and S. Moradi and F. Khosh-Ahang [Moradi and Khosh-Ahang 2016], can be deduced from Theorem 1.1 (see Corollary 3.4).

Theorem 1.1 can also be used to show that all geometrically vertex decomposable ideals are "almost" Hilbertian. For a homogeneous ideal $I \subseteq R$, let $HP_{R/I}(t)$ denote the Hilbert polynomial of $R/I$, and let $HF_{R/I}(t)$ denote its Hilbert function. An ideal is *Hilbertian* if $HP_{R/I}(t) = HF_{R/I}(t)$ for all $t \geq 0$; this definition is attributed to S. Abhyankar (see [Abhyankar and Kulkarni 1989]). Recently, A. Stelzer and A. Yong [Stelzer and Yong 2023] proved that Schubert determinantal ideals are Hilbertian. This result is related to our situation because Schubert determinantal ideals are geometrically vertex decomposable [Klein and Rajchgot 2021, Section 5]. Because the $a$-invariant is intimately linked to when $HF_{R/I}(t)$ and $HP_{R/I}(t)$ agree, we can contribute the following result.

**Theorem 1.2** (Corollary 5.8). *Let $I \subset R$ be a proper, homogeneous, geometrically vertex decomposable ideal. Then $a(R/I) \leq 0$. Consequently, $HF_{R/I}(t) = HP_{R/I}(t)$ for all $t \geq 1$.*

Note that Theorem 1.2 implies that, except possibly at $t = 0$, the Hilbert function and the Hilbert polynomial of a geometrically vertex decomposable ideal agree. Under extra hypotheses, we are able to determine when there is also agreement at $t = 0$, and consequently, the ideal $I$ is Hilbertian.

In the last part of the paper, we apply our results to the class of toric ideals of bipartite graphs, which are known to be geometrically vertex decomposable by [Cummings et al. 2023] (which builds on [Constantinescu and Gorla 2018]). Our results complement and extend recent work on invariants of toric ideals of graphs; for example, see [Almousa et al. 2022; Biermann et al. 2017; Corso and Nagel 2009; D'Alì 2015; Galetto et al. 2019; Hà et al. 2019; Ohsugi and Hibi 1999; Tatakis and Thoma 2011; Villarreal 1995]. Recall that if $G = (V, E)$ is a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_q\}$, the *toric ideal of* $G$, denoted $I_G$, is the kernel of the $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}[e_1, \ldots, e_q] \to \mathbb{K}[x_1, \ldots, x_n]$ given by $\varphi(e_i) = x_j x_k$, where $e_i = \{x_i, x_j\} \in E$. Theorems 1.1 and 1.2 allow us to show the following results.

**Theorem 1.3** (Theorems 6.7 and 6.20). *Let $H$ be any subgraph of a bipartite graph $G$. Then*

(1) $\operatorname{reg}(I_H) \leq \operatorname{reg}(I_G)$,

(2) $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G)$, *and*

(3) $e(\mathbb{K}[E(G)]/I_H) \leq e(\mathbb{K}[E(G)]/I_G)$.

*Furthermore, if $G$ is connected, then $I_G$ is Hilbertian.*

Theorem 1.3 (1) was recently shown in [Almousa et al. 2022, Theorem 6.11] by A. Almousa, A. Dochtermann, and B. Smith using combinatorial techniques involving root polytopes and also in [Pinto and Villarreal 2023, Corollary 8.16] by M.V. Pinto and R.H. Villarreal using edge polytopes. Note that one could use the above results to obtain the upper bounds in terms of graph-theoretic invariants as those invariants of the complete bipartite graphs can be exactly computed (by the technique of geometric vertex decomposition or other techniques). Our technique not only gives a new proof for the regularity bound but can also be used to recover the results on precise values of the regularity, $a$-invariant, as well as multiplicity of toric ideals of Ferrer graphs (including complete bipartite graphs) in [Corso and Nagel 2009] by A. Corso and U. Nagel.

**Theorem 1.4** (Theorem 6.12). *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let $T_\lambda$ be the associated Ferrers graph. Write $I_\lambda = I_{T_\lambda}$ and $R = \mathbb{K}[E(T_\lambda)]$.*

(1) *If $n = 1$ or $\lambda_2 = 1$, then* $\operatorname{reg}(R/I_\lambda) = 0$.

(2) *If $\lambda_2 \geq 2$, and suppose that $\lambda = (\lambda_1, \ldots, \lambda_s, 1, 1, \ldots, 1)$ where $\lambda_s \geq 2$, then*

$$\operatorname{reg}(R/I_\lambda) = \min\{s-1, \{\lambda_j + j - 3 \mid 2 \leq j \leq s\}\}.$$

*In particular, if $G = K_{n,m}$ is a complete bipartite graph, then* $\operatorname{reg}(R/I_G) = \min\{n, m\} - 1$.

In addition to the above results, we consider the "gluing" procedure of G. Favacchio, J. Hofscheier, G. Keiper, and the last author [Favacchio et al. 2021] which "glues" an even cycle to a graph $G$ to form a new graph $H$. We relate the regularity, the $a$-invariant, and the multiplicity of $I_G$, when this ideal is geometrically vertex decomposable, to that of $I_H$ (see Theorem 6.9 and Corollary 6.10). To further illustrate the usefulness of our results, our techniques are used to explicitly compute all three invariants for all toric ideals of graphs that belong to a family first considered in [Galetto et al. 2019].

Our paper is structured as follows. In Section 2 we recall the relevant background on geometrically vertex decomposable ideals. In Sections 3–5, we consider the regularity, the multiplicity, and the $a$-invariant, respectively. In Section 6, we apply our results to study the invariants of toric ideals of (bipartite) graphs.

## 2. Background on geometrically vertex decomposable ideals

In this section we recall the notion of geometric vertex decomposition, introduced by Knutson, Miller and Yong in [Knutson et al. 2009], and geometrically vertex decomposable ideals, introduced by Klein and the second author in [Klein and Rajchgot 2021]. Due to the recursive nature of the theory of geometric vertex decomposition, this technique provides us with a convenient inductive set-up for studying properties and invariants of certain classes of ideals, as expanded upon in later sections.

Hereafter, we let $\mathbb{K}$ denote an arbitrary field. We let $R = \mathbb{K}[x_1, \dots, x_n]$ be a standard graded polynomial ring in $n$ variables. Fix a variable $y = x_j$. Then, for any $f \in R$, we can write $f = \sum_{i=0}^{d} \alpha_i y^i$, where, for each $i$, $\alpha_i$ is a polynomial in $\mathbb{K}[x_1, \dots, \hat{x}_j, \dots, x_n]$. For $f \neq 0$, we define the *initial y-form* denoted $\mathrm{in}_y(f)$ to be the sum of all the nonzero terms of $f$ having the highest power of $y$, that is, $\mathrm{in}_y(f) = \alpha_d y^d$. For an ideal $J \subset R$, define $\mathrm{in}_y(J) = \langle \mathrm{in}_y(f) \mid f \in J \rangle$. A monomial order $<$ on $R$ is said to be *y-compatible* if it satisfies $\mathrm{in}_<(\mathrm{in}_y(f)) = \mathrm{in}_<(f)$ for all $f \in R$, where $\mathrm{in}_<(f)$ is the initial term of $f$ with respect to $<$. It follows that, for such an order, we have $\mathrm{in}_<(\mathrm{in}_y(I)) = \mathrm{in}_<(I)$ for all ideals $I$.

Consider an ideal $I$ and a $y$-compatible monomial order on $R$. Suppose that $\mathcal{G} = \{g_1, \dots, g_m\}$ is a Gröbner basis of $I$ with respect to this monomial order, and for each $i = 1, \dots, m$, write $g_i = y^{d_i} q_i + r_i$, where $\mathrm{in}_y(g_i) = y^{d_i} q_i$ and $y$ does not divide any term of $q_i$. It follows that $\mathrm{in}_y(I) = \langle y^{d_i} q_i \mid 1 \le i \le m \rangle$. We define the ideals

$$C_{y,I} = \langle q_1, \dots, q_m \rangle \quad \text{and} \quad N_{y,I} = \langle q_i \mid d_i = 0 \rangle.$$

It is important to observe that the ideals $C_{y,I}$ and $N_{y,I}$ do not depend on the choice of Gröbner basis, and in particular do not depend on the choice of $y$-compatible monomial order, since $C_{y,I} = \mathrm{in}_y(I) : \langle y \rangle^\infty$ and $N_{y,I} + \langle y \rangle = \mathrm{in}_y(I) + \langle y \rangle$ by [Knutson et al. 2009, Theorem 2.1]; see also [Klein and Rajchgot 2021, Section 2].

**Definition 2.1.** When $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, we call this a *geometric vertex decomposition* of $I$ with respect to $y$. We say that the geometric vertex decomposition is *degenerate* if $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$ or if $C_{y,I} = \langle 1 \rangle$, and *nondegenerate* otherwise.

Recall that an ideal $I$ is *unmixed* if $I$ satisfies $\dim(R/I) = \dim(R/P)$ for all associated primes $P \in \mathrm{Ass}_R(R/I)$. We define the main object of study in this paper.

**Definition 2.2.** An ideal $I$ of $R = \mathbb{K}[x_1, \dots, x_n]$ is *geometrically vertex decomposable* if $I$ is unmixed and

(1) $I = \langle 1 \rangle$, or $I$ is generated by a (possibly empty) subset of variables of $R$, or

(2) there exists a variable $y = x_j$ of $R$ such that we have a geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, and the contractions of the ideals $C_{y,I}$ and $N_{y,I}$ to the ring $\mathbb{K}[x_1, \dots, \hat{x}_j, \dots, x_n]$ are geometrically vertex decomposable.

Thus, given a geometrically vertex decomposable ideal $I$, one can perform a geometric vertex decomposition with respect to some variable $y$ to obtain a geometrically vertex decomposable ideal $C_{y,I}$ and a geometrically vertex decomposable ideal $N_{y,I}$. Each of these ideals can then be decomposed into their own geometrically vertex decomposable ideals, and so on, until all ideals have the form of item (1) of Definition 2.2. We refer to such a process of repeatedly performing geometric vertex decompositions, where all ideals at all stages are geometrically vertex decomposable, as a *geometric vertex decomposition process*. We note that an ideal $I$ may have multiple different geometric vertex decomposition processes. Many well-known ideals are geometrically vertex decomposable. These include Stanley–Reisner ideals of vertex decomposable simplicial complexes, classes of generalized determinantal ideals (e.g., classical determinantal ideals, Schubert determinantal ideals, ideals of varieties of complexes), defining ideals of lower bound cluster algebras, and toric ideals of bipartite graphs [Cummings et al. 2023; Klein and Rajchgot 2021].

Our main results depend upon a relationship between the $h$-polynomials of $R/I$, $R/C_{y,I}$, and $R/N_{y,I}$. Recall that the *Hilbert series* of a graded $R$-module $M = \bigoplus_{i=1}^{\infty} M_i$ is the generating function

$$H_M(t) = \sum_{i=0}^{\infty} (\dim_{\mathbb{K}} M_i) t^i.$$

By the Hilbert–Serre theorem [Bruns and Herzog 1993, Corollary 4.1.8], the Hilbert series can be expressed as a rational function

$$H_M(t) = \frac{h_M(t)}{(1-t)^d},$$

where $h_M(t)$, the *h-polynomial* of $M$, is a polynomial with integer coefficients and $d = \dim M$. The following result, which is [Knutson et al. 2009, Theorem 2.1 (e)], relates the Hilbert series of an ideal to that of its geometric vertex decomposition; for completeness, we have included a proof.

**Theorem 2.3.** *Suppose that $I \subseteq R$ is a homogeneous ideal that has a geometric vertex decomposition with respect to $y$, that is,* $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. *Then*

$$H_{R/I}(t) = H_{R/(N_{y,I}+\langle y \rangle)}(t) + t H_{R/C_{y,I}}(t).$$

*Proof.* We have the short exact sequence

$$0 \longrightarrow \frac{R}{C_{y,I} \cap (N_{y,I} + \langle y \rangle)} \longrightarrow \frac{R}{C_{y,I}} \oplus \frac{R}{N_{y,I} + \langle y \rangle} \longrightarrow \frac{R}{C_{y,I} + N_{y,I} + \langle y \rangle} \longrightarrow 0.$$

Furthermore, note that $C_{y,I} + N_{y,I} + \langle y \rangle = C_{y,I} + \langle y \rangle$. Because $\mathrm{in}_y(I)$ and $I$ have the same Hilbert series (since $\mathrm{in}_<(\mathrm{in}_y(I)) = \mathrm{in}_<(I)$), the Hilbert series of $R/I$

satisfies

$$H_{R/I}(t) = H_{R/(N_{y,I}+\langle y\rangle)}(t) + H_{R/C_{y,I}}(t) - H_{R/(C_{y,I}+\langle y\rangle)}(t).$$

Since $y$ is a nonzero divisor on $R/C_{y,I}$, we have the short exact sequence

$$0 \longrightarrow \frac{R}{C_{y,I}}(-1) \xrightarrow{\times y} \frac{R}{C_{y,I}} \longrightarrow \frac{R}{C_{y,I}+\langle y\rangle} \longrightarrow 0,$$

which then implies that

$$H_{R/C_{y,I}}(t) - H_{R/(C_{y,I}+\langle y\rangle)}(t) = tH_{R/C_{y,I}}(t).$$

Consequently, $H_{R/I}(t) = H_{R/(N_{y,I}+\langle y\rangle)}(t) + tH_{R/C_{y,I}}(t)$, as desired. □

Note that Theorem 2.3 implies a relationship among the $h$-polynomials $h_{R/I}(t)$, $h_{R/(N_{y,I}+\langle y\rangle)}(t)$, and $h_{R/C_{y,I}}(t)$. To explicitly determine this relationship, one would need to know the dimension of the three rings. As shown below, we know the dimensions in the case that the homogeneous ideal $I \subseteq R$ is geometrically vertex decomposable. In fact, instead of the $h$-polynomial of $R/(N_{y,I}+\langle y\rangle)$, we use the $h$-polynomial of $R/N_{y,I}$.

**Theorem 2.4.** *Suppose that $I \subseteq R$ is a homogeneous ideal and $R/I$ is equidimensional. If there is a variable $y$ such that $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ is a nondegenerate geometric vertex decomposition, then the $h$-polynomial of $R/I$ satisfies*

$$h_{R/I}(t) = h_{R/N_{y,I}}(t) + th_{R/C_{y,I}}(t).$$

*Proof.* Because $R/I$ is equidimensional and the decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ is nondegenerate, we may apply [Klein and Rajchgot 2021, Lemma 2.8] to conclude that $\mathrm{ht}(I) = \mathrm{ht}(C_{y,I}) = \mathrm{ht}(N_{y,I}) + 1$.

Since $\dim(R) = \dim(R/J) + \mathrm{ht}(J)$ for any ideal $J$ in the polynomial ring $R$, we have

$$\dim(R/I) = \dim(R/C_{y,I}) = \dim(R/N_{y,I}) - 1 = \dim(R/(N_{y,I}+\langle y\rangle)) = d.$$

Therefore, by Theorem 2.3, we have

$$\frac{h_{R/I}(t)}{(1-t)^d} = \frac{h_{R/(N_{y,I}+\langle y\rangle)}(t)}{(1-t)^d} + \frac{th_{R/C_{y,I}}(t)}{(1-t)^d}.$$

To complete the proof, it suffices to show that $h_{R/(N_{y,I}+\langle y\rangle)}(t) = h_{R/N_{y,I}}(t)$. Because the generators of $N_{y,I}$ do not involve $y$, we then have a short exact sequence

$$0 \longrightarrow \frac{R}{N_{y,I}}(-1) \xrightarrow{\times y} \frac{R}{N_{y,I}} \longrightarrow \frac{R}{(N_{y,I}+\langle y\rangle)} \longrightarrow 0.$$

Consequently, the Hilbert series of $R/(N_{y,I}+\langle y\rangle)$ satisfies

$$\frac{h_{R/(N_{y,I}+\langle y\rangle)}(t)}{(1-t)^{d-1}} = H_{R'/N_{y,I}}(t) = (1-t)H_{R/N_{y,I}}(t) = \frac{(1-t)h_{R/N_{y,I}}(t)}{(1-t)^d}.$$

Comparing the numerator of both sides now gives the conclusion. □

## 3. Regularity of geometrically vertex decomposable ideals

We consider the (Castelnuovo–Mumford) regularity of geometrically vertex decomposable ideals. We derive a recursive formula for the regularity for this family that allows us to compute the regularity of various classes of ideals, e.g., Stanley–Reisner ideals of pure vertex decomposable simplicial complexes, and toric ideals of bipartite graphs in Section 6.

The (*Castelnuovo–Mumford*) *regularity* of a graded $R$-module $M$ is given by

$$\operatorname{reg}(M) = \max\{j-i \mid \beta_{i,j}(M) \neq 0\},$$

where $\beta_{i,j}(M)$ denotes the $(i,j)$-th graded Betti number that appears in the minimal graded free resolution of $M$. The following property, which relates the regularity to the degree of the $h$-polynomial in the Hilbert series, shall be of great use.

**Lemma 3.1** [Vasconcelos 1998, Corollary B.28]. *Let $I \subseteq R$ be a homogeneous ideal such that $R/I$ is Cohen–Macaulay. Then* $\operatorname{reg}(R/I) = \deg h_{R/I}(t)$.

Because all geometrically vertex decomposable ideals are Cohen–Macaulay, for this family of ideals, we can informally define the regularity of $R/I$ to be the degree of the $h$-polynomial.

We come to the main result of this section, which describes a recursion between regularity values of a geometrically vertex decomposable $I$ and the corresponding ideals $C_{y,I}$ and $N_{y,I}$. The proof relies on Theorem 2.4.

**Theorem 3.2.** *Suppose $I \subseteq R$ is a homogeneous and radical ideal such that $R/I$ is Cohen–Macaulay. Suppose $y$ is a variable such that* $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ *is a geometric vertex decomposition. If the decomposition is nondegenerate and $R/C_{y,I}$ and $R/N_{y,I}$ are Cohen–Macaulay, then*

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I)) = \max\{\operatorname{reg}(N_{y,I}),\ \operatorname{reg}(C_{y,I})+1\}.$$

*Otherwise, if the decomposition is degenerate, then*

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I)) = \operatorname{reg}(C_{y,I}) = \operatorname{reg}(N_{y,I}) \quad \text{if } C_{y,I} \neq \langle 1\rangle$$

*and*

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I)) = \operatorname{reg}(N_{y,I}) \quad \text{if } C_{y,I} = \langle 1\rangle.$$

*Proof.* Since $R/I$ is Cohen–Macaulay, $\operatorname{reg}(R/I) = \deg h_{R/I}(t)$ by Lemma 3.1.

If the geometric vertex decomposition is nondegenerate and $R/C_{y,I}$ and $R/N_{y,I}$ are Cohen–Macaulay, then by Theorem 2.4 we have

$$\begin{aligned}\operatorname{reg}(R/I) = \deg h_{R/I}(t) &= \max\{\deg h_{R/N_{y,I}}(t), \deg h_{R/C_{y,I}}(t) + 1\}\\ &= \max\{\operatorname{reg}(R/N_{y,I}), \operatorname{reg}(R/C_{y,I}) + 1\}.\end{aligned}$$

Note that the second equality follows from the fact the $h$-polynomial of a Cohen–Macaulay ring always has nonnegative coefficients (e.g., see [Stanley 1978, Corollary 3.11]), so there is no cancellation among the top-degree terms when the polynomials of Theorem 2.4 are added. To recover the statement of the theorem, use the fact that $\operatorname{reg}(R/J) = \operatorname{reg}(J) - 1$ for any proper homogeneous ideal $J$.

To show that $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I))$, note that, by [Klein and Rajchgot 2021, Corollary 4.11], we have $\operatorname{in}_y(I)$ is also Cohen–Macaulay. Since $\operatorname{in}_y(I)$ and $I$ have the same Hilbert series (as the Hilbert series of $\operatorname{in}_<(\operatorname{in}_y(I)) = \operatorname{in}_<(I)$), we get

$$\operatorname{reg}(R/I) = \deg h_{R/I}(t) = \deg h_{R/\operatorname{in}_y(I)}(t) = \operatorname{reg}(R/\operatorname{in}_y(I)),$$

as desired.

Now suppose that the decomposition is degenerate. If $C_{y,I} = \langle 1\rangle$, we have $R/I \cong R/(N_{y,I} + \langle y\rangle)$, and thus the claim follows. Otherwise, if $C_{y,I} \neq \langle 1\rangle$, the result follows directly from the fact that $I = \operatorname{in}_y(I) = C_{y,I} = N_{y,I}$, as shown in [Klein and Rajchgot 2021, Proposition 2.4]. □

**Corollary 3.3.** *Theorem 1.1, part* (1) *is true.*

*Proof.* Let $I$ be a geometrically vertex decomposable ideal. Then, by definition, there exists a variable $y$ such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is a geometric vertex decomposition and $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable. Since geometrically vertex decomposable ideals are Cohen–Macaulay and radical by [Klein and Rajchgot 2021, Corollary 4.5 and Proposition 2.10], the result is now immediate from Theorem 3.2. □

The above result recovers the recursive formula for the regularity of the Stanley–Reisner ideal of a pure vertex decomposable complex. Since this is our only result concerning vertex decomposable simplicial complexes and Stanley–Reisner ideals, we point the reader to [Klein and Rajchgot 2021, Section 2.1] for notation and terminology that is not explained. We want to highlight that our result is for *pure* simplicial complexes; the following result can be seen as giving new proofs for special cases of [Hà and Woodroofe 2014, Theorem 4.2] and [Moradi and Khosh-Ahang 2016, Corollary 2.11]; in particular, [Hà and Woodroofe 2014; Moradi and Khosh-Ahang 2016] do not require the vertex decomposable simplicial complex to be pure.

**Corollary 3.4.** *Let $\Delta$ be a pure vertex decomposable simplicial complex, and let $v$ be a shedding vertex of $\Delta$. If $I_\Delta$ is the Stanley–Reisner ideal of $\Delta$, then*

$$\operatorname{reg}(R/I_\Delta) = \max\{\operatorname{reg}(R/I_{\Delta_1}), \operatorname{reg}(R/I_{\Delta_2}) + 1\},$$

*where $\Delta_1 = \operatorname{del}_\Delta(v)$ is the deletion of $v$, and $\Delta_2 = \operatorname{lk}_\Delta(v)$ is the link of $v$.*

*Proof.* As pointed out in [Klein and Rajchgot 2021, Proposition 2.9], $I_\Delta$ is geometrically vertex decomposable, and from [Klein and Rajchgot 2021, Remark 2.5], we have a geometric vertex decomposition with $N_{y,I} + \langle y\rangle = I_{\Delta_1}$ and $C_{y,I} = I_{\operatorname{star}_\Delta(v)}$, where $y$ is the variable corresponding to the vertex $v$. The decomposition is nondegenerate since $v$ is a shedding vertex. Now, as $I_{\Delta_2} = I_{\operatorname{star}_\Delta(v)} + \langle y\rangle$ and since $y$ is not in the support of $C_{y,I}$ and $N_{y,I}$, we have $\operatorname{reg}(C_{y,I}) = \operatorname{reg}(I_{\Delta_2})$ and $\operatorname{reg}(N_{y,I}) = \operatorname{reg}(I_{\Delta_1})$. Hence, the above formula follows from Theorem 3.2. □

**Remark 3.5.** Under the hypotheses that $\Delta$ is a pure simplicial complex, the decomposition $C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is degenerate if and only if $\Delta$ is a cone from $v$ on $\Delta_2$ (where $y$ is the variable corresponding to $v$), and in this case, $\operatorname{reg}(I_\Delta) = \operatorname{reg}(I_{\Delta_1}) = \operatorname{reg}(I_{\Delta_2})$.

**Remark 3.6.** The proof of Corollary 3.4, as given in [Hà and Woodroofe 2014], uses tools from combinatorial topology, like the Mayer–Vietoris sequence. On the other hand, Corollary 3.4 is proved in [Moradi and Khosh-Ahang 2016] by first computing the projective dimension of $I_\Delta^\vee$, the corresponding Alexander dual of $I_\Delta$, and then using the fact that this value equals $\operatorname{reg}(R/I_\Delta)$. Our proof of Corollary 3.4 provides an entirely new approach in the case of pure vertex decomposable simplicial complexes.

**Example 3.7.** The ideal $I = \langle y(zs - x^2), ywr, wr(z^2 + zx + wr + s^2)\rangle$ is geometrically vertex decomposable, with

$$C_{y,I} = \langle zs - x^2, wr\rangle \quad \text{and} \quad N_{y,I} = \langle wr(z^2 + zx + wr + s^2)\rangle,$$

and the geometric vertex decomposition is nondegenerate; see [Klein and Rajchgot 2021, Example 2.16]. The ideal $N_{y,I}$ is generated by one polynomial of degree 4, so $\operatorname{reg}(N_{y,I}) = 4$. For the ideal $C_{y,I}$, since its two generators have separate variables, we have

$$\operatorname{reg}(C_{y,I}) = \operatorname{reg}(\langle zs - x^2\rangle) + \operatorname{reg}(\langle wr\rangle) - 1 = 2 + 2 - 1 = 3.$$

Therefore, by Theorem 3.2, $\operatorname{reg}(I) = 4$.

**Remark 3.8.** As defined in [Klein and Rajchgot 2021, Definition 4.6], an ideal $I \subseteq R$ is called *weakly geometrically vertex decomposable* if $I$ is unmixed and if

(1) $I = \langle 1\rangle$, or $I$ is generated by a (possibly empty) subset of variables of $R$, or

(2) for some variable $y = x_j$ of $R$, there is a *degenerate* geometric vertex decomposition $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$, and the contraction of $N_{y,I}$ to the ring $\mathbb{K}[x_1, \ldots, \hat{x}_j, \ldots, x_n]$ is weakly geometrically vertex decomposable, or

(3) for some variable $y = x_j$ of $R$, there is a *nondegenerate* geometric vertex decomposition $\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, the contraction of $C_{y,I}$ to the ring $\mathbb{K}[x_1, \ldots, \hat{x}_j, \ldots, x_n]$ is weakly geometrically vertex decomposable, and $N_{y,I}$ is radical and Cohen–Macaulay.

The proofs of Theorems 2.4 and 3.2 can be adapted easily to weakly geometrically vertex decomposable ideals. In these proofs, we only used the geometrically vertex decomposable property of $I$ to obtain that the ideals $I$, $\text{in}_y(I)$, $C_{y,I}$, and $N_{y,I}$ are Cohen–Macaulay, which is true by [Klein and Rajchgot 2021, Corollaries 4.8 and 4.11] without the geometrically vertex decomposable assumption. Furthermore, the height lemma [Klein and Rajchgot 2021, Lemma 2.8] and the fact that $I$ is radical [Klein and Rajchgot 2021, Corollary 4.8] are also true in the weakly geometrically vertex decomposable setting. Therefore, we have the same recursive formula for weakly geometrically vertex decomposable ideals.

**Example 3.9.** The ideal $I = \langle y(zs - x^2), ywr, wr(x^2 + z^2 + wr + s^2) \rangle$ is weakly geometrically vertex decomposable, with

$$C_{y,I} = \langle zs - x^2, wr \rangle \quad \text{and} \quad N_{y,I} = \langle wr(x^2 + z^2 + wr + s^2) \rangle,$$

but $I$ is not geometrically vertex decomposable; see [Klein and Rajchgot 2021, Example 4.10]. Nevertheless, by Remark 3.8, we still have

$$\text{reg}(I) = \max\{\text{reg}(N_{y,I}), \text{reg}(C_{y,I}) + 1\} = \max\{4, 3 + 1\} = 4.$$

**Remark 3.10.** More generally, the proofs of Theorems 2.4 and 3.2 can be adapted easily to the case when we only require $I$ to be a homogeneous ideal that possesses a geometric vertex decomposition and the ideals $I$ and $N_{y,I}$ are Cohen–Macaulay. As in the proofs of Theorems 2.4 and 3.2, we only need that the ideals $I$, $\text{in}_y(I)$, $C_{y,I}$, and $N_{y,I}$ are Cohen–Macaulay, which is true by [Klein and Rajchgot 2021, Corollaries 4.8 and 4.11], and the height lemma [Klein and Rajchgot 2021, Lemma 2.8] is also true when $I$ is Cohen–Macaulay and the geometric vertex decomposition is nondegenerate. If, in addition, $I$ is radical then the formula in the degenerate case also works by the same argument.

**Example 3.11.** Consider the ideal $I = \langle yz - xw, xy \rangle$. One can check that $I$ is not geometrically vertex decomposable. Nevertheless, using the lexicographical order $x > y > z > w$, the Gröbner basis of $I$ is $\{yz - xw, xy, y^2z\}$. One can check that $I$ has a nondegenerate geometric vertex decomposition with $C_{x,I} = \langle y, w \rangle$ and $N_{x,I} = \langle y^2z \rangle$. Since $I$ and $N_{x,I}$ are Cohen–Macaulay, we get

$$\text{reg}(R/I) = \max\{\text{reg}(R/N_{x,I}), \text{reg}(R/C_{x,I}) + 1\} = 2.$$

## 4. Multiplicity of geometrically vertex decomposable ideals

This short section considers the multiplicity of geometrically vertex decomposable ideals. As with Theorem 3.2, our results rely on using Theorem 2.4 to relate the $h$-polynomial of $I$ to the $h$-polynomials of $C_{y,I}$ and $N_{y,I}$.

**Definition 4.1.** Let $M$ be an $R$-module with Hilbert series $H_M(t) = h_M(t)/(1-t)^d$, where $d = \dim M$. Then the *multiplicity* of $M$ is $e(M) = h_M(1)$.

The leading coefficient of the *Hilbert polynomial* of $M$ is given by $e(M)/d!$, and when $M = R/I$, where $I$ is the defining ideal of a projective variety, then $e(M)$ is the degree of the variety.

**Theorem 4.2.** *Let $I \subseteq R$ be a homogeneous and radical ideal such that $R/I$ is equidimensional. Suppose there is a variable $y$ such that $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition. If the decomposition is nondegenerate*, *then*

$$e(R/I) = e(R/N_{y,I}) + e(R/C_{y,I}).$$

*Otherwise*, *if the decomposition is degenerate*, *we have*

$$e(R/I) = e(R/N_{y,I}) = e(R/C_{y,I}) \quad \text{if } C_{y,I} \neq \langle 1 \rangle$$

*and*

$$e(R/I) = e(R/N_{y,I}) \quad \text{if } C_{y,I} = \langle 1 \rangle.$$

*Proof.* By Theorem 2.4, if the decomposition is nondegenerate, we have

$$h_{R/I}(t) = h_{R/N_{y,I}}(t) + t h_{R/C_{y,I}}(t).$$

Evaluating at $t = 1$ now gives the result. The result in the degenerate case again follows from $R/I \cong R/(N_{y,I} + \langle y \rangle)$ if $C_{y,I} = \langle 1 \rangle$, and $I = \mathrm{in}_y(I) = C_{y,I} = N_{y,I}$ (by [Klein and Rajchgot 2021, Proposition 2.4]) if $C_{y,I} \neq \langle 1 \rangle$. □

The following corollary is now immediate. Its proof is essentially the same as the proof of Corollary 3.3, so we omit it.

**Corollary 4.3.** *Theorem 1.1*, *part* (2) *is true.*

**Example 4.4.** (1) Referring to Example 3.7,

$$I = \langle y(zs - x^2),\ ywr,\ wr(z^2 + zx + wr + s^2) \rangle$$

is geometrically vertex decomposable with

$$C_{y,I} = \langle zs - x^2,\ wr \rangle \quad \text{and} \quad N_{y,I} = \langle wr(z^2 + zx + wr + s^2) \rangle.$$

Hence, $e(R/I) = e(R/N_{y,I}) + e(R/C_{y,I}) = 4 + 4 = 8$.

(2) Referring to Example 3.11, $I = \langle yz - xw, xy\rangle$ has a nondegenerate geometric vertex decomposition with $C_{x,I} = \langle y, w\rangle$ and $N_{x,I} = \langle y^2 z\rangle$. Since $I$ and $N$ are Cohen–Macaulay, the argument as in Remark 3.10 applies. Hence, $e(R/I) = e(R/N_{x,I}) + e(R/C_{x,I}) = 3 + 1 = 4$.

## 5. The $a$-invariant and the Hilbertian property

In this section, we study the $a$-invariant, as well as the related *Hilbertian property*, of a geometrically vertex decomposable ideal. As in Theorems 3.2 and 4.2, our results rely on relating the $h$-polynomial of $I$ to the $h$-polynomials of $C_{y,I}$ and $N_{y,I}$.

We begin by recalling the definition of the $a$-invariant.

**Definition 5.1.** Let $M$ be an $R$-module with Hilbert series $H_M(t) = h_M(t)/(1-t)^d$, where $d = \dim M$. Then the *a-invariant* of $M$ is $a(M) = \deg h_M(t) - d$, that is, the degree of $H_M(t)$ as a rational function.

Among other things, the $a$-invariant equals the degree of the largest nonzero graded piece of the local cohomology module $H_{\mathfrak{m}}^{\dim M}(M)$, i.e.,

$$a(M) = \max\{t \mid \dim_k(H_{\mathfrak{m}}^{\dim M}(M))_t \neq 0\}.$$

If $M$ is Cohen–Macaulay, it is well known that $a(M) = \operatorname{reg}(M) + \dim(M)$. In addition, for Cohen–Macaulay ideals, the $a$-invariant can also be used to characterize the following property.

**Definition 5.2.** A homogeneous ideal $I \subseteq R$ is *Hilbertian* if $HP_{R/I}(t) = HF_{R/I}(t)$ for all $t \geq 0$. The ideal $I$ is *almost Hilbertian* if $HP_{R/I}(t) = HF_{R/I}(t)$ for all $t \geq 1$.

By using Serre's formula for $HF_{R/I}(t) - HP_{R/I}(t)$ (see, for instance, [Bruns and Herzog 1993, Theorem 4.4.3]), when $R/I$ is Cohen–Macaulay,

$$HF_{R/I}(t) - HP_{R/I}(t) = (-1)^{\dim R/I} \dim_k(H_{\mathfrak{m}}^{\dim R/I}(R/I))_t \quad \text{for all } t \in \mathbb{Z}.$$

Consequently, the Hilbertian property can be defined by means of the $a$-invariant as follows.

**Lemma 5.3.** *A homogeneous, Cohen–Macaulay ideal $I \subseteq R$ is Hilbertian if and only if $a(R/I) < 0$. The ideal $I$ is almost Hilbertian if $a(R/I) \leq 0$.*

We now have a result similar to Theorems 3.2 and 4.2.

**Theorem 5.4.** *Suppose that $I \subseteq R$ is a homogeneous and radical ideal such that $R/I$ is Cohen–Macaulay. Suppose that $y$ is a variable such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is a geometric vertex decomposition. If the decomposition is nondegenerate and $R/C_{y,I}$ and $R/N_{y,I}$ are Cohen–Macaulay, then*

$$a(R/I) = \max\{a(R/N_{y,I}) + 1, a(R/C_{y,I}) + 1\}.$$

*Otherwise, if the decomposition is degenerate, we have*

$$a(R/I) = a(R/N_{y,I}) = a(R/C_{y,I}) \quad \textit{if } C_{y,I} \neq \langle 1\rangle$$

*and*

$$a(R/I) = a(R/N_{y,I}) \quad \textit{if } C_{y,I} = \langle 1\rangle.$$

*Proof.* Recall that, by Lemma 3.1, for any homogeneous ideal $J$ such that $R/J$ is Cohen–Macaulay, we have $\operatorname{reg}(R/J) = a(R/J) + \dim(R/J)$. So, if the geometric vertex decomposition is nondegenerate, then when applying Theorem 3.2 we have

$$a(R/I) + \dim(R/I) \\ = \max\{a(R/N_{y,I}) + \dim(R/N_{y,I}),\ a(R/C_{y,I}) + \dim(R/C_{y,I}) + 1\}.$$

We have $\dim(R/I) = \dim(R/C_{y,I}) = \dim(R/N_{y,I}) - 1$, as shown in the proof of Theorem 3.2. The conclusion now follows.

The degenerate case uses the same reasoning as in the proof of Theorem 3.2. □

**Remark 5.5.** In Theorem 5.4, we are viewing $C_{y,I}$ and $N_{y,I}$ as ideals of $R$. But by Definition 2.2, we can also view these ideals as ideals of $R' = R/\langle y\rangle$. Since no generator of $N_{y,I}$ is divisible by $y$, we have $\dim(R/N_{y,I}) - 1 = \dim(R'/N_{y,I})$, and since no generator of $C_{y,I}$ is divisible by $y$, we have $\dim(R/C_{y,I}) - 1 = \dim(R'/C_{y,I})$. Theorem 5.4 thus implies that $a(R/I) = \max\{a(R'/N_{y,I}), a(R'/C_{y,I})\}$ if the geometric vertex decomposition $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is nondegenerate. If the decomposition is degenerate, then $a(R/I) = a(R'/N_{y,I}) - 1$.

**Remark 5.6.** There are situations where $a(R/I) = a(R/C_{y,I}) + 1$, and other situations where $a(R/I) = a(R/N_{y,I}) + 1$. Indeed, by Lemma 3.1, this is equivalent to saying that there are situations where the regularity of $I$ is equal to $\operatorname{reg}(C_{y,I}) + 1$, and other situations where the regularity of $I$ is equal to $\operatorname{reg}(N_{y,I})$ (see Theorem 3.2). We refer the readers to Example 6.15 for concrete examples using the results of Theorem 6.12 in Section 6.

Theorem 5.4 immediately implies Theorem 1.1 (3). The proof is omitted as it is essentially the same as the proof of Corollary 3.3.

**Corollary 5.7.** *Theorem 1.1* (3) *is true.*

Using Theorem 5.4 and Remark 5.5, we can easily prove that the $a$-invariant of a geometrically vertex decomposable ideal is always nonpositive. In particular, geometrically vertex decomposable ideals are almost Hilbertian.

**Corollary 5.8.** *Let $I \subset R$ be a proper homogeneous geometrically vertex decomposable ideal. Then $a(R/I) \leq 0$. In particular, $I$ is almost Hilbertian.*

*Proof.* We induct on the number of variables in $R = \mathbb{K}[x_1, \dots, x_n]$. If $n = 0$, the result is trivial. If $n = 1$, then the only proper homogeneous geometrically vertex decomposable ideals are $\langle x_1 \rangle$ and $\langle 0 \rangle$. Thus, the result holds.

More generally, consider a homogeneous geometrically vertex decomposable ideal $I \subset R = \mathbb{K}[x_1, \dots, x_n]$, with $n \geq 2$. Then there is a variable $y = x_j$ and a geometric vertex decomposition $\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. If this is a degenerate geometric vertex decomposition, then, by Remark 5.5 and the induction hypothesis, we have $a(R/I) = a(R'/N_{y,I}) - 1 < 0$. If $\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a nondegenerate geometric vertex decomposition, then, by Remark 5.5, we have that $a(R/I) = \max\{a(R'/N_{y,I}), a(R'/C_{y,I})\}$. Hence the desired result follows from the induction hypothesis. □

Inspired by the definition of Hilbertian ideals, we next study geometrically vertex decomposable ideals for which the $a$-invariant is always negative. We single out the following class first, as the proof is straightforward.

**Corollary 5.9.** *Let $I \subset R$ be a proper homogeneous geometrically vertex decomposable ideal, and suppose that there is a minimal generating set of $I$ that does not involve all the variables in $R$. Then $a(R/I) < 0$. In particular, $I$ is Hilbertian.*

*Proof.* Suppose that there is some minimal generating set of $I$ which does not involve the variable $y = x_i$. Then the geometric vertex decomposition $\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is degenerate. Thus, by Remark 5.5 and Corollary 5.8, $a(R/I) = a(R'/N_{y,I}) - 1 < 0$. □

We can use this corollary to recover the result of [Stelzer and Yong 2023] that Schubert determinantal ideals are Hilbertian.

**Example 5.10.** Given a permutation $w \in S_n$, there is an associated generalized determinantal ideal $I_w \in \mathbb{K}[x_{ij}, 1 \leq i, j \leq n]$ called a *Schubert determinantal ideal*. By construction, $x_{nn}$ doesn't appear in any term of any minimal generator of $I_w$. Hence by Corollary 5.9, $I_w$ is Hilbertian.

We now consider a larger subclass of geometrically vertex decomposable ideals for which the $a$-invariant is always negative.

**Definition 5.11.** A proper homogeneous ideal $I$ of $R = \mathbb{K}[x_1, \dots, x_n]$ is *$C$-saturated geometrically vertex decomposable* if $I$ is saturated and unmixed and

(1) $I$ is generated by a (possibly empty) subset of variables of $R$, or

(2) there exists a variable $y = x_j$ of $R$ and a $y$-compatible monomial order such that we have a degenerate geometric vertex decomposition

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

where $N_{y,I}$ is $C$-saturated geometrically vertex decomposable, or

(3) there exists a variable $y = x_j$ of $R$ and a $y$-compatible monomial order such that we have a nondegenerate geometric vertex decomposition

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

and the contractions of the ideals $C_{y,I}$ and $N_{y,I}$ to the ring $\mathbb{K}[x_1, \dots, \hat{x}_j, \dots, x_n]$ are $C$-saturated geometrically vertex decomposable.

**Remark 5.12.** Let $I$ be a proper, saturated, homogeneous geometrically vertex decomposable ideal. We will now check that $I$ is $C$-saturated geometrically vertex decomposable if and only if there is some geometric vertex decomposition process of $I$ in which every $C_{y,I}$ ideal that appears in this decomposition process is *not* an irrelevant ideal (contracted to its appropriate polynomial ring). The forward direction is immediate by Definition 5.11.

Conversely, suppose that there is some decomposition process for $I$ in which no ideal of the form $C_{y,I}$ is the irrelevant ideal (in its appropriate polynomial ring). To verify that $I$ is $C$-saturated geometrically vertex decomposable, it suffices to check that every ideal of the form $N_{y,I}$ that appears in the given decomposition process is saturated. Furthermore, since $N_{y,I}$ is geometrically vertex decomposable, and hence radical, this is equivalent to checking that each $N_{y,I}$ ideal is not the irrelevant ideal. So, consider a geometric vertex decomposition $\text{in}_{x_i}(J) = C_{x_i,J} \cap (N_{x_i,J} + \langle x_i \rangle)$ in the given decomposition process, where $J$ is one of the ideals that appears in the decomposition of $I$. Since $C_{x_i,J}$ is not the irrelevant ideal by our assumption, and $N_{x_i,J} \subseteq C_{x_i,J}$, we have that $N_{x_i,J}$ is also not the irrelevant ideal.

**Example 5.13.** Consider the homogeneous and unmixed ideal $I = \langle yz, x+z \rangle \subseteq \mathbb{K}[y, x, z]$. The given generators are a Gröbner basis for the $y$-compatible monomial order Lex with $y > x > z$. We have the geometric vertex decomposition

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

where $C_{y,I} = \langle z, x \rangle$ and $N_{y,I} = \langle x + z \rangle$. Observe that $C_{y,I}$ is the irrelevant ideal in $\mathbb{K}[x, z]$. Alternatively, there are geometric vertex decompositions

$$\text{in}_x(I) = C_{x,I} \cap (N_{x,I} + \langle x \rangle) \quad \text{or} \quad \text{in}_z(I) = C_{z,I} \cap (N_{z,I} + \langle z \rangle).$$

In each case, $C_{x,I}$ or $C_{z,I}$ are irrelevant ideals (in their respective polynomial rings $\mathbb{K}[y, z]$ and $\mathbb{K}[x, y]$). Hence, for each potential decomposition process of $I$, one encounters irrelevant ideals. Thus, $I$ is not $C$-saturated geometrically vertex decomposable.

The next result shows that $C$-saturated geometrically vertex decomposable ideals are always Hilbertian.

**Proposition 5.14.** *Let $I$ be a proper homogeneous geometrically vertex decomposable ideal in a polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$, with $n \geq 1$. If $I$ is $C$-saturated geometrically vertex decomposable, then $a(R/I) < 0$. In particular, $I$ is Hilbertian.*

*Proof.* Our argument is nearly identical to the proof of Corollary 5.8. We induct on the number of variables in $R = \mathbb{K}[x_1, \dots, x_n]$. If $n = 1$, the only $C$-saturated geometrically vertex decomposable ideal is $\langle 0\rangle$, and the result holds.

More generally, consider a homogeneous, $C$-saturated geometrically vertex decomposable ideal $I \subset R = \mathbb{K}[x_1, \dots, x_n]$, with $n \geq 2$. Then there is a variable $y = x_j$ and a geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$. If this is a degenerate geometric vertex decomposition, then, by Remark 5.5 and the induction hypothesis, we have $a(R/I) = a(R'/N_{y,I}) - 1 < 0$ as $N_{y,I} \subseteq R'$ is $C$-saturated geometrically vertex decomposable.

If $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is a nondegenerate $C$-saturated geometric vertex decomposition then, by Remark 5.5, we have

$$a(R/I) = \max(a(R'/N_{y,I}), a(R'/C_{y,I})).$$

Hence the desired result follows from the induction hypothesis. □

**Remark 5.15.** A Stanley–Reisner ideal of a vertex decomposable simplicial complex is Hilbertian if and only if, in its vertex decomposition process, at every step (each step corresponds to removing one vertex from the vertex set) except the last one (where all simplicial complexes are at most one point), taking the link gives all nonempty simplicial complexes. Note that a connected vertex decomposable simplicial complex can have links that are $\{\varnothing\}$ in its vertex decomposition process. For example, consider the graph $C_3$, the three cycle. The link of any vertex is a simplicial complex consisting of two disconnected points, and the link of this simplicial complex is the complex $\{\varnothing\}$. Its Stanley–Reisner ideal $I = \langle xyz\rangle$ is vertex decomposable, but $R/I$ is not Hilbertian.

## 6. Applications to toric ideals of graphs

Here we apply Theorems 3.2, 4.2 and 5.4 to study the invariants of toric ideals of (bipartite) graphs. By leveraging the result that the toric ideals of bipartite graphs are geometrically vertex decomposable (see [Cummings et al. 2023, Theorem 5.8]), we can give new proofs for a number of known results (e.g., [Almousa et al. 2022; Corso and Nagel 2009; Favacchio et al. 2021]) using our techniques.

**6.1.** ***Background on toric ideals of graphs.*** We begin with the relevant background on toric ideals of graphs. Let $G = (V, E)$ be a finite simple graph with vertex set $V = \{x_1, \dots, x_n\}$ and edge set $E = \{e_1, \dots, e_q\}$. If we need to highlight the graph, we sometimes write $V(G)$ and $E(G)$ for the vertices and edges of $G$. Abusing notation, we let $x_i$ and $e_j$ also denote variables, and let $\mathbb{K}[E] = \mathbb{K}[e_1, \dots, e_q]$ and $\mathbb{K}[V] = \mathbb{K}[x_1, \dots, x_n]$. We define a $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}[E] \to \mathbb{K}[V]$ by $\varphi(e_i) = x_j x_k$, where $e_i = \{x_j, x_k\} \in E$. The kernel of $\varphi$, denoted $I_G$, is the *toric ideal of $G$*.

The ideal $I_G$ is a toric ideal because it is a prime binomial ideal; for this fact and for more details about $I_G$, see [Herzog et al. 2018, Chapter 5] or [Villarreal 2015, Chapter 10]. A (nonminimal) set of generators of $I_G$ can be described in terms of closed even walks of the graph. A sequence of distinct edges $\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_t})$ is a *walk* if $e_{i_j} \cap e_{i_{j+1}} \neq \varnothing$ for $1 \leq j \leq t-1$. The walk is *closed* if $e_{i_t} \cap e_{i_1} \neq \varnothing$. The walk is *even* if $t$ is even. A closed walk $\Gamma$ is a *cycle* if no edges in $\Gamma$ are repeated. We can associate with every closed even walk $\Gamma = (e_{i_1}, \ldots, e_{i_{2m}})$ a binomial of the form

$$f_\Gamma = e_{i_1} e_{i_3} \cdots e_{i_{2m-1}} - e_{i_2} e_{i_4} \cdots e_{i_{2m}}.$$

Recall that a graph $G$ is *bipartite* if the vertex set $V$ can be partitioned into two disjoint sets, $V = V_1 \cup V_2$, such that every edge $e \in E$ satisfies $e \cap V_1 \neq \varnothing$ and $e \cap V_2 \neq \varnothing$. The next result now gives a set of generators for toric ideals of (bipartite) graphs. For integers $m, n \geq 1$, the *complete bipartite graph* $K_{m,n}$ is defined to be the graph with vertex set $V = \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$ and edge set $E = \{\{x_i, y_j\} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

**Theorem 6.1** [Villarreal 1995, Proposition 3.1]. *If $G$ is a finite simple graph with toric ideal $I_G$, then*

$$I_G = \langle f_\Gamma \mid \Gamma \textit{ is a closed even walk of } G \rangle.$$

*In addition, if $G$ is bipartite, then $I_G = \langle f_\Gamma \mid \Gamma$ is a even cycle of $G \rangle$.*

A binomial $f = u - v \in I_G$ is called *primitive* if there is no other binomial $g = u' - v'$ in $I_G$ such that $u'|u$ and $v'|v$. A closed even walk (or cycle) $\Gamma$ is a *primitive walk* (*or cycle*) if the corresponding binomial $f_\Gamma$ is a primitive binomial. We have the following refinement of the previous result.

**Theorem 6.2** [Ene and Herzog 2012, Proposition 5.19]. *If $G$ is a finite simple graph with toric ideal $I_G$, then $\{f_\Gamma \mid \Gamma$ is a primitive walk$\}$ forms a universal Gröbner basis for $I_G$, and, in particular, forms a set of generators of $I_G$.*

For bipartite graphs, computing the $a$-invariant of $\mathbb{K}[E]/I_G$ is equivalent to computing the regularity of $R/I_G$.

**Lemma 6.3.** *Let $G = (V, E)$ be a finite simple bipartite graph with toric ideal $I_G$. Then*

$$a(\mathbb{K}[E]/I_G) = \operatorname{reg}(\mathbb{K}[E]/I_G) - (|V| - 1).$$

*Proof.* If $G$ is bipartite, then $\mathbb{K}[E]/I_G$ is Cohen–Macaulay (see [Herzog et al. 2018, Corollary 5.26]). By Lemma 3.1, we have $\operatorname{reg}(\mathbb{K}[E]/I_G) = \deg h_{\mathbb{K}[E]/I_G}(t)$. By [Villarreal 2015, Corollary 10.1.21], $\dim(\mathbb{K}[E]/I_G) = |V| - 1$ when $G$ is bipartite. Thus

$$a(\mathbb{K}[E]/I_G) = \deg h_{\mathbb{K}[E]/I_G}(t) - \dim(\mathbb{K}[E]/I_G) = \operatorname{reg}(\mathbb{K}[E]/I_G) - (|V| - 1),$$

as desired. □

**6.2.** ***Toric ideals of bipartite graphs.*** As our first application, we will show how Theorem 3.2 can be used to give a new proof for a result of [Almousa et al. 2022] about the regularity of bipartite graphs and their subgraphs.

We first recall some more relevant graph theory. Given a graph $G = (V, E)$, we say $H = (W, F)$ is a *subgraph* of $G$ if $W \subseteq V$ and $F \subseteq E$. In the special case $H = (V, E \setminus \{e\})$ for some edge $e$, we write $G \setminus \{e\}$ to denote the graph $G$ with the edge $e$ removed. The *degree* of a vertex $x$ is given by $\deg(x) = |\{y \in V \mid \{x, y\} \in E\}|$. An edge $e = \{x, y\} \in E$ is a *leaf* if $\deg(x) = 1$ or $\deg(y) = 1$, and a vertex $x$ is *isolated* if $\deg(x) = 0$. If $x$ is an isolated vertex of $G$, then it can be shown (e.g., see [Cummings et al. 2023, Lemma 3.2]) that $I_G = I_{G'}$, where $G' = (V \setminus \{x\}, E)$. Similarly, if $e$ is a leaf of $G$, then $I_G = I_{G'}$, where $G' = G \setminus \{e\}$.

The next lemma applies to all toric ideals of graphs, not just bipartite graphs.

**Lemma 6.4** [Cummings et al. 2023, Lemma 3.5]. *Let $G$ be a finite simple graph with toric ideal $I_G$. If $<$ is any $y$-compatible monomial order with $y = e$ for some edge $e \in E$ of $G$, then $N_{y,I_G} = I_{G\setminus\{e\}}$.*

Now suppose that $\mathcal{G}$ is a family of graphs such that, for every $G \in \mathcal{G}$, the toric ideal $I_G$ is geometrically vertex decomposable, and, for all $e \in E$, there is an $e$-compatible monomial order such that there is a geometric vertex decomposition with respect to $e$. Furthermore, suppose that $G \setminus \{e\} \in \mathcal{G}$ for any edge $e$ of $G$. For such a family, we have the following result.

**Theorem 6.5.** *Let $G$ be any graph in the family $\mathcal{G}$ given above. Then, for any subgraph $H$ of $G$, we have*

(1) $\operatorname{reg}(I_H) \le \operatorname{reg}(I_G)$,

(2) $a(\mathbb{K}[E(G)]/I_H) \le a(\mathbb{K}[E(G)]/I_G)$, *and*

(3) $e(\mathbb{K}[E(G)]/I_H) \le e(\mathbb{K}[E(G)]/I_G)$.

*Proof.* For any edge $e$, there is an $e$-compatible monomial order such that $I_G$ has the decomposition $\operatorname{in}_e(I_G) = (N_{e,I_G} + \langle e\rangle) \cap C_{e,I_G}$, and, by Lemma 6.4, we can write the decomposition as

$$\operatorname{in}_e(I_G) = (I_{G\setminus\{e\}} + \langle e\rangle) \cap C_{e,I_G}.$$

As $G \setminus \{e\} \in \mathcal{G}$, $I_{G\setminus\{e\}}$ is geometrically vertex decomposable, and hence is Cohen–Macaulay and radical. Because $I_G$ is geometrically vertex decomposable, we have $\operatorname{reg}(I_{G\setminus\{e\}}) \le \operatorname{reg}(I_G)$ by Theorem 3.2 and Remark 3.10. Similarly, Theorem 5.4 gives

$$a(\mathbb{K}[E(G)]/I_{G\setminus\{e\}}) < a(\mathbb{K}[E(G)]/I_{G\setminus\{e\}}) + 1 \le a(\mathbb{K}[E(G)]/I_G).$$

Moreover, by Theorem 4.2, we have $e(\mathbb{K}[E(G)]/I_{G\setminus\{e\}}) \le e(\mathbb{K}[E(G)]/I_G)$, since multiplicity is a nonnegative integer.

Since $G \setminus \{e\}$ is a graph in $\mathcal{G}$, $I_{G\setminus\{e\}}$ again has a geometric vertex decomposition with respect to any edge $f$. Since any subgraph $H$ of $G$ can be obtained by removing edges and vertices, by repeating this argument (and possibly removing leaves and isolated vertices when needed), we get the desired conclusion. □

**Remark 6.6.** Note that we showed that $a(\mathbb{K}[E]/I_{G\setminus\{e\}}) \leq a(\mathbb{K}[E]/I_G) - 1$, when we remove the edge $e$ from $G$. If $t = |E(G)| - |E(H)|$, that is, the number of edges we remove from $H$ to form $G$, we actually have the stronger result $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G) - t$.

We can recover [Almousa et al. 2022, Theorem 6.11], which was proved using combinatorial techniques involving root polytopes. We also derive results about the $a$-invariant and multiplicity.

**Theorem 6.7.** *Let $H$ be any subgraph of a bipartite graph $G$. Then*

(1) $\operatorname{reg}(I_H) \leq \operatorname{reg}(I_G)$,

(2) $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G)$, *and*

(3) $e(\mathbb{K}[E(G)]/I_H) \leq e(\mathbb{K}[E(G)]/I_G)$.

*Proof.* Let $\mathcal{G}$ be the family of bipartite graphs. By [Cummings et al. 2023, Theorem 5.8], for all $G \in \mathcal{G}$, the toric ideal $I_G$ is geometrically vertex decomposable. It also follows that, for any $G \in \mathcal{G}$, we have $G \setminus \{e\} \in \mathcal{G}$, since removing edges does not destroy the bipartite property.

In addition, by [Cummings et al. 2023, Proposition 5.4], there is a geometric vertex decomposition

$$\operatorname{in}_e(I_G) = (I_{G\setminus\{e\}} + \langle e \rangle) \cap I_e^G,$$

where $\{e\}$ is a path-ordered matching of $G$ and $I_e^G = I_{G\setminus\{e\}} + \langle M_e^G \rangle$, and

$$M_e^G = \{m \mid me - n \text{ is a binomial that corresponds to a cycle in } G\}.$$

Since any edge of a bipartite graph can be regarded as a path-ordered matching, the above geometric vertex decomposition holds for the toric ideal of any bipartite graph and any edge. Hence, Theorem 6.5 applies to any subgraph of $H$ of $G$. □

**Remark 6.8.** Theorem 6.7 (1) gives another proof of [Almousa et al. 2022, Theorem 6.11] using the geometric vertex decomposability property of toric ideals of bipartite graphs. Villarreal pointed out to the third author that regularity could also be deduced from the edge polytope of $G$ and Stanley's monotonicity property [1993, Theorem 3.3]. In particular, one can use the strategy given just after [Almousa et al. 2022, Question 6.12]. Another proof of Theorem 6.7 (1) can be found in the recent paper of Pinto and Villarreal [2023, Corollary 8.17] using normal monomial ideals. Additionally, there is a similar result to Theorem 6.7 (1) for the regularity of *induced* graphs that can be found in [Hà et al. 2019, Theorem 3.6].

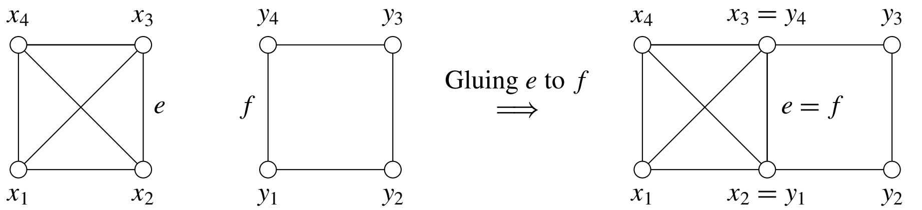


**Figure 1.** Two graphs $G$ and $C_4$ glued along edges $e$ and $f$.

**6.3.** ***Regularity and gluing cycles.*** We can use Theorem 3.2 to give a different proof for [Favacchio et al. 2021, Corollary 3.11] which describes how regularity behaves with respect to a "gluing" operation on graphs.

We first recall the notion of gluing a cycle to a graph along an edge, following [Favacchio et al. 2021, Construction 4.1]. A *cycle of length* $m$ is the graph with vertex set $\{x_1, \dots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{m-1}, x_m\}, \{x_m, x_1\}\}$; we denote this graph by $C_m$. Let $G = (V, E)$ be any graph. Fix an edge $e \in E$ and an edge $f$ of $C_m$. The graph $H$ obtained from $G$ by *gluing a cycle of length* $m$ *along an edge* is the graph $G \bigcup_{e=f} C_m$, where we identify the edges and vertices of $e$ and $f$. An example of gluing is given in Figure 1. When $C_m$ has even length, the regularity of $I_H$, the toric ideal of the glued graph $H$, is related to that of $I_G$ if $I_G$ is geometrically vertex decomposable.

**Theorem 6.9.** *Suppose that $G$ is a graph such that $I_G$ is geometrically vertex decomposable in $\mathbb{K}[E(G)]$. Let $H$ be the graph obtained from $G$ by gluing a cycle of length $2d$ $(d \geq 2)$ along an edge of $G$. Then*

$$\operatorname{reg}(\mathbb{K}[E(H)]/I_H) = \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-1).$$

*Proof.* Let $E(G) = \{e_1, \dots, e_q\}$ denote the edges of $G$, and let $E(C) = \{f_1, \dots, f_{2d}\}$ denote the edges of the cycle $C = C_{2d}$. We assume that the cycle is glued to $G$ along $f_{2d}$ and any edge of $G$. By [Cummings et al. 2023, Theorem 3.11] and its proof, $I_H$ is geometrically vertex decomposable and, moreover, the geometric decomposition is given by

$$N_{y,I_H} = I_G \quad \text{and} \quad C_{y,I_H} = I_G + \langle f_3 f_5 \cdots f_{2d-1} \rangle,$$

where $y = f_1$, and some $y$-compatible monomial order. Since $I_G \subset \mathbb{K}[E(G)]$, $\sqrt{C_{y,I_H}} \neq \sqrt{N_{y,I_H}}$, and $C_{y,I_H} \neq \langle 1 \rangle$, the decomposition is nondegenerate.

Note that $\mathbb{K}[E(H)] = \mathbb{K}[E(G)] \otimes T$, where $T = \mathbb{K}[f_1, \dots, f_{2d-1}]$. So

$$\begin{aligned}
&\operatorname{reg}(\mathbb{K}[E(H)]/(I_G + \langle f_3 f_5 \cdots f_{2d-1} \rangle)) \\
&\qquad = \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + \operatorname{reg}(T/\langle f_3 f_5 \cdots f_{2d-1} \rangle) \\
&\qquad = \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-2).
\end{aligned}$$

So, by Theorem 3.2, we have

$$\begin{aligned}\operatorname{reg}(\mathbb{K}[E(H)]/I_H) &= \max\{\operatorname{reg}(\mathbb{K}[E(G)]/I_G), \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-2) + 1\} \\ &= \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-1),\end{aligned}$$

thus completing the proof. □

We can now derive the following corollary for gluing even cycles to graphs $G$ such that $I_G$ is geometrically vertex decomposable.

**Corollary 6.10.** *Suppose that $G$ is a graph such that $I_G$ is geometrically vertex decomposable in $\mathbb{K}[E(G)]$. Let $H$ be the graph obtained from $G$ by gluing a cycle of length $2d$ ($d \geq 2$) along an edge of $G$. Then*

(1) $a(\mathbb{K}[E(H)]/I_H) = a(\mathbb{K}[E(G)]/I_G) - (d-1)$,

(2) $e(\mathbb{K}[E(H)]/I_H) = d \cdot e(\mathbb{K}[E(G)]/I_G)$.

*Proof.* Note that $|V(H)| = |V(G)| + (2d-1)$. By Lemma 6.3 and Theorem 6.9, we have

$$\begin{aligned}a(\mathbb{K}[E(H)]/I_H) &= \operatorname{reg}(\mathbb{K}[E(H)]/I_H) - (|V(H)|-1) \\ &= (\operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-1)) - ((|V(G)| + (2d-2)) - 1) \\ &= a(\mathbb{K}[E(G)]/I_G) - (d-1),\end{aligned}$$

and hence part (1) follows.

For part (2), by Theorem 4.2,

$$e(\mathbb{K}[E(H)]/I_H) = e(\mathbb{K}[E(H)]/I_G) + e(\mathbb{K}[E(H)]/(I_G + \langle f_3 f_5 \cdots f_{2d-1}\rangle)).$$

Since $e(\mathbb{K}[E(H)]/I_G) = e(\mathbb{K}[E(G)]/I_G)$ and

$$\begin{aligned}&e(\mathbb{K}[E(H)]/(I_G + \langle f_3 f_5 \cdots f_{2d-1}\rangle)) \\ &\qquad = e(\mathbb{K}[E(G)]/I_G) \cdot e(\mathbb{K}[f_1, f_2, \ldots, f_{2d-1}]/\langle f_3 f_5 \cdots f_{2d-1}\rangle) \\ &\qquad = (d-1)e(\mathbb{K}[E(G)]/I_G),\end{aligned}$$

it follows that $e(\mathbb{K}[E(H)]/I_H) = d \cdot e(\mathbb{K}[E(G)]/I_G)$, as desired. □

**6.4. *Regularity of toric ideals of Ferrers graphs.*** Recall that a *Ferrers graph* is a bipartite graph on the vertex set $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_1, u_2, \ldots, u_m\}$ such that $\{v_1, u_m\}$ and $\{v_n, u_1\}$ are edges, and if $\{v_i, u_j\}$ is an edge, then so are all the edges $\{v_k, u_l\}$ with $1 \leq k \leq i$ and $1 \leq l \leq j$. We also associate a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, to a Ferrers graph where $\lambda_i = \deg v_i$, and we denote the Ferrers graph by $T_\lambda$. See Figure 2 on page 384 for an example. We show how to use Theorem 3.2 to give a different proof to a result of Corso and Nagel [2009].

We first require a lemma; in the statement below, a bipartite graph is a *chordal bipartite* graph if every cycle of length $\geq 6$ has a chord, that is, an edge that joins two nonconsecutive vertices of the cycle.

**Lemma 6.11.** *Suppose that $T_\lambda$ is a Ferrers graph. Then $T_\lambda$ is a chordal bipartite graph. Consequently, the toric ideal $I_{T_\lambda}$ is generated by quadratics.*

*Proof.* Let $X = \{v_1, \dots, v_n\}$ and $Y = \{u_1, \dots, u_m\}$ be the partition of the vertices of $T_\lambda$. Suppose that $(v_{i_1}, u_{j_1}, v_{i_2}, u_{j_2}, \dots, v_{i_s}, u_{j_s})$ are the vertices of a cycle of length $2s \geq 6$ in $T_\lambda$.

The three edges $\{v_{i_1}, u_{j_1}\}$, $\{v_{i_1}, u_{j_s}\}$, and $\{v_{i_s}, u_{j_s}\}$ appear in this cycle. Consider the indices of the two $u$ vertices. If $j_1 < j_s$, then by the definition of $T_\lambda$, the edge $\{v_{i_s}, u_{j_1}\}$ is also an edge of $T_\lambda$. So the cycle has a chord. If $j_s < j_1$, note that $\{v_{i_2}, u_{j_1}\}$ is the next edge in the cycle. Since $j_s < j_1$, the edge $\{v_{i_2}, u_{j_s}\}$ is also an edge of $T_\lambda$. But then $(v_{i_1}, u_{j_s}, v_{i_2}, u_{j_1})$ is a four cycle of $T_\lambda$, that is, $\{v_{i_2}, u_{j_s}\}$ is a chord.

The final statement follows from the main result of [Ohsugi and Hibi 1999], which showed that the toric ideals of all chordal bipartite graphs are generated by quadratics. □

**Theorem 6.12** [Corso and Nagel 2009, Proposition 5.7]. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be a partition, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let $T_\lambda$ be the associated Ferrers graph. Write $I_\lambda = I_{T_\lambda}$ and $R = \mathbb{K}[E(T_\lambda)]$.*

(1) *If $n = 1$ or $\lambda_2 = 1$, then* $\operatorname{reg}(R/I_\lambda) = 0$.

(2) *If $\lambda_2 \geq 2$, and suppose that $\lambda = (\lambda_1, \dots, \lambda_s, 1, 1, \dots, 1)$ where $\lambda_s \geq 2$, then*

$$\operatorname{reg}(R/I_\lambda) = \min\{s - 1, \{\lambda_j + j - 3 \mid 2 \leq j \leq s\}\}.$$

*Proof.* If $n = 1$, we have $I_\lambda = \langle 0 \rangle$; hence $\operatorname{reg}(R/I_\lambda) = 0$. When $\lambda_2 = 1$, all the edges $\{v_i, u_1\}$ with $i \geq 2$ are leaves; hence, after removing them, we have $I_\lambda = \langle 0 \rangle$.

Now suppose that $\lambda_2 \geq 2$. Write $e_{ij} = \{v_i, u_j\}$, and let $e = e_{n\lambda_n} = \{v_n, u_{\lambda_n}\}$. Let $G = T_\lambda$ and $I_G = I_\lambda$. By [Cummings et al. 2023, Theorem 5.8], $I_G$ is geometrically vertex decomposable since $G$ is bipartite. Moreover, by [Cummings et al. 2023, Proposition 5.4], there is a geometric vertex decomposition

$$\operatorname{in}_e(I_G) = (I_{G \setminus \{e\}} + \langle e \rangle) \cap I_e^G,$$

where $I_e^G = I_{G\setminus\{e\}} + \langle M_e^G \rangle$ and $M_e^G = \{m_1 \mid m_1 e - m_2$ corresponds to a cycle in $G\}$ (the description of $M_e^G$ appears directly after [Cummings et al. 2023, Lemma 5.1]).

We claim that $\langle M_e^G \rangle = \langle \{e_i \mid \{e, e_j, e_i, e_k\}$ is a cycle in $G\}\rangle$. Note that it suffices to verify that $\langle M_e^G \rangle \subseteq \langle \{e_i \mid \{e, e_j, e_i, e_k\}$ is a cycle in $G\}\rangle$, since the reverse containment is immediate. Suppose that the binomial $m_1 e - m_2$ corresponds to a cycle of $G$. Since the cycle contains the edge $e = \{v_n, u_{\lambda_n}\}$, we can write this cycle as

$$(v_{i_1}, u_{j_1}, v_{i_2}, u_{j_2}, \dots, v_{i_{s-1}}, u_{i_{s-1}}, v_n, u_{\lambda_n}),$$

where $i_s = n$ and $j_s = \lambda_n$. Furthermore, we denote the edges in the cycle as follows: $e_k = \{v_{i_k}, u_{j_k}\}$ for $k = 1, \dots, s$ and $f_k = \{u_{j_k}, v_{i_{k+1}}\}$ for $k = 1, \dots, s$, where $v_{i_{s+1}} = v_{i_1}$. Note that $e = e_{i_s}$, and we have $m_1 e - m_2 = e_1 e_2 \cdots e_{s-1} e - f_1 f_2 \cdots f_s$ with this notation. In this cycle, we now consider the three consecutive edges $e_{s-1} = \{v_{i_{s-1}}, u_{i_{s-1}}\}$, $f_{s-1} = \{u_{i_{s-1}}, v_n\}$, and $e = \{v_n, u_{\lambda_n}\}$. Since $T_\lambda$ is a Ferrers graph, and since $i_{s-1} < n$, the edge $f = \{v_{i_{s-1}}, u_{\lambda_n}\}$ also belongs to $T_\lambda$. This gives a four cycle $(e_{s-1}, f_{s-1}, e, f)$, and thus $e_{s-1}e - f_{s-1}f$ is a binomial that corresponds to a cycle of $G$. But this means that $e_{s-1} \in M_e^G$, and since $e_{s-1}$ divides $m_1 = e_1 \cdots e_{s-1}$, we have that $m_1$ is in the ideal on the right-hand side.

Note that the graph $G \setminus \{e\}$ is the Ferrers graph on the same vertex set associated to the partition $\lambda'' = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n - 1)$. Hence, $N_{e,I_G} = I_{G\setminus\{e\}} = I_{\lambda''}$. On the other hand,

$$\langle M_e^G \rangle = \langle e_{ij} \mid 1 \le i \le n-1,\ 1 \le j \le \lambda_n - 1 \rangle.$$

Since $G\setminus\{e\}$ is a Ferrers graph, the generators of $I_{G\setminus\{e\}}$ are quadratics by Lemma 6.11. Moreover these generators are of the form $f = e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_1} e_{i_1 j_2}$, with $i_1 < i_2$ and $j_1 < j_2$. If $i_2 \le n-1$ and $j_2 \le \lambda_n - 1$, then we have $f \in \langle M_e^G \rangle$. If $i_2 = n$, then $j_1 < j_2 \le \lambda_n - 1$, and hence $f \in \langle M_e^G \rangle$. Therefore,

$$\begin{aligned} C_{e,I_G} &= I_{G\setminus\{e\}} + \langle M_e^G \rangle \\ &= \langle M_e^G \rangle + \langle e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_1} e_{i_1 j_2} \mid 1 \le i_1 < i_2 \le n-1, \lambda_n \le j_1 < j_2 \rangle. \end{aligned}$$

Note that the ideal $\langle e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_1} e_{i_1 j_2} \mid 1 \le i_1 < i_2 \le n-1, \lambda_n \le j_1 < j_2 \rangle$ is the toric ideal of the Ferrers graph on the vertex set $X' = \{v_1, v_2, \dots, v_{n-1}\}$ and $Y' = \{u_{\lambda_n}, u_2, \dots, u_m\}$ associated to the partition

$$\lambda' = (\lambda_1 - \lambda_n + 1, \lambda_2 - \lambda_n + 1, \dots, \lambda_{n-1} - \lambda_n + 1).$$

Thus, we can write $C_{e,I_G} = \langle M_e^G \rangle + I_{\lambda'}$, where the generators of the two ideals in the right-hand side are in separate sets of variables. Moreover, since $\langle M_e^G \rangle$ is generated by variables, $\operatorname{reg}(R/\langle M_e^G \rangle + I_{\lambda'}) = \operatorname{reg}(R/I'_\lambda)$. Thus, by Theorem 3.2,

$$\operatorname{reg}(R/I_\lambda) = \max\{\operatorname{reg}(R/I_{\lambda''}), \operatorname{reg}(R/I_{\lambda'}) + 1\},$$

where $\lambda'$ and $\lambda''$ are defined as above.

We will now apply the above recursive formula to derive the formula of $\operatorname{reg}(R/I_\lambda)$ as claimed. First, if $\lambda = (\lambda_1, \dots, \lambda_s, 1, 1, \dots, 1)$ with $\lambda_s \ge 2$, then the edges $v_i u_1$ with $s+1 \le i \le n$ are all leaves; hence, we can remove them without changing the toric ideal. In other words, $\operatorname{reg}(I_\lambda) = \operatorname{reg}(I_{\tilde{\lambda}})$, where $\tilde{\lambda} = (\lambda_1, \dots, \lambda_s)$. We will use induction on $n$. Suppose $n = 2$ and $\lambda_1 \geq \lambda_2 \geq 2$. Then, by the recursive formula,

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1,\lambda_2)}) &= \max\{\operatorname{reg}(R/I_{(\lambda_1,\lambda_2-1)}),\ \operatorname{reg}(R/I_{(\lambda_1-\lambda_2+1)}) + 1\} \\ &= \max\{\operatorname{reg}(R/I_{(\lambda_1,\lambda_2-1)}),\ 1\}.\end{aligned}$$

By induction on $\lambda_2$, if $\lambda_2 = 2$, we have

$$\operatorname{reg}(R/I_{(\lambda_1,\lambda_2)}) = \max\{\operatorname{reg}(R/I_{(\lambda_1,1)}),\ 1\} = 1 = \min\{2-1,\ \lambda_2 + 2 - 3\},$$

and if $\lambda_2 > 2$, we have

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1,\lambda_2)}) &= \max\{\operatorname{reg}(R/I_{(\lambda_1,\lambda_2-1)}),\ 1\} \\ &= \max\{\min\{2-1,\ (\lambda_2-1)+2-3\}\},\ 1\} = 1 \\ &= \min\{2-1,\ \lambda_2+2-3\}.\end{aligned}$$

In both cases, the regularity agrees with the formula in our claim.

Now suppose that $n \geq 3$ and that the formula holds for any $k \leq n-1$ and for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. We will show that it holds for $n$ and any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Now if $\lambda = (\lambda_1, \ldots, \lambda_s, 1, 1, \ldots, 1)$ with $\lambda_s \geq 2$, as shown above, we have $\operatorname{reg}(R/I_\lambda) = \operatorname{reg}(R/I_{\tilde{\lambda}})$, where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_s)$. Hence, if $s \leq n-1$, the result follows by the induction hypothesis. Also, if $\lambda_n = 1$, as $\operatorname{reg}(R/I_\lambda) = \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1})})$, the result again follows by the induction hypothesis. Thus, we can assume that $s = n$ and $\lambda_n \geq 2$. Then, by the recursive formula,

$$\operatorname{reg}(R/I_\lambda) = \max\{\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}),\ \operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) + 1\}.$$

**Case 1:** If $\lambda_n = 2$, then $\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}) = \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1})})$. So, by the induction hypothesis

$$\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}) = \min\{n-2,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq n-1\}\}.$$

Now if $\lambda_{n-1} \geq 3$, then, by the induction hypothesis,

$$\begin{aligned}&\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) + 1 \\ &\qquad = \min\{n-2,\ \{\lambda_j - \lambda_n + j - 2 \mid 2 \leq j \leq n-1\}\} + 1 \\ &\qquad = \min\{n-1,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq n-1\}\} \\ &\qquad \geq \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}).\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{reg}(R/I_\lambda) &= \operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) + 1 \\ &= \min\{n-1,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq n\}\},\end{aligned}$$

where the last equality holds since $\lambda_n + n - 3 = n - 1$.

Otherwise, if $\lambda_{n-1}=2$, assume that $\lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_{n-1}=2$ and $\lambda_k\geq 3$ for some $k\leq n-2$. Then $(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)=(\lambda_1-1,\ldots,\lambda_k-1,1,\ldots,1)$. Again, by the induction hypothesis,

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})+1&=\min\{k-1,\,\{\lambda_j-1+j-3\mid 2\leq j\leq k\}\}+1\\&=\min\{k,\,\{\lambda_j+j-3\mid 2\leq j\leq k\}\}\end{aligned}$$

if $k\geq 2$, or the regularity equals 0 if $k=1$. Moreover, for $k+1\leq j\leq n-1$, we have $k\leq\lambda_j+j-3\leq n-2$, and thus

$$\begin{aligned}\min\{k,\,\{\lambda_j+j-3\mid 2\leq j\leq k\}\}&=\min\{k,\,\{\lambda_j+j-3\mid 2\leq j\leq n-1\}\}\\&\leq\min\{n-2,\,\{\lambda_j+j-3\mid 2\leq j\leq n-1\}\}\\&=\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}),\end{aligned}$$

where the last equality follows since $\lambda_j+j-3\leq n-2<n-1$ for $k+1\leq j\leq n-1$. Therefore,

$$\operatorname{reg}(R/I_\lambda)=\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)})=\min\{n-1,\,\{\lambda_j+j-3\mid 2\leq j\leq n\}\},$$

where the last equality follows since $\lambda_n+n-3=n-1$.

**Case 2:** If $\lambda_n\geq 3$, by induction on $\lambda_n$, we have

$$\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)})=\min\{n-1,\,\{\lambda_j+j-3\mid 2\leq j\leq n-1\},\,\lambda_n+n-4\}.$$

In addition, by the induction hypothesis (on $n$), we have

$$\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})=\min\{k-1,\,\{\lambda_j-\lambda_n+j-2\mid 2\leq j\leq k\}\},$$

where $2\leq k\leq n-1$ is the maximum integer such that $\lambda_k\geq\lambda_n+1$ (hence, $\lambda_k-\lambda_n+1\geq 2$), or the regularity equals 0 if $k=1$. On the other hand, as $\lambda_j=\lambda_n$ for $j\geq k+1$, we have $\lambda_j-\lambda_n+j-2=j-2\geq k-1$ for all $j\geq k+1$. Thus,

$$\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})=\min\{k-1,\,\{\lambda_j-\lambda_n+j-2\mid 2\leq j\leq n\}\}.$$

Therefore,

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})+1&=\min\{k,\,\{\lambda_j-\lambda_n+j-1\mid 2\leq j\leq n\}\}\\&\leq\min\{n-1,\,\{\lambda_j+j-3\mid 2\leq j\leq n\}\}\\&=\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}),\end{aligned}$$

where the last equality holds since $\lambda_n+n-3>\lambda_n+n-4\geq n-1$. Therefore,

$$\operatorname{reg}(R/I_\lambda)=\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)})=\min\{n-1,\,\{\lambda_j+j-3\mid 2\leq j\leq n\}\},$$

finishing our proof. □

**Remark 6.13.** If $y$ is the variable that corresponds to the edge $e = \{v_n, \lambda_n\}$, by Theorem 2.3, we have

$$H_{R/I_\lambda}(t) = H_{R/(N_{y,I}+\langle y\rangle)}(t) + tH_{R/C_{y,I}}(t) = H_{R/I''_\lambda}(t) + tH_{R/I'_\lambda}(t).$$

This recovers [Corso and Nagel 2009, Lemma 5.3].

**Remark 6.14.** As shown in the proof of Theorem 6.12, we record the formulae for regularity of $I_\lambda$ avoiding taking the maximum:

(1) If $\lambda_n = 2$ and $\lambda_{n-1} \geq 3$, then $\operatorname{reg}(I_\lambda) = \operatorname{reg}(C_{e,I_\lambda}) + 1 = \operatorname{reg}(I'_\lambda) + 1$.

(2) If $\lambda_{n-1} = \lambda_n = 2$, then $\operatorname{reg}(I_\lambda) = \operatorname{reg}(N_{e,I_\lambda}) = \operatorname{reg}(I''_\lambda)$.

(3) If $\lambda_n \geq 3$, then $\operatorname{reg}(I_\lambda) = \operatorname{reg}(N_{e,I_\lambda}) = \operatorname{reg}(I''_\lambda)$.

As mentioned in Remark 5.6, the following example shows that the $a$-invariant of a geometrically vertex decomposable ideal $I$ can be either that of the $C$ ideal or the $N$ ideal of its decomposition, or equivalently, the regularity of $I$ can be either $\operatorname{reg}(C) + 1$ or $\operatorname{reg}(N)$.

**Example 6.15.** • Consider the toric ideal $I = I_\lambda$ of the Ferrers graph associated to $\lambda = (3, 3, 3, 3)$. Then $I$ has a nondegenerate geometric vertex decomposition with $N = I_{\lambda''}$ and $C = I_{\lambda'}$, where $\lambda'' = (3, 3, 3, 2)$ and $\lambda' = (1, 1, 1)$; see the proof of Theorem 6.12. Thus, by Theorem 6.12, $\operatorname{reg}(N) = 3$, $\operatorname{reg}(C) = 1$, and $\operatorname{reg}(I) = \operatorname{reg}(N) = 3 > \operatorname{reg}(C) + 1$. Note also that it is not hard to construct an example with a degenerate decomposition with $C = \langle 1\rangle$; in this case, we also have $\operatorname{reg}(I) = \operatorname{reg}(N) > \operatorname{reg}(C) + 1$.

• If $\lambda = (4, 4, 3, 2)$, we have $\lambda'' = (4, 4, 3, 1)$ and $\lambda' = (3, 3, 2)$. By Theorem 6.12, $\operatorname{reg}(N) = 3$, $\operatorname{reg}(C) = 3$, and $\operatorname{reg}(I) = \operatorname{reg}(C) + 1 = 4 > \operatorname{reg}(N)$.

• If $\lambda = (3, 3, 2, 2)$, we have $\lambda'' = (3, 3, 2, 1)$ and $\lambda' = (2, 2, 1)$. By Theorem 6.12, $\operatorname{reg}(N) = 3$, $\operatorname{reg}(C) = 2$, and $\operatorname{reg}(I) = \operatorname{reg}(C) + 1 = \operatorname{reg}(N) = 3$.

**Remark 6.16.** Two formulae for the regularity of $R/I_\lambda$ are given in [Corso and Nagel 2009, Proposition 5.7]; however these two formulas to do not agree in general:

$$\min\{s-1, \{\lambda_j + j - 3 \mid 2 \leq j \leq s\}\} \neq \begin{cases} s-1 & \text{if } \lambda_s \geq 3, \\ \min\{j-1 \mid \lambda_j = 2\} & \text{if } \lambda_s = 2, \end{cases}$$

where $s$ is the number of $\lambda_i \geq 2$. For example, let $s \geq 4$, $\lambda_1 = \cdots = \lambda_{s-1} = 3$ and $\lambda_s = 2$, then the left-hand side is 2, whereas the right-hand side is $s - 1 \geq 3$. Or if $\lambda_1 = \lambda_2 = \cdots = \lambda_s = 3$ and $s \geq 4$, then the left-hand side is 2 and the right-hand side is $s - 1 \geq 3$. This error occurs in [Corso and Nagel 2009] because the authors claim the equality between the two sides follows from the equality $\min\{\lambda_j + j - 3 \mid 2 \leq j \leq s\} = \lambda_s + s - 2$, which is not true in general. For a more specific example, consider the Ferrers graph $T_\lambda$ for $\lambda = (3, 3, 3, 2)$ as given in Figure 2.

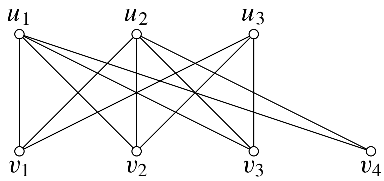


**Figure 2.** The graph $T_\lambda$ for $\lambda = (3, 3, 3, 2)$.

Then the minimal graded free resolution of $R/I_\lambda$ has Betti table

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| total: | 1 | 12 | 25 | 21 | 10 | 3 |
| 0: | 1 | - | - | - | - | - |
| 1: | - | 12 | 25 | 15 | - | - |
| 2: | - | - | - | 6 | 10 | 3 |

In particular, $\mathrm{reg}(R/I_\lambda) = 2$.

**Corollary 6.17.** *With the same notation as in Theorem 6.12, we have*

(1) $a(R/I_\lambda) =$
$$\begin{cases} -(n+m-1) & \text{if } n = 1 \text{ or } \lambda_2 = 1, \\ -(n+m-1) + \min\{s-1, \{\lambda_j + j - 3 \mid 2 \le j \le s\}\} & \text{otherwise}. \end{cases}$$

(2)
$$e(R/I_\lambda) = \sum_{j_{n-2}=\lambda_2-\lambda_n+1}^{\lambda_2} \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{j_{n-2}} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1.$$

*Proof.* Part (1) follows from Lemma 6.3 and Theorem 6.12, while part (2) follows by induction using Theorem 4.2, as

$$\begin{aligned} &e(R/I_\lambda) \\ &= e(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}) + e(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) \\ &= \sum_{j_{n-2}=\lambda_2-\lambda_n}^{\lambda_2} \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{j_{n-2}} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1 + \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{\lambda_2-\lambda_n+1} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1 \\ &= \sum_{j_{n-2}=\lambda_2-\lambda_n+1}^{\lambda_2} \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{j_{n-2}} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1. \end{aligned}$$ □

**Remark 6.18.** Let $G = K_{n,m}$ be the complete bipartite graph. Then $I_G = I_\lambda$, where $\lambda = (m, m, m, \ldots, m)$. Hence, $\mathrm{reg}(R/I_G) = \min\{n-1, m-1\} = \min\{n, m\} - 1$.

We recover the bounds for the regularity of bipartite graphs in [Almousa et al. 2022, Theorem 6.13] for connected bipartite graphs and [Biermann et al. 2017, Theorem 4.9] for chordal bipartite graphs, and the $a$-invariant of bipartite graphs in [Villarreal 2015, Proposition 11.5.1].

**Corollary 6.19.** *Let $G$ be a connected bipartite graph with bipartition $V_1 \cup V_2$, with $|V_1| = n$ and $|V_2| = m$. Let $r = |\{v_i \in V_1 \mid \deg(v_i) = 1\}|$ and $s = |\{u_i \in V_2 \mid \deg(u_i) = 1\}|$. Then*

(1) $\operatorname{reg}(I_G) \leq \min\{n - r, m - s\}$, *and*

(2) $a(\mathbb{K}[E(G)]/I_G) \leq \min\{-m, -n\}$.

*Proof.* (1) Since $r$ vertices in vertex set $X$ and $s$ vertices in vertex set $Y$ belong to leaves, we can remove these vertices without changing the toric ideal $I_G$. Thus, we can assume that $r = s = 0$; that is $G$ does not have any leaves. As $G$ is a subgraph of $K_{n,m}$, by Theorem 6.7 and Remark 6.18, we have $\operatorname{reg}(I_G) \leq \operatorname{reg}(I_{K_{n,m}}) = \min\{n, m\}$ as desired.

(2) By Lemma 6.3, $a(\mathbb{K}[E(G)]/I_G) = \operatorname{reg}(\mathbb{K}[E(G)]/I_G) - (|V(G)| - 1)$. Because the graph is connected, $|V(G)| = m + n$. Also, by part (1), $\operatorname{reg}(\mathbb{K}[E(G)]/I_G) \leq \min\{n, m\} + 1$. The result now follows. □

We can now prove a very interesting property about toric ideals of bipartite graphs, which does not seem to have been observed before.

**Theorem 6.20.** *If $G$ is a connected bipartite graph, then $I_G$ is Hilbertian.*

*Proof.* As shown in Corollary 6.19 (2), the $a$-invariant of $\mathbb{K}[E(G)]/I_G$ is always negative; hence $I_G$ is Hilbertian. □

**6.5. *Invariants for a class of bipartite graphs.*** In this section we apply Theorems 3.2, 5.4, and 4.2 to a family of toric ideals of bipartite graphs first studied in [Galetto et al. 2019] to further illustrate our techniques. The graphs studied in that paper were defined as follows.

**Definition 6.21.** Let $d, r$ be integers such that $d \geq 1$ and $r \geq 3$. Let $G = G_{r,d}$ be the graph with $V(G) = \{x_1, x_2, y_1, \ldots, y_d, z_1, \ldots, z_{2r-3}\}$ and

$$E(G) = \{\{x_i, y_j\} \mid 1 \leq i \leq 2,\ 1 \leq j \leq d\} \\ \cup \{\{x_1, z_1\}, \{z_1, z_2\}, \{z_2, z_3\}, \ldots, \{z_{2r-4}, z_{2r-3}\}, \{z_{2r-3}, x_2\}\}.$$

We label edges as follows: for $i = 1, \ldots, d$, let $a_i = \{x_1, y_i\}$ and $b_i = \{x_2, y_i\}$. Also, let $e_1 = \{x_1, z_1\}$, $e_{2r-2} = \{x_2, z_{2r-3}\}$, and let

$$e_2 = \{z_1, z_2\}, \quad e_3 = \{z_2, z_3\}, \quad \ldots, \quad e_{2r-3} = \{z_{2r-4}, z_{2r-3}\}.$$

Informally, the graph $G_{r,d}$ is constructed by starting with a complete bipartite graph $K_{2,d}$ (with $d \geq 1$). Then one connects the vertex $x_1$ to $x_2$ (the two vertices of degree $d$) with a path of length $2r - 2$. Figure 3 shows the graph $G_{6,5}$, where we start with the graph $K_{2,5}$ (the graph on $\{x_1, x_2, y_1, \ldots, y_5\}$), and then add a path of length $2 \cdot 6 - 2$ between $x_1$ and $x_2$, the two vertices of degree five in $K_{2,5}$. Figure 3 also illustrates our edge labeling.

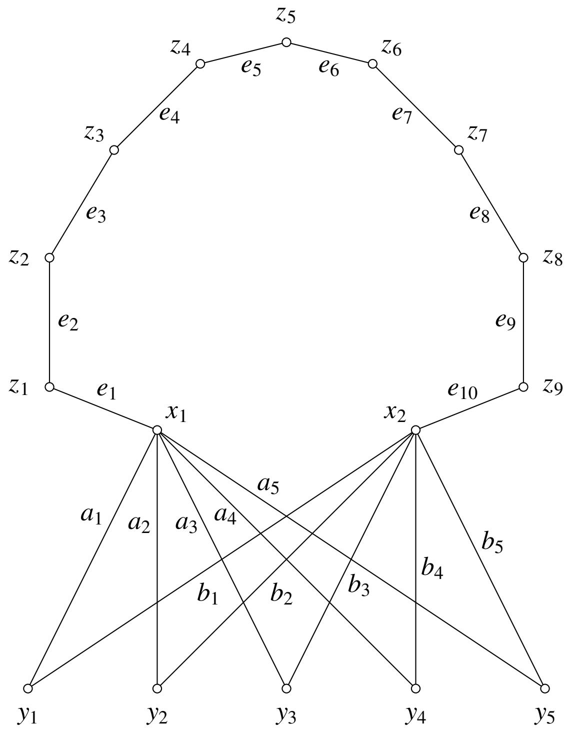


**Figure 3.** Illustration of $G_{r,d} = G_{6,5}$. This graph is $K_{2,5}$ with a path of length $2 \cdot 6 - 2 = 10$ connecting the two vertices of degree 5 in $K_{2,5}$.

**Remark 6.22.** Note Definition 6.21 still makes sense if we allow $r = 2$. However, if we allow $r = 2$, then we are adding a path of length $2r - 2 = 2$ between $x_1$ and $x_2$. Then the graph $G_{2,d}$ and the graph $K_{2,d+1}$ are isomorphic. It thus makes sense to restrict to the case $r \geq 3$.

The toric ideals of these graphs were studied in [Galetto et al. 2019] (see also [Nandi and Nanduri 2019] for a more general family); in particular, all the graded Betti numbers of $I_G$ were determined, and consequently, one can determine the regularity (see [Galetto et al. 2019, Theorem 3.9]). The approach taken in that paper is to consider a careful analysis of the initial ideal of $I_G$ and exploiting the fact that a universal Gröbner basis of $I_G$ could be explicitly described.

**Lemma 6.23** [Galetto et al. 2019, Corollary 3.3]. *Let $d, r$ be integers such that $d \geq 1$ and $r \geq 3$. Then a universal Gröbner basis for $I_{G_{r,d}}$ is given by*

$$\{a_i b_j - a_j b_i \mid 1 \leq i < j \leq d\} \cup \{a_i e_2 e_4 \cdots e_{2r-2} - b_i e_1 e_3 \cdots e_{2r-3} \mid 1 \leq i \leq d\},$$

*where, if $d = 1$, the first set is empty.*

On the other hand, since $G = G_{r,d}$ is bipartite, we know that $I_G$ is geometrically vertex decomposable. Using this fact, Lemma 6.23, and Theorem 3.2, we can give an alternative proof for the calculation of $\operatorname{reg}(I_{G_{r,d}})$.

**Theorem 6.24.** *Let $d \geq 1$ and $r \geq 3$ be integers, and let $G = G_{r,d}$. Then* $\operatorname{reg}(I_G) = r$.

*Proof.* We first consider the case that $d = 1$ and $r \geq 3$. In this case, $G_{r,1} = C_{2r}$, that is, the cycle on $2r$ vertices. By Lemma 6.23, $I_{G_{r,1}} = \langle a_1e_2 \cdots e_{2r-2} - b_1e_1 \cdots e_{2r-3}\rangle$ is generated by a single polynomial of degree $r$. The conclusion then follows.

We now proceed by induction on the tuple $(d, r)$, where we assume the result holds for all graphs $G_{r',d'}$ with $d > d'$. Let $G = G_{r,d}$. Let $>$ denote the lexicographical monomial order on $\mathbb{K}[E(G)]$ with

$$a_d > a_{d-1} > \cdots > a_1 > e_1 > e_2 > \cdots > e_{2r-2} > b_d > \cdots > b_1.$$

If $y = a_d$, then $>$ is a $y$-compatible monomial order.

By Lemma 6.4, $N_{y,I_G}$ is the toric ideal of $G \setminus \{a_d\}$. If we remove $a_d$ from $G$, then $b_d$ is a leaf of $G \setminus \{a_d\}$. Consequently, $N_{y,I_G}$ is the toric ideal of the graph $G$ with both edges $a_d$ and $b_d$ removed. But if we remove $a_d$ and $b_d$ from $G$, we obtain the graph $G_{r,d-1}$. Thus $N_{y,I_G} = I_{G_{r,d-1}}$.

By using the universal Gröbner basis of Lemma 6.23, we have

$$\begin{aligned} C_{y,I_G} &= \langle b_{d-1}, \ldots, b_1, e_2e_4 \cdots e_{2r-2}\rangle + \langle a_ib_j - a_jb_i \mid 1 \leq i < j \leq d-1\rangle \\ &\qquad + \langle a_ie_2 \cdots e_{2r-2} - b_ie_1 \cdots e_{2r-3} \mid 1 \leq i \leq d-1\} \\ &= \langle b_{d-1}, \ldots, b_1, e_2e_4 \cdots e_{2r-2}\rangle. \end{aligned}$$

The last equality follows from the fact that each term of the generators in the other two ideals is either divisible by some $b_i$ with $i \in \{1, \ldots, d-1\}$ or $e_2e_4 \cdots e_{2r-2}$. Consequently, $C_{y,I_G}$ is a monomial ideal that is a complete intersection (since the monomials have disjoint support) with regularity

$$\underbrace{1 + \cdots + 1}_{d-1} + (r-1) - (d-1) = r - 1.$$

So, by Theorem 3.2 and induction, we have

$$\operatorname{reg}(I_G) = \max\{\operatorname{reg}(I_{G_{r,d-1}}), \operatorname{reg}(C_{y,I_G}) + 1\} = \max\{r, (r-1) + 1\} = r,$$

as desired. □

We can now compute the $a$-invariant and the multiplicity of the rings $\mathbb{K}[E]/I_{G_{r,d}}$.

**Corollary 6.25.** *Let $d \geq 1$ and $r \geq 3$ be integers, and let $G = G_{r,d}$. Then*

(1) $a(\mathbb{K}[E(G)]/I_G) = 1 - d - r$, *and*

(2) $e(\mathbb{K}[E(G)]/I_G) = dr - (d-1)$.

*Proof.* We can use Theorem 5.4 to prove (1), but it is more direct to use Theorem 6.24, Lemma 6.3, and the fact that $G_{r,d}$ is a bipartite graph on $2+d+(2r-3)$ vertices.

To prove (2), we do induction on $d$. If $d=1$, then $G_{r,d}=C_{2r}$, and so $I_{G_{r,d}}$ is a principal ideal generated by a single generator of degree $r$. So $e(\mathbb{K}[E]/I_{G_{r,d}})=r$.

So suppose $d>1$. If $I=I_{G_{r,d}}$, then by Theorem 4.2, we have

$$e(\mathbb{K}[E]/I)=e(\mathbb{K}[E]/N_{y,I})+e(\mathbb{K}[E]/C_{y,I}).$$

As shown in the proof of Theorem 6.24, $C_{y,I}$ is a complete intersection generated by $d-1$ generators of degree one and one generator of degree $r-1$. Consequently,

$$e(\mathbb{K}[E]/C_{y,I})=1^{d-1}\cdot(r-1)=r-1.$$

On the other hand, as also shown in the proof of Theorem 6.24, $N_{y,I}=I_{G_{r,d-1}}$. Hence, by induction, $e(\mathbb{K}[E]/N_{y,I})=(d-1)r-(d-2)$. Thus

$$e(\mathbb{K}[E]/I)=(d-1)r-(d-2)+r-1=dr-(d-1). \qquad \square$$

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