Pacific Journal of Mathematics

Volume 333 No. 2 December 2024

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.197

MULTIPLIER ALGEBRAS OF L^p-OPERATOR ALGEBRAS

ANDREY BLINOV, ALONSO DELFÍN AND ELLEN WELD

It is known that the multiplier algebra of an approximately unital and nondegenerate L^p-operator algebra is again an L^p-operator algebra. In this paper we investigate examples that drop both hypotheses. In particular, we show that the multiplier algebra of T_2^p, the algebra of strictly upper triangular 2×2 matrices acting on ℓ_2^p, is still an L^p-operator algebra for any p. To contrast this result, we first provide a thorough study of the augmentation ideal of $\ell^1(G)$ for a discrete group G. We use this ideal to define a family of nonapproximately unital degenerate L^p-operator algebras, $F_0^p(\mathbb{Z}/3\mathbb{Z})$, whose multiplier algebras cannot be represented on any L^q-space for any $q \in [1, \infty)$ as long as $p \in [1, p_0] \cup [p_0', \infty)$, where $p_0 = 1.606$ and p_0' is its Hölder conjugate.

1. Introduction

In 1971, C. S. Herz studied certain operators acting on L^p-spaces (see [17]). In fact, Herz constructed group algebras of operators acting on L^p-spaces, those were referred to as algebras of p-pseudofunctions (see [18] and also [9] for a modern approach). About 15 years ago, the study of these operators regained interest from the community thanks to the general research works of M. Daws, E.

MSC2020: primary 46H15, 46H35; secondary 47L10.
Keywords: multiplier algebra, augmentation ideal, representations on L^p-spaces, L^p-group algebras, p-operator norms.

Gardella, N. C. Phillips, N. Spronk, and H. Thiel, among others (see [5; 6; 12; 13; 14; 21], for instance). This area is now known as L^p-operator algebras. Given that L^p-operator algebras include the large and well studied class of C*-algebras, part of the research in this area focuses on understanding what C*-properties and constructions can and cannot be extended to the L^p-case. In particular it is worth mentioning that L^2-operator algebras include the class of C*-algebras, but also the larger class of nonselfadjoint operator algebras. Three big differences between C*-algebras and L^p-operator algebras, even when $p = 2$, which will be exploited here are: (1) L^p-operator norms are not unique, (2) some L^p-operator algebras cannot be nondegenerately represented on any Banach space, and (3) some L^p-operator algebras do not have contractive approximate identities (which will be henceforth abbreviated as cai's). A usually hard problem in L^p-operator algebras, arising in some sense due to the difficulty of computing L^p-operator norms, is to determine whether a given Banach algebra can be isometrically represented on an L^p-space. Different techniques have been employed to show that certain Banach algebras cannot be represented on L^p-spaces; see for instance Theorem 2.5 in [11] and Example 4.7 in [3], where it is shown that the class of L^p-operator algebras is not closed under taking quotients when $p \neq 2$. As a main result of this paper, we look at certain algebras with no contractive approximate identities to conclude that, for certain values of $p \neq 2$, the class of L^p-operator algebras is not closed under taking multiplier algebras.

This paper is organized as follows: We start with a brief review of the multiplier algebra construction (as double centralizers) for general Banach algebras. In Section 3, we present main results for the construction of the multiplier algebra, $M(A)$, of a general Banach algebra A. It is known that if a Banach algebra A has a cai and is nondegenerately representable on a Banach space E, then $M(A)$ can also be nondegenerately represented on E. In fact, in Corollary 3.6, we prove that for the known representation of $M(A)$ on E the action is given by two-sided multipliers, agreeing with the well known C*-result. Next, we investigate general properties of $M(A)$ when A has a nonunital identity which means that A an algebraic multiplicative identity with norm greater than 1 (see Definition 2.5). This condition automatically prevents A from having a cai and also from being nondegenerately represented on any Banach space, making these algebras suitable candidates to investigate whether their multiplier algebras can be nondegenerately represented on Banach spaces.

Starting in Section 4, we work only with multiplier algebras of L^p-operator algebras. As before, when A is a nondegenerate L^p-operator algebra with a cai, its multiplier algebra is also a nondegenerate L^p-operator algebra. We begin Section 4 by exploring whether these assumptions are necessary for the multiplier algebra of an L^p-operator algebra to be itself an L^p-operator algebra. Consequently, we

produce two contrasting examples dealing with the multiplier algebra of degenerate L^p-operator algebras without cai's:

The first one, explored in Section 4B, deals with strictly upper triangular matrices, denoted T_d^p, acting on $\ell_d^p = \ell^p(\{1, \ldots, d\})$. When $d = 2$, even though T_2^p is itself a degenerate L^p-operator algebra without a cai, we have the following main result:

Theorem 4.2. *For any $p \in [1, \infty)$, $M(T_2^p)$ is isometrically isomorphic to $\mathbb{C}^2$ with the supremum norm. In particular, for any $p, q \in [1, \infty)$, $M(T_2^p)$ is an L^q-operator algebra that is nondegenerately representable on ℓ_2^q.*

The second one is explored in Section 4C, where we focus on $\ell_0^1(G)$, the augmentation ideal of $\ell^1(G)$ for a discrete group G. As an algebra, $\ell_0^1(G)$ is an L^1-operator algebra that has an identity with norm strictly greater than 1 when G is finite and $\text{card}(G) > 2$. The augmentation ideal $\ell_0^1(G)$ is defined as the kernel of the map $\ell^1(G) \to \mathbb{C}$ given by $a \mapsto \sum_{g \in G} a(g)$. We then study the main properties of this ideal and show that, when G is finite, $\ell_0^1(G)$ has no cai and cannot be nondegenerately represented on any Banach space (see Proposition 4.13(7)). For our main result, we construct a family of L^p-operator algebras whose multiplier algebras cannot be represented on any L^q-space for any $q \in [1, \infty)$ as long as $p \in [1, p_0] \cup [p_0', \infty)$, where $p_0 = 1.606$ and p_0' is its Hölder conjugate. This is done by looking at the image of the augmentation ideal $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ in the group algebra $F^p(\mathbb{Z}/3\mathbb{Z})$ under the left regular representation $\lambda_p : \ell^1(\mathbb{Z}/3\mathbb{Z}) \to \mathcal{L}(\ell^p(\mathbb{Z}/3\mathbb{Z}))$. That is, for each $p \in [1, \infty)$, we put

$$F_0^p(\mathbb{Z}/3\mathbb{Z}) = \lambda_p(\ell_0^1(\mathbb{Z}/3\mathbb{Z})),$$

and then denote its multiplier algebra as $M_0^p(\mathbb{Z}/3\mathbb{Z}) = M(F_0^p(\mathbb{Z}/3\mathbb{Z}))$. The Banach algebra $F_0^p(\mathbb{Z}/3\mathbb{Z})$ has an algebraic identity, denoted $\mathbb{1}_0$, whose norm is strict greater than 1. However, in Proposition 4.18 we show that $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is simply $F_0^p(\mathbb{Z}/3\mathbb{Z})$ equipped with a different norm in which $\mathbb{1}_0$ now has norm 1. Our main result in this subsection is then the following:

Theorem 4.21. *Let $p_0 = 1.606$ and let $p_0' = \frac{p_0}{p_0 - 1}$. Take any $p \in [1, p_0] \cup [p_0', \infty)$. Then $M_0^p(\mathbb{Z}/3\mathbb{Z})$, the multiplier algebra of $F_0^p(\mathbb{Z}/3\mathbb{Z})$, is not an L^q-operator algebra for any $q \in [1, \infty)$.*

The main ingredient in the proof of Theorem 4.21 follows a technique previously used by Blecher and Phillips in Section 4 of [3] to show that the class of L^p-operator algebras is not closed under taking quotients. To be more precise, we exhibit a bicontractive idempotent $e \in M_0^p(\mathbb{Z}/3\mathbb{Z})$ such that $\mathbb{1}_0 - 2e$ is not an invertible isometry when $p \in [1, p_0] \cup [p_0', \infty)$, contradicting a structural result of Bernau and Lacey, Theorem 2.9 below, regarding bicontractive idempotent acting on L^p-spaces. Unfortunately, the bicontractive idempotent argument might not work for

groups other that $\mathbb{Z}/3\mathbb{Z}$. For instance, when $G = \mathbb{Z}/4\mathbb{Z}$ or $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the bicontractive idempotents in $M_0^p(G) = M(F_0^p(G))$ behave as expected for an L^p-operator algebra. The essential difficulty in deciding whether $M_0^p(G)$ is an L^p-operator algebra for a general finite group G is that its norm is generally hard to compute (see [16] for instance).

In Section 5, we restrict our attention to nontrivial finite abelian groups and show that when $n = \operatorname{card}(G) > 1$, the algebra $\ell_0^1(G)$ is algebraically isomorphic to $\mathbb{C}^{n-1}$ with pointwise multiplication via the Gelfand map. This yields a nice description of the idempotents of $M_0^p(\mathbb{Z}/n\mathbb{Z})$, see Proposition 5.8, that can be compared with Rudin's more general work for idempotents in $L^1(G)$ when G is a locally compact group (see W. Rudin's work [25] and [24]).

In the last section we present a brief discussion on different norms that make $\mathbb{C}^n$ an L^p-operator algebra, posing a final question regarding what properties a norm on $\mathbb{C}^n$ must satisfy to be an L^p-operator algebra. Answering this general question might provide a tool to decide when $M_0^p(G)$ is representable on an L^q-space.

We end our introduction with a brief overview of the general notation that will be used throughout the paper. When E is a Banach space, we denote by $\mathcal{L}(E)$ the space of all bounded linear maps $E \to E$. For $p \in [1, \infty)$, we sometimes write $L^p(\mu)$ for $L^p(\Omega, \mu)$, the L^p-space of a measure space (Ω, μ). In particular, when ν is the counting measure on Ω we simply write $\ell^p(\Omega)$ for $L^p(\Omega, \nu)$ and, for $d \in \mathbb{Z}_{\geq 2}$, we write ℓ_d^p for $\ell^p(\{1, \dots, d\})$.

2. Preliminaries

We first establish standard definitions for representations of Banach algebras on Banach spaces. The definition of L^p-operator algebras will be given using this terminology. We then present basic facts about Banach algebras with either contractive approximate units or with nonunital identities.

Definition 2.1. Let A be a Banach algebra and let E be a Banach space. A *representation* of A on E is a continuous algebra homomorphism $\pi : A \to \mathcal{L}(E)$.

(1) We say that π is *contractive* if $\|\pi(a)\| \leq \|a\|$ for all $a \in A$.

(2) We say that π is *isometric* if $\|\pi(a)\| = \|a\|$ for all $a \in A$.

(3) We say that π is *nondegenerate* if

$$\pi(A)E = \operatorname{span}(\{\pi(a)\xi : a \in A \text{ and } \xi \in E\})$$

is dense in E, and that A is *nondegenerately representable* if it has a nondegenerate isometric representation.

Definition 2.2. Let $p \in [1, \infty)$. A Banach algebra A is an *L^p-operator algebra* if there is a measure space (Ω, μ) and an isometric representation of A on $L^p(\mu)$.

Definition 2.3. Let A be a Banach algebra. We say that A has a *contractive approximate identity* (*cai*) if there is a net $(e_\lambda)_{\lambda\in\Lambda}$ in A such that $\|e_\lambda\| \le 1$ for all $\lambda \in \Lambda$ and for all $a \in A$,

$$\lim_{\lambda\in\Lambda} \|ae_\lambda - a\| = \lim_{\lambda\in\Lambda} \|e_\lambda a - a\| = 0.$$

Lemma 2.4. *Let A be a Banach algebra with a cai. Then for any $a \in A$,*

$$\|a\| = \sup_{\|b\|=1} \|ab\| = \sup_{\|b\|=1} \|ba\|.$$

Proof. Let $(e_\lambda)_{\lambda\in\Lambda}$ be a cai of A. Then for any $a \in A$,

$$\|a\| \ge \sup_{\|b\|=1} \|ba\| \ge \sup_{\lambda\in\Lambda} \frac{\|e_\lambda a\|}{\|e_\lambda\|} \ge \sup_{\lambda\in\Lambda} \|e_\lambda a\| \ge \|a\|.$$

The other equality is proved analogously. □

Definition 2.5. Let A be a Banach algebra. We say that A has an *identity element* if there is an element $1_A \in A$ such that, for all $a \in A$,

$$1_A \cdot a = a = a \cdot 1_A.$$

If $\|1_A\| = 1$, we call 1_A a *unit* of A; otherwise we say 1_A is a *nonunital identity* of A.

We will work below with a Banach algebra that has a nonunital identity, see Section 4C. The following two lemmas show that such Banach algebras cannot have cai's and cannot be nondegenerately represented on any Banach space.

Lemma 2.6. *Let A be a Banach algebra with a nonunital identity $1_A \in A$ (that is $\|1_A\| \ne 1$). Then, A cannot have a cai.*

Proof. It is clear that $\|1_A\| > 1$. Now suppose A has a cai, say $(e_\lambda)_{\lambda\in\Lambda}$. Then, for each $\lambda \in \Lambda$

$$0 \le |\|e_\lambda\| - \|1_A\|| \le \|e_\lambda - 1_A\| = \|e_\lambda 1_A - 1_A\|.$$

Therefore, by the squeeze theorem

$$\lim_\lambda |\|e_\lambda\| - \|1_A\|| = 0.$$

This implies that $\lim_\lambda \|e_\lambda\| = \|1_A\|$, whence $\|1_A\| \le 1$ which is a contradiction. □

Lemma 2.7. *Let A be a Banach algebra with a nonunital identity $1_A \in A$ (that is $\|1_A\| \ne 1$). Then A cannot be nondegenerately represented on any Banach space.*

Proof. Assume on the contrary that there is a Banach space E and an isometric nondegenerate representation $\pi : A \to \mathcal{L}(E)$. Then, for any $a \in A$ and any $\xi \in E$ we have

$$\pi(1_A)\pi(a)\xi = \pi(1_A a)\xi = \pi(a)\xi.$$

Nondegeneracy now gives that $\pi(1_A) = \mathrm{id}_E$, but this implies

$$1 = \|\mathrm{id}_E\| = \|\pi(1_A)\| = \|1_A\| > 1,$$

a contradiction. □

In Section 4, we will take the multiplier algebra of certain L^p-operator algebras in order to construct a family of unital Banach algebras that are not representable on any L^q-space for any $q \in [1, \infty)$. Our technique, which was also used by Blecher and Phillips in Section 4 of [3], will be to show that some bicontractive idempotents on the algebra do not behave as expected for L^q-operator algebras. We record precise definitions and known facts below.

Definition 2.8. Let A be a unital Banach algebra with unit 1_A.

(1) An idempotent $e \in A$ is *bicontractive* if $\|e\| \leq 1$ and $\|1_A - e\| \leq 1$.

(2) An element $s \in A$ is an *invertible isometry* if $\|s\| = 1$ and $\|s^{-1}\| = 1$.

The following is a combination of Theorem 2.1 in [1] and part (c) of Lemma 2.29 in [3].

Theorem 2.9. *Let $p \in [1, \infty)$ and let (Ω, μ) be a measure space. Then an idempotent $e \in \mathcal{L}(L^p(\mu))$ is bicontractive if and only if $1 - 2e$ is an invertible isometry, where $1 = \mathrm{id}_{L^p(\mu)}$.*

3. Multiplier algebras

Below we present general definitions and some results about multiplier algebras of Banach algebras.

Parts of following definition come from Section 2.5 in [4].

Definition 3.1. Let A be a Banach algebra.

(1) A *left multiplier of A* is a map $L \in \mathcal{L}(A)$ satisfying

$$L(ab) = L(a)b \quad \text{for all} \quad a, b \in A.$$

The set of left multipliers of A is denoted by $LM(A)$.

(2) A *right multiplier of A* is a map $R \in \mathcal{L}(A)$ satisfying

$$R(ab) = aR(b) \quad \text{for all} \quad a, b \in A.$$

The set of right multipliers of A is denoted by $RM(A)$.

(3) A *double centralizer of A* is a pair (L, R) with $L \in LM(A)$, $R \in RM(A)$, and

$$aL(b) = R(a)b \quad \text{for all} \quad a, b \in A.$$

We define $M(A)$, the *multiplier algebra of A*, to be the subset of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$ (equipped with the supremum norm) consisting of double centralizers.

It is clear that $M(A)$ is a unital Banach subalgebra of $\mathcal{L}(A) \times \mathcal{L}(A)^{\mathrm{op}}$.

If A is a C*-algebra, $M(A)$ is equivalently defined as the set of two sided multipliers of A on any Hilbert space as long as A acts nondegenerately on it; see Definition 2.2.2 in [27] for instance. This will be also the case for a Banach algebra that has a cai and that can be nondegenerately represented on a Banach space. We start with the definition of a map from A to $M(A)$ that will be a canonical inclusion when A has a cai.

Definition 3.2. Let A be a Banach algebra. For each $a \in A$ we define maps $L_a : A \to A$ and $R_a : A \to A$ by left and right multiplication respectively, that is for any $b \in A$

$$L_a(b) = ab \quad \text{and} \quad R_a(b) = ba.$$

We get a map $\iota_A : A \to M(A)$ given by $\iota_A(a) = (L_a, R_a)$.

Lemma 3.3. *Let A be a Banach algebra with a cai, and let $(L, R) \in M(A)$. Then $\|L\| = \|R\|$. Furthermore, the map $\iota_A : A \to M(A)$ from Definition 3.2 is an isometric algebra homomorphism and $\iota_A(A)$ is a closed two-sided ideal in $M(A)$.*

Proof. Lemma 2.4 gives

$$\|L(a)\| = \sup_{\|b\|=1} \|bL(a)\| = \sup_{\|b\|=1} \|R(b)a\| \leq \|R\|\|a\|.$$

Thus, $\|L\| \leq \|R\|$. Similarly,

$$\|R(a)\| = \sup_{\|b\|=1} \|R(a)b\| = \sup_{\|b\|=1} \|aL(b)\| \leq \|a\|\|L\|,$$

whence $\|R\| \leq \|L\|$ and therefore $\|L\| = \|R\|$. Note that ι_A is clearly a linear map. To check it is multiplicative take $a, b \in A$ and compute

$$\iota_A(ab) = (L_{ab}, R_{ab}) = (L_a L_b, R_b R_a) = (L_a, R_a)(L_b, R_b) = \iota_A(a)\iota_A(b).$$

Once again, Lemma 2.4 gives $\|L_a\| = \|R_a\| = \|a\|$, showing that ι_A is isometric and that $\iota_A(A)$ is closed in $M(A)$. Finally, direct computations show that for any $a \in A$ and any $(L, R) \in M(A)$, $\iota_A(a)(L, R) = \iota_A(R(a))$ and $(L, R)\iota_A(a) = \iota_A(L(a))$, whence $\iota_A(A)$ is indeed a two-sided ideal in $M(A)$, finishing the proof. □

A well known and useful fact is that whenever A is a Banach algebra with a cai that is nondegenerately represented on a Banach space E, then $M(A)$ can be nondegenerately represented on E. For convenience, we state such result below and refer the reader to Theorem 4.1 and Remark 4.2 in [13] for a complete proof.

Theorem 3.4. *Let A be a Banach algebra with a cai and let $\pi : A \to \mathcal{L}(E)$ be a contractive nondegenerate representation of A on a Banach space E. Then π induces a unique nondegenerate, contractive, and unital representation $\widehat{\pi} : M(A) \to$*

$\mathcal{L}(E)$ such that, if ι_A is as in Definition 3.2, then $\pi = \widehat{\pi} \circ \iota_A$. Furthermore, $\widehat{\pi}$ is isometric when π is.

Remark 3.5. Theorem 3.4 can also be proved by either appropriately modifying the proof of Proposition 2.6.12 in [2] or by the methods described in Section 6 of [19]. We are grateful to both David Blecher and Hannes Thiel for pointing these references to us. An alternative way is to use Corollary 4.1.7 in [7]. This alternative proof uses that, just as in the C*-case, $M(A)$ is isometrically isomorphic to the two-sided multipliers of A on any Banach space E where A acts nondegenerately. This fact is actually equivalent to Theorem 3.4, as we see below in Corollary 3.6.

Corollary 3.6. *Let A be a Banach algebra with a cai and that is nondegenerately represented on a Banach space E via $\pi : A \to \mathcal{L}(E)$. Then the Banach algebra $M(A)$ is isometrically isomorphic to*

$$\{t \in \mathcal{L}(E) : t\pi(A) \subseteq \pi(A), \pi(A)t \subseteq \pi(A)\},$$

the algebra of two sided multipliers of $\pi(A)$, via the map $\widehat{\pi}$ from Theorem 3.4.

Proof. Theorem 3.4 gives that $\widehat{\pi} : M(A) \to \mathcal{L}(E)$ is an isometric and unital representation satisfying $\widehat{\pi} \circ \iota_A = \pi$. Set

$$B = \{t \in \mathcal{L}(E) : t\pi(A) \subseteq \pi(A), \pi(A)t \subseteq \pi(A)\}.$$

It only remains to show that the image of $\widehat{\pi}$ is indeed B. To do so, notice that for any $(L, R) \in M(A)$ and $a \in A$ we have $\widehat{\pi}(L, R)\pi(a) = \pi(L(a))$ and $\pi(a)\widehat{\pi}(L, R) = \pi(R(a))$, whence $\widehat{\pi}(M(A)) \subseteq B$. To check the reverse inclusion, for any $t \in B$ we define $L_t, R_t \in \mathcal{L}(A)$ as follows:

$$L_t(a) = \pi^{-1}(t\pi(a)), \;\; R_t(a) = \pi^{-1}(\pi(a)t),$$

where $\pi^{-1} : \pi(A) \to A$ is understood as the inverse of the invertible isometry $\pi : A \to \pi(A)$. Since $t \in B$, we have $(L_t, R_t) \in M(A)$. Furthermore, for any $a \in A$, $\xi \in E$ we have

$$\widehat{\pi}(L_t, R_t)\pi(a)\xi = \pi(L_t(a))\xi = t\pi(a)\xi.$$

Hence, nondegeneracy implies that $\widehat{\pi}(L_t, R_t) = t$ and therefore we conclude that $B \subseteq \widehat{\pi}(M(A))$, finishing the proof. □

We will be mostly interested in $M(A)$ when A is a Banach algebra that has a nonunital identity as in Definition 2.5. This implies that A does not have a cai, as shown in Lemma 2.6, whence the previous two results cannot be used. However, the following proposition shows that in such cases $M(A)$ is actually the Banach algebra A equipped with a new norm.

Proposition 3.7. *Let A be a Banach algebra with identity $1_A \in A$, as in Definition 2.5, and let $\iota_A : A \to M(A)$ be as in Definition 3.2. Then the map $\varphi : M(A) \to A$ given by*

$$\varphi(L, R) = L(1_A)$$

is an isometric isomorphism between the Banach algebras $M(A)$ and $(A, |||-|||)$, where $|||a||| = \|\iota_A(a)\|$. In particular, $|||a||| = \|a\|$ for all $a \in A$ when 1_A is a unit.

Proof. It is routine to check that $\varphi : M(A) \to A$ is a linear map. Further, it is clear that $\varphi(\iota_A(a)) = a$, whence φ is surjective. If $\varphi(L, R) = 0$, then for any $a \in A$ we get $L(a) = L(1_A a) = L(1_A)a = \varphi(L, R)a = 0$ and $R(a) = R(a)1_A = aL(1_A) = a\varphi(L, R) = 0$, from where it follows that φ is injective. Next, to show that φ is multiplicative we take any $(L, R), (L', R') \in M(A)$, and compute

$$\begin{aligned}\varphi\big((L, R)(L', R')\big) &= \varphi(LL', R'R)\\ &= L(L'(1_A))\\ &= L(1_A L'(1_A))\\ &= L(1_A)L'(1_A) = \varphi(L, R)\varphi(L', R').\end{aligned}$$

Next, for any $(L, R) \in M(A)$ we obtain

$$|||\varphi(L, R)||| = \|\iota_A(L(1_A))\| = \|(L_{L(1_A)}, R_{L(1_A)})\| = \|(L, R)\|,$$

so that $\varphi : M(A) \to A$ is indeed isometric when A is equipped with $|||a||| = \|\iota_A(a)\|$. Finally, if 1_A is a unit, then A has a trivial cai and therefore Lemma 3.3 implies that ι_A is an isometry. □

Remark 3.8. Section 2 of [26] defines the multiplier algebra of a commutative Banach algebra A without absolute zero divisors other than 0 by

$$N(A) = \{L \in \mathcal{L}(A) : aL(b) = L(a)b \ \text{ for all } \ a, b \in A\}.$$

When A is commutative and $(L, R) \in M(A)$, we have $L = R$ and $(L, R) \mapsto L$ is an isometric algebra isomorphism from $M(A)$ to $N(A)$. Hence, both definitions are equivalent. If A is a commutative Banach algebra with identity $1_A \in A$ as in Definition 2.5, then A has no absolute zero divisors other than 0 and therefore Proposition 3.7 above implies that $M(A)$ is isometrically isomorphic, as a Banach algebra, to A equipped with the norm $a \mapsto \|L_a\|$.

4. Main results

In this section, we explore whether the conditions in Theorem 3.4 are necessary to guarantee that the multiplier algebra of an L^p-operator algebra is again an L^p-operator algebra. We do so via two contrasting examples presented in Sections 4B

and 4C. On the one hand, we show that the multiplier algebra of 2×2 strictly upper triangular matrices acting on ℓ_2^p is an L^q-operator algebra for any $p, q \in [1, \infty)$. On the other hand, for each $p \in [1, \infty)$ we define a two dimensional L^p-operator algebra whose multiplier algebra is not an L^q-operator algebra for any $q \in [1, \infty)$ as long as $p \in [1, p_0] \cup [p_0', \infty)$ where $p_0 = 1.606$ and p_0' is its Hölder conjugate given by $p_0' = \frac{p_0}{p_0 - 1} = 2.65016501\ldots = 2.\overline{6501}$.

4A. ***Approximately unital and nondegenerate case.*** The following result is simply a consequence of Theorem 3.4 being a particular case of Corollary 3.6.

Theorem 4.1. *Let A be an L^p-operator algebra that has a cai and such that there is a nondegenerate representation π of A on $L^p(\mu)$. Then $M(A)$ is an L^p-operator algebra that is isometrically isomorphic to the algebra of two sided multipliers of $\pi(A)$, that is $M(A) \cong \{t \in L^p(\mu) : t\pi(a) \in \pi(A), \pi(a)t \in \pi(A)$ for all $a \in A\}$.*

Theorem 4.1 raises a natural question: Are the properties of having a cai and being nondegenerately represented necessary for $M(A)$ to be an L^p-operator algebra? Below, we present two examples of families of L^p-operator algebras that have neither property. In Section 4B, we investigate strictly upper triangular 2×2 matrices and show its multiplier algebra can be nondegenerately represented on an L^p-space. In Section 4C, we investigate the group L^p-operator algebras associated to the augmentation ideal $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ and show that for certain values of p, its multiplier algebra fails to be an L^q-operator algebra for any $q \in [1, \infty)$.

4B. ***Strictly upper triangular matrices.*** Let $p \in [1, \infty)$. We start by considering T_2^p, the algebra of strictly upper triangular 2×2 matrices acting on ℓ_2^p, that is,

$$T_2^p = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbb{C} \right\} \subset M_2^p(\mathbb{C}) = \mathcal{L}(\ell_2^p).$$

This is a degenerate L^p-operator algebra without identity, unit, nor cai. In fact, T_2^p is isometrically isomorphic to $\mathbb{C}$ with the trivial multiplication $zw = 0$ (see Example 3.6 in [9]). Notice that T_2^p is commutative, but all its elements are absolute zero divisors, whence Remark 3.8 does not apply. However, it is clear that $\mathcal{L}(T_2^p) = \mathcal{L}(\mathbb{C}) \cong \mathbb{C}$ and that $aL(b) = 0 = R(a)b$ for any $(L, R) \in M(T_2^p)$ and any $a, b \in T_2^p$. This, together with Definition 3.1, imply at once that

$$M(T_2^p) = LM(T_2^p) \times RM(T_2^p).$$

Furthermore, notice that $LM(T_2^p) = RM(T_2^p) = \mathcal{L}(T_2^p) \cong \mathbb{C}$. Therefore, $M(T_2^p)$ is isometrically isomorphic to $\mathbb{C}^2$ with the supremum norm, which is in turn isometrically identified with $C(\{1, 2\})$, the space of continuous functions on $\{1, 2\}$. Finally, for any $q \in [1, \infty)$, we have the map $\varphi_q : C(\{1, 2\}) \to \mathcal{L}(\ell_2^q)$ given by

$$(\varphi_q(\xi)x)(j) = \xi(j)x(j),$$

where $\xi \in C(\{1,2\})$, $x \in \ell_2^q$, and $j \in \{1,2\}$. This allows us to represent continuous functions on $\{1,2\}$ as multiplication operators on ℓ_2^q. It is well known and easy to see that φ_q is an isometric nondegenerate representation. Therefore, we have shown the following result:

Theorem 4.2. *For any $p \in [1,\infty)$, $M(T_2^p)$ is isometrically isomorphic to $\mathbb{C}^2$ with the supremum norm. In particular, for any $p, q \in [1,\infty)$, $M(T_2^p)$ is an L^q-operator algebra that is nondegenerately representable on ℓ_2^q.*

A more complicated example is T_3^p, the algebra of strictly upper triangular 3×3 matrices acting on ℓ_3^p. That is

$$T_3^p = \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} : a_{12}, a_{13}, a_{23} \in \mathbb{C} \right\} \subset M_3^p(\mathbb{C}) = \mathcal{L}(\ell_3^p).$$

Notice that T_3^p is identified with $\mathbb{C}^3$ with multiplication given by

$$ab = (a_{12}, a_{13}, a_{23})(b_{12}, b_{13}, b_{23}) = (0, a_{12}b_{23}, 0).$$

If $p = 1$, then T_3^p is isometrically isomorphic to $\mathbb{C}^3$ with norm

$$\|a\| = \max\{|a_{12}|, |a_{13}| + |a_{23}|\}.$$

However, when $p \neq 1, 2, \infty$ it is known that computing the p-operator norm of a matrix is an NP-hard problem (see [16]). This complicates the study of $M(T_3^p)$ and at this moment we have not investigated whether $M(T_3^p)$ can be isometrically represented on an L^q-space.

4C. ***Augmentation ideals.*** We will now construct, for certain values of p, a class of L^p-operator algebras that have no cai, cannot be nondegenerately represented (on any Banach space), and whose multiplier algebra is not an L^q-operator algebra for any $q \in [1,\infty)$. We begin with the general construction of the augmentation ideal for an arbitrary discrete group G and eventually specialize to the finite case. In particular we will work with $G = \mathbb{Z}/3\mathbb{Z}$. The augmentation ideal of a group ring is a well known algebraic object (see [20, Section 11.2], [8, Example 7, p. 245] for instance) and is easily generalized to operator algebras arising from discrete groups. We include details here for completeness.

Let G be a discrete group with identity element denoted by $1_G \in G$. For any $g \in G$, the indicator function $\delta_g : G \to \mathbb{C}$ is given by

$$\delta_g(h) = \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

For each $g \in G$, we define the function $\Delta_g : G \to \mathbb{C}$ by

$$\Delta_g = \delta_g - \delta_{1_G}. \tag{4-1}$$

For each $p \in [1, \infty)$, recall that the Banach space $\ell^p(G)$ is the set of complex valued functions $a : G \to \mathbb{C}$ satisfying

$$\|a\|_p^p = \sum_{g \in G} |a(g)|^p < \infty.$$

Further, we define the convolution of any pair $(a, b) \in \ell^1(G) \times \ell^p(G)$ as the complex valued function $a * b : G \to \mathbb{C}$ defined at each $g \in G$ by

$$(a * b)(g) = \sum_{h \in G} a(h) b(h^{-1} g). \tag{4-2}$$

It is well known that $\|a * b\|_p \leq \|a\|_1 \|b\|_p$, whence $a * b \in \ell^p(G)$. Moreover, when $p = 1$, convolution gives rise to a multiplication on $\ell^1(G)$ that makes it a unital Banach algebra with unit δ_{1_G}.

Remark 4.3. Given the large presence of multiplication by convolution in this paper, when $a \in \ell^1(G)$ and $b \in \ell^p(G)$ for some $p \in [1, \infty)$ we will simply write ab instead of $a * b$ from (4-2). It will be understood by context when convolution is the multiplication that is being used.

It is also well known that $C_c(G)$, the subspace of compactly supported functions on G, lies densely in $\ell^1(G)$ with respect to the $\| - \|_1$ norm. Clearly, $\delta_g \in C_c(G)$ for each $g \in G$.

Lemma 4.4. *The map $\iota_0 : \ell^1(G) \to \mathbb{C}$ given by*

$$\iota_0(a) = \sum_{g \in G} a(g) \tag{4-3}$$

is a contractive algebra homomorphism.

Proof. It is clear that ι_0 is a well-defined linear map and that $|\iota_0(a)| \leq \|a\|_1$, whence $\|\iota_0\| \leq 1$. By Fubini's Theorem, we see that

$$\iota_0(ab) = \sum_{g \in G} (ab)(g) = \sum_{h \in G} a(h) \sum_{g \in G} b(h^{-1} g) = \iota_0(a) \iota_0(b),$$

finishing the proof. □

The main object of interest for this subsection is the augmentation ideal of $\ell^1(G)$, which is defined as the kernel of the map $\iota_0 : \ell^1(G) \to \mathbb{C}$ from (4-3). We record a precise definition below.

Definition 4.5. Let G be a discrete group. The *augmentation ideal* of $\ell^1(G)$ is defined as

$$\ell_0^1(G) = \ker(\iota_0) = \left\{ a \in \ell^1(G) : \iota_0(a) = 0 \right\}.$$

Similarly, we define $\mathbb{C}_0[G] \subseteq C_c(G)$ as

$$\mathbb{C}_0[G] = \{ a \in C_c(G) : \iota_0(a) = 0 \}.$$

Next we establish the main properties of $\ell^1_0(G)$, which we present as a series of Propositions.

Proposition 4.6. *Let G be a discrete group.*

(1) *The augmentation ideal $\ell^1_0(G)$ is a closed two-sided proper ideal of $\ell^1(G)$.*

(2) *The subspace $\mathbb{C}_0[G]$ is dense in $\ell^1_0(G)$ with respect to the $\|-\|_1$ norm.*

(3) *Let Δ_g be as in (4-1). Then* $\operatorname{span}\{\Delta_g : g \in G\}$ *is dense in $\ell^1_0(G)$. In fact,* $\operatorname{span}\{\Delta_g : g \in G\} = \mathbb{C}_0[G]$.

Proof. That $\ell^1_0(G)$ is a closed two-sided ideal follows from Lemma 4.4 above. Since $\delta_{1_G} \notin \ell^1_0(G)$, we also get that $\ell^1_0(G)$ is proper, so part (1) is done. For part (2), let $a \in \ell^1_0(G)$ and let $\varepsilon > 0$. Then, there is $b \in C_c(G)$ such that $\|a - b\|_1 < \frac{\varepsilon}{2}$ and observe that $b - \iota_0(b)\delta_{1_G} \in \mathbb{C}_0[G]$. Further, note that $|\iota_0(b)| = |\iota_0(a-b)| \leq \|a-b\|_1$. Therefore,

$$\|a - (b - \iota_0(b)\delta_{1_G})\|_1 \leq \|a-b\|_1 + |\iota_0(b)| \leq 2\|a-b\|_1 < \varepsilon.$$

This proves that $\mathbb{C}_0[G]$ is dense in $\ell^1_0(G)$, as wanted. Finally, for part (3), note first that $\operatorname{span}\{\Delta_g : g \in G\} \subseteq \mathbb{C}_0[G]$ because for each $g \in G$, it is clear that $\Delta_g \in C_c(G)$ and $\iota_0(\Delta_g) = \iota_0(\delta_g) - \iota_0(\delta_{1_G}) = 1 - 1 = 0$. For the reverse inclusion, take any $a \in \mathbb{C}_0[G]$, that is $\iota_0(a) = 0$ and $F = \operatorname{supp}(a)$ is a finite subset of G. Therefore,

$$a = \sum_{g \in F} a(g)\delta_g = \left(\sum_{g \in F} a(g)\delta_g\right) - \iota_0(a)\delta_{1_G} = \sum_{g \in F} a(g)\Delta_g.$$

This proves that $a \in \operatorname{span}\{\Delta_g : g \in G\}$ and we are done with part (3). □

The previous proposition shows that $\ell^1_0(G)$ is a Banach algebra in its own right with the 1-norm inherited from $\ell^1(G)$. When $p \in [1, \infty)$, we will mainly be interested in the image of $\ell^p_0(G)$ in $F^p(G)$, the full L^p operator algebra of G, which we define below in the discrete group case setting. We refer the interested reader to Definition 2.1 in [10] for a general definition of $F^p(G)$ for locally compact groups.

Definition 4.7. Let G be a discrete group and let $p \in [1, \infty)$.

(1) We define the *full group L^p-operator algebra of G*, denoted by $F^p(G)$, as the completion of $\ell^1(G)$ in the norm

$$a \mapsto \|a\|_{F^p(G)} = \sup\{\|\pi(a)\| : \pi \in \operatorname{Rep}_{L^p}(G)\},$$

where $\operatorname{Rep}_{L^p}(G)$ is the class of all nondegenerate contractive representations of $\ell^1(G)$ on L^p-spaces.

(2) We define *the augmentation ideal of $F^p(G)$*, denoted by $F^p_0(G)$, as the completion of $\ell^1_0(G)$ in the $\|-\|_{F^p(G)}$ norm.

The algebra $F^p(G)$ is in fact an L^p-operator algebra, this follows from the general crossed product case, see for instance Theorem 3.6 in [22]. We next show that $F_0^p(G)$ is actually a closed proper ideal in $F^p(G)$, making it also an L^p-operator algebra in its own right.

Proposition 4.8. *Let G be a discrete group and let $p \in [1, \infty)$. Then $F_0^p(G)$ is a nonzero closed two-sided proper ideal in $F^p(G)$.*

Proof. That $F_0^p(G)$ is a nonzero closed two-sided ideal follows at once from the fact that $\ell_0^1(G)$ is an ideal in $\ell^1(G)$. To show that it is proper, it suffices to prove that δ_{1_G}, the unit in $F^p(G)$, does not belong to $F_0^p(G)$. Assume on the contrary that there is a sequence $(a_n)_{n=1}^{\infty}$ in $\ell_0^1(G)$ such that a_n converges to δ_{1_G} in $F^p(G)$. Next, by Lemma 4.4, ι_0 is a contractive representation of $\ell^1(G)$ on the L^p-space $\ell_1^p = \mathbb{C}$, so it follows from definition of $\|-\|_{F^p(G)}$ that $|\iota_0(a)| \leq \|a\|_{F^p(G)}$ for all $a \in \ell_0^1(G)$. Therefore ι_0 extends continuously to a contraction $\iota_0 : F^p(G) \to \mathbb{C}$. In particular, since $\iota_0(a_n) = 0$ for all $n \in \mathbb{Z}_{\geq 1}$, it follows that

$$\begin{aligned} 1 = |\iota_0(\delta_{1_G})| &= |\iota_0(\delta_{1_G}) - \iota_0(a_n)| \\ &\leq \|\delta_{1_G} - a_n\|_{F^p(G)}. \end{aligned}$$

Taking the limit as $n \to \infty$ shows that $1 \leq 0$, a contradiction. Thus, $\delta_{1_G} \notin F_0^p(G)$, as wanted. □

For each $p \in [1, \infty)$, we will also let $\ell_0^1(G)$ act on $\ell^p(G)$ via left multiplication by convolution. That way the completion of the image of $\ell_0^1(G)$ in $\mathcal{L}(\ell^p(G))$ becomes an L^p-operator algebra. In particular, when $p = 1$, the algebra $\ell_0^1(G)$ is isometrically isomorphic to its image in $\mathcal{L}(\ell^1(G))$, making $\ell_0^1(G)$ an L^1-operator algebra. More precisely, this will be done by looking at the image of $\ell_0^1(G)$ in $F_{\mathrm{r}}^p(G)$, the reduced L^p-operator algebra of G. The algebra $F_{\mathrm{r}}^p(G)$ was first introduced by Herz in [18, Section 8], where it was called the algebra of p-pseudofunctions on G, and then reintroduced by Phillips in [22, Section 3] as a particular case of reduced L^p-operator crossed products. For convenience we state below the definition of $F_{\mathrm{r}}^p(G)$ for G discrete. We refer the interested reader to Definition 3.1 in [10] for the general definition of $F_{\mathrm{r}}^p(G)$ when G is locally compact.

Definition 4.9. Let G be a discrete group and let $p \in [1, \infty)$.

(1) Let $\lambda_p : \ell^1(G) \to \mathcal{L}(\ell^p(G))$ denote left multiplication by convolution; that is, for $a \in \ell^1(G)$, $b \in \ell^p(G)$, we put $\lambda_p(a)b = ab \in \ell^p(G)$ as in (4-2) (see Remark 4.3). We define the *reduced L^p-operator algebra of G*, denoted by $F_{\mathrm{r}}^p(G)$, as the closure of $\lambda_p(\ell^1(G))$ in $\mathcal{L}(\ell^p(G))$.

(2) We define *the augmentation ideal of $F_{\mathrm{r}}^p(G)$*, denoted by $F_{\mathrm{r},0}^p(G)$, as the closure of $\lambda_p(\ell_0^1(G))$ in $\mathcal{L}(\ell^p(G))$.

Since G is discrete, $\|a\|_p \leq \|a\|_1$ for all $a \in \ell^1(G)$, in fact the canonical inclusion $\ell^1(G) \hookrightarrow \ell^p(G)$ has norm 1. Furthermore, observe that for any $a \in \ell^1(G)$ we have

$$\|a\|_p \leq \|a\|_{F_r^p(G)} \leq \|a\|_{F^p(G)} \leq \|a\|_1. \tag{4-4}$$

Hence, when $p = 1$, the map $\lambda_1 : \ell^1(G) \to \mathcal{L}(\ell^1(G))$ is isometric and therefore we have isometric isomorphisms of Banach algebras

$$F^1(G) \cong F_{\mathrm{r}}^1(G) \cong \ell^1(G) \text{ and } F_0^1(G) \cong F_{\mathrm{r},0}^1(G) \cong \ell_0^1(G). \tag{4-5}$$

For general $p \in [1, \infty)$, the map $\lambda_p : \ell^1(G) \to \mathcal{L}(\ell^p(G))$ is an injective contractive nondegenerate representation of $\ell^1(G)$ on $\ell^p(G)$. Thus, $\lambda_p(\ell^1(G))$ is a dense subalgebra of $F_{\mathrm{r}}^p(G)$ that is algebraically isomorphic to $\ell^1(G)$. Moreover, by definition $F_{\mathrm{r}}^p(G)$ is a closed unital subalgebra of $\mathcal{L}(\ell^p(G))$, making it an L^p-operator algebra as in Definition 2.2.

Remark 4.10. As in Proposition 4.8, $F_{\mathrm{r},0}^p(G)$ is a nonzero closed two-sided ideal in $F_{\mathrm{r}}^p(G)$ and therefore an L^p-operator algebra. However, unlike Proposition 4.8, we no longer have that the ideal $F_{\mathrm{r},0}^p(G)$ is proper in general. This is explained due to the fact that in many instances $F_r^p(G)$ is a simple Banach algebra. Indeed, by Theorem 2.5 in [23], if $F_{\mathrm{r}}^2(G)$ is simple, then $F_r^p(G)$ is simple for any $p \in (1, \infty)$. In particular, for $p \in (1, \infty)$, $F_{\mathrm{r}}^p(G)$ will be simple when G has the Powers property (see also Theorem 3.7 in [15]). Thus, we can guarantee that for any $n \in \mathbb{Z}_{>1}$, $F_{r,0}^p(\mathbb{F}_n) = F_r^p(\mathbb{F}_n)$ where $\mathbb{F}_n$ is the free group in n generators.

However, Theorem 3.7 in [10] shows that, for any $p \in (1, \infty)$, if G is an amenable group, then $F^p(G)$ is isometrically isomorphic to $F_{\mathrm{r}}^p(G)$ as Banach algebras. Therefore, this fact combined with (4-5) give at once the following result:

Proposition 4.11. *Let G be a discrete group and let $p \in [1, \infty)$. If either $p = 1$ or G is amenable, then $F_{\mathrm{r},0}^p(G)$ is a proper ideal in $F_{\mathrm{r}}^p(G)$ that is degenerately represented on $\ell^p(G)$.*

In any case, it is more natural to work with the ideal $F_0^p(G)$, rather than $F_{\mathrm{r},0}^p(G)$, for $F_0^p(G)$ is guaranteed to be a proper ideal for any discrete group G. Moreover, for a finite nontrivial group G, we will show in Corollary 4.17 below that $F_0^p(G)$ cannot be nondegenerately represented on any Banach space, already voiding one of the assumptions in Theorem 4.1. Part of our work below is to show that, for a finite nontrivial group G, the assumption of having a cai is also not met for $F_0^p(G)$.

Remark 4.12. We mention that since finite groups are amenable, $F_0^p(G)$ and $F_{\mathrm{r},0}^p(G)$ are isometrically isomorphic as Banach algebras (see Theorem 3.7 in [10]). Therefore, if G is a nontrivial finite group, we will keep using the simpler notation $F_0^p(G)$, but for convenience we will use that $F_0^p(G)$ can be defined, in this case, as the closure of $\lambda_p(\ell_0^1(G))$ in $\mathcal{L}(\ell^p(G))$.

For the rest of the paper, we will concentrate on the properties of $F_0^p(G)$ when G is a finite nontrivial group.

Proposition 4.13. *Let G be a finite group with* $\operatorname{card}(G) \geq 2$ *and let $p \in [1, \infty)$. Then, for $n = \operatorname{card}(G)$:*

(1) *As vector spaces,*

$$F_0^p(G) = \ell_0^1(G) = \operatorname{span}\{\Delta_g : g \in G\} = \mathbb{C}_0[G] \cong \mathbb{C}^{n-1}.$$

(2) *For any $a \in \ell_0^1(G)$,*

$$a\left(\sum_{g\in G} \delta_g\right) = \left(\sum_{g\in G} \delta_g\right)a = 0.$$

(3) *$F_0^p(G)$ has an identity element* (see Definition 2.5) *given by*

$$\mathbb{1}_0 = -\frac{1}{n}\sum_{g\in G} \Delta_g.$$

(4) $$\frac{((n-1)^p + n - 1)^{1/p}}{n} \leq \|\mathbb{1}_0\|_{F^p(G)} \leq 2 - \frac{2}{n}.$$

(5) $\|\mathbb{1}_0\|_1 = 2 - \frac{2}{n}$.

(6) *$\mathbb{1}_0$ is not a unit in $F_0^1(G)$ when $n > 2$* (see Definition 2.5).

(7) *$F_0^1(G)$ has no cai and cannot be nondegenerately represented on any Banach space when $n > 2$.*

Proof. In what follows we write $n = \operatorname{card}(G)$ and $G = \{g_1, \ldots, g_n\}$ where $g_1 = 1_G$. Since G is finite, $C_c(G) = \ell^1(G)$ and $\ell^1(G) \cong \mathbb{C}^n$ as vector spaces via the map $\delta_{g_j} \mapsto \mathbf{e}_j \in \mathbb{C}^n$ for $j = 1, \ldots, n$. Thus using Proposition 4.6 we now get $\mathbb{C}_0[G] = \ell_0^1(G)$ and $\ell_0^1(G) \cong \mathbb{C}^{n-1}$ as vector spaces via the map $\Delta_{g_j} \mapsto \mathbf{e}_{j-1} \in \mathbb{C}^{n-1}$ for $j = 2, \ldots, n$. Then $\ell_0^1(G)$ is a finite dimensional vector space and therefore already complete in the $\|-\|_{F^p(G)}$ norm. This gives that, for any $p \in [1, \infty)$, $F_0^p(G) = \ell_0^1(G)$ which is in turn identified with $\mathbb{C}^{n-1}$, equipped with a different norm for each p, proving part (1). Next, to show parts (2) and (3), we take any $a \in \ell_0^1(G)$ so that $\iota_0(a) = 0$. Then, since $(a\delta_g)(h) = a(hg^{-1})$ for any $g, h \in G$, it follows that for any $k \in \{1, \ldots, n\}$

$$\left(a\left(\sum_{j=1}^n \delta_{g_j}\right)\right)(g_k) = \sum_{j=1}^n a(g_k g_j^{-1}) = \iota_0(a) = 0,$$

proving that $a\left(\sum_{j=1}^n \delta_{g_j}\right) = 0$. A similar computation shows that $\left(\sum_{j=1}^n \delta_{g_j}\right)a = 0$, so part (2) follows. For part (3), notice that for any $j, k \in \{1, \ldots, n\}$,

$$(a\Delta_{g_j})(g_k) = (a\delta_{g_j})(g_k) - a(g_k) = a(g_k g_j^{-1}) - a(g_k).$$

Therefore, for any $k \in \{1, \dots, n\}$ we obtain

$$(a\mathbb{1}_0)(g_k) = \frac{1}{n}\sum_{j=1}^{n}\big(-a(g_k g_j^{-1}) + a(g_k)\big) = \frac{1}{n}(-\iota_0(a) + na(g_k)) = a(g_k),$$

from where it follows that $a\mathbb{1}_0 = a$. An analogous computation gives that $\mathbb{1}_0 a = a$ and therefore part (3) is done. Part (4) follows at once from (4-4) and the following direct computation for each $p \in [1, \infty)$:

$$\|\mathbb{1}_0\|_p = \frac{1}{n}\left\|\sum_{j=2}^{n}(\delta_{g_j} - \delta_{g_1})\right\|_p = \frac{1}{n}\big(|n-1|^p + (n-1)|-1|^p\big)^{1/p}.$$

Part (5) is simply (4) when $p = 1$. Next, notice that part (6) is implied by part (5) for $\|\mathbb{1}_0\|_1 > 1$ when $n > 2$. Finally, when $n > 2$, part (6) together with Lemma 2.6 shows that $F_0^1(G)$ does not have a cai. Combined with Lemma 2.7, this shows that $F_0^1(G)$ cannot be nondegenerately represented on any Banach space, so part (7) follows. □

Our final goal for this subsection is to investigate whether $M(F_0^p(G))$, the multiplier algebra of $F_0^p(G)$ (see Definition 4.9), is an L^p-operator algebra when G is finite and abelian. Proposition 4.13 (7) shows that, at least when $p = 1$, the hypotheses of Theorem 4.1 are not met for $A = F_0^p(G)$. We will show that for all $p \in [1, \infty) \setminus \{2\}$ and any finite group G with $\mathrm{card}(G) > 2$, $F_0^p(G)$ still does not meet the hypotheses in Theorem 4.1. In contrast to the algebra T_2^p (see Section 4B), we will see that for certain values of p, including 1, $M(F_0^p(\mathbb{Z}/3\mathbb{Z}))$ is not an L^q-operator algebra for any $q \in [1, \infty)$.

We start with a computational lemma that will be needed twice in the rest of the paper.

Lemma 4.14. *Let $p \in (1, \infty) \setminus \{2\}$ and let G be a finite group with $n = \mathrm{card}(G) > 2$. Define $a_0 : G \to \mathbb{C}$ by*

$$a_0(g) = \begin{cases} n-1 & \text{if } g = 1_G, \\ -1 & \text{if } g \neq 1_G. \end{cases}$$

For each $\varepsilon \in \mathbb{R}$, we define $b_\varepsilon : G \to \mathbb{C}$ by

$$b_\varepsilon = a_0 + \varepsilon \sum_{g \in G} \delta_g.$$

Then:

(1) *$a_0 \in \ell_0^1(G)$ and $b_\varepsilon \in \ell^p(G) \setminus \ell_0^1(G)$ for any $\varepsilon \neq 0$.*

(2) *For any $\varepsilon \in \mathbb{R}$, $\mathbb{1}_0 b_\varepsilon = a_0$ (when $\varepsilon \neq 0$ we think of b_ε as a perturbation of a_0 that leaves $\ell_0^1(G)$).*

(3) *There exists $\varepsilon_p \neq 0$ such that $\|a_0\|_p > \|b_{\varepsilon_p}\|_p$.*

(4) *In particular, when* $n = 3$, *we get*

$$\frac{\|a_0\|_p}{\|b_{\varepsilon_p}\|_p} = \frac{(2^p+2)(2^{\frac{1}{p-1}}+1)^p}{3^p(2^{\frac{p}{p-1}}+2)} > 1.$$

Proof. Part (1) is readily verified. Part (2) follows at once from parts (2) and (3) in Proposition 4.13. For part (3), note that asking for $\|a_0\|_p > \|b_\varepsilon\|_p$ is equivalent to

$$(n-1)^p + n - 1 > |n-1+\varepsilon|^p + (n-1)|\varepsilon - 1|^p. \tag{4-6}$$

Next, we define $f_p : (-1, 1) \to \mathbb{R}$ by $f_p(\varepsilon) = \|a_0\|_p - \|b_\varepsilon\|_p$. That is,

$$f_p(\varepsilon) = (n-1)^p + n - 1 - (n-1+\varepsilon)^p - (n-1)(1-\varepsilon)^p.$$

We need to show that exists $\varepsilon_p \neq 0$ such that $f_p(\varepsilon_p) > 0$. Clearly $f_p(0) = 0$. Furthermore, it is routine to verify that f_p is a differentiable concave function, that f_p has a maximum at

$$\varepsilon_p = \frac{(n-1)^{\frac{1}{p-1}} - (n-1)}{(n-1)^{\frac{1}{p-1}} + 1} \neq 0,$$

and that

$$f_p'(0) = \frac{1}{n-1}((n-1)^2 - (n-1)^p)p.$$

Thus, since $n > 2$, if $p \in (1, 2)$ we have $f_p'(0) > 0$ and $\varepsilon_p > 0$, which gives $f_p(\varepsilon_p) > 0$. Similarly, $p \in (2, \infty)$ implies $f_p'(0) < 0$ and $\varepsilon_p < 0$, from where it follows that $f_p(\varepsilon_p) > 0$. This now gives part (3). Finally, part (4) follows from plugging in $n = 3$ and ε_p in (4-6). □

We now obtain a generalization of part (6) in Proposition 4.13 for any $p \neq 2$. The proof for $p = 1$ was immediate, but since the right hand side of the inequality in part (4) of Proposition 4.13 is not always strictly greater than 1, a different argument is needed that works for any $p \in (1, \infty) \setminus \{2\}$ and any finite group G with $\operatorname{card}(G) > 2$.

Proposition 4.15. *Let* $p \in [1, \infty) \setminus \{2\}$ *and let* G *be a finite group with* $\operatorname{card}(G) > 2$. *Then* $\mathbb{1}_0$ *is a nonunital identity of* $F_0^p(G)$ (see Definition 2.5).

Proof. As before, we write $n = \operatorname{card}(G)$ and $G = \{g_1, \ldots, g_n\}$ where $g_1 = 1_G$. Proposition 4.13(6) gives the result for $p = 1$. Take $p \in (1, \infty) \setminus \{2\}$ and recall that

$$\|\mathbb{1}_0\|_{F^p(G)} = \|\lambda_p(\mathbb{1}_0)\| = \sup\left\{\frac{\|\mathbb{1}_0 b\|_p}{\|b\|_p} : b \in \ell^p(G), b \neq 0\right\}.$$

Now by Lemma 4.14 there is $a_0 \in \ell^1_0(G)$ and $b_{\varepsilon_p} \in \ell^p(G) \setminus \ell^1_0(G)$ such that $\mathbb{1}_0 b_{\varepsilon_p} = a_0$ and $\|a_0\|_p > \|b_{\varepsilon_p}\|_p$ for any $p \in (1,\infty)\setminus\{2\}$. Therefore,

$$\|\mathbb{1}_0\|_{F^p(G)} \geq \frac{\|\mathbb{1}_0 b_{\varepsilon_p}\|_p}{\|b_{\varepsilon_p}\|_p} = \frac{\|a_0\|_p}{\|b_{\varepsilon_p}\|_p} > 1,$$

finishing the proof. □

Remark 4.16. Notice that the argument given in the proof of Proposition 4.15 does not work when $p = 2$. Indeed when $p = 2$, with notation as in the proof of Lemma 4.14, we get $\varepsilon_2 = 0$ and $f_2(0) = 0$. In fact, f_2 is a parabola with vertex at the origin. The reason for this is that, when $p = 2$, the C*-equation gives that $\|\mathbb{1}_0\|_{F^2(G)} = 1$. Furthermore, it follows from Corollary 3.20 of [10] that

$$\|\mathbb{1}_0\|_{F^{p_2}(G)} \leq \|\mathbb{1}_0\|_{F^{p_1}(G)}$$

when either $1 \leq p_1 < p_2 \leq 2$ or $2 \leq p_2 < p_1 < \infty$. Proposition 4.15 seems to suggest that $\|\mathbb{1}_0\|_{F^p(G)}$ strictly decreases to 1 as p goes from 1 to 2 and that $\|\mathbb{1}_0\|_{F^p(G)}$ is strictly increasing as p goes from 2 to infinity. However, we have not investigated these potential strict inequalities for all p any further.

An immediate consequence of Proposition 4.15, whose proof is identical to that of part (7) in Proposition 4.13, is the following.

Corollary 4.17. *Let $p \in [1,\infty)\setminus\{2\}$ and let G be a finite group with* $\mathrm{card}(G) > 2$. *Then $F^p_0(G)$ has no cai and cannot be nondegenerately represented on any Banach space.*

Notice that the work done in Section 3 already gives us a clear description of what $M(F^p_0(G))$ is when G is finite and abelian:

Proposition 4.18. *Let G be a finite abelian group and let $p \in [1,\infty)$. Then, $M(F^p_0(G))$ is isometrically isomorphic to $(\ell^1_0(G), |||-|||_p)$ where*

$$|||a|||_p = \|L_a\|_{F^p_0(G)\to F^p_0(G)} = \sup\{\|ab\|_{F^p(G)} : b \in \ell^1_0(G), \|b\|_{F^p(G)} = 1\}.$$

Proof. Since G is abelian, $\ell^1_0(G)$ is commutative and, since G is finite, $\ell^1_0(G)$ has a multiplicative identity and therefore the only absolute zero divisor in $\ell^1_0(G)$ is 0. Thus, Remark 3.8 together with Proposition 4.13(1) give that $(L, R) \mapsto L$ is an isometric isomorphism from $M(F^p_0(G))$ to $\{L \in \mathcal{L}(F^p_0(G)) : aL(b) = L(a)b\}$. Proposition 3.7 now shows that $M(F^p_0(G))$ is indeed isometrically isomorphic to $(\ell^1_0(G), |||-|||_p)$ via $L_a \mapsto a$. □

The multiplier algebra of $F^p_0(G)$ will be our main object of study for the rest of the section. We introduce a compact notation that will be used henceforward:

$$M^p_0(G) = M(F^p_0(G)).$$

By Proposition 4.18, when G is finite abelian, then $M_0^p(G)$ is simply $\ell_0^1(G)$ equipped with the norm $|||-|||_p$. When $p=1$, the norm in $M_0^1(G) = M(F_0^1(G)) = M(\ell_0^1(G))$ has a nicer description:

$$|||a|||_1 = \sup\{\|ab\|_1 : b \in \ell_0^1(G),\ \|b\|_1 = 1\}.$$

This norm is equivalent to the 1-norm in $\ell_0^1(G)$ via the following straightforward inequalities:

$$\frac{\|a\|_1}{\|\mathbb{1}_0\|_1} \leq |||a|||_1 \leq \|a\|_1 \tag{4-7}$$

where we clearly have equality if and only if $\mathrm{card}(G) = 2$. In fact, these inequalities might be strict when $\mathrm{card}(G) \neq 2$:

Example 4.19. Let $G = \mathbb{Z}/3\mathbb{Z} = \{[0], [1], [2]\}$. Here, $\|\mathbb{1}_0\|_1 = \frac{4}{3}$ and it is easy to check that $|||\Delta_{[j]}|||_1 = 2 = \|\Delta_{[j]}\|_1$ for $j \in \{1, 2\}$. However, $|||\Delta_{[1]} + \Delta_{[2]}|||_1 = 3 \neq 4 = \|\Delta_{[1]} + \Delta_{[2]}\|_1$.

For general $p \in [1, \infty)$ we get, using (4-4), the following bounds for the norm in $M_0^p(G)$:

$$\frac{\|a\|_p}{\|\mathbb{1}_0\|_1} \leq \frac{\|a\|_{F^p(G)}}{\|\mathbb{1}_0\|_{F^p(G)}} \leq |||a|||_p \leq \|a\|_{F^p(G)} \leq \|a\|_1 \tag{4-8}$$

We end this section by explicitly showing, via a bicontractive idempotent argument, that for certain values of $p \in [1, \infty) \setminus \{2\}$, $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is not an L^q-operator algebra for any $q \in [1, \infty)$. Let $G = \mathbb{Z}/3\mathbb{Z} = \{[0], [1], [2]\}$ and recall that $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ is the span of elements $\Delta_{[1]}$ and $\Delta_{[2]}$. Furthermore, the multiplication in $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ obeys the following table:

	$\Delta_{[1]}$	$\Delta_{[2]}$
$\Delta_{[1]}$	$-2\Delta_{[1]}+\Delta_{[2]}$	$-\Delta_{[1]}-\Delta_{[2]}$
$\Delta_{[2]}$	$-\Delta_{[1]}-\Delta_{[2]}$	$-\Delta_{[1]}+2\Delta_{[2]}$

Recall that the element $\mathbb{1}_0 = -\frac{1}{3}\Delta_{[1]} - \frac{1}{3}\Delta_{[2]}$ is the identity in $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$. For any $a \in \ell_0^1(\mathbb{Z}/3\mathbb{Z})$, there are $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $a = \alpha_1\Delta_{[1]} + \alpha_2\Delta_{[2]}$. Therefore, for any $p \in [1, \infty)$ we have

$$\|a\|_p = \|\alpha_1\Delta_{[1]} + \alpha_2\Delta_{[2]}\|_p = (|-\alpha_1 - \alpha_2|^p + |\alpha_1|^p + |\alpha_2|^p)^{1/p}.$$

In particular, $\|\mathbb{1}_0\|_1 = \frac{4}{3}$.

Proposition 4.18 shows that $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is isometrically isomorphic to $\ell_0^1(\mathbb{Z}/3\mathbb{Z})$ equipped with the norm $|||-|||_p$. For $a = \alpha_1\Delta_{[1]} + \alpha_2\Delta_{[2]} \in M_0^p(\mathbb{Z}/3\mathbb{Z})$, the inequalities in (4-8) become

$$|||a|||_p \leq \|a\|_1 = |-\alpha_1 - \alpha_2| + |\alpha_1| + |\alpha_2| \tag{4-9}$$

and

$$|||a|||_p \geq \frac{\|a\|_p}{\|\mathbb{1}_0\|_1} = \frac{3}{4}(|-\alpha_1-\alpha_2|^p + |\alpha_1|^p + |\alpha_2|^p)^{1/p}. \tag{4-10}$$

We note that the norm on $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is an operator norm induced from a proper ideal of an L^1-space and is generally difficult to compute. Fortunately, there do exist certain elements a for which we can compute $|||a|||_p$ easily (see Example 4.19). In particular, we can compute it for the idempotents of $M_0^p(\mathbb{Z}/3\mathbb{Z})$.

Proposition 4.20. *Let $p \in [1,\infty)$ and let $\gamma = \frac{1}{3}\exp(2\pi i/3) \in \mathbb{C}$. We define $e = \gamma\Delta_{[1]} + \overline{\gamma}\Delta_{[2]}$. Then*:

(1) $|\gamma| = |\overline{\gamma}| = |\gamma+\overline{\gamma}| = \frac{1}{3}$, $|\gamma-\overline{\gamma}| = \frac{\sqrt{3}}{3}$.

(2) $e^2 = e$ *and* $\|e\|_1 = 1$.

(3) $\mathbb{1}_0 - e = \overline{\gamma}\Delta_{[1]} + \gamma\Delta_{[2]}$ *and* $\|\mathbb{1}_0 - e\|_1 = 1$.

(4) $|||e|||_p = 1$.

(5) $|||\mathbb{1}_0 - e|||_p = 1$.

(6) $|||\mathbb{1}_0 - 2e|||_1 = \frac{2\sqrt{3}}{3}$.

(7) $|||\mathbb{1}_0 - 2e|||_2 = 1$.

(8) $|||\mathbb{1}_0 - 2e|||_p \leq \|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})} \leq \left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}$ *if* $p \in [1,2]$.

(9) $|||\mathbb{1}_0 - 2e|||_p \leq \|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})} \leq \left(\frac{2\sqrt{3}}{3}\right)^{1-\frac{2}{p}}$ *if* $p \in [2,\infty)$.

Proof. Each of the parts (1), (2), and (3) are checked by direct computations. For part (4), observe that the bounds (4-9) and (4-10) give

$$0 < |||e|||_p \leq \|e\|_1 = 1.$$

Since $e^2 = e$, submultiplicativity gives

$$|||e|||_p^2 \geq |||e^2|||_p = |||e|||_p.$$

Dividing both sides by $|||e|||_p$, we obtain that $|||e|||_p \geq 1$. The proof of part (5) is analogous to the part (4). For part (6), first notice that

$$\|\mathbb{1}_0 - 2e\|_1 = \|(\overline{\gamma}-\gamma)\Delta_{[1]} + (\gamma-\overline{\gamma})\Delta_{[2]}\|_1 = |0| + 2|\overline{\gamma}-\gamma| = \frac{2\sqrt{3}}{3}.$$

Then, the upper bound from (4-9) gives us that

$$|||\mathbb{1}_0 - 2e|||_1 \leq \|\mathbb{1}_0 - 2e\|_1 = \frac{2\sqrt{3}}{3}.$$

On the other hand, since $\left\|\frac{1}{2}\Delta_{[1]}\right\|_1 = 1$, we compute

$$\begin{aligned}\left\|(\mathbb{1}_0-2e)\left(\tfrac{1}{2}\Delta_{[1]}\right)\right\|_1 &= \|((\overline{\gamma}-\gamma)\Delta_{[1]}+(\gamma-\overline{\gamma})\Delta_{[2]})(\tfrac{1}{2}\Delta_{[1]})\|_1\\ &= \tfrac{1}{2}\cdot|\overline{\gamma}-\gamma|\cdot\|(\Delta_{[1]}-\Delta_{[2]})\Delta_{[1]}\|_1\\ &= \tfrac{1}{2}\cdot\tfrac{\sqrt{3}}{3}\cdot\|-\Delta_{[1]}+2\Delta_{[2]}\|_1=\tfrac{2\sqrt{3}}{3}.\end{aligned}$$

Therefore, $|||\mathbb{1}_0-2e|||_1=\frac{2\sqrt{3}}{3}$. For part (7), first notice that

$$\mathbb{1}_0-2e=(\overline{\gamma}-\gamma)\Delta_{[1]}+(\gamma-\overline{\gamma})\Delta_{[2]}=(\overline{\gamma}-\gamma)(-\Delta_{[1]}+\Delta_{[2]})=\frac{-i\sqrt{3}}{3}(-\Delta_{[1]}+\Delta_{[2]}).$$

Then, by computing $(\mathbb{1}_0-2e)\delta_g$ for each $g\in\mathbb{Z}/3\mathbb{Z}$, we find that the standard matrix of $\mathbb{1}_0-2e$ is

$$\frac{-i\sqrt{3}}{3}\begin{pmatrix}0&1&-1\\-1&0&1\\1&-1&0\end{pmatrix}. \tag{4-11}$$

Since the $\ell^2_d\to\ell^2_d$ operator norm is the maximum singular value of the matrix, (4-11) shows that $\|\mathbb{1}_0-2e\|_{F^2(\mathbb{Z}/3\mathbb{Z})}=1$. Combining this with the inner inequalities in (4-8) and the fact that $\|\mathbb{1}_0\|_{F^2(\mathbb{Z}/3\mathbb{Z})}=1$ (see Remark 4.16) we conclude that $|||\mathbb{1}_0-2e|||_2=1$, as wanted. Finally, for parts (8) and (9), the result is clear for $p=1$ and $p=2$. Next assume that $p\in(1,2)$ with $\frac{1}{p}=(1-\theta)+\frac{\theta}{2}$ for $\theta\in(0,1)$. Then, the Riesz–Thorin interpolation theorem shows that

$$\|\mathbb{1}_0-2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}\le\|\mathbb{1}_0-2e\|^{1-\theta}_{F^1(\mathbb{Z}/3\mathbb{Z})}\|\mathbb{1}_0-2e\|^{\theta}_{F^2(\mathbb{Z}/3\mathbb{Z})}=\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}.$$

Now, if $p'=\frac{p}{1-p}\in(2,\infty)$, Proposition 4.11 in [12] shows that the $F^{p'}(\mathbb{Z}/3\mathbb{Z})$ and $F^p(\mathbb{Z}/3\mathbb{Z})$ norms of $\mathbb{1}_0-2e$ agree and therefore

$$\|\mathbb{1}_0-2e\|_{F^{p'}(\mathbb{Z}/3\mathbb{Z})}\le\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}=\left(\frac{2\sqrt{3}}{3}\right)^{1-\frac{2}{p'}}.$$

The desired results now follow from the right inner bound in (4-8). □

The previous proposition gives all the necessary tools to show that for a certain $p_0\in(1,2)$, when $p\in[1,p_0)\cup(p_0',\infty)$, then the element $\mathbb{1}_0-2e\in M_0^p(\mathbb{Z}/3\mathbb{Z})$ is not an invertible isometry, which is essentially the proof of the following main result.

Theorem 4.21. *Let $p_0=1.606$ and let $p_0'=\frac{p_0}{p_0-1}$. Take $p\in[1,p_0]\cup[p_0',\infty)$. Then $M_0^p(\mathbb{Z}/3\mathbb{Z})$, the multiplier algebra of $F_0^p(\mathbb{Z}/3\mathbb{Z})$, is not an L^q-operator algebra for any $q\in[1,\infty)$.*

Proof. Suppose on the contrary that $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is an L^q-operator algebra for some $q\in[1,\infty)$. Then, since $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is a multiplier algebra it is unital (see Definition 3.1), whence Proposition 3.7 in [9] guarantees the existence of

an isometric unital representation $\pi : M_0^p(\mathbb{Z}/3\mathbb{Z}) \to \mathcal{L}(L^q(\mu))$ for some measure space (Ω, μ).

Take $e = \gamma\Delta_{[1]} + \overline{\gamma}\Delta_{[2]}$ as in Proposition 4.20. Parts (2),(4), and (5) in Proposition 4.20 show that e is a bicontractive element in $M_0^p(\mathbb{Z}/3\mathbb{Z})$. We claim the following:

Claim 1. $|||\mathbb{1}_0 - 2e|||_p > 1$ when $p \in [1, p_0]$ or $p \in [p_0', \infty)$.

Once the claim above is proven, it will follow that $\mathbb{1}_0 - 2e$ is not an invertible isometry in $M_0^p(\mathbb{Z}/3\mathbb{Z})$. Thus, since π is a unital isometry, we would have found a bicontractive idempotent $\pi(e)$ in $\mathcal{L}(L^q(\mu))$ such that $1 - 2\pi(e)$ is not an invertible isometry when $p \in [1, p_0] \cup [p_0', \infty)$, contradicting Theorem 2.9.

Therefore it only remains to establish Claim 1, which already follows when $p = 1$ by Part (6) in Proposition 4.20. Otherwise, observe that when $p \in (1, \infty) \setminus \{2\}$ we have

$$|||\mathbb{1}_0 - 2e|||_p \geq \frac{\|(\mathbb{1}_0 - 2e)(\mathbb{1}_0 - 2e)\|_{F^p(\mathbb{Z}/3\mathbb{Z})}}{\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}} = \frac{\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})}}{\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}} \tag{4-12}$$

Next, let $a_0 \in \ell_0^1(\mathbb{Z}/3\mathbb{Z})$ and $b_{\varepsilon_p} \in \ell^p(\mathbb{Z}/3\mathbb{Z})$ be as in part (4) of Lemma 4.14. That is, for any $p \in (1, \infty) \setminus \{2\}$ we have

$$\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})} \geq \frac{\|a_0\|_p}{\|b_{\varepsilon_p}\|_p} = \frac{(2^p + 2)(2^{\frac{1}{p-1}} + 1)^p}{3^p(2^{\frac{p}{p-1}} + 2)} > 1. \tag{4-13}$$

Now take any $p \in (1, 2)$ and set $p' = \frac{p}{p-1} \in (2, \infty)$. Proposition 4.11 in [12] gives that the norms of $\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$ and $\|\mathbb{1}_0\|_{F^{p'}(\mathbb{Z}/3\mathbb{Z})}$ agree, so together with Part (8) in Proposition 4.20, combining the bounds from (4-12) and (4-13), we now get

$$|||\mathbb{1}_0 - 2e|||_p \geq \frac{\|\mathbb{1}_0\|_{F^{p'}(\mathbb{Z}/3\mathbb{Z})}}{(\frac{2\sqrt{3}}{3})^{\frac{2}{p}-1}} \geq \frac{\frac{\|a_0\|_{p'}}{\|b_{\varepsilon_{p'}}\|_{p'}}}{\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}} = \frac{(2^{\frac{p}{p-1}} + 2)(2^{p-1} + 1)^{\frac{p}{p-1}}}{3^{\frac{p}{p-1}}(2^p + 2)\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}}.$$

Next we define $h : (1, 2) \to \mathbb{R}$ using the right hand side in the inequality above, that is

$$h(p) = \frac{(2^{\frac{p}{p-1}} + 2)(2^{p-1} + 1)^{\frac{p}{p-1}}}{3^{\frac{p}{p-1}}(2^p + 2)\left(\frac{2\sqrt{3}}{3}\right)^{\frac{2}{p}-1}} = \frac{(2^{\frac{p}{p-1}} + 2)\left(2^{p-1} + 1\right)^{\frac{1}{p-1}}}{3^{\frac{1}{2}-\frac{1}{p}+\frac{p}{p-1}} \cdot 2^{\frac{2}{p}}}.$$

It can be checked that $p \mapsto h(p)$ is strictly decreasing on $(1, p_0]$, we omit the lengthy calculations. A direct computation now shows that $h(p_0) > 1.000098$, so it follows that $|||\mathbb{1}_0 - 2e|||_p > 1$ for any $p \in (1, p_0]$. Similarly, when $p \in (2, \infty)$ we

get using Part (9) in Proposition 4.20

$$|||\mathbb{1}_0 - 2e|||_p \geq \frac{(2^p+2)(2^{\frac{1}{p-1}}+1)^p}{3^p(2^{\frac{p}{p-1}}+2)\left(\frac{2\sqrt{3}}{3}\right)^{1-\frac{2}{p}}} = h\Big(\frac{p}{p-1}\Big).$$

It follows from the $p \in (1, p_0]$ case that $|||\mathbb{1}_0 - 2e|||_p > 1$ for any $p \in [p_0', \infty)$, proving Claim 1, and therefore finishing the proof. □

Remark 4.22. It is likely that, for all $p \in [1, \infty) \setminus \{2\}$, the algebra $M_0^p(\mathbb{Z}/3\mathbb{Z})$ is not an L^q-operator algebra for any $q \in [1, \infty)$. A proof of this fact could potentially be obtained by the same method employed in the proof of Theorem 4.21 above provided that we have either better bounds or exact values for the norms of $\|\mathbb{1}_0\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$ and $\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$, which are generally hard to compute (see [16]). For instance, we believe that the actual value for $\|\mathbb{1}_0 - 2e\|_{F^p(\mathbb{Z}/3\mathbb{Z})}$ when $p \in [1, 2]$ is $\frac{\sqrt{3}}{3}(1+2^{p-1})^{\frac{1}{p}}$ which, if true, improves the value of p_0 to $p_0 = 1.675$.

Remark 4.23. Unfortunately, the bicontractive idempotent argument does not work when $G \in \{\mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\}$. Indeed, in this case if $e \in M_0(G)$ is any bicontractive idempotent, then it can be shown that $\mathbb{1}_0 - 2e \in M_0(G)$ is in fact an invertible isometry as expected for an L^p-operator algebra. Due to the difficulty of computing the norm of an element in $M_0^p(G)$, a different argument is likely needed to argue a version of Theorem 4.21 for higher order groups. At this time, for a general finite group G with $\mathrm{card}(G) > 3$ and $p \neq 2$, we do not know whether $M_0^p(G)$ can be isometrically represented on an L^q-space.

5. Idempotents and Gelfand transform for $\ell_0^1(G)$

In what follows we assume that G is a finite abelian group. In Proposition 4.18, for each $p \in [1, \infty)$, we obtained a unital Banach algebra $M_0^p(G) = M(F_0^p(G))$ whose underlying space is the augmentation ideal $\ell_0^1(G)$ normed with $|||-|||_p$. In addition, $M_0^p(G)$ is both a unital and a commutative Banach algebra when G is abelian. When $\mathrm{card}(G) > 2$ and $p \neq 2$, notice that $M_0^p(G)$ equipped with the inherited involution from $\ell^1(G)$ cannot be a C*-algebra. In fact, in such cases $M_0^p(G)$ is not isomorphically isometric to any closed subalgebra of $C(\Omega)$ where $\Omega = \mathrm{Max}(M_0^p(G))$ because $|||\Delta_g|||_p^2 \neq |||\Delta_g \Delta_g^*|||_p$ for all $g \in G \setminus \{1_G\}$, so the Gelfand map is not isometric. We will see, however, that the Gelfand map is an algebraic isomorphism.

Lemma 5.1. *For $n \in \mathbb{Z}_{>1}$, let*

$$G = \mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$$

and $(x_{[1]}, \ldots, x_{[n-1]}) \in \mathbb{C}^{n-1}$ with $x_{[0]} = 0$. Consider the system of equations

$$x_{[j]+[k]} = x_{[j]}x_{[k]} + x_{[j]} + x_{[k]}, \quad 1 \leq j, k \leq n-1. \tag{5-1}$$

Then the only nontrivial solutions for (5-1) are of the form

$$x_{[k]} = (x_{[1]}+1)^k - 1 \text{ for all } 1 \le k \le n-1 \text{ where } (x_{[1]}+1)^n = 1. \tag{5-2}$$

In particular, this means that (5-1) has exactly $n-1$ nontrivial solutions.

Proof. Given a solution for (5-1), we show that $x_{[k]} = (x_{[1]}+1)^k - 1$ for all $1 \le k \le n-1$. We do so by induction on k. When $k = 1$,

$$x_{[1]} = (x_{[1]}+1)^1 - 1 = x_{[1]}.$$

When $k = 2$,

$$x_{[2]} = x_{[1]+[1]} = x_{[1]}^2 + 2x_{[1]} = (x_{[1]}+1)^2 - 1$$

Suppose $x_{[j]} = (x_{[1]}+1)^j - 1$. Then

$$\begin{aligned} x_{[j]+[1]} &= x_{[j]}x_{[1]} + x_{[j]} + x_{[1]} \\ &= \left((x_{[1]}+1)^j - 1\right)x_{[1]} + \left((x_{[1]}+1)^j - 1\right) + x_{[1]} \\ &= x_{[1]}(x_{[1]}+1)^j - x_{[1]} + x_{[1]} + (x_{[1]}+1)^j - 1 \\ &= (x_{[1]}+1)(x_{[1]}+1)^j - 1 \\ &= (x_{[1]}+1)^{j+1} - 1 \end{aligned}$$

Now since $x_{[0]} = 0$, it follows that

$$0 = x_{[0]} = x_{[n-1]+[1]} = x_{[n-1]}x_{[1]} + x_{[n-1]} + x_{[1]} = (x_{[1]}+1)^n - 1,$$

so indeed any solution of the system (5-1) is of the form (5-2).

Next, we show that $x_{[k]} = (x_{[1]}+1)^k - 1$ for all $1 \le k \le n-1$ and $(x_{[1]}+1)^n = 1$ satisfies (5-1). Let $1 \le j, k \le n-1$. Then,

$$\begin{aligned} x_{[j]}x_{[k]} &= \left((x_{[1]}+1)^j - 1\right)\left((x_{[1]}+1)^k - 1\right) \\ &= (x_{[1]}+1)^{j+k} - (x_{[1]}+1)^k - (x_{[1]}+1)^j + 1, \end{aligned}$$

and since $x_{[j]} = (x_{[1]}+1)^j - 1$ for any $1 \le j \le n-1$, we conclude

$$x_{[j]}x_{[k]} + x_{[j]} + x_{[k]} = (x_{[1]}+1)^{j+k} - 1 = x_{[j+k]} = x_{[j]+[k]}.$$

Clearly, $(x_{[1]}+1)^n = 1$ satisfies $(x_{[1]}+1)^n - 1 = 0 = x_{[0]}$, so we are done. □

Maschke's and Wedderburn's theorems (see Chapter 16 of [8] for an elementary introduction) imply that the only nilpotent element of $C_c(G)$ for G finite abelian is $0 \in C_c(G)$. In particular, this means that $C_c(G)$ and all of its (two-sided) ideals have trivial nilradical. This leads to the following important observation.

Lemma 5.2. *Let I be any (two-sided) ideal in $\ell^1(G)$ for G finite abelian. Then the Gelfand transform $\Gamma_I : I \to C(\mathrm{Max}\,(I))$ is an algebraic isomorphism.*

We now begin with a technical lemma.

Lemma 5.3. *Let G be any discrete group and define for $m \in \mathbb{Z}_{>1}$*

$$\mathcal{B}^m = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \ : \ \varepsilon_j \in \{0, 1\} \text{ for all } 1 \le j \le m\} \setminus \{(0, 0, \ldots, 0)\}.$$

Then for any collection of m elements $g_1, g_2, \ldots, g_m \in G \setminus \{1_G\}$,

$$\Delta_{g_1 g_2 \cdots g_m} = \sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) \in \mathcal{B}^m} \Delta_{g_1}^{\varepsilon_1} \Delta_{g_2}^{\varepsilon_2} \cdots \Delta_{g_m}^{\varepsilon_m} \tag{5-3}$$

where we define $\Delta_{g_j}^0 = 1$ for all $1 \le j \le m$. In particular, this means

$$\Delta_{gg'} = \Delta_g \Delta_{g'} + \Delta_g + \Delta_{g'} \text{ for all } g, g' \in G \setminus \{1_G\}. \tag{5-4}$$

Proof. Fix $g, g' \in G \setminus \{1_G\}$. We argue by induction. A direct computation gives

$$\Delta_{gg'} = \Delta_g \Delta_{g'} + \Delta_{g'} + \Delta_g$$

Next, assume equation (5-3) holds for the product of $m-1$ nonidentity elements of G. Choose $g_1, g_2, \ldots, g_{m-1}, g_m \in G \setminus \{1_G\}$. Then, using the induction hypothesis at the last step,

$$\begin{aligned} \Delta_{g_1 g_2 \cdots g_{m-1} g_m} &= \Delta_{(g_1 g_2 \cdots g_{m-1}) g_m} = \Delta_{g_1 g_2 \cdots g_{m-1}} \Delta_{g_m} + \Delta_{g_1 g_2 \cdots g_{m-1}} + \Delta_{g_m} \\ &= \sum_{(\varepsilon_j) \in \mathcal{B}^m} \Delta_{g_1}^{\varepsilon_1} \Delta_{g_2}^{\varepsilon_2} \cdots \Delta_{g_{m-1}}^{\varepsilon_{m-1}} \Delta_{g_m}^{\varepsilon_m}. \end{aligned}$$

□

We are now ready to prove the cyclic group case.

Proposition 5.4. *Let $G = \mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{Z}_{>1}$. Then $\ell_0^1(G)$ is algebraically isomorphic to $\mathbb{C}^{n-1}$ equipped with pointwise multiplication.*

Proof. We have already established that $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ is an ideal in $\ell^1(\mathbb{Z}/n\mathbb{Z})$ and so we immediately have an algebraic isomorphism between $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ and continuous functions on its maximal ideal space. Thus, it suffices to establish that the maximal ideal space of $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ consists of $n-1$ elements.

Let $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\}$. Thanks to (5-4), any multiplicative linear functional $\omega : \ell_0^1(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ must satisfy

$$\omega(\Delta_{[j]+[k]}) = \omega(\Delta_{[j]})\omega(\Delta_{[k]}) + \omega(\Delta_{[j]}) + \omega(\Delta_{[k]}) \tag{5-5}$$

for all $1 \le j, k \le n-1$. Letting $\omega(\Delta_{[j]}) = x_{[j]}$, we recover the system described in Lemma 5.1. We observe that each solution of the system (5-1) is completely determined by the choice of $x_{[1]}$, which must be chosen to satisfy $(x_{[1]} + 1)^n = 1$. For each $1 \le j \le n-1$, let

$$\omega_j(\Delta_{[1]}) = \exp(2\pi i \cdot j/n) - 1 = x_{j,[1]}. \tag{5-6}$$

As discussed, this choice determines the remaining $\omega_j(\Delta_{[k]}) = x_{j,[k]}$ and so we have a one-to-one correspondence between the multiplicative linear functionals of

the ideal $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$ and the solutions of System (5-1). Of course, $\mathrm{Max}(\ell^1_0(\mathbb{Z}/n\mathbb{Z}))$ is in bijection with $\Omega = \{\omega_j : 1 \le j \le n-1\}$. Lemma 5.2 gives that

$$\Gamma : \ell^1_0(\mathbb{Z}/n\mathbb{Z}) \to C(\Omega)$$

is an algebraic isomorphism. Finally since the canonical map $C(\Omega) \to \mathbb{C}^{n-1}$ given by

$$f \mapsto (f(\omega_1), f(\omega_2), \ldots, f(\omega_{n-1}))$$

is also an algebraic isomorphism, we have indeed shown that $\ell^1_0(\mathbb{Z}/n\mathbb{Z})$ and $\mathbb{C}^{n-1}$ are algebraically isomorphic, completing the proof. □

We now extend Proposition 5.4 to any finite abelian group. Consider

$$G = (\mathbb{Z}/n_1\mathbb{Z}) \times (\mathbb{Z}/n_2\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_m\mathbb{Z}),$$

where $n_1, \ldots, n_m \in \mathbb{Z}_{>1}$. Then any $a \in \ell^1_0(G)$ may be written as

$$a = \sum_{(g_1, g_2, \ldots, g_m) \in G} a(g_1, g_2, \ldots, g_m)\Delta_{(g_1, g_2, \ldots, g_m)}.$$

We note that $(g_1, g_2, \ldots, g_m)(g_1', g_2', \ldots, g_m') = (g_1 + g_1', g_2 + g_2', \ldots, g_m + g_m')$ and we will abuse notation by writing g for $(0, \ldots, g, \ldots 0)$ and $g_1 g_2 \cdots g_m$ for $(g_1, g_2, \ldots, g_m)$.

For each cyclic group $\mathbb{Z}/n_j\mathbb{Z}$, let $\{\omega_j^{(k)}\}_{0 \le k \le n_j - 1}$ be the multiplicative linear functionals associated with $\mathrm{Max}\,(\ell^1_0(\mathbb{Z}/n_j\mathbb{Z}))$ where $\omega_j^{(0)}$ represents the zero function on $\ell^1_0(\mathbb{Z}/n_j\mathbb{Z})$. We show that all multiplicative linear functionals of $\ell^1_0(G)$ are of the form

$$\omega(\Delta_{g_1 g_2 \cdots g_m}) = \sum_{(\varepsilon_j) \in \mathcal{B}^m} \omega_1^{(k_1)}(\Delta_{g_1}^{\varepsilon_1})\omega_2^{(k_2)}(\Delta_{g_2}^{\varepsilon_2}) \cdots \omega_m^{(k_m)}(\Delta_{g_m}^{\varepsilon_m}) \tag{5-7}$$

for all $0 \le k_i \le n_i$, $1 \le i \le m$, where we interpret $\omega_{n_j}^{(k_j)}(\Delta_{g_k}^0)$ as equaling 1 when in a product with other nonzero terms.

Suppose $\omega : \ell^1_0(G) \to \mathbb{C}$ is a multiplicative linear functional. When ω is restricted to linear sums of Δ_{g_j} for $g_j \in \mathbb{Z}/n_j\mathbb{Z}$, which we will denote by ω_j, then ω_j is a multiplicative linear functional of $\ell^1_0(\mathbb{Z}/n_j\mathbb{Z})$. Thus, there must exist $k_j \in \{1, 2, \ldots, n_j - 1\}$ such that $\omega_j = \omega_j^{(k_j)}$. Equation (5-7) then follows by previous comments and Lemma 5.3. Note that ω is the zero map exactly when all the $\omega_j^{(k_j)}$ are zero maps. Thus, we see that we have $n_1 n_2 \cdots n_m = \mathrm{card}(G)$ possible choices for ω, including the zero map.

This leads to the following theorem and corollary.

Theorem 5.5. *Let G be a finite abelian group with* $\mathrm{card}(G) > 1$. *Then the Gelfand transform from $\ell^1_0(G)$ to $C(\{1, 2, \ldots, n-1\})$ is an algebraic isomorphism.*

Corollary 5.6. *Let G be a finite abelian group with* $\operatorname{card}(G) > 1$. *Then the algebras $\ell_0^1(G)$, $\ell^1(\mathbb{Z}/(n-1)\mathbb{Z})$, $C(\{1, 2, \ldots, n-1\})$, and $\mathbb{C}^{n-1}$ are all algebraically isomorphic.*

Recall that we used a bicontractive idempotent as the primary tool to show that for certain values of $p \in [1, \infty) \setminus \{2\}$, the algebra $M_0^p(\mathbb{Z}/3\mathbb{Z})$ cannot be isometrically represented on any L^q-space. To this end, we present a description of the idempotents of $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ and, in consequence, those of $M_0^p(\mathbb{Z}/n\mathbb{Z})$ since the algebraic structures are the same. We start with a short definition.

Definition 5.7. Let $a \in \ell_0^1(\mathbb{Z}/n\mathbb{Z})$ for $n \in \mathbb{Z}_{>1}$. Then, since a is uniquely determined by $[a] = (a([1]), \ldots, a([n-1])) \in \mathbb{C}^{n-1}$, we define a canonical map $\ell_0^1(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}^{n-1}$ via $a \mapsto [a] \in \mathbb{C}^{n-1}$.

Let $n \in \mathbb{Z}_{>1}$. We define the *translation matrix of* $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ by

$$X = \begin{pmatrix} x_{1,[1]} & x_{1,[2]} & \cdots & x_{1,[n-1]} \\ x_{2,[1]} & x_{2,[2]} & \cdots & x_{2,[n-1]} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1,[1]} & x_{n-1,[2]} & \cdots & x_{n-1,[n-1]} \end{pmatrix}. \tag{5-8}$$

Proposition 5.4 implies that for any $v \in \mathbb{C}^{n-1}$ there exists a unique $a \in \ell_0^1(\mathbb{Z}/n\mathbb{Z})$ such that $X[a] = v$, where $[a]$ is as in Definition 5.7. In particular, if we let $\{e_k\}_{k=1}^{n-1}$ represent the standard basis of $\mathbb{C}^{n-1}$, then all nonzero idempotents of $\mathbb{C}^{n-1}$ are of the form

$$\varepsilon_1 e_1 + \varepsilon_2 e_2 + \cdots + \varepsilon_{n-1} e_{n-1} \quad \text{for} \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) \in \mathcal{B}^{n-1}$$

where $\mathcal{B}^{n-1}$ is as in Lemma 5.3. Then our discussion above yields the following Proposition.

Proposition 5.8. *Fix $n \in \mathbb{Z}_{>1}$ and let $a_k \in \ell_0^1(\mathbb{Z}/n\mathbb{Z})$ be the unique element such that $X[a_k] = e_k$. Then*:

(1) *Each a_k is an idempotent of norm at least one.*

(2) *$a_k a_j = 0$ for all $k \neq j$.*

(3) *$a_1 + a_2 + \cdots + a_{n-1} = \mathbb{1}_0$.*

(4) *$\{a_k\}_{k=1}^{n-1}$ forms a basis for $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$.*

(5) *Each nonzero idempotent of $\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ is of the form*

$$\varepsilon_1 a_1 + \varepsilon_2 a_2 + \cdots + \varepsilon_{n-1} a_{n-1} \quad \text{for} \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}) \in \mathcal{B}^{n-1}.$$

(6) *$\ell_0^1(\mathbb{Z}/n\mathbb{Z})$ has precisely 2^{n-1} idempotents.*

6. L^p-operator norms on $\mathbb{C}^n$

We have shown that the conclusion of Theorem 4.1 can go either way when dropping the two hypotheses. As shown in Section 4B, for each $p \in [1, \infty)$, the algebra of strictly upper triangular matrices acting on ℓ_2^p, T_2^p, is a degenerate L^p-operator algebra without a cai and yet its multiplier algebra is nondegenerately representable on an L^q-space for any $q \in [1, \infty)$. In contrast, as shown in Section 4C, $F_0^p(\mathbb{Z}/3\mathbb{Z})$ is a degenerate L^p-operator algebra without a cai, but when $p \in [1, p_0] \cup [p_0', \infty)$, its multiplier algebra cannot be isometrically represented on an L^q-space for any $q \in [1, \infty)$. At this time, it is unclear what conditions are necessary to guarantee that the multiplier algebra of a degenerate L^p-operator algebra without a cai is again an L^p-operator algebra.

Given that the norms of L^p-operator algebras are not unique, a natural question is to characterize norms on $\mathbb{C}^n$ that make it into an L^p-operator algebra. This problem is already interesting for $p = 1$, for which we know of at least two norms on $\mathbb{C}^n$ which make it an L^1-operator algebra:

(1) $\mathbb{C}^n$ with the supremum norm is an L^1-operator algebra acting on ℓ_n^1 via multiplication operators.

(2) Let $\mathcal{F} : \ell^1(\mathbb{Z}/n\mathbb{Z}) \to C(\mathbb{Z}/n\mathbb{Z})$ the Fourier transform. Then we have algebra isomorphisms

$$\mathbb{C}^n \cong C(\mathbb{Z}/n\mathbb{Z}) \cong \mathcal{F}^{-1}(C(\mathbb{Z}/n\mathbb{Z})) = \ell^1(\mathbb{Z}/n\mathbb{Z})$$

which make $\mathbb{C}^n$ an L^1-operator algebra with norm coming from the identification with $\ell^1(\mathbb{Z}/n\mathbb{Z})$.

Question 6.1. Are these the only two norms that make $\mathbb{C}^n$ an L^1-operator algebra?

Theorem 4.21 exhibits a norm in $\mathbb{C}^2$ for which $\mathbb{C}^2$ fails to be an L^q-operator algebra for any $q \in [1, \infty)$, which is to say the identification of $M_0^1(\mathbb{Z}/3\mathbb{Z})$ with $\mathbb{C}^2$ carries neither of the norms listed above. This leads to a more general question.

Question 6.2. Let $n \in \mathbb{Z}_{>1}$ and consider $\mathbb{C}^n$ with pointwise multiplication. Is it possible to find all the norms on $\mathbb{C}^n$ that make it an L^1-operator algebra? An L^p-operator algebra for some $p \in [1, \infty)$?

Acknowledgments

The authors thank David Blecher and Eusebio Gardella for their encouragement and discussions, particularly regarding Question 6.2 above. We are also thankful to Hannes Thiel for pointing out several references and for encouraging us to generalize our main results to $p > 1$. Delfín also thanks N. Christopher Phillips, who read earlier versions of Section 3 and gave suggestions to improve the presentation.

Finally, we would like to thank the anonymous referee for a careful and thorough reading of the initial draft and for sending useful references and suggestions that greatly improved some content in this paper. In particular, we thank the referee for pointing out that it made more sense to look at the augmentation ideal of $F^p(G)$ instead of $F_{\mathrm{r}}^p(G)$ as we had initially considered.

References

[1] S. J. Bernau and H. E. Lacey, "Bicontractive projections and reordering of L_p-spaces", *Pacific J. Math.* **69**:2 (1977), 291–302. MR Zbl

[2] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules: an operator space approach*, London Mathematical Society Monographs. New Series **30**, Oxford University Press, 2004. MR Zbl

[3] D. P. Blecher and N. C. Phillips, "L^p-operator algebras with approximate identities, I", *Pacific J. Math.* **303**:2 (2019), 401–457. MR Zbl

[4] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs (New Series) **24**, Oxford University Press, 2000. MR Zbl

[5] M. Daws, "p-operator spaces and Figà-Talamanca–Herz algebras", *J. Operator Theory* **63**:1 (2010), 47–83. MR Zbl

[6] M. Daws and N. Spronk, "On convoluters on L^p-spaces", *Studia Math.* **245**:1 (2019), 15–31. MR Zbl

[7] A. Delfin Ares de Parga, *C*-correspondences, Hilbert bimodules, and their L_p versions*, Ph.D. thesis, University of Oregon, 2023, available at https://www.proquest.com/docview/2842733535. MR Zbl

[8] D. S. Dummit and R. M. Foote, *Abstract algebra*, 3rd ed., Wiley, Hoboken, NJ, 2004. MR Zbl

[9] E. Gardella, "A modern look at algebras of operators on L^p-spaces", *Expo. Math.* **39**:3 (2021), 420–453. MR Zbl

[10] E. Gardella and H. Thiel, "Group algebras acting on L^p-spaces", *J. Fourier Anal. Appl.* **21**:6 (2015), 1310–1343. MR Zbl

[11] E. Gardella and H. Thiel, "Quotients of Banach algebras acting on L^p-spaces", *Adv. Math.* **296** (2016), 85–92. MR Zbl

[12] E. Gardella and H. Thiel, "Representations of p-convolution algebras on L^q-spaces", *Trans. Amer. Math. Soc.* **371**:3 (2019), 2207–2236. MR Zbl

[13] E. Gardella and H. Thiel, "Extending representations of Banach algebras to their biduals", *Math. Z.* **294**:3-4 (2020), 1341–1354. MR Zbl

[14] E. Gardella and H. Thiel, "Isomorphisms of algebras of convolution operators", *Ann. Sci. Éc. Norm. Supér.* (4) **55**:5 (2022), 1433–1471. MR Zbl

[15] S. Hejazian and S. Pooya, "Simple reduced L^p-operator crossed products with unique trace", *J. Operator Theory* **74**:1 (2015), 133–147. MR Zbl

[16] J. M. Hendrickx and A. Olshevsky, "Matrix p-norms are NP-hard to approximate if $p \neq 1, 2, \infty$", *SIAM J. Matrix Anal. Appl.* **31**:5 (2010), 2802–2812. MR Zbl

[17] C. Herz, "The theory of p-spaces with an application to convolution operators", *Trans. Amer. Math. Soc.* **154** (1971), 69–82. MR Zbl

[18] C. Herz, "Harmonic synthesis for subgroups", *Ann. Inst. Fourier* (*Grenoble*) **23**:3 (1973), 91–123. MR Zbl

[19] B. E. Johnson, "An introduction to the theory of centralizers", *Proc. London Math. Soc.* (3) **14** (1964), 299–320. MR Zbl

[20] D. L. Johnson, *Presentations of groups*, 2nd ed., London Mathematical Society Student Texts **15**, Cambridge University Press, 1997. MR Zbl

[21] N. C. Phillips, "Analogs of Cuntz algebras on L^p spaces", 2012. Zbl arXiv 1201.4196

[22] N. C. Phillips, "Crossed products of L^p operator algebras and the K-theory of Cuntz algebras on L^p spaces", 2013. Zbl arXiv 1309.6406

[23] N. C. Phillips, "Simplicity of reduced group Banach algebras", 2019. Zbl arXiv 1909.11278v1

[24] W. Rudin, *Fourier analysis on groups*, Tracts in pure and applied mathematics **12**, Interscience, New York, 1962. Reprinted Wiley, 1990. MR Zbl

[25] W. Rudin, "Idempotents in group algebras", *Bull. Amer. Math. Soc.* **69** (1963), 224–227. MR Zbl

[26] J.-k. Wang, "Multipliers of commutative Banach algebras", *Pacific J. Math.* **11** (1961), 1131–1149. MR Zbl

[27] N. E. Wegge-Olsen, *K-theory and C*-algebras*: *a friendly approach*, Oxford University Press, 1993. MR Zbl

Received March 25, 2024. Revised November 12, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.229

TWIST EQUIVALENCE AND NICHOLS ALGEBRAS OVER COXETER GROUPS

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Fomin–Kirillov algebras are quadratic approximations of Nichols algebras associated with the conjugacy class of transpositions in a symmetric group and a (rack) 2-cocycle q^+ with values in $\{\pm 1\}$. Bazlov generalized their construction replacing the class of transpositions by the classes of reflections in an arbitrary finite Coxeter group. We prove that Bazolv's cocycle q^+ is twist equivalent to the constant cocycle $q^- \equiv -1$, generalizing a result of Vendramin. As a consequence, the Nichols algebras associated with the two different cocycles have the same Hilbert series and one is quadratic if and only if the other is quadratic. We further apply recent results of Heckenberger, Meir and Vendramin and Andruskiewitsch, Heckenberger and Vendramin to complete the classification of the finite-dimensional Nichols algebras of Yetter–Drinfeld modules over the dihedral groups.

1. Introduction

Racks and rack cocycles appear in the study of knot theory, as they naturally provide a solution of the braid equation. It became apparent that they are key basic objects in the study of Nichols (shuffle) algebras, and the classification of pointed Hopf algebras [2], since Nichols algebras are graded braided Hopf algebras that depend only on the datum of a braided vector space. Several tools have been developed in order to detect finiteness properties of Nichols algebras (such as finite dimension, finite GK-dimension, finite presentation) that depend only on the rack and not on the cocycle, and properties that are preserved under suitable transformations of the cocycle; see, among others, [2; 5; 6; 7; 8; 9; 25; 31].

Relevant racks in the study of Nichols algebras can be described as subsets X of a group G that are stable by conjugation [2, Definition 1.3]. For such racks, *twisting* a cocycle by a 2-cocycle of G with trivial coefficients corresponds, at the level of Nichols algebras, to performing a (dual) Drinfeld twist [7; 34]. As a consequence,

MSC2020: primary 16T05, 16T30, 20F55; secondary 16P10.
Keywords: Coxeter groups, racks, Nichols algebras.

Nichols algebras associated with the same rack and twist-equivalent cocycles have the same Hilbert series, same dimension and same GK-dimension.

We first focus on finitely generated Coxeter groups and a special family of racks and 2-cocycles, that is of interest due to its relation with the cohomology of the flag variety. Here the rack is the class (or union of classes) of reflections in a Coxeter group W, and the 2-cocycle is a cocycle with values in $\{\pm 1\}$. If W is a Weyl group, the quadratic approximation of the corresponding Nichols algebra is a noncommutative algebra containing the cohomology algebra of the flag variety of the corresponding algebraic group. For $W = \mathbb{S}_n$ this is the so-called Fomin–Kirillov algebra introduced in [21] for creating an algebraic and combinatorial framework to perform Schubert calculus. The construction was later generalized to the case of arbitrary Weyl groups in [12]. For $W = \mathbb{S}_3$, $\mathbb{S}_4$ and $\mathbb{S}_5$ the Nichols algebra in question is quadratic and finite dimensional [22; 36]. For $n > 5$ a proof of infinite dimensionality of the Fomin–Kirillov algebras appeared in [11], but it is not known whether the Nichols algebras are quadratic. This property and dimension are preserved by twists.

A novel approach to the analysis of the dimension and of the degree of a minimal set of generating relations may come through geometry, using the category equivalence in [33]. Through this equivalence, Nichols algebras correspond to intersection cohomology (IC) complexes on the infinite-dimensional space $\mathrm{Sym}(\mathbb{C})$ of monic polynomials, with a so-called factorization datum. One may expect that the Nichols algebras associated with a conjugacy class C of a finite group G and the constant cocycle equal to -1 might be easier to handle because the restriction of the corresponding IC complexes to the dense open subset $\mathrm{Sym}_{\neq}(\mathbb{C})$ of multiplicity-free polynomials is the push-forward through the covering map of the constant sheaf on the Hurwitz space with Galois group G and local monodromy C [19, §2.4, 3.3; 33, §3.3 A]. These spaces are widely studied. For this reason, it becomes relevant to know whether the cocycle associated with the rack of reflections in Coxeter groups as in [12; 21; 36] are twist equivalent to the -1 constant cocycle.

For $W = \mathbb{S}_n$ this problem was answered in the affirmative in [39], and for the dihedral group an attempt was done in [17]. We provide a unified approach to give a positive answer that applies to all finitely generated Coxeter groups as long as the entries of its Coxeter matrix $A(W)$ are all finite. We also show that the cocycles are cohomologous if and only if the entries in $A(W)$ are all odd. Hence, for dihedral groups of regular polygons with an odd number of edges, we recover the well-known fact that the Nichols algebras corresponding to these two cocycles are isomorphic [36, §5].

We next combine the information we gained in order to reorganize and complement what is known about the dimension of Nichols algebras associated with

reflections in Coxeter groups, showing that the key case to be studied is type A_5. Finally, we address the case of Coxeter groups of rank 2, that is, the dihedral groups $I_2(n)$ of order $2n$. This is of particular interest because all finite groups generated by two involutions are dihedral. Finite-dimensional Nichols algebras over $I_2(n)$ for $4 \mid n$ were classified in [13; 20]. The case of $n = 3$ is to be found in [6] and $I_2(2)$ is not irreducible. Using the main results in [10; 31] and the work in [1] we address the remaining open case, i.e., when 4 does not divide n and $n \neq 2, 3$, completing the classification of finite-dimensional Nichols algebras over dihedral groups. In particular, we show that there is no finite-dimensional Nichols algebra over $I_2(n)$ when n is odd and different from 3. Hence, the only finite-dimensional complex pointed Hopf algebra with group of grouplikes isomorphic to $I_2(n)$ for such an n is the group algebra.

The paper is structured as follows: In Section 2 we introduce the basic notions on Coxeter groups, racks, cocycles and Nichols algebras; in Section 2.4 we define the cocycles q^+ and q^- on the rack of reflections in an arbitrary Coxeter group and translate twist equivalence of q^+ and q^- into existence of a section, generalizing the strategy in [39]. The core argument is in Section 3 where we apply results in [38] in order to show existence of the sought section (Theorem 2.8). In Section 4 we collect and complement to what is known on Nichols algebras associated with the rack of reflections in finite Coxeter groups, we apply Theorem 2.8 and the main results in [10; 31] in order to list the Coxeter groups for which a conjugacy class of reflections could possibly support a finite-dimensional Nichols algebra. Section 4.1 is devoted to the completion of the classification of finite-dimensional Nichols algebras over dihedral groups.

2. Basic notions

2.1. ***Coxeter groups and root systems.*** Let (W, S) be a Coxeter system, with preferred generating set $S = \{s_1, \dots, s_l\}$, Coxeter graph $\Gamma(W)$, length function $\ell\colon W \to \mathbb{N}$ and Coxeter matrix $A(W) = (m_{ij})_{i,j=1,\dots,l}$, where $|s_i s_j| = m_{ij}$. We further assume that $m_{ij} < \infty$ for any i, j. Following [14] we fix a basis $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of $\mathbb{R}^l$ and define the bilinear form $(\cdot\,,\cdot)$ on $\mathbb{R}^l$ by setting $(\alpha_i, \alpha_j) := -\cos\frac{\pi}{m_{ij}}$ for all $i, j \in \{1, \dots, l\}$. We identify W with the image of the geometric representation obtained mapping $s_i \in S$ to the reflection with respect to α_i. Then $\Phi = W\Delta$ is the set of roots and we denote by Φ^+ and Φ^- the set of positive and negative roots, respectively. We also set

$$T := \{s_\alpha \mid \alpha \in \Phi^+\} = \{w s_\alpha w^{-1} \mid \alpha \in \Delta,\ w \in W\}$$

for the set of reflections in W so $\Phi = \{\pm\alpha \mid s_\alpha \in T\}$. For $x \in T$ we indicate by α_x the unique root in Φ^+ satisfying $x = s_{\alpha_x}$.

We will be mainly interested in W-conjugacy classes in T: there are as many as the number of connected components of the graph obtained from $\Gamma(W)$ by removing the edges with even label [15, Proposition 3, Chapitre IV, §1, Théorème 1, Chapitre VI, §4].

We denote by $\Sigma = S^{(\mathbb{N})}$ the free group on S whose elements are words in the alphabet S and by $\pi : \Sigma \to W$ the projection. We denote by a^{op} the word *opposite* to a, that is, the word obtained from a by reversing the order of its letters. Clearly, $\pi(a^{\mathrm{op}}) = \pi(a)^{-1}$.

For $x \in W$ the subset $\mathcal{R}(x) \subset \Sigma$ stands for the set of reduced expressions of x. An element $x \in W$ lies in T if and only if it has a nontrivial palindromic reduced expression, that is, a nonempty reduced expression $r(x)$ of x such that $r(x) = r(x)^{\mathrm{op}}$; see [38, Proposition 2.4]. For $x \in T$ we denote by $\mathcal{P}(x)$ the set of *palindromic* reduced expressions of x.

For the reader's convenience we recollect some very basic facts on length of conjugate reflections that will be needed in the sequel.

Lemma 2.1. *Let $\beta \in \Phi^+$ and $\alpha \in \Delta$. Then*:

(i) $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) + 2 \iff s_\beta(\alpha) \in \Phi^+ \setminus \{\alpha\} \iff (\alpha, \beta) < 0$ *and* $\beta \neq \alpha$.

(ii) $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) - 2 \iff s_\beta(\alpha) \in \Phi^- \setminus \{-\alpha\} \iff (\alpha, \beta) > 0$ *and* $\beta \neq \alpha$.

(iii) $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) \iff s_\beta s_\alpha = s_\alpha s_\beta \iff (\alpha, \beta) = 0$ *or* $\beta = \alpha$.

(iv) *If $\beta \in \Phi^+ \setminus \Delta$, there is always $\alpha \in \Delta$ such that $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) - 2$.*

Proof. We prove (i). In virtue of [15, Exercise V.4.8], the condition $\ell(s_\alpha s_\beta s_\alpha) = \ell(s_\beta) + 2$ is equivalent to $s_\beta(\alpha) \in \Phi^+$ and $s_\alpha s_\beta(\alpha) \in \Phi^+$, that is, equivalent to $s_\beta(\alpha) \in \Phi^+ \setminus \{\alpha\}$ by [14, Lemma 4.4.3]. Now, $s_\beta(\alpha) = \alpha - 2(\alpha, \beta)\beta \in \Phi^+ \setminus \{\alpha\}$ if and only if $(\alpha, \beta) < 0$ and $\beta \neq \alpha$. Then, (ii) follows similarly and (iii) by exclusion. To prove (iv) assume that (ii) never occurs for β. Then $(\beta, \alpha) \leqslant 0$ for any $\alpha \in \Delta$ and thus also for any $\alpha \in \Phi^+$, whence $(\beta, \beta) \leqslant 0$, a contradiction. □

2.2. *Racks, cocycles and Nichols algebras.*

Definition 2.2. A rack is a pair $(X, \triangleright)$ where X is a nonempty set and $\triangleright : X \times X \to X$ is a function, such that the map $x \mapsto i \triangleright x$ is bijective for all $i \in X$, and $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$.

We will be mainly interested in racks of the form $X \subset G$, where G is a group, X is stable by conjugation and $x \triangleright y = xyx^{-1}$ for $x, y \in X$.

Definition 2.3 [2, §2.2; 16, §2]. Let X be a rack and let A be an abelian group. A map $q : X \times X \to A$ is a 2-cocycle (or just a cocycle) if

$$q(x, y \triangleright z)q(y, z) = q(x \triangleright y, x \triangleright z)q(x, z) \tag{2-1}$$

for all $x, y, z \in X$. We call $Z^2_R(X, A)$ the set of rack cocycles. Two cocycles $q, q' \in Z^2_R(X, A)$ are said to be cohomologous if there exists $\gamma : X \to A$ such that

$$q(x, y) = \gamma(x \triangleright y)^{-1} q'(x, y)\gamma(y) \quad \text{for all } x, y \in X. \tag{2-2}$$

Our main motivation for studying racks and cocycles comes from Nichols algebras, whose construction we briefly recall. The base field here is $\mathbb{C}$.

Given a rack X and a cocycle q, the operator

$$c_q : \mathbb{C}X \otimes \mathbb{C}X \to \mathbb{C}X \otimes \mathbb{C}X, \quad x \otimes y \mapsto q(x, y)x \triangleright y \otimes x,$$

satisfies the braid equation on $(\mathbb{C}X)^{\otimes 3}$:

$$(c_q \otimes \mathrm{id})(\mathrm{id} \otimes c_q)(c_q \otimes \mathrm{id}) = (\mathrm{id} \otimes c_q)(c_q \otimes \mathrm{id})(\mathrm{id} \otimes c_q). \tag{2-3}$$

We call any pair (V, c), where V is a vector space and $c \in \mathrm{GL}(V^{\otimes 2})$ satisfies (2-3), a braided vector space, and c a braiding for V.

We recall that, for $n \geqslant 2$, the n-th braid group B_n is generated by σ_i for $i = 1, \dots, n-1$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i \leqslant n-2$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i, j = 1, \dots, n-1$ with $|i-j| > 1$. There is a unique section, called the Matsumoto section, $M : \mathbb{S}_n \to B_n$ satisfying $M(\sigma\tau) = M(\sigma)M(\tau)$ whenever $\ell(\sigma\tau) = \ell(\sigma)\ell(\tau)$.

Given a braided vector space (V, c), for any $n \geqslant 2$, there is a representation

$$\rho_n : B_n \to \mathrm{GL}(V^{\otimes n}), \quad \sigma_i \mapsto \mathrm{id}^{\otimes(i-1)} \otimes c \otimes \mathrm{id}^{\otimes(n-i-1)},$$

that combined with the Matsumoto section, gives rise to the n-th quantum symmetrizer $\Omega_n := \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)) \in \mathrm{End}(V^{\otimes n})$.

The Nichols algebra associated with (V, c) is the quotient of the tensor algebra $\mathcal{B}(V, c) := T(V)/\bigoplus_{n \geqslant 0} \mathrm{Ker}\, \Omega_n$. If (V, c) comes from a rack-cocycle pair (X, c) we will also denote the Nichols algebra by $\mathcal{B}(X, q)$. It is usually very difficult to detect for which pairs (X, q) the corresponding Nichols algebra is finite dimensional, or has finite GK-dimension, or whether it is finitely presented. Big progresses have been obtained in order to detect properties of $\mathcal{B}(X, q)$ from the knowledge of X only. However, there are cases on which dependence on the cocycle is apparent [31, Table 1] and some cocycles are easier to handle than others.

Remark 2.4. (1) If X is a rack and q, q' are cohomologous rack cocycles, then $\mathcal{B}(X, q) \simeq \mathcal{B}(X, q')$; see [2, Theorem 4.14].

(2) Let (V, c) be a braided vector space. If V' is a subspace of V such that $c(V' \otimes V') \subset V' \otimes V'$, then the inclusion $V' \subset V$ induces an injective algebra homomorphism $\mathcal{B}(V', c|_{V'\otimes V'}) \to \mathcal{B}(V, c)$ [4, Remark 1.1]. Hence, for any pair (X, q) and any inclusion $X' \to X$ such that $X' \triangleright X' \subset X'$ (we call such an X' a subrack of X), we have an algebra inclusion $\mathcal{B}(X', q|_{X'\times X'}) \subseteq \mathcal{B}(X, q)$.

(3) Let G be a group. A Yetter–Drinfeld module for G is the datum of a G-graded representation $V = \bigoplus_{g\in G} V_g$ of G such that $hV_g \subset V_{hgh^{-1}}$ for any $h, g \in G$. Setting $c_V(v \otimes w) := gw \otimes v$ for any $v \in V_g$ and $w \in V$ gives a braiding on V. Yetter–Drinfeld modules decompose as direct sums of irreducible ones. The latter are, as representations, of the form $V = \mathrm{Ind}_H^G U$, where H is the centralizer of some $g \in G$ and $\eta\colon H \to \mathrm{GL}(U)$ is an irreducible representation of H. The G-grading on V is determined by setting $V_g := U$. If η is 1-dimensional, then there is a rack cocycle q on the conjugacy class X of g, with values in $\mathbb{C}^*$ such that $(V, c_V) = (\mathbb{C}X, q)$. For general η, one needs the more general notion of cocycle with values in $\mathrm{GL}(U)$ [7, Section 2.4]. If V is a Yetter–Drinfeld module for G, we will write $\mathcal{B}(V)$ instead of $\mathcal{B}(V, c_V)$.

(4) If $\tau \in \mathrm{Aut}(G)$ and $V = \bigoplus_{g\in G} V_g$ is a Yetter–Drinfeld module for a group G, then twisting V by τ gives another Yetter–Drinfeld module structure V^τ by setting $g.v := \tau(g)v$ and $(V^\tau)_g := V_{\tau(g)}$ for any $g \in G$ and $v \in V$. The braidings c_V and c_{V^τ} coincide, so $\mathcal{B}(V) \simeq \mathcal{B}(V^\tau)$.

(5) Let X be a rack admitting a subrack decomposition $X = X_1 \coprod X_2$ such that $X_i \triangleright X_j = X_j$ for $i, j \in \{1, 2\}$, and let $q \in Z^2_R(X, \mathbb{C})$. Assume, in addition, that $c_q^2 = \mathrm{id}$ on $\mathbb{C}X_i \otimes \mathbb{C}X_j$, for $i \neq j$, and that either all Nichols algebras involved are finite dimensional or that $(X_i, q|_{X_i\times X_i})$, for $i = 1, 2$, are as in point (3). Then, there is an algebra isomorphism $\mathcal{B}(X_1, q|_{X_1\times X_1}) \widetilde{\otimes} \mathcal{B}(X_2, q|_{X_2\times X_2}) \simeq \mathcal{B}(X, q)$, where $\widetilde{\otimes}$ stands for the tensor product of algebras where the multiplication is twisted via c_q [24, Theorem 2.2; 27, Proposition 1.10.12].

2.3. ***Group 2-cocycles and sections.*** We recall that if G and A are groups, with A abelian, the group $Z^2(G, A)$ of 2-cocycles of G with values in A consists of those maps $\phi : G \times G \to A$ satisfying

$$\phi(xy, z)\phi(x, y) = \phi(x, yz)\phi(y, z) \quad \text{for all } x, y, z \in G.$$

If E is a central extension of G, with conjugation action $\blacktriangleright$, projection $\pi_G : E \to G$ and $\mathrm{Ker}(\pi_G) = A$, then for any section $\rho : G \to E$ of π_G, we will denote by ϕ_ρ the 2-cocycle defined by $\phi_\rho(x, y) := \rho(xy)\rho(y)^{-1}\rho(x)^{-1}$ for $x, y \in G$. A direct

calculation shows that

$$\phi_\rho(x, y)(\rho(x) \blacktriangleright \rho(y)) = \phi_\rho(x \triangleright y, x)\rho(x \triangleright y) \quad \text{for all } x, y \in G. \tag{2-4}$$

Definition 2.5 [7, §3.4]. Let G be a group, let $X \subset G$ be stable by conjugation and let $q, q' \in Z^2_R(X, A)$. We say that q and q' are twist equivalent if there exists $\phi \in Z^2(G, A)$ such that

$$q(x, y) = \phi(x, y)\phi(x \triangleright y, x)^{-1} q'(x, y) \quad \text{for all } x, y \in X. \tag{2-5}$$

Remark 2.6. If X is a rack and q, q' are twist-equivalent rack cocycles, then $\mathcal{B}(X, q)$ and $\mathcal{B}(X, q')$ are cocycle twist-equivalent, in particular, they have the same Hilbert series [7, §3.4; 34, §2.7, 3.4].

Lemma 2.7. *Let G be a group and let X be a subset of G stable by conjugation. Let E be a central extension of G with kernel A and conjugation $\blacktriangleright$ and let $q_1, q_2 \in Z^2_R(X, A)$. If there is a section $\rho : G \to E$ satisfying*

$$\rho(x) \blacktriangleright \rho(y) = q_1(x, y)^{-1} q_2(x, y)\rho(x \triangleright y) \quad \textit{for all } x, y \in X, \tag{2-6}$$

then q_1 and q_2 are twist equivalent. If, in addition, $E \simeq A \times G$ is a trivial extension, then q_1 and q_2 are cohomologous.

Proof. The first statement follows from (2-4) and (2-5). If $E \simeq A \times G$ is trivial and π_A is the projection on A, then applying π_A to (2-6) gives (2-2) with $\gamma := \pi_A \circ \rho$. □

2.4. ***Twist equivalence for the racks of reflections.*** We now introduce two specific cocycles q^+ and $q^- \in Z^2_R(T, \mathbb{C}^*)$, as restriction to $T \times T$ of the following functions on $W \times T$. Let $w \in W$ and $y \in T$. Recall that α_y is the positive root associated with y. We set

$$q^+(w, y) = \begin{cases} 1 & \text{if } w(\alpha_y) \in \Phi^+, \\ -1 & \text{if } w(\alpha_y) \in \Phi^-, \end{cases} \qquad \text{and} \qquad q^-(w, y) = \det(w). \tag{2-7}$$

The function q^+ was introduced in [12] for finite W, generalizing the 2-cocycle for reflections in $\mathbb{S}_n$ [21; 36]. A direct calculation shows that q^+ and q^- satisfy

$$q^\pm(w_1 w_2, x) = q^\pm(w_1, w_2 \triangleright x) q^\pm(w_2, x) \quad \text{for all } w_1, w_2 \in W \text{ and } x \in T, \tag{2-8}$$

which implies (2-1); see [36, Section 5].

The main result in [39, Theorem 3.8] states that if $W = \mathbb{S}_n$, then q^+ and q^- are cohomologous for $n = 3$ and twist equivalent for $n \geqslant 4$. In the next section we will prove the following generalization to (possibly infinite) Coxeter groups of this result.

Theorem 2.8. *Assume that $A(W)$ has all its entries in $\mathbb{N}$. Then, the cocycles q^+ and q^- on T are twist equivalent.*

3. Proof of Theorem 2.8

3.1. ***Adapting Vendramin's construction.*** In [39] the group cocycle inducing the twist equivalence is obtained by constructing a suitable section from $\mathbb{S}_n$ to the Schur covering group of $\mathbb{S}_n$. This approach cannot be slavishly reproduced to the case of an arbitrary Coxeter group W because the Schur covering group of W may be trivial or too large for our purposes [32]. We replace the Schur covering group with a suitable, possibly trivial, central extension of W.

Let $\widetilde{W}$ be the group generated by $t_1, \dots, t_l, z$ with relations

$$z^2 = (t_i z)^2 = 1 \quad \text{and} \quad (t_i t_j)^{m_{ij}} = z^{m_{ij}+1} \qquad \text{for all } 1 \le i, j \le l. \tag{3-1}$$

By construction, z is central and the assignment $t_i \mapsto s_i$ for $i = 1, \dots, l$ and $z \mapsto 1$ defines a surjective homomorphism $\pi_W : \widetilde{W} \to W$. Making use of [32, page 264] one sees that $\ker(\pi_W) = \langle z \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

We define the cocycles $q_z^+, q_z^- \in Z_R^2(T, \langle z \rangle)$ by replacing the -1 by z, so that for the nontrivial group isomorphism $\psi : \langle z \rangle \to \{\pm 1\}$ we have $q^\pm = \psi \circ q_z^\pm$. It then suffices to show the twist equivalence of q_z^+ and q_z^- by means of a suitable section $W \to \widetilde{W}$.

Lemma 3.1. *Let $\rho : W \to \widetilde{W}$ be a section of π_W. Assume that ρ satisfies*

$$\rho(s) \blacktriangleright \rho(y) = \begin{cases} \rho(s \triangleright y) z & \text{if } s \neq y, \\ \rho(s \triangleright y) & \text{if } s = y \end{cases} \qquad \text{for all } s \in S \text{ and } y \in T. \tag{3-2}$$

Then

$$\rho(w) \blacktriangleright \rho(y) = q_z^+(w, y) q_z^-(w, y)^{-1} \rho(w \triangleright y) \quad \text{for all } w \in W \text{ and } y \in T. \tag{3-3}$$

Hence, q^+ and q^- are twist equivalent.

Proof. We prove (3-3) by induction on the length of $w \in W$, the case $\ell(w) = 1$ being (3-2) in virtue of [14, Lemma 4.4.3]. Assume $\ell(w) > 1$. Then $w = sw'$ for some $s \in S$ and $w' \in W$ such that $\ell(w') = \ell(w) - 1$. Applying in sequence the definition of ϕ_ρ from (2-4); centrality of z; the inductive hypothesis to w' and s, and (2-8) we obtain

$$\begin{aligned}
\rho(w) \blacktriangleright \rho(y) &= (\phi_\rho(s, w')\rho(s)\rho(w')) \blacktriangleright \rho(y) \\
&= \rho(s) \blacktriangleright (\rho(w') \blacktriangleright \rho(y)) \\
&= \rho(s) \blacktriangleright (q_z^+(w', y) q_z^-(w', y)^{-1} \rho(w' \triangleright y)) \\
&= q_z^+(w', y) q_z^-(w', y)^{-1} \rho(s) \blacktriangleright \rho(w' \triangleright y) \\
&= q_z^+(w', y) q_z^-(w', y)^{-1} q_z^+(s, w' \triangleright y) q_z^-(s, w' \triangleright y)^{-1} \rho(s \triangleright (w' \triangleright y)) \\
&= q_z^+(sw', y) q_z^-(sw', y)^{-1} \rho(sw' \triangleright y).
\end{aligned}$$

The last statement follows from Lemma 2.7. □

3.2. ***Towards the definition of a section.*** We build a section verifying (3-2) using a modified version of the conjugacy graph of (W, S) in [23, Section 3.2].

Definition 3.2. The *reflection conjugacy graph* $\widetilde{\Gamma}(W)$ of (W, S) is the labeled, directed graph having the elements of T as vertices, and labeled edges $x \xrightarrow{s} y$ whenever $x, y \in T$ satisfy $y = s \triangleright x$ with $s \in S$ and $\ell(x) = \ell(y) + 2$.

To simplify notation we write $x \xrightarrow{i} y$ instead of $x \xrightarrow{s_i} y$. More generally, for a sequence $s_{j_r}, \dots, s_{j_1}$ of elements in S, with $a = s_{j_r} \dots s_{j_1} \in \Sigma$, we write $x \xrightarrow{a} y$ if there is a path in $\widetilde{\Gamma}(W)$ from x to y, with labels $j_r, j_{r-1}, \dots, j_1$, or, equivalently, if $x = \pi(a) \triangleright y$ with $\ell(x) = \ell(y) + 2r$. Inductive application of Lemma 2.1(iv) guarantees that for any $x \in T$ there is always a path in $\widetilde{\Gamma}(W)$ starting in x and ending at some $s \in S$.

Remark 3.3. By [38, Proposition 2.4], the set of reduced expressions of $x \in T$ consists of all words of the form $a_L s a_R$ with $s \in S$ and $a_L, a_R \in \Sigma$, such that $x \xrightarrow{a_L} s$ and $x \xrightarrow{a_R^{\mathrm{op}}} s$. Hence, $\mathcal{P}(x)$ consists of all words of the form $a_L s a_L^{\mathrm{op}}$ where $s \in S$ and $x \xrightarrow{a_L} s$. In other terms, $\mathcal{P}(x)$ is in bijection with the paths in $\widetilde{\Gamma}(W)$ starting from x and ending in some $s \in S$. By construction all such paths have the same length.

We aim at defining a section of π_W inductively. To do this we need to compare different elements in $\mathcal{P}(x)$ for $x \in T$. First of all, we compare different elements in $\mathcal{R}(x)$. By [14, Theorem 3.3.1(ii)] it suffices to look at reduced expressions that differ by application of one braid relation, a situation that was considered by Stembridge.

Proposition 3.4 [38, Lemma 2.5]. *Let $x \in T$ and let $a = a_L s_i a_R$ and $b = b_L s_j b_R \in \mathcal{R}(x)$, with $s_i, s_j \in S$, and $a_L, a_R, b_L, b_R \in \Sigma$. Assume that a and b differ by one braid relation. Then, one of the following three alternatives holds*:

Case $a \rightsquigarrow_L b$: *$a_R = b_R$, $i = j$, $\pi(a_L) = \pi(b_L)$ and a_L and b_L differ by one braid relation.*

Case $a \rightsquigarrow_R b$: *$a_L = b_L$, $i = j$, $\pi(a_R) = \pi(b_R)$ and a_R and b_R differ by one braid relation.*

Case $a \rightsquigarrow_C b$: *m_{ij} is odd, $i \neq j$, and there exist $a'_L, a'_R \in \Sigma$ such that*

$$a = a'_L(\underbrace{\dots s_i s_j s_i s_j s_i \dots}_{m_{ij} \text{ terms}})a'_R \quad \text{and} \quad b = a'_L(\underbrace{\dots s_j s_i s_j s_i s_j \dots}_{m_{ij} \text{ terms}})a'_R.$$

We now focus on $\mathcal{P}(x)$. For any $x \in T$ and any $a = a_L s_i a_R \in \mathcal{R}(x)$ we consider the mirrored expression $\mu(a) = a_L s_i a_L^{\mathrm{op}} \in \Sigma$. By [18, (2.7)], the assignment μ gives a well-defined function $\mu\colon \mathcal{R}(x) \to \mathcal{P}(x)$, which restricts to the identity on $\mathcal{P}(x)$.

Proposition 3.5. *Let $x \in T$, and let $a = a_L s_i a_L^{\mathrm{op}}$ and $b = b_L s_j b_L^{\mathrm{op}} \in \mathcal{P}(x)$. Then there exists a sequence $a_d := a_{d,L} s_{i_d} a_{d,L}^{\mathrm{op}} \in \mathcal{P}(x)$, for $d = 0, \dots, r$, with $a_0 = a$ and $a_r = b$ and each $s_{i_d} \in S$ such that for every $d = 0, \dots, r-1$ either $a_{d+1,L}$ differs from $a_{d,L}$ by one braid move, or else $a_d \rightsquigarrow_C a_{d+1}$.*

Proof. By Proposition 3.4 there is a sequence $b_c := b_{c,L} s_{i_c} b_{c,R} \in \mathcal{R}(x)$, for $c = 0, \dots, q$, with $b_0 = a$ and $b_q = b$ and each $s_{i_c} \in S$ such that $b_c \rightsquigarrow_{D_c} b_{c+1}$ for some $D_c \in \{L, R, C\}$ and for all $c \in \{0, \dots, q-1\}$. Applying the mirroring function μ to all terms we obtain a sequence $\mu(b_0) = \mu(a) = a, \mu(b_1), \dots, \mu(b_q) = b \in \mathcal{P}(x)$. Now, if $D_c = R$, then $b_c = b_{c,L} s_{i_c} b_{c,R} \rightsquigarrow_R b_{c,L} s_{i_c} b_{c+1,R} = b_{c+1}$ and so $\mu(b_c) = \mu(b_{c+1})$. If $D_c = L$, then $b_c = b_{c,L} s_{i_c} b_{c,R} \rightsquigarrow_R b_{c+1,L} s_{i_c} b_{c,R} = b_{c+1}$ with $b_{c,L}$ and $b_{c+1,L}$ differing by one braid move. Hence, $\mu(b_c) = b_{c,L} s_{i_c} b_{c,L}^{\mathrm{op}}$ and $\mu(b_{c+1}) = b_{c+1,L} s_{i_c} b_{c+1,L}^{\mathrm{op}}$ with $b_{c,L}$ and $b_{c+1,L}$ differing by one braid move. Finally, if $D_c = C$, then

$$b_c = b'_{c,L}(\dots s_{i_c} s_{i_{c+1}} s_{i_c} s_{i_{c+1}} s_{i_c} \dots) b'_{c,R} \rightsquigarrow_C b'_{c,L}(\dots s_{i_{c+1}} s_{i_c} s_{i_{c+1}} s_{i_c} s_{i_{c+1}} \dots) b'_{c,R} = b_{c+1}$$

for some $b'_{c,L}, b'_{c,R} \in \Sigma$. Thus, $\mu(b_c) = b'_{c,L}(\dots s_{i_{c+1}} s_{i_c} s_{i_{c+1}} \dots) b_{c,L}^{'\mathrm{op}}$ and $\mu(b_{c+1}) = b'_{c,L}(\dots s_{i_c} s_{i_{c+1}} s_{i_c} \dots) b_{c,L}^{'\mathrm{op}}$, so $\mu(b_c) \rightsquigarrow_C \mu(b_{c+1})$. Removing redundancy, we obtain the desired sequence in $\mathcal{P}(x)$. □

We are now able to inductively define a section of π_W with good properties.

Lemma 3.6. *Let $\rho_0 : W \to \widetilde{W}$ be any set-theoretic section of π_W. The assignment*

$$\rho\colon W \to \widetilde{W}, \qquad x \mapsto \begin{cases} \rho_0(x) & \text{if } x \notin T, \\ t_i & \text{if } x = s_i \text{ for some } 1 \le i \le l, \\ t_i \blacktriangleright \rho(y) z & \text{if } x \xrightarrow{i} y \text{ for some } 1 \le i \le l, \end{cases} \tag{3-4}$$

determines a well-defined section of π_W.

Proof. For $x \in T$, let

$$x = x_0 \xrightarrow{i_1} x_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{r-1}} x_{r-1} \xrightarrow{i_r} s_i$$

be a path in $\widetilde{\Gamma}(W)$ ending at some $s_i \in S$. Applying (3-4), using that $z \in Z(\widetilde{W})$, gives

$$\rho(x) = t_{i_1} \blacktriangleright (t_{i_2} \blacktriangleright (\cdots \blacktriangleright t_{i_r} \blacktriangleright t_i) \cdots) z^r = t_{i_1} t_{i_2} \dots t_{i_r} t_i t_{i_r} \dots t_{i_2} t_{i_1} z^r, \tag{3-5}$$

and ρ is well defined if the term on the right-hand side is independent from the chosen path, or, equivalently, independent from the corresponding reduced expression $a = s_{i_1} \dots s_{i_r} s_i s_{i_r} \dots s_{i_1}$ in $\mathcal{P}(x)$. By Proposition 3.5, it is enough to verify that $\rho(x)$ does not change if a is modified by applying either one braid move to

$a_L = s_{i_1} \dots s_{i_r}$ (and consequently to a_L^{op}) or by applying the move $\rightsquigarrow_C$ to a. In the first scenario, a subword of the form

$$\underbrace{s_a s_b \dots}_{m_{ab}} \text{ in } a_L \quad \text{is replaced by} \quad \underbrace{s_b s_a \dots}_{m_{ab}}.$$

In this case, a subword of the form

$$\underbrace{t_a t_b \dots}_{m_{ab}} \text{ occurring before } t_i \text{ in (3-5)} \quad \text{is replaced by} \quad \underbrace{t_b t_a \dots}_{m_{ab}},$$

and symmetrically a subword of the form

$$\underbrace{t_b t_a \dots}_{m_{ab}} \text{ occurring after } t_i \text{ in (3-5)} \quad \text{is replaced by} \quad \underbrace{t_a t_b \dots}_{m_{ab}}.$$

Since

$$\underbrace{t_a t_b \dots}_{m_{ab}} = z^{m_{ab}+1} \underbrace{t_b t_a \dots}_{m_{ab}}$$

and z is central, the value of $\rho(x)$ does not change in this case. In the second scenario, the term

$$a = a_L s_i a_R = a'_L(\underbrace{\dots s_j s_i s_j \dots}_{m_{ij}})a'_R \quad \text{is replaced by} \quad a'_L(\underbrace{\dots s_i s_j s_i \dots}_{m_{ij}})a'_R$$

and m_{ij} is odd. In this case, the central term

$$\underbrace{\dots t_j t_i t_j \dots}_{m_{ij}} \text{ in (3-5)} \quad \text{is replaced by} \quad \underbrace{\dots t_i t_j t_i \dots}_{m_{ij}},$$

while the rest is unmodified. As

$$\underbrace{\dots t_j t_i t_j \dots}_{m_{ij}} = z^0 \underbrace{\dots t_i t_j t_i \dots}_{m_{ij}}$$

in $\widetilde{W}$, the value of $\rho(x)$ is unaltered, so ρ is well defined. Applying π_W to (3-5) shows that it is a section. □

3.3. ***Good properties of the section.*** The next step is to prove that the section ρ as in (3-4) satisfies Vendramin's condition. We fix some further notation.

Definition 3.7. The sequence $(U_n(X))_{n\geqslant 0}$ of Chebychev polynomials of the second kind is the sequence of polynomials in $\mathbb{Z}[X]$ defined recursively by

$$U_0(X) = 1, \quad U_1(X) = 2X, \quad U_{n+1}(X) = 2XU_n(X) - U_{n-1}(X). \tag{3-6}$$

We recall the well-known formula

$$\sin(\theta)U_n(\cos\theta) = \sin((n+1)\theta). \tag{3-7}$$

The following lemma is key for proving inductively the good properties of ρ.

Lemma 3.8. *Let $\beta \in \Phi^+ \setminus \Delta$ and assume there are $\alpha_i, \alpha_j \in \Delta$ satisfying $\delta := (\alpha_j, \beta) > 0$ and $(\alpha_i, \beta) = 0$. For $p \geqslant 0$, let*

$$\alpha_{(p)} = \begin{cases} \alpha_i & \text{if } p \text{ is even,} \\ \alpha_j & \text{if } p \text{ is odd,} \end{cases} \qquad s_{(p)} := s_{\alpha_{(p)}}, \qquad \beta_0 := \beta, \qquad \beta_{p+1} := s_{(p+1)}\beta_{(p)}.$$

Then,

$$(\beta_p, \alpha_{(p+1)}) > 0 \quad \text{for } p = 0, \dots, m_{ij} - 2, \qquad (\beta_{m_{ij}-1}, \alpha_{(m_{ij})}) = 0 \tag{3-8}$$

and either

$$\ell(s_{\beta_{m_{ij}-1}}) = \ell(s_\beta) - 2m_{ij} + 2, \quad s_{(m_{ij})}s_{\beta_{m_{ij}-1}} = s_{\beta_{m_{ij}-1}}s_{(m_{ij})}, \quad \alpha_{(m_{ij})} \neq \beta_{m_{ij}-1} \tag{3-9}$$

or else

$$m_{ij} \text{ is even}, \quad \beta_{m_{ij}/2-1} = \alpha_{(m_{ij}/2)} \in \Delta, \quad \ell(s_\beta) = 2m_{ij} - 1. \tag{3-10}$$

Proof. For $p \geqslant 0$ we set

$$u_p := \frac{(\beta_p, \alpha_{(p+1)})}{\delta} = \frac{(\beta_p, \alpha_{(p-1)})}{\delta}, \quad \gamma := -(\alpha_i, \alpha_j) = \cos\frac{\pi}{m_{ij}}.$$

Then $u_0 = 1$. For $p \geqslant 1$ we compute

$$\begin{aligned} u_1 &= \frac{(s_j\beta, \alpha_i)}{\delta} = \frac{(\beta - 2(\beta, \alpha_j)\alpha_j, \alpha_i)}{\delta} = 0 + 2\gamma, \\ u_{p+1} &= \frac{(s_{(p+1)}\beta_p, \alpha_{(p)})}{\delta} = \frac{(\beta_p - 2(\beta_p, \alpha_{(p+1)})\alpha_{(p+1)}, \alpha_{(p)})}{\delta} \\ &= \frac{(\beta_p, \alpha_{(p)})}{\delta} - \frac{2(\beta_p, \alpha_{(p-1)})(\alpha_{(p+1)}, \alpha_{(p)})}{\delta} = \frac{(s_{(p)}\beta_{p-1}, \alpha_{(p)})}{\delta} + 2\gamma u_p \\ &= \frac{(\beta_{p-1}, s_{(p)}\alpha_{(p)})}{\delta} + 2\gamma u_p = -u_{p-1} + 2\gamma u_p. \end{aligned}$$

Therefore $(\beta_p, \alpha_{(p+1)})/\delta = U_p(\gamma)$, the p-th Chebychev's polynomial evaluated at $\gamma = \cos(\pi/m_{ij})$. Then, (3-7) gives (3-8) and Lemma 2.1(i) and (ii) applied to $\beta_{(p)}$ and $\alpha_{(p+1)}$ imply that either

$$\ell(s_{\beta_{p+1}}) < \ell(s_{\beta_p}) \quad \text{for } p = 0, \dots, m_{ij} - 2, \tag{3-11}$$

giving (3-9), or else $\ell(s_{\beta_{q+1}}) = \ell(s_{\beta_q})$ for some $q \in \{0, \dots, m_{ij} - 2\}$. In this case, Lemma 2.1(iii) gives $\beta_q = \alpha_{(q+1)}$, whence

$$s_\beta = s_{(1)} s_{(2)} \dots s_{(q)} s_{(q+1)} s_{(q)} \dots s_{(1)}. \tag{3-12}$$

Applying the equality $s_i s_\beta s_i = s_\beta$ yields

$$s_i s_{(1)} s_{(2)} \dots s_{(q)} s_{(q+1)} s_{(q)} \dots s_{(1)} = s_{(1)} s_{(2)} \dots s_{(q)} s_{(q+1)} s_{(q)} \dots s_{(1)} s_i,$$

where both sides are products of $2q+2$ simple reflections. Bearing in mind that $s_{(1)} = s_j$, this gives $(s_i s_j)^{2q+2} = 1$ and so m_{ij} divides $2(q+1)$. By construction $m_{ij} > q+1$; hence $m_{ij} = 2(q+1)$ is even and $\beta_{m_{ij}/2-1} = \alpha_{(m_{ij}/2)}$. The claim on the length follows because $\ell(s_{\beta_{p+1}}) = \ell(s_{\beta_p}) - 2$ for $p \in \{0, \dots, q-1\}$. □

Lemma 3.9. *The section ρ defined by (3-4) satisfies (3-2).*

Proof. Let $y \in T$ and $s_i \in S$. Then we fall in one of the following situations: $y = s_i$, or $\ell(s_i \triangleright y) = \ell(y) \pm 2$, or $(\alpha_y, \alpha_i) = 0$, that we analyze separately.

- If $y = s_i$, then $\rho(s_i) \blacktriangleright \rho(y) = t_i \blacktriangleright t_i = t_i = \rho(y)$.
- If $\ell(s_i \triangleright y) = \ell(y) + 2$, then by construction $\rho(s_i \triangleright y) = t_i \blacktriangleright \rho(y) z = \rho(s_i) \blacktriangleright \rho(y) z$. Multiplying both sides by z gives (3-2).
- If $\ell(s_i \triangleright y) = \ell(y) - 2$, then we invoke the previous case applied to $x = s_i \triangleright y$ and use that t_i and z are involutions and that z is central.
- If $(\alpha_y, \alpha_i) = 0$ then $y \neq s_i$ and $s_i \triangleright y = y$. We proceed by induction on the length of y. If $\ell(y) = 1$, then $y = s_j$ for some $j \in \{1, \dots, l\}$, with $m_{ij} = 2$ and $\rho(s_i \triangleright y) = \rho(y) = t_j = z t_i \blacktriangleright t_j = \rho(s_i) \blacktriangleright \rho(s_j) z$, confirming (3-2) in this case. Assume now that $\ell(y) > 1$. By Lemma 2.1, there always is a $j \in \{1, \dots, l\}$ satisfying $\ell(s_j y s_j) = \ell(y) - 2$. Setting $\beta = \alpha_y$, Lemma 3.8, from which we retain notation, gives either (3-9) or else (3-10).

If (3-10) holds, then $y = s_{(1)} \triangleright (s_{(2)} \triangleright \dots \triangleright (s_{(m_{ij}/2-1)} \triangleright s_{(m_{ij}/2)}))$ and $\ell(y) = 2m_{ij} - 1$, so

$$\rho(s_i \triangleright y) = \rho(y) = (t_j t_i)^{m_{ij}/2-1} t_j z^{m_{ij}/2-1}.$$

As $(t_i t_j)^{m_{ij}} = z$, we have

$$(t_i t_j)^{m_{ij}/2} = z (t_j t_i)^{m_{ij}/2}.$$

Then,

$$\begin{aligned} \rho(s_i) \blacktriangleright \rho(y) z &= t_i (t_j t_i)^{m_{ij}/2-1} t_j t_i z^{m_{ij}/2} = t_i (t_j t_i)^{m_{ij}/2} z^{m_{ij}/2} \\ &= t_i (t_i t_j)^{m_{ij}/2} z^{m_{ij}/2+1} = (t_j t_i)^{m_{ij}/2-1} t_j z^{m_{ij}/2-1} = \rho(s_i \triangleright y). \end{aligned}$$

If instead, (3-9) holds, then $y = s_{(1)} \triangleright (s_{(2)} \triangleright \cdots \triangleright (s_{(m_{ij}-1)} \triangleright s_{\beta_{m_{ij}-1}}) \cdots)$ with $\ell(s_{\beta_{m_{ij}-1}}) = \ell(y) - 2m_{ij} + 2 < \ell(y)$ and $s_{(m_{ij})} \triangleright s_{\beta_{m_{ij}-1}} = s_{\beta_{m_{ij}-1}}$, so by induction $t_{(m_{ij})} \blacktriangleright \rho(s_{\beta_{m_{ij}-1}}) = \rho(s_{\beta_{m_{ij}-1}})z$. By definition of ρ we have

$$\rho(s_i \triangleright y) = \rho(y) = \underbrace{t_j t_i \dots}_{m_{ij}-1 \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\dots t_i t_j}_{m_{ij}-1 \text{ terms}} z^{m_{ij}-1}.$$

Therefore

$$\begin{aligned}\rho(s_i) \blacktriangleright \rho(y)z &= \underbrace{t_i t_j t_i \dots}_{m_{ij} \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\dots t_i t_j t_i}_{m_{ij} \text{ terms}} z^{m_{ij}} \\ &= \underbrace{t_j t_i t_j \dots}_{m_{ij} \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\dots t_j t_i t_j}_{m_{ij} \text{ terms}} z^{m_{ij}+2m_{ij}+2},\end{aligned}$$

where the last factor in $t_i t_j \dots$ is now $\rho(s_{(m_{ij})-2}) = \rho(s_{(m_{ij})})$. By induction,

$$\rho(s_i) \blacktriangleright \rho(y)z = \underbrace{t_j t_i \dots}_{m_{ij}-1 \text{ terms}} \rho(s_{\beta_{m_{ij}-1}}) \underbrace{\dots t_i t_j}_{m_{ij}-1 \text{ terms}} z^{m_{ij}+1} = \rho(s_i \triangleright y)$$

concluding the proof. □

We end this section characterizing when q^+ and q^- are cohomologous. For the case of $\mathbb{S}_n$, see [39, Remark 2.2].

Theorem 3.10. *Assume that $A(W)$ has all its entries in $\mathbb{N}$. Then, the following are equivalent*:

(1) *All the coefficients in $A(W)$ are odd.*

(2) *The group $\widetilde{W}$ is the trivial extension $\widetilde{W} = W \times \langle z \rangle$.*

(3) *The cocycles q^+ and q^- are cohomologous.*

Proof. The implication (1) ⇒ (2) is immediate from the definition of $\widetilde{W}$, and (2) ⇒ (3) follows from Lemmata 2.7, 3.1, and 3.9. We prove that (3) ⇒ (1). Suppose for a contradiction that $m := m_{ij}$ is even for some $i, j \in \{1, \dots, l\}$, and that there exists $\gamma : W \to \mathbb{C}^*$ such that $q^-(x, y) = \gamma(x \triangleright y)^{-1} q^+(x, y)\gamma(y)$ for all $x, y \in T$. Set $s = s_i$, $s' = s_j$ and let $\sigma := (s's)^{m/2-1}s'$, so that $\sigma \in T$ and $s\sigma s = \sigma$. Then $\gamma(s \triangleright \sigma) = \gamma(\sigma)$. Thus $-1 = q^-(s, \sigma) = q^+(s, \sigma)$, contradicting (2-7). □

Remark 3.11. If all coefficients in $A(W)$ are odd, then the underlying graph of $\Gamma(W)$ is complete. Hence, the corresponding group W is finite if and only if $l \leqslant 2$. If $l = 1$, then $T = \{s_\alpha\}$ and $q^+ = q^-$ so the statement is trivial. If $l = 2$, then W is of type $I_2(2m + 1)$, for $m \geqslant 1$, i.e., it is the dihedral group of a regular $(2m+1)$-gon [15, Chapitre VI, §4, Théorème 1]. In this case, the cohomology of q^+ and q^- is proved in [36, Example 5.4(a)]. Notice that, even though the

extension $\widetilde{W}$ is trivial, by [32], the Schur multiplier of W is elementary abelian of order $2^{(l-1)(l-2)/2}$, whence nontrivial whenever W is infinite.

4. Applications to Nichols algebras

Here, W is finite and the base field is $\mathbb{C}$. The Nichols algebra associated with (T, q^+) is of particular interest because it contains the coinvariant algebra of W [12; 21]. If W is crystallographic, the latter is isomorphic to the cohomology of the flag variety of the algebraic group with associated Weyl group W. It is in general an open question whether $\mathcal{B}(T, q^+)$ is finite dimensional, quadratic, or finitely presented [12; 21; 22; 34]. The main result in [11] states that the quadratic cover of $\mathcal{B}(T, q^+)$ is infinite dimensional for $W = \mathbb{S}_n$ and $n \geqslant 6$.

Combining Theorem 2.8 and Remark 2.6 readily gives the following.

Corollary 4.1. *If W is an arbitrary finite Coxeter group, then $\mathcal{B}(T, q^+)$ and $\mathcal{B}(T, q^-)$ have the same Hilbert series. This also holds for their quadratic approximations. In addition, $\mathcal{B}(T, q^+)$ is quadratic if and only if $\mathcal{B}(T, q^-)$ is quadratic.*

It also follows from Theorem 3.10 and Remark 2.4(1) that if W is of type $I_2(2m+1)$, for $m \geqslant 1$, that is, the dihedral group of a regular $(2m+1)$-gon, then $\mathcal{B}(T, q^+) \simeq \mathcal{B}(T, q^-)$, a result that was already present in [36, Example 5.4(b)]. By [36, Remark 5.2 part 2)] the braided vector spaces attached to $(T, q^\pm)$ are constructed as in Remark 2.4(3), so results in this setting apply. We summarize below some facts on the Nichols algebras $\mathcal{B}(T, q^\pm)$. Some of these results are known, we focus on giving a uniform point of view.

Remark 4.2. (1) If (W, S) is not irreducible, let $(W_1, S_1), \dots, (W_r, S_r)$ be its irreducible factors. Setting $T_i := W \triangleright S_i = W_i \triangleright S_i$ for $i = 1, \dots, r$ we have a rack decomposition $T = \coprod_{i=1}^r T_i$ where $t \triangleright t' = t'$ and $q^+(t, t') = q^+(t', t) = 1$ if $t \in T_i$ and $t' \in T_j$ with $i \neq j$. Hence $c^2_{q^+}$ and $c^2_{q^-}$ act as the identity on $\mathbb{C}T_i \otimes \mathbb{C}T_j$ whenever $i \neq j$. Remark 2.4(5) then gives $\mathcal{B}(T, q^+) \simeq \bigotimes_{i=1}^r \mathcal{B}(T_i, q_i^+)$ and $\mathcal{B}(T, q^-) \simeq \bigotimes_{i=1}^r \mathcal{B}(T_i, q_i^-)$, where $q_i^\pm$ stands for the restriction of $q^\pm$ to $T_i \times T_i$. Therefore, it is enough to study the case of irreducible Coxeter groups.

(2) Coxeter graph inclusions imply inclusions of the corresponding racks of reflections, hence a braided vector space inclusion for the cocycle q^-, and therefore a Nichols algebra inclusion by Remark 2.4(2).

(3) Let (W, S) be irreducible. If $\Gamma(W)$ has no even labeled edges, then the reflections form a single conjugacy class. Otherwise, W is of type B_l for $l \geqslant 3$, F_4, or $I_2(2m)$ for $m \geqslant 2$ and T is the union of two classes, represented by any s and $s' \in S$ that are joined in $\Gamma(W)$ by an even labeled edge. Setting $T_1 := W \triangleright s$ and

type of W	$\mathcal{B}(T_1, q^{\pm})$	$\mathcal{B}(T_2, q^{\pm})$	$\mathcal{B}(T, q^{\pm})$
$B_l,\ l \geqslant 3$	$\mathcal{B}(T_{D_l}, q^{\pm})$	$\left(\bigwedge k\right)^{\otimes l},\ \bigwedge \mathbb{C}^l$	infinite dimensional
F_4	$\mathcal{B}(T_{D_4}, q^{\pm})$	$\mathcal{B}(T_{D_4}, q^{\pm})$	infinite dimensional
$I_2(4)$	$\left(\bigwedge \mathbb{C}\right)^{\otimes 2}$	$\left(\bigwedge \mathbb{C}\right)^{\otimes 2}$	dimension 64
$I_2(2m),\ m > 2$	$\mathcal{B}(T_{I_2(m)}, q^{\pm})$	$\mathcal{B}(T_{I_2(m)}, q^{\pm})$	infinite dimensional

Table 1. Nichols algebras of (T, q^+) for W such that $\Gamma(W)$ has an even labeled edge.

$T_2 := W \triangleright s'$ we have a rack decomposition $T = T_1 \coprod T_2$ with $T_i \triangleright T_j = T_j$ for $i, j \in \{1, 2\}$. Observe that $c^2_{q^{\pm}}(s \otimes s') = \pm(s \triangleright s') \triangleright s \otimes s \triangleright s' \neq s \otimes s'$, so $c^2_{q^{\pm}} \neq \mathrm{id}$ on $\mathbb{C}T_1 \otimes \mathbb{C}T_2$. If $\min(|T_1|, |T_2|) > 2$ or $\max(|T_1|, |T_2|) > 4$, then $\dim \mathcal{B}(T, q) = \infty$ for *any* cocycle q, by [9, Theorem 2.9] applied to $Y = T$. By construction, for $i = 1, 2$, we have twist equivalence of the restrictions to T_i of q^+ and q^-.

(4) If W is of type B_l with $l \geqslant 3$, up to renumbering, T_1 is isomorphic to the rack T_{D_l} of reflections for W of type D_l, so $|T_1| = l^2 - l$ and T_2 is abelian, that is, it has trivial action, and $|T_2| = l$. By [9, Theorem 2.9], $\dim \mathcal{B}(T, q) = \infty$ for any cocycle q. On the other hand, $\mathcal{B}(T_2, q^-) = \bigwedge \mathbb{C}^l$ and $\mathcal{B}(T_2, q^+) = \left(\bigwedge \mathbb{C}\right)^{\otimes l}$, and hence $\dim \mathcal{B}(T_2, q^{\pm}) = 2^l$.

(5) If W is of type F_4 or $I_2(2m)$ for $m \geqslant 2$, then the classes T_1 and T_2 are interchanged by the automorphism of W coming from the symmetry of $\Gamma(W)$, so $T_1 \simeq T_2$ as racks. Using the descriptions of roots in [15, Planches IV, VIII], one sees that in type F_4 the racks T_1 and T_2 are isomorphic to T_{D_4}, so $|T_1| = |T_2| > 4$ whence $\dim \mathcal{B}(T, q) = \infty$ for *any* cocycle q.

(6) If W is of type $I_2(2m)$ for $m \geqslant 2$, the racks T_1 and T_2 are isomorphic to the rack $T_{I_2(m)}$ of reflections of type $I_2(m)$. In addition, it was shown in [5, Lemma 2.1] using [8, Theorem 3.6] that if $m > 2$, then $\dim \mathcal{B}(T_{I_2(2m)}, q) = \infty$ for *any* cocycle q. Several considerations concerning the rack T for dihedral groups are present in [36, Sections 5, 6]. In particular, [36, Example 6.5] shows that $\dim \mathcal{B}(T_{I_2(4)}, q^{\pm}) = 64$. The Nichols algebras and/or their quadratic covers for the rack of reflections in $I_2(2p)$ for p an odd prime has been studied in [2; 3, Example 3.3.5; 35].

In Table 1 we summarize what is currently known about the Nichols algebras of the pair (T, q^+) for W such that $\Gamma(W)$ has an even labeled edge, up to twist equivalence.

We now focus on the irreducible, finite Coxeter groups with one conjugacy class of reflections.

Proposition 4.3. *Let W be of type $I_2(2m+1)$ for $m > 1$. Then* $\dim \mathcal{B}(T, q^{\pm}) = \infty$.

Proof. Let $2m+1 = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorization and let $W = \langle s, s' \rangle$. Then

$$T = \{s(s's)^j \mid j = 0, \dots, 2m+1\}$$

and for any n dividing $2m+1$, the subset $T_n = \{s(s's)^j \in T \mid n \text{ divides } j\}$ is a subrack of T because $s(s's)^j \triangleright s(s's)^l = s(s's)^{2j-l}$; see [1, Remark 3.3]. In particular, T contains a subrack of size p_i for any prime divisor of $2m+1$. The pair $(T_{p_i}, q^+|_{T_{p_i}})$ is precisely the pair corresponding to the rack of reflections of the dihedral group of size $2p_i$ and T_{p_i} is an indecomposable affine rack [2, Section 1.3.8]. If $p_i > 7$ for some i, then [31, Theorem 1.6] implies that $\dim \mathcal{B}(T_{p_i}, q^+|_{T_{p_i}}) = \infty$ and, a fortiori, $\dim \mathcal{B}(T, q^{\pm}) = \infty$. Assume now that $p_i = 5$ or 7 for some i. In the notation of [31], the rack T_{p_i} is $\mathrm{Aff}(p_i, p_i - 1)$. Therefore, if $p_i = 5, 7$, the rack T_{p_i} does not occur in [31, Table 1] and so $\dim \mathcal{B}(T, q^{\pm}) = \dim \mathcal{B}(T_{p_i}, q^{\pm}) = \infty$. We are left with the case $2m+1 = 3^b$ for some $b \geqslant 2$. In this case T is an indecomposable affine rack, and $\dim \mathcal{B}(T_{3^b}, q^{\pm}) = \infty$ by [10, Theorem 1.3]. □

Corollary 4.4. *If W is of type H_3 or H_4, then* $\dim \mathcal{B}(T, q^{\pm}) = \infty$.

Proof. The rack T contains a subrack isomorphic to $T_{I_2(5)}$, so the statement follows from Theorem 2.8, Remark 4.2(2) and Proposition 4.3. □

In the remaining cases W is in one of the crystallographic, simply laced families of groups A_n, $n \geqslant 1$, D_n for $n \geqslant 4$, and E_6, E_7, E_8. They afford a crystallographic root system $\widetilde{\Phi}$ as in [15, Chapitre VI.1.1], and all roots in $\widetilde{\Phi}$ have the same length. Here $T = \{s_\alpha \mid \alpha \in \widetilde{\Phi}^+\}$ and by [15, Chapitre VI.1.3] the subgroup generated by any pair of noncommuting reflections is isomorphic to $\mathbb{S}_3$. It is well known that $\dim \mathcal{B}(T_{A_n}, q^{\pm}) < \infty$ for $n \leqslant 4$ [22; 36], whilst infinite dimensionality for $n \geqslant 5$ is still open. Observe that none of the splitting criteria in [9, Sections 2.1, 2.2] and [8, Section 3.2] apply to the rack T: in the terminology of [9], the rack T is kthulhu.

Remark 4.5. It was kindly pointed out to us by I. Heckenberger that $\mathcal{B}(T_{D_4}, q^{\pm})$ is infinite dimensional as a consequence of [10, Theorem 6.14] because the Coxeter group of type D_4 is solvable, noncyclic, and generated by T, which has size > 7. Hence, $\dim \mathcal{B}(T_{D_n}, q^{\pm}) = \infty$ for any $n \geqslant 4$.

In all remaining cases $\Gamma(W)$ contains a graph of type A_5 (that is, $\mathbb{S}_6 \leqslant W$). The case of $\mathbb{S}_6$ has been addressed by several authors: by the main result in [11] it would be infinite dimensional provided $\mathcal{B}(T, -1)$ is quadratic, but the latter property has not been established despite several attempts. Summarizing we have:

Corollary 4.6. *Assume that* $\dim \mathcal{B}(T_{A_5}, q^{\pm}) = \infty$. *If* $\dim \mathcal{B}(T, q^{\pm}) < \infty$ *then W is either the dihedral group of order* 8, *the cyclic group of order* 2, $\mathbb{S}_3$, $\mathbb{S}_4$ *or* $\mathbb{S}_5$.

Through the equivalence of categories in [33], the Nichols algebra $\mathcal{B}(T, q^-)$ corresponds to a factorizable perverse sheaf whose underlying perverse sheaf is the intermediate extension of the local system on the ind-variety of configuration spaces in $\mathbb{C}$, corresponding to the collection of braid group representations $V^{\otimes n}$, for $n \geqslant 0$ associated with T and the trivial cocycle. The latter are obtained as the push-forward of the constant sheaf on the Hurwitz space with Galois group W and local monodromy T. One may hope to retrieve further information on these algebras (e.g., if they are quadratic) by using this geometric interpretation.

4.1. ***Nichols algebras over dihedral groups.*** Here, we restrict to the case in which W is of rank 2, that is, W is of type $I_2(n)$ for $n > 2$ and we write $S = \{s, s'\}$. The analysis of the finite-dimensional Nichols algebras over $I_2(n)$ when $4 \mid n$ and $n \geqslant 12$ was obtained in [20], the cases $n = 4, 8$ were then completed in [13, Chapter 2]. Thus in this section n is not divisible by 4 and we separate the analysis according to its parity.

Corollary 4.7. *Assume $n = 2m + 1$ for $m > 1$. Then, any Nichols algebra of a Yetter–Drinfeld module of W is infinite dimensional. Therefore, if H is a finite-dimensional complex, pointed Hopf algebra with group of grouplikes isomorphic to W, then $H = \mathbb{C}W$, the group algebra of W.*

Proof. By [6, Theorem 4.8], the only possible finite-dimensional Nichols algebra coming from a Yetter–Drinfeld module for W could come from the irreducible Yetter–Drinfeld module as in Remark 2.4(3) with $g \in S$ and η the nontrivial irreducible representation of $H = \langle g \rangle$. However, this corresponds precisely to the pair $(T_{I_2(2m+1)}, q^+)$, see [36, Section 5], which has an infinite-dimensional Nichols algebra by Proposition 4.3. The second statement follows from [2, §0.3]. □

We will now look at the case $n = 2r$, where r is odd. Let $\zeta \in \mathbb{C}^*$ be a primitive n-th root of 1 and let $C := \langle ss' \rangle$. The following irreducible Yetter–Drinfeld modules over W have a finite-dimensional associated Nichols algebra [1, Theorem 3.1]. The action can be extracted from [37, 5.3]. The grading of all of them is supported in C, thus to understand the braiding, it is enough to consider the C-action. This way, we can regard them as Yetter–Drinfeld modules over C. They are:

- $V_0 := \mathbb{C}v_0$, concentrated in degree $(ss')^r$ and with action $(ss')v_0 = -v_0$. The Nichols algebra is $\bigwedge V_0$;
- $V_{r,j} := \mathbb{C}v_{+r,j} \oplus \mathbb{C}v_{-r,j}$, for $j \in \{1, \ldots, r-2\}$ and odd, with grading concentrated in degree $(ss')^r$ and action $(ss')v_{\pm r,j} = \zeta^{\pm j} v_{\pm r,j}$. The Nichols algebra is $\bigwedge V_{r,j}$;

• $V_{h,j} := \mathbb{C}v_{+h,j} \oplus \mathbb{C}v_{-h,j}$, for $j, h \in \{1, \dots, r-2\}$, both odd, and such that $r \mid jh$, with $v_{\pm h,j}$ in degree $(ss')^{\pm h}$, respectively, and action $(ss')v_{\pm h,j} = \zeta^{\pm j} v_{\pm h,j}$. The Nichols algebra is $\bigwedge V_{h,j}$.

Theorem 4.8. *Assume $n = 2r$ for $r > 3$ and odd. Let V be a Yetter–Drinfeld module of W. Then $\dim \mathcal{B}(V) < \infty$ if and only if as a Yetter–Drinfeld module over C,*

$$V \simeq V_0^{\oplus k} \oplus \bigoplus_{d=1}^{N} V_{h_d, j_d}^{\oplus k_d}$$

for some $k, k_d, N \geqslant 0$, and some distinct pairs (h_d, j_d) for $d \in \{1, \dots, N\}$ with $h_d \in \{1, \dots, r\}$ odd, $j_d \in \{1, \dots, r-2\}$ odd and such that $r \mid (h_d j_{d'} + h_{d'} j_d)$ for all $d, d' \in \{1, \dots, N\}$. In this case,

$$\mathcal{B}(V) \simeq \bigwedge V_0^{\oplus k} \,\widetilde{\otimes}\, \left(\widetilde{\bigotimes}_{d=1}^{N} \bigwedge V_{h_d, j_d}^{\oplus k_d}\right).$$

where $\widetilde{\otimes}$ stands for the tensor product of algebras twisted by the braiding of V.

Proof. The analysis in [1, Theorem 3.1], combined with Proposition 4.3 and Remark 4.2(3), shows that the only irreducible Yetter–Drinfeld modules of W with finite-dimensional Nichols algebras are V_0 and $V_{h,j}$ with $h \in \{1, \dots, r\}$ odd, $j \in \{1, \dots, r-2\}$ odd and such that $r \mid jh$. Now we look at direct sums.

A direct calculation shows that the braiding c satisfies $c(x \otimes y) = -y \otimes x$ for $x \in V_0$, $y \in U$ or $x \in U$ and $y \in V_0$. Then we have the isomorphism $\mathcal{B}(V_0^{\oplus k} \oplus U) \simeq \bigwedge V_0^{\oplus k} \,\widetilde{\otimes}\, \mathcal{B}(U)$ for any $k \geqslant 0$ after an iterated application of Remark 2.4(5).

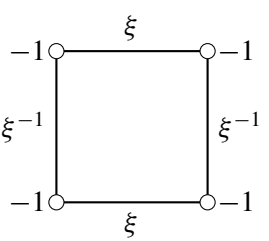
We now look at the braiding on $V_{h,j} \oplus V_{h',j'}$ where h, j, h', j' are odd; $1 \leqslant h, h' \leqslant r$, $1 \leqslant j, j' \leqslant r-2$, and $r \mid hj$ and $r \mid h'j'$. For $\epsilon, \epsilon' \in \{\pm 1\}$ we have

$$c(v_{\epsilon h,j} \otimes v_{\epsilon' h',j'}) = \zeta^{\epsilon\epsilon' hj'} v_{\epsilon' h',j'} \otimes v_{\epsilon h,j}. \tag{4-1}$$

Let $\xi := \zeta^{hj'+jh'}$. If $\xi = 1$, that is, if $r \mid (hj' + jh')$, then $c^2|_{V_{h,j} \oplus V_{h',j'}} = \mathrm{id}$, and so

$$\mathcal{B}(V_{h,j} \oplus V_{h',j'}) \simeq \mathcal{B}(V_{h,j}) \,\widetilde{\otimes}\, \mathcal{B}(V_{h',j'}).$$

If, instead, $\xi \neq 1$, then we compute the generalized Dynkin diagram associated to the braiding on $V_{h,j} \oplus V_{h',j'}$ according to the recipe in [26, §2] obtaining



The above diagram does not occur in [26, Table 3], hence $\dim \mathcal{B}(V_{h,j} \oplus V_{h',j'}) = \infty$ by [25, Theorem 3, Corollary 5].

Now let $U = \bigoplus_{d=1}^{N} V_{h_d,j_d}^{\oplus k_d}$ for some $k_d, N \geqslant 0$, $h_d \in \{1, \dots, r\}$ odd, $j_d \in \{1, \dots, r-2\}$ odd and such that $r \mid h_d j_d$ for all d. If r does not divide $h_d j_{d'} + h_{d'} j_d$ for some d, d', then $\dim \mathcal{B}(U) = \infty$ by Remark 2.4(2). If, instead, $r \mid (h_d j_{d'} + h_{d'} j_d)$ for all d, d', then the square of the braiding between any irreducible component of U and the sum of the others is the identity. Thus, by Remark 2.4(5) and induction on the number of irreducible components, we obtain

$$\mathcal{B}(U) \simeq \widetilde{\bigotimes}_{d=1}^{N} \mathcal{B}(V_{h_d,j_d}^{\oplus k_d}) \simeq \widetilde{\bigotimes}_{d=1}^{N} \big(\bigwedge V_{h_d,j_d} \big)^{\widetilde{\otimes} k_d}.$$

Finally, observe that if $d = d'$ then $\zeta^{\epsilon\epsilon' h_d j_{d'}} = -1$, so $\big(\bigwedge V_{h_d,j_d} \big)^{\widetilde{\otimes} k_d} = \bigwedge V_{h_d,j_d}^{\oplus k_d}$. □

We now complement [1, Table 2] for $W = I_2(6)$ with the analysis of nonsimple Yetter–Drinfeld modules. Here we have to take into account also the simple Yetter–Drinfeld modules supported in $T_1 = W \triangleright s$ and $T_2 = W \triangleright s'$, whose corresponding Nichols algebra is the Fomin–Kirillov algebra FK_3 of dimension 12.

The Yetter–Drinfeld modules whose associated braided spaces correspond to $(T_1, q^{\pm})$ are the modules U_j for $j \in \{0, 1\}$ whose underlying vector space is $\mathbb{C}e_s^j \oplus \mathbb{C}e_{s'ss'}^j \oplus \mathbb{C}e_{ss'ss's}^j$ where e_g^j is in degree g for $g \in W$, and the action is given by

$$\begin{aligned} s \cdot e_s^j &= -e_s^j, & s \cdot e_{s'ss'}^j &= (-1)^{j+1} e_{ss'ss's}^j, & s \cdot e_{ss'ss's}^j &= -(-1)^j e_{s'ss'}^j, \\ (ss') \cdot e_s^j &= e_{ss'ss's}^j, & (ss') \cdot e_{s'ss'}^j &= (-1)^j e_s^j, & (ss') \cdot e_{ss'ss's}^j &= e_{s'ss'}^j. \end{aligned}$$

Similarly, the Yetter–Drinfeld modules whose associated braided spaces correspond to $(T_2, q^{\pm})$ are the modules U_j' for $j \in \{0, 1\}$ whose underlying vector space is $\mathbb{C}e_{s'}^j \oplus \mathbb{C}e_{ss's}^j \oplus \mathbb{C}e_{s'ss'ss'}^j$ where e_g is in degree g for $g \in W$, and the action is given by

$$\begin{aligned} s \cdot e_{s'}^j &= -e_{ss's}^j, & s \cdot e_{ss's}^j &= -e_{s'}^j, & s \cdot e_{s'ss'ss'}^j &= (-1)^{j+1} e_{s'ss'ss's}^j, \\ (ss') \cdot e_{s'}^j &= e_{ss's}^j, & (ss') \cdot e_{ss's}^j &= e_{s'ss'ss'}^j, & (ss') \cdot e_{ss'ss's}^j &= (-1)^j e_{s'}^j. \end{aligned}$$

In addition, to consider the braiding on sums of irreducible modules, we need to take into account the full action of W on the modules V_0 and $V_{3,1}$ following [1, Theorem 3.1]. According to [37, 5.3] we have $s \cdot v_{\pm 1,1} = v_{\mp 1,1}$ on $V_{3,1}$ and two possibilities for extending the action on V_0 from C to W: we denote the two extensions by $V_0^j := \mathbb{C}v_0^j$ for $j \in \{0, 1\}$ and set $s \cdot v_0^j = (-1)^j v_0^j$.

Twisting Yetter–Drinfeld modules as in Remark 2.4(4), by the automorphism τ of W that swaps s and s', preserves the isomorphism class of $V_{3,1}$ and interchanges V_0^0 and V_0^1. For $j = 0, 1$, it interchanges U_j and U_j'.

Proposition 4.9. *Let $W = I_2(6)$ and V be a Yetter–Drinfeld module for which* $\dim \mathcal{B}(V) < \infty$. *Then V is isomorphic to one of the modules*

- $V_{3,1}^{\oplus a} \oplus (V_0^0)^{\oplus b} \oplus (V_0^1)^{\oplus c}$ *for* $a, b, c \geqslant 0$ *and* $a+b+c \geqslant 1$;
- $U_j \oplus (V_0^j)^{\oplus a}$ *or* $U_j' \oplus (V_0^{1-j})^{\oplus a}$ *for* $j \in \{1, 2\}$ *and* $a \geqslant 0$;
- $U \oplus V_{3,1}$ *for* $U \in \{U_0, U_1, U_0', U_1'\}$;

and $\mathcal{B}(V)$ is, respectively, $\big(\bigwedge V_{3,1}\big)^{\widetilde{\otimes} a} \widetilde{\otimes} \big(\bigwedge V_0^0\big)^{\widetilde{\otimes} b} \otimes \big(\bigwedge V_0^1\big)^{\widetilde{\otimes} c}$; $\mathrm{FK}_3 \widetilde{\otimes} \big(\bigwedge V_0^0\big)^{\widetilde{\otimes} a}$ *and the* 2304*-dimensional Nichols algebra in* [30, Theorem 8.2], *where $\widetilde{\otimes}$ denotes a twisted tensor product.*

Proof. By [1, Table 2], if V is irreducible then it is either $V_{3,1}$, V_0^j, U_j, or U_j' for $j \in \{0, 1\}$. Sums of copies of V_0^0, V_0^1 and of $V_{3,1}$ can be handled as in the proof of Theorem 4.8 because their grading is supported in C. Observe that the supports of U_0 and U_1 generate a group isomorphic to $\mathbb{S}_3$. Thus, by [6, Theorem 4.8] if $\dim \mathcal{B}(U_0^a \oplus U_1^b) < \infty$ for some $(a, b) \neq (0, 0)$, then $a+b=1$. The same argument applies to U_0' and U_1'. For $\sigma \in T_1$ and $j, j' \in \{0, 1\}$ we have

$$c^2(e_\sigma^j \otimes v_0^{j'}) = (-1)^{j+j'} e_\sigma^j \otimes v_0^{j'}$$

so $\mathcal{B}((V_0^{j'})^{\oplus k} \oplus U_j)$ is a twisted tensor product of $\mathcal{B}((V_0^{j'})^{\oplus k})$ and $\mathcal{B}(U_j)$ if and only if $j = j'$. Remark 2.4(4) implies that the same property holds for the pair $V_0^{1-j} = (V_0^j)^\tau$ and $U_j' = U_j^\tau$.

For all other pairs (X, Y) of irreducible modules the support of $X \oplus Y$ generates W and $(c^2 - \mathrm{id})(X \oplus Y) \neq 0$. Under these conditions, [28, Corollary 7.2] gives a precise list of the possible supports for $X \oplus Y$ such that $\dim \mathcal{B}(X \oplus Y) < \infty$ and for each support, further conditions on W. In particular, if the size of the support is 6, then W must be a quotient of the group Γ_4 generated by a, b, ν, such that $\nu^4 = 1$, $\nu \mathrm{a} = \mathrm{a}\nu^{-1}$, $\nu \mathrm{b} = \mathrm{b}\nu^{-1}$, and $\mathrm{ba} = \nu \mathrm{ab}$. A direct calculation shows that this is not the case. Hence, $\dim \mathcal{B}(X \oplus Y) = \infty$ for $(X, Y) = (U_j, U_{j'}')$ for any $j, j' \in \{0, 1\}$. We are left with the pairs (U_0, V_0^1), (U_1, V_0^0), $(U_0, V_{3,1})$, and $(U_1, V_{3,1})$ and their twistings by τ. As a rack, their support is isomorphic to the one denoted by $Z_3^{3,1}$ in loc. cit. By [30, Theorem 2.1], the only pair of Yetter–Drinfeld modules X and Y of dimension, respectively, 3 and 1 and such that $X \oplus Y$ is supported by $Z_3^{3,1}$ is the one occurring in [30, Example 1.10]. There, it is required that, for z in the support of Y and g in the support of X, there holds $1 - \rho(z)\sigma(g) + (\rho(z)\sigma(g))^2 = 0$, where ρ denotes the action on the homogeneous component X_g and σ denotes the action on the homogeneous component Y_z. This cannot apply to the pairs (U_0, V_0^1) and (U_1, V_0^0) because the supports contain only involutions. The pair $(U_0, V_{3,1})$ occurs in [30, Example 1.11] for $g = s$, $z = (ss')^3$

and $\epsilon = (ss')^2$, hence $\mathcal{B}(U_0 \oplus V_{3,1})$ is the algebra described in [30, Theorem 8.4] whose dimension is 2304. Now, as (T_1, q^-) and (T_1, q^+) are twist equivalent by Theorem 2.8, the cocycles corresponding to U_0 and U_1 are twist equivalent, and therefore the cocycle corresponding to $U_0 \oplus V_{3,1}$ is twist equivalent to the cocycle corresponding to $U_1 \oplus V_{3,1}$, so $\mathcal{B}(U_0 \oplus V_{3,1})$ and $\mathcal{B}(U_1 \oplus V_{3,1})$ are twist equivalent. We finally need to consider the Yetter–Drinfeld modules of the form $U_0 \oplus V_{3,1}^{\oplus k}$ for $k > 1$. They are all braid indecomposable because $(\mathrm{id} - c^2)(U_0 \otimes V_{3,1}) \neq 0$, and $V_{3,1}$ is induced from a 2-dimensional representation of W, that is the centralizer of $(ss')^3$. Hence this case is ruled out by [29, Theorem 2.5]. □

Since the cases of $I_2(3) = \mathbb{S}_3$ and $I_2(4m)$ are to be found in [6; 13; 20], Corollary 4.7, Theorem 4.8 and Proposition 4.9 conclude the classification of finite-dimensional Nichols algebras over dihedral groups.

Acknowledgements

We wish to thank N. Andruskiewitsch for useful discussions and I. Heckenberger for pointing out the content of Remark 4.5.

This research has been carried out during an internship of Gabriel Maret at the Scuola Galileiana di Studi Superiori of the University of Padova, in the framework of the Programme Italie of the Ecole Normale Supérieure de Paris.

This project has been funded by the European Union - Next Generation EU under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.1 - Call PRIN 2022 No. 104 of February 2, 2022 of Italian Ministry of University and Research; Project 2022S8SSW2 (subject area: PE - Physical Sciences and Engineering) “Algebraic and geometric aspects of Lie theory”.

References

[1] N. Andruskiewitsch and F. Fantino, “On pointed Hopf algebras associated with alternating and dihedral groups”, *Rev. Un. Mat. Argentina* **48**:3 (2007), 57–71. MR Zbl

[2] N. Andruskiewitsch and M. Graña, “From racks to pointed Hopf algebras”, *Adv. Math.* **178**:2 (2003), 177–243. MR Zbl

[3] N. Andruskiewitsch and M. Graña, “Braided Hopf algebras over non-abelian finite groups”, pp. 45–78 , 1999. MR Zbl

[4] N. Andruskiewitsch and S. Zhang, “On pointed Hopf algebras associated to some conjugacy classes in $\mathbb{S}_n$”, *Proc. Amer. Math. Soc.* **135**:9 (2007), 2723–2731. MR Zbl

[5] N. Andruskiewitsch, F. Fantino, G. A. García, and L. Vendramin, “On twisted homogeneous racks of type D”, *Rev. Un. Mat. Argentina* **51**:2 (2010), 1–16. MR Zbl

[6] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, “The Nichols algebra of a semisimple Yetter–Drinfeld module”, *Amer. J. Math.* **132**:6 (2010), 1493–1547. MR Zbl

[7] N. Andruskiewitsch, F. Fantino, G. A. García, and L. Vendramin, "On Nichols algebras associated to simple racks", pp. 31–56 in *Groups, algebras and applications*, edited by C. P. Milies, Contemp. Math. **537**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[8] N. Andruskiewitsch, F. Fantino, M. Graña, and L. Vendramin, "Finite-dimensional pointed Hopf algebras with alternating groups are trivial", *Ann. Mat. Pura Appl.* (4) **190**:2 (2011), 225–245. MR Zbl

[9] N. Andruskiewitsch, G. Carnovale, and G. A. García, "Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type, III: Semisimple classes in $\mathbf{PSL}_n(q)$", *Rev. Mat. Iberoam.* **33**:3 (2017), 995–1024. MR Zbl

[10] N. Andruskiewitsch, I. Heckenberger, and L. Vendramin, "Pointed Hopf algebras of odd dimension and Nichols algebras over solvable groups", preprint, 2024. arXiv 2411.02304v1

[11] C. Bärligea, "On the dimension of the Fomin–Kirillov algebra and related algebras", preprint, 2024. arXiv 2001.04597v2

[12] Y. Bazlov, "Nichols–Woronowicz algebra model for Schubert calculus on Coxeter groups", *J. Algebra* **297**:2 (2006), 372–399. MR Zbl

[13] S. Beltrán Cubillos, *Álgebras de Nichols sobre grupos diedrales y pecios kthulhu en grupos esporádicos*, Ph.D. thesis, Universidad Nacional de Córdoba, 2020, available at http://hdl.handle.net/11086/17698.

[14] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer, 2005. MR Zbl

[15] N. Bourbaki, *Groupes et algèbres de Lie*, *Chapitres* 4–6, Masson, Paris, 1981. MR Zbl

[16] J. S. Carter, S. Kamada, and M. Saito, "Diagrammatic computations for quandles and cocycle knot invariants", pp. 51–74 in *Diagrammatic morphisms and applications* (San Francisco, CA, 2000), edited by D. E. Radford et al., Contemp. Math. **318**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[17] C. Dold, *On the twist-equivalence of certain quadratic algebras associated to finite irreducible Coxeter groups*, Ph.D. thesis, The University of Manchester, 2016, available at https://pure.manchester.ac.uk/ws/portalfiles/portal/54584027/FULL_TEXT.PDF.

[18] M. Dyer, "Reflection subgroups of Coxeter systems", *J. Algebra* **135**:1 (1990), 57–73. MR Zbl

[19] J. S. Ellenberg, T. Tran, and C. Westerland, "Fox–Neuwirth–Fuks cells, quantum shuffle algebras, and Malle's conjecture for function fields", preprint, 2023. arXiv 1701.04541v2

[20] F. Fantino and G. A. Garcia, "On pointed Hopf algebras over dihedral groups", *Pacific J. Math.* **252**:1 (2011), 69–91. MR Zbl

[21] S. Fomin and A. N. Kirillov, "Quadratic algebras, Dunkl elements, and Schubert calculus", pp. 147–182 in *Advances in geometry*, edited by J.-L. Brylinski et al., Progr. Math. **172**, Birkhäuser, Boston, 1999. MR Zbl

[22] G. A. García and A. García Iglesias, "Finite-dimensional pointed Hopf algebras over $\mathbb{S}_4$", *Israel J. Math.* **183** (2011), 417–444. MR Zbl

[23] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Mathematical Society Monographs. New Series **21**, Oxford University Press, New York, 2000. MR Zbl

[24] M. Graña, "A freeness theorem for Nichols algebras", *J. Algebra* **231**:1 (2000), 235–257. MR Zbl

[25] I. Heckenberger, "Rank 2 Nichols algebras with finite arithmetic root system", *Algebr. Represent. Theory* **11**:2 (2008), 115–132. MR Zbl

[26] I. Heckenberger, "Classification of arithmetic root systems", *Adv. Math.* **220**:1 (2009), 59–124. MR Zbl

[27] I. Heckenberger and H.-J. Schneider, *Hopf algebras and root systems*, Mathematical Surveys and Monographs **247**, American Mathematical Society, Providence, RI, 2020. MR Zbl

[28] I. Heckenberger and L. Vendramin, "Nichols algebras over groups with finite root system of rank two III", *J. Algebra* **422** (2015), 223–256. MR Zbl

[29] I. Heckenberger and L. Vendramin, "A classification of Nichols algebras of semisimple Yetter–Drinfeld modules over non-abelian groups", *J. Eur. Math. Soc.* (*JEMS*) **19**:2 (2017), 299–356. MR Zbl

[30] I. Heckenberger and L. Vendramin, "The classification of Nichols algebras over groups with finite root system of rank two", *J. Eur. Math. Soc.* (*JEMS*) **19**:7 (2017), 1977–2017. MR Zbl

[31] I. Heckenberger, E. Meir, and L. Vendramin, "Finite-dimensional Nichols algebras of simple Yetter–Drinfeld modules (over groups) of prime dimension", *Adv. Math.* **444** (2024), art. id. 109637. MR Zbl

[32] R. B. Howlett, "On the Schur multipliers of Coxeter groups", *J. London Math. Soc.* (2) **38**:2 (1988), 263–276. MR Zbl

[33] M. Kapranov and V. Schechtman, "Shuffle algebras and perverse sheaves", *Pure Appl. Math. Q.* **16**:3 (2020), 573–657. MR Zbl

[34] S. Majid and R. Oeckl, "Twisting of quantum differentials and the Planck scale Hopf algebra", *Comm. Math. Phys.* **205**:3 (1999), 617–655. MR Zbl

[35] G. Michel, *The cohomology of dihedral Nichols algebras using Koszul complexes*, 2020. MR

[36] A. Milinski and H.-J. Schneider, "Pointed indecomposable Hopf algebras over Coxeter groups", pp. 215–236 in *New trends in Hopf algebra theory* (La Falda, Argentina, 1999), edited by N. Andruskiewitsch et al., Contemp. Math. **267**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[37] J.-P. Serre, *Représentations linéaires des groupes finis*, Hermann, Paris, 1971. MR Zbl

[38] J. R. Stembridge, "Quasi-minuscule quotients and reduced words for reflections", *J. Algebraic Combin.* **13**:3 (2001), 275–293. MR Zbl

[39] L. Vendramin, "Nichols algebras associated to the transpositions of the symmetric group are twist-equivalent", *Proc. Amer. Math. Soc.* **140**:11 (2012), 3715–3723. MR Zbl

Received August 11, 2024. Revised November 10, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.253

GELFAND–CETLIN ABELIANIZATIONS OF SYMPLECTIC QUOTIENTS

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We show that generic symplectic quotients of a Hamiltonian G-space M by the action of a compact connected Lie group G are also symplectic quotients of the same manifold M by a compact torus. The torus action in question arises from certain integrable systems on $\mathfrak{g}^*$, the dual of the Lie algebra of G. Examples of such integrable systems include the Gelfand–Cetlin systems of Guillemin and Sternberg (1980; 1983) in the case of unitary and special orthogonal groups, and certain integrable systems constructed for all compact connected Lie groups by Hoffman and Lane (2023). Our abelianization result holds for smooth quotients, and more generally for quotients which are stratified symplectic spaces in the sense of Sjamaar and Lerman (1991).

1. Introduction

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that M is a Hamiltonian G-space, i.e., a symplectic manifold equipped with a symplectic action of G and equivariant moment map $\mu : M \to \mathfrak{g}^*$. The symplectic or Marsden–Weinstein [9] quotient of M by G at level $\xi \in \mathfrak{g}^*$ is the topological space

$$M/\!/_\xi G := \mu^{-1}(\xi)/G_\xi,$$

where $G_\xi \subset G$ is the G-stabilizer of ξ. If G acts freely on $\mu^{-1}(\xi)$, then $M/\!/_\xi G$ is a smooth symplectic manifold. In the absence of this freeness assumption, $M/\!/_\xi G$ is a stratified symplectic space in the sense of Sjamaar and Lerman [10].

The purpose of this paper is to show that certain integrable systems on $\mathfrak{g}^*$ allow us to express generic symplectic quotients of a Hamiltonian G-space M as symplectic quotients of the same manifold M by the action of a compact torus. Such integrable systems include the Gelfand–Cetlin systems constructed by Guillemin and Sternberg [3; 4] for unitary and special orthogonal groups, as well as the more recent generalizations of Gelfand–Cetlin systems by Hoffman and Lane [8] to arbitrary Lie type.

MSC2020: primary 53D20; secondary 17B80.
Keywords: symplectic quotient, Gelfand–Cetlin system, stratified symplectic space.

An example of our main result arises in classical mechanics [2]. Suppose that we are given a Hamiltonian SO(3)-space M with an invariant Hamiltonian function $H : M \to \mathbb{R}$. The SO(3)-action gives rise to two Poisson-commuting conserved quantities: the total angular momentum, and the angular momentum in some fixed direction in the Lie algebra of SO(3) corresponding to a choice of maximal torus. These quantities give the components of a moment map for a densely defined 2-torus action on M, coming from the Gelfand–Cetlin system of Guillemin and Sternberg for the case of SO(3).[1] Our main result shows that for a nonzero value ξ of the angular momentum, the symplectic quotient $M/\!/_{\xi}\,\mathrm{SO}(3)$ coincides with an appropriate symplectic quotient of M under the densely defined torus action. See [2] for more examples of these techniques.

1.1. ***Main result.*** We introduce the notion of a Gelfand–Cetlin datum $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ in Definition 1. This amounts to λ_{big} being a continuous map on $\mathfrak{g}^*$ that restricts to a Poisson moment map for a Hamiltonian action of a compact torus $\mathbb{T}_{\text{big}}$ on an open dense subset $\mathfrak{g}^*_{\text{s-reg}} \subset \mathfrak{g}^*$, along with some extra conditions that capture salient properties of the classical Gelfand–Cetlin systems. One of these conditions is that the open, symplectic submanifold $M_{\text{s-reg}} := \mu^{-1}(\mathfrak{g}^*_{\text{s-reg}}) \subset M$ be a Hamiltonian $\mathbb{T}_{\text{big}}$-space with moment map $\lambda_M := (\lambda_{\text{big}} \circ \mu)|_{M_{\text{s-reg}}}$ for any Hamiltonian G-space M with moment map $\mu : M \to \mathfrak{g}^*$. The results of Guillemin and Sternberg [3; 4] imply that Gelfand–Cetlin data exist for all unitary and special orthogonal groups, while more recent results of Hoffman and Lane [8] imply that such data exist in all Lie types.

The following is the main result of our paper.

Theorem. *Let G be a compact connected Lie group, and M a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ is a Gelfand–Cetlin datum, and consider a point $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *The torus $\mathbb{T}_{\text{big}}$ acts freely on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ if and only if G_ξ acts freely on $\mu^{-1}(\xi)$. In this case, there is a canonical symplectomorphism $M/\!/_{\xi}G \cong M_{\text{s-reg}}/\!/_{\lambda_{\text{big}}(\xi)}\mathbb{T}_{\text{big}}$.*

(ii) *There is a canonical isomorphism $M/\!/_{\xi}\, G \cong M_{\text{s-reg}}\, /\!/_{\lambda_{\text{big}}(\xi)}\, \mathbb{T}_{\text{big}}$ of stratified symplectic spaces.*

Part (ii) is strictly more general than (i). Part (i) is included for the sake of exposition and accessibility. One may regard this theorem as an approach to abelianizing the generic symplectic quotients of a Hamiltonian G-space M, i.e., to presenting such quotients as symplectic quotients by a compact torus.

[1] The square of the total angular momentum is a smooth function, but the orbits of its Hamiltonian flow do not have constant period, and so it does not generate a circle action. Taking the square root gives a function whose Hamiltonian flow generates a circle action, but which is only continuous, not differentiable, at zero. It therefore does not define a Hamiltonian flow at zero.

1.2. ***Comparison to symplectic implosion.*** By means of *symplectic implosion*, Guillemin, Jeffrey, and Sjamaar [5] offer a different approach to abelianizing symplectic quotients. Their construction replaces G and a Hamiltonian G-space M with a maximal torus $T \subset G$ and the *imploded cross-section* M_{impl}, respectively; the latter is a stratified symplectic Hamiltonian T-space whose dimension is generally less than that of M. Symplectic quotients of M by G are then shown to be realizable as symplectic quotients of M_{impl} by T. Our construction retains the same manifold M and replaces G with a torus $\mathbb{T}_{\mathrm{big}}$.

The imploded cross-section $(T^*G)_{\mathrm{impl}}$ features prominently in [5]. This stratified symplectic space also features prominently in [8], where Hoffman and Lane show Gelfand–Cetlin data to exist in arbitrary Lie type. It is natural to suspect that this coincidence is a shadow of some deeper connection between our work and symplectic implosion.

1.3. ***Organization.*** Section 2 briefly establishes some of our conventions concerning Lie theory and Hamiltonian geometry. Section 3 subsequently motivates and contextualizes the notion of a Gelfand–Cetlin datum. Our main result is then proved in Section 4 for smooth quotients. A generalization to stratified symplectic spaces is formulated and proved in Section 5.

2. Background and conventions

This section establishes some of our notation and conventions regarding Lie theory and Hamiltonian geometry.

2.1. ***Tori.*** The Lie algebra of the unitary group $\mathrm{U}(1)$ is the real vector space $i\mathbb{R} \subset \mathbb{C}$ of purely imaginary numbers. We will identify this vector space with $\mathbb{R}$ in the obvious way. It follows that $\mathbb{R}^k$ is the Lie algebra of $\mathrm{U}(1)^k$ for all nonnegative integers k, and that

$$\mathbb{R}^k \to \mathrm{U}(1)^k, \quad (x_1, \ldots, x_k) \mapsto (e^{ix_1}, \ldots, e^{ix_k})$$

is the exponential map for $\mathrm{U}(1)^k$. In certain contexts, we will implicitly use the dot product to regard $\mathbb{R}^k$ as the dual of the Lie algebra of $\mathrm{U}(1)^k$.

2.2. ***General compact connected Lie groups.*** Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and rank ℓ. One has the adjoint representations $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ and $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, as well as the coadjoint representations $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ and $\mathrm{ad}^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$. The G-representations induce G-actions on $\mathfrak{g}$ and $\mathfrak{g}^*$, and thereby give rise to stabilizer subgroups

$$G_x := \{g \in G : \mathrm{Ad}_g(x) = x\} \quad \text{and} \quad G_\xi := \{g \in G : \mathrm{Ad}^*_g(\xi) = \xi\}$$

of G for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. On the other hand, the $\mathfrak{g}$-representations allow us to define centralizers

$$\mathfrak{g}_x := \{y \in \mathfrak{g} : \mathrm{ad}_y(x) = 0\} \quad \text{and} \quad \mathfrak{g}_\xi := \{y \in \mathfrak{g} : \mathrm{ad}_y^*(\xi) = 0\}$$

for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. It follows that $\mathfrak{g}_x$ (resp. $\mathfrak{g}_\xi$) is the Lie algebra of G_x (resp. G_ξ). Let us also note that $\dim \mathfrak{g}_x \geq \ell$ and $\dim \mathfrak{g}_\xi \geq \ell$ for all $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. The regular loci

$$\mathfrak{g}_{\mathrm{reg}} := \{x \in \mathfrak{g} : \dim \mathfrak{g}_x = \ell\} \quad \text{and} \quad \mathfrak{g}^*_{\mathrm{reg}} := \{\xi \in \mathfrak{g}^* : \dim \mathfrak{g}_\xi = \ell\}$$

are open, dense, G-invariant subsets of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. An element $x \in \mathfrak{g}$ (resp. $\xi \in \mathfrak{g}^*$) then belongs to $\mathfrak{g}_{\mathrm{reg}}$ (resp. $\mathfrak{g}^*_{\mathrm{reg}}$) if and only if $\mathfrak{g}_x$ (resp. $\mathfrak{g}_\xi$) is a Cartan subalgebra of $\mathfrak{g}$. This is equivalent to G_x (resp. G_ξ) being a maximal torus of G.

A few remarks on maximal tori and Cartan subalgebras are warranted. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra, and write $T \subset G$ for the maximal torus with Lie algebra $\mathfrak{t}$. The exponential map $\exp : \mathfrak{g} \to G$ then restricts to a surjective homomorphism $\exp|_{\mathfrak{t}} : \mathfrak{t} \to T$ of abelian groups. The kernel of the latter is a free $\mathbb{Z}$-submodule of $\mathfrak{t}$ with rank equal to ℓ. It follows that the same is true of

$$\Lambda_{\mathfrak{t}} := \frac{1}{2\pi} \ker(\exp|_{\mathfrak{t}}) \subset \mathfrak{t}.$$

2.3. ***Hamiltonian geometry.*** Suppose that (M, σ) is a Poisson manifold, that is, $\sigma \in H^0(M, \Lambda^2 TM)$ is a Poisson bivector field on the manifold M. Note that σ may be regarded as a skew-symmetric bilinear map from two copies of T^*M to the trivial rank-1 vector bundle over M. Contracting σ with cotangent vectors in the first argument then determines a vector bundle morphism $\sigma^\vee : T^*M \to TM$. One calls (M, σ) *nondegenerate* if $\sigma^\vee$ is an isomorphism. In this case, $(\sigma^\vee)^{-1} = \omega^\vee$ for a unique symplectic form $\omega \in H^0(M, \Lambda^2 T^*M)$, where $\omega^\vee : TM \to T^*M$ is the vector bundle morphism obtained by contracting ω with tangent vectors in the first argument. This process gives rise to a bijective correspondence between symplectic structures on M and nondegenerate Poisson structures on M. We will thereby make no distinction between symplectic manifolds and nondegenerate Poisson manifolds.

If (M, σ) is a Poisson manifold, then σ can be recovered from the Poisson bracket $\{\cdot, \cdot\}$ that it induces. This bracket associates to smooth functions $f_1, f_2 : M \to \mathbb{R}$ the smooth function

$$\{f_1, f_2\} := \sigma(\mathrm{d}f_1 \wedge \mathrm{d}f_2) : M \to \mathbb{R}.$$

At the same time, one defines the Hamiltonian vector field of a smooth function $f : M \to \mathbb{R}$ by $X_f := -\sigma^\vee(\mathrm{d}f) \in H^0(M, TM)$. It follows that

$$\{f_1, f_2\} = -X_{f_1}(f_2) = X_{f_2}(f_1)$$

for all smooth functions $f_1, f_2 : M \to \mathbb{R}$.

Now let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and exponential map $\exp : \mathfrak{g} \to G$. If G acts smoothly on a manifold M, then each $\eta \in \mathfrak{g}$ determines a generating vector field $\eta_M \in H^0(M, TM)$ by

$$(\eta_M)_m := \frac{d}{dt}\Big|_{t=0} \exp(-t\eta) \cdot m$$

for all $m \in M$. A Poisson manifold (M, σ) with a smooth G-action will be called a *Poisson Hamiltonian G-space* if σ is G-invariant and M comes equipped with a *moment map*. This last term refers to G-equivariant smooth map $\mu : M \to \mathfrak{g}^*$ satisfying $X_{\mu^\eta} = \eta_M$ for all $\eta \in \mathfrak{g}$, where $\mu^\eta : M \to \mathbb{R}$ is the result of pairing μ with η pointwise. We will reserve the term *Hamiltonian G-space* for a Poisson Hamiltonian G-space whose underlying Poisson structure is symplectic.

It will be advantageous to recall the Poisson Hamiltonian G-space structure on $\mathfrak{g}^*$. The Poisson bracket on $\mathfrak{g}^*$ is given by

$$\{f_1, f_2\}(\xi) = \xi([\mathrm{d}_\xi f_1, \mathrm{d}_\xi f_2])$$

for all smooth functions $f_1, f_2 : \mathfrak{g}^* \to \mathbb{R}$ and points $\xi \in \mathfrak{g}^*$, where

$$\mathrm{d}_\xi f_1, \mathrm{d}_\xi f_2 \in (\mathfrak{g}^*)^* = \mathfrak{g}$$

denote the differentials of f_1, f_2 at ξ, respectively. One finds that $\mathfrak{g}^*$ is a Poisson Hamiltonian G-space with respect to the coadjoint action, and with the identity $\mathfrak{g}^* \to \mathfrak{g}^*$ serving as the Poisson moment map.

3. Gelfand–Cetlin data

In this section, we define Gelfand–Cetlin data and introduce their main properties. This begins with the definition itself in Section 3.1. The existence of Gelfand–Cetlin data is addressed in Section 3.2, while concrete techniques for constructing such data are discussed in Sections 3.3 and 3.4. In Section 3.5, we describe concrete Gelfand–Cetlin data for unitary groups.

3.1. ***Definition and relation to integrable systems.*** Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and rank ℓ. Consider the quantities

$$\mathrm{u} := \tfrac{1}{2}(\dim \mathfrak{g} - \ell) \quad \text{and} \quad \mathrm{b} := \tfrac{1}{2}(\dim \mathfrak{g} + \ell),$$

and introduce the following tori of *small*, *intermediate*, and *big* ranks:

$$\mathbb{T}_{\mathrm{small}} := \mathrm{U}(1)^\ell, \quad \mathbb{T}_{\mathrm{int}} := \mathrm{U}(1)^{\mathrm{u}}, \quad \text{and} \quad \mathbb{T}_{\mathrm{big}} := \mathbb{T}_{\mathrm{small}} \times \mathbb{T}_{\mathrm{int}} \cong \mathrm{U}(1)^{\mathrm{b}}.$$

The respective Lie algebras of these tori are

$$\mathbb{R}_{\mathrm{small}} := \mathbb{R}^\ell, \quad \mathbb{R}_{\mathrm{int}} := \mathbb{R}^{\mathrm{u}}, \quad \text{and} \quad \mathbb{R}_{\mathrm{big}} := \mathbb{R}_{\mathrm{small}} \times \mathbb{R}_{\mathrm{int}} \cong \mathbb{R}^{\mathrm{b}}.$$

Definition 1. A *Gelfand–Cetlin datum* is a pair $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$, consisting of a continuous map $\lambda_{\text{big}} = (\lambda_1, \dots, \lambda_b) : \mathfrak{g}^* \to \mathbb{R}_{\text{big}}$ and open dense subset $\mathfrak{g}^*_{\text{s-reg}} \subset \mathfrak{g}^*$ that satisfy the following conditions:

(i) $\lambda_1, \dots, \lambda_\ell$ are G-invariant on $\mathfrak{g}^*$ and smooth on $\mathfrak{g}^*_{\text{reg}}$.

(ii) $\{\mathrm{d}_\xi \lambda_1, \dots, \mathrm{d}_\xi \lambda_\ell\}$ is a $\mathbb{Z}$-basis of the lattice $\Lambda_{\mathfrak{g}_\xi} \subset \mathfrak{g}_\xi$ for all $\xi \in \mathfrak{g}^*_{\text{reg}}$.

(iii) $\mathfrak{g}^*_{\text{s-reg}} \subset \mathfrak{g}^*_{\text{reg}}$.

(iv) $\lambda_{\text{big}}|_{\mathfrak{g}^*_{\text{s-reg}}} : \mathfrak{g}^*_{\text{s-reg}} \to \mathbb{R}_{\text{big}}$ is a smooth submersion and moment map for a Poisson Hamiltonian $\mathbb{T}_{\text{big}}$-space structure on $\mathfrak{g}^*_{\text{s-reg}}$.

(v) $\lambda_{\text{big}}|_{\mathfrak{g}^*_{\text{s-reg}}} : \mathfrak{g}^*_{\text{s-reg}} \to \lambda_{\text{big}}(\mathfrak{g}^*_{\text{s-reg}})$ is a principal $\mathbb{T}_{\text{int}}$-bundle.

(vi) If M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$, then

$$(\lambda_{\text{big}} \circ \mu)|_{\mu^{-1}(\mathfrak{g}^*_{\text{s-reg}})} : \mu^{-1}(\mathfrak{g}^*_{\text{s-reg}}) \to \mathbb{R}_{\text{big}}$$

is a moment map for a Hamiltonian $\mathbb{T}_{\text{big}}$-space structure on $\mu^{-1}(\mathfrak{g}_{\text{s-reg}})$.

In this case, we adopt the notation

$$\lambda_{\text{small}} := (\lambda_1, \dots, \lambda_\ell) : \mathfrak{g}^* \to \mathbb{R}_{\text{small}}, \quad \lambda_{\text{int}} := (\lambda_{\ell+1}, \dots, \lambda_b) : \mathfrak{g}^* \to \mathbb{R}_{\text{int}},$$
$$M_{\text{s-reg}} := \mu^{-1}(\mathfrak{g}^*_{\text{s-reg}}), \quad \text{and} \quad \lambda_M := (\lambda_{\text{big}} \circ \mu)|_{M_{\text{s-reg}}} : M_{\text{s-reg}} \to \mathbb{R}_{\text{big}}.$$

We also refer to the elements of $\mathfrak{g}^*_{\text{s-reg}}$ as the *strongly regular* elements of $\mathfrak{g}^*$.

Remark 2. Condition (v) in Definition 1 is only slightly weaker than the existence of global action-angle coordinates on $\mathfrak{g}^*_{\text{s-reg}}$. This existence question features prominently in [1; 8].

It is instructive to consider this definition in relation to the theory of completely integrable systems. One is thereby led to the following result.

Proposition 3. *Let $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ be a Gelfand–Cetlin datum. If $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit and $\mathcal{O}_{\text{s-reg}} := \mathcal{O} \cap \mathfrak{g}^*_{\text{s-reg}}$, then*

$$\lambda_{\text{int}}|_{\mathcal{O}_{\text{s-reg}}} : \mathcal{O}_{\text{s-reg}} \to \lambda_{\text{int}}(\mathcal{O}_{\text{s-reg}}) \subset \mathbb{R}_{\text{int}}$$

is a completely integrable system, principal $\mathbb{T}_{\text{int}}$-bundle, and moment map for a Hamiltonian action of $\mathbb{T}_{\text{int}}$ on $\mathcal{O}_{\text{s-reg}}$.

Proof. Note that the Hamiltonian vector field of any smooth function $\mathfrak{g}^* \to \mathbb{R}$ is tangent to $\mathcal{O}$. It follows that $\mathcal{O}_{\text{s-reg}}$ is stable under the action of $\mathbb{T}_{\text{big}}$ on $\mathfrak{g}^*_{\text{s-reg}}$. Definition 1(iv) now implies that $\mathcal{O}_{\text{s-reg}}$ is a Hamiltonian $\mathbb{T}_{\text{big}}$-space with moment map $\lambda_{\text{big}}|_{\mathcal{O}_{\text{s-reg}}}$. We conclude that $\lambda_{\text{int}}|_{\mathcal{O}_{\text{s-reg}}}$ is a moment map for the Hamiltonian action of $\mathbb{T}_{\text{int}} \subset \mathbb{T}_{\text{big}}$ on $\mathcal{O}_{\text{s-reg}}$.

Since λ_{small} is constant-valued on $\mathcal{O}$, Definition 1(v) tells us that

$$\lambda_{\text{int}}|_{\mathcal{O}_{\text{s-reg}}} : \mathcal{O}_{\text{s-reg}} \to \lambda_{\text{int}}(\mathcal{O}_{\text{s-reg}})$$

is a principal $\mathbb{T}_{\text{int}}$-bundle. It therefore remains only to prove that

$$\dim \mathbb{T}_{\text{int}} = \tfrac{1}{2}\dim \mathcal{O}_{\text{s-reg}}.$$

This follows immediately from the fact that $\dim \mathbb{T}_{\text{int}} = \mathrm{u} = \frac{1}{2}(\dim \mathfrak{g} - \ell)$. □

3.2. ***Existence of Gelfand–Cetlin data.*** It is natural to wonder about the generality in which Gelfand–Cetlin data exist. The earliest constructions are due to Guillemin and Sternberg [3; 4], and apply to all unitary groups $\mathrm{U}(n)$ and special orthogonal groups $\mathrm{SO}(n)$. The underlying techniques are based on Thimm's method, as described in [4]. Further details are outlined in Sections 3.3–3.5. The case of symplectic groups is considerably more subtle and addressed in Harada's paper [6].

Some recent work of Hoffman and Lane [8] implies the existence of Gelfand–Cetlin data for an arbitrary compact connected Lie group G. In more detail, one may consider completely integrable systems on an arbitrary Poisson manifold. A completely integrable system on $\mathfrak{g}^*$ is thereby a continuous map $\phi : \mathfrak{g}^* \to \mathbb{R}_{\text{big}}$ with the following property: there exists an open dense subset $U \subset \mathfrak{g}^*_{\text{reg}}$ such that $\phi|_U$ is a smooth submersion with pairwise Poisson-commuting component functions. A Gelfand–Cetlin datum is thereby a completely integrable system on $\mathfrak{g}^*$. On the other hand, suppose that M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. One might hope for $\phi \circ \mu : M \to \mathbb{R}_{\text{big}}$ to be a completely integrable system on an open subset of M; such a system would be called *collective*. Another aspiration would be for $\phi \circ \mu : M \to \mathbb{R}_{\text{big}}$ to be the moment map of a Hamiltonian torus action on an open subset of M. This central theme of [8] is reflected in Definition 1: Gelfand–Cetlin data are defined in such a way as to induce Hamiltonian torus actions on open subsets of Hamiltonian G-spaces. The construction of Gelfand–Cetlin data is done in Sections 6.2 and 6.3 of [8]. Hoffman and Lane [8] begin by associating completely integrable systems on $\mathfrak{g}^*$ to *good valuations* on the *base affine space*; see [8, Data 5.2] and [8, Theorem 6.2]. Propositions 6.4 and 6.5 of [8] then show such systems to have several properties; these imply the existence of pairs $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ satisfying parts (i)–(iv) of Definition 1. To address part (v) of Definition 1, Hoffman and Lane introduce the notion of *toric contraction* for Hamiltonian G-spaces [8, Definition 6.7]. The existence of pairs $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ satisfying parts (i)–(v) then follows from Propositions 6.8 and 6.9 of [8].

The Hoffman–Lane paper is part of a broader program aimed at generalizing the results of Harada and Kaveh [7].

3.3. ***Construction of Gelfand–Cetlin data: integrality.*** We now discuss the construction of functions $\lambda_1, \ldots, \lambda_\ell : \mathfrak{g}^* \to \mathbb{R}$ satisfying conditions (i) and (ii) in Definition 1. Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and rank ℓ. Choose a G-invariant inner product on $\mathfrak{g}$, a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$, and a closed, fundamental Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$. The chamber $\mathfrak{t}_+$ is known to be a

fundamental domain for the adjoint action of G on $\mathfrak{g}$. Our inner product thereby identifies $\mathfrak{t}_+$ with a fundamental domain $\mathfrak{t}_+^* \subset \mathfrak{g}^*$ for the G-action on $\mathfrak{g}^*$. We may therefore define a continuous surjection $\pi : \mathfrak{g}^* \to \mathfrak{t}_+^*$ by the property that

$$(G \cdot \xi) \cap \mathfrak{t}_+^* = \{\pi(\xi)\} \quad \text{for all } \xi \in \mathfrak{g}^*,$$

where $G \cdot \xi \subset \mathfrak{g}^*$ is the coadjoint orbit of ξ. One sometimes calls π the *sweeping map* on $\mathfrak{g}^*$ with respect to $\mathfrak{t}$; its fibers are exactly the coadjoint orbits of G, and $\pi(\mathfrak{g}_{\text{reg}}^*)$ is the interior $(\mathfrak{t}_+^*)^\circ$ of $\mathfrak{t}_+^*$. One also finds that the commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_{\text{reg}}^* & \hookrightarrow & \mathfrak{g}^* \\ {\scriptstyle \pi|_{\mathfrak{g}_{\text{reg}}^*}}\downarrow & & \downarrow{\scriptstyle \pi} \\ (\mathfrak{t}_+^*)^\circ & \hookrightarrow & \mathfrak{t}_+^* \end{array}$$

is Cartesian. The left vertical map

$$\pi|_{\mathfrak{g}_{\text{reg}}^*} : \mathfrak{g}_{\text{reg}}^* \to (\mathfrak{t}_+^*)^\circ$$

is easily seen to be a smooth, surjective submersion.

Now choose a $\mathbb{Z}$-basis $\{\phi_1, \ldots, \phi_\ell\}$ of the $\mathbb{Z}$-submodule $\Lambda_{\mathfrak{t}} \subset \mathfrak{t}$. Note that the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ allows one to regard $\phi_1, \ldots, \phi_\ell$ as functions on $\mathfrak{t}_+^*$. With this in mind, each $k \in \{1, \ldots, \ell\}$ determines a function

$$\lambda_k := \phi_k \circ \pi : \mathfrak{g}^* \to \mathbb{R}.$$

The previous paragraph implies that $\lambda_1, \ldots, \lambda_\ell$ are smooth on $\mathfrak{g}_{\text{reg}}^*$, while being G-invariant and continuous as functions on $\mathfrak{g}^*$. Given any $\xi \in \mathfrak{g}_{\text{reg}}^*$, the differentials $\mathrm{d}_\xi \lambda_1, \ldots, \mathrm{d}_\xi \lambda_\ell \in (\mathfrak{g}^*)^* = \mathfrak{g}$ may be described as follows.

Proposition 4. *If $\xi \in \mathfrak{g}_{\text{reg}}^*$, then $\{\mathrm{d}_\xi \lambda_1, \ldots, \mathrm{d}_\xi \lambda_\ell\}$ is a $\mathbb{Z}$-basis of the lattice $\Lambda_{\mathfrak{g}_\xi} \subset \mathfrak{g}_\xi$.*

Proof. Choose $g \in G$ for which $\mathrm{Ad}_g^*(\xi) \in (\mathfrak{t}_+^*)^\circ$. Since each function λ_k is G-invariant, one has

$$\mathrm{d}_\xi \lambda_k = \mathrm{d}_{\mathrm{Ad}_g^*(\xi)} \lambda_k \circ \mathrm{Ad}_g^* = \mathrm{Ad}_{g^{-1}}(\mathrm{d}_{\mathrm{Ad}_g^*(\xi)} \lambda_k)$$

for all $k \in \{1, \ldots, \ell\}$. We also observe that $\mathrm{Ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ restricts to a $\mathbb{Z}$-module isomorphism $\lambda_{\mathfrak{t}} \xrightarrow{\cong} \lambda_{\mathfrak{g}_\xi}$. It therefore suffices to take $\xi \in (\mathfrak{t}_+^*)^\circ$ and prove that $\mathrm{d}_\xi \lambda_k = \phi_k$ for all $k \in \{1, \ldots, \ell\}$.

Assume that $\xi \in (\mathfrak{t}_+^*)^\circ$, and fix $k \in \{1, \ldots, \ell\}$. Our invariant inner product allows us to regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$. We then note that $\mathfrak{g}^* = \mathfrak{t}^* \oplus T_\xi(G \cdot \xi)$, and that $T_\xi(G \cdot \xi)$ is contained in the kernel of $\mathrm{d}_\xi \lambda_k$. Let us also note that $T_\xi(G \cdot \xi)$ is the annihilator of $\mathfrak{t}$ in $\mathfrak{g}^*$, and as such is contained in the kernel of ϕ_k. These last two sentences reduce us to proving that $\mathrm{d}_\xi \lambda_k(\eta) = \phi_k(\eta)$ for all $\eta \in \mathfrak{t}^*$. On the other hand, we have $\mathrm{d}_\xi \lambda_k = \phi_k \circ \mathrm{d}_\xi \pi$. It therefore suffices to prove that $\mathrm{d}_\xi \pi(\eta) = \eta$ for

all $\eta \in \mathfrak{t}^*$. But this is an immediate consequence of the following two observations: $(\mathfrak{t}_+^*)^\circ$ is an open subset of $\mathfrak{t}^*$, and $\pi(\eta) = \eta$ for all $\eta \in (\mathfrak{t}_+^*)^\circ$. □

3.4. ***Construction of Gelfand–Cetlin data: Thimm's method.*** Retain the objects and notation discussed in Section 3.3. Let $G = G_0 \supset G_1 \supset \cdots \supset G_m$ be a descending filtration of G by connected closed subgroups with respective Lie algebras $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m$. Let us also choose a Cartan subalgebra $\mathfrak{t}_j \subset \mathfrak{g}_j$ and closed, fundamental Weyl chamber $(\mathfrak{t}_j)_+ \subset \mathfrak{t}_j$ for each $j \in \{0, \ldots, m\}$. Our G-invariant inner product on $\mathfrak{g}$ gives rise to a G_j-module isomorphism $\mathfrak{g}_j \cong \mathfrak{g}_j^*$, by means of which $\mathfrak{t}_j$ and $(\mathfrak{t}_j)_+$ correspond to subsets $\mathfrak{t}_j^*$ and $(\mathfrak{t}_j^*)_+$ of $\mathfrak{g}_j^*$. As in Section 3.3, one may define a continuous surjection $\pi_j : \mathfrak{g}_j^* \to (\mathfrak{t}_j^*)_+$ by the condition that $(G_j \cdot \xi) \cap (\mathfrak{t}_j^*)_+ = \{\pi_j(\xi)\}$ for all $\xi \in \mathfrak{g}_j^*$.

Let $\ell = \ell_0 \geq \ell_1 \geq \cdots \geq \ell_m$ be the ranks of $G = G_0 \supset G_1 \supset \cdots \supset G_m$, respectively. Let us also choose a $\mathbb{Z}$-basis $\{\phi_{j1}, \ldots, \phi_{j\ell_j}\}$ of the lattice $\Lambda_{\mathfrak{t}_j} \subset \mathfrak{t}_j$ for each $j \in \{0, \ldots, m\}$. As in Section 3.3, we define the functions

$$\nu_{jk} := \phi_{jk} \circ \pi_j : \mathfrak{g}_j^* \to \mathbb{R}$$

for $k \in \{1, \ldots, \ell_j\}$. The same section implies that $\nu_{j1}, \ldots, \nu_{j\ell_j}$ are G_j-invariant and continuous on $\mathfrak{g}_j^*$, as well as smooth on $(\mathfrak{g}_j^*)_{\mathrm{reg}}$. We also have the following equivalent version of Proposition 4.

Proposition 5. *If $j \in \{0, \ldots, m\}$ and $\xi \in (\mathfrak{g}_j^*)_{\mathrm{reg}}$, then $\{\mathrm{d}_\xi \nu_{j1}, \ldots, \mathrm{d}_\xi \nu_{j\ell_j}\}$ is a $\mathbb{Z}$-basis of the lattice $\Lambda_{(\mathfrak{g}_j)_\xi} \subset (\mathfrak{g}_j)_\xi$.*

Let $\sigma_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ be the transpose of the inclusion $\mathfrak{g}_j \hookrightarrow \mathfrak{g}$ for each $j \in \{0, \ldots, m\}$. Consider the functions on $\mathfrak{g}^*$ defined by

$$\lambda_{jk} := \nu_{jk} \circ \sigma_j : \mathfrak{g}^* \to \mathbb{R} \quad \text{for } j \in \{0, \ldots, m\} \text{ and } k \in \{1, \ldots, \ell_j\}.$$

It will be convenient to enumerate these functions as

$$\begin{aligned} \lambda_{\mathrm{big}} &:= (\lambda_1, \ldots, \lambda_{\mathrm{c}}) \\ &:= (\lambda_{01}, \ldots, \lambda_{0\ell}, \lambda_{11}, \ldots, \lambda_{1\ell_1} \ldots, \lambda_{m1}, \ldots, \lambda_{m\ell_m}) : \mathfrak{g}^* \to \mathbb{R}^{\mathrm{c}}, \end{aligned} \tag{3-1}$$

where $\mathrm{c} := \ell_0 + \cdots + \ell_m$.

The discussion preceding Proposition 5 implies that λ_{big} is smooth on the open subset

$$\mathcal{U} := \bigcap_{j=0}^{m} \sigma_j^{-1}((\mathfrak{g}_j^*)_{\mathrm{reg}}) \subset \mathfrak{g}^*.$$

Let us also define

$$\mathfrak{g}^*_{\text{s-reg}} := \{\xi \in \mathcal{U} : \mathrm{d}_\xi \lambda_{\mathrm{big}} \text{ is surjective}\}.$$

These last few sentences give context for the following consequence of [3, Theorem 3.4].

Proposition 6. *Let* λ_{big} *and* $\mathfrak{g}^*_{\text{s-reg}}$ *be as defined above.*

(i) *The restriction* $\lambda_{\text{big}}|_{\mathfrak{g}^*_{\text{s-reg}}} : \mathfrak{g}^*_{\text{s-reg}} \to \mathbb{R}^{\text{c}}$ *is a moment map for a Poisson Hamiltonian* $\mathrm{U}(1)^{\text{c}}$*-space structure on* $\mathfrak{g}^*_{\text{s-reg}}$.

(ii) *If* M *is a Hamiltonian* G*-space with moment map* $\mu : M \to \mathfrak{g}^*$*, then*

$$(\lambda_{\text{big}} \circ \mu)|_{\mu^{-1}(\mathfrak{g}^*_{\text{s-reg}})} : \mu^{-1}(\mathfrak{g}^*_{\text{s-reg}}) \to \mathbb{R}^{\text{c}}$$

is a moment map for a Hamiltonian $\mathrm{U}(1)^{\text{c}}$*-space structure on* $\mu^{-1}(\mathfrak{g}^*_{\text{s-reg}})$.

This result has the following immediate connection to Definition 1: the pair $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ is a Gelfand–Cetlin datum if and only if $\text{c} = \text{b}$, $\mathfrak{g}^*_{\text{s-reg}}$ is dense in $\mathfrak{g}^*$, and $\lambda_{\text{big}}|_{\mathfrak{g}^*_{\text{s-reg}}} : \mathfrak{g}^*_{\text{s-reg}} \to \lambda_{\text{big}}(\mathfrak{g}^*_{\text{s-reg}})$ is a principal $\mathbb{T}_{\text{int}}$-bundle. Guillemin and Sternberg [3; 4] explicitly show these conditions to be achievable for $G = \mathrm{U}(n)$ and $G = \mathrm{SO}(n)$. Our next section outlines the details of the Guillemin–Sternberg construction for $G = \mathrm{U}(n)$.

3.5. ***Example of Gelfand–Cetlin datum: the Gelfand–Cetlin system on*** $\mathfrak{u}(n)^*$**.** Fix a positive integer n. Consider the Lie group $G := \mathrm{U}(n)$ of unitary $n \times n$ matrices, and its Lie algebra $\mathfrak{g} := \mathfrak{u}(n)$ of skew-Hermitian $n \times n$ matrices. Let us also consider the real $\mathrm{U}(n)$-module $\mathcal{H}(n)$ of Hermitian $n \times n$ matrices. In what follows, we will freely identify $\mathfrak{u}(n)^*$ with $\mathcal{H}(n)$ by means of the nondegenerate, G-invariant bilinear form

$$\mathfrak{u}(n) \otimes_{\mathbb{R}} \mathcal{H}(n) \to \mathbb{R}, \quad \eta \otimes \xi \mapsto -i\,\mathrm{tr}(\eta\xi). \tag{3-2}$$

Given an integer $j \in \{0, \dots, n-1\}$, define the subgroup

$$G_j := \left\{ \left[\begin{array}{c|c} I_j & 0 \\ \hline 0 & A \end{array}\right] : A \in \mathrm{U}(n-j) \right\} \subset G = \mathrm{U}(n).$$

The descending chain $\mathrm{U}(n) = G = G_0 \supset G_1 \supset \cdots \supset G_{n-1}$ then induces such a chain $\mathfrak{u}(n) = \mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_{n-1}$ on the level of Lie algebras. We have

$$\mathfrak{g}_j = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & x \end{array}\right] : x \in \mathfrak{u}(n-j) \right\} \subset \mathfrak{u}(n) \quad \text{and} \quad \mathfrak{g}_j^* = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \xi \end{array}\right] : \xi \in \mathcal{H}(n-j) \right\} \subset \mathcal{H}(n)$$

for all $j \in \{0, \dots, n-1\}$, where the second equation implicitly uses (3-2). The transpose $\sigma_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ of $\mathfrak{g}_j \subset \mathfrak{g}$ then sends $\xi \in \mathfrak{g}^* = \mathcal{H}(n)$ to the $(n-j) \times (n-j)$ submatrix in the bottom right-hand corner of ξ.

Now note that

$$\mathfrak{t}_j := \left\{ \left[\begin{array}{c|ccc} 0 & 0 & & \\ \hline & ia_1 & & \\ 0 & & \ddots & \\ & & & ia_{n-j} \end{array}\right] : a_1, \dots, a_{n-j} \in \mathbb{R} \right\} \subset \mathfrak{g}_j$$

is a Cartan subalgebra for each $j \in \{0, \dots, n-1\}$. Let $\phi_{jk} \in \mathfrak{t}_j$ be the result of setting $a_k = 1$ and $a_p = 0$ for $p \neq k$, noting that $\{\phi_{j1}, \dots, \phi_{j(n-j)}\}$ is a $\mathbb{Z}$-basis of $\Lambda_{\mathfrak{t}_j} \subset \mathfrak{t}_j$. At the same time, consider the fundamental Weyl chamber

$$(\mathfrak{t}_j)_+ := \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} ia_1 & & \\ & \ddots & \\ & & ia_{n-j} \end{matrix} \end{array}\right] : a_1, \dots, a_{n-j} \in \mathbb{R},\ a_1 \geq \cdots \geq a_{n-j} \right\} \subset \mathfrak{t}_j$$

for each $j \in \{0, \dots, n-1\}$. Under the pairing (3-2), $(\mathfrak{t}_j)_+$ corresponds to the cone

$$(\mathfrak{t}_j^*)_+ := \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_{n-j} \end{matrix} \end{array}\right] : a_1, \dots, a_{n-j} \in \mathbb{R},\ a_1 \geq \cdots \geq a_{n-j} \right\} \subset \mathcal{H}(n).$$

We then have sweeping maps $\pi_j : \mathfrak{g}_j^* \to (\mathfrak{t}_j^*)_+$ and compositions

$$\nu_{jk} := \phi_{jk} \circ \pi_j : \mathfrak{g}_j^* \to \mathbb{R}$$

for $j \in \{0, \dots, n-1\}$ and $k \in \{1, \dots, n-j\}$, as in Section 3.4. The functions

$$\lambda_{jk} := \nu_{jk} \circ \sigma_j : \mathcal{H}(n) \to \mathbb{R}$$

from Section 3.4 are therefore given by the following condition: if $\xi \in \mathcal{H}(n)$ and $j \in \{0, \dots, n-1\}$, then $\lambda_{j1}(\xi) \geq \lambda_{j2}(\xi) \geq \cdots \geq \lambda_{j(n-j)}(\xi)$ are the eigenvalues of the $(n-j) \times (n-j)$ submatrix in the bottom right-hand corner of ξ.

Observe that the number of maps λ_{jk} is

$$n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2} = \tfrac{1}{2}(\dim \mathfrak{u}(n) + n).$$

Our enumeration (3-1) therefore takes the form

$$\begin{aligned} \lambda_{\mathrm{big}} &:= (\lambda_1, \dots, \lambda_{\frac{n(n+1)}{2}}) \\ &:= \big(\lambda_{01}, \dots, \lambda_{0n}, \lambda_{11}, \dots, \lambda_{1(n-1)}, \dots, \lambda_{(n-2)1}, \lambda_{(n-2)2}, \lambda_{(n-1)1}\big) : \mathcal{H}(n) \to \mathbb{R}^{\frac{n(n+1)}{2}}. \end{aligned}$$

Let us also consider the open dense subset

$$\mathcal{H}(n)_{\text{s-reg}} := \left\{ \xi \in \mathcal{H}(n) : \begin{array}{l} \lambda_{j1}(\xi) > \cdots > \lambda_{j(n-j)}(\xi) \text{ for all } j \in \{0, \dots, n-1\} \\ \lambda_{0k}(\xi) > \cdots > \lambda_{(n-k)k}(\xi) \text{ for all } k \in \{1, \dots, n\} \end{array} \right\}$$

of $\mathfrak{g}^* = \mathcal{H}(n)$. By the paragraph following Proposition 6, $(\lambda_{\mathrm{big}}, \mathcal{H}(n)_{\text{s-reg}})$ is a Gelfand–Cetlin datum if and only if $\lambda_{\mathrm{big}}|_{\mathcal{H}(n)_{\text{s-reg}}} : \mathcal{H}(n)_{\text{s-reg}} \to \lambda_{\mathrm{big}}(\mathcal{H}(n)_{\text{s-reg}})$ is a principal bundle for $\mathbb{T}_{\mathrm{int}} = \mathrm{U}(1)^{\frac{n(n-1)}{2}}$; this latter condition is verified in [3, Section 5].

4. The abelianization theorem

This section is devoted to the proof of our abelianization theorem for smooth quotients. Some preliminary results are established in Sections 4.1 and 4.2, while the main proof appears in Section 4.3.

4.1. ***The universal maximal torus.*** Adopt the notation and conventions given in Section 3.1, and let $(\lambda_{\text{big}}, \mathfrak{g}^*_{\text{s-reg}})$ be a Gelfand–Cetlin datum. Given any $\xi \in \mathfrak{g}^*_{\text{reg}}$, Definition 1(ii) tells us that $\{\mathrm{d}_\xi \lambda_1, \ldots, \mathrm{d}_\xi \lambda_\ell\}$ is a basis of $\mathfrak{g}_\xi$. This basis determines a vector space isomorphism

$$\kappa_\xi : \mathfrak{g}_\xi \xrightarrow{\cong} \mathbb{R}^\ell, \quad x_1 \mathrm{d}_\xi \lambda_1 + \cdots + x_\ell \mathrm{d}_\xi \lambda_\ell \mapsto (x_1, \ldots, x_\ell).$$

The torus $\mathbb{T}_{\text{small}}$ is then a universal maximal torus in the following sense.

Proposition 7. *If $\xi \in \mathfrak{g}^*_{\text{reg}}$, then κ_ξ integrates to a Lie group isomorphism*

$$\tau_\xi : G_\xi \xrightarrow{\cong} \mathbb{T}_{\text{small}}.$$

Proof. It suffices to prove that κ_ξ restricts to a $\mathbb{Z}$-module isomorphism from the kernel of $\exp|_{\mathfrak{g}_\xi} : \mathfrak{g}_\xi \to G_\xi$ to $(2\pi\mathbb{Z})^\ell \subset \mathbb{R}^\ell$. This is an immediate consequence of Proposition 4. □

Proposition 8. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$. Then $g \cdot m = \tau_\xi(g) \cdot m$ for all $g \in G_\xi$ and $m \in \mu^{-1}(\xi)$, where the left- and right-hand sides denote the actions of $G_\xi \subset G$ on M and $\mathbb{T}_{\text{small}} \subset \mathbb{T}_{\text{big}}$ on $M_{\text{s-reg}}$, respectively.*

Proof. Let X_ζ be the generating vector field on $M_{\text{s-reg}}$ determined by $\zeta \in \mathbb{R}_{\text{small}}$ via the action of $\mathbb{T}_{\text{small}}$ on $M_{\text{s-reg}}$. Write Y_η for the generating vector field on M determined by $\eta \in \mathfrak{g}_\xi$ through the action of $G_\xi \subset G$ on M. It suffices to prove that $(X_{\kappa_\xi(\eta)})_m = (Y_\eta)_m$ for all $\eta \in \mathfrak{g}_\xi$ and $m \in \mu^{-1}(\xi)$. Setting $\gamma_j := d_\xi \lambda_j$ and letting $e_j \in \mathbb{R}^\ell = \mathbb{R}_{\text{small}}$ denote the j-th standard basis vector, this is equivalent to establishing that $(X_{e_j})_m = (Y_{\gamma_j})_m$ for all $j \in \{1, \ldots, \ell\}$ and $m \in \mu^{-1}(\xi)$. On the other hand, Y_{γ_j} (resp. X_{e_j}) is the Hamiltonian vector field on M (resp. $M_{\text{s-reg}}$) associated to μ^{γ_j} (resp. the j-th component $\mu^*\lambda_j$ of $\lambda \circ \mu$). This further reduces us to proving that $\mathrm{d}_m \mu^{\gamma_j} = \mathrm{d}_m \mu^* \lambda_j$. But it is clear that

$$\mathrm{d}_m \mu^* \lambda_j = \mathrm{d}_\xi \lambda_j \circ \mathrm{d}_m \mu = \mathrm{d}_m \mu^{\gamma_j}$$

for all $m \in \mu^{-1}(\xi)$ and $j \in \{1, \ldots, \ell\}$. □

4.2. ***Some supplementary results.*** We now prove two supplementary facts needed to establish the main result of this paper. We continue with the notation and conventions of Sections 3.1 and 4.1.

Proposition 9. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *If $m \in \mu^{-1}(\xi)$ and $t \in \mathbb{T}_{\text{int}}$ satisfy $t \cdot m \in \mu^{-1}(\xi)$, then $t = e$.*

(ii) *The saturation of $\mu^{-1}(\xi)$ under the action of $\mathbb{T}_{\text{int}}$ on $M_{\text{s-reg}}$ is $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$.*

Proof. To verify (i), let $m \in \mu^{-1}(\xi)$ and $t \in \mathbb{T}_{\text{int}}$ be such that $t \cdot m \in \mu^{-1}(\xi)$. Let us also observe that μ is $\mathbb{T}_{\text{int}}$-equivariant when restricted to a map $M_{\text{s-reg}} \to \mathfrak{g}^*_{\text{s-reg}}$. These last two sentences imply that $\xi = t \cdot \xi$. Since $\mathbb{T}_{\text{int}}$ acts freely on $\mathfrak{g}^*_{\text{s-reg}}$, we must have $t = e$.

We now verify (ii). To this end, note that $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ is a $\mathbb{T}_{\text{int}}$-invariant subset of $M_{\text{s-reg}}$ that contains $\mu^{-1}(\xi)$. This implies that the saturation of $\mu^{-1}(\xi)$ is contained in $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$. For the opposite inclusion, suppose that $m \in \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$. Definition 1(v) tells us that $t \cdot \mu(m) = \xi$ for some $t \in \mathbb{T}_{\text{int}}$. By the equivariance property of μ mentioned in the previous paragraph, we must have $t \cdot m \in \mu^{-1}(\xi)$. This completes the proof of (ii). □

Fix $\xi \in \mathfrak{g}^*_{\text{s-reg}}$. In light of the previous proposition, we may define the map

$$\delta_\xi : \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}} \to \lambda_M^{-1}(\lambda_{\text{big}}(\xi)), \quad (m, t) \mapsto t \cdot m.$$

The following result is immediate consequence of the previous proposition.

Corollary 10. *If $\xi \in \mathfrak{g}^*_{\text{s-reg}}$, then δ_ξ is a homeomorphism.*

4.3. ***Proof of the abelianization theorem.*** Let us continue with the notation and conventions set in Sections 3.1, 4.1, and 4.2.

Theorem 11. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *The stabilizer G_ξ acts freely on $\mu^{-1}(\xi)$ if and only if $\mathbb{T}_{\text{big}}$ acts freely on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$.*

(ii) *In the case of* (i), *there is a canonical symplectomorphism*

$$M/\!/_\xi G \cong M_{\text{s-reg}} /\!/_{\lambda_{big}(\xi)} \mathbb{T}_{\text{big}}.$$

Proof. We begin by verifying (i). In light of Proposition 7, the multiplication map

$$\rho_\xi : G_\xi \times \mathbb{T}_{\text{int}} \to \mathbb{T}_{\text{big}}, \quad (g, t) \mapsto \tau_\xi(g)\, t$$

is a Lie group isomorphism. We also note that the action of G_ξ on $\mu^{-1}(\xi)$ and multiplication action of $\mathbb{T}_{\text{int}}$ on itself define an action $G_\xi \times \mathbb{T}_{\text{int}}$ on $\mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}$. By Proposition 8, the homeomorphism δ_ξ is equivariant in the sense that

$$\delta_\xi((g, t) \cdot x) = \rho_\xi(g, t) \cdot \delta_\xi(x)$$

for all $(g,t) \in G_\xi \times \mathbb{T}_{\text{int}}$ and $x \in \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}$. It follows that G_ξ acts freely on $\mu^{-1}(\xi)$ if and only if $\mathbb{T}_{\text{big}}$ acts freely on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$.

We now prove (ii). By Corollary 10, the inclusion $\mu^{-1}(\xi) \hookrightarrow \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ descends to a diffeomorphism

$$\mathrm{f} : \mu^{-1}(\xi) \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}}.$$

We also note that the $\mathbb{T}_{\text{big}}$-action on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ induces a residual action of the subtorus $\mathbb{T}_{\text{small}}$ on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}}$. Proposition 8 then tells us that

$$\mathrm{f}(g \cdot m) = \tau_\xi(g) \cdot \mathrm{f}(m)$$

for all $g \in G_\xi$ and $m \in \mu^{-1}(\xi)$. The map f therefore descends to a diffeomorphism

$$\varphi : M/\!/_\xi G \xrightarrow{\cong} M_{\text{s-reg}}/\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}.$$

It therefore suffices to prove that φ pulls the symplectic form β on $M_{\text{s-reg}}/\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}$ back to the symplectic form α on $M/\!/_\xi G$.

We have a commutative diagram

$$\begin{array}{ccc} \mu^{-1}(\xi) & \xrightarrow{\mathrm{j}} & \lambda_M^{-1}(\lambda_{\text{big}}(\xi)) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\theta} \\ M/\!/_\xi G & \xrightarrow[\varphi]{} & M_{\text{s-reg}}/\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}} \end{array}$$

where

$$\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G_\xi = M/\!/_\xi G$$

and

$$\theta : \lambda_M^{-1}(\lambda_{\text{big}}(\xi)) \to \lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{big}} = M_{\text{s-reg}}/\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}$$

are the canonical quotient maps and $\mathrm{j} : \mu^{-1}(\xi) \hookrightarrow \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ is the inclusion. We also have inclusion maps $\mathrm{k} : \mu^{-1}(\xi) \hookrightarrow M$ and $\mathrm{l} : \lambda_M^{-1}(\lambda_{\text{big}}(\xi)) \hookrightarrow M$. Another consideration is that α (resp. β) is the unique 2-form on $M/\!/_\xi G$ (resp. $M_{\text{s-reg}}/\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}}$) for which $\pi^*\alpha = \mathrm{k}^*\omega$ (resp. $\theta^*\beta = \mathrm{l}^*\omega$), where ω is the symplectic form on M. It therefore suffices to prove that $\pi^*(\varphi^*\beta) = \mathrm{k}^*\omega$. On the other hand, our commutative diagram implies that

$$\pi^*(\varphi^*\beta) = \mathrm{j}^*(\theta^*\beta) = \mathrm{j}^*(\mathrm{l}^*\omega) = \mathrm{k}^*\omega. \qquad \square$$

5. Generalization to stratified symplectic spaces

We now provide a generalization of Theorem 11 in the realm of stratified symplectic spaces [10]. In Section 5.1, we recall the immediately pertinent parts of Sjamaar and Lerman's more general theory of stratified symplectic spaces. The generalization of Theorem 11 to stratified symplectic spaces appears in Section 5.2.

5.1. ***Stratified symplectic spaces.*** Let X be a topological space on which a compact torus T acts continuously. Given a closed subgroup $H \subset T$, let

$$X_H := \{x \in X : T_x = H\}$$

be the locus of points with T-stabilizer T_x equal to H. Denote by $\mathrm{Stab}(T, X)$ the set of all closed subgroups $H \subset T$ for which $X_H \neq \varnothing$.

Now let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that M is a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. As discussed in the introduction to this paper, $M/\!/_\xi G$ is a stratified symplectic space [10] for all $\xi \in \mathfrak{g}^*$. This means that $M/\!/_\xi G$ is naturally partitioned into symplectic manifolds satisfying certain compatibility conditions. While we refer the reader to [10, Definition 1.12] for a precise definition and description of stratified symplectic spaces, the following exposition will be sufficient for our purposes.

Fix a point $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$, and recall that $G_\xi \subset G$ is a maximal torus. Adopt the more parsimonious notation

$$\mathrm{Stab}(G, \xi) := \mathrm{Stab}(G_\xi, \mu^{-1}(\xi)),$$

and note that $\mu^{-1}(\xi)$ is the disjoint union

$$\mu^{-1}(\xi) = \bigsqcup_{H \in \mathrm{Stab}(G,\xi)} \mu^{-1}(\xi)_H.$$

The arguments in the proof of [10, Theorem 2.1] imply that each subset $\mu^{-1}(\xi)_H$ is a locally closed, G_ξ-invariant submanifold of M. These arguments also imply that the topological quotient $(\mu^{-1}(\xi)_H)/G_\xi$ carries a unique manifold structure for which the canonical map $\pi : \mu^{-1}(\xi)_H \to (\mu^{-1}(\xi)_H)/G_\xi$ is a surjective submersion. One further consequence of [10, Theorem 2.1] is the existence of a symplectic form $\bar{\omega}$ on $(\mu^{-1}(\xi)_H)/G_\xi$ such that $\pi^*\bar{\omega}$ is the pullback of ω along the inclusion $\mu^{-1}(\xi)_H \hookrightarrow M$. It follows that $M/\!/_\xi G = \mu^{-1}(\xi)/G_\xi$ is a disjoint union

$$M/\!/_\xi G = \bigsqcup_{H \in \mathrm{Stab}(G,\xi)} (\mu^{-1}(\xi)_H)/G_\xi \tag{5-1}$$

of symplectic manifolds, called the *symplectic strata* of $M/\!/_\xi G$.

Remark 12. The quotients $(\mu^{-1}(\xi)_H)/G_\xi$ need not be manifolds in the traditional sense of the term; each may have connected components of different dimensions. To obtain a stratification into genuine symplectic manifolds, one must refine (5-1) and declare the symplectic strata to be the connected components of the quotients $(\mu^{-1}(\xi)_H)/G_\xi$. The distinction between (5-1) and this refined stratification will not materially affect any argument in this paper.

Definition 13. Let G and K be compact connected Lie groups with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$. Suppose that M (resp. N) is a Hamiltonian G-space (resp. Hamiltonian K-space) with moment map $\mu : M \to \mathfrak{g}^*$ (resp. $\nu : N \to \mathfrak{k}^*$). Take $\xi \in \mathfrak{g}^*_{\mathrm{reg}}$ and

$\eta \in \mathfrak{k}^*_{\mathrm{reg}}$. A pair of maps $\varphi : M/\!/_\xi G \to N/\!/_\eta K$ and $\phi : \mathrm{Stab}(G, \xi) \to \mathrm{Stab}(K, \eta)$ will be called an *isomorphism of stratified symplectic spaces* if the following conditions are satisfied:

(i) φ is a homeomorphism.

(ii) ϕ is a bijection.

(iii) φ restricts to a symplectomorphism $(\mu^{-1}(\xi)_H)/G_\xi \to (\nu^{-1}(\eta)_{\phi(H)})/K_\eta$ for each $H \in \mathrm{Stab}(G, \xi)$.

Remark 14. Assume that this definition is satisfied. Equip $M/\!/_\xi G$ and $N/\!/_\eta K$ with the refined stratifications discussed in Remark 12. By (ii) and (iii), the association $S \mapsto \varphi(S)$ defines a bijection from the set of symplectic strata $S \subset M/\!/_\xi G$ to the set of symplectic strata in $N/\!/_\eta K$. Property (i) implies that this bijection is an isomorphism of partially ordered sets, i.e., any symplectic strata $S, T \subset M/\!/_\xi G$ satisfying $S \subset \overline{T}$ must also satisfy $\varphi(S) \subset \overline{\varphi(T)}$. We also know that φ restricts to a symplectomorphism $S \to \varphi(S)$ for all symplectic strata $S \subset M/\!/_\xi G$, as follows from (iii). In other words, an isomorphism in the sense of Definition 13 gives rise to an isomorphism between the refined symplectic stratifications on $M/\!/_\xi G$ and $N/\!/_\eta K$.

5.2. ***A more general abelianization theorem.*** Let us continue with the notation and conventions set in Section 4, as well as those in Section 5.1 concerning stratified symplectic spaces.

In preparation for our next proposition, we encourage the reader to recall Proposition 7 and Corollary 10.

Proposition 15. *Let M be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$. Suppose that $\xi \in \mathfrak{g}^*_{\text{s-reg}}$.*

(i) *The association $H \mapsto \tau_\xi(H)$ defines a bijection*

$$\mathrm{Stab}(G, \xi) \xrightarrow{\cong} \mathrm{Stab}(\mathbb{T}_{\mathrm{big}}, \lambda_{\mathrm{big}}(\xi)).$$

(ii) *If $H \subset G_\xi$ is a closed subgroup, then δ_ξ restricts to a diffeomorphism*

$$\mu^{-1}(\xi)_H \times \mathbb{T}_{\mathrm{int}} \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)}.$$

Proof. As in the proof of Theorem 11(i), we have

$$\delta_\xi((g, t) \cdot x) = \rho_\xi(g, t) \cdot \delta_\xi(x)$$

for all $(g, t) \in G_\xi \times \mathbb{T}_{\mathrm{int}}$ and $x \in \mu^{-1}(\xi) \times \mathbb{T}_{\mathrm{int}}$. It follows that $K \mapsto \rho_\xi(K)$ defines a bijection

$$\mathrm{Stab}(G_\xi \times \mathbb{T}_{\mathrm{int}}, \mu^{-1}(\xi) \times \mathbb{T}_{\mathrm{int}}) \xrightarrow{\cong} \mathrm{Stab}(\mathbb{T}_{\mathrm{big}}, \lambda_{\mathrm{big}}(\xi)),$$

and that δ_ξ restricts to a homeomorphism

$$(\mu^{-1}(\xi) \times \mathbb{T}_{\text{int}})_K \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\rho_\xi(K)}$$

for all closed subgroups $K \subset G_\xi \times \mathbb{T}_{\text{int}}$. On the other hand, we clearly have a bijection

$$\text{Stab}(G, \xi) \xrightarrow{\cong} \text{Stab}(G_\xi \times \mathbb{T}_{\text{int}}, \mu^{-1}(\xi) \times \mathbb{T}_{\text{int}}), \quad H \mapsto H \times \{e\} \subset G_\xi \times \mathbb{T}_{\text{int}}.$$

We also note that

$$(\mu^{-1}(\xi) \times \mathbb{T}_{\text{int}})_K = \mu^{-1}(\xi)_H \times \mathbb{T}_{\text{int}} \quad \text{and} \quad \rho_\xi(K) = \tau_\xi(H)$$

for $K = H \times \{e\}$. These last three sentences combine to imply the desired results. □

The following is our generalization of Theorem 11 to stratified symplectic spaces.

Theorem 16. *If $\xi \in \mathfrak{g}^*_{\text{s-reg}}$, then there is a canonical isomorphism*

$$M /\!\!/_\xi G \cong M_{\text{s-reg}} /\!\!/_{\lambda_{big}(\xi)} \mathbb{T}_{\text{big}}$$

of stratified symplectic spaces.

Proof. By Corollary 10 and Proposition 15, the inclusion $\mu^{-1}(\xi) \hookrightarrow \lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ descends to a homeomorphism

$$\text{f} : \mu^{-1}(\xi) \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}},$$

whose restriction to $\mu^{-1}(\xi)_H$ is a diffeomorphism

$$\mu^{-1}(\xi)_H \xrightarrow{\cong} \lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\tau_\xi(H)}/\mathbb{T}_{\text{int}}$$

for all $H \in \text{Stab}(G, \xi)$. We also note that the $\mathbb{T}_{\text{big}}$-action on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))$ induces a residual action of the subtorus $\mathbb{T}_{\text{small}}$ on $\lambda_M^{-1}(\lambda_{\text{big}}(\xi))/\mathbb{T}_{\text{int}}$. Proposition 8 then tells us that

$$\text{f}(g \cdot m) = \tau_\xi(g) \cdot \text{f}(m)$$

for all $g \in G_\xi$ and $m \in \mu^{-1}(\xi)$. The map f therefore descends to a homeomorphism

$$\varphi : M /\!\!/_\xi G \xrightarrow{\cong} M_{\text{s-reg}} /\!\!/_{\lambda_{\text{big}}(\xi)} \mathbb{T}_{\text{big}},$$

whose restriction to $(\mu^{-1}(\xi)_H)/G_\xi$ is a diffeomorphism

$$\varphi_H : (\mu^{-1}(\xi)_H)/G_\xi \xrightarrow{\cong} (\lambda_M^{-1}(\lambda_{\text{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\text{big}}$$

for all $H \in \text{Stab}(G, \xi)$.

Now consider the bijection

$$\phi : \text{Stab}(G, \xi) \xrightarrow{\cong} \text{Stab}(\mathbb{T}_{\text{big}}, \lambda_{\text{big}}(\xi)), \quad H \mapsto \tau_\xi(H)$$

from Proposition 15(i). We claim that φ and ϕ define an isomorphism of stratified symplectic spaces, in the sense of Definition 13. In light of the previous

paragraph, it suffices to prove the following for all $H \in \mathrm{Stab}(G, \xi)$: φ_H pulls the symplectic form β on $(\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}$ back to the symplectic form α on $(\mu^{-1}(\xi)_H)/G_\xi$.

Proposition 15(ii) implies that $\mu^{-1}(\xi)_H \subset \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))$. This leads to the commutative diagram

$$\begin{array}{ccc} \mu^{-1}(\xi)_H & \xrightarrow{\mathrm{j}} & \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)} \\ \pi \downarrow & & \downarrow \theta \\ (\mu^{-1}(\xi)_H)/G_\xi & \xrightarrow[\varphi_H]{} & (\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}} \end{array}$$

where

$$\pi : \mu^{-1}(\xi)_H \to (\mu^{-1}(\xi)_H)/G_\xi$$

and

$$\theta : \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)} \to (\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}$$

are the canonical quotient maps and $\mathrm{j} : \mu^{-1}(\xi)_H \hookrightarrow \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))$ is the inclusion. We also have inclusion maps $\mathrm{k} : \mu^{-1}(\xi) \hookrightarrow M$ and $\mathrm{l} : \lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi)) \hookrightarrow M$. Another consideration is that α (resp. β) is the unique 2-form on $(\mu^{-1}(\xi)_H)/G_\xi$ (resp. $(\lambda_M^{-1}(\lambda_{\mathrm{big}}(\xi))_{\tau_\xi(H)})/\mathbb{T}_{\mathrm{big}}$) for which $\pi^*\alpha = \mathrm{k}^*\omega$ (resp. $\theta^*\beta = \mathrm{l}^*\omega$). It therefore suffices to prove that $\pi^*(\varphi_H^*\beta) = \mathrm{k}^*\omega$. On the other hand, our commutative diagram implies that

$$\pi^*(\varphi_H^*\beta) = \mathrm{j}^*(\theta^*\beta) = \mathrm{j}^*(\mathrm{l}^*\omega) = \mathrm{k}^*\omega. \qquad \square$$

Acknowledgements

The authors would like to thank Megumi Harada and Jeremy Lane for exceedingly useful conversations, as well as the referee for constructive feedback. Crooks acknowledges support from a Utah State University startup grant and the Simons Foundation Grant MPS-TSM-00002292. Weitsman acknowledges support from Simons Collaboration Grant 579801.

References

[1] J. J. Duistermaat, "On global action-angle coordinates", *Comm. Pure Appl. Math.* **33**:6 (1980), 687–706. MR Zbl

[2] V. Guillemin and S. Sternberg, "The moment map and collective motion", *Ann. Physics* **127**:1 (1980), 220–253. MR Zbl

[3] V. Guillemin and S. Sternberg, "The Gelfand–Cetlin system and quantization of the complex flag manifolds", *J. Funct. Anal.* **52**:1 (1983), 106–128. MR Zbl

[4] V. Guillemin and S. Sternberg, "On collective complete integrability according to the method of Thimm", *Ergodic Theory Dynam. Systems* **3**:2 (1983), 219–230. MR Zbl

[5] V. Guillemin, L. Jeffrey, and R. Sjamaar, "Symplectic implosion", *Transform. Groups* **7**:2 (2002), 155–184. MR Zbl

[6] M. Harada, "The symplectic geometry of the Gelfand–Cetlin–Molev basis for representations of $\mathrm{Sp}(2n, \mathbb{C})$", *J. Symplectic Geom.* **4**:1 (2006), 1–41. MR Zbl

[7] M. Harada and K. Kaveh, "Integrable systems, toric degenerations and Okounkov bodies", *Invent. Math.* **202**:3 (2015), 927–985. MR Zbl

[8] B. Hoffman and J. Lane, "Stratified gradient Hamiltonian vector fields and collective integrable systems", preprint, 2023. arXiv 2008.13656

[9] J. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetry", *Rep. Mathematical Phys.* **5**:1 (1974), 121–130. MR Zbl

[10] R. Sjamaar and E. Lerman, "Stratified symplectic spaces and reduction", *Ann. of Math.* (2) **134**:2 (1991), 375–422. MR Zbl

Received March 6, 2023. Revised December 7, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.273

HILBERT POLYNOMIALS FOR FINITARY MATROIDS

ANTONGIULIO FORNASIERO AND ELLIOT KAPLAN

We consider a tuple $\Phi = (\phi_1, \dots, \phi_m)$ of commuting maps on a finitary matroid X. We show that if Φ satisfies certain conditions, then for any finite set $A \subseteq X$, the rank of $\{\phi_1^{r_1} \cdots \phi_m^{r_m}(a) : a \in A \text{ and } r_1 + \cdots + r_m = t\}$ is eventually a polynomial in t (we also give a multivariate version of the polynomial). This allows us to easily recover Khovanskii's theorem on the growth of sumsets, the existence of the classical Hilbert polynomial, and the existence of the Kolchin polynomial. We also prove some new Kolchin polynomial results for differential exponential fields and derivations on o-minimal fields, as well as a new result on the growth of Betti numbers in simplicial complexes.

Introduction

Eventual polynomial growth is a common theme in combinatorics and commutative algebra. The quintessential example is the Hilbert function, which measures the K-linear dimension of the graded pieces M_t of a finitely generated $K[x_1, \dots, x_m]$-module $M = \bigoplus_t M_t$. This function is eventually equal to a polynomial in t, called the *Hilbert polynomial*. Another example is due to Kolchin, who showed that

MSC2020: primary 05B35; secondary 03C64, 05E40, 12H05, 12H10, 13D40.
Keywords: finitary matroid, Hilbert polynomial, Kolchin polynomial, exponential field, o-minimal field, sumset.

given a partial differential field $(F, \delta_1, \dots, \delta_m)$ of characteristic 0 and a tuple $\bar{a}$ in a differential field extension of F, the transcendence degree

$$\operatorname{trdeg}_F F\big((\delta_1^{r_1} \cdots \delta_m^{r_m} \bar{a})_{r_1+\cdots+r_m \le t}\big)$$

is eventually equal to a polynomial in t [14]. This polynomial, called the *Kolchin polynomial*, is a foundational object in differential algebra. In the area of additive combinatorics, Khovanskii showed that for finite subsets A and B of a commutative semigroup, the size of the sumset $A + tB$ is eventually polynomial in t [12].

In this paper, we show that the Hilbert polynomial, the Kolchin polynomial, and Khovanskii's polynomial admit a common generalization in terms of finitary matroids. A *matroid* is a combinatorial structure that axiomatizes the notion of independence. While most matroids studied are finite, it is useful for our purposes to allow infinite matroids, while still requiring that any instance of dependence is witnessed by a finite set. There are many equivalent ways to define a finitary matroid, but for the purposes of this paper, we most frequently use closure operators (where the closure of a set consists of all elements which are not independent from the set) and ranks (where the rank of a finite set is the size of a maximal independent subset).

We consider how the rank of a finite set grows as one applies commuting operators to the set. We show that, under certain assumptions on the operators, this rank is eventually polynomial in the total number of times the operators are applied. In the case of the Hilbert polynomial, the closure operator is the K-linear span in M, the rank is the K-linear dimension, and the operators are scalar multiplication by the elements x_i. For the Kolchin polynomial, the closure operator is the algebraic closure over F, the rank is the transcendence degree over F, and the operators are $\delta_1, \dots, \delta_m$, and the identity map. For Khovanskii's polynomial, the underlying matroid is the semigroup with trivial closure, so the rank coincides with cardinality, and the operators are addition by elements of B.

We are also able to apply our result in several other settings. We recover a result of Maclagan and Rincón on Hilbert polynomials for homogeneous tropical ideals in the semiring of tropical polynomials [22], we prove that the Betti numbers of a finite subcomplex of a simplicial complex $\mathcal{K}$ grow polynomially under an action of $\mathbb{N}^m$ on $\mathcal{K}$ by simplicial endomorphisms, and we establish an analog of the Kolchin polynomial for difference-differential exponential fields and derivations on o-minimal fields. This last application was the original motivation behind this project, and we believe that our analog can serve the same role in the model theory of o-minimal fields with derivations that the classical Kolchin polynomial serves in the model theory of differential fields.

Khovanskii proved his result by constructing an appropriate graded module and using the existence of the Hilbert polynomial. Kolchin's result can also be proven using the Hilbert polynomial, as was shown by Johnson [9]. Other known

examples of eventual polynomial growth, such as Maclagan and Rincón's result on Hilbert polynomials for homogeneous tropical ideals [22], are also often proved by reduction to the classical Hilbert polynomial. Nathanson and Ruzsa later gave a more elementary proof of Khovanskii's result [26] by reducing the problem to showing that the number of elements in an upward-closed subset of $\mathbb{N}^m$ of a given height t is eventually polynomial in t (where the *height* of $(r_1, \dots, r_m) \in \mathbb{N}^m$ is the sum $r_1 + \cdots + r_m$). This approach is more in line with Kolchin's original proof of the existence of the Kolchin polynomial [15, Chapter II, Theorem 6].

In some sense, the proof of our main result also reduces to counting the number of elements in an upward-closed subset of $\mathbb{N}^m$ or, more precisely, to looking at decreasing functions $\mathbb{N}^m \to \mathbb{N}$. However, the framework of finitary matroids — a fundamentally novel aspect of our approach — allows our main theorem to be quickly and readily applied. Many proofs of the existence of the classical Hilbert polynomial make use of generating functions, exploiting the relationship between rational generating functions and eventual polynomial growth through results like Lemma 1.2 below. We also make use of this relationship in our proof. Key to our approach is Proposition 1.3, which describes the generating function G_f associated to a decreasing function $f : \mathbb{N}^m \to \mathbb{N}$. This result appears to be new.

While our proof is self-contained and fairly elementary, we show in Proposition 4.6 that once we isolate an appropriate decreasing function, our main theorem can be established à la Khovanskii by constructing a graded module and using the classical Hilbert polynomial. Consequently, we do not obtain any new numerical polynomials in our setting; see Corollary 4.7.

Let us state our main results more precisely. Let $(X, \mathrm{c}\ell)$ be a finitary matroid and let rk be the corresponding rank function; see Section 1B for definitions. Let $m \in \mathbb{N}^{>0}$, and let $\Phi := (\phi_1, \dots, \phi_m)$ be a finite tuple of commuting maps $X \to X$. The tuple Φ is said to be a *triangular system* if

$$a \in \mathrm{c}\ell(B) \Longrightarrow \phi_i a \in \mathrm{c}\ell(\phi_1 B \cup \cdots \cup \phi_i B)$$

for all $i \in \{1, \dots, m\}$, all $a \in X$, and all $B \subseteq X$. For $\bar{r} = (r_1, \dots, r_m) \in \mathbb{N}^m$, we let $|\bar{r}| = r_1 + \cdots + r_m$, and we let $\phi^{\bar{r}} : X \to X$ be the composite map $\phi^{\bar{r}} = \phi_1^{r_1} \cdots \phi_m^{r_m}$. For $t \in \mathbb{N}$ and $A \subseteq X$, put

$$\Phi^{(t)}(A) := \left\{\phi^{\bar{r}}(a) : a \in A \text{ and } |\bar{r}| = t\right\}.$$

We prove the following:

Theorem. *Suppose that Φ is a triangular system, and let $A, B \subseteq X$ with A finite. Then there is a polynomial $P \in \mathbb{Q}[Y]$ of degree at most $m - 1$ such that*

$$\mathrm{rk}(\Phi^{(t)}(A) | \Phi^{(t)}(B)) = P(t)$$

for all sufficiently large $t \in \mathbb{N}$.

The above theorem is a special case of a multivariate version, which can be used to recover multivariate generalizations of the results considered above, such as Nathanson's generalization of Khovanskii's sumset theorem [25] and Levin's multivariate generalizations of the Kolchin polynomial [20]. Fix

$$0 = m_1 < m_2 < \cdots < m_k < m_{k+1} = m,$$

and set $d_i := m_{i+1} - m_i$ for $i \in \{1, \ldots, k\}$. For each i, we set

$$\Phi_i := (\phi_{m_i+1}, \phi_{m_i+2}, \ldots, \phi_{m_i+d_i}),$$

and we call the tuple $(\Phi_1, \ldots, \Phi_k)$ a *partition of* Φ. For a tuple $\bar{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$, we set

$$\|\bar{r}\| := (r_{m_1+1} + \cdots + r_{m_1+d_1}, \ldots, r_{m_k+1} + \cdots + r_{m_k+d_k}) \in \mathbb{N}^k,$$

and for $\bar{s} \in \mathbb{N}^k$ and $A \subseteq X$, we put $\Phi^{(\bar{s})}(A) := \left\{\phi^{\bar{r}}(a) : a \in A \text{ and } \|\bar{r}\| = \bar{s}\right\}$.

Theorem A. *Suppose that each* Φ_i *is a triangular system, and let* $A, B \subseteq X$ *with* A *finite. Then there is a polynomial* $P^{\Phi}_{A|B} \in \mathbb{Q}[Y_1, \ldots, Y_k]$ *of degree at most* $d_i - 1$ *in each variable* Y_i *such that*

$$\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B)) = P^{\Phi}_{A|B}(\bar{s})$$

for $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ *with* $\min\{s_1, \ldots, s_k\}$ *sufficiently large.*

We call the polynomial $P^{\Phi}_{A|B}$ in Theorem A the *dimension polynomial of* A *over* B *with respect to the partition* $(\Phi_1, \ldots, \Phi_k)$. To apply Theorem A to the case of differential fields, we need to consider a slightly more general framework than a triangular system. We say that the system $\Phi = (\phi_1, \ldots, \phi_m)$ is *quasi-triangular* if the augmented system $(\mathrm{id}, \phi_1, \ldots, \phi_m)$ is triangular, where $\mathrm{id}\colon X \to X$ is the identity map. Let $\preceq$ denote the product order on $\mathbb{N}^k$, and for $\bar{s} \in \mathbb{N}^k$ and $A \subseteq X$, put $\Phi^{\preceq(\bar{s})}(A) := \left\{\phi^{\bar{r}}(a) : a \in A \text{ and } \|\bar{r}\| \preceq \bar{s}\right\}$.

Corollary A. *Suppose that each* Φ_i *is a quasi-triangular system, and let* $A, B \subseteq X$ *with* A *finite. Then there is a polynomial* $Q^{\Phi}_{A|B} \in \mathbb{Q}[Y_1, \ldots, Y_k]$ *of degree at most* d_i *in each variable* Y_i *such that*

$$\mathrm{rk}(\Phi^{\preceq(\bar{s})}(A)|\Phi^{\preceq(\bar{s})}(B)) = Q^{\Phi}_{A|B}(\bar{s})$$

for $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ *with* $\min\{s_1, \ldots, s_k\}$ *sufficiently large.*

We call the polynomial $Q^{\Phi}_{A|B}$ in Corollary A the *cumulative dimension polynomial of* A *over* B *with respect to the partition* $(\Phi_1, \ldots, \Phi_k)$. Any triangular system is quasi-triangular, so Corollary A applies in a strictly broader context than Theorem A. Indeed, there are quasi-triangular systems for which the conclusion of Theorem A

doesn't hold; see Section 5B. As we will see, derivations on fields form a quasi-triangular system with respect to the matroid of algebraic closure.

In addition to the dimension and cumulative dimension polynomials, we define closure operators $\mathrm{c}\ell^{\Phi}$ and $\mathrm{c}\ell^{\Phi}_{*}$ on X, called the Φ-*closure* and Φ_*-*closure*, respectively, with respect to the partition $(\Phi_1, \ldots, \Phi_k)$ as follows:

$$a \in \mathrm{c}\ell^{\Phi}(B) :\Longleftrightarrow \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) < |\Phi^{(\bar{s})}| \text{ for some } \bar{s} \in \mathbb{N}^k$$
$$a \in \mathrm{c}\ell^{\Phi_*}(B) :\Longleftrightarrow \mathrm{rk}(\Phi^{\preceq(\bar{s})}(a)|\Phi^{\preceq(\bar{s})}(B)) < |\Phi^{\preceq(\bar{s})}| \text{ for some } \bar{s} \in \mathbb{N}^k.$$

Our second theorem relates these closure operators to the dimension and cumulative dimension polynomials.

Theorem B. *Let $A, B \subseteq X$ with A finite. If each Φ_i is a triangular system, then $(X, \mathrm{c}\ell^{\Phi})$ is a finitary matroid. The rank function rk^{Φ} corresponding to this matroid satisfies the identity*

$$\mathrm{rk}^{\Phi}(A|B) = \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|}.$$

We also have

$$P^{\Phi}_{A|B}(Y_1, \ldots, Y_k) = \frac{\mathrm{rk}^{\Phi}(A|B)}{(d_1-1)!\cdots(d_k-1)!} Y_1^{d_1-1}\cdots Y_k^{d_k-1} + \textit{lower degree terms.}$$

Likewise, if each Φ_i is quasi-triangular, then $(X, \mathrm{c}\ell^{\Phi_*})$ is a finitary matroid with rank function rk^{Φ_*} satisfying the identities

$$\mathrm{rk}^{\Phi_*}(A|B) = \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{\preceq(\bar{s})}(A)|\Phi^{\preceq(\bar{s})}(B))}{|\Phi^{\preceq(\bar{s})}|},$$
$$Q^{\Phi}_{A|B}(Y_1, \ldots, Y_k) = \frac{\mathrm{rk}^{\Phi_*}(A|B)}{d_1!\cdots d_k!} Y_1^{d_1}\cdots Y_k^{d_k} + \textit{lower degree terms.}$$

The Φ_*-closure can be thought of as an analog of differentially algebraic closure; see [15, Chapter II, Section 8]. In fact, if each ϕ_i is a field derivation, then Φ_*-closure is exactly the differentially algebraic closure, and the corresponding rank rk^{Φ_*} is differential transcendence degree.

Outline. After some preliminaries in Section 1, we prove our main theorems in Sections 2 and 3. In Section 4, we collect some classical consequences of Theorem A, and in Section 5, we consider an application to simplicial complexes. Some applications of Corollary A for difference-differential fields (as well as difference-differential exponential fields and differential o-minimal fields) are considered in Section 6.

1. Preliminaries

1A. ***Notation and conventions.*** Throughout, $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$. Let $d \in \mathbb{N}^{>0}$, and let $\bar{r} = (r_1, \ldots, r_d)$ and $\bar{s} = (s_1, \ldots, s_d)$ range over $\mathbb{N}^d$. We write $\min(\bar{r})$ to mean $\min\{r_1, \ldots, r_d\}$. Let $\preceq$ denote the product order on $\mathbb{N}^d$, so

$$\bar{r} \preceq \bar{s} :\Longleftrightarrow r_i \leq s_i \text{ for each } i = 1, \ldots, d,$$

and let $<_{\text{lex}}$ denote the lexicographic order on $\mathbb{N}^d$ with emphasis on the last coordinate, so

$$\bar{r} <_{\text{lex}} \bar{s} :\Longleftrightarrow \text{there is } i \in \{1, \ldots, d\} \text{ such that } r_i < s_i \text{ and } r_j = s_j \text{ for } i < j \leq d.$$

Let $\bar{0}_d$ be the tuple $(0, \ldots, 0)$ consisting of d zeros, so $\bar{0}_d$ is the minimal element of $\mathbb{N}^d$ with respect to both of the orders $\preceq$ and $\leq_{\text{lex}}$. For $i \in \{1, \ldots, d\}$, let $\hat{e}_{i,d} := (0, \ldots, 1, \ldots, 0)$ be the tuple which consists of a 1 in the ith spot and zeros everywhere else.

We fix, for the remainder of this paper, numbers $0 < k \leq m \in \mathbb{N}$, as well as a partition

$$0 = m_1 < m_2 < \cdots < m_k < m_{k+1} = m.$$

We set $d_i := m_{i+1} - m_i$ for $i \in \{1, \ldots, k\}$, and for a tuple $\bar{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$, we set

$$\|\bar{r}\| := (r_{m_1+1} + \cdots + r_{m_1+d_1}, \ldots, r_{m_k+1} + \cdots + r_{m_k+d_k}) \in \mathbb{N}^k.$$

1B. ***Finitary matroids, triangular systems, and quasi-triangular systems.*** Recall that a *finitary matroid* consists of a set X, together with a map $\mathrm{c}\ell\colon \mathcal{P}(X) \to \mathcal{P}(X)$ which satisfies the following conditions:

(1) Reflexivity: $A \subseteq \mathrm{c}\ell(A)$.

(2) Monotonicity: if $A \subseteq B \subseteq X$, then $cl(A) \subseteq \mathrm{c}\ell(B)$.

(3) Idempotence: $\mathrm{c}\ell(\mathrm{c}\ell(A)) = \mathrm{c}\ell(A)$ for $A \subseteq X$.

(4) Finite character: if $A \subseteq X$ and $a \in \mathrm{c}\ell(A)$, then $a \in \mathrm{c}\ell(A_0)$ for some finite subset $A_0 \subseteq A$.

(5) Steinitz exchange: For $a, b \in X$ and $A \subseteq X$, if $a \in \mathrm{c}\ell(A \cup \{b\}) \setminus \mathrm{c}\ell(A)$, then $b \in \mathrm{c}\ell(A \cup \{a\})$.

More on finitary matroids can be found in [27], where they are called *independence spaces*. Finitary matroids often appear in model theory, where they are called *pregeometries*; see [28, Appendix C.1]. For the remainder of this paper, we fix a finitary matroid $(X, \mathrm{c}\ell)$, a positive natural number $m \in \mathbb{N}^{>0}$, and a finite tuple $\Phi := (\phi_1, \ldots, \phi_m)$ of commuting maps $X \to X$. We will usually use a, b to denote

elements of X and A, B to denote subsets of X. We will often abuse notation and write things like "$a \in \mathrm{c}\ell(ABb)$" to mean "$a \in \mathrm{c}\ell(A \cup B \cup \{b\})$."

For $A \subseteq X$, we let $\mathrm{c}\ell_A$ denote the following closure operator:

$$a \in \mathrm{c}\ell_A(B) :\Longleftrightarrow a \in \mathrm{c}\ell(AB).$$

Then $(X, \mathrm{c}\ell_A)$ is also a finitary matroid, called the *relativization of* $(X, \mathrm{c}\ell)$ *at* A. We say that B is $\mathrm{c}\ell$*-independent over* A if $b \notin \mathrm{c}\ell_A(B \setminus \{b\})$ for all $b \in B$. A *basis for* B *over* A is a subset $B_0 \subseteq B$ which is $\mathrm{c}\ell$-independent over A such that $B \subseteq \mathrm{c}\ell_A(B_0)$. Steinitz exchange ensures that any two bases for B over A have the same cardinality, called the *rank of* B *over* A and denoted $\mathrm{rk}(B|A)$. We just write $\mathrm{rk}(B)$ for $\mathrm{rk}(B|\varnothing)$, and we use rk_A for the rank corresponding to the relativization $(X, \mathrm{c}\ell_A)$, so $\mathrm{rk}(B|A) = \mathrm{rk}_A(B)$.

Let Θ be the free commutative monoid on Φ, so Θ consists of all operators $\phi^{\bar{r}} := \phi_1^{r_1} \cdots \phi_m^{r_m}$ for $\bar{r} \in \mathbb{N}^m$. Note that $\phi^{\bar{0}_m}$ is the identity map on X (and also the identity element of Θ) and that $\phi^{\hat{e}_{i,m}} = \phi_i$ for $i = 1, \ldots, m$. For $\theta \in \Theta$, we let $\theta(A) := \{\theta a : a \in A\}$, and for $\Theta_0 \subseteq \Theta$, we let

$$\Theta_0(A) := \bigcup_{\theta \in \Theta} \theta(A).$$

Recall from the introduction that Φ is a *triangular system* for $(X, \mathrm{c}\ell)$ if

$$a \in \mathrm{c}\ell(B) \Longrightarrow \phi_i a \in \mathrm{c}\ell(\phi_1 B \cdots \phi_i B)$$

for every $B \subseteq X$ and for each $i \in \{1, \ldots, m\}$. If Φ is a triangular system for $(X, \mathrm{c}\ell)$ and $A \subseteq X$ is closed under each map ϕ_i, then one can easily verify that Φ is a triangular system for the relativization $(X, \mathrm{c}\ell_A)$. The following lemma on triangular systems will be used in the proof of the main theorem.

Lemma 1.1. Φ *is a triangular system if and only if for any* $A, B \subseteq X$ *and for each* $i \in \{1, \ldots, m\}$, *we have*

$$\mathrm{rk}\big(\phi_i(A)\big|\phi_1(AB) \cdots \phi_{i-1}(AB)\phi_i(B)\big) \leq \mathrm{rk}(A|B).$$

Proof. Suppose that Φ is a triangular system. Let A_0 be a $\mathrm{c}\ell$-basis for A over B, so $A \subseteq \mathrm{c}\ell(A_0 B)$. Since Φ is a triangular system, we have

$$\phi_i(A) \subseteq \mathrm{c}\ell\big(\phi_1(A_0 B) \cdots \phi_i(A_0 B)\big) \subseteq \mathrm{c}\ell\big(\phi_1(AB) \cdots \phi_{i-1}(AB)\phi_i(A_0 B)\big).$$

This gives

$$\mathrm{rk}\big(\phi_i(A)\big|\phi_1(AB) \cdots \phi_{i-1}(AB)\phi_i(B)\big) \leq |\phi_i(A_0)| \leq |A_0| = \mathrm{rk}(A|B).$$

For the converse, let $a \in \mathrm{c}\ell(B)$. Then,

$$\mathrm{rk}\big(\phi_i(a)\big|\phi_1(aB) \cdots \phi_{i-1}(aB)\phi_i(B)\big) \leq \mathrm{rk}(a|B) = 0.$$

By induction on $i = 1, \dots, m$, we conclude that $\phi_i a \in \mathrm{c}\ell(\phi_1 B \cdots \phi_{i-1} B \phi_i B)$. $\square$

For the remainder of the paper, we let $(\Phi_1, \dots, \Phi_k)$ be the partition of Φ given in the introduction, so $\Phi_i := (\phi_{m_i+1}, \phi_{m_i+2}, \dots, \phi_{m_i+d_i})$ for each i. For $\bar{s} \in \mathbb{N}^k$ and $A \subseteq X$, we recall the sets

$$\Phi^{(\bar{s})}(A) := \big\{\phi^{\bar{r}}(a) : a \in A \text{ and } \|\bar{r}\| = \bar{s}\big\}, \quad \Phi^{\preceq(\bar{s})}(A) := \big\{\phi^{\bar{r}}(a) : a \in A \text{ and } \|\bar{r}\| \preceq \bar{s}\big\}.$$

For each $i \in \{1, \dots, k\}$, let

$$(\Phi_i)_* := (\mathrm{id}, \phi_{m_i+1}, \phi_{m_i+2}, \dots, \phi_{m_i+d_i}),$$

where $\mathrm{id}\colon X \to X$ is the identity map. Then $\big((\Phi_1)_*, (\Phi_2)_*, \dots, (\Phi_k)_*\big)$ is a partition of the augmented system

$$\Phi_* := (\mathrm{id}, \phi_{m_1+1}, \dots, \phi_{m_1+d_1}; \dots; \mathrm{id}, \phi_{m_k+1}, \dots, \phi_{m_k+d_k}).$$

Given $\bar{s} \in \mathbb{N}^k$ and $\bar{r} \in \mathbb{N}^m$ with $\|\bar{r}\| \preceq \bar{s}$, the map $\phi^{\bar{r}} \in \Phi^{\preceq(\bar{s})}$ acts the same way on X as the map $\mathrm{id}^{\bar{s}-\|\bar{r}\|}\, \phi^{\bar{r}} \in \Phi_*^{(\bar{s})}$, so we may identify $\Phi^{\preceq(\bar{s})}$ and $\Phi_*^{(\bar{s})}$. For each i, if Φ_i is quasi-triangular, then $(\Phi_i)_*$ is triangular, so many of our results on quasi-triangular systems will follow from the corresponding result on triangular systems, applied with Φ_* in place of Φ.

The main examples of (quasi)-triangular systems studied in this paper are tuples of (quasi)-endomorphisms. Let $\psi\colon X \to X$ be a map. We say that ψ is an *endomorphism of* $(X, \mathrm{c}\ell)$ (also called a "strong map" in the matroid literature) if

$$a \in \mathrm{c}\ell(B) \Longrightarrow \psi a \in \mathrm{c}\ell(\psi B).$$

We say that ψ is a *quasi-endomorphism of* $(X, \mathrm{c}\ell)$ if

$$a \in \mathrm{c}\ell(B) \Longrightarrow \psi a \in \mathrm{c}\ell(B \psi B).$$

If Φ is a (quasi)-triangular system, then ϕ_1 is necessarily a (quasi)-endomorphism, and if $\phi_1, \dots, \phi_m$ are (quasi)-endomorphisms, then Φ is (quasi)-triangular. Quasi-endomorphisms were first considered in [7, Section 3.1].

1C. ***Generating functions.*** Let $\bar{Y} = (Y_1, \dots, Y_k)$ be a tuple of variables. Given $\bar{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$, we write $\bar{Y}^{\bar{s}}$ for the monomial $Y_1^{s_1} \cdots Y_k^{s_k}$. A polynomial $P \in \mathbb{Q}[\bar{Y}]$ is said to have *degree at most* $\bar{s}$ if P has degree at most s_i in each variable Y_i, that is, if

$$P(\bar{Y}) = a\bar{Y}^{\bar{s}} + \text{lower degree terms}$$

for some $a \in \mathbb{Q}$.

Let $f : \mathbb{N}^k \to \mathbb{N}$ be a function. The *generating function of* f is the multivariate power series

$$G_f(\bar{Y}) = \sum_{\bar{s}\in\mathbb{N}^k} f(\bar{s})\bar{Y}^{\bar{s}} \in \mathbb{Z}[\![\bar{Y}]\!].$$

The following is well-known; we include here essentially the same proof given in [11, Lemma 2.1].

Lemma 1.2. *Suppose that G_f is a rational function with numerator $R(\bar{Y}) \in \mathbb{Q}[\bar{Y}]$ of degree at most $\bar{m}$ and denominator $(1-Y_1)^{d_1}\cdots(1-Y_k)^{d_k}$. Then there is $P \in \mathbb{Q}[\bar{Y}]$ of degree at most $(d_1-1,\ldots,d_k-1)$ such that $f(\bar{s}) = P(\bar{s})$ whenever $\bar{s} \succeq \bar{m}$. This polynomial P has the form*

$$P(\bar{Y}) = \frac{R(1,\ldots,1)}{(d_1-1)!\cdots(d_k-1)!}Y_1^{d_1-1}\cdots Y_k^{d_k-1} + \textit{lower degree terms}.$$

Proof. Write $R(\bar{Y}) = \sum_{\bar{r}\preceq\bar{m}} a_{\bar{r}}\bar{Y}^{\bar{r}}$, so

$$G_f(\bar{Y}) = \frac{R(\bar{Y})}{(1-Y_1)^{d_1}\cdots(1-Y_k)^{d_k}} = \sum_{\bar{r}\preceq\bar{m}} \frac{a_{\bar{r}}\bar{Y}^{\bar{r}}}{(1-Y_1)^{d_1}\cdots(1-Y_k)^{d_k}}.$$

For each $\bar{r}$, a simple computation gives

$$\frac{a_{\bar{r}}\bar{Y}^{\bar{r}}}{(1-Y_1)^{d_1}\cdots(1-Y_k)^{d_k}} = \sum_{\bar{s}\succeq\bar{r}} a_{\bar{r}}\binom{s_1-r_1+d_1-1}{d_1-1}\cdots\binom{s_k-r_k+d_k-1}{d_k-1}\bar{Y}^{\bar{s}}$$

Comparing coefficients, we get for each $\bar{s} \succeq \bar{m}$ that

$$f(\bar{s}) = \sum_{\bar{r}\preceq\bar{m}} a_{\bar{r}}\binom{s_1-r_1+d_1-1}{d_1-1}\cdots\binom{s_k-r_k+d_k-1}{d_k-1}.$$

Putting $P(\bar{Y}) := \sum_{\bar{r}\preceq\bar{m}} a_{\bar{r}}\binom{Y_1-r_1+d_1-1}{d_1-1}\cdots\binom{Y_k-r_k+d_k-1}{d_k-1}$, we have $f(\bar{s}) = P(\bar{s})$ for $\bar{s} \succeq \bar{m}$. Note that

$$\begin{aligned} P(\bar{Y}) &= \frac{\sum_{\bar{r}\preceq\bar{m}} a_{\bar{r}}}{(d_1-1)!\cdots(d_k-1)!}Y_1^{d_1-1}\cdots Y_k^{d_k-1} + \text{lower degree terms} \\ &= \frac{R(1,\ldots,1)}{(d_1-1)!\cdots(d_k-1)!}Y_1^{d_1-1}\cdots Y_k^{d_k-1} + \text{lower degree terms.} \end{aligned}$$

□

The function f is said to be *decreasing* if $f(\bar{r}) \le f(\bar{s})$ whenever $\bar{r} \succeq \bar{s}$. Suppose that f is decreasing. For $n \in \mathbb{N}$, set

$$S_n(f) := \{\bar{s} \in \mathbb{N}^k : f(\bar{s}) \le n\},$$

so each $S_n(f)$ is a $\preceq$-upward closed subset of $\mathbb{N}^k$ and $S_n(f) = \mathbb{N}^k$ for $n \ge f(\bar{0}_k)$. Let $M_n(f)$ be the set of $\preceq$-minimal elements of $S_n(f)$, so each $M_n(f)$ is finite by

Dickson's lemma. Set $M(f) := \bigcup_{n\in\mathbb{N}} M_n(f)$, and let $\bar{m}(f)$ be the $\preceq$-least upper bound of $M(f)$ (notice that the set $M(f)$ is finite).

Proposition 1.3. *If f is decreasing, then G_f is a rational function with numerator of degree at most $\bar{m}(f)$ and denominator $(1-Y_1)\cdots(1-Y_k)$.*

Proof. Suppose that f is decreasing, set $H := (1-Y_1)\cdots(1-Y_k)G_f$, and let $\bar{m}(f) = (m_1, \ldots, m_k)$. We need to show for each $i \in \{1, \ldots, k\}$ and each $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$ that if $s_i > m_i$, then the coefficient of $\bar{Y}^{\bar{s}}$ in H is zero. By symmetry, it suffices to consider the case $i = k$. Take power series $H_0, H_1, H_2, \ldots$ in the variables $(Y_1, \ldots, Y_{k-1})$ such that

$$H = H_0 + H_1 Y_k + H_2 Y_k^2 + \cdots.$$

We fix $t > m_k$, and we will show that $H_t = 0$. Distributing $(1-Y_k)$ through the series G_f, we see that

$$H_t(Y_1, \ldots, Y_{k-1}) = (1-Y_1)\cdots(1-Y_{k-1}) \sum_{\bar{r}\in\mathbb{N}^{k-1}} (f(\bar{r}, t) - f(\bar{r}, t-1)) Y_1^{r_1} \cdots Y_{k-1}^{r_{k-1}},$$

so it suffices to show that $f(\bar{r}, t) = f(\bar{r}, t-1)$ for each $\bar{r} \in \mathbb{N}^{k-1}$. Let $\bar{r}$ be given and let $n = f(\bar{r}, t)$. Take $\bar{s} \in M_n(f)$ with $\bar{s} \preceq (\bar{r}, t)$. Since $s_k \leq m_k < t$, we see that $(\bar{r}, t-1) \succeq \bar{s}$ as well, so $f(\bar{r}, t-1) \leq n$. Since f is decreasing, we conclude that $f(\bar{r}, t-1) = n = f(\bar{r}, t)$. □

Remark 1.4. We have another proof of Proposition 1.3, using that the partial order on decreasing functions $\mathbb{N}^k \to \mathbb{N}$ given by $f \leq g :\iff f(\bar{s}) \leq g(\bar{s})$ for all $\bar{s} \in \mathbb{N}^k$ is well-founded. The proof is as follows: let $f\colon \mathbb{N}^k \to \mathbb{N}$ be decreasing and assume that Proposition 1.3 holds for all decreasing functions $\mathbb{N}^k \to \mathbb{N}$ less than f, as well as all decreasing functions $\mathbb{N}^{k-1} \to \mathbb{N}$ (both base cases hold trivially). Let $g\colon \mathbb{N}^k \to \mathbb{N}$ be the function $(\bar{r}, t) \mapsto f(\bar{r}, t+1)$, so $g \leq f$, and let $h\colon \mathbb{N}^{k-1} \to \mathbb{N}$ be the function $\bar{r} \mapsto f(\bar{r}, 0)$. First, consider the case that $g = f$. Then $f(\bar{r}, t) = f(\bar{r}, 0) = h(\bar{r})$ for all $\bar{r} \in \mathbb{N}^{k-1}$ and all $t \in \mathbb{N}$ and so $\bar{m}(f) = (\bar{m}(h), 0)$. We have

$$\begin{aligned} G_f(\bar{Y}) &= \sum_{t\in\mathbb{N}} \sum_{\bar{r}\in\mathbb{N}^{k-1}} f(\bar{r}, 0)\bar{Y}^{(\bar{r},t)} \\ &= (1-Y_k)^{-1} \sum_{\bar{r}\in\mathbb{N}^{k-1}} h(\bar{r}) Y_1^{r_1} \cdots Y_{k-1}^{r_{k-1}} = \frac{G_h(Y_1, \ldots, Y_{k-1})}{(1-Y_k)}, \end{aligned}$$

so Proposition 1.3 holds for f by our induction hypothesis. Now, consider the case that $g < f$. In this case, $\bar{m}(f)$ is the $\preceq$-least upper bound of $(\bar{m}(h), 0)$ and $\bar{m}(g) + \hat{e}_{k,k}$. We have

$$G_f(\bar{Y}) = \sum_{\bar{r}\in\mathbb{N}^{k-1}} f(\bar{r},0)\bar{Y}^{(\bar{r},0)} + \sum_{t\in\mathbb{N}}\sum_{\bar{r}\in\mathbb{N}^{k-1}} f(\bar{r},t+1)\bar{Y}^{(\bar{r},t+1)}$$
$$= G_h(Y_1,\dots,Y_{k-1}) + Y_k G_g(\bar{Y}),$$

and we again conclude that Proposition 1.3 holds for f by our induction hypothesis.

2. The dimension and cumulative polynomials

In this section, we prove Theorem A and Corollary A. For the remainder of this section, let $A, B \subseteq X$ with A finite.

2A. ***The dimension polynomial.*** In this subsection, we prove Theorem A, and we sketch a slight generalization in Remark 2.2. We assume for the remainder of the subsection that each part of the partition Φ_i is a triangular system. For $\bar{u}\in\mathbb{N}^m$, set

$$\Theta_{\bar{u}} := \{\phi^{\bar{r}} : \|\bar{r}\| = \|\bar{u}\| \text{ and } \bar{r} <_{\text{lex}} \bar{u}\}, \qquad f^{\Phi}_{A|B}(\bar{u}) := \operatorname{rk}\big(\phi^{\bar{u}}(A)\big|\Theta_{\bar{u}}(A)\Phi^{(\|u\|)}(B)\big).$$

Then $f^{\Phi}_{A|B}$ is bounded above by $|A|$, and $\sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{A|B}(\bar{u}) = \operatorname{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))$ for each $\bar{s}\in\mathbb{N}^k$.

Lemma 2.1. *The function $f^{\Phi}_{A|B}$ is decreasing.*

Proof. Let $\bar{u}'\succeq\bar{u}$ be given. We may assume that

$$\bar{u}' = \bar{u} + \hat{e}_{m_i+d,m}$$

for some $i\in\{1,\dots,k\}$ and some $d\in\{1,\dots,d_i\}$. Since $\Phi_i = (\phi_{m_i+1},\dots,\phi_{m_i+d_i})$ is a triangular system, Lemma 1.1 tells us that

$$\operatorname{rk}\big(\phi_{m_i+d}\phi^{\bar{u}}(A)\big|\{\phi_{m_i+j}\phi^{\bar{u}}(A) : 0<j<d\}\cup\{\phi_{m_i+j}(\Theta_{\bar{u}}(A)\Phi^{(\|\bar{u}\|)}(B)) : 0<j\le d\}\big)$$

is at most $f^{\Phi}_{A|B}(\bar{u})$. For $j\in\{1,\dots,d-1\}$, we have $\|\hat{e}_{m_i+j,m}\| = \|\hat{e}_{m_i+d,m}\|$ and $\hat{e}_{m_i+j,m} <_{\text{lex}} \hat{e}_{m_i+d,m}$, so

$$\|\hat{e}_{m_i+j,m}+\bar{u}\| = \|\hat{e}_{m_i+d,m}+\bar{u}\| = \|\bar{u}'\|, \qquad \hat{e}_{m_i+j,m}+\bar{u} <_{\text{lex}} \hat{e}_{m_i+d,m}+\bar{u} = \bar{u}'.$$

It follows that $\phi_{m_i+j}\phi^{\bar{u}}\in\Theta_{\bar{u}'}$ for $j\in\{1,\dots,d-1\}$. Likewise, for $\phi^{\bar{r}}\in\Theta_{\bar{u}}$ and $j\in\{1,\dots,d\}$, we have

$$\|\hat{e}_{m_i+j,m}+\bar{r}\| = \|\hat{e}_{m_i+j,m}+\bar{u}\| = \|\bar{u}'\|, \qquad \hat{e}_{m_i+j,m}+\bar{r} <_{\text{lex}} \hat{e}_{m_i+j,m}+\bar{u} \le_{\text{lex}} \bar{u}',$$

so $\phi_{m_i+j}\Theta_{\bar{u}}\subseteq\Theta_{\bar{u}'}$. Finally, we have $\phi_{m_i+j}(\Phi^{(\|\bar{u}\|)})\subseteq\Phi^{(\|\bar{u}+\hat{e}_{m_i+j,m}\|)} = \Phi^{(\|\bar{u}'\|)}$ for $j\in\{1,\dots,d\}$, so $f^{\Phi}_{A|B}(\bar{u}')$ is at most

$$\operatorname{rk}\big(\phi_{m_i+d}\phi^{\bar{u}}(A)\big|\{\phi_{m_i+j}\phi^{\bar{u}}(A) : 0<j<d\}\cup\{\phi_{m_i+j}(\Theta_{\bar{u}}(A)\Phi^{(\|\bar{u}\|)}(B)) : 0<j\le d\}\big).$$

Thus $f^{\Phi}_{A|B}(\bar{u}')\le f^{\Phi}_{A|B}(\bar{u})$, as desired. □

Proof of Theorem A. We let $G^{\Phi}_{A|B}$ denote the generating function of the function $\bar{s} \mapsto \mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))$. We have

$$G^{\Phi}_{A|B}(\bar{Y}) = \sum_{\bar{s}\in\mathbb{N}^k} \mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))\bar{Y}^{\bar{s}}$$
$$= \sum_{\bar{s}\in\mathbb{N}^k} \sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{A|B}(\bar{u})\bar{Y}^{\bar{s}} = \sum_{\bar{u}\in\mathbb{N}^m} f^{\Phi}_{A|B}(\bar{u})\bar{Y}^{\|\bar{u}\|}.$$

The rightmost sum above is just the generating function of $f^{\Phi}_{A|B}$ with Y_1 substituted for the first d_1 variables, Y_2 substituted for the next d_2 variables, and so on. By Proposition 1.3 and Lemma 2.1, we have that $G^{\Phi}_{A|B}$ is a rational function with numerator of degree at most $\|\bar{m}(f^{\Phi}_{A|B})\|$ and denominator $(1-Y_1)^{d_1}\cdots(1-Y_k)^{d_k}$. By Lemma 1.2, we conclude that there is a polynomial $P^{\Phi}_{A|B}(\bar{Y}) \in \mathbb{Q}[\bar{Y}]$ of degree at most $(d_1-1,\ldots,d_k-1)$ such that

$$\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B)) = P^{\Phi}_{A|B}(\bar{s})$$

for $\bar{s} \in \mathbb{N}^k$ with $\bar{s} \succeq \|\bar{m}(f^{\Phi}_{A|B})\|$. □

Remark 2.2. Let $\Psi = (\psi_1,\ldots,\psi_n)$ be another tuple of commuting maps $X \to X$, and let $(\Psi_1,\ldots,\Psi_k)$ be a partition of Ψ. Suppose that for each $i \in \{1,\ldots,k\}$, the tuple Φ_i is a subtuple of Ψ_i. One can prove the following generalization of Theorem A:

There is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ of degree at most $d_i - 1$ in each variable Y_i such that

$$\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Psi^{(\bar{s})}(B)) = P(\bar{s})$$

for $\bar{s} \in \mathbb{N}^k$ with $\min(\bar{s})$ sufficiently large.

Note that no assumptions beyond commutativity are imposed on the maps in $\Psi_i \setminus \Phi_i$. All that is required is that each tuple Φ_i is triangular. To prove this generalization, replace the function $f^{\Phi}_{A|B}$ in the above proof with a function $f^{\Phi,\Psi}_{A|B}$, where

$$f^{\Phi,\Psi}_{A|B}(\bar{u}) := \mathrm{rk}\big(\phi^{\bar{u}}(A)\big|\Theta_{\bar{u}}(A)\Psi^{(\|u\|)}(B)\big)$$

for $\bar{u} \in \mathbb{N}^m$. With the obvious changes, the proof of Lemma 2.1 shows that $f^{\Phi,\Psi}_{A|B}$ is decreasing.

2B. ***The cumulative dimension polynomial.*** Now we turn to Corollary A. We will also investigate the dominant terms of the cumulative dimension polynomial. We assume for the remainder of the subsection that each part of the partition Φ_i is a quasi-triangular system.

Proof of Corollary A. Recall the augmented system $\Phi_* = \big((\Phi_1)_*, (\Phi_2)_*, \ldots, (\Phi_k)_*\big)$, where $(\Phi_i)_* = (\mathrm{id}, \phi_{m_i+1}, \phi_{m_i+2}, \ldots, \phi_{m_i+d_i})$ for each $i \in \{1,\ldots,k\}$. Applying

Theorem A with Φ_* in place of Φ, we get a polynomial $P^{\Phi_*}_{A|B}(\bar{Y}) \in \mathbb{Q}[\bar{Y}]$ of degree at most d_i in each variable Y_i such that

$$\mathrm{rk}(\Phi^{\preceq(\bar{s})}(A)|\Phi^{\preceq(\bar{s})}(B)) = \mathrm{rk}(\Phi_*^{(\bar{s})}(A)|\Phi_*^{(\bar{s})}(B)) = P^{\Phi_*}_{A|B}(\bar{s})$$

for $\bar{s} \in \mathbb{N}^k$ with $\min(\bar{s})$ sufficiently large. Take $Q^{\Phi}_{A|B} := P^{\Phi_*}_{A|B}$. □

Now we turn to the dominant terms. Given a polynomial

$$Q(\bar{Y}) = \sum_{\bar{e}\in\mathbb{N}^k} a_{\bar{e}}\bar{Y}^{\bar{e}} \in \mathbb{Q}[\bar{Y}],$$

the *support of* Q is the set $\mathrm{supp}(Q) := \{\bar{e} : a_{\bar{e}} \neq 0\} \subseteq \mathbb{N}^k$. We extend the partial order $\preceq$ on $\mathbb{N}^k$ to all of $\mathbb{R}^k$, so for $\bar{x}, \bar{y} \in \mathbb{R}^k$, we have $\bar{x} \preceq \bar{y} \iff x_i \leq y_i$ for $i = 1, \ldots, k$. We set

$$\mathcal{M}(Q) := \left\{\bar{y} \in \mathbb{R}^k : \bar{y} \preceq \bar{e} \text{ for some } \bar{e} \in \mathrm{supp}(Q)\right\}.$$

That is, $\mathcal{M}(Q)$ is the Minkowski sum of the orthant $(-\infty, 0]^k$ and the Newton polytope of Q (the convex hull of the support of Q). We define the *dominant terms of* Q to be the terms $a_{\bar{d}}\bar{Y}^{\bar{d}}$ such that $\bar{d}$ is a vertex of $\mathcal{M}(Q)$.

As explained in [8, Section 3], the dominant terms of Q can be recovered using limits: the term $a_{\bar{d}}\bar{Y}^{\bar{d}}$ is a dominant term of Q if and only if

$$\lim_{t\to\infty} \frac{Q(t^{w_1}, t^{w_2}, \ldots, t^{w_k})}{a_{\bar{d}}t^{\bar{d}\cdot\bar{w}}} = 1. \tag{2-1}$$

for some positive vector $\bar{w} \in \mathbb{R}^k$ with positive $\mathbb{Q}$-linearly independent entries. Indeed, for such a vector $\bar{w}$, the map $\bar{e} \mapsto \bar{e}\cdot\bar{w} \colon \mathbb{N}^k \to \mathbb{R}$ is injective, and equation (2-1) can be readily verified for $\bar{d} \in \mathrm{supp}(Q)$ with $\bar{d}\cdot\bar{w} = \max\{\bar{e}\cdot\bar{w} : \bar{e} \in \mathrm{supp}(Q)\}$. It remains to note that $\bar{d} \in \mathrm{supp}(Q)$ is a vertex of $\mathcal{M}(Q)$ if and only if the sector

$$\left\{\bar{w} \in (\mathbb{R}^{>0})^k : (\bar{d} - \bar{e})\cdot\bar{w} > 0 \text{ for all } \bar{e} \in \mathrm{supp}(Q) \setminus \{\bar{d}\}\right\}$$

is nonempty. The vectors $\bar{w}$ in this sector are exactly those vectors for which (2-1) holds.

In the case that $B = \Theta(B)$, the dominant terms of $Q^{\Phi}_{A|B}$ only depend on $\mathrm{c}\ell(\Theta(A)B)$:

Proposition 2.3. *Let A, A' be finite subsets of X, let $B \subseteq X$ with $\Theta(B) = B$, and suppose that $\mathrm{c}\ell(\Theta(A)B) = \mathrm{c}\ell(\Theta(A')B)$. Then $Q^{\Phi}_{A|B}$ and $Q^{\Phi}_{A'|B}$ have the same dominant terms.*

Proof. Take $\bar{s}_0$ with

$$A' \subseteq \mathrm{c}\ell(\Phi^{\preceq(\bar{s}_0)}(A)B), \quad A \subseteq \mathrm{c}\ell(\Phi^{\preceq(\bar{s}_0)}(A')B).$$

Since each Φ_i is quasi-triangular, a routine induction on $|\bar{s}| = s_1 + \cdots + s_k$ gives

$$\Phi^{\preceq(\bar{s})}(A') \subseteq \mathrm{c}\ell(\Phi^{\preceq(\bar{s}_0+\bar{s})}(A)B), \quad \Phi^{\preceq(\bar{s})}(A) \subseteq \mathrm{c}\ell(\Phi^{\preceq(\bar{s}_0+\bar{s})}(A')B),$$

for all $\bar{s} \in \mathbb{N}^k$. It follows that

$$\mathrm{rk}(\Phi^{\preceq(\bar{s})}(A')|B) \leq \mathrm{rk}(\Phi^{\preceq(\bar{s}_0+\bar{s})}(A)|B), \quad \mathrm{rk}(\Phi^{\preceq(\bar{s})}(A)|B) \leq \mathrm{rk}(\Phi^{\preceq(\bar{s}_0+\bar{s})}(A')|B)$$

for each $\bar{s}$. Taking $\min(\bar{s})$ to be sufficiently large, we get

$$Q^{\Phi}_{A'|B}(\bar{s}) \leq Q^{\Phi}_{A|B}(\bar{s}_0 + \bar{s}), \quad Q^{\Phi}_{A|B}(\bar{s}) \leq Q^{\Phi}_{A'|B}(\bar{s}_0 + \bar{s}).$$

The proposition follows, using that the limits (2-1) agree for $Q^{\Phi}_{A|B}$ and $Q^{\Phi}_{A'|B}$. □

3. The Φ-closure and Φ_*-closure operators

As in the previous section, we fix subsets $A, B \subseteq X$ with A finite. Recall that the Φ-*closure* operator is given by

$$a \in \mathrm{c}\ell^{\Phi}(B) :\Longleftrightarrow \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) < |\Phi^{(\bar{s})}| \text{ for some } \bar{s} \in \mathbb{N}^k.$$

The Φ_*-*closure operator* is defined identically, but with Φ_* in place of Φ, so

$$a \in \mathrm{c}\ell^{\Phi_*}(B) \Longleftrightarrow \mathrm{rk}(\Phi^{\preceq(\bar{s})}(a)|\Phi^{\preceq(\bar{s})}(B)) < |\Phi^{\preceq(\bar{s})}| \text{ for some } \bar{s} \in \mathbb{N}^k.$$

In this section, we prove Theorem B. We also examine whether the rank associated to these operators depends on our choice of partition, and we discuss an extension of the Φ_*-closure operator to more general monoid actions by matroid endomorphisms. In order to prove Theorem B, we need the following lemma.

Lemma 3.1. *Suppose that each Φ_i is a triangular system. Then for $a \in X$, we have*

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|} = \begin{cases} 0 & \text{if } a \in \mathrm{c}\ell^{\Phi}(B), \\ 1 & \text{if } a \notin \mathrm{c}\ell^{\Phi}(B). \end{cases}$$

Proof. If $a \notin \mathrm{c}\ell^{\Phi}(B)$, then $\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) = |\Phi^{(\bar{s})}|$ for all $\bar{s} \in \mathbb{N}^k$, so

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|} = 1.$$

Suppose a belongs to $\mathrm{c}\ell^{\Phi}(B)$, as witnessed by $\bar{s}_0 \in \mathbb{N}^k$. Then we have

$$\sum_{\|\bar{u}\|=\bar{s}_0} f^{\Phi}_{a|B}(\bar{u}) = \mathrm{rk}(\Phi^{(\bar{s}_0)}(a)|\Phi^{(\bar{s}_0)}(B)) < |\Phi^{(\bar{s}_0)}|.$$

It follows that $f^{\Phi}_{a|B}(\bar{u}_0) = 0$ for some $\bar{u}_0 \in \mathbb{N}^m$ with $\|\bar{u}_0\| = \bar{s}_0$. By Lemma 2.1, we

have $f^{\Phi}_{a|B}(\bar{u}) = 0$ whenever $\bar{u} \succeq \bar{u}_0$. Let $\bar{s} \in \mathbb{N}^k$ with $\bar{s} \succeq \bar{s}_0$. Then

$$\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) = \sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{a|B}(\bar{u}) = \sum_{\substack{\|\bar{u}\|=\bar{s}\\ \bar{u}\not\succeq\bar{u}_0}} f^{\Phi}_{a|B}(\bar{u}) \le |\Phi^{(\bar{s})}| - |\Phi^{(\bar{s}-\bar{s}_0)}|.$$

Since $\dfrac{|\Phi^{(\bar{s}-\bar{s}_0)}|}{|\Phi^{(\bar{s})}|}$ approaches 1 as $\min(\bar{s})$ grows, we have

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|} = 0. \qquad \square$$

Proof of Theorem B. Let us first assume that each Φ_i is a triangular system. We need to show that $(X, \mathrm{c}\ell^{\Phi})$ is a finitary matroid. Monotonicity and finite character are both clear. For idempotence, let $a \in \mathrm{c}\ell^{\Phi}(\mathrm{c}\ell^{\Phi}(B))$ and, using finite character, take elements $b_1, \dots, b_n \in \mathrm{c}\ell^{\Phi}(B)$ with $a \in \mathrm{c}\ell^{\Phi}(b_1, \dots, b_n)$. We have

$$\begin{aligned}\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) &\le \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(b_1, \dots, b_n)) + \mathrm{rk}(\Phi^{(\bar{s})}(b_1, \dots, b_n)|\Phi^{(\bar{s})}(B))\\ &\le \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(b_1, \dots, b_n)) + \sum_{i=1}^{n} \mathrm{rk}(\Phi^{(\bar{s})}(b_i)|\Phi^{(\bar{s})}(B)).\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|}\\ &\quad\le \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(b_1, \dots, b_n)) + \sum_{i=1}^{n} \mathrm{rk}(\Phi^{(\bar{s})}(b_i)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|}.\end{aligned}$$

We conclude by Lemma 3.1 that $a \in \mathrm{c}\ell^{\Phi}(B)$. Finally, for exchange, suppose that $a \in \mathrm{c}\ell^{\Phi}(Bb) \setminus \mathrm{c}\ell^{\Phi}(B)$. Since $\mathrm{c}\ell^{\Phi}$ has finite character, we may assume that B is finite. Idempotence tells us that $b \notin \mathrm{c}\ell^{\Phi}(B)$, so

$$\mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) = |\Phi^{(\bar{s})}| = \mathrm{rk}(\Phi^{(\bar{s})}(b)|\Phi^{(\bar{s})}(B))$$

for all $\bar{s}$. It follows that

$$\begin{aligned}\mathrm{rk}(\Phi^{(\bar{s})}(b)|\Phi^{(\bar{s})}(Ba)) &= \mathrm{rk}(\Phi^{(\bar{s})}(Bab)) - \mathrm{rk}(\Phi^{(\bar{s})}(Ba))\\ &= \mathrm{rk}(\Phi^{(\bar{s})}(Bab)) - \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(B)) - \mathrm{rk}(\Phi^{(\bar{s})}(B))\\ &= \mathrm{rk}(\Phi^{(\bar{s})}(Bab)) - \mathrm{rk}(\Phi^{(\bar{s})}(b)|\Phi^{(\bar{s})}(B)) - \mathrm{rk}(\Phi^{(\bar{s})}(B))\\ &= \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(Bb)).\end{aligned}$$

Since $a \in \mathrm{c}\ell^{\Phi}(Bb)$, we see that $b \in \mathrm{c}\ell^{\Phi}(Ba)$.

Now we turn to the properties of the rank function rk^{Φ} associated to the matroid $(X, \mathrm{c}\ell^{\Phi})$. Let us show that

$$\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|} = \mathrm{rk}^{\Phi}(A|B). \tag{3-1}$$

We prove this by induction on $|A|$, with the case $A = \varnothing$ holding trivially. Suppose that this holds for a given A, and let $a \in X \setminus A$. We have

$$\begin{aligned}\lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(Aa)|\Phi^{(\bar{s})}(B))}{|\Phi^{(\bar{s})}|} &= \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B)) + \mathrm{rk}(\Phi^{(\bar{s})}(a)|\Phi^{(\bar{s})}(AB))}{|\Phi^{(\bar{s})}|} \\ &= \mathrm{rk}^{\Phi}(A|B) + \mathrm{rk}^{\Phi}(a|AB) = \mathrm{rk}^{\Phi}(Aa|B),\end{aligned}$$

where the second equality follows from the induction hypothesis and Lemma 3.1.

Finally, we will show that

$$P^{\Phi}_{A|B}(\bar{Y}) = \frac{\mathrm{rk}^{\Phi}(A|B)}{(d_1-1)!\cdots(d_k-1)!} Y_1^{d_1-1}\cdots Y_k^{d_k-1} + \text{lower degree terms}.$$

As $\min(\bar{s})$ grows, we have

$$P^{\Phi}_{A|B}(\bar{s}) = \mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B)), \quad |\Phi^{(\bar{s})}| = \frac{s_1^{d_1-1}\cdots s_k^{d_k-1}}{(d_1-1)!\cdots(d_k-1)!} + o(s_1^{d_1-1}\cdots s_k^{d_k-1}),$$

so the leading coefficient of $P^{\Phi}_{A|B}$ is equal to the limit

$$\begin{aligned}\lim_{\min(\bar{s})\to\infty} \frac{P^{\Phi}_{A|B}(\bar{s})}{s_1^{d_1-1}\cdots s_k^{d_k-1}} &= \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))}{(d_1-1)!\cdots(d_k-1)!|\Phi^{(\bar{s})}|} \\ &= \frac{\mathrm{rk}^{\Phi}(A|B)}{(d_1-1)!\cdots(d_k-1)!},\end{aligned}$$

where the last equality follows from (3-1).

The second part of Theorem B, involving quasi-triangular systems and the Φ_*-closure, follows from the first part with Φ_* in place of Φ. □

Remark 3.2. By Theorem B and Lemma 1.2, the Φ-rank $\mathrm{rk}^{\Phi}(A|B)$ coincides with the numerator of $G^{\Phi}_{A|B}(\bar{Y})$, evaluated at the tuple $(1,\ldots,1)$.

3A. ***Dependence on our choice of partition.*** The Φ-closure operator is dependent on our partition $(\Phi_1,\ldots,\Phi_k)$, as the following example illustrates:

Example 3.3. Let $(X, \mathrm{c}\ell)$ be the set $\mathbb{Z}$ with the trivial closure operator, so the rank of any subset of $\mathbb{Z}$ is its cardinality. Let $a \in \mathbb{Z}$, let ϕ_1 be the map $x \mapsto x+1$, let $\phi_2 = \phi_1$, and let $\Phi = (\phi_1, \phi_2)$. First, suppose Φ is partitioned trivially (so $k = 1$). Then $\Phi^{(t)}(a) = \{a + t\}$ for any $t \in \mathbb{N}$, so $\mathrm{rk}(\Phi^{(t)}(a)) = 1 < |\Phi^{(t)}|$ for $t > 0$ and $\mathrm{rk}^{\Phi}(a) = 0$. Now, suppose Φ is given the partition (Φ_1, Φ_2), where $\Phi_1 = (\phi_1)$ and $\Phi_2 = (\phi_2)$. Then $\Phi^{(s_1,s_2)}(a) = \{a + s_1 + s_2\}$ for $s_1, s_2 \in \mathbb{N}$, so $\mathrm{rk}(\Phi^{(s_1,s_2)}(a)) = 1 = |\Phi^{(s_1,s_2)}|$ and $\mathrm{rk}^{\Phi}(a) = 1$.

On the other hand, the Φ_*-closure operator is more robust, as it does not depend on our partition:

Proposition 3.4. *For $a \in X$, we have $a \in \mathrm{c}\ell^{\Phi_*}(B)$ if and only if $(\theta a)_{\theta\in\Theta}$ is not $\mathrm{c}\ell$-independent over $\Theta(B)$.*

Proof. Clearly, if $a \in \mathrm{c}\ell^{\Phi_*}(B)$, then $(\theta a)_{\theta\in\Theta}$ is not $\mathrm{c}\ell$-independent over $\Theta(B)$. Suppose that $(\theta a)_{\theta\in\Theta}$ is not $\mathrm{c}\ell$-independent over $\Theta(B)$, and take a finite subset $\Theta_0 \subseteq \Theta$ with $\mathrm{rk}(\Theta_0(a)|\Theta(B)) < |\Theta_0|$. Since $\mathrm{c}\ell$ has finite character, we can take $\bar{s} \in \mathbb{N}^k$ with $\Theta_0 \subseteq \Phi^{\preceq(\bar{s})}$ such that $\mathrm{rk}(\Theta_0(a)|\Phi^{\preceq(\bar{s})}(B)) < |\Theta_0|$. Set $\Theta_1 := \Phi^{\preceq(\bar{s})} \setminus \Theta_0$, so

$$\begin{aligned}\mathrm{rk}(\Phi^{\preceq(\bar{s})}(a)|\Phi^{\preceq(\bar{s})}(B)) &\leq \mathrm{rk}(\Theta_0(a)|\Phi^{\preceq(\bar{s})}(B)) + \mathrm{rk}(\Theta_1(a)|\Phi^{\preceq(\bar{s})}(B))\\ &< |\Theta_0| + |\Theta_1| = |\Phi^{\preceq(\bar{s})}|.\end{aligned}$$

We conclude that $a \in \mathrm{c}\ell^{\Phi_*}(B)$. □

Corollary 3.5. *Suppose that each Φ_i is a quasi-triangular system. Then $\mathrm{rk}^{\Phi_*}(A|B)$ is the maximal size of a subset $A_0 \subseteq \mathrm{c}\ell(\Theta(AB))$ such that $(\theta a)_{\theta\in\Theta, a\in A_0}$ is $\mathrm{c}\ell$-independent over $\Theta(B)$.*

Proof. Proposition 3.4 tells us that $\mathrm{rk}^{\Phi_*}(A|B)$ is the maximal size of a subset $A_0 \subseteq A$ such that $(\theta a)_{\theta\in\Theta, a\in A_0}$ is $\mathrm{c}\ell$-independent over $\Theta(B)$. Let A' be a finite subset of $\mathrm{c}\ell(\Theta(AB))$ such that $(\theta a)_{\theta\in\Theta, a\in A'}$ is $\mathrm{c}\ell$-independent over $\Theta(B)$. We will show that $|A'| \leq \mathrm{rk}^{\Phi_*}(A|B)$. Again using Proposition 3.4, we may assume that $B = \Theta(B)$. Arguing as in Proposition 2.3, we find $\bar{s}_0 \in \mathbb{N}^k$ such that $Q^{\Phi}_{A'|B}(\bar{s}) \leq Q^{\Phi}_{A|B}(\bar{s}_0 + \bar{s})$ for all $\bar{s}$ with $\min(\bar{s})$ sufficiently large. Then $\mathrm{rk}^{\Phi_*}(A'|B) \leq \mathrm{rk}^{\Phi_*}(A|B)$ by Theorem B, and it remains to note that $|A'| = \mathrm{rk}^{\Phi_*}(A'|B)$. □

3B. ***Monoid actions by endomorphisms.*** Suppose that each ϕ_i is an endomorphism, that is, suppose

$$a \in \mathrm{c}\ell(B) \Longrightarrow \phi_i a \in \mathrm{c}\ell(\phi_i B)$$

for each i. Then $\phi^{\bar{r}}$ is an endomorphism for each $\bar{r} \in \mathbb{N}^m$, so $(\bar{r}, a) \mapsto \phi^{\bar{r}}(a)$ gives us a monoid action $\mathbb{N}^m \curvearrowright (X, \mathrm{c}\ell)$ by endomorphisms. In this case, each Φ_i is a triangular system, so $(X, \mathrm{c}\ell^{\Phi_*})$ is also a finitary matroid and

$$\mathrm{rk}^{\Phi_*}(A) = \lim_{\min(\bar{s})\to\infty} \frac{\mathrm{rk}(\Phi^{\preceq(\bar{s})}(A))}{|\Phi^{\preceq(\bar{s})}|}$$

by Theorem B. Thus, we can view $\mathrm{rk}^{\Phi_*}(A)$ as an average of $\mathrm{rk}(A)$ over the action $\mathbb{N}^m \curvearrowright (X, \mathrm{c}\ell)$, the averaging being with respect to the net $\big(\{\bar{r} \in \mathbb{N}^m : \|\bar{r}\| \preceq \bar{s}\}\big)_{\bar{s}\in\mathbb{N}^k}$ of subsets of $\mathbb{N}^m$.

While our result on polynomial growth seems limited to the context of $\mathbb{N}^m$ acting on $(X, \mathrm{c}\ell)$, one can make sense of this "averaging" for the right action of any

cancellative left-amenable monoid, using the main theorem from [4]. Let M be a cancellative left-amenable semigroup and let $\alpha\colon X \times M \to X$ be a right action $(X, \mathrm{c}\ell) \curvearrowleft M$ by endomorphisms. Given finite subsets $A \subseteq X$ and $S \subseteq M$, we put $r_A(S) := \mathrm{rk}(\alpha(A, S))$, where $\alpha(A, S) = \bigcup_{(a,s)\in A\times S} \alpha(a, s)$. It is routine to check that r_A, as a map from finite subsets of M to $\mathbb{N}$, satisfies the three conditions in the statement of [4, Theorem 1.1]; for the second condition, one needs to use that s is a matroid endomorphism and that our action is on the *right*. It follows that there is a number $\mathrm{rk}^M(A)$, depending only on M and A, such that

$$\lim_i \frac{\mathrm{rk}(\alpha(A, F_i))}{|F_i|} = \lim_i \frac{r_A(F_i)}{|F_i|} = \mathrm{rk}^M(A)$$

for any left-Følner net $(F_i)_{i\in I}$ of M (such nets always exist). Then $\mathrm{rk}^M(A)$ serves as this "averaged" rank.

4. Applications I: Some classical results

In this section, we give a handful of applications in the case that each ϕ_i is an endomorphism of $(X, \mathrm{c}\ell)$.

4A. ***Polynomial growth of sumsets.*** Let G be a commutative semigroup. In [12], Khovanskii showed that for finite subsets $A, B \subseteq G$, the size of the sumset $A + tB$ is given by a polynomial in t for t sufficiently large (here, $A + tB$ is the set of all elements $a + b_1 + \cdots + b_t$, where $a \in A$ and each $b_i \in B$). Moreover, the degree of this polynomial is less than the size of B. Using Theorem A, we can recover a generalization of Khovanskii's theorem, originally proven by Nathanson [25].

Corollary 4.1 (Nathanson). *Let $A, B_1, \ldots, B_k$ be finite subsets of G. Then there is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$|A + s_1B_1 + \cdots + s_kB_k| = P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large. Moreover, the degree of P in each variable Y_i is less than $|B_i|$.*

Proof. Let $(X, \mathrm{c}\ell)$ be the underlying set of the semigroup G with the trivial closure operator, so the rank of any subset of G is its cardinality. Let $b_1, \ldots, b_{m_1}$ be an enumeration of B_1, let $b_{m_1+1}, \ldots, b_{m_2}$ be an enumeration of B_2, and so on. For each $i \in \{1, \ldots, m\}$, let $\phi_i\colon G \to G$ be the map $x \mapsto x + b_i$, so each ϕ_i is an endomorphism. Then

$$\Phi^{(\bar{s})}(A) = A + s_1B_1 + \cdots + s_kB_k$$

for each $\bar{s} = (s_1, \ldots, s_k) \in \mathbb{N}^k$, and it remains to invoke Theorem A. □

The set B in Theorem A makes no appearance in the argument above. This suggests a slight improvement:

Corollary 4.2. *Let $A, B_1, \ldots, B_k$ be finite subsets of G and let B be an arbitrary subset of G. Then there is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$|(A + s_1B_1 + \cdots + s_kB_k) \setminus (B + s_1B_1 + \cdots + s_kB_k)| = P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

4B. ***Counting elements in an ideal.*** In this next application, we make use of the fact that if ϕ is an endomorphism of $(X, \mathrm{c}\ell)$ and $\phi(C) \subseteq C$ for some $C \subseteq X$, then ϕ is also an endomorphism of $(X, \mathrm{c}\ell_C)$.

Corollary 4.3. *Let I be an ideal of $\mathbb{N}^m$, that is, a $\preceq$-downward closed subset of $\mathbb{N}^m$. For each $\bar{s} \in \mathbb{N}^k$, let $H_I(\bar{s})$ be the number of elements $\bar{r} \in I$ with $\|\bar{r}\| = \bar{s}$. Then there is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$H_I(\bar{s}) = P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Proof. Let $C := \mathbb{N}^m \setminus I$, let $(X, \mathrm{c}\ell)$ be the set $\mathbb{N}^m$ with the trivial closure operator, and let $(X, \mathrm{c}\ell_C)$ be the relativization of $(X, \mathrm{c}\ell)$ at C. Then $\mathrm{rk}_C(Y) = |Y \setminus C| = |Y \cap I|$ for any subset $Y \subseteq \mathbb{N}^m$. For each $i \in \{1, \ldots, m\}$, let $\phi_i \colon \mathbb{N}^m \to \mathbb{N}^m$ be the map $\bar{r} \mapsto \bar{r} + \hat{e}_{i,m}$, so

$$\mathrm{rk}_C(\Phi^{(\bar{s})}(\bar{0}_m)) = |\{\bar{r} \in \mathbb{N}^m : \|\bar{r}\| = \bar{s}\} \cap I| = H_I(\bar{s}).$$

Since I is $\preceq$-downward closed, we have $\phi_i(C) \subseteq C$ for each $i \in \{1, \ldots, m\}$. Thus, $\phi_1, \ldots, \phi_m$ are commuting endomorphisms of $(X, \mathrm{c}\ell_C)$, and the corollary follows from Theorem A. □

The following variant of Corollary 4.3 was established by Kondratieva, Levin, Mikhalev, and Pankratiev [16], with the $k = 1$ case first proven by Kolchin [15, Chapter 0, Lemma 16]. This variant can be deduced in the same way as Corollary 4.3; just use Corollary A in place of Theorem A:

Corollary 4.4 [16, Theorem 2.2.5]. *Let I be an ideal of $\mathbb{N}^m$, and for each $\bar{s} \in \mathbb{N}^k$, let $H_I^*(\bar{s})$ be the number of elements $\bar{r} \in I$ with $\|\bar{r}\| \preceq \bar{s}$. Then there is a polynomial $Q \in \mathbb{Q}[\bar{Y}]$ such that*

$$H_I^*(\bar{s}) = Q(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

4C. ***The Hilbert polynomial.*** Let K be a field and let $R := K[x_1, \dots, x_m]$, where $x_1, \dots, x_m$ are indeterminates. Using our partition $(\Phi_1, \dots, \Phi_k)$, we associate to R an $\mathbb{N}^k$-grading $R = \bigoplus_{\bar{s}\in\mathbb{N}^k} R_{\bar{s}}$ as follows: the graded part $R_{\bar{s}}$ is the K-vector space generated by monomials $x_1^{r_1} \cdots x_m^{r_m}$ with $\|\bar{r}\| = \bar{s}$. Let $M = \bigoplus_{\bar{s}\in\mathbb{Z}^k} M_{\bar{s}}$ be a finitely generated graded R-module. Then each graded piece $M_{\bar{s}}$ is a finite-dimensional K-vector space. The *Hilbert function of* M, denoted $H_M \colon \mathbb{Z}^k \to \mathbb{N}$, is given by $H_M(\bar{s}) := \dim_K(M_{\bar{s}})$. The following result is classical:

Corollary 4.5. *There is a polynomial* $P \in \mathbb{Q}[\bar{Y}]$ *such that*

$$H_M(\bar{s}) = P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Proof. Let $(X, \mathrm{c}\ell)$ be the underlying set of our R-module M with the K-linear closure operator (that is, the closure of any subset of M is its K-linear span). For each $i \in \{1, \dots, m\}$, let $\phi_i \colon M \to M$ be the map $a \mapsto x_i \cdot a$. Since M is finitely generated, we can find an index $\bar{s}_0$ and a finite set $A \subseteq M_{\bar{s}_0}$ such that the R-submodule $\bigoplus_{\bar{s}\geq\bar{s}_0} M_{\bar{s}}$ is generated by A. By re-indexing, we may assume that $\bar{s}_0 = \bar{0}_k$. Then for each $\bar{s} \in \mathbb{N}^k$, the graded part $M_{\bar{s}}$ is generated as a K-vector space by $\Phi^{(\bar{s})}(A)$, so

$$\dim_K(M_{\bar{s}}) = \mathrm{rk}(M_{\bar{s}}) = \mathrm{rk}(\Phi^{(\bar{s})}(A)).$$

The corollary follows from Theorem A. □

While we deduce Corollary 4.5 from Theorem A, one can also deduce Theorem A from Corollary 4.5 using the decreasing function $f^{\Phi}_{A|B}$ that we constructed and exploited in Section 2A:

Proposition 4.6. *Let* A, B, *and* $\Phi = (\Phi_1, \dots, \Phi_k)$ *be as in Theorem A. There is a finitely generated graded* R*-module* $M = \bigoplus_{\bar{s}\in\mathbb{N}^k} M_{\bar{s}}$ *such that* $\dim_K M_{\bar{s}} = \mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B))$ *for all* $\bar{s} \in \mathbb{N}^k$.

Proof. Let $f^{\Phi}_{A|B} \colon \mathbb{N}^m \to \mathbb{N}$ be as in the proof of Theorem A, so $f^{\Phi}_{A|B}$ is decreasing and bounded above by $|A|$. For each $n \geq 1$, we put $I_n = \{\bar{u} \in \mathbb{N}^m : f^{\Phi}_{A|B}(\bar{u}) \geq n\}$, so I_n is an ideal of $\mathbb{N}^m$, as defined in Corollary 4.3, and $I_n = \varnothing$ whenever $n > |A|$. Let χ_n be the indicator function of I_n, so $f^{\Phi}_{A|B} = \chi_1 + \chi_2 + \cdots$ and for $\bar{s} \in \mathbb{N}^k$, we have

$$\mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B)) = \sum_{\|\bar{u}\|=\bar{s}} f^{\Phi}_{A|B}(\bar{u}) = \sum_{n\geq 1} \sum_{\|\bar{u}\|=\bar{s}} \chi_n(\bar{u}).$$

For each $n \geq 1$, let $M_n := R/(\bar{X}^{\bar{u}} : \bar{u} \notin I_n)$, so $M_n = \bigoplus_{\bar{s}\in\mathbb{N}^k} (M_n)_{\bar{s}}$ is a graded R-module with

$$\dim_K(M_n)_{\bar{s}} = |\{\bar{u} \in I_n : \|\bar{u}\| = \bar{s}\}| = \sum_{\|\bar{u}\|=\bar{s}} \chi_n(\bar{u}).$$

Let $M := \bigoplus_{n\geq 1} M_n$. Then

$$\dim_K M_{\bar{s}} = \sum_{n\geq 1} \dim_K (M_n)_{\bar{s}} = \mathrm{rk}(\Phi^{(\bar{s})}(A)|\Phi^{(\bar{s})}(B)). \qquad \square$$

Much work has been done on characterizing exactly which numerical polynomials arise as Hilbert polynomials of finitely generated graded R-modules; see [11] for our present multivariate setting. It follows from the above proposition that we don't obtain any new dimension polynomials in our framework:

Corollary 4.7. *The dimension polynomials we obtain in Theorem A are exactly the Hilbert polynomials of finitely generated graded R-modules, as in Corollary 4.5.*

4D. *Tropical ideals.* Let $\bar{\mathbb{R}}$ be the tropical semiring $\mathbb{R} \cup \{\infty\}$ with operations $a \oplus b := \min(a, b)$ and $a \odot b = a + b$. Let $\bar{\mathbb{R}}[\bar{x}] = \bar{\mathbb{R}}[x_1, \ldots, x_m]$ consist of the tropical polynomials $f(\bar{x}) = \bigoplus_{\bar{r}} f_{\bar{r}} \odot \bar{x}^{\bar{r}}$. Then $\bar{\mathbb{R}}[\bar{x}]$ is a tropical semiring in its own right, where $\oplus$ and $\odot$ are extended to $\bar{\mathbb{R}}[\bar{x}]$ in the natural way. We associate to each tropical polynomial f the subset of monomials $\mathrm{supp}(f) := \{\bar{x}^{\bar{r}} : f_{\bar{r}} \neq \infty\}$.

An *ideal* is a subset $I \subseteq \bar{\mathbb{R}}[\bar{x}]$ which is closed under sums and multiplication by elements of $\bar{\mathbb{R}}[\bar{x}]$. Following Maclagan and Rincón [22], we say that an ideal I is *tropical* if it satisfies the 'monomial elimination axiom': for any $f, g \in I$ and any $\bar{r} \in \mathbb{N}^m$ for which $f_{\bar{r}} = g_{\bar{r}} \neq \infty$, there is $h \in I$ such that $h_{\bar{r}} = \infty$ and $h_{\bar{u}} \geq f_{\bar{u}} \oplus g_{\bar{u}}$ for all $\bar{u} \in \mathbb{N}^m$, with equality whenever $f_{\bar{u}} \neq g_{\bar{u}}$. Using our partition $(\Phi_1, \ldots, \Phi_k)$, we associate to $\bar{\mathbb{R}}[\bar{x}]$ an $\mathbb{N}^k$-grading, where the graded part $\bar{\mathbb{R}}[\bar{x}]_{\bar{s}}$ consists of all homogeneous tropical polynomials of degree $\bar{s}$. We call an ideal I *homogeneous* if it is homogeneous with respect to this grading.

Let I be a homogeneous tropical ideal. For $\bar{s} \in \mathbb{N}^k$, let $I_{\bar{s}} = I \cap \bar{\mathbb{R}}[\bar{x}]_{\bar{s}}$. Let $\mathrm{Mon}_{\bar{s}} := \{\bar{x}^{\bar{r}} : \|\bar{r}\| = \bar{s}\}$ be the set of monomials of degree $\bar{s}$, and define the closure operator $\mathrm{c}\ell_I : \mathcal{P}(\mathrm{Mon}_{\bar{s}}) \to \mathcal{P}(\mathrm{Mon}_{\bar{s}})$ as follows: for $A \subseteq \mathrm{Mon}_{\bar{s}}$, let $\mathrm{c}\ell_I(A)$ consist all elements of A along with all $\bar{x}^{\bar{r}} \in \mathrm{Mon}_{\bar{s}}$ such that $A_0 \cup \{\bar{x}^{\bar{r}}\} = \mathrm{supp}(f)$ for some $A_0 \subseteq A$ and some $f \in I_{\bar{s}}$.

Lemma 4.8. $(\mathrm{Mon}_{\bar{s}}, \mathrm{c}\ell_I)$ *is a finite matroid.*

Proof. Monotonicity is clear. For idempotence, let $\bar{x}^{\bar{r}} \in \mathrm{c}\ell_I(\mathrm{c}\ell_I(A))$, and take $f \in I_{\bar{s}}$ with $\mathrm{supp}(f) = B \cup \{\bar{x}^{\bar{r}}\}$ for some $B \subseteq \mathrm{c}\ell_I(A)$. We may assume that $|B \setminus A|$ is minimal, and we want to show that $|B \setminus A| = 0$. Suppose that this is not the case, let $\bar{x}^{\bar{u}} \in B \setminus A$, and take $g \in I_{\bar{s}}$ with $\mathrm{supp}(g) = A_0 \cup \{\bar{x}^{\bar{u}}\}$ for some $A_0 \subseteq A$. By multiplying g with an appropriate element of $\bar{\mathbb{R}}$, we may arrange that $g_{\bar{u}} = f_{\bar{u}}$. By the monomial elimination axiom, there is $h \in I_{\bar{s}}$ such that $\mathrm{supp}(h) = B^* \cup \{\bar{x}^{\bar{r}}\}$ for some $B^* \subseteq (B \cup A_0) \setminus \{\bar{x}^{\bar{u}}\}$. But then $|B^* \setminus A| < |B \setminus A|$, contradicting our choice of B. Finally, for exchange, let $\bar{x}^{\bar{r}}, \bar{x}^{\bar{u}} \in \mathrm{Mon}_{\bar{s}}$ and $A \subseteq \mathrm{Mon}_{\bar{s}}$ with $\bar{x}^{\bar{r}} \in \mathrm{c}\ell_I(\{\bar{x}^{\bar{u}}\} \cup A) \setminus \mathrm{c}\ell_I(A)$. Take $f \in I_{\bar{s}}$ with $\mathrm{supp}(f) = B \cup \{\bar{x}^{\bar{r}}\}$ for some

$B \subseteq \{\bar{x}^{\bar{u}}\} \cup A$. If $B \subseteq A$, then we would have $\bar{x}^{\bar{r}} \in \mathrm{c}\ell_I(A)$, so B must contain $\bar{x}^{\bar{u}}$. Then $\bar{x}^{\bar{u}} \in \mathrm{c}\ell_I(\{\bar{x}^{\bar{r}}\} \cup A)$, as witnessed by f. □

Corollary 4.9. *Let I be a homogeneous tropical ideal. There is a polynomial $P \in \mathbb{Q}[\bar{Y}]$ such that*

$$\mathrm{rk}_I(\mathrm{Mon}_{\bar{s}}) = P(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large, where* rk_I *is the rank associated to* $\mathrm{c}\ell_I$.

Proof. Let $\mathrm{Mon} := \{\bar{x}^{\bar{r}} : \bar{r} \in \mathbb{N}^m\}$ be the collection of all monomials. We extend the closure operators on the sets $\mathrm{Mon}_{\bar{s}}$ to a closure operator $\mathrm{c}\ell_I$ on Mon as follows:

$$\bar{x}^{\bar{r}} \in \mathrm{c}\ell_I(A) :\Longleftrightarrow \bar{x}^{\bar{r}} \in \mathrm{c}\ell_I(A \cap \mathrm{Mon}_{\|\bar{r}\|}).$$

Then $(\mathrm{Mon}, \mathrm{c}\ell_I)$ is the disjoint union of the finite matroids $(\mathrm{Mon}_{\bar{s}}, \mathrm{c}\ell_I)$, so one easily verifies that $(\mathrm{Mon}, \mathrm{c}\ell_I)$ is a finitary (infinite) matroid. It is also routine to check that the map $\bar{x}^{\bar{r}} \mapsto x_i \odot \bar{x}^{\bar{r}}$ is an endomorphism of $(\mathrm{Mon}, \mathrm{c}\ell_I)$. Note that

$$\mathrm{rk}_I(\mathrm{Mon}_{\bar{s}}) = \mathrm{rk}_I(\Phi^{(\bar{s})}(\bar{x}^{\bar{0}_m})),$$

so the corollary follows from Theorem A. □

Remark 4.10. Maclagan and Rincón [22] established Corollary 4.9 in the case $k = 1$. In their definition of a tropical ideal, the underlying matroid is defined in terms of circuits (minimal dependent sets). The circuits are exactly the minimal nonempty elements of the set $\{\mathrm{supp}(f) : f \in I_{\bar{s}}\}$, ordered by inclusion. In fact, Maclagan and Rincón equip the underlying sets $\mathrm{Mon}_{\bar{s}}$ with the richer structure of a *valuated* matroid (though the Hilbert polynomial is defined in terms of the underlying classical matroid).

5. Applications II: Simplicial complexes and Betti numbers

Let $\mathcal{K}$ be a simplicial complex. In this section, we consider commuting simplicial endomorphisms $\phi_1, \ldots, \phi_m$ of $\mathcal{K}$. We will use Theorem A to show that for a finite subcomplex $A \subseteq \mathcal{K}$, the Betti numbers $b_n(\Phi^{(\bar{s})}(A))$ are all eventually polynomial in $\bar{s}$.

Let $(C_\bullet, \partial_\bullet)$ be the chain complex associated to $\mathcal{K}$, so C_n is the free abelian group generated by the n-simplices in $\mathcal{K}$ and $\partial_n : C_n \to C_{n-1}$ is the boundary map. For each n, we define two ranks on C_n as follows: given a finite subset $B \subseteq C_n$, we let $\mathrm{rk}_n(B)$ be the rank of the group generated by B, and we let $\mathrm{rk}_n^\partial(B)$ be the rank of the image of this group under ∂_n. Note that rk_0^∂ is always zero.

Given a finite subcomplex A of $\mathcal{K}$, we have an associated chain subcomplex $(C_\bullet(A), \partial_\bullet) \subseteq (C_\bullet, \partial_\bullet)$. We write $\mathrm{rk}_n(A)$ and $\mathrm{rk}_n^\partial(A)$ in place of $\mathrm{rk}_n(C_n(A))$ and $\mathrm{rk}_n^\partial(C_n(A))$. Then $\mathrm{rk}_n(A) - \mathrm{rk}_n^\partial(A)$ is the rank of the kernel of $\partial_n : C_n(A) \to$

$C_{n-1}(A)$, and $\mathrm{rk}^{\partial}_{n+1}(A)$ is the rank of the image of $\partial_{n+1}\colon C_{n+1}(A) \to C_n(A)$. It follows that the n-th Betti number of A — that is, the rank of the n-th simplicial homology group $H_n(A)$ — is exactly $\mathrm{rk}_n(A) - \mathrm{rk}^{\partial}_n(A) - \mathrm{rk}^{\partial}_{n+1}(A)$. It is routine to verify the following:

Lemma 5.1. *For each n, the functions rk_n and rk^{∂}_n both are rank functions of finitary matroids on C_n. That is, if we define the closure operators $\mathrm{c}\ell_n$ and $\mathrm{c}\ell^{\partial}_n$ on C_n by putting*

$$a \in \mathrm{c}\ell_n(B) :\Longleftrightarrow \mathrm{rk}_n(B_0 \cup \{a\}) = \mathrm{rk}_n(B_0) \text{ for some finite } B_0 \subseteq B$$
$$a \in \mathrm{c}\ell^{\partial}_n(B) :\Longleftrightarrow \mathrm{rk}^{\partial}_n(B_0 \cup \{a\}) = \mathrm{rk}^{\partial}_n(B_0) \text{ for some finite } B_0 \subseteq B,$$

then $(C_n, \mathrm{c}\ell_n)$ is a finitary matroid with corresponding rank function rk_n, and likewise for $(C_n, \mathrm{c}\ell^{\partial}_n)$.

Let $V(\mathcal{K})$ denote the vertex set of $\mathcal{K}$ (that is, the set of zero-dimensional simplices). A *simplicial endomorphism* of $\mathcal{K}$ is a map $f\colon V(\mathcal{K}) \to V(\mathcal{K})$ that maps every simplex in $\mathcal{K}$ to a (possibly lower-dimensional) simplex in $\mathcal{K}$. Explicitly, if $\sigma \in \mathcal{K}$ is an n-simplex with vertex set $\{v_0, \ldots, v_n\} \subseteq V(\mathcal{K})$, then we require that $\{f(v_0), \ldots, f(v_n)\}$ be the vertex set of a simplex in $\mathcal{K}$, which we denote $f(\sigma)$. If A is a subcomplex of $\mathcal{K}$, then $f(A) := \{f(\sigma) : \sigma \in A\}$ is also a subcomplex of $\mathcal{K}$. Any simplicial endomorphism of $\mathcal{K}$ induces a chain endomorphism $f_\bullet\colon (C_\bullet, \partial_\bullet) \to (C_\bullet, \partial_\bullet)$. Thus:

Lemma 5.2. *Let f be a simplicial endomorphism of $\mathcal{K}$, and let $f_\bullet\colon (C_\bullet, \partial_\bullet) \to (C_\bullet, \partial_\bullet)$ be the chain endomorphism induced by f. Then f_n is an endomorphism of the finitary matroids $(C_n, \mathrm{c}\ell_n)$ and $(C_n, \mathrm{c}\ell^{\partial}_n)$ for each n.*

We can now state our result on the growth of Betti numbers:

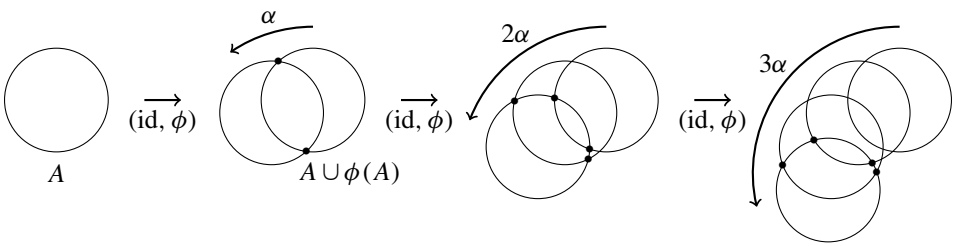
Corollary 5.3. *Let A be a finite subcomplex of $\mathcal{K}$ and suppose that each ϕ_i is a simplicial endomorphism of $\mathcal{K}$. Then for each n, the n-th Betti number $b_n(\Phi^{(\bar{s})}(A))$ is eventually polynomial in $\bar{s}$.*

Proof. For a fixed n, we have $b_n(\Phi^{(\bar{s})}(A)) = \mathrm{rk}_n(\Phi^{(\bar{s})}(A)) - \mathrm{rk}^{\partial}_n(\Phi^{(\bar{s})}(A)) - \mathrm{rk}^{\partial}_{n+1}(\Phi^{(\bar{s})}(A))$. These three ranks are eventually polynomial in $\bar{s}$ by Theorem A, so $b_n(\Phi^{(\bar{s})}(A))$ is as well. □

5A. ***Topological dynamical systems.*** Let B be a topological space, let $\phi_1, \ldots, \phi_m$ be commuting continuous maps $B \to B$, and let A be a compact subspace of B. We say that the topological dynamical system $(B, A; \phi_1, \ldots, \phi_m)$ is *triangulable* if there is a simplicial complex $\mathcal{K}$ with underlying space $|\mathcal{K}|$, a finite subcomplex $\mathcal{K}_0 \subseteq \mathcal{K}$, simplicial maps $f_1, \ldots, f_m\colon \mathcal{K} \to \mathcal{K}$, and a homeomorphism $\tau\colon |\mathcal{K}| \to B$ such that $\tau(|\mathcal{K}_0|) = A$ and such that $\tau \circ f_i = \phi_i \circ \tau$ for each i. If $(B, A; \phi_1, \ldots, \phi_m)$

is triangulable, then n-th Betti number $b_n(\Phi^{(\bar{s})}(A))$ is eventually polynomial in $\bar{s}$ by Corollary 5.3.

This leads to the question: if $(B, A; \phi_1, \dots, \phi_m)$ is a nontriangulable topological dynamical system, can the Betti numbers $b_n(\Phi^{(\bar{s})}(A))$ fail to grow eventually polynomially? The following example shows that even for relatively nice topological dynamical systems, this can fail. Let $A \subseteq \mathbb{R}^2$ be the circle of radius $\sqrt{2}/2$, centered at the point $(0, 1)$. Take $\alpha \in (0, \pi/2)$ such that α/π is irrational, and let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by α about the origin. We consider the (clearly nontriangulable) system $(\mathbb{R}^2, A; \mathrm{id}, \phi)$, pictured below, where id is the identity map.



For each t, put $A_t := A \cup \cdots \cup \phi^t(A)$. We apply Mayer–Vietoris to $A_{t+1} = A_t \cup \phi^{t+1}(A)$, using that $H_1(A_t \cap \phi^{t+1}(A)) = 0$, to get an exact sequence

$$0 \to H_1(A_t) \oplus H_1(A) \to H_1(A_{t+1}) \to H_0(A_t \cap \phi^{t+1}(A)) \\ \to H_0(A_t) \oplus H_0(A) \to H_0(A_{t+1}) \to 0.$$

With the exception of $H_1(A_{t+1})$, $H_1(A_t)$, and $H_0(A_t \cap \phi^{t+1}(A))$, all the homology groups above have rank 1, so we compute that

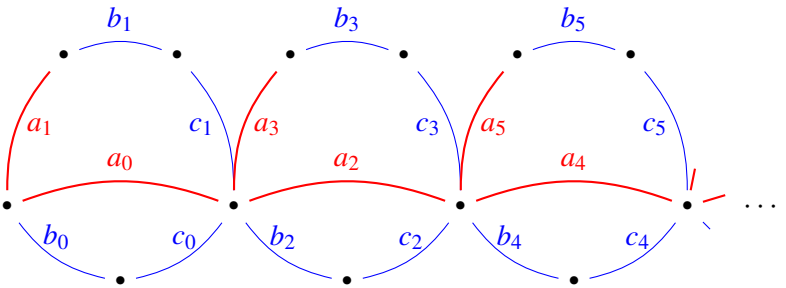
$$b_1(A_{t+1}) - b_1(A_t) = b_0(A_t \cap \phi^{t+1}(A)). \tag{5-1}$$

The intersection $A_t \cap \phi^{t+1}(A)$ is indicated by dots in the picture above. To determine the number of connected components in this intersection (that is, zeroth Betti number), let us first compute the intersection of two circles $\phi^i(A)$ and $\phi^j(A)$. If the angle between these two circles is less than $\pi/2$ in absolute value, then $\phi^i(A)$ and $\phi^j(A)$ intersect in exactly two points. If the angle is larger than $\pi/2$, then $\phi^i(A)$ and $\phi^j(A)$ are disjoint. The case that $\phi^i(A)$ and $\phi^j(A)$ intersect in exactly one point never occurs, as α/π is irrational. It follows that for t large enough, the Betti number $b_0(A_t \cap \phi^{t+1}(A))$ is an even number which is approximately equal to $t + 1$. In particular $b_0(A_t \cap \phi^{t+1}(A))$ is *not* eventually polynomial in t. Using (5-1), we see that $b_1(A_t)$ cannot be eventually polynomial in t either.

5B. ***The graphic matroid.*** Let $\mathcal{G}$ be an undirected graph with vertex set $V(\mathcal{G})$ and edge set $E(\mathcal{G})$. Then $\mathcal{G}$ is a simplicial complex with only zero and one-dimensional

simplices. The rank rk_0 corresponds to the cardinality of a set of vertices, the rank rk_1 gives the cardinality of a set of edges, and rk_1^∂ is the rank with respect to the graphic matroid on $E(\mathcal{G})$. It follows from Corollary 5.3 that for a finite subgraph $A \subseteq \mathcal{G}$ and commuting graph endomorphisms $\phi_1, \dots, \phi_m$ of $\mathcal{G}$, the first two Betti numbers $b_0(\Phi^{(\bar{s})}(A))$ and $b_1(\Phi^{(\bar{s})}(A))$ are eventually polynomial in $\bar{s}$.

Using the graphic matroid, we can construct an example that shows that our assumption in Theorem A that each Φ_i is triangular is, in some sense, necessary. Consider the following graph $\mathcal{G}$:



Let ϕ be the map sending a_i to a_{i+1}, sending b_i to b_{i+1}, and sending c_i to c_{i+1} for each i. Let $\mathrm{c}\ell$ be the closure operator associated to the graphic matroid on $E(\mathcal{G})$ (explicitly, an edge a is in the closure of a set of edges B if there are $b_1, \dots, b_n \in B$ which connect the endpoints of a), and let rk be the corresponding rank function. One can check that $\phi a \in \mathrm{c}\ell(B\phi B)$ whenever $a \in \mathrm{c}\ell(B)$, so ϕ is a quasi-endomorphism of the graphic matroid $E(\mathcal{G})$. We have

$$\mathrm{rk}(\phi^t(a_0, b_0, c_0)) = \mathrm{rk}(a_t, b_t, c_t) = \begin{cases} 2 & \text{if } t \text{ is even,} \\ 3 & \text{if } t \text{ is odd,} \end{cases}$$

so the conclusion of Theorem A fails for the single map ϕ. Of course, since ϕ is a quasi-endomorphism, Corollary A tells us that $\mathrm{rk}(\{a_i, b_i, c_i : i \leq t\})$ is eventually polynomial in t, as can easily be verified.

6. Applications III: Difference-differential fields

In this final section, we examine endomorphisms of and derivations on (expansions of) fields. Recall that a *derivation* on a field K is a map $\delta\colon K \to K$ that satisfies the identities

$$\delta(a+b) = \delta a + \delta b, \quad \delta(ab) = a\delta b + b\delta a$$

for all $a, b \in K$.

6A. ***Kolchin polynomials for difference-differential fields.*** Let K be a field of characteristic zero. We let acl be algebraic closure in K, so for $A \subseteq K$, the

set $\mathrm{acl}(A)$ consists of all $a \in K$ which are algebraic over the field $\mathbb{Q}(A)$. Then (K, acl) is a finitary matroid, and the rank of $A \subseteq K$ with respect to acl is equal to $\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}(A)$, the transcendence degree of $\mathbb{Q}(A)$ over $\mathbb{Q}$. Clearly, any field endomorphism $\sigma : K \to K$ is an endomorphism of (K, acl).

Lemma 6.1. *Let δ be a derivation on K. Then δ is a quasi-endomorphism of (K, acl).*

Proof. Let $a \in K$ and $B \subseteq K$ with $a \in \mathrm{acl}(B)$. Let P be a polynomial of minimal degree witnessing this, so $P(a) = 0$ and $P'(a) \neq 0$. Write $P(X) = \sum_d b_d X^d$, where each $b_d \in \mathbb{Q}[B]$. We have

$$0 = \delta P(a) = \sum_d \delta(b_d a^d) = \sum_d \Big(\delta(b_d) a^d + d b_d a^{d-1} \delta a\Big) = \sum_d \delta(b_d) a^d + P'(a)\delta a,$$

so $\delta a \in \mathrm{acl}(a B \delta B) \subseteq \mathrm{acl}(B \delta B)$, as desired. □

Suppose now that $\Phi = (\phi_1, \dots, \phi_m)$ is a collection of commuting maps $K \to K$, each of which is either a derivation or a field endomorphism (hence, a quasi-endomorphism of (K, acl) by Lemma 6.1). Then $(K, \phi_1, \dots, \phi_m)$ is called a *difference-differential field* (or a d-*field* for short). Let F be a d-subfield of K, that is, a subfield of K which is closed under each ϕ_i. Then each ϕ_i is a quasi-endomorphism of the relativization (K, acl_F) of (K, acl), and the rank of a subset $A \subseteq K$ with respect to acl_F coincides with $\mathrm{trdeg}_F F(A)$. Applying Corollary A, Proposition 2.3, and Corollary 3.5 to (K, acl_F), we get the following:

Corollary 6.2. *Let $\bar{a}$ be a tuple from K. Then there is a polynomial $Q^{\Phi}_{\bar{a}} \in \mathbb{Q}[\bar{Y}]$ such that*

$$\mathrm{trdeg}_F F(\Phi^{\preceq(\bar{s})}(\bar{a})) = Q^{\Phi}_{\bar{a}}(\bar{s})$$

whenever $\min(\bar{s})$ is sufficiently large. Moreover:

(i) *The dominant terms of $Q^{\Phi}_{\bar{a}}$ only depend on the algebraic closure of the* d-*field extension $F(\Theta(\bar{a}))$, that is, if $\bar{b} \in K$ and if $F(\Theta(\bar{a}))$ and $F(\Theta(\bar{b}))$ have the same algebraic closure in K, then $Q^{\Phi}_{\bar{a}}$ and $Q^{\Phi}_{\bar{b}}$ have the same dominant terms.*

(ii) *The coefficient of $\bar{Y}^{\bar{d}}$ in $Q^{\Phi}_{\bar{a}}$ times $d_1! \cdots d_k!$ is equal to the maximal size of a subset $B \subseteq \mathrm{acl}_F(\Theta(\bar{a}))$ such that the tuple $(\theta b)_{\theta \in \Theta, b \in B}$ is algebraically independent over F.*

The case when $k = 1$ and each ϕ_i is a derivation was established by Kolchin [14]. The multivariate differential case (each ϕ_i is a derivation but k is arbitrary) was shown by Levin [18]. The case where each ϕ_i is an endomorphism is also due to Levin [19], as is the most general case to date: where each ϕ_i may be either a derivation or an endomorphism, but each part of the partition Φ_i must consist of only derivations or endomorphisms; see [20]. Our Corollary 6.2 is slightly more

general than the result in [20], since the parts of the partition can consist of both derivations and endomorphisms.

6B. ***Modules over a difference-differential field.*** Let (K, Φ) be a difference-differential field, and let $K[\Phi]$ denote the ring of linear difference-differential operators over K, so $K[\Phi]$ consists of elements $\sum_{\bar{r}} \lambda_{\bar{r}} \phi^{\bar{r}}$ with finitely many nonzero coefficients $\lambda_{\bar{r}}$. The multiplication on $K[\Phi]$ satisfies the identity

$$\phi_i \lambda = \begin{cases} \phi_i(\lambda)\phi_i & \text{if } \phi_i \text{ is an endomorphism of } K, \\ \phi_i(\lambda) + \lambda\phi_i & \text{if } \phi_i \text{ is a derivation on } K, \end{cases}$$

for $i \in \{1, \ldots, m\}$ and $\lambda \in K$. Let V be a left $K[\Phi]$-module. For each i and each $v \in V$, we let $\phi_i(v) := \phi_i \cdot v$, so each ϕ_i is a map $V \to V$.

Lemma 6.3. *Let $\mathrm{c}\ell$ denote the K-linear closure operator on V and let $i \in \{1, \ldots, m\}$. If ϕ_i is an endomorphism of K, then the map $v \mapsto \phi_i(v)$ is an endomorphism of $(V, \mathrm{c}\ell)$. If ϕ_i is a derivation on K, then the map $v \mapsto \phi_i(v)$ is a quasi-endomorphism of $(V, \mathrm{c}\ell)$.*

Proof. Let $v_1, \ldots, v_n \in V$ and suppose that v is in the K-linear span of $v_1, \ldots, v_n$. Take scalars $\lambda_1, \ldots, \lambda_n \in K$ with $v = \sum_{j=1}^n \lambda_j \cdot v_j$. If ϕ_i is an endomorphism of K, then

$$\phi_i(v) = \sum_{j=1}^n \phi_i(\lambda_j \cdot v_j) = \sum_{j=1}^n \phi_i(\lambda_j) \cdot \phi_i(v_j),$$

so $\phi_i(v)$ is in the K-linear span of $\phi_i(v_1), \ldots, \phi_i(v_n)$. If ϕ_i is a derivation on K, then

$$\phi_i(v) = \sum_{j=1}^n \phi_i(\lambda_j \cdot v_j) = \sum_{j=1}^n \phi_i(\lambda_j) \cdot v_j + \lambda_j \cdot \phi_i(v_j),$$

so $\phi_i(v)$ is in the K-linear span of $v_1, \ldots, v_n, \phi_i(v_1), \ldots, \phi_i(v_n)$. □

We now associate to $K[\Phi]$ an $\mathbb{N}^k$-filtration as follows: for $\bar{s} \in \mathbb{N}^k$, let $K[\Phi]_{\bar{s}}$ consist of the linear operators $\sum_{\|\bar{r}\| \preceq \bar{s}} \lambda_{\bar{r}} \phi_1^{r_1} \cdots \phi_m^{r_m}$. Let $V = \bigcup_{\bar{s} \in \mathbb{Z}^k} V_{\bar{s}}$ be a filtered $K[\Phi]$-module. Following Johnson [9], we say that the filtration $(V_{\bar{s}})_{\bar{s} \in \mathbb{Z}^k}$ is *excellent* if each $V_{\bar{s}}$ is finite-dimensional as a K-vector space and if there is $\bar{s}_0 \in \mathbb{Z}^k$ such that

$$V_{\bar{s}} = \{D(v) : D \in K[\Phi]_{\bar{s} - \bar{s}_0} \text{ and } v \in V_{\bar{s}_0}\}$$

for all $\bar{s} \succeq \bar{s}_0$.

Corollary 6.4. *Let $V = \bigcup_{\bar{s} \in \mathbb{Z}^k} V_{\bar{s}}$ be an excellently filtered $K[\Phi]$-module. There is a polynomial $Q \in \mathbb{Q}[\bar{Y}]$ such that*

$$\dim_K(V_{\bar{s}}) = Q(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Proof. Let $(X, \mathrm{c}\ell)$ be the underlying set of V with the K-linear closure operator. Since V is excellent, we can find an index $\bar{s}_0$ and a finite set $A \subseteq V_{\bar{s}_0}$ such that the $K[\Phi]$-module $\bigcup_{\bar{s}\succeq\bar{s}_0} V_{\bar{s}}$ is generated by A. By re-indexing, we may assume that $\bar{s}_0 = \bar{0}_k$. Then for each $\bar{s} \in \mathbb{N}^k$, the graded part $V_{\bar{s}}$ is generated as a K-vector space by

$$\Phi^{\preceq(\bar{s})}(A) = \{\phi^{\bar{r}}(a) : \|\bar{r}\| \preceq \bar{s} \text{ and } a \in A\}.$$

Thus, $\dim_K(V_{\bar{s}}) = \mathrm{rk}(\Phi^{\preceq(\bar{s})}(A))$ for $\bar{s} \in \mathbb{N}^k$. The claim follows from Corollary A. □

Remark 6.5. In the case that each ϕ_i is a derivation on K and $k = 1$, Corollary 6.4 is due to Johnson [9], who used this result to establish the existence of the Kolchin polynomial. Variations of this result were later given by Levin; see [17] for the case where each ϕ_i is a field automorphism. The dimension polynomial for modules over rings of differential operators also appears in work of Bernshtein [3].

6C. ***Higher derivations and $\mathcal{D}$-fields.*** Let K be a field. A *higher derivation on K* is a tuple $\Delta = (\delta_0, \delta_1, \ldots, \delta_m)$ where $\delta_0 \colon K \to K$ is the identity map and where $\delta_i \colon K \to K$ is an additive map satisfying the identity

$$\delta_i(xy) = \sum_{j\le i} \delta_j(x)\delta_{i-j}(y)$$

for each $i \le m$; see [23, Section 27]. Note that δ_1 is a classical derivation on K. We may relax the assumption that δ_0 is the identity map and ask only that it is an endomorphism. In this case, we call Δ a *twisted higher derivation on K*.

We will consider higher derivations as a particular case of $\mathcal{D}$-fields in the sense of [24]. For the remainder of this subsection, fix a subfield A of K, and let $\Delta = (\delta_0, \delta_1, \ldots, \delta_m)$ be a tuple of A-linear maps from K to K. We say that (K, Δ) is a *twisted $\mathcal{D}$-field* if δ_0 is an endomorphism of K and, for every $k = 1, \ldots, m$, there are coefficients $a_{i,j,k} \in A$ such that

$$\delta_k(xy) = \sum_{i,j\le m} a_{i,j,k}\delta_i(x)\delta_j(y). \tag{6-1}$$

We refer the reader to [24] for a more precise description of $\mathcal{D}$-fields. We do note that in [24], the coefficients $a_{i,j,k}$ only depend on the ring scheme $\mathcal{D}$, and that the first map δ_0 is required to be the identity map, not just a ring endomorphism. This is why we refer to $(K, \delta_0, \ldots, \delta_m)$ as a "twisted" $\mathcal{D}$-field.

We say that the twisted $\mathcal{D}$-field (K, Δ) is *pyramidal* if the coefficients $a_{i,j,k}$ satisfy the following additional conditions:

(1) $a_{k,0,k} = a_{0,k,k} = 1$;

(2) if $i > 0$, then $a_{k,i,k} = a_{i,k,k} = 0$;

(3) if $i > k$ or $j > k$, then $a_{i,j,k} = 0$.

If (K, Δ) is a pyramidal twisted $\mathcal{D}$-field, then we can rewrite (6-1) as

$$\delta_0(xy) = \delta_0(x)\delta_0(y),$$
$$\delta_k(xy) = \delta_k(x)\delta_0(y) + \sum_{i,j<k} a_{i,j,k}\delta_i(x)\delta_j(y) + \delta_0(x)\delta_k(y) \quad \text{for } k = 1, \ldots, m.$$

The following lemma can be established via a straightforward induction on k and d:

Lemma 6.6. *Let (K, Δ) be a pyramidal twisted $\mathcal{D}$-field. Then for each $0 < k \leq m$ and each $d \in \mathbb{N}$, there is a polynomial $R_{k,d}$ over A such that*

$$\delta_k(a^d) = d\delta_0(a)^{d-1}\delta_k(a) + R_{k,d}(\delta_0 a, \ldots, \delta_{k-1}a).$$

for all $a \in K$.

Lemma 6.7. *Let (K, Δ) be a pyramidal twisted $\mathcal{D}$-field of characteristic zero. Let $a \in K$, and let $B \subseteq K$. If $a \in \mathrm{acl}_A(B)$, then $\delta_k a \in \mathrm{acl}_A(\delta_0 B \cdots \delta_k B)$ for each $k \leq m$.*

Proof. Take a nonzero polynomial $P(X)$ of minimal degree witnessing that $a \in \mathrm{acl}_A(B)$, so $P(a) = 0$ and $P'(a) \neq 0$. Write $P(X) = \sum_d b_d X^d$ where each $b_d \in A[B]$. We will show by induction on $k \leq m$ that $\delta_k a \in \mathrm{acl}_A(\delta_0 B \cdots \delta_k B)$. This holds for $k = 0$ since δ_0 is an endomorphism of K. Let $0 < k \leq m$, and assume that $\delta_j a \in \mathrm{acl}_A(\delta_0 B \cdots \delta_j B)$ for all $j < k$. By Lemma 6.6, we have

$$\begin{aligned}
0 &= \delta_k P(a) \\
&= \sum_d \delta_k(b_d a^d) = \sum_d \Big(\delta_k(b_d)\delta_0(a^d) + \sum_{i,j<k} a_{i,j,k}\delta_i(b_d)\delta_j(a^d) + \delta_0(b_d)\delta_k(a^d)\Big) \\
&= \sum_d \Big(\delta_k(b_d)\delta_0(a^d) + \sum_{i,j<k} a_{i,j,k}\delta_i(b_d)\delta_j(a^d) + \delta_0(b_d)R_{k,d}(\delta_0 a, \ldots, \delta_{k-1}a) \\
&\qquad + \delta_0(b_d)d\delta_0(a)^{d-1}\delta_k(a)\Big) \\
&= \sum_d \Big(\delta_k(b_d)\delta_0(a^d) + \sum_{i,j<k} a_{i,j,k}\delta_i(b_d)\delta_j(a^d) + \delta_0(b_d)R_{k,d}(\delta_0 a, \ldots, \delta_{k-1}a)\Big) \\
&\qquad + \delta_0(P'(a))\delta_k(a).
\end{aligned}$$

Since $P'(a) \neq 0$, this shows that $\delta_k a$ is algebraic over $\delta_0 B \cdots \delta_k B$, together with $\delta_j(a^d)$ for $j < k$. Applying Lemma 6.6 again and using our induction hypothesis, we see that

$$\delta_j(a^d) \in \mathrm{acl}_A(\delta_0 a, \ldots, \delta_j a) \subseteq \mathrm{acl}_A(\delta_0 B \cdots \delta_k B)$$

for $j < k$, so $\delta_k a \in \mathrm{acl}_A(\delta_0 B \cdots \delta_k B)$. □

Corollary 6.8. *Let K be a field of characteristic zero, let F be a subfield of K containing A, let $\bar{a}$ be a tuple from K, and let $\Phi = (\Phi_1, \ldots, \Phi_k)$ be a partitioned set of commuting maps $K \to K$. Suppose for each $i = 1, \ldots, k$ that (K, Φ_i) is a*

pyramidal twisted $\mathcal{D}_i$-field, and that (F, Φ_i) is a $\mathcal{D}_i$-subfield of K. Then there is a polynomial $P_{\bar{a}}^{\Phi} \in \mathbb{Q}[\bar{Y}]$ such that

$$\operatorname{trdeg}_F F(\Phi^{(\bar{s})}(\bar{a})) = P_{\bar{a}}^{\Phi}(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large.*

Question 6.9. Does Corollary 6.8 hold for all $\mathcal{D}$-fields (in the sense of [24]) with commuting operators?

6D. ***Higher derivations in positive characteristic.*** Let K be a field of characteristic $p > 0$. Then any derivation δ on K is trivial on the subfield $K^p \subseteq K$, as $\delta a^p = pa^{p-1}\delta a = 0$ for $a \in K$. It follows that δ is not generally a quasi-endomorphism of (K, acl). For example, let $K = \mathbb{F}_p(x, y)$ with x, y mutually transcendental over $\mathbb{F}_p$, and consider the derivation δ on K that satisfies $\delta x = 1, \delta y = x$. We have $y \in \operatorname{acl}(y^p)$, but

$$\delta y = x \notin \operatorname{acl}(y^p, \delta y^p) = \operatorname{acl}(y^p).$$

One may attempt to remedy this by using the separable closure scl in place of the algebraic closure, but then we run into a different issue: (K, scl) need not be a finitary matroid. For example, with K as above, we have $x^p \in \operatorname{scl}(x) \setminus \operatorname{scl}(\varnothing)$ but $x \notin \operatorname{scl}(x^p)$. This second issue can be solved by relativizing at K^p.

Before checking that $(K, \operatorname{scl}_{K^p})$ is indeed a finitary matroid, we note that for any $P(X) \in K^p[X]$ and any $a \in K$, there is some polynomial $Q(X) \in K^p[X]$ of degree at most $p-1$ with $P(a) = Q(a)$ and $P'(a) = Q'(a)$. To obtain Q, we simply replace any monomial X^{np+d} appearing in P with $a^{np}X^d \in K^p[X]$. Thus, we have $a \in \operatorname{scl}_{K^p}(B)$ if and only if $P(a) = 0$ for some nonzero polynomial $P(X) \in K^p[B][X]$ of degree at most $p-1$.

Lemma 6.10. $(K, \operatorname{scl}_{K^p})$ *is a finitary matroid.*

Proof. Since monotonicity, idempotence, and finite character hold for scl, and since these properties are preserved by relativization, we need only check Steinitz exchange. Let $a, b \in K$ and $A \subseteq K$, and suppose that $a \in \operatorname{scl}_{K^p}(Ab) \setminus \operatorname{scl}_{K^p}(A)$. Let $P(X, Y)$ be a nonzero polynomial over $K^p[A]$ of degree at most $p-1$ in both X and Y with $P(a, b) = 0$. We write

$$P(X, Y) = \sum_{d<p} Q_d(X)Y^d$$

where each $Q_d(X)$ is a polynomial over $K^p[A]$ of degree at most $p-1$. As P is nonzero, some Q_d is nonzero, so $Q_d(a) \neq 0$ as $a \notin \operatorname{scl}_{K^p}(A)$. Thus, $P(a, b) = 0$ and $P(a, Y) \neq 0$, so $b \in \operatorname{scl}_{K^p}(Aa)$. □

Given a subfield $F \subseteq K$, the rank $\operatorname{rk}(K|F)$ corresponding to the closure scl_{K^p} is the cardinality of a p-basis for F over K; see [23, Section 26].

We return to the setting of pyramidal twisted $\mathcal{D}$-fields. Again, we fix a subfield $A \subseteq K$ and a tuple $\Delta = (\delta_0, \ldots, \delta_m)$ of A-linear maps $K \to K$. Lemma 6.6 still applies. Therefore, the following lemma can be proven in the same way as Lemma 6.7 above, so long as one takes the polynomial P in that proof to have degree at most $p-1$.

Lemma 6.11. *Let (K, Δ) be a pyramidal twisted $\mathcal{D}$-field of characteristic $p > 0$. Let $a \in K$, and let $B \subseteq K$. If $a \in \mathrm{scl}_{K^p[A]}(B)$, then $\delta_k a \in \mathrm{scl}_{K^p[A]}(\delta_0 B \cdots \delta_k B)$ for each $k \leq m$. In particular,*

(i) *Any field endomorphism of K is an endomorphism of (K, scl_{K^p}),*

(ii) *Any derivation on K is a quasi-endomorphism of (K, scl_{K^p}).*

(iii) *Any commuting twisted higher derivation on K is a triangular system with respect to the matroid (K, scl_{K^p}).*

Accordingly, we can prove a positive characteristic analog of Corollary 6.8.

Corollary 6.12. *Let K be a field of characteristic $p > 0$, let F be a subfield of K containing $K^p[A]$, let $\bar{a}$ be a tuple from K, and let $\Phi = (\Phi_1, \ldots, \Phi_k)$ be a partitioned set of commuting maps $K \to K$. Suppose for each $i = 1, \ldots, k$ that (K, Φ_i) is a pyramidal twisted $\mathcal{D}_i$-field, and that (F, Φ_i) is a $\mathcal{D}_i$-subfield of K. Then there is a polynomial $P^{\Phi}_{\bar{a}} \in \mathbb{Q}[\bar{Y}]$ such that the size of a p-basis for $F(\Phi^{(\bar{s})}(\bar{a}))$ is equal to $P^{\Phi}_{\bar{a}}(\bar{s})$, whenever $\min(\bar{s})$ is sufficiently large.*

6E. *Kolchin polynomials for difference-differential exponential fields.* Let K be a field of characteristic zero. An *exponential on K* is a group homomorphism $E\colon A(K) \to K^\times$, where $A(K)$ is a divisible subgroup of the additive group of K. If E is an exponential on K, then the pair (K, E) is called an *exponential field*. A subfield F of K is an *exponential subfield* if $E(a) \in F$ for all $a \in A(F) := A(K) \cap F$. The fields $\mathbb{R}$ and $\mathbb{C}$ with their usual exponential functions are exponential fields (where the domain of the exponential is the entire field).

Let (K, E) be an exponential field. An *E-term* is a partial function given by arbitrary compositions of E and polynomials over $\mathbb{Z}$. Model theoretically speaking, an E-term is a term in the language $(+, \cdot, -, 0, 1, E)$; to avoid partially defined functions, one may take E to be identically zero away from $A(K)$. Let $B \subseteq K$. A tuple $\bar{a} = (a_1, \ldots, a_n)$ is said to be a *regular solution to a Khovanskii system over B* if there is a tuple $\bar{b} = (b_1, \ldots, b_m)$ from B and E-terms $t_1, \ldots, t_n$ in $n+m$ variables such that

$$t_1(\bar{a}, \bar{b}) = \cdots = t_n(\bar{a}, \bar{b}) = 0, \quad \det \begin{pmatrix} \frac{\partial t_1}{\partial X_1}(\bar{a}, \bar{b}) & \cdots & \frac{\partial t_1}{\partial X_n}(\bar{a}, \bar{b}) \\ \vdots & \ddots & \vdots \\ \frac{\partial t_n}{\partial X_1}(\bar{a}, \bar{b}) & \cdots & \frac{\partial t_n}{\partial X_n}(\bar{a}, \bar{b}) \end{pmatrix} \neq 0.$$

The *exponential-algebraic closure of* B, written $\mathrm{ecl}(B)$, consists of all components of any regular solution to a Khovanskii system over B. The exponential-algebraic closure was first defined by Macintyre [21], and Kirby later showed that (K, ecl) is always a finitary matroid [13, Theorem 1.1], extending earlier work of Wilkie [29]. If F is an exponential subfield of K and A is a subset of K, then we let $F(A)^E$ denote the exponential subfield of K generated by F and A, and we define the *exponential transcendence degree of* $F(A)^E$ *over* F, denoted $\mathrm{etrdeg}_F F(A)^E$, to be the rank $\mathrm{rk}(F(A)^E \,|\, F) = \mathrm{rk}(A|F)$ given by the matroid (K, ecl).

An *exponential endomorphism of* K is a field endomorphism $\sigma\colon K \to K$ such that $\sigma E(a) = E(\sigma a)$ for all $a \in A(K)$. An *exponential derivation on* K is a derivation $\delta\colon K \to K$ which satisfies the identity $\delta E(a) = E(a)\delta a$ for all $a \in A(K)$.

Lemma 6.13. *Any exponential endomorphism of* K *is an endomorphism of* (K, ecl). *Any exponential derivation on* K *is a quasi-endomorphism of* (K, ecl).

Proof. Let σ be an exponential endomorphism of K, let δ be an exponential derivation on K, and let $B \subseteq K$. If $a \in K$ is a component of a regular solution to a Khovanskii system over B, then $\sigma(a)$ is a component to a regular solution to a Khovanskii system over $\sigma(B)$, namely, the same Khovanskii system but with the parameters from B replaced with the corresponding parameters from $\sigma(B)$. Thus, σ is an endomorphism of (K, ecl).

To see that δ is a quasi-endomorphism of (K, ecl), we use [13, Theorem 1.1], which states that $a \in K$ belongs to $\mathrm{ecl}(B)$ if and only if every exponential derivation on K which vanishes on B also vanishes at a. Suppose $a \in \mathrm{ecl}(B)$, and let ϵ be an exponential derivation on K which vanishes on $B \cup \delta(B)$. We need to show that $\epsilon\delta a = 0$. Consider the map $\epsilon\delta - \delta\epsilon\colon K \to K$, where $(\epsilon\delta - \delta\epsilon)(y) = \epsilon\delta y - \delta\epsilon y$. It is routine to show that $\epsilon\delta - \delta\epsilon$ is an exponential derivation. For $b \in B$, we have $\epsilon\delta b - \delta\epsilon b = 0$, since ϵ vanishes on $B \cup \delta(B)$. Thus, $\epsilon\delta - \delta\epsilon$ vanishes on B, so it also vanishes at a. Since ϵ also vanishes at a, we see that

$$0 = \epsilon\delta a - \delta\epsilon a = \epsilon\delta a. \qquad \square$$

Suppose now that $\Phi = (\phi_1, \ldots, \phi_m)$ is a collection of commuting maps $K \to K$, each of which is either an exponential derivation or an exponential endomorphism. The structure $(K, E, \phi_1, \ldots, \phi_m)$ is called a *difference-differential exponential field* (or a d-*exponential field* for short). Let F be a d-exponential subfield of K, that is, an exponential subfield of K which is closed under each ϕ_i. Applying Corollary A, Proposition 2.3, and Corollary 3.5 to the relativized matroid (K, ecl_F) gives us the following:

Corollary 6.14. *Let* $\bar{a}$ *be a tuple from* K. *Then there is a polynomial* $Q^{\Phi}_{\bar{a}} \in \mathbb{Q}[\bar{Y}]$ *such that*

$$\mathrm{etrdeg}_F F(\Phi^{\preceq(\bar{s})}(\bar{a}))^E = Q^{\Phi}_{\bar{a}}(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large. Moreover*:

(i) *The dominant terms of* $Q_{\bar{a}}^{\Phi}$ *only depend on the exponential-algebraic closure of the* d*-field extension* $F(\Theta(\bar{a}))$.

(ii) *The coefficient of* $\bar{Y}^{\bar{d}}$ *in* $Q_{\bar{a}}^{\Phi}$ *times* $d_1! \cdots d_k!$ *is equal to the maximal size of a subset* $B \subseteq \mathrm{ecl}_F(\Theta(\bar{a}))$ *such that the tuple* $(\theta b)_{\theta\in\Theta, b\in B}$ *is exponential-algebraically independent over* F.

Remark 6.15. With the obvious changes, Corollary 6.14 may be applied to j-fields. These fields, introduced in [6], are equipped with partially defined functions which behave like the modular j-function. The relevant closure operator in this setting is the $j$cl-closure, defined in [6], and the tuple Φ should consist of commuting j-field endomorphisms and j-derivations, also defined in [6]. See [1] for more on the j-closure operator. Thanks to Vincenzo Mantova for bringing this to our attention.

6F. ***Derivations on o-minimal fields.*** Let T be an o-minimal theory extending the theory of real closed ordered fields, and let K be a model of T; see [5] for definitions and background. The *definable closure operator on* K, denoted dcl, is given by

$$a \in \mathrm{dcl}(B) \ :\Longleftrightarrow \ a = f(\bar{b}) \text{ for some tuple } \bar{b} \text{ from } B \text{ and some } \varnothing\text{-definable function } f.$$

It is well-known that (K, dcl) is a finitary matroid, and we denote the corresponding rank function by rk_T. Given an elementary substructure F of K and a set $A \subseteq K$, we let $F\langle A\rangle$ denote the definable closure of $F \cup A$ in K. Then $F\langle A\rangle$ is also a model of T, and $\mathrm{rk}_T(F\langle A\rangle|F) = \mathrm{rk}_T(A|F)$.

A *T-derivation on K* is a map $\delta\colon K \to K$ such that for each tuple $\bar{a} = (a_1, \ldots, a_n)$ in K^n and each $\varnothing$-definable function f which is $\mathcal{C}^1$ in a neighborhood of $\bar{a}$, we have

$$\delta f(\bar{a}) = \frac{\partial f}{\partial Y_1}(\bar{a})\delta a_1 + \cdots + \frac{\partial f}{\partial Y_n}(\bar{a})\delta a_n. \tag{6-2}$$

The study of T-derivations was initiated by the authors in [7] and was expanded on by the second author [10]. The link between compatible derivations on o-minimal structures and definable closure in o-minimal structures has long been used by Wilkie and others; see [29; 2]. The following fact can be proven along the lines of Lemma 6.1, using (6-2) above:

Fact 6.16. *Any T-derivation on K is a quasi-endomorphism of* (K, dcl).

Now suppose that $\Phi = (\phi_1, \ldots, \phi_m)$ is a collection of commuting T-derivations on K. Then $(K, \phi_1, \ldots, \phi_m)$ is called a *T-differential field*. Let F be a T-differential

subfield of K, that is, an elementary substructure of K which is closed under each ϕ_i. Then each ϕ_i is a quasi-endomorphism of the relativization (K, dcl_F).

Corollary 6.17. *Let $\bar{a}$ be a tuple from K. Then there is a polynomial $Q^{\Phi}_{\bar{a}} \in \mathbb{Q}[\bar{Y}]$ such that*

$$\mathrm{rk}_T(\Phi^{\preceq(\bar{s})}(\bar{a})|F) = Q^{\Phi}_{\bar{a}}(\bar{s})$$

whenever $\min(\bar{s})$ *is sufficiently large. Moreover*:

(1) *The dominant terms of $Q^{\Phi}_{\bar{a}}$ only depend on the T-differential field extension $F\langle\Theta(\bar{a})\rangle$.*

(2) *The coefficient of $\bar{Y}^{\bar{d}}$ in $Q^{\Phi}_{\bar{a}}$ times $d_1! \cdots d_k!$ is equal to the maximal size of a subset $B \subseteq F\langle\Theta(\bar{a})\rangle$ such that the tuple $(\theta b)_{\theta\in\Theta, b\in B}$ is* dcl*-independent over F.*

Acknowledgements

Fornasiero was partially supported by GNSAGA–INdAM. Kaplan was located at the Fields Institute for Research in Mathematical Sciences and was supported by the National Science Foundation under Award No. 2103240.

The authors thank Piotr Kowalski, George Shakan, Francesco Gallinaro, James Freitag, and Matthias Aschenbrenner for helpful conversations. We also thank the referees for useful feedback, including an improvement to Proposition 2.3.

References

[1] V. Aslanyan, S. Eterović, and J. Kirby, "A closure operator respecting the modular j-function", *Israel J. Math.* **253**:1 (2023), 321–357. MR Zbl

[2] M. Bays, J. Kirby, and A. J. Wilkie, "A Schanuel property for exponentially transcendental powers", *Bull. Lond. Math. Soc.* **42**:5 (2010), 917–922. MR Zbl

[3] I. N. Bernshtein, "Modules over a ring of differential operators: an investigation of the fundamental solutions of equations with constant coefficients", *Funkcional. Anal. i Priložen.* **5**:2 (1971), 1–16. In Russian; translated in *Funct. Anal. Appl.* **5**:2 (1971), 89–101. MR Zbl

[4] T. Ceccherini-Silberstein, M. Coornaert, and F. Krieger, "An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups", *J. Anal. Math.* **124** (2014), 59–81. MR Zbl

[5] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series **248**, Cambridge University Press, 1998. MR Zbl

[6] S. Eterović, "A Schanuel property for j", *Bull. London Math. Soc.* **50**:2 (2018), 293–315. Zbl

[7] A. Fornasiero and E. Kaplan, "Generic derivations on o-minimal structures", *J. Math. Log.* **21**:2 (2021), art. id. 2150007, 45 pp. MR Zbl

[8] J. D. Hauenstein and F. Sottile, "Newton polytopes and witness sets", *Math. Comput. Sci.* **8**:2 (2014), 235–251. MR Zbl

[9] J. Johnson, "Differential dimension polynomials and a fundamental theorem on differential modules", *Amer. J. Math.* **91** (1969), 239–248. MR Zbl

[10] E. Kaplan, *Derivations on o-minimal fields*, PhD thesis, University of Illinois at Urbana-Champaign, 2021, available at https://www.ideals.illinois.edu/items/118302. Zbl

[11] L. Katthän, J. J. Moyano-Fernández, and J. Uliczka, "Which series are Hilbert series of graded modules over standard multigraded polynomial rings?", *Math. Nachr.* **293**:1 (2020), 129–146. MR Zbl

[12] A. G. Khovanskii, "The Newton polytope, the Hilbert polynomial and sums of finite sets", *Funktsional. Anal. i Prilozhen.* **26**:4 (1992), 57–63. In Russian; translated in *Funct. Anal. Appl.* bf26:4 (1992), 276–281. MR Zbl

[13] J. Kirby, "Exponential algebraicity in exponential fields", *Bull. Lond. Math. Soc.* **42**:5 (2010), 879–890. MR Zbl

[14] E. R. Kolchin, "The notion of dimension in the theory of algebraic differential equations", *Bull. Amer. Math. Soc.* **70** (1964), 570–573. MR Zbl

[15] E. R. Kolchin, *Differential algebra and algebraic groups*, Pure and Applied Mathematics **54**, Academic Press, New York, 1973. MR

[16] M. V. Kondratieva, A. B. Levin, A. V. Mikhalev, and E. V. Pankratiev, *Differential and difference dimension polynomials*, Mathematics and its Applications **461**, Kluwer, Dordrecht, 1999. MR Zbl

[17] A. B. Levin, "Characteristic polynomials of filtered difference modules and extensions of difference fields", *Uspehi Mat. Nauk* **33**:3(201) (1978), 177–178. In Russian; translated in Russian Math. Surv. **33**:3 (1978), 165-166. MR Zbl

[18] A. B. Levin, "Multivariable dimension polynomials and new invariants of differential field extensions", *Int. J. Math. Math. Sci.* **27**:4 (2001), 201–214. MR Zbl

[19] A. B. Levin, "Multivariable difference dimension polynomials", *Sovrem. Mat. Prilozh.* **14**:Algebra (2004), 48–70. In Russian; translated in *J. Math. Sci.* (*New York*), **131**:9 (2005), 6060–6082. MR Zbl

[20] A. Levin, "Multivariate difference-differential dimension polynomials", *Math. Comput. Sci.* **14**:2 (2020), 361–374. MR Zbl

[21] A. J. Macintyre, "Exponential algebra", pp. 191–210 in *Logic and algebra* (Pontignano, 1994), Lecture Notes in Pure and Appl. Math. **180**, Dekker, New York, 1996. MR Zbl

[22] D. Maclagan and F. Rincón, "Tropical ideals", *Compos. Math.* **154**:3 (2018), 640–670. Zbl

[23] H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, 1986. MR Zbl

[24] R. Moosa and T. Scanlon, "Model theory of fields with free operators in characteristic zero", *J. Math. Log.* **14**:2 (2014), art. id. 1450009, 43pp. MR Zbl

[25] M. B. Nathanson, "Growth of sumsets in abelian semigroups", *Semigroup Forum* **61**:1 (2000), 149–153. MR Zbl

[26] M. B. Nathanson and I. Z. Ruzsa, "Polynomial growth of sumsets in abelian semigroups", *J. Théor. Nombres Bordeaux* **14**:2 (2002), 553–560. MR Zbl

[27] J. Oxley, "Infinite matroids", pp. 73–90 in *Matroid applications*, Encyclopedia Math. Appl. **40**, Cambridge Univ. Press, 1992. MR Zbl

[28] K. Tent and M. Ziegler, *A course in model theory*, Lecture Notes in Logic **40**, Cambridge University Press, 2012. MR Zbl

[29] A. J. Wilkie, "Some local definability theory for holomorphic functions", pp. 197–213 in *Model theory with applications to algebra and analysis*, vol. 1, London Math. Soc. Lecture Note Ser. **349**, Cambridge Univ. Press, 2008. MR Zbl

Received May 30, 2024. Revised December 19, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.309

DIVERGENCE FUNCTIONS OF HIGHER-DIMENSIONAL THOMPSON'S GROUPS

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We prove that higher-dimensional Thompson's groups have linear divergence functions. By the work of Druțu, Mozes, and Sapir, this implies none of the asymptotic cones of nV has a cut-point.

1. Introduction

Thompson's groups F, T, and V are finitely presented infinite groups defined by Richard Thompson in the 1960s. They are all known to be mysterious groups. For example, T and V are the first examples of finitely presented, infinite, and simple, and it is known that the amenability of F is a difficult open problem. Because they have several unpredictable properties, by focusing on such properties, many "generalized" Thompson's groups were also defined. Higher-dimensional Thompson's groups, denoted by $2V, 3V, \ldots$, are some such groups defined by Brin [3]. The group V acts on the Cantor set $\mathfrak{C}$, and the group nV acts on the powers of the Cantor set $\mathfrak{C}^n$. It is known that nV is also finitely presented [4; 14] and simple [3; 5]. In addition, it was shown that for $n, m \in \mathbb{Z}_{>0}$, the group nV is isomorphic to mV if and only if $n = m$ holds [1]. In [16], it was proved that nV has Serre's property FA, and hence is one-ended.

In 2018, Golan and Sapir showed that the divergence functions of F, T, and V are linear [12]. This function was first mentioned by Gromov [13], and later, Gersten gave the formal definition [11] as a quasi-isometric invariant of geodesic metric spaces. Roughly speaking, the order of the function indicates whether the Cayley graphs of the group are "close" to the Euclidean or hyperbolic spaces. In fact, the orders of the functions of the direct powers of the infinite cyclic group $\mathbb{Z}^2, \mathbb{Z}^3, \ldots$ are linear, and it is known that the orders of the functions of hyperbolic groups are at least exponential [2]. In [12], they asked whether their proof could be extended to generalized Thompson's groups. In recent years, similar results have been obtained for some groups by extending the original arguments [17; 19; 18]. For recent results on functions of groups other than generalized Thompson's groups, see [15].

MSC2020: 20F65.
Keywords: divergence function, higher-dimensional Thompson's groups, asymptotic cones.

In this paper, we also extend the argument given by Golan and Sapir. That is, we show the following:

Theorem 1.1. *Higher-dimensional Thompson's groups have linear divergence functions.*

This paper is organized as follows. In Section 2, we summarize the definition of the higher-dimensional Thompson's group $2V$ and the notion of divergence functions. In Section 3, we first prepare estimates of the word length of elements in $2V$ to ensure that the condition required by the definition of the divergence function is satisfied. Subsequently, we construct a "good" path from any $g \in 2V$ to a specific element in $2V$, where the element is determined only from the word length of g. It should be noted that we assume $n = 2$ in most of this paper; however, all the proofs can be generalized to n with the appropriate modifications.

2. Preliminaries

2.1. ***Definition of the higher-dimensional Thompson's group 2V.*** In this section, we define only the higher-dimensional Thompson's group $2V$. Readers unfamiliar with the Thompson's groups are referred to [8].

2.1.1. *Homeomorphisms on the direct product of Cantor sets.* Let $\mathfrak{C}$ be the Cantor set $\{0,1\} \times \{0,1\} \times \cdots$. For a finite word w on $\{0,1\}$ and a finite or infinite word ζ on $\{0,1\}$, let $w\zeta$ denote their concatenation. Following [3], we will describe partitions of $\mathfrak{C}^2$ by using subdivisions of the unit square $[0,1]^2$. Subsequently, we will define homeomorphisms from $\mathfrak{C}$ to itself that are obtained from such partitions.

Firstly, we call $[0,1]^2$ itself the *trivial pattern*. Let us consider a rectangle $[a_1, a_2] \times [b_1, b_2] \subset [0,1]^2$. By dividing this rectangle in half, we obtain two new rectangles. The way of obtaining rectangles $[a_1, (a_1+a_2)/2] \times [b_1, b_2]$ and $[(a_1+a_2)/2, a_2] \times [b_1, b_2]$ is called *vertical subdivision*, and the way of obtaining rectangles $[a_1, a_2] \times [b_1, (b_1+b_2)/2]$ and $[a_1, a_2] \times [(b_1+b_2)/2, b_2]$ is called *horizontal subdivision*. A *pattern* is defined as a finite set of rectangles in $[0,1]^2$ obtained from the trivial pattern by applying finitely many vertical and horizontal subdivisions. See Figure 1. For a rectangle $[a_1, a_2] \times [b_1, b_2]$, the lengths $a_2 - a_1$ and $b_2 - b_1$ of its horizontal and vertical edges are called its *horizontal* and *vertical lengths*, respectively.

Next, by using patterns, we define partitions of $\mathfrak{C}^2$ inductively. The trivial pattern corresponds to $\mathfrak{C}^2$ itself. Assume that a rectangle in a pattern corresponds to a subset $\{w\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'\zeta_2 \mid \zeta_2 \in \mathfrak{C}\} \subset \mathfrak{C}^2$ where w and w' are finite words on $\{0,1\}$. For the vertical subdivision, we define that the left rectangle corresponds to the subset $\{w0\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and the right rectangle corresponds to the subset $\{w1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$. Similarly, for the horizontal subdivision, we define that the bottom rectangle corresponds to the

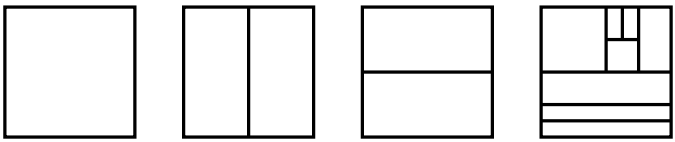


Figure 1. The trivial pattern, the pattern once divided vertically, the pattern once divided horizontally, and a pattern.

subset $\{w\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'0\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and the top rectangle corresponds to the subset $\{w\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w'1\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$. Consequently, a set of rectangles of a pattern gives a partition of $\mathfrak{C}^2$. For example, the pattern once divided vertically illustrated in Figure 1 corresponds to $\{0\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \mathfrak{C} \cup \{1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \mathfrak{C}$. We often identify a partition of $\mathfrak{C}^2$ with the corresponding partition of $[0, 1]^2$.

For a pattern with m rectangles, we assign a number from 0 to $m-1$ to each rectangle. Such a pattern is called *numbered pattern*. Let P and P' be numbered patterns with m rectangles. Let R_i (resp. R'_i) be a rectangle of P (resp. P') numbered i. Then there exist two subsets $\{a_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and $\{a'_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b'_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ corresponding to R_i and R'_i, respectively. By mapping each element $(a_i\zeta_1, b_i\zeta_2) \in \{a_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ to $(a'_i\zeta_1, b'_i\zeta_2) \in \{a'_i\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{b'_i\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, we obtain a homeomorphism on $\mathfrak{C}^2$.

Definition 2.1. *The higher-dimensional Thompson's group* $2V$ is the subgroup of $\mathrm{Homeo}(\mathfrak{C}^2)$ consisting of all homeomorphisms obtained from pairs of numbered patterns with the same number of rectangles.

Following a familiar convention, we write fg for $g \circ f$; namely, we always consider the right action of $2V$ on $\mathfrak{C}^2$.

The *domain* and *target pattern* of a pair of numbered patterns are defined as the patterns that determine the partition of the domain and range set of $\mathfrak{C}^2$, respectively.

Remark 2.2. The group nV is defined similarly as a subgroup of $\mathrm{Homeo}(\mathfrak{C}^n)$. Using the unit n-cube instead of the unit square, n-subdivisions are defined, which yield partitions of $\mathfrak{C}^n$.

Note that two distinct pairs of numbered patterns may give the same map. Let (P, P') be a pair of numbered patterns with the same number of rectangles (such a pair is just called a *pair of numbered patterns*). Let R_i and R'_i be rectangles of P and P' numbered i, respectively. Apply a vertical subdivision to R_i, and assign i_1 to the left rectangle and i_2 to the right rectangle. Do the same to R'_i and assign i_1 and i_2. Consequently, the obtained maps from the two pairs are the same. The same also holds for the case of horizontal subdivisions. A *reduced* pair of numbered patterns is a pair of numbered patterns where the inverse operations (called *vertical* and *horizontal reductions*) cannot be applied to any rectangles.

In order to multiply two pairs of numbered patterns (P_+, P_-) and (Q_+, Q_-), take a common refinement P of P_- and Q_+, and by using the operations, construct pairs (P'_+, P) and (P, Q'_-) such that they give the same maps as (P_+, P_-) and (Q_+, Q_-), respectively. Then the pair (P'_+, Q'_-) is the desired one.

Unfortunately, unlike the original Thompson's groups, there is no known way to define a unique reduced numbered pair for each element in $2V$. The notion of grid diagrams provides a solution to this issue. A detailed explanation will be given in Section 2.1.3.

2.1.2. *Pairs of colored binary trees.* Just as elements of the Thompson's groups are represented by pairs of binary trees, there exists a similar approach for $2V$. Like numbered patterns, the binary trees do not give unique representatives for elements of $2V$. However, because they can represent elements more simply than numbered patterns, we will frequently use colored binary trees in this paper.

We always assume that binary trees are rooted; namely, they have one specific vertex called the *root*. Vertices whose degree is one are called *leaves*. A graph with only one vertex (and no edges) is also regarded as a binary tree. We define a *caret* to be the binary tree consisting of three vertices, where the degree of the root is two and of the remaining vertices is one. See Figure 2.

Any binary tree can be constructed inductively by attaching carets to leaves. A *colored binary tree* is a binary tree where each caret is colored. We use two colors $\{a, b\}$ for $2V$ and use n colors for nV. We first explain the relationship between colored binary trees and patterns.

As previously stated, any pattern can be obtained inductively from the trivial pattern. Similarly, corresponding binary trees are also defined inductively. In the case of the trivial pattern, the corresponding binary tree is the one that consists of only the root. More precisely, consider the root as a leaf, the trivial pattern as a rectangle, and then this rectangle corresponds to this leaf. Since there exists no caret in this tree, we do not color it. Assume that for a pattern, there exists a colored binary tree with leaves such that there is a one-to-one correspondence between the set of rectangles and the set of leaves. Let R be a rectangle of this

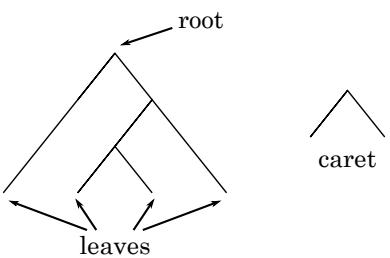


Figure 2. A binary tree and a caret.

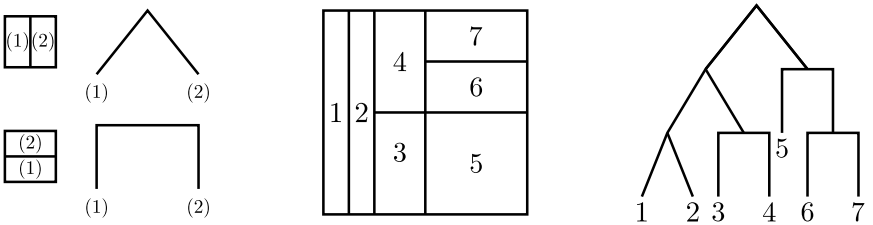


Figure 3. The two types of carets and a colored binary tree obtained from a pattern.

pattern, and assume that the corresponding leaf is i-th (from left to right). For the vertical subdivision of R, let R_1 be the left rectangle and R_2 be the right rectangle. Subsequently, we attach a caret colored by a to the i-th leaf and let R_1 correspond to the newly created left leaf and R_2 to the right leaf. Similarly, for the horizontal subdivision, we attach a caret colored by b to the i-th leaf and correspond the bottom rectangle to the left leaf and the top rectangle to the right leaf.

Following [6], in this paper, we represent carets colored by a by the "triangular carets" and carets colored by b by "square carets." See Figure 3.

The numbers in the rectangles and under the leaves represent the one-to-one correspondence between the set of leaves and rectangles. Observe that numbered patterns can also be represented by colored binary trees with numbers by writing numbers under the leaves in the same way. Consequently, each element in $2V$ can be represented by a pair of colored binary trees with numbers. A tree corresponding to a domain (resp. target) pattern is called a *domain* (resp. *target*) *tree*.

Note that, in general, more than one colored binary tree may give the same pattern. This is due to the fact that applying the following operations to a rectangle in a pattern are the same: the pattern obtained by subdividing vertically once and then subdividing two rectangles horizontally; subdividing horizontally once and then subdividing two rectangles vertically. See Figure 4.

As with pairs of numbered patterns, it is difficult to easily obtain a "good" pair of colored binary trees for each element of $2V$. However, we can use this to give an estimation of the word length, as we will see in Section 2.2. For a colored binary tree T, we define a *branch* of T to be a path from the root of T to a leaf of T. The

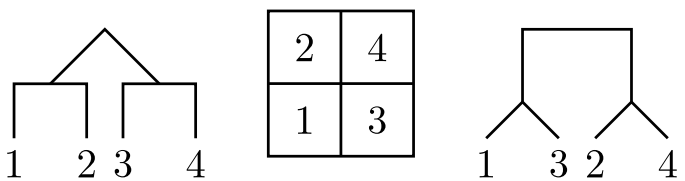


Figure 4. Two colored binary trees give the same pattern.

depth of T is then defined as the maximum length of the branches of T (with each edge having length one). It is clear that for any element g in $2V$, there exists a pair of colored binary trees giving g. Therefore, we may define the minimality of such pairs. First, we define a pair of colored binary trees to be *shallow* if it is a pair of colored binary trees whose target tree has the smallest depth among pairs that give the same map. A pair of colored binary trees is then defined as *minimal* if it is a shallow pair of binary trees with the smallest number of carets. It should be noted that such pairs may not be uniquely determined.

Remark 2.3. To make the relationship between grid diagrams and pairs of colored binary trees clearer, the definition of minimality is slightly modified from that in [6].

2.1.3. *Grid diagrams.* In this section, we explain how to represent each element in $2V$ using a "grid" based on [7]. However, unlike [7], the "grid" is constructed on the target patterns for the sake of our proof. Note that in [7, the second paragraph of Section 6], it was also pointed out that the same result holds in our setting. Indeed, we use only the fact that there exists a unique representative for each element in $2V$ (and nV). In this case, we only need to change the pattern we focus on from domain to target. Hence we omit most of the proofs in this section.

Definition 2.4. A *grid pattern* is defined as a pattern obtained by subdividing the unit square using only line segments of length one.

The pattern illustrated in Figure 3 is not a grid pattern, since the lengths of the horizontal line segment on $[1/4, 1] \times \{1/2\}$ and the horizontal line segment between rectangles numbered 6 and 7 are less than one. An example is illustrated in Figure 5.

Definition 2.5. A *grid diagram* is a pair of numbered patterns with the same number of rectangles, and the target pattern is a grid pattern.

Proposition 2.6 [7, Proposition 2.3]. *Each element in* $2V$ *admits a grid diagram as a representative.*

Sketch of proof. We can repeat horizontal and vertical subdivisions (so that the pairs of numbered patterns obtained always give the same map) until the target pattern becomes a grid pattern. □

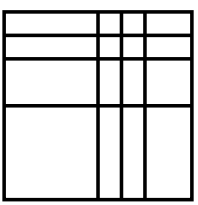


Figure 5. A grid pattern.

Moreover, it is known that there exists a unique representative for any element in $2V$. In order to give such a representative, we first recall the definition that a grid diagram is reduced.

Let (P, G) be a grid diagram where G is a grid pattern. In G, take a strip $I_i \times [0,1] \subset [0,1] \times [0,1]$ where I_i are horizontal edges of the rectangles in G. In other words, take a rectangle of G where the rectangle is $I_i \times J_j \subset [0,1] \times [0,1]$, and then consider $I_i \times [0,1]$. We apply vertical subdivisions to all rectangles in $I_i \times [0,1]$, and then apply vertical subdivisions to the corresponding rectangles in P such that the obtained grid pattern and (P, G) give the same homeomorphism on $\mathfrak{C}^2$. We call this operation a *vertical global subdivision*. We say a grid diagram is *vertically reduced* if the inverse operation of the vertical global subdivision cannot be applied to any two adjacent strips of the target pattern. Similarly, we can define a *horizontal global subdivision* and a grid diagram to be *horizontally reduced*. Finally, a grid diagram is said to be *reduced* if it is vertically and horizontally reduced. Then we have the following:

Theorem 2.7 [7, Theorem 3.2]. *Any element in $2V$ has a unique reduced grid diagram.*

Note that since each reduced grid diagram is a pair of numbered patterns, there exist corresponding pairs of colored binary trees.

2.2. ***A generating set of $2V$ and estimations of word length.*** Consider a finite set

$$X_{2V} := \left\{ x_0, x_1, x_2, y_i, B_i, C_i, \hat{x}_j, \hat{y}_1, \pi_i, \overline{\pi_i}, \alpha_i, \beta_i, \hat{B}_0, \gamma_0, {}_h x_j, {}_h \hat{x}_j \;\middle|\; i \in \{0,1\}, j \in \{1,2\} \right\}$$

defined in Figures 6, 7 and 8. The colored binary trees without numbers are all assigned $1, 2, \dots$ to the leaves from left to right. Because this set contains well-known generating sets (cf. [3; 4; 6]), this also generates $2V$. Note that this set is an inefficient set specialized for calculating the divergence function of $2V$.

Following [6], we also express an element of $2V$ as "$P \Pi Q^{-1}$ form" and use a part of the form in the proof of the main theorem. We define $A_i := A_0^{-(i-1)} A_1 A_0^{(i-1)}$ for $i \geq 1$ and $B_i := B_0^{-(i-1)} B_1 B_0^{(i-1)}$ for $i \geq 1$. Then the following holds:

Theorem 2.8 [6, Theorem 2.2]. *For an element g of $2V$, we have its expression $P \Pi Q^{-1}$, where*

(1) *P and Q are represented by words of the form*

$$C_{m_1} \cdots C_{m_p} W_{i_1} \cdots W_{i_r},$$

where C_i are in Figure 7, W_i are words on $\{A_i, B_i\}$ without inverse elements, $m_1 < m_2 < \cdots < m_p$, and $i_1 < i_2 < \cdots < i_r$.

(2) *Π is represented by a word on $\{\pi_0, \pi_1, \dots\} \cup \{\overline{\pi_0}, \overline{\pi_1}, \dots\}$.*

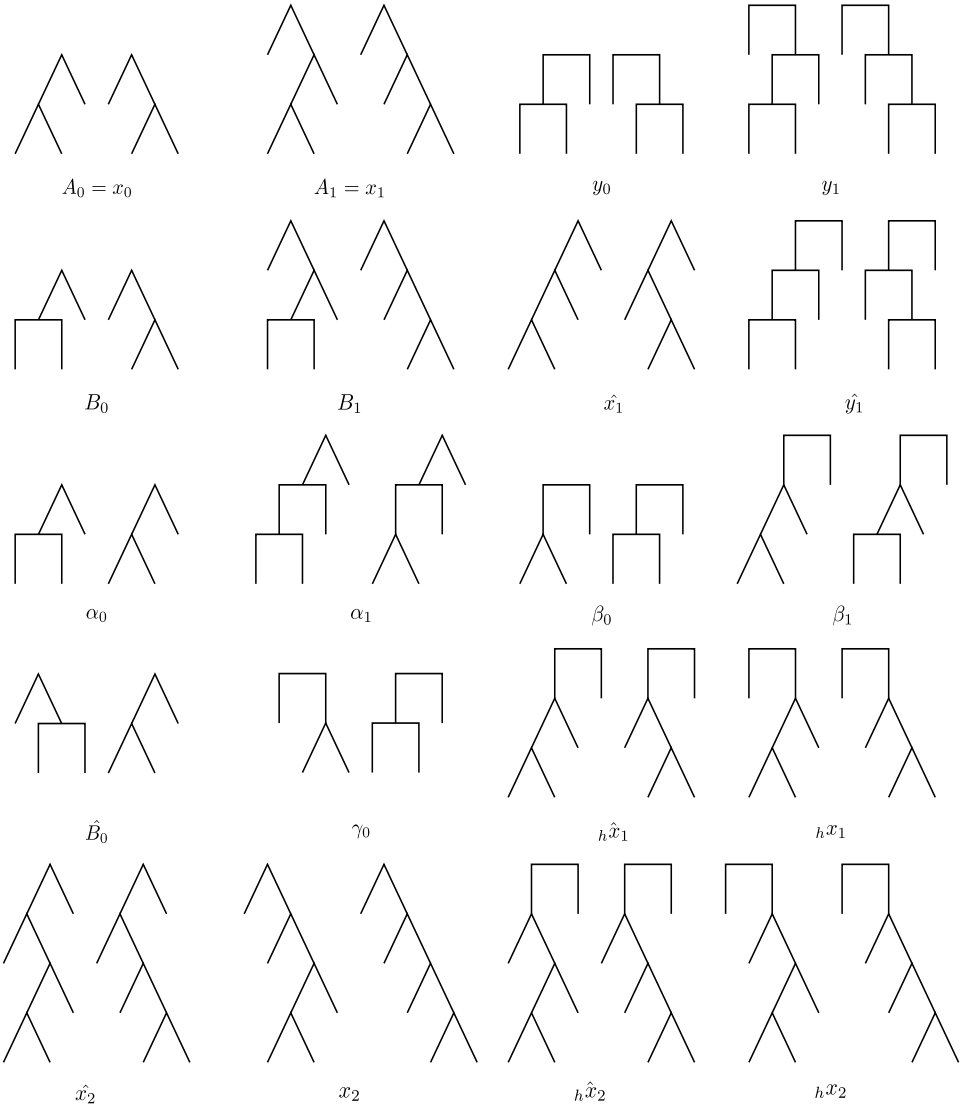


Figure 6. Generators of $2V$ except C_0, C_1, π_0, π_1, $\overline{\pi_0}$, and $\overline{\pi_1}$.

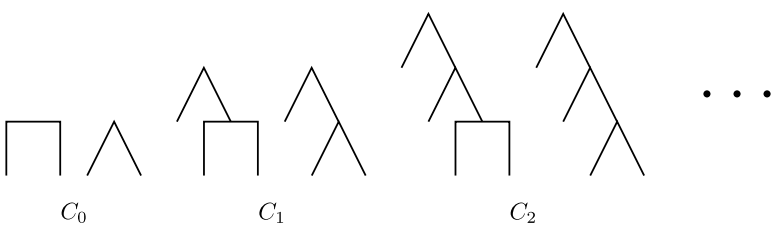


Figure 7. Generators $C_0, C_1, \ldots$.

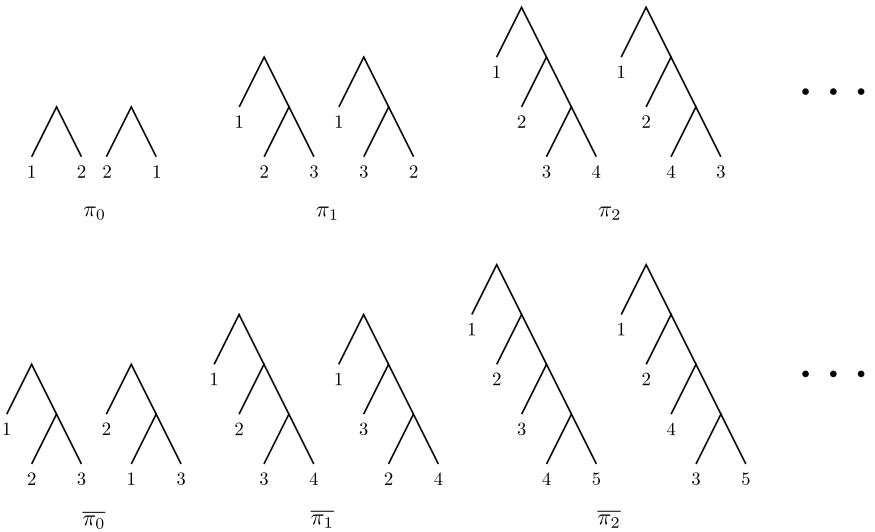


Figure 8. Generators $\pi_0, \pi_1, \ldots$ and $\overline{\pi_0}, \overline{\pi_1}, \ldots$.

In [6], Burillo and Cleary constructed this expression for each pair of colored binary trees. We also use this construction. See the proof of [6, Theorem 2.1] for details. Note that for element $g \in 2V$, if we take an expression $P\Pi Q^{-1}$ corresponding to a minimal pair of colored binary trees (T_+, T_-), then we have $P = (T_+, R_k)$, $\Pi = (R_k, \pi, R_k)$, and $Q = (T_-, R_k)$ where π is the corresponding permutation and R_k is the all-right tree with only vertical carets and k is the depth of g. Here, an *all-right tree* is a binary tree obtained by a finite number of attaching a caret only to the right leaf of a caret, and an all-right tree consisting of two types of carets is called *all-right colored binary tree*.

We will consider the word metric with respect to the above generating set. We first recall a known result of an estimation of the word length for pairs of binary trees.

Proposition 2.9 (cf. [6, Lemma 4.1]). *For an element of* $2V$ *which is represented by a minimal pair of colored binary trees with depth* D*, its word length with respect to* X_{2V} *is at least* $D/4$.

Remark 2.10. The difference between the denominators in [6, Lemma 4.1] and this lemma arises from the difference in the generating sets. The definition of minimality is also different, yet this lemma can still be shown similarly: if we take one of the shortest words with length n, we can say that the depth is at most $4n$ since each process of multiplying a generator increases the length of the branches by at most four.

As previously stated, a minimal pair of colored binary trees may not be uniquely determined for an element in $2V$. Hence, we use the notion of grid diagrams. We define the *size* of a rectangle $R = \{[a_i, a_{i+1}] \times [b_j, b_{j+1}]\}$ of a pattern as $-(\log_2(a_{i+1} - a_i) + \log_2(b_{j+1} - b_j))$ and write it as $\|R\|$. For a pattern P, the *fineness* of P is defined as the largest size of the rectangles in P and written as $\|P\|$. For an element $g \in 2V$, the *fineness* of g is defined as $\|G\|$, where G is the target numbered pattern of the reduced grid diagram of g. Note that the fineness of g is the same as the depth of the target tree of a pair of colored binary trees obtained from the grid diagram of g. Indeed, at the end of Section 2.1.2, we defined the depth of a pair of trees as the depth of the target tree. In the proof of the main theorem, we also use the following estimations:

Proposition 2.11. *Let g be in $2V$ with fineness k. Then the word length of g with respect to X_{2V} is at least $k/8$.*

Proof. Consider a minimal pair of colored binary trees of g and let D be its depth. Take a pair of numbered patterns (P_1, P_2) corresponding to this pair of colored binary trees. Then, we have $\|P_2\| = D$. We construct a grid diagram (P, G) of g from (P_1, P_2) without vertical and horizontal global subdivisions. Note that (P, G) may not be reduced. Since each rectangle in G has horizontal and vertical lengths of at least $1/2^D$, the fineness of G is at most $2D$. By considering the reduced grid diagram of g, the fineness of g is at most $\|G\|$, namely, $k \leq \|G\|$ holds. By Proposition 2.9, the word length of g is at least $D/4$. Hence, the desired result is obtained. □

By a similar argument, the following also holds:

Corollary 2.12. *Let (P_1, P_2) be a pair of numbered patterns corresponding to $g \in 2V$, and $R = \{w_1 x_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2 x_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ be a rectangle of P_2 where w_1, w_2 are words on $\{0, 1\}$ and x_1, x_2 are in $\{0, 1\}$. Assume that neither of the following conditions* (1) *nor* (2) *is satisfied*:

(1)
- $g^{-1}(R) = \{w_1' x_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2' \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, *where w_1' and w_2' are some word on $\{0, 1\}$, and*
- $g^{-1}((w_1 \hat{x}_1 \zeta_1, w_2 x_2 \zeta_2)) = (w_1' \hat{x}_1 \zeta_1, w_2' \zeta_2)$ *for every $(w_1 \hat{x}_1 \zeta_1, w_2 x_2 \zeta_2) \in \{w_1 \hat{x}_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2 x_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, where $\hat{x}_1$ is in $\{0, 1\}$ with $\hat{x}_1 \neq x_1$.*

(2)
- $g^{-1}(R) = \{w_1' \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2' x_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, *where w_1' and w_2' are some word on $\{0, 1\}$, and*
- $g^{-1}((w_1 x_1 \zeta_1, w_2 \hat{x}_2 \zeta_2)) = (w_1' \zeta_1, w_2' \hat{x}_2 \zeta_2)$ *for every $(w_1 x_1 \zeta_1, w_2 \hat{x}_2 \zeta_2) \in \{w_1 x_1 \zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2 \hat{x}_2 \zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, where $\hat{x}_2$ is in $\{0, 1\}$ with $\hat{x}_2 \neq x_2$.*

Then, the word length of g is at least $\|R\|/8$.

Proof. From the assumptions, the fineness of g is at least $\|R\|$. By Proposition 2.11, we have the desired result. □

We say a rectangle R is an *essential* rectangle if it satisfies the assumptions of Corollary 2.12. We call *horizontal* and *vertical twin* the subsets $R^v := \{w_1\hat{x}_1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2x_2\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ and $R^h := \{w_1x_1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2\hat{x}_2\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$, respectively, which are defined from $R = \{w_1x_1\zeta_1 \mid \zeta_1 \in \mathfrak{C}\} \times \{w_2x_2\zeta_2 \mid \zeta_2 \in \mathfrak{C}\}$ in the assumptions of Corollary 2.12. Then the assumptions mean that we cannot apply vertical and horizontal reductions to R and "congruent rectangles" adjacent to R. In other words, the assumptions are that R^v is not mapped to the $(g^{-1}(R))^v$ with keeping the orientation and R^h is not mapped to $(g^{-1}(R))^h$ with keeping the orientation.

2.3. ***Divergence functions of finitely generated groups.*** For finitely generated groups, the property of having linear divergence functions is a quasi-isometric invariant. Since we see asymptotic properties of functions, we introduce an equivalence relation on functions from $\mathbb{R}_{>0}$ to itself as follows: let f and g be functions from $\mathbb{R}_{>0}$ to itself. We first define $f \preceq g$ if there exist $A, B, C, D, E \geq 0$ such that

$$f(x) \leq Ag(Bx + C) + Dx + E$$

holds for all $x \in \mathbb{R}_{>0}$. Then we define $f \approx g$ if $f \preceq g$ and $g \preceq f$ hold. This is an equivalence relation on the set of functions from $\mathbb{R}_{>0}$ to itself. Note that all linear functions and constant functions are equivalent.

Let G be a finitely generated group with a finite generating set X. Let Γ be the Cayley graph of G with respect to X. For $\delta \in (0, 1)$, we first define the δ-divergence function of Γ as follows: let $\Omega(g_1, g_2)$ be the set of all paths connecting $g_1, g_2 \in \Gamma$. Let $\|\omega\|$ denote the length of the path ω. Then for $x \in \mathbb{R}_{>0}$, define the function ϕ_δ by setting

$$\phi_\delta(x) := \max\left\{ \min\{\|\omega\| \mid \omega \in \Omega(g_1, g_2) \text{ avoiding } B(e, \delta x)\} \;\middle|\; |g_1| = |g_2| = x \right\},$$

where $B(e, \delta x)$ denotes the open ball of radius δx with a center at identity of G, and $|g_1|, |g_2|$ denote the lengths from identity of G to g_1, g_2 in Γ. If there does not exist such a path, take $\phi_\delta(x) = \infty$. For each $\delta \in (0, 1)$, the equivalence class of ϕ_δ is invariant under quasi-isometries [11]. Hence, the δ-divergence function of G is well-defined as an equivalence class of functions.

Definition 2.13. We say *a group G has a linear divergence function* if there exists $\delta \in (0, 1)$ such that the δ-divergence function of G is in the equivalence class of linear maps.

From this definition, it is clear that if the δ-divergence function of G is in the equivalence class of linear maps, then the δ'-divergence function of G is also in the equivalence class of linear functions for every $0 < \delta' \leq \delta$. In fact, Druţ, Mozes, and Sapir [9; 10] showed the following theorem, which establishes a relationship between divergence functions and a topological property of asymptotic cones.

Theorem 2.14 [9; 10]. *The following are equivalent*:

(1) *G has a linear divergence function.*

(2) *For every $\delta \in (0, 1/54)$, the function ϕ_δ is in the equivalence class of linear functions.*

(3) *None of the asymptotic cones of G has a cut point.*

3. Proof of Theorem 1.1

3.1. ***Size of the bottom left rectangle.*** For a given pattern P, let $R_0(P)$ be defined as the rectangle of P containing the point $(0, 0) \in [0, 1] \times [0, 1]$. We first study the change of the word length when we multiply some of the generators in X_{2V}.

Lemma 3.1. *Let (P_+, P_-) be a pair of numbered patterns representing $g \in 2V$. Assume that $R_0(P_-)$ is essential and a subset of $[0, 1/4] \times [0, 1]$. Then*:

(1) *There exists a pair of numbered patterns $(P_+(gx_0^{-1}), P_-(gx_0^{-1}))$ representing gx_0^{-1} such that $R_0(P_-(gx_0^{-1}))$ is essential, $R_0(P_-(gx_0^{-1})) \subset [0, 1/8] \times [0, 1]$ and $\|R_0(P_-(gx_0^{-1}))\| = \|R_0(P_-)\| + 1$.*

(2) *There exists a pair of numbered patterns $(P_+(g\hat{B}_0), P_-(g\hat{B}_0))$ representing $g\hat{B}_0$ such that $R_0(P_-(g\hat{B}_0))$ is essential, $R_0(P_-(g\hat{B}_0)) \subset [0, 1/8] \times [0, 1]$ and $\|R_0(P_-(g\hat{B}_0))\| = \|R_0(P_-)\| + 1$.*

(3) *If $R_0(P_-)$ is a subset of $[0, 1/4] \times [0, 1/2]$, then there exists a pair of numbered patterns $(P_+(gC_0), P_-(gC_0))$ representing gC_0 such that $R_0(P_-(gC_0))$ is essential, $R_0(P_-(gC_0)) \subset [0, 1/8] \times [0, 1]$ and $\|R_0(P_-(gC_0))\| = \|R_0(P_-)\|$.*

Proof. For part (1), we have $R_0(P_-) \cup (R_0(P_-))^v \cup (R_0(P_-))^h \subset [0, 1/2] \times [0, 1]$ if $(R_0(P_-))^h$ is defined. Consider the composition of pairs of patterns. The rectangle $R_0(P_-)$ is unchanged even if the common refinement is taken. Hence, $x_0^{-1}(R_0(P_-))$ is an essential rectangle of a pair of patterns representing gx_0^{-1}. The remaining claims are clear from the definition of x_0^{-1} and because if $(R_0(P_-))^h$ is not defined, neither is $(x_0^{-1}(R_0(P_-)))^h$. By the same argument, part (2) also follows.

For part (3), if $R_0(P_-)$ is a subset of $[0, 1/4] \times [0, 1/4]$, then it is also followed by a similar argument of the proof of part (1). If $R_0(P_-)$ is $[0, a] \times [0, 1/2]$ for some a with $a \leq 1/4$, then $C_0(R_0(P_-))^h$ is not defined. Hence, we also have the desired result. □

By a similar argument, we also have the following:

Lemma 3.2. *Let (P_+, P_-) be a pair of numbered patterns representing $g \in 2V$. Assume that $R_0(P_-)$ is essential and a subset of $[0, 1] \times [0, 1/4]$. Then*:

(1) *There exists a pair of numbered patterns* $(P_+(gy_0^{-1}), P_-(gy_0^{-1}))$ *representing* gy_0^{-1} *such that* $R_0(P_-(gy_0^{-1}))$ *is essential,* $R_0(P_-(gy_0^{-1})) \subset [0,1] \times [0,1/8]$ *and* $\|R_0(P_-(gy_0^{-1}))\| = \|R_0(P_-)\| + 1$.

(2) *There exists a pair of numbered patterns* $(P_+(g\gamma_0), P_-(\gamma_0))$ *representing* $g\gamma_0$ *such that* $R_0(P_-(g\gamma_0))$ *is essential,* $R_0(P_-(g\gamma_0)) \subset [0,1] \times [0,1/8]$ *and* $\|R_0(P_-(g\gamma_0))\| = \|R_0(P_-)\| + 1$.

(3) *If* $R_0(P_-)$ *is a subset of* $[0,1/2] \times [0,1/4]$, *there exists a pair of numbered patterns* $(P_+(gC_0^{-1}), P_-(gC_0^{-1}))$ *representing* gC_0^{-1} *such that* $R_0(P_-(gC_0^{-1}))$ *is essential,* $R_0(P_-(gC_0^{-1})) \subset [0,1] \times [0,1/8]$ *and* $\|R_0(P_-(gC_0^{-1}))\| = \|R_0(P_-)\|$.

3.2. ***Construction of the path.*** In the rest of this paper, we write $|\cdot|$ for the word length of $2V$ with respect to X_{2V}. For a word w and w', we write $w \equiv w'$ when w and w' are the same as words, and $\|w\|$ denotes the length of w. If we have $w \equiv w'w''$ for some words w, w' and w'' (the word w'' may be the empty word), then w' is said to be a *prefix* of w and denoted by $w' \leq w$. Theorem 1.1 immediately follows from the following proposition:

Proposition 3.3. *There exist constants* δ, D *and a positive integer* Q *such that the following holds*: *let* $g \in 2V$ *with* $|g| \geq 4$. *Then there exists a path of length at most* $D|g|$ *in the Cayley graph of* $2V$ *which avoids a* $\delta|g|$*-neighborhood of the identity and which has the initial vertex* g *and the terminal vertex* $\hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$.

In other words, there exists a word ω *on the generating set such that* $\|\omega\| < D|g|$; *for any prefix* ω' *of* ω, *we have* $g\omega' > \delta|g|$ *such that*

$$g\omega = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}.$$

Proof. Let $(P_+(g), P_-(g))$ be a reduced pair of numbered patterns representing g such that $R_0(P_-(g))$ is essential. Note that $R_0(P_-(g))$ can be made essential by changing the choice of rectangles to be reduced if necessary. We will define six subwords, denoted by $\omega_1, \ldots, \omega_6$, and then the concatenation of these subwords, $\omega_1 \cdots \omega_6$, will be the desired word, ω.

In subpath 1, we may assume that $R_0(P_-(g))$ is a subset of $[0,1/2] \times [0,1]$. Indeed, since g is not the identity map, if not, we have $R_0(P_-(g)) \subset [0,1] \times [0,1/2]$. Then by replacing the "vertical argument" with the "horizontal one", it is possible to join the argument from subpath 3 onward. More precisely, we replace all vertical and horizontal carets in generators in subpaths 1, 2 with horizontal and vertical carets, respectively, and replace all the set $I_1 \times I_2 \subset [0,1] \times [0,1]$ in the definition of subpath 1 with $I_2 \times I_1$, and construct the $P\Pi Q^{-1}$ expression of g_1 by using an all-right tree with only horizontal carets instead of vertical carets. There are no generators in Figure 6 which generate Π constructed above, but this is

not a problem since all we need is Q and the required generators are defined in Figure 6.

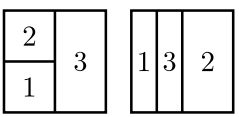
Subpath 1. We define ω_1 to be one of the following:

(a) If $R_0(P_-(g))$ is a subset of $[0, 1/4] \times [0, 1]$, then define ω_1 to be the empty word.

(b) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1]$, then define ω_1 to be $\hat{x}_1$.

(c) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2]$ and $\alpha_0(R_0(P_-(g)))$ is an essential rectangle of $g\alpha_0$, then define ω_1 to be α_0.

(d) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2]$ and $\alpha_0(R_0(P_-(g)))$ is not an essential rectangle of $g\alpha_0$, then define ω_1 to be $B_0\pi_1 x_0^{-1}$, as in the figure:



$B_0\pi_1 x_0^{-1}$

(e) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2^i]$ where $i \geq 2$ and $\alpha_0(R_0(P_-(g)))$ is an essential rectangle of $g\alpha_0$, then define ω_1 to be α_0.

(f) If $R_0(P_-(g))$ is $[0, 1/2] \times [0, 1/2^i]$ where $i \geq 2$ and $\alpha_0(R_0(P_-(g)))$ is not an essential rectangle of $g\alpha_0$, then define ω_1 to be α_1.

Lemma 3.4. *In any case, g_1 has a reduced pair $(P_+(g_1), P_-(g_1))$ of numbered patterns such that $R_0(P_-(g_1))$ is essential and a subset of $[0, 1/4] \times [0, 1]$. Moreover, $\|\omega_1\| \leq 3$ holds and for any prefix $\omega' \leq \omega_1$, we have $|g\omega'| \geq |g|/4$.*

Proof. Since the latter statements are obvious, it is sufficient to show the first statement. But since cases (a), (c), (d), (e), and (f) of the remaining claim are also clear, we consider only case (b). Let $\{w_1\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$ be a rectangle corresponding to $g^{-1}(R_0(P_-(g))$. In the process of multiplying $\hat{x}_1$, the rectangle $g^{-1}(R_0(P_-(g))$ is subdivided into $\{w_1 00\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$, $\{w_1 01\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$, and $\{w_1 1\zeta \mid \zeta \in \mathfrak{C}\} \times \{w_2\zeta \mid \zeta \in \mathfrak{C}\}$. Since the rectangles of a target pattern of $g\hat{x}_1$ corresponding to the first two are $\{00\zeta \mid \zeta \in \mathfrak{C}\} \times \{\zeta \mid \zeta \in \mathfrak{C}\}$ and $\{010\zeta \mid \zeta \in \mathfrak{C}\} \times \{\zeta \mid \zeta \in \mathfrak{C}\}$, respectively, the rectangle $\{00\zeta \mid \zeta \in \mathfrak{C}\} \times \{\zeta \mid \zeta \in \mathfrak{C}\}$ of $g\hat{x}_1$ is essential.

□

The idea of the following subpath comes from [12; 18].

Subpath 2. We fix an integer $M \geq 100$. Consider the expression $P\Pi Q^{-1}$ of a minimal pair of colored binary trees of g_1 and let $C_{m_1} \cdots C_{m_p}$ be the maximal

words on $\{C_i\}$ contained in Q, where $0 \le m_1 < \cdots < m_p$. When m_1 is not zero, we have

$$\begin{aligned} C_{m_1} \cdots C_{m_p} &= (x_0^{-(m_1-1)} C_1 x_0^{(m_1-1)} \cdots x_0^{-(m_p-1)} C_1 x_0^{(m_p-1)}) \\ &= x_0^{-(m_1-1)} C_1 x_0^{-(m_2-m_1)} \cdots C_1 x_0^{-(m_p-m_{p-1})} C_1 x_0^{(m_p-1)} \\ &= x_0^{-(m_1-1)} \hat{B}_0 x_0^{-(m_2-m_1-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 x_0^{m_p}, \end{aligned}$$

and define

$$\begin{aligned} \omega_2 \equiv\ & x_0^{-(m_1-1)} \hat{B}_0 x_0^{-(m_2-m_1-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 x_0^{-(M|g_1|-m_p)} \\ & \cdot x_1 \cdot x_0^{(M|g_1|-m_p)} \hat{B}_0^{-1} x_0^{(m_p-m_{p-1}-1)} \hat{B}_0^{-1} \cdots x_0^{(m_2-m_1-1)} \hat{B}_0^{-1} x_0^{(m_1-1)}. \end{aligned}$$

When m_1 is zero, define

$$\begin{aligned} \omega_2 \equiv\ & C_0 x_0^{-(m_2-1)} \hat{B}_0 x_0^{-(m_3-m_2-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 x_0^{-(M|g_1|-m_p)} \\ & \cdot x_1 \cdot x_0^{(M|g_1|-m_p)} \hat{B}_0^{-1} x_0^{(m_p-m_{p-1}-1)} \hat{B}_0^{-1} \cdots x_0^{(m_3-m_2-1)} \hat{B}_0^{-1} x_0^{(m_2-1)} C_0^{-1}. \end{aligned}$$

Let $g_2 = g_1\omega_2$.

See Figure 9 for example.

Lemma 3.5. (1) *We have* $\|\omega_2\| < 4M|g|$.

(2) *For every prefix* ω' *of* ω_2, *we have* $|g_1\omega'| > |g|/64$.

Proof. The proof is only provided when m_1 is not zero; however, it can be shown similarly when m_1 is zero. We first note that $m_p \le 4|g_1|$ holds by Proposition 2.9.

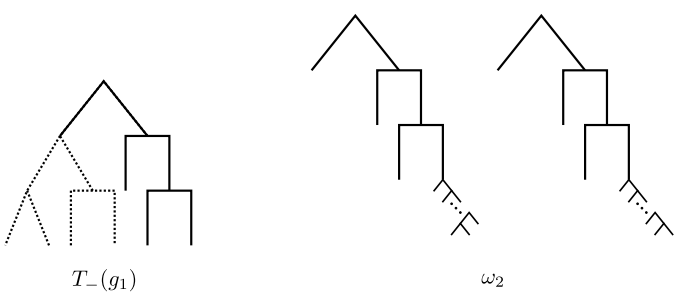


Figure 9. An example of the construction of ω_2. The dotted carets in $T_-(g_1)$ are not used when constructing ω_2. The all-right tree $T'_-(g_1)$ is obtained by removing the dotted carets from $T_-(g_1)$. Note that the tree $T'_-(g_1)$ is the domain tree of the product $C_{m_1} \cdots C_{m_p}$. By using $T'_-(g_1)$, all we needed to define ω_2 are the generators $\hat{B}_0$, x_0, and x_1. See also Figure 10.

A straightforward calculation yields the upper bound of $\|\omega_2\|$. Indeed, we have

$$\begin{aligned}&\big((m_1-1)+1+(m_2-m_1-1)+1+\cdots+(m_p-m_{p-1}-1)+1+(M|g_1|-m_p)\big)\times 2+1\\&=2M|g_1|+1\\&\le 2M(|g|+3)+1\\&<4M|g|.\end{aligned}$$

For part (2), we first consider $\omega' \le \omega_2$ which does not contain x_1. When $\|\omega'\| \le \lfloor |g_1|/2 \rfloor$, we have $|g_1\omega'| \ge |g_1|/2$. Indeed, if not, we have

$$|g_1| = |g_1\omega'(\omega')^{-1}| \le |g_1\omega'| + \|\omega'\| < \frac{|g_1|}{2} + \frac{|g_1|}{2}.$$

Hence we have $|g_1\omega'| \ge |g_1|/2 \ge (|g|-3)/2 \ge |g|/8$. Next, we assume that ω' does not contain x_1 and $\|\omega'\| > |g_1|/2$ holds. Then by Lemmas 3.4 and 3.1, there exists a pair of numbered patterns $(P_+(g_1\omega'), P_-(g_1\omega'))$ representing $g_1\omega'$ such that $R_0(P_-(g_1\omega'))$ is essential. Then we have

$$\|R_0(P_-(g_1\omega'))\| = \|R_0(P_-(g_1))\| + \|\omega'\| > \|\omega'\| > \frac{|g_1|}{2}. \tag{3-1}$$

By Corollary 2.12, we have

$$|g_1\omega'| \ge \frac{\|R_0(P_-(g_1\omega'))\|}{8} > \frac{|g_1|}{16} \ge \frac{|g|}{64}. \tag{3-2}$$

Next, we assume ω' contains x_1 and no $\hat{B}_0^{-1}$. Then there exists $i \ge 0$ such that

$$g_1\omega' = g_1 x_0^{-(m_1-1)} \hat{B}_0 x_0^{-(m_2-m_1-1)} \cdots \hat{B}_0 x_0^{-(m_p-m_{p-1}-1)} \hat{B}_0 (x_0^{-(M|g_1|-m_p)} x_1 x_0^{(M|g_1|-m_p)}) x_0^{-i}$$

as an element in $2V$. Since $x_0^{-(M|g_1|-m_p)} x_1 x_0^{(M|g_1|-m_p)}$ is identity on $[0, 1/2]\times[0, 1]$, the essentiality of the rectangle is preserved when $i = 0$. This means that when $i = 0$, we have $\|R_0(P_-(g_1\omega'))\| > |g_1|/2$ by inequality (3-1). According to Lemma 3.1(1), multiplying x_0^{-1} from the right increases the size of the rectangle. Then for any $i \ge 0$, we have $\|R_0(P_-(g_1\omega'))\| > |g_1|/2$. This implies that we have $|g_1\omega'| > |g|/64$ for the same reason as in inequality (3-2).

Finally, consider the case where ω' contains x_1 and $\hat{B}_0^{-1}$. From the construction of ω_2, it can be observed that g_2 is obtained by attaching carets to the minimal pair of colored binary trees $(T_+(g_1), T_-(g_1))$. This implies that there exists a pair of numbered patterns of g_2 such that the rectangle of the target pattern that contains a point $(1, 1) \in [0, 1] \times [0, 1]$ is essential, and its size is at least $M|g_1| + 3 - 4|g_1|$. Indeed, by Proposition 2.9, the length of the right most branch of $T_-(g_1)$ is at most $4|g_1|$. Hence the number of attached vertical carets to the

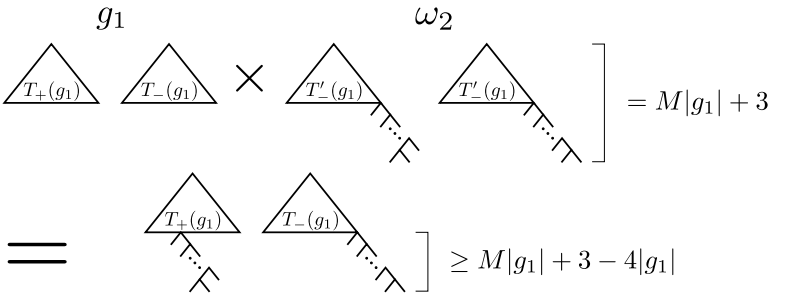


Figure 10. An illustration of the multiplication of g_1 and ω_2, where $(T_+(g_1), T_-(g_1))$ is a minimal pair of colored binary trees of g_1 and $T'_-(g_1)$ is the maximal all-right colored binary tree contained in $T_-(g_1)$.

rightmost branch is at least $M|g_1| + 3 - 4|g_1|$. Also, from the construction, vertical reduction cannot be applied. Therefore the size of the rectangle containing $(1, 1)$ is at least $M|g_1| + 3 - 4|g_1|$ by focusing only on the vertical subdivisions, since the size of a rectangle is defined as the sum of the number of vertical and horizontal subdivisions. See Figure 10 for an illustration of this argument.

We now consider $g_1\omega'$ as $g_2\omega''$ by a certain word ω''. Note that we have $\|\omega''\| \leq m_p$ by a straightforward calculation. This means that $\|\omega''\|$ is sufficiently smaller than $|g_2|$, so we estimate $|g_1\omega'|$ based on g_2. Since $m_p \leq 4|g_1|$ holds, by Corollary 2.12 and Lemma 3.4, we have

$$\begin{aligned} |g_1\omega'| = |g_2\omega''| \geq |g_2| - \|\omega''\| &\geq \frac{M|g_1| + 3 - 4|g_1|}{8} - 4|g_1| \\ &> \tfrac{1}{8}(M - 36)|g_1| \geq 8|g_1| \geq 2|g|. \end{aligned} \tag{3-3}$$

Hence for any prefix $\omega' \leq \omega_2$, we have $|g_1\omega'| > |g|/64$.

□

Subpath 3. Let ω_3 be a minimal word on X_{2V} such that $\omega_3 = g_1^{-1}$ holds. Let $g_3 = g_2\omega_3$.

Lemma 3.6. (1) *We have* $\|\omega_3\| \leq 2|g|$.

(2) *For every prefix* ω' *of* ω_3, *we have* $|g_2\omega'| > 2|g|$.

Proof. The first claim is obvious by Lemma 3.4. For the second claim, we note that $|\omega'| \leq |g_1|$ holds since ω_3 is a minimal word. Then we have

$$|g_2\omega'| \geq |g_2| - |\omega'| \geq |g_2| - |g_1| > 2|g|,$$

as estimated in inequality (3-3).

□

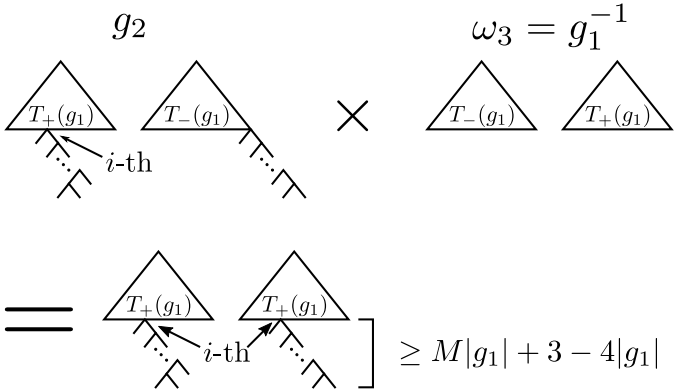


Figure 11. An illustration of g_3. Observe that g_3 is the identity map on the subset corresponding to all leaves except the i-th leaf.

Subpath 4. We fix an integer $Q \geq 48M$. We define ω_4 based on a subset $[0, 1] \times [0, 1]$ where g_3 is the identity map. From the construction, one of the following holds (see also Figure 11):

(a) g_3 is the identity map on $[0, 1] \times [0, 1/2]$;

(b) g_3 is the identity map on $[0, 1] \times [1/2, 1]$;

(c) g_3 is the identity map on $[0, 1/2] \times [0, 1]$; and

(d) g_3 is the identity map on $[1/2, 1] \times [0, 1]$.

Then we define ω_4 as the word determined from one of the following cases of capital letters corresponding to each small letter:

(A) let $\omega_4 \equiv {}_h\hat{x}_1{}^{-(Q|g|+1)}{}_h\hat{x}_{2h}\hat{x}_1{}^{Q|g|+1}$;

(B) let $\omega_4 \equiv {}_hx_1{}^{-(Q|g|+1)}{}_hx_{2h}x_1{}^{Q|g|+1}$;

(C) let $\omega_4 \equiv \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}$; and

(D) let $\omega_4 \equiv x_1{}^{-Q|g|}x_2x_1{}^{Q|g|}$.

Let $g_4 = g_3\omega_4$.

Lemma 3.7. (1) *We have* $\|\omega_4\| \leq 3Q|g|$.

(2) *For every prefix* ω' *of* ω_4, *we have* $|g_3\omega'| > 3|g|$.

(3) *As elements in* $2V$, g_3 *and* ω_4 *commute.*

Proof. Part (1) follows from a straightforward estimation, and part (3) is obvious since the supports of g_3 and ω_4 are disjoint. For part (2), in any case, we note that generators in ω_4 preserve the rectangle with its size at least $M|g_1| + 3 - 4|g_1|$. Since the process of obtaining the essential rectangle from this rectangle requires only at most one horizontal reduction, $g_3\omega_4$ is also represented by a pair

of numbered patterns with an essential rectangle of size at least $M|g_1|+2-4|g_1|$. Indeed, when ω_4 is defined in the case of (A), since ${}_h\hat{x}_1, {}_h\hat{x}_2$ is the identity on $[0,1]\times[0,1/2]$, the vertical reduction cannot be applied in the process and the horizontal reduction may be possible. Also, since any generator in ω_4 vertically divides $[0,1]\times[0,1/2]$, the number of this horizontal reduction is at most one. For case (B), only at most one reduction is applied for the same reason as in case (A), and no such reduction is possible in cases (C) and (D). Hence the size is at least $M|g_1|+2-4|g_1|$ in any cases. By Corollary 2.12 and Lemma 3.4, we have

$$|g_3\omega'| \geq \frac{M|g_1|+2-4|g_1|}{8} > 12|g_1| \geq 3|g|,$$

which is the desired result.

□

Subpath 5. Let ω_5 be a minimal word on X_{2V} such that $\omega_5 = g_3^{-1}$ holds. Let $g_5 = g_4\omega_5$.

Lemma 3.8. (1) *We have* $\|\omega_5\| \leq 5M|g|$.

(2) *For every prefix* ω' *of* ω_5, *we have* $|g_4\omega'| > M|g|$.

Proof. For part (1), by Lemmas 3.4, 3.5 and 3.6, we have

$$\|\omega_5\| \leq |g| + \|\omega_1\| + \|\omega_2\| + \|\omega_3\| \leq |g| + 3 + 4M|g| + 2|g| < 5M|g|.$$

For part (2), we first note that ω_4 is represented by a pair of patterns with an essential rectangle of size at least $Q|g|+4$. In particular, by a similar argument to the proof of Lemma 3.7, the horizontal length of this rectangle is unchanged for $g_4 = g_3\omega_4$. Hence by Corollary 2.12, we have $|g_4| \geq (Q|g|+3)/8$. Therefore we have

$$|g_4\omega'| \geq |g_4| - \|\omega_5\| > 6M|g| - 5M|g| = M|g|.$$

This completes the proof.

□

We may now obtain $g\omega_1\omega_2\omega_3\omega_4\omega_5 = \omega_4$, which only depends on $|g|$.

The final step is to connect any of cases (A) to (D) defined in subpath 4 to $\hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$ by a final subpath. In order to define this subpath, we write the subpaths defined in cases (A) to (D) as $\omega_4(A)$, $\omega_4(B)$, $\omega_4(C)$ and $\omega_4(D)$, respectively.

Subpath 6. If the path ω_4 is $\omega_4(A)$, let $\omega_6 \equiv \omega_4(B)\omega_4(C)$. If the path ω_4 is $\omega_4(B)$, let $\omega_6 \equiv \omega_4(A)\omega_4(C)$. If the path ω_4 is $\omega_4(C)$, let $\omega_6 \equiv \omega_4(D)$. Finally, if the path ω_4 is $\omega_4(D)$, let $\omega_6 \equiv \omega_4(C)$.

Lemma 3.9. (1) *In any case, we have* $g_5\omega_6 = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$ *as an element in* $2V$.

(2) *We have* $\|\omega_6\| \leq 6Q|g|$.

(3) *For every prefix* ω' *of* ω_6, *we have* $|g_5\omega'| > 6M|g|$.

Proof. For part (1), we note that $\omega_4(A)\omega_4(B) = \omega_4(D)$ holds as elements in $2V$. Observe that the supports of $\omega_4(A)$ and $\omega_4(B)$ are disjoint, and the same holds for $\omega_4(C)$ and $\omega_4(D)$. Since we have $\omega_4(C)\omega_4(D) = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$, we obtain the desired result. Part (2) follows from Lemma 3.7.

Finally, part (3) follows from the following observation: in the process of multiplying generators of ω_6, there always exists a pair of numbered patterns with an essential rectangle whose horizontal length is at least $Q|g|+2$. Hence by Corollary 2.12, we have $|g_5\omega'| \geq (Q|g|+2)/8 > 6M|g|$.

□

Now, we define ω as $\omega_1 \cdots \omega_6$. From Lemmas 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, take $D = 10Q$ and $\delta = 1/64$. Then we have that $g\omega = \hat{x}_1^{-Q|g|}\hat{x}_2\hat{x}_1^{Q|g|}x_1^{-Q|g|}x_2x_1^{Q|g|}$, $\|\omega\| < D|g|$, and $|g\omega'| > \delta|g|$ for any prefix ω' of ω. This completes the proof of Proposition 3.3.

□

Acknowledgements

The author would like to thank Professor Tomohiro Fukaya for his helpful comments. He also appreciates Professor Lucy Lifschitz for teaching him about the divergence function of the Lodha–Moore group. Last but not least, he expresses his gratitude to the referee for his or her kind comments. The author was supported by JSPS KAKENHI Grand number 24K22836.

References

[1] C. Bleak and D. Lanoue, "A family of non-isomorphism results", *Geom. Dedicata* **146** (2010), 21–26. MR Zbl

[2] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren Math. Wiss. **319**, Springer. Zbl

[3] M. G. Brin, "Higher dimensional Thompson groups", *Geom. Dedicata* **108** (2004), 163–192. MR Zbl

[4] M. G. Brin, "Presentations of higher dimensional Thompson groups", *J. Algebra* **284**:2 (2005), 520–558. MR Zbl

[5] M. G. Brin, "On the baker's map and the simplicity of the higher dimensional Thompson groups nV", *Publ. Mat.* **54**:2 (2010), 433–439. MR Zbl

[6] J. Burillo and S. Cleary, "Metric properties of higher-dimensional Thompson's groups", *Pacific J. Math.* **248**:1 (2010), 49–62. MR Zbl

[7] J. Burillo, S. Cleary, and B. Nucinkis, "Grid diagrams for higher-dimensional Thompson's groups", preprint, 2024. arXiv 2403.02562

[8] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Introductory notes on Richard Thompson's groups", *Enseign. Math.* (2) **42**:3-4 (1996), 215–256. MR Zbl

[9] C. Druţu, S. Mozes, and M. Sapir, "Divergence in lattices in semisimple Lie groups and graphs of groups", *Trans. Amer. Math. Soc.* **362**:5 (2010), 2451–2505. MR Zbl

[10] C. Druţu, S. Mozes, and M. Sapir, "Corrigendum to "Divergence in lattices in semisimple Lie groups and graphs of groups"", *Trans. Amer. Math. Soc.* **370**:1 (2018), 749–754. MR Zbl

[11] S. M. Gersten, "Quadratic divergence of geodesics in CAT(0) spaces", *Geom. Funct. Anal.* **4**:1 (1994), 37–51. MR Zbl

[12] G. Golan and M. Sapir, "Divergence functions of Thompson groups", *Geom. Dedicata* **201**, 227–242. MR Zbl

[13] M. Gromov, "Asymptotic invariants of infinite groups", pp. 1–295 in *Geometric group theory* (Sussex, 1991), London Math. Soc. Lecture Note Ser. **182**, Cambridge Univ. Press, 1993. MR Zbl

[14] J. Hennig and F. Matucci, "Presentations for the higher-dimensional Thompson groups nV", *Pacific J. Math.* **257**:1 (2012), 53–74. MR Zbl

[15] L. Issini, "On linear divergence in finitely generated groups", preprint, 2023. arXiv 2311.12938

[16] M. Kato, "Higher dimensional Thompson groups have Serre's property FA", preprint, 2015. Zbl arXiv 1504.06680

[17] Y. Kodama, "Divergence function of the braided Thompson group", *Kyoto J. Math.* **63**:2 (2023), 435–470. MR Zbl

[18] L. Lifschitz, "Divergence of the Lodha–Moore group", *Proc. Amer. Math. Soc.* (published online 2024).

[19] X. Sheng, "Divergence property of the Brown–Thompson groups and braided Thompson groups", *Transform. Groups* (published online January 2024).

Received June 7, 2024. Revised December 16, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.331

VOGAN'S CONJECTURE ON LOCAL ARTHUR PACKETS OF p-ADIC GL_n AND A COMBINATORIAL LEMMA

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For GL_n over a p-adic field, Cunningham and Ray proved Vogan's conjecture, that is, local Arthur packets are the same as ABV packets. They used endoscopic theory to reduce the general case to a combinatorial lemma for irreducible local Arthur parameters, and their proof implies that one can also prove Vogan's conjecture for p-adic GL_n by proving a generalized version of this combinatorial lemma. Riddlesden recently proved this generalized lemma. We give a new proof of it, which has its own interest.

1. Introduction

Let F be a non-Archimedean field of characteristic zero and let W_F denote the Weil group of F. Let G be a connected reductive group defined over F. We define $G := \mathrm{G}(F)$ and $\Pi(G)$ the isomorphism classes of smooth irreducible representations of G. A local Arthur parameter ψ is a continuous homomorphism

$$\psi : W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to {}^L G,$$

such that

(1) the restriction of ψ to W_F has bounded image;

(2) the restrictions of ψ to both $\mathrm{SL}_2^D(\mathbb{C})$ and $\mathrm{SL}_2^A(\mathbb{C})$ are analytic;

(3) ψ commutes with the projections $W_F \times \mathrm{SL}_2^D(\mathbb{C}) \times \mathrm{SL}_2^A(\mathbb{C}) \to W_F$ and ${}^L G \to W_F$.

Here $\mathrm{SL}_2^D(\mathbb{C})$ is called the Deligne-SL_2 and $\mathrm{SL}_2^A(\mathbb{C})$ is called the Arthur-SL_2.

In his fundamental work [2], Arthur attached a local Arthur packet Π_ψ for each local Arthur parameter ψ of quasisplit classical groups. This is a finite (multi)set of smooth irreducible representations, satisfying certain regular and twisted endoscopic character identities [2, Theorem 2.2.1]. Assuming the Ramanujan conjecture, Arthur showed that the union of these local Arthur packets contains the local components of all discrete square-integrable automorphic representations.

The research of the author is partially supported by the NSF Grant DMS-1848058.
MSC2020: primary 11F70, 22E50; secondary 11F85.
Keywords: local Arthur packets, local Arthur parameters, Vogan's conjecture.

An interesting question is to construct each local Arthur packet Π_ψ over p-adic fields besides the abstract definition. In a series of work [16; 17; 18; 19; 20], Mœglin explicitly constructed each local Arthur packet and showed that it is multiplicity-free. However, there are difficulties in her construction when trying to compute the representations in the local Arthur packets in terms of the Langlands classification. To remedy this, for symplectic and split odd special orthogonal groups, Atobe gave a reformulation of Mœglin's construction [3] using extended multisegments, and gave an explicit algorithm to compute the Langlands classification data for the representations in a local Arthur packet. The main tools in these results are (partial) Aubert–Zelevinsky involution and partial Jacquet module (called derivatives in [4]), which are representation theoretic.

On the other hand, in [8], Cunningham, Fiori, Moussaoui, Mracek and Xu aimed to construct local Arthur packets over p-adic fields using a geometric approach. They extended [1] to p-adic reductive groups: For any L-parameter ϕ of a p-adic quasisplit connected reductive group G^*, they defined the ABV-packet Π_ϕ^{ABV}, a finite set of irreducible representations of pure inner forms of G^*, using microlocal vanishing cycle functors. For a fixed pure inner form G of G^*, we shall write $\Pi_\phi^{\mathrm{ABV}}(G) := \Pi_\phi^{\mathrm{ABV}} \cap \Pi(G)$. It is expected that the ABV-packets recover the local Arthur packets in the following sense. For each local Arthur parameter ψ of G, we associate an L-parameter ϕ_ψ by

$$\phi_\psi(w, x) := \psi\left(w, x, \begin{pmatrix} |w|^{\frac{1}{2}} & \\ & |w|^{\frac{-1}{2}} \end{pmatrix}\right).$$

We say ϕ is of Arthur type if $\phi = \phi_\psi$ for some local Arthur parameter ψ. *Vogan's conjecture* is stated as follows.

Conjecture 1.1 [8, Conjecture 8.3.1(a)]. *For any local Arthur parameter ψ of $\mathrm{G}(F)$,*

$$\Pi_\psi = \Pi_{\phi_\psi}^{\mathrm{ABV}}(\mathrm{G}(F))$$

There are more precise statements matching the distributions associated with local Arthur packets and ABV-packets; we refer to [8, Conjecture 8.3.1] for details.

Conjecture 1.1 remains widely open. The only known case is $\mathrm{GL}_n(F)$, proved by Cunningham and Ray [5; 6]. To introduce our main result, we sketch their main idea in the following subsection.

1.1. ***Proof of Vogan's conjecture of $\mathbf{GL}_n(F)$ and the combinatorial lemma.*** First, we introduce a crucial ingredient of their proof, the *Pyasetskii involution* on L-parameters, which we denote by

$$\phi \mapsto \widehat{\phi}.$$

This involution is defined via the geometric structure on Vogan varieties. See [8, §6.4] or [23, §4.3] for a precise definition. When $G = \mathrm{GL}_n(F)$, there are bijections among irreducible representations of G, L-parameters of G, and a collection of multisegments (see Section 2.1 for details). In [21], Mœglin and Waldspurger showed that under these bijections, Pyasetskii involution on L-parameters matches the *Zelevinsky involution* on irreducible representations defined in [23, §4.1]. They gave a combinatorial algorithm on multisegments to realize these involutions (see Section 2.3). Later in [14], Knight and Zelevinsky gave a closed formula for the involution on multisegments, which is proved using the theory of flows in network.

For $\mathrm{GL}_n(F)$, the structure of L-packets and local Arthur packets are simple: they are all singletons. Therefore, the inclusion $\Pi_\psi = \Pi_{\phi_\psi} \subseteq \Pi_{\phi_\psi}^{\mathrm{ABV}}$ automatically holds. With this observation and the geometric structure of ABV-packets of $\mathrm{GL}_n(F)$ [7, Proposition 3.2.1], Cunningham and Ray demonstrated in the proof of [6, Theorem 5.3] that for an irreducible local Arthur parameter ψ of $\mathrm{GL}_n(F)$,

$$\Pi_\psi = \Pi_{\phi_\psi}^{\mathrm{ABV}}(\mathrm{GL}_n(F))$$

holds if the following lemma holds for ψ.

Lemma 1.2 [6, Lemma 4.8]. *Let ψ be an irreducible local Arthur parameter of $\mathrm{GL}_n(F)$. If ϕ is an L-parameter of $\mathrm{GL}_n(F)$ satisfying $\phi \geq \phi_\psi$ and $\widehat{\phi} \geq \widehat{\phi_\psi}$, then*

$$\phi = \phi_\psi.$$

Here the inequality is the closure ordering of L-parameters defined via the geometric structure on Vogan varieties, which is equivalent to the partial ordering on multisegments considered in [23] (see Section 2.2 for details). Cunningham and Ray proved the above lemma, and hence established Vogan's conjecture in this case. Later in [5], they used endoscopic lifting to reduce the general case to the case of irreducible parameters. This proved Vogan's conjecture for $\mathrm{GL}_n(F)$ completely.

On the other hand, the proof in [6, Theorem 5.3] implies that for an *arbitrary* local Arthur parameter ψ of $\mathrm{GL}_n(F)$, not necessarily irreducible, the equality

$$\Pi_\psi = \Pi_{\phi_\psi}^{\mathrm{ABV}}(\mathrm{GL}_n(F))$$

holds if the following generalized lemma holds for ψ.

Lemma 1.3. *Let ψ be an arbitrary local Arthur parameter of $\mathrm{GL}_n(F)$. If ϕ is an L-parameter of $\mathrm{GL}_n(F)$ satisfying $\phi \geq \phi_\psi$ and $\widehat{\phi} \geq \widehat{\phi_\psi}$, then*

$$\phi = \phi_\psi.$$

Riddlesden [22] proved Lemma 1.3, mainly using the network description of the Zelevinsky involution as in [14]. Therefore, combining with the proof of [6, Theorem 5.3], this provides a combinatorial approach to Vogan's conjecture for $\mathrm{GL}_n(F)$.

1.2. ***Main result.*** We present a new proof of Lemma 1.3, thereby offering an alternative approach to Vogan's conjecture for $\mathrm{GL}_n(F)$. Compared to [22], our proof is elementary and only involves the Mœglin–Waldspurger algorithm. This reflects an important technique in the study of general representations of Arthur type, and hence has its own interest and is expected to have applications, as outlined below.

The main idea of the proof is to verify that the ABV-packets of $\mathrm{GL}_n(F)$ satisfy an analogue of the following property for local Arthur packets of $G_n = \mathrm{Sp}_{2n}(F)$ or split $\mathrm{SO}_{2n+1}(F)$ proved in [11]. Suppose π is a representation of G_n of Arthur type. Then the L-parameter ϕ_π shares a specific common summand with the L-parameter $\phi_{\psi^{\max}(\pi)}$, where

$$\psi^{\max}(\pi) \in \Psi(\pi) := \{\psi \mid \pi \in \Pi_\psi\}$$

is "the" local Arthur parameter of π considered in [10]. If π is not tempered, one may define a representation π^- and a local Arthur parameter $(\psi^{\max}(\pi))^-$ of G_{n^-} with $n^- < n$ by removing this common summand from the L-parameters. Then $\pi^- \in \Pi_{(\psi^{\max}(\pi))^-}$. Conversely, repeating this process (with slight modifications) gives an algorithm to determine whether π is of Arthur type or not. See [12, §6] for a precise statement.

For $\mathrm{GL}_n(F)$, motivated by the phenomenon above, we show that the dual inequalities

$$\phi \geq \phi_\psi \quad \text{and} \quad \widehat{\phi} \geq \widehat{\phi_\psi}$$

imply that ϕ and ϕ_ψ must share certain common summands in Proposition 4.3. Define ϕ^- and ψ^- by removing these common summands. The dual inequalities are preserved, i.e.,

$$\phi^- \geq \phi_{\psi^-} \quad \text{and} \quad \widehat{\phi^-} \geq \widehat{\phi_{\psi^-}}.$$

Repeating the above procedure, we conclude that $\phi = \phi_\psi$, which proves the main result. Note that $\Psi(\pi)$ is always a singleton for $\mathrm{GL}_n(F)$, and hence $\psi^{\max}(\pi)$ does not play an essential role here.

1.3. ***Applications of the combinatorial lemma.*** Finally, we discuss some applications of the combinatorial lemma. We say that an L-parameter ϕ is *atomic* if

$$\{\phi' \mid \phi' \geq \phi,\ \widehat{\phi'} \geq \widehat{\phi}\} = \{\phi\}.$$

In upcoming work [9; 15], we show that this atomic property enables us to explicitly study the associated ABV-packet, as we will now explain.

First, suppose ϕ is an atomic L-parameter of $\mathrm{GL}_n(F)$ and let π be the corresponding irreducible representation. Note that the atomic property implies that $\Pi_\phi^{\mathrm{ABV}} = \{\pi\}$. In [15], we give an algorithm to express π as a linear combination of standard modules.

Next, we consider L-parameters of classical groups $\mathrm{Sp}_{2n}(F)$ or split $\mathrm{SO}_{2n+1}(F)$. On one hand, we can associate a "stable standard" distribution η_ϕ^{std} by summing over all standard modules in the L-packet Π_ϕ, which is stable. On the other hand, there is another distribution η_ϕ^{ABV} defined via the ABV-packet Π_ϕ^{ABV}, which is conjectured to be stable [8, Conjecture 8.4.2]. Assuming this conjecture, for each atomic L-parameter ϕ, we give an algorithm to express η_ϕ^{ABV} as a linear combination of stable standard distributions in [15].

Finally, in a joint work currently in progress [9], we are going to establish the conjecture that η_ϕ^{ABV} is always stable for classical groups under certain assumptions. As a nontrivial corollary, we show that if ϕ is atomic, then the ABV-packet Π_ϕ^{ABV} is atomically stable, meaning that no proper subset supports any stable distribution. This justifies the use of the term "atomic".

With the above results in mind, it is an interesting and important question to characterize/classify the set of atomic L-parameters. The combinatorial lemma (Lemma 1.3) implies that L-parameters of Arthur type of $\mathrm{GL}_n(F)$ are always atomic. The same holds for classical groups $\mathrm{SO}_{2n+1}(F)$, $\mathrm{Sp}_{2n}(F)$ and $\mathrm{O}_{2n}(F)$ by the explicit computation of Pyasetskii involution for these groups in [13]. Therefore, the above results can be applied to these ABV-packets of Arthur type. We remark that not all atomic L-parameters are of Arthur type. For example, ϕ_π is atomic but not of Arthur type for any generic nontempered representation π of $\mathrm{GL}_n(F)$. We expect that the careful study of the combinatorial aspect of Mœglin–Waldspurger algorithm in this paper will play important roles toward the classification of atomic L-parameters not necessarily of Arthur type.

Here is the structure of this paper. In Section 2, we recall the necessary notation and preliminaries. We recall the notion of multisegments in Section 2.1, the partial ordering on multisegments in Section 2.2, and the Mœglin–Waldspurger algorithm in Section 2.3. In Section 3, we rephrase the Mœglin–Waldspurger algorithm and develop certain notation and lemmas for the proofs in Section 4. Then we prove Lemma 1.3 in Section 4. We rephrase Lemma 1.3 in terms of multisegments in Section 4.1. We prove the key of reduction, Proposition 4.3, in Section 4.2. Finally, we prove Lemma 1.3 in Section 4.3.

2. Preliminaries

Let F be a non-Archimedean field of characteristic zero, and let W_F denote the Weil group of F. Let $|\cdot|$ represent the normalized absolute value on F, which we also regard as a character of $\mathrm{GL}_n(F)$ via composition with the determinant.

We denote by $\Pi(\mathrm{GL}_n(F))$ the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$, and by $\Phi(\mathrm{GL}_n(F))$ the set of equivalence classes of L-parameters for $\mathrm{GL}_n(F)$. We define

$$\Pi(\mathrm{GL}(F)) := \bigsqcup_{n\geq 1} \Pi(\mathrm{GL}_n(F)), \quad \Phi(\mathrm{GL}(F)) := \bigsqcup_{n\geq 1} \Phi(\mathrm{GL}_n(F)).$$

We denote by $+$ the sum of multisets (disjoint union), and by $\setminus$ the difference of multisets.

2.1. ***Langlands classification for*** $\mathbf{GL_n(F)}$**.** We recall the Langlands classification for $\mathrm{GL}_n(F)$, and the bijection among $\Pi(\mathrm{GL}_n(F))$, $\Phi(\mathrm{GL}_n(F))$ and multisegments of correct rank.

Let $\mathcal{C}(\mathrm{GL}_n(F))$ denote the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}_n(F)$. By local Langlands correspondence for $\mathrm{GL}_n(F)$, we may identify $\mathcal{C}(\mathrm{GL}_n(F))$ as the set of isomorphism classes of n-dimensional irreducible representations of W_F. We let

$$\mathcal{C} := \bigsqcup_{n\geq 1} \mathcal{C}(\mathrm{GL}_n(F)),$$

and denote by $\mathcal{C}_{\mathrm{unit}}$ the subset of $\mathcal{C}$ that consists of unitary supercuspidal representations.

Let P be a standard parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi subgroup $L \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_s}(F)$. An irreducible representation σ of L can be identified with

$$\sigma = \sigma_1 \otimes \cdots \otimes \sigma_s,$$

where $\sigma_i \in \Pi(\mathrm{GL}_{n_i}(F))$. We set

$$\sigma_1 \times \cdots \times \sigma_s := \mathrm{Ind}_P^{\mathrm{GL}_n(F)} \sigma,$$

which is the normalized parabolic induction.

A segment Δ is a set of the form

$$\{\rho|\cdot|^b, \rho|\cdot|^{b+1}, \ldots, \rho|\cdot|^e\},$$

where $\rho \in \mathcal{C}_{\mathrm{unit}}$, $b, e \in \mathbb{R}$ such that $e - b \in \mathbb{Z}_{\geq 0}$. We shall write $\Delta = [b, e]_\rho$ for short and call b the base value of Δ, e the end value of Δ, and $e - b + 1$ the length of Δ.

We also write

$$b(\Delta) := b, \quad e(\Delta) := e, \quad l(\Delta) := e - b + 1.$$

A multisegment, which we usually denote by α, β, γ or δ, is a finite multiset of segments. We denote the collection of segments by $\underline{\mathrm{Seg}}$ and the collection of multisegments by $\underline{\mathrm{Mseg}}$. For each $\rho \in \mathcal{C}_{\mathrm{unit}}$, let $\underline{\mathrm{Seg}}_\rho$ denote the subset of $\underline{\mathrm{Seg}}$ consisting of segments of the form $[b, e]_\rho$, and let $\underline{\mathrm{Mseg}}_\rho$ denote the subset of $\underline{\mathrm{Mseg}}$ consisting of multisets of segments in $\underline{\mathrm{Seg}}_\rho$. For each $\alpha \in \underline{\mathrm{Mseg}}$, there is a unique decomposition

$$\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho,$$

where $\alpha_\rho \in \underline{\mathrm{Mseg}}_\rho$ and $\alpha_\rho = \varnothing$ except for a finite number of $\rho \in \mathcal{C}_{\mathrm{unit}}$.

For each segment $[y, x]_\rho$, let $\Delta_\rho[x, y]$ be the unique irreducible subrepresentation of the parabolic induction

$$\rho|\cdot|^x \times \rho|\cdot|^{x-1} \times \cdots \times \rho|\cdot|^y.$$

We recall the Langlands classification for $\mathrm{GL}_n(F)$ now. Any irreducible representation $\pi \in \Pi(\mathrm{GL}_n(F))$ can be realized as the unique irreducible subrepresentation of a parabolic induction of the form

$$\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_f}[x_f, y_f],$$

where

- $n = \sum_{i=1}^{f} \dim(\rho_i)(x_i - y_i + 1)$,
- $\rho_i \in \mathcal{C}_{\mathrm{unit}}$, and
- $x_1 + y_1 \leq \cdots \leq x_f + y_f$.

Here $\dim(\rho_i)$ is the dimension of ρ_i as an irreducible representation of W_F. The multiset

$$\{\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_f}[x_f, y_f]\}$$

with the above requirement is unique. With this notation, we may write down the L-parameter of π as

$$\phi_\pi = \rho_1|\cdot|^{\frac{x_1+y_1}{2}} \otimes \mathrm{Sym}^{x_1-y_1} \oplus \cdots \oplus \rho_f|\cdot|^{\frac{x_f+y_f}{2}} \otimes \mathrm{Sym}^{x_f-y_f},$$

where Sym^{b+1} is the unique b-dimensional irreducible analytic representation of $\mathrm{SL}_2(\mathbb{C})$. Also, we associate, to π, the multisegment

$$\delta_\pi := \{[y_1, x_1]_{\rho_1}, \ldots, [y_f, x_f]_{\rho_f}\}.$$

Thus, with the above notation, we obtain the correspondence

$$\begin{array}{ccccc} \Pi(\mathrm{GL}(F)) & \longleftrightarrow & \Phi(\mathrm{GL}(F)) & \longleftrightarrow & \underline{\mathrm{Mseg}}, \\ \pi & \longmapsto & \phi_\pi & \longmapsto & \delta_\pi. \end{array}$$

Suppose ϕ is an L-parameter of $\mathrm{GL}_n(F)$. We shall simply write $\delta_\phi := \delta_\pi$ where π is the unique representation of $\Pi(\mathrm{GL}_n(F))$ such that $\phi_\pi = \phi$.

2.2. ***A partial ordering on multisegments.*** We recall the partial ordering on $\underline{\mathrm{Mseg}}$ introduced in [23].

Suppose $\Delta_1 = [b_1, e_1]_{\rho_1}$ and $\Delta_2 = [b_2, e_2]_{\rho_2}$ are two segments. We say Δ_1 and Δ_2 are *linked* if the union $\Delta_1 \cup \Delta_2$ (as a set) is also a segment, and $\Delta_1 \not\supseteq \Delta_2$, $\Delta_2 \not\supseteq \Delta_1$. In particular, Δ_1 and Δ_2 are linked only if $\rho_1 \cong \rho_2$ (recall that we require ρ_1, ρ_2 to be unitary).

Now let α, β be two multisegments. We say β is obtained from α by performing a single *elementary operation* if we can form β from α by replacing a submultiset $\{\Delta_1, \Delta_2\}$ of α by

$$\begin{cases} \{\Delta_1, \Delta_2\} & \text{if } \Delta_1, \Delta_2 \text{ are not linked,} \\ \{\Delta_1 \cup \Delta_2, \Delta_1 \cap \Delta_2\} & \text{if } \Delta_1, \Delta_2 \text{ are linked and } \Delta_1 \cap \Delta_2 \neq \varnothing, \\ \{\Delta_1 \cup \Delta_2\} & \text{if } \Delta_1, \Delta_2 \text{ are linked and } \Delta_1 \cap \Delta_2 = \varnothing. \end{cases}$$

Definition 2.1. Let α, β be two multisegments. We define $\alpha \geq \beta$ if α can be obtained from β by performing a sequence of elementary operations. This gives a partial ordering on $\underline{\mathrm{Mseg}}$.

For any multisegment $\delta = \{\Delta_1, \ldots, \Delta_r\}$, we define

$$\mathrm{supp}(\delta) := \sum_{i=1}^{r} \Delta_i,$$

which is a multiset of supercuspidal representations. Then, it follows from the definition that $\alpha \geq \beta$ only if $\mathrm{supp}(\alpha) = \mathrm{supp}(\beta)$. Also, if $\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho$ and $\beta = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \beta_\rho$, then $\alpha \geq \beta$ if and only if $\alpha_\rho \geq \beta_\rho$ for every ρ.

It is proved in [23, §2] that the partial ordering in Definition 2.1 exactly corresponds to the closure ordering on the orbits of Vogan varieties.

Theorem 2.2 [23, Theorem 2.2]. *Let ϕ_1, ϕ_2 be two L-parameters of $\mathrm{GL}_n(F)$. The following are equivalent*:

(a) $\delta_{\phi_1} \geq \delta_{\phi_2}$.

(b) $\phi_1 \geq \phi_2$.

Here the $\geq$ in part (b) is the closure ordering on the associated Vogan variety.

2.3. ***Mœglin–Waldspurger algorithm.*** We recall the statement of the Mœglin–Waldspurger algorithm and introduce the related notation. For simplicity, we shall assume any multisegment in this section is in $\underline{\mathrm{Mseg}}_\rho$ for a fixed $\rho \in \mathcal{C}_{\mathrm{unit}}$, and omit ρ in the notation.

Suppose $\Delta_1 = [b_1, e_1]$ and $\Delta_2 = [b_2, e_2]$ are linked. We say Δ_1 *precedes* Δ_2 if $b_1 < b_2$ and $e_1 < e_2$. For $\Delta = [b, e]$, define

$$\Delta^- := \begin{cases} [b, e-1] & \text{if } b \neq e, \\ \varnothing & \text{otherwise.} \end{cases}$$

With these definitions, we are ready to state the Mœglin–Waldspurger algorithm.

Algorithm 2.3 (Mœglin–Waldspurger algorithm). Suppose α is a multisegment. We associate a segment $M(\alpha)$ as follows.

(1) Set e to be the largest end value of segments in α. Set $m := e$.

(2) Consider all segments in α with end value m. Among these, choose a segment with the largest base value and call it Δ_m.

(3) Consider the set of all segments in α that precede Δ_m with end value $m-1$. If this is empty, go to step (5). Otherwise, choose a segment from this set with the largest base value and call it Δ_{m-1}.

(4) Set $m := m-1$ and go to step (3).

(5) Return $M(\alpha) = [m, e]$.

Next, we inherit the following notation from the above procedure. Define $\alpha \setminus M(\alpha)$ to be the multisegment obtained from α by replacing Δ_i with Δ_i^- for all $m \leq i \leq e$ and removing empty sets.

Finally, we set

$$\widetilde{\alpha} := \{M(\alpha)\} + \widetilde{(\alpha \setminus M(\alpha))}$$

if $\alpha \neq \varnothing$ and $\widetilde{\alpha} := \varnothing$ otherwise.

For a general multisegment $\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho$, we define

$$\widetilde{\alpha} := \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \widetilde{\alpha_\rho}.$$

The main result of [21] is that Algorithm 2.3 computes the Zelevinsky involution on multisegments and also the Pyasetskii involution on L-parameters of $\mathrm{GL}_n(F)$.

Theorem 2.4 [21, Theorem II.13]. *For any L-parameter ϕ of $\mathrm{GL}_n(F)$, we have*

$$\delta_{\widehat{\phi}} = \widetilde{\delta_\phi}.$$

3. Rephrasing Mœglin–Waldspurger algorithm

We introduce certain notation and a useful observation (Lemma 3.3 below) for the Mœglin–Waldspurger algorithm. Then, we rephrase Algorithm 2.3 in Corollary 3.4.

We inherit the notation in Algorithm 2.3 in the following discussion. For any multisegment β, set $\beta^0 := \beta$ and $\beta^i := \beta^{i-1} \setminus M(\beta^{i-1})$, so that for any i, we have

$$\widetilde{\beta} = \{M(\beta^l)\}_{l=0}^{i-1} + \widetilde{\beta^i}.$$

Write $\beta = \{\Delta_j\}_{j\in J}$, and $\beta^i = \{\Delta_j^i\}_{j\in J^i}$. For $i > 0$, we fix an injection $J^i \hookrightarrow J^{i-1}$, which identifies J^i as a subset of J^{i-1}, with the conditions

- $\Delta_j^i \subseteq \Delta_j^{i-1}$, and
- if $\Delta_j^{i-1} \neq \Delta_j^i$, then $\Delta_j^i = (\Delta_j^{i-1})^-$.

In this way, we identify each J^i as a subset of $J = J^0$. Define

$$K^i := \{j \in J^i \mid \Delta_j^i \neq \Delta_j^{i+1}\}.$$

Then,

$$M(\beta^i) = \{e(\Delta_j^i) \mid j \in K^i\}.$$

Write $M(\beta^i) = [m^i, e^i]$. For $m^i \le l \le e^i$, let k_l^i be the unique index in K^i such that $e(\Delta_{k_l^i}^i) = l$. Now we rephrase Algorithm 2.3.

Lemma 3.1. *With the above notation, the following properties uniquely characterize* $M(\beta^i) = [m^i, e^i]$ *and* $\{\Delta_{k_l^i}^i\}_{m^i \le l \le e^i}$:

(1) $e^i = \max\{e(\Delta_j^i) \mid j \in J^i\}$.

(2) $b(\Delta_{k_{e^i}^i}^i) = \max\{b(\Delta_j^i) \mid j \in J^i,\ e(\Delta_j^i) = e^i\}$.

(3) $b(\Delta_{k_l^i}^i) = \max\{b(\Delta_j^i) \mid j \in J^i,\ e(\Delta_j^i) = l,\ b(\Delta_j^i) < b(\Delta_{k_{l+1}^i}^i)\}$

(4) $\{j \in J^i \mid e(\Delta_j^i) = m^i - 1,\ b(\Delta_j^i) < b(\Delta_{k_{m^i}^i}^i)\} = \varnothing$.

Finally, let $e := e^0 = \max_{j\in J}\{e(\Delta_j)\}$ and set

$$t := \max\{i \in \mathbb{Z}_{\ge 0} \mid e^i = e(M(\beta^i)) = e\} + 1.$$

Note that $t = \#\{j \in J \mid e(\Delta_j) = e\}$. When there are more than one multisegment involved in the argument, we write $J^i = J^i(\beta)$, $K^i = K^i(\beta)$, $e = e(\beta)$ and $t = t(\beta)$ to specify the multisegment β.

We give an example to demonstrate this notation.

Example 3.2. Consider $\beta = \{\Delta_j\}_{j\in J}$, where $J = \{1, 2, \dots, 8\}$, and

$$\Delta_1 = [2, 2], \quad \Delta_2 = [0, 1], \quad \Delta_3 = [-2, 0], \quad \Delta_4 = [-3, -1],$$
$$\Delta_5 = [1, 2], \quad \Delta_6 = [-1, 1], \quad \Delta_7 = [-2, 0], \quad \Delta_8 = [-2, 1].$$

We have $e(\beta) = 2$ and $t(\beta) = 2$. Applying Algorithm 2.3 once, we obtain $M(\beta) = [m^0, e^0] = [-1, 2]$, and $\beta^1 = \beta \setminus M(\beta)$ is obtained from β by replacing $\{\Delta_j\}_{j=1}^4$ by $\{\Delta_j^-\}_{j=1}^4$. Thus we can choose $K^0 = \{1, 2, 3, 4\}$ and $(k_2^0, k_1^0, k_0^0, k_{-1}^0) = (1, 2, 3, 4)$. Note that one can also choose $K^0 = \{1, 2, 7, 4\}$ since $\Delta_3 = \Delta_7$.

With the choice $K^0 = \{1, 2, 3, 4\}$, we obtain that $J^1 = \{2, 3, \dots, 8\} \subseteq J$ and

$$\beta^1 = \{\Delta_j^1\}_{j=2}^8 = \{\Delta_j^-\}_{j=2}^4 \sqcup \{\Delta_j\}_{j=5}^8.$$

Apply Algorithm 2.3 again, we obtain $M(\beta^1) = [m^1, e^1] = [0, 2]$ and the only choice of K^1 is $\{5, 6, 7\}$ with $(k_2^1, k_1^1, k_0^1) = (5, 6, 7)$. We obtain $J^2 = \{2, 3, \dots, 8\} \subseteq J^1 \subseteq J$ and

$$\beta^2 = \{\Delta_j^2\}_{j=2}^8 = \{\Delta_j^-\}_{j=2}^4 \sqcup \{\Delta_j^-\}_{j=5}^7 \sqcup \{\Delta_8\}.$$

Observe that in the above example, $K^0 \cap K^1 = \varnothing$, or in other words, the index sets $\{K^i(\beta)\}_{i=0}^{t(\beta)-1}$ are mutually disjoint. One can also check this on the multisegment in Example 4.6, which is slightly more complicated. We prove this interesting phenomenon for all multisegments along with other properties in the following lemma.

Lemma 3.3. *With the notation developed in this section, the following holds for any multisegment β:*

(a) $m^0 \le m^1 \le \cdots \le m^{t-1}$.

(b) *For any $0 \le i \le t-1$ and $m^i \le l \le e$, we have containment*

$$\Delta_{k_l^0}^0 \subseteq \Delta_{k_l^1}^1 \subseteq \cdots \subseteq \Delta_{k_l^i}^i.$$

(c) *The sets $\{K^i\}_{i=0}^{t-1}$ are mutually disjoint.*

(d) *For $0 \le i \le t-1$ and $m^i \le l \le e$, we have*

$$\Delta_{k_l^i} = \Delta_{k_l^i}^0 = \Delta_{k_l^i}^1 = \cdots = \Delta_{k_l^i}^i \supsetneq (\Delta_{k_l^i}^i)^- = \Delta_{k_l^i}^{i+1} = \cdots = \Delta_{k_l^i}^{t-1}.$$

Proof. First, observe that part (a) implies that

$$m^0 \le m^1 \le \cdots \le m^i \le l \le e,$$

and hence the indices $k_l^0, \dots, k_l^{i-1}$ in part (b) are well defined. Also, part (d) is a direct consequence of part (c). To prove parts (a), (b), (c), we apply induction on $t = t(\beta)$. If $t = 1$, the conclusions trivially hold. We assume $t > 1$ from now on.

We claim that for any $l \in \{e, e-1, \dots, m^{t-1}\}$, the following hold for any $0 \le i < t-1$:

(i) $m^i \le l$.

(ii) $\Delta^i_{k^i_l} \subseteq \Delta^{t-1}_{k^{t-1}_l}$.

(iii) k^{t-1}_l is not in K^i.

We demonstrate that the claims imply the desired conclusion before we verify the claims. Set $J' := J \setminus K^{t-1}$ and $\beta' := \{\Delta_j\}_{j \in J'}$. Then,

$$t(\beta') = \#\{j \in J' \mid e(\Delta_j) = e\} = t-1.$$

Also, for any $0 \le i < t-1$, claim (iii) implies $K^{t-1} \cap K^i = \varnothing$, and hence claim (ii) implies that for $m^i \le l \le e^i$,

$$\Delta^i_{k^i_l} \subseteq \Delta^{t-1}_{k^{t-1}_l} = \Delta^i_{k^{t-1}_l}.$$

This gives $b(\Delta^i_{k^i_l}) \ge b(\Delta^i_{k^{t-1}_l})$. From this inequality, for $0 \le i < t-1$ and $m^i \le l \le e^i$, we inductively check that $[m^i, e^i]$ and $\Delta^i_{k^i_l} \in (\beta')^i$ satisfy the properties in Lemma 3.1 for β'. Thus, we can choose $K^i(\beta') = K^i(\beta)$. The induction hypothesis for β' then implies that:

(a') $m^0 \le \cdots \le m^{t-2}$.

(b') For any $0 \le i \le t-2$ and $m^i \le l \le e$, we have containment

$$\Delta^0_{k^0_l} \subseteq \Delta^1_{k^1_l} \subseteq \cdots \subseteq \Delta^i_{k^i_l}.$$

(c') The sets $\{K^i\}_{i=0}^{t-2}$ are mutually disjoint.

Thus, together with the claims, this verifies conditions (a), (b) and (c) for β.

Now we prove the claims by applying induction on l. When $l = e$, claim (i) trivially holds. Claim (ii) follows from Lemma 3.1(2). For claim (iii), write $\Delta^0_{k^{t-1}_e} = [x, y]$. Then, $\Delta^t_{k^{t-1}_e} = [x, y-s]$ where $s = \#\{0 \le i < t-1 \mid k^{t-1}_e \in K^i\}$. Since $e(\Delta^{t-1}_{k^{t-1}_e}) = e$ and

$$y = e(\Delta^0_{k^{t-1}_e}) \le \max_{\Delta^0 \in \beta}\{e(\Delta^0)\} = e$$

by definition, we obtain that

$$e = y - s \le y \le e,$$

which implies that $s = 0$. This completes the verification of the claims for $l = e$.

Suppose $e > r \geq m^{t-1}$ and the claims are already verified for $l = r+1$. We are going to verify the claims for $l = r$.

First, for any $0 \leq i < t-1$, claim (ii) for $l = r+1$ gives $\Delta^i_{k^i_{r+1}} \subseteq \Delta^{t-1}_{k^{t-1}_{r+1}}$. Combining with Lemma 3.1(3), we obtain that

$$b(\Delta^{t-1}_{k^{t-1}_r}) < b(\Delta^{t-1}_{k^{t-1}_{r+1}}) \leq b(\Delta^i_{k^i_{r+1}}). \tag{3-1}$$

On the other hand, claim (iii) for $l = r+1$ implies that $k^{t-1}_{r+1} \neq k^j_r$ for any $0 \leq j < t-1$, and hence

$$\Delta^0_{k^{t-1}_r} = \Delta^1_{k^{t-1}_r} = \cdots = \Delta^i_{k^{t-1}_r} = \cdots = \Delta^{t-1}_{k^{t-1}_r}. \tag{3-2}$$

In particular, their base values are all the same. As a consequence, the set

$$\{j \in J^i \mid e(\Delta^i_j) = r,\ b(\Delta^i_j) < b(\Delta^i_{k^i_{r+1}})\}$$

is nonempty since it contains k^{t-1}_r. We conclude that $m^i \leq r$ by Lemma 3.1(4). This proves claim (i) for $l = r$.

Next, for any $0 \leq i \leq t-1$, (3-1) and (3-2) give

$$e(\Delta^i_{k^{t-1}_r}) = e(\Delta^{t-1}_{k^{t-1}_r}) = r, \quad b(\Delta^i_{k^{t-1}_r}) = b(\Delta^{t-1}_{k^{t-1}_r}) < b(\Delta^i_{k^i_{r+1}}).$$

Therefore, k^{t-1}_r is in the set

$$\{j \in J^i \mid e(\Delta^i_j) = r,\ b(\Delta^i_j) < b(\Delta^i_{k^i_{r+1}})\}.$$

We conclude that $b(\Delta^i_{k^{t-1}_r}) \leq b(\Delta^i_{k^i_r})$ by Lemma 3.1(3). This verifies claim (ii) for $l = r$.

Finally for claim (iii), suppose the contrary that $k^{t-1}_r \in K^i$ for some $0 \leq i < t$ and take maximal such i. We must have

$$\Delta^{t-1}_{k^{t-1}_r} = \Delta^{t-2}_{k^{t-1}_r} = \cdots = \Delta^{i+1}_{k^{t-1}_r} = (\Delta^i_{k^{t-1}_r})^-,$$

and hence $k^{t-1}_r = k^i_{r+1}$. However, claim (ii) for $l = r+1$ gives

$$\Delta^{t-1}_{k^{t-1}_r} = \Delta^{i+1}_{k^i_{r+1}} = (\Delta^i_{k^i_{r+1}})^- \subsetneq \Delta^i_{k^i_{r+1}} \subseteq \Delta^{t-1}_{k^{t-1}_{r+1}}.$$

In particular, $b(\Delta^{t-1}_{k^{t-1}_r}) \geq b(\Delta^{t-1}_{k^{t-1}_{r+1}})$, contradicting Lemma 3.1(3). This completes the verification of claim (iii) for $l = r$ and the proof of the lemma. □

As a corollary, we can improve the statement of Lemma 3.1 when $0 \leq i \leq t-1$ as follows.

Corollary 3.4. *Let $\{K^i\}_{i=0}^{t(\beta)-1}$ be a collection of mutually disjoint subsets of $J(\beta)$. The following are equivalent*:

(a) *K^i can be labeled as $K^i = \{k_l^i\}_{m^i \le l \le e(\beta)}$ such that the following holds*:

(1) $e(\Delta_{k_l^i}) = l$.

(2) $b(\Delta_{k_e^i}) = \max\{b(\Delta_j) \mid j \in J \setminus \big(\bigsqcup_{0\le r<i} K^r\big),\ e(\Delta_j) = e\}$.

(3) $b(\Delta_{k_l^i}) = \max\{b(\Delta_j) \mid j \in J \setminus \big(\bigsqcup_{0\le r<i} K^r\big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^i})\}$

(4) $\{j \in J \setminus \big(\bigsqcup_{0\le r<i} K^r\big) \mid e(\Delta_j) = m^i - 1,\ b(\Delta_j) < b(\Delta_{k_{m^i}^i})\} = \varnothing$.

(b) *The multisegment β^i can be obtained from β^{i-1} by replacing $\{\Delta_{k^i}\}_{k^i \in K^i}$ with $\{\Delta^-_{k^i}\}_{k^i \in K^i}$.*

Proof. The descriptions of parts (a) and (b) both determine the collection of segments $\{\Delta_{k_l^i}\}_{k^i \in K^i}$ uniquely. Thus, it suffices to examine the index sets $\{K^i\}_{i=0}^{t-1}$ chosen in the beginning of this section, where part (b) holds, satisfying all of the conditions in part (a).

Condition (1) holds by definition. Condition (2) holds since $e(\Delta_j^i) = e$ if and only if $\Delta_j^i = \Delta_j$ and $j \notin \bigsqcup_{0\le r<i} K^r$. For condition (3), observe that $\Delta_j^i = \Delta_j$ for $j \in J \setminus \bigsqcup_{0\le r<i} K^r$, and hence Lemma 3.1(3) implies that

$$b(\Delta_{k_l^i}) \ge \max\Big\{b(\Delta_j) \;\Big|\; j \in J \setminus \Big(\bigsqcup_{0\le r<i} K^r\Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^i})\Big\}.$$

On the other hand, Lemma 3.3(c) shows that

$$k_l^i \in J \setminus \Big(\bigsqcup_{0\le r<i} K^r\Big),$$

so the equality holds. By the same observation, Lemma 3.1(4) implies condition (4). This completes the proof of the corollary. □

4. Proof of Lemma 1.3

4.1. ***Rephrasing Lemma 1.3 by multisegments.*** We rephrase Lemma 1.3 using the notation of multisegments recalled in Section 2. First, we define multisegments of *Arthur type*.

Definition 4.1. We say a multisegment α is of Arthur type if $\alpha = \delta_\phi$ for some L-parameter ϕ of Arthur type. The following is an equivalent but more explicit description.

1. For an irreducible local Arthur parameter $\rho \otimes \mathrm{Sym}^d \otimes \mathrm{Sym}^a$, we associate a multisegment

$$\delta_{\rho,d,a} := \Big\{\Big[\frac{a-d}{2}, \frac{a+d}{2}\Big]_\rho, \Big[\frac{a-d}{2} - 1, \frac{a+d}{2} - 1\Big]_\rho, \dots, \Big[\frac{-a-d}{2}, \frac{-a+d}{2}\Big]_\rho\Big\}.$$

2. For a local Arthur parameter of the form

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}, \tag{4-1}$$

we associate a multisegment

$$\delta_\psi := \sum_{\rho} \sum_{i \in I_\rho} \delta_{\rho, d_i, a_i}.$$

3. We say a multisegment α is of *Arthur type* if $\alpha = \delta_\psi$ for some ψ of the form (4-1).

With this definition, we may rephrase Lemma 1.3 as follows.

Lemma 4.2. *Let α be a multisegment of Arthur type. If β is a multisegment such that $\beta \geq \alpha$ and $\widetilde{\beta} \geq \widetilde{\alpha}$, then $\beta = \alpha$.*

4.2. ***Preparation for reduction.*** In this subsection we prove Proposition 4.3 below, which is the key to the reduction argument in the proof of Lemma 1.3. The motivation for the formulation of Proposition 4.3 is discussed in the introduction. Note that some of the statements in this subsection are similar to those in [22], but the proofs are different.

Throughout this subsection, we fix $\rho \in \mathcal{C}_{\mathrm{unit}}$ and write $\delta_{d,a} := \delta_{\rho,d,a}$, $[b, e] := [b, e]_\rho$ to simplify the notation.

Proposition 4.3. *Suppose α is of Arthur type with $\alpha = \delta_\psi$, where*

$$\psi = \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}.$$

Set

$$a + d := \max\{a_i + d_i \mid i \in I_\rho\},$$
$$d := \min\{d_i \mid i \in I_\rho,\ a_i + d_i = a + d\}.$$

Then, for any multisegment β such that $\beta \geq \alpha$ and $\widetilde{\beta} \geq \widetilde{\alpha}$, the following holds:

(i) *The multisegment β must contain a copy of $\delta_{d,a}$.*

(ii) *If we define α^- and β^- by removing a copy of $\delta_{d,a}$ from α and β, respectively, then $\alpha^- = \delta_{\psi^-}$ where*

$$\psi^- := \psi - \rho \otimes \mathrm{Sym}^d \otimes \mathrm{Sym}^a,$$

and $\beta^- \geq \alpha^-$ and $\widetilde{\beta^-} \geq \widetilde{\alpha^-}$.

We first show that the choice of $a + d$ and d in Proposition 4.3 gives the following bounds. The definition of β^i and other notation in the proof can be found in Section 3.

Lemma 4.4. *Under the setting of Proposition 4.3, for any $i \geq 0$ and $[x, y] \in \beta^i$, the following holds*:

(a) $x \geq \frac{-a-d}{2}$.

(b) $y \leq \frac{a+d}{2}$. *If equality holds, then $[x, y] \in \beta^l$ for any $0 \leq l \leq i$ and $x \leq \frac{a-d}{2}$.*

Proof. For arbitrary multisegments α, β, we have the following observations:

- If $\beta \geq \alpha$, then $\mathrm{supp}(\beta) = \mathrm{supp}(\alpha)$.
- For $i \geq 0$, $\mathrm{supp}(\beta^i) = \mathrm{supp}(M(\beta^i)) + \mathrm{supp}(\beta^{i+1})$. In particular, we have $\mathrm{supp}(\beta) \supseteq \mathrm{supp}(\beta^i)$.

Now we return to the setting of Proposition 4.3. The choice of $a + d$ implies that

$$\begin{aligned} \tfrac{1}{2}(a+d) &= \max\{x \in \mathbb{R} \mid \rho|\cdot|^x \in \mathrm{supp}(\alpha) = \mathrm{supp}(\beta)\}, \\ \tfrac{1}{2}(-a-d) &= \min\{x \in \mathbb{R} \mid \rho|\cdot|^x \in \mathrm{supp}(\alpha) = \mathrm{supp}(\beta)\}, \end{aligned} \tag{4-2}$$

which shows part (a) and the first part of part (b) by the observations above.

Next, we show the second part of part (b) with the notation developed in Section 3 for β. Take a $j \in J^i$ such that $\Delta^i_j = [x, y]$. We have

$$\Delta^0_j \supseteq \Delta^1_j \supseteq \cdots \supseteq \Delta^i_j = [x, y].$$

Moreover, if any of the inclusions is strict, then $e(\Delta^0_j) > y$. If $y = \frac{a+d}{2}$, this cannot happen by (4-2). Therefore, we conclude that $[x, y] = \Delta^l_j \in \beta^l$ for any $0 \leq l \leq i$.

Finally, we verify the property that if $\left[x, \frac{a+d}{2}\right] \in \beta$ then $x \leq \frac{a-d}{2}$. If $\beta = \alpha$, then

$$\left[x, \tfrac{1}{2}(a+d)\right] \in \left\{\left[\tfrac{1}{2}(a+d) - d_i, \tfrac{1}{2}(a+d)\right] \mid i \in I_\rho,\ a_i + d_i = a + d\right\},$$

and hence $x \leq \frac{a-d}{2}$ by the definition of d. In general, β is obtained from α by performing a sequence of elementary operations. By (4-2), the desired property is preserved under each elementary operation, and hence β also has this property. This completes the proof of the lemma. □

For part (ii) in Proposition 4.3, we need the following computation, which works for a general multisegment β. We obtain the same conclusion as [22, Lemma 3.18] but with slightly weaker conditions.

Lemma 4.5. *Suppose a multisegment β contains a copy of*

$$\delta_{b,e,s} := \{[b, e], [b-1, e-1], \ldots, [b-s, e-s]\},$$

and any segment $[x, y] \in \beta$ satisfies

(i) $x \geq b - s$,

(ii) $y \leq e$.

Define β^- by removing a copy of $\delta_{b,e,s}$ from β. Then we have

$$\widetilde{\beta} = \widetilde{\beta^-} + \widetilde{\delta}_{b,e,s},$$

where

$$\widetilde{\delta}_{b,e,s} = \{[e-s, e], [e-s-1, e-1], \dots, [b-s, b]\}.$$

We give an example to illustrate the reduction process in the proof.

Example 4.6. Let $\beta = \sum_{i=0}^{4} \{\Delta_{k^i}\}_{k^i \in K^i}$, where

$$\begin{aligned}
\{\Delta_{k^0}\}_{k^0 \in K^0} &= \{[2,2], [1,1], [0,0], [-1,-1], [-2,-2]\},\\
\{\Delta_{k^1}\}_{k^1 \in K^1} &= \{[1,2], [-1,1], [\mathbf{-2},\mathbf{0}], [\mathbf{-3},\mathbf{-1}]\},\\
\{\Delta_{k^2}\}_{k^2 \in K^2} &= \{[0,2], [\mathbf{-1},\mathbf{1}], [-3,0]\},\\
\{\Delta_{k^3}\}_{k^3 \in K^3} &= \{[\mathbf{0},\mathbf{2}], [-3,1]\},\\
\{\Delta_{k^4}\}_{k^4 \in K^4} &= \{[-2,2]\}.
\end{aligned} \tag{4-3}$$

Note that β contains $\delta_{0,2,3} = \{[0,2], [-1,1], [-2,0], [-3,-1]\}$, and the pair $(\beta, \delta_{0,2,3})$ satisfies assumptions (i), (ii) in the above lemma. Also, the subsets $\{K^i\}_{i=0}^4$ of $J(\beta) = \bigsqcup_{i=0}^4 K^i$ satisfy all of the conditions in Corollary 3.4(a), where recall that we let k_l^i denote the index in K^i such that $e(\Delta_{k_l^i}) = l$. Therefore, $\beta^4 = \sum_{i=0}^4 \{\Delta^-_{k^i}\}_{k^i \in K^i}$, and

$$\widetilde{\beta} = \{M(\beta^i)\}_{i=0}^4 + \widetilde{\beta^4} = \{[-2,2], [-1,2], [0,2], [1,2], [2,2]\} + \widetilde{\beta^4}.$$

For $-1 = e-s \le l \le e = 2$, we let $r_l := \max\{0 \le r \le 4 \mid \Delta_{k_l^r} = [e-s-l, e-l]\}$. Thus, $(r_2, r_1, r_0, r_{-1}) = (3, 2, 1, 1)$. The corresponding segments $\Delta_{k_l^{r_l}}$ are displayed in bold text in (4-3).

We may construct the segment β^- from β by removing $\{\Delta_{k_l^{r_l}}\}_{-1 \le l \le 2}$. Let $J^- := J \setminus \{k_l^{r_l}\}_{-1 \le l \le 2}$, which we identify with $J(\beta^-)$. We define the mutually disjoint subsets $\{(K^i)^-\}_{i=0}^3$ of J^- by removing $\{\Delta_{k_l^{r_l}}\}_{-1 \le l \le 2}$ in (4-3) and then "push" the segments below them upward accordingly. That is, we define

$$\begin{aligned}
\{\Delta_{(k^0)^-}\}_{(k^0)^- \in (K^0)^-} &= \{[2,2], [1,1], [0,0], [-1,-1], [-2,-2]\},\\
\{\Delta_{(k^1)^-}\}_{(k^1)^- \in (K^1)^-} &= \{[1,2], [-1,1], [-3,0]\},\\
\{\Delta_{(k^2)^-}\}_{(k^2)^- \in (K^2)^-} &= \{[0,2], [-3,1]\},\\
\{\Delta_{(k^3)^-}\}_{(k^3)^- \in (K^3)^-} &= \{[-2,2]\}.
\end{aligned}$$

It follows that the mutually disjoint subsets $\{(K^i)^-\}_{i=0}^3$ of J^- satisfy all of the conditions in Corollary 3.4(a). Therefore, $(\beta^-)^3 = \sum_{i=0}^3 \{\Delta^-_{(k^i)^-}\}_{(k^i)^- \in (K^i)^-}$, and

$$\widetilde{\beta^-} = \{M((\beta^-)^i)\}_{i=0}^3 + \widetilde{(\beta^-)^3} = \{[-2,2], [0,2], [1,2], [2,2]\} + \widetilde{(\beta^-)^3}.$$

Now observe that β^4 contains a copy of $\{\Delta^-_{k_l^{r_l}}\}_{-1\le l\le 3} = \delta_{0,1,3}$ and the pair $(\beta^4, \delta_{0,1,3})$ satisfies assumptions (i), (ii) in the lemma. Moreover, $(\beta^4)^-$, which is obtained from β^4 by removing a copy of $\delta_{0,1,3}$, is exactly $(\beta^-)^3$, and $\{M(\beta^i)\}_{i=0}^4$ equals $\{M((\beta^-)^i)\}_{i=0}^3 \sqcup \{[2,-1]\}$. We conclude that the lemma holds for the pair $(\beta, \delta_{0,2,3})$ if it holds for the pair $(\beta^4, \delta_{0,1,3})$.

Now we apply the reduction process in the above specific example to prove the lemma in general cases.

Proof of Lemma 4.5. We use the notation developed in Section 3 for β, which we recall briefly as follows:

- $\beta = \{\Delta_j\}_{j\in J(\beta)}$.
- $e(\beta) := \max\{e(\Delta_j)\}_{j\in J(\beta)}$.
- $t(\beta) := \#\{j \in J(\beta) \mid e(\Delta_j) = e(\beta)\}$.
- $\{K^i(\beta)\}_{i=0}^{t-1}$ is a collection of (mutually disjoint) subsets of $J(\beta)$ satisfying all conditions in part (a) of Corollary 3.4.

Note that assumption (ii) implies that the number e matches $e(\beta)$. We shall simply write $J = J(\beta)$, $t = t(\beta)$, $K^i = K^i(\beta)$ in the following discussion.

We apply induction on $\ell := e - b + 1$ to prove the lemma. When $\ell = 1$, it follows directly from Algorithm 2.3 that $M(\beta) = \widetilde{\delta}_{b,e,s}$, $\beta^1 = \beta^-$, and hence

$$\widetilde{\beta} = M(\beta) + \widetilde{\beta^1} = \widetilde{\beta^-} + \widetilde{\delta}_{b,e,s}.$$

From now on, we assume $\ell > 1$ and the conclusions are already established for the case that $e - b + 1 = \ell - 1$.

First, we give the following observations:

- $\Delta_{k_e^i} = [b, e]$ for some $0 \le i \le t-1$.
- If $\Delta_{k^i_{e-r}} = [b-r, e-r]$, and β^i contains a $[b-r-1, e-r-1]$ for some $0 \le r < s$, then $\Delta_{k^i_{e-r-1}} = [b-r-1, e-r-1]$.

As a consequence, $\bigsqcup_{i=0}^{t-1}\{\Delta_{k^i}\}_{k^i\in K^i}$ must contain a copy of $\delta_{b,e,s}$. Thus, for each $e - s \le l \le e$, the set

$$\{0 \le r \le t-1 \mid \Delta_{k_l^r} = [l - e + b, l]\}$$

is nonempty, and we define r_l to be the maximum of this set.

Next, we show that

$$r_{e-s} \le r_{e-s+1} \le \cdots \le r_e. \tag{4-4}$$

Indeed, for $e-s<l\le e$, parts (b) and (d) of Lemma 3.3 and the definition of r_l imply that for any $r_l<i\le t-1$,

$$[l-e+b,l]=\Delta_{k_l^{r_l}}\subsetneq\Delta_{k_l^i}.$$

Thus, Corollary 3.4(a)(3) shows that if $l-1\ge m^i$, then

$$b(\Delta_{k_{l-1}^i})<b(\Delta_{k_l^i})\le l-e+b-1,$$

and hence $\Delta_{k_{l-1}^i}\ne[(l-1)-e+b,l-1]$. This shows that $r_{l-1}\le r_l$, and hence (4-4) holds.

Now set $J^-:=J\setminus\{k_l^{r_l}\}_{l=e-s}^{e}$. We have $\beta^-=\{\Delta_j\}_{j\in J^-}$. For $0\le i\le t-2$, define mutually disjoint subsets $(K^i)^-$ of J^- as follows.

$$(K^i)^-:=\begin{cases}K^i & \text{if } i<r_{e-s},\\ \{k_e^i,\dots,k_{l(i)+1}^i\}\sqcup\{k_{l(i)}^{i+1},\dots,k_{m^{i+1}}^{i+1}\} & \text{if } r_{l(i)}\le i<r_{l(i)+1},\\ K^{i+1} & \text{if } i\ge r_e.\end{cases}$$

Note that if i is not in the first or the third case above, i.e., $r_e>i\ge r_{e-s}$, then the index $l(i)=\max\{e>l\ge e-s\mid i\ge r_{l(i)}\}$ is the unique index such that $r_{l(i)}\le i<r_{l(i)+1}$ holds.

We claim that the collection of index sets $\{(K^i)^-\}_{i=0}^{t-2}$ satisfies all of the conditions in Corollary 3.4(a) for β^-, and hence we can compute $\widetilde{\beta^-}$ using these index sets as described in part (b) of the same corollary.

Before verifying the claim, we demonstrate that the claim implies the desired conclusion. Define

$$K^-:=\bigsqcup_{i=0}^{t-2}(K^i)^-,\qquad K:=\bigsqcup_{i=0}^{t-1}K^i.$$

Observe that $b(\Delta_{k_{e-s}^{r_{e-s}}})=b-s$, and hence by assumption (i), $M(\beta^{r_{e-s}})=[e-s,e]$. This implies that

$$K^-\sqcup\{k_l^{r_l}\}_{l=e-s}^{e}=K. \tag{4-5}$$

As a consequence of the above equality, we have

$$\beta^t=(\beta^-)^{t-1}\sqcup\delta_{b,e-1,s},$$

and hence β^t contains a copy of $\delta_{b,e-1,s}$. Also, β^t satisfies assumptions (i), (ii) with e replaced by $e-1$ according to its construction. Thus, the induction hypothesis implies that

$$\widetilde{\beta^t}=\widetilde{(\beta^-)^{t-1}}\sqcup\widetilde{\delta}_{b,e-1,s}. \tag{4-6}$$

Another consequence of (4-5) is that

$$\{M((\beta^-)^i)\}_{i=0}^{t-2} \sqcup \{[e-s, e]\} = \{M(\beta^i)\}_{i=0}^{t-1}. \tag{4-7}$$

Indeed, observe that as multisets over $\mathcal{C}$,

$$\sum_{i=0}^{t-1} M(\beta^i) = \sum_{k\in K} \{e(\Delta_k)\},$$

and the left-hand side of the above equality uniquely determines the collection $\{M(\beta^i)\}_{i=0}^{t-1}$. The same argument works for β^-. Therefore, (4-5) implies (4-7).

Combining (4-6) and (4-7), we obtain that

$$\begin{aligned}\widetilde{\beta} &= \{M(\beta^i)\}_{i=0}^{t-1} + \widetilde{\beta}^t = \{M((\beta^-)^i)\}_{i=0}^{t-2} \sqcup \{[e-s, e]\} + \widetilde{(\beta^-)^{t-1}} \sqcup \widetilde{\delta}_{b,e-1,s} \\ &= \widetilde{\beta^-} + \widetilde{\delta}_{e,b-1,s}.\end{aligned}$$

This gives the desired conclusion of the lemma.

Now we prove the claim. Write $(K^i)^- = \{(k_l^i)^-\}_{(m^i)^- \le l \le e}$, where $e(\Delta_{(k_l^i)^-}) = l$. In other words,

$$(m^i)^- = \begin{cases} m^i & \text{if } i < r_{e-s}, \\ m^{i+1} & \text{if } i \ge r_{e-s}, \end{cases} \qquad (k_l^i)^- = \begin{cases} k_l^i & \text{if } i < r_l, \\ k_l^{i+1} & \text{if } l \ge (m^i)^- \text{ and } i \ge r_l. \end{cases}$$

Conditions (1) and (2) of Corollary 3.4(a) for $\{(K^i)^-\}_{i=0}^{t-2}$ hold by the construction.

Now we verify condition (3) for each $\Delta_{(k_l^i)^-}$, where $0 \le i \le t-2$ and $e > l \ge (m^i)^-$. We separate into three cases: $r_{l+1} \ge r_l > i$, $r_{l+1} > i \ge r_l$, and $i \ge r_{l+1} \ge r_l$. The key observation is

$$\begin{aligned}\left\{ j \in J \setminus \Big(\bigsqcup_{0\le r<i} K^r \Big) \,\middle|\, e(\Delta_j) = l \right\} = \left\{ j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^- \Big) \,\middle|\, e(\Delta_j) = l \right\} \\ \sqcup \{k_l^{\max(i,r_l)}\}.\end{aligned} \tag{4-8}$$

Case 1. Suppose that $r_{l+1} \ge r_l > i$. Then, $(k_l^i)^- = k_l^i$, $(k_{l+1}^i)^- = k_{l+1}^i$, and $b(\Delta_{k_l^i}) \ge b(\Delta_{k_l^{r_l}})$ by Lemma 3.3(b) and (d). Thus Corollary 3.4(a)(3) for $\Delta_{k_l^i}$ and (4-8) give

$$\begin{aligned}&b(\Delta_{(k_l^i)^-}) \\ &= b(\Delta_{k_l^i}) = \max\Big\{ b(\Delta_j) \,\Big|\, j \in J \setminus \Big(\bigsqcup_{0\le r<i} K^r \Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^i}) \Big\} \\ &= \max\Big(\Big\{ b(\Delta_j) \,\Big|\, j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^- \Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^i}) \Big\} \\ &\qquad\qquad \sqcup \{b(\Delta_{k_l^{r_l}})\} \Big) \\ &= \max\Big\{ b(\Delta_j) \,\Big|\, j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^- \Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{(k_{l+1}^i)^-}) \Big\}.\end{aligned}$$

This verifies condition (3) in this case.

Case 2. Suppose that $r_{l+1} > i \geq r_l$. Then,

$$(k_l^i)^- = k_l^{i+1} \quad \text{and} \quad (k_{l+1}^i)^- = k_{l+1}^i.$$

We first check that for any $j^- \in J^- \setminus \left(\bigsqcup_{0\leq r\leq i-1}(K^r)^-\right)$ such that $e(\Delta_{j^-}) = l$, $b(\Delta_{j^-}) < b(\Delta_{k_{l+1}^i})$ if and only if $b(\Delta_{j^-}) < b(\Delta_{k_{l+1}^{i+1}})$. Indeed, Lemma 3.3(b) and (d) imply that

$$b(\Delta_{k_{l+1}^i}) = b(\Delta^i_{k_{l+1}^i}) \geq b(\Delta^{i+1}_{k_{l+1}^{i+1}}) = b(\Delta_{k_{l+1}^{i+1}}),$$

which shows one direction. For the other direction, Corollary 3.4(a)(3) for $\Delta_{k_l^i}$ and (4-8) imply that

$$\begin{aligned} b(\Delta_{k_l^i}) &= \max\left\{b(\Delta_j) \mid j \in J \setminus \left(\bigsqcup_{0\leq r\leq i-1} K^r\right),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^i})\right\} \\ &\geq b(\Delta_{j^-}). \end{aligned}$$

Then, since $r_{l+1} \geq i+1 > i \geq r_l$, Lemma 3.3(b) and (d) imply that

$$b(\Delta_{j^-}) \leq b(\Delta_{k_l^i}) \leq b(\Delta_{k_l^{r_l}}) = l - e + b < l + 1 - e + b = b(\Delta_{k_{l+1}^{r_l+1}}) \leq b(\Delta_{k_{l+1}^{i+1}}).$$

In particular,

$$b(\Delta_{j^-}) < b(\Delta_{k_{l+1}^{i+1}}).$$

As a consequence, we obtain that

$$\begin{aligned} &b(\Delta_{(k_l^i)^-}) \\ &= b(\Delta_{k_l^{i+1}}) = \max\left\{b(\Delta_j) \mid j \in J \setminus \left(\bigsqcup_{0\leq r<i+1} K^r\right), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\right\} \\ &= \max\left(\left\{b(\Delta_j) \mid j \in J \setminus \left(\bigsqcup_{0\leq r<i} K^r\right), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\right\} \setminus \{b(\Delta_{k_l^i})\}\right) \\ &= \max\left\{b(\Delta_j) \mid j \in J^- \setminus \left(\bigsqcup_{0\leq r<i} (K^r)^-\right), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\right\} \\ &= \max\left\{b(\Delta_j) \mid j \in J^- \setminus \left(\bigsqcup_{0\leq r<i} (K^r)^-\right), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{k_{l+1}^{i}})\right\} \\ &= \max\left\{b(\Delta_j) \mid j \in J^- \setminus \left(\bigsqcup_{0\leq r<i} (K^r)^-\right), e(\Delta_j) = l, b(\Delta_j) < b(\Delta_{(k_{l+1}^{i})^-})\right\}. \end{aligned}$$

This verifies condition (3) in this case.

Case 3. Suppose that $i \geq r_{l+1} \geq r_l$. Then, $(k_l^i)^- = k_l^{i+1}$ and $(k_{l+1}^i)^- = k_{l+1}^{i+1}$. Corollary 3.4(a)(3) for $\Delta_{k_l^{i+1}}$ and (4-8) imply that

$$\begin{aligned}
&b(\Delta_{(k_l^i)^-}) \\
&= b(\Delta_{k_l^{i+1}}) = \max\Big\{b(\Delta_j) \,\Big|\, j \in J \setminus \Big(\bigsqcup_{0\leq r<i+1} K^r\Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\Big\} \\
&= \max\Big(\Big\{b(\Delta_j) \,\Big|\, j \in J^- \setminus \Big(\bigsqcup_{0\leq r<i+1} (K^r)^-\Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{k_{l+1}^{i+1}})\Big\} \\
&\qquad\qquad \sqcup \{b(\Delta_{k_l^{i+1}})\}\Big) \\
&= \max\Big\{b(\Delta_j) \,\Big|\, j \in J^- \setminus \Big(\bigsqcup_{0\leq r<i} (K^r)^-\Big),\ e(\Delta_j) = l,\ b(\Delta_j) < b(\Delta_{(k_{l+1}^i)^-})\Big\}.
\end{aligned}$$

This verifies condition (3) in this case.

Finally, we verify condition (4). We separate into three cases: $i < r_{e-s}$, $r_{l(i)+1} > i \geq r_{l(i)}$ for some $e - s \leq l(i) < e$, and $e \geq r_e$.

Case 1. Suppose that $i < r_{e-s}$. Then, $(m^i)^- = m^i$ and $(k^i_{(m^i)^-})^- = k^i_{m^i}$. Therefore,

$$\begin{aligned}
&\Big\{j \in J^- \setminus \Big(\bigsqcup_{0\leq r<i} (K^r)^-\Big) \,\Big|\, e(\Delta_j) = (m^i)^- - 1,\ b(\Delta_j) < b(\Delta_{(k^i_{(m^i)^-})^-})\Big\} \\
&\qquad = \Big\{j \in J^- \setminus \Big(\bigsqcup_{0\leq r<i} (K^r)^-\Big) \,\Big|\, e(\Delta_j) = m^i - 1,\ b(\Delta_j) < b(\Delta_{k^i_{m^i}})\Big\} \\
&\qquad \subseteq \Big\{j \in J \setminus \Big(\bigsqcup_{0\leq r<i} K^r\Big) \,\Big|\, e(\Delta_j) = m^i - 1,\ b(\Delta_j) < b(\Delta_{k^i_{m^i}})\Big\}.
\end{aligned}$$

Since the last set is empty by Corollary 3.4(a)(4) for $K^i(\beta)$, this verifies condition (4) in this case.

Case 2. Suppose that $r_{l(i)+1} > i \geq r_{l(i)}$ for some $e - s \leq l(i) < e$. We write $l = l(i)$ for simplicity in the following discussion. We have $(m^i)^- = m^{i+1}$ and $(k^i_{(m^i)^-})^- = k^{i+1}_{m^{i+1}}$. Observe that $\bigsqcup_{0\leq r<i} (K^r)^- \subseteq \bigsqcup_{0\leq r<i+1} K^r$. Moreover, the difference set can be written down explicitly:

$$\Big(\bigsqcup_{0\leq r<i+1} K^r\Big) \setminus \Big(\bigsqcup_{0\leq r<i} (K^r)^-\Big) = \{k_{e-s}^{r_{e-s}}, \ldots, k_l^{r_l}\} \sqcup \{k_{l+1}^i, \ldots, k_e^i\}.$$

As a consequence, if $j \in J^- \setminus \big(\bigsqcup_{0\leq r<i} (K^r)^-\big)$ but $j \notin J \setminus \big(\bigsqcup_{0\leq r<i+1} K^r\big)$, then $j \in \{k_{l+1}^i, \ldots, k_e^i\}$. In particular, since $i + 1 \leq r_{l+1}$, Lemma 3.3(a) implies

$$e(\Delta_j) \geq l + 1 \geq m^{r_{l+1}} \geq m^{i+1} > m^{i+1} - 1.$$

Therefore,

$$\begin{aligned}\Big\{j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) \mid e(\Delta_j) = (m^i)^- - 1,\ b(\Delta_j) < b(\Delta_{(k^i_{(m^i)^-})^-})\Big\}\\ = \Big\{j \in J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) \mid e(\Delta_j) = m^{i+1} - 1,\ b(\Delta_j) < b(\Delta_{k^{i+1}_{m^{i+1}}})\Big\}\\ = \Big\{j \in J \setminus \Big(\bigsqcup_{0\le r<i+1} K^r\Big) \mid e(\Delta_j) = m^{i+1} - 1,\ b(\Delta_j) < b(\Delta_{k^{i+1}_{m^{i+1}}})\Big\},\end{aligned}$$

which is empty by Corollary 3.4(a)(4) for $K^{i+1}(\beta)$, this verifies condition (4) in this case.

Case 3. Suppose $i \ge r_e$. Then $(m^i)^- = m^{i+1}$ and $(k^i_{(m^i)^-})^- = k^{i+1}_{m^{i+1}}$. Similar to the previous case, we have

$$\Big(\bigsqcup_{0\le r<i+1} K^r\Big) \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) = \{k^{r_{e-s}}_{e-s}, \dots, k^{r_e}_e\},$$

and hence

$$J^- \setminus \Big(\bigsqcup_{0\le r<i} (K^r)^-\Big) = J \setminus \Big(\bigsqcup_{0\le r<i+1} K^r\Big).$$

Thus a similar argument as in the previous case verifies condition (4) in this case. This completes the verification of the claim and the proof of the lemma. □

We remark that we have

$$\delta_{d,a} = \delta_{\frac{a-d}{2},\frac{a+d}{2},a}, \qquad \delta_{a,d} = \widetilde{\delta}_{\frac{a-d}{2},\frac{a+d}{2},a}.$$

As a corollary, this gives an alternate proof of the following fact.

Lemma 4.7. *Consider a local Arthur parameter of the form*

$$\psi = \bigoplus_{i\in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i},$$

and let $\alpha = \delta_\psi$. *Then* $\widetilde{\alpha} = \delta_{\widehat{\psi}}$, *where*

$$\widehat{\psi} = \bigoplus_{i\in I_\rho} \rho \otimes \mathrm{Sym}^{a_i} \otimes \mathrm{Sym}^{d_i}.$$

Proof. We apply induction on $|I_\rho|$. Define $a+d, d$ as in Proposition 4.3. We apply Lemma 4.5 on α, which contains a copy of $\delta_{d,a} = \delta_{\frac{a-d}{2},\frac{a+d}{2},a}$. Note that the assumptions are verified by Lemma 4.4 with $\beta = \alpha$ and $i = 0$. We have

$$\widetilde{\alpha} = \widetilde{\alpha^-} + \widetilde{\delta}_{\frac{a-d}{2},\frac{a+d}{2},a} = \widetilde{\alpha^-} + \delta_{a,d},$$

where $\alpha^- = \alpha - \delta_{d,a} = \delta_{\psi^-}$ and

$$\psi^- = \psi - \rho \otimes \mathrm{Sym}^d \otimes \mathrm{Sym}^a.$$

If $|I_\rho| = 1$, then α^- is empty, and hence $\widetilde{\alpha} = \delta_{a,d} = \delta_{\widehat{\psi}}$. If $|I_\rho| > 1$, then the induction hypothesis shows that $\widetilde{\alpha^-} = \delta_{\widehat{\psi^-}}$, which implies that $\widetilde{\alpha} = \delta_{\widehat{\psi}}$. This completes the proof of the lemma. □

Proof of Proposition 4.3. We use the notation $e = e(\beta), t = t(\beta), K^i = K^i(\beta)$ defined in Section 3. We have $e = \frac{a+d}{2}$ by Lemma 4.4(b). Let

$$r := \min\{0 \le i \le t-1 \mid M(\beta^i) \supseteq [(-a+d)/2, (a+d)/2]\}.$$

Note that the right-hand side is nonempty since

$$\widetilde{\beta} \ge \widetilde{\alpha} = \delta_{\widehat{\psi}} \supseteq \delta_{a,d} \ni [(-a+d)/2, (a+d)/2].$$

Recall our notation that $M(\beta^r) = [m^r, e]$ and $K^r = \{k^r_e, \dots, k^r_{m^r}\}$. Corollary 3.4(a)(3) implies

$$b(\Delta_{k^r_{m^r}}) < b(\Delta_{k^r_{m^r+1}}) < \cdots < b(\Delta_{k^r_e}),$$

and hence

$$b(\Delta_{k^r_{m^r}}) \le b(\Delta_{k^r_e}) - (e - m^r) \le b(\Delta_{k^r_e}) - \frac{a+d}{2} + \frac{-a+d}{2} = b(\Delta_{k^r_e}) - a, \tag{4-9}$$

where we use $m^r \le (-a+d)/2$ given by the containment $[m^r, e] \supseteq [(-a+d)/2, e]$ in the second inequality. On the other hand, Lemma 4.4(b) gives

$$b(\Delta_{k^r_e}) \le \frac{a-d}{2}. \tag{4-10}$$

Therefore,

$$b(\Delta_{k^r_{m^r}}) \le \frac{a-d}{2} - a = \frac{-a-d}{2}.$$

By Lemma 4.4(a), the equality must hold, and hence all the inequalities in (4-9) and (4-10) are indeed equalities. In particular, for $m^r \le l \le e$, we have $\Delta_{k^r_l} = [l-d, l]$, We conclude that

$$\beta = \{\Delta_j\}_{j\in J} \supseteq \{\Delta_{k^r}\}_{k^r \in K^r} = \{[l-d, l]\}_{\frac{-a-d}{2} \le l \le \frac{a+d}{2}} = \delta_{d,a}.$$

This proves part (i).

For part (ii), it is clear that $\alpha^- = \delta_{\psi^-}$ and $\beta^- \ge \alpha^-$. It remains to show that $\widetilde{\beta^-} \ge \widetilde{\alpha^-}$. We apply Lemma 4.5 on β, which contains a copy of $\delta_{d,a} = \delta_{\frac{a-d}{2}, \frac{a+d}{2}, a}$. Note that the assumptions in Lemma 4.5 are verified by Lemma 4.4 with $i = 0$. Thus, we obtain that

$$\widetilde{\alpha} = \widetilde{\alpha^-} + \delta_{a,d}, \ \widetilde{\beta} = \widetilde{\beta^-} + \delta_{a,d},$$

which implies that $\widetilde{\beta^-} \ge \widetilde{\alpha^-}$ since $\widetilde{\beta} \ge \widetilde{\alpha}$. This completes the proof of part (ii) and the proposition. □

4.3. ***Proof of Lemma 1.3.*** It is equivalent to prove Lemma 4.2. Write $\alpha = \delta_\psi$ where

$$\psi = \bigoplus_{\rho \in \mathcal{C}_{\mathrm{unit}}} \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}.$$

Let

$$\psi_\rho := \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}.$$

We have the decomposition $\alpha = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \alpha_\rho$, where $\alpha_\rho = \delta_{\psi_\rho}$ and $\beta = \sum_{\rho \in \mathcal{C}_{\mathrm{unit}}} \beta_\rho$. The pair of inequalities $\beta \geq \alpha$ and $\widetilde{\beta} \geq \widetilde{\alpha}$ is equivalent to the pairs of inequalities $\beta_\rho \geq \alpha_\rho$, $\widetilde{\beta}_\rho \geq \widetilde{\alpha}_\rho$ for every ρ. Therefore, we may assume that

$$\psi = \psi_\rho = \bigoplus_{i \in I_\rho} \rho \otimes \mathrm{Sym}^{d_i} \otimes \mathrm{Sym}^{a_i}$$

for some $\rho \in \mathcal{C}_{\mathrm{unit}}$, and adopt the notation in Proposition 4.3.

Apply induction on $k := |I_\rho|$. When $k = 1$, Proposition 4.3(i) implies that $\alpha = \delta_{d,a} = \beta$. Suppose that $k > 1$. We construct α^-, β^- as in Proposition 4.3(ii). Then, the induction hypothesis implies that $\alpha^- = \beta^-$, and hence

$$\alpha = \alpha^- + \delta_{d,a} = \beta^- + \delta_{d,a} = \beta.$$

This completes the proof of the lemma. □

Acknowledgments

I would like to thank Professor Baiying Liu and Professor Freydoon Shahidi for the constant support and encouragement. I also wish to thank Professor Clifton Cunningham, Mishty Ray, and Connor Riddlesden for their insightful discussions and helpful correspondence. I am grateful to the referee for the careful reading of the manuscripts and helpful comments and suggestions.

References

[1] J. Adams, D. Barbasch, and D. A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics **104**, Birkhäuser, Boston, 1992. MR Zbl

[2] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications **61**, American Mathematical Society, Providence, RI, 2013. MR Zbl

[3] H. Atobe, "Construction of local A-packets", *J. Reine Angew. Math.* **790** (2022), 1–51. MR Zbl

[4] H. Atobe and A. Mínguez, "The explicit Zelevinsky–Aubert duality", *Compos. Math.* **159**:2 (2023), 380–418. MR Zbl

[5] C. Cunningham and M. Ray, "Proof of Vogan's conjecture on Arthur packets for GL_n over p-adic fields", preprint, 2023. arXiv 2302.10300

[6] C. Cunningham and M. Ray, "Proof of Vogan's conjecture on Arthur packets: irreducible parameters of p-adic general linear groups", *Manuscripta Math.* **173**:3-4 (2024), 1073–1097. MR Zbl

[7] C. Cunningham, A. Fiori, and N. Kitt, "Appearance of the Kashiwara–Saito singularity in the representation theory of p-adic GL(16)", *Pacific J. Math.* **321**:2 (2022), 239–282. MR Zbl

[8] C. L. R. Cunningham, A. Fiori, A. Moussaoui, J. Mracek, and B. Xu, *Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples*, Mem. Amer. Math. Soc. **1353**, American Mathematical Society, Providence, RI, 2022. MR Zbl

[9] C. Cunningham, A. Hazeltine, B. Liu, C.-H. Lo, M. Ray, and B. Xu, "The Vogan conjecture on local Arthur packets for split odd orthogonal groups", In preparation.

[10] A. Hazeltine, B. Liu, and C.-H. Lo, "On the intersection of local Arthur packets for classical groups", preprint, 2022. arXiv 2201.10539

[11] A. Hazeltine, B. Liu, C.-H. Lo, and Q. Zhang, "The closure ordering conjecture on local Arthur packets of classical groups", preprint, 2022. arXiv 2209.03816

[12] A. Hazeltine, D. Jiang, B. Liu, C.-H. Lo, and Q. Zhang, "On representations of Arthur type and unitary dual for classical groups", preprint, 2024. arXiv 2410.11806

[13] A. Hazeltine, B. Liu, C.-H. Lo, and Q. Zhang, "Computation of Pyasetskii involution for classical groups", In preparation.

[14] H. Knight and A. Zelevinsky, "Representations of quivers of type A and the multisegment duality", *Adv. Math.* **117**:2 (1996), 273–293. MR Zbl

[15] C.-H. Lo, "On the closure ordering of standard modules of classical groups", In preparation.

[16] C. Mœglin, "Paquets d'Arthur pour les groupes classiques; point de vue combinatoire", preprint, 2006. arXiv math/0610189v1

[17] C. Mœglin, "Sur certains paquets d'Arthur et involution d'Aubert–Schneider–Stuhler généralisée", *Represent. Theory* **10** (2006), 86–129. MR Zbl

[18] C. Mœglin, "Paquets d'Arthur discrets pour un groupe classique p-adique", pp. 179–257 in *Automorphic forms and L-functions*, *II*: *Local aspects*, edited by D. Ginzburg et al., Contemp. Math. **489**, Amer. Math. Soc., Providence, RI, 2009. MR Zbl

[19] C. Mœglin, "Holomorphie des opérateurs d'entrelacement normalisés à l'aide des paramètres d'Arthur", *Canad. J. Math.* **62**:6 (2010), 1340–1386. MR Zbl

[20] C. Mœglin, "Multiplicité 1 dans les paquets d'Arthur aux places p-adiques", pp. 333–374 in *On certain L-functions*, edited by J. Arthur et al., Clay Math. Proc. **13**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[21] C. Mœglin and J.-L. Waldspurger, "Sur l'involution de Zelevinski", *J. Reine Angew. Math.* **372** (1986), 136–177. MR Zbl

[22] C. Riddlesden, "Combinatorial approach to ABV-packets for $\mathbf{GL_n}$", preprint, 2023. arXiv 2304.09598

[23] A. V. Zelevinsky, "The p-adic analogue of the Kazhdan–Lusztig conjecture", *Funktsional. Anal. i Prilozhen.* **15**:2 (1981), art. id. 96. MR Zbl

Received January 9, 2024. Revised December 16, 2024.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 333, No. 2, 2024
https://doi.org/10.2140/pjm.2024.333.357

THREE INVARIANTS OF GEOMETRICALLY VERTEX DECOMPOSABLE IDEALS

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We study three invariants of geometrically vertex decomposable ideals: the Castelnuovo–Mumford regularity, the multiplicity, and the a-invariant. We show that these invariants can be computed recursively using the ideals that appear in the geometric vertex decomposition process.

As an application, we prove that the a-invariant of a geometrically vertex decomposable ideal is nonpositive. We also recover some previously known results in the literature including a formula for the regularity of the Stanley–Reisner ideal of a pure vertex decomposable simplicial complex, and proofs that some well-known families of ideals are Hilbertian. Finally, we apply our recursions to the study of toric ideals of bipartite graphs. Included among our results on this topic is a new proof for a known bound on the a-invariant of a toric ideal of a bipartite graph.

1. Introduction

Vertex decomposable simplicial complexes and their associated Stanley–Reisner ideals have been extensively studied in the fields of combinatorial algebraic topology and combinatorial commutative algebra; for example, see [Dochtermann and Engström 2009; Hà and Woodroofe 2014; Knutson et al. 2009; Provan and Billera 1980; Woodroofe 2009]. These complexes are known to have many nice combinatorial properties. Such complexes are defined recursively via vertex decompositions into subcomplexes, and this suggests that one can study their structure and invariants by means of those recursions. A generalization of this concept, *geometric vertex decomposition*, was introduced by A. Knutson, E. Miller and A. Yong in [Knutson et al. 2009]. They used this computational-algebraic technique to study Schubert determinantal ideals associated to vexillary permutations.

Building upon this work, P. Klein and the second author in [Klein and Rajchgot 2021] introduced the notion of a *geometrically vertex decomposable ideal*, which is a generalization of the Stanley–Reisner ideal of a vertex decomposable simplicial

MSC2020: 05E40, 13P10, 14M25.

Keywords: geometrically vertex decomposable, Castelnuovo–Mumford regularity, a-invariant, multiplicity, Hilbertian, toric ideals of graphs.

complex. In fact, a geometrically vertex decomposable squarefree monomial ideal is precisely the Stanley–Reisner ideal of a vertex decomposable simplicial complex, with the geometric vertex decomposition given by the vertex decomposition of the complex. Other well-known families of geometrically vertex decomposable ideals include Schubert determinantal ideals [Klein and Rajchgot 2021] and toric ideals of bipartite graphs [Cummings et al. 2023]. The technique of geometric vertex decomposition has been increasingly useful in various algebro-geometric contexts including applications in liaison theory [Klein and Rajchgot 2021], Gröbner geometry of matrix Schubert varieties [Klein 2023; Klein and Weigandt 2022; Knutson et al. 2009], and the study of Hessenberg varieties [Cummings et al. 2024; Da-Silva and Harada 2023]. It was also shown in [Cummings et al. 2023; Klein and Rajchgot 2021] that geometrically vertex decomposable ideals have many algebraic properties in common with Stanley–Reisner ideals of vertex decomposable simplicial complexes.

The purpose of this paper is to study algebraic invariants of geometrically vertex decomposable ideals. We shall exploit their inherently recursive structure to derive recursive formulae for their invariants. Our recursions reduce the study of our original geometrically vertex decomposable ideal to the study of two related geometrically vertex decomposable ideals, each of which is in one less variable. The following theorem summarizes our main results about these invariants. In the statement below, the ideals $C_{y,I}$ and $N_{y,I}$ refer to ideals formed from the decomposition of I; complete definitions are found in Section 2. Furthermore, $\operatorname{reg}(R/I)$, $e(R/I)$, and $a(R/I)$ refer to the Castelnuovo–Mumford regularity, multiplicity, and a-invariant of R/I, respectively.

Theorem 1.1. *Suppose that $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ is a homogeneous, geometrically vertex decomposable ideal. Then there exists a variable y such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition, and $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable. If the geometric vertex decomposition is nondegenerate, then*

(1) (Corollary 3.3) $\operatorname{reg}(R/I) = \max\{\operatorname{reg}(R/N_{y,I}), \operatorname{reg}(R/C_{y,I}) + 1\}$,

(2) (Corollary 4.3) $e(R/I) = e(R/N_{y,I}) + e(R/C_{y,I})$, *and*

(3) (Corollary 5.7) $a(R/I) = \max\{a(R/N_{y,I}) + 1, a(R/C_{y,I}) + 1\}$.

When the geometric vertex decomposition is degenerate, we have

$$\operatorname{reg}(R/I) = \operatorname{reg}(R/N_{y,I}), \quad e(R/I) = e(R/N_{y,I}), \quad \text{and} \quad a(R/I) = a(R/N_{y,I}).$$

A key observation of Knutson, Miller, and Yong [Knutson et al. 2009] is that the Hilbert series of an ideal I that has a geometric vertex decomposition is related to the Hilbert series of the smaller ideals (see Theorem 2.3). We use this result to

show that, in the case that I is also geometrically vertex decomposable, there is a relation among the associated h-polynomials (see Theorem 2.4). The proof of Theorem 1.1 then relies heavily on this relation among the h-polynomials.

Applying these recursions to various classes of ideals that are previously known to be geometrically vertex decomposable (see [Cummings et al. 2023; Klein and Rajchgot 2021]), we are able to offer a new approach to recover several known results. As an example, the recursive formula for the regularity of Stanley–Reisner ideals of pure vertex decomposable simplicial complexes, independently found by H.T. Hà and R. Woodroofe [Hà and Woodroofe 2014] and S. Moradi and F. Khosh-Ahang [Moradi and Khosh-Ahang 2016], can be deduced from Theorem 1.1 (see Corollary 3.4).

Theorem 1.1 can also be used to show that all geometrically vertex decomposable ideals are "almost" Hilbertian. For a homogeneous ideal $I \subseteq R$, let $HP_{R/I}(t)$ denote the Hilbert polynomial of R/I, and let $HF_{R/I}(t)$ denote its Hilbert function. An ideal is *Hilbertian* if $HP_{R/I}(t) = HF_{R/I}(t)$ for all $t \geq 0$; this definition is attributed to S. Abhyankar (see [Abhyankar and Kulkarni 1989]). Recently, A. Stelzer and A. Yong [Stelzer and Yong 2023] proved that Schubert determinantal ideals are Hilbertian. This result is related to our situation because Schubert determinantal ideals are geometrically vertex decomposable [Klein and Rajchgot 2021, Section 5]. Because the a-invariant is intimately linked to when $HF_{R/I}(t)$ and $HP_{R/I}(t)$ agree, we can contribute the following result.

Theorem 1.2 (Corollary 5.8). *Let $I \subset R$ be a proper, homogeneous, geometrically vertex decomposable ideal. Then $a(R/I) \leq 0$. Consequently, $HF_{R/I}(t) = HP_{R/I}(t)$ for all $t \geq 1$.*

Note that Theorem 1.2 implies that, except possibly at $t = 0$, the Hilbert function and the Hilbert polynomial of a geometrically vertex decomposable ideal agree. Under extra hypotheses, we are able to determine when there is also agreement at $t = 0$, and consequently, the ideal I is Hilbertian.

In the last part of the paper, we apply our results to the class of toric ideals of bipartite graphs, which are known to be geometrically vertex decomposable by [Cummings et al. 2023] (which builds on [Constantinescu and Gorla 2018]). Our results complement and extend recent work on invariants of toric ideals of graphs; for example, see [Almousa et al. 2022; Biermann et al. 2017; Corso and Nagel 2009; D'Alì 2015; Galetto et al. 2019; Hà et al. 2019; Ohsugi and Hibi 1999; Tatakis and Thoma 2011; Villarreal 1995]. Recall that if $G = (V, E)$ is a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_q\}$, the *toric ideal of* G, denoted I_G, is the kernel of the $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}[e_1, \ldots, e_q] \to \mathbb{K}[x_1, \ldots, x_n]$ given by $\varphi(e_i) = x_j x_k$, where $e_i = \{x_i, x_j\} \in E$. Theorems 1.1 and 1.2 allow us to show the following results.

Theorem 1.3 (Theorems 6.7 and 6.20). *Let H be any subgraph of a bipartite graph G. Then*

(1) $\operatorname{reg}(I_H) \leq \operatorname{reg}(I_G)$,

(2) $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G)$, *and*

(3) $e(\mathbb{K}[E(G)]/I_H) \leq e(\mathbb{K}[E(G)]/I_G)$.

Furthermore, if G is connected, then I_G is Hilbertian.

Theorem 1.3 (1) was recently shown in [Almousa et al. 2022, Theorem 6.11] by A. Almousa, A. Dochtermann, and B. Smith using combinatorial techniques involving root polytopes and also in [Pinto and Villarreal 2023, Corollary 8.16] by M.V. Pinto and R.H. Villarreal using edge polytopes. Note that one could use the above results to obtain the upper bounds in terms of graph-theoretic invariants as those invariants of the complete bipartite graphs can be exactly computed (by the technique of geometric vertex decomposition or other techniques). Our technique not only gives a new proof for the regularity bound but can also be used to recover the results on precise values of the regularity, a-invariant, as well as multiplicity of toric ideals of Ferrer graphs (including complete bipartite graphs) in [Corso and Nagel 2009] by A. Corso and U. Nagel.

Theorem 1.4 (Theorem 6.12). *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let T_λ be the associated Ferrers graph. Write $I_\lambda = I_{T_\lambda}$ and $R = \mathbb{K}[E(T_\lambda)]$.*

(1) *If $n = 1$ or $\lambda_2 = 1$, then $\operatorname{reg}(R/I_\lambda) = 0$.*

(2) *If $\lambda_2 \geq 2$, and suppose that $\lambda = (\lambda_1, \ldots, \lambda_s, 1, 1, \ldots, 1)$ where $\lambda_s \geq 2$, then*

$$\operatorname{reg}(R/I_\lambda) = \min\{s-1, \{\lambda_j + j - 3 \mid 2 \leq j \leq s\}\}.$$

In particular, if $G = K_{n,m}$ is a complete bipartite graph, then $\operatorname{reg}(R/I_G) = \min\{n, m\} - 1$.

In addition to the above results, we consider the "gluing" procedure of G. Favacchio, J. Hofscheier, G. Keiper, and the last author [Favacchio et al. 2021] which "glues" an even cycle to a graph G to form a new graph H. We relate the regularity, the a-invariant, and the multiplicity of I_G, when this ideal is geometrically vertex decomposable, to that of I_H (see Theorem 6.9 and Corollary 6.10). To further illustrate the usefulness of our results, our techniques are used to explicitly compute all three invariants for all toric ideals of graphs that belong to a family first considered in [Galetto et al. 2019].

Our paper is structured as follows. In Section 2 we recall the relevant background on geometrically vertex decomposable ideals. In Sections 3–5, we consider the regularity, the multiplicity, and the a-invariant, respectively. In Section 6, we apply our results to study the invariants of toric ideals of (bipartite) graphs.

2. Background on geometrically vertex decomposable ideals

In this section we recall the notion of geometric vertex decomposition, introduced by Knutson, Miller and Yong in [Knutson et al. 2009], and geometrically vertex decomposable ideals, introduced by Klein and the second author in [Klein and Rajchgot 2021]. Due to the recursive nature of the theory of geometric vertex decomposition, this technique provides us with a convenient inductive set-up for studying properties and invariants of certain classes of ideals, as expanded upon in later sections.

Hereafter, we let $\mathbb{K}$ denote an arbitrary field. We let $R = \mathbb{K}[x_1, \dots, x_n]$ be a standard graded polynomial ring in n variables. Fix a variable $y = x_j$. Then, for any $f \in R$, we can write $f = \sum_{i=0}^{d} \alpha_i y^i$, where, for each i, α_i is a polynomial in $\mathbb{K}[x_1, \dots, \hat{x}_j, \dots, x_n]$. For $f \neq 0$, we define the *initial y-form* denoted $\mathrm{in}_y(f)$ to be the sum of all the nonzero terms of f having the highest power of y, that is, $\mathrm{in}_y(f) = \alpha_d y^d$. For an ideal $J \subset R$, define $\mathrm{in}_y(J) = \langle \mathrm{in}_y(f) \mid f \in J \rangle$. A monomial order $<$ on R is said to be *y-compatible* if it satisfies $\mathrm{in}_<(\mathrm{in}_y(f)) = \mathrm{in}_<(f)$ for all $f \in R$, where $\mathrm{in}_<(f)$ is the initial term of f with respect to $<$. It follows that, for such an order, we have $\mathrm{in}_<(\mathrm{in}_y(I)) = \mathrm{in}_<(I)$ for all ideals I.

Consider an ideal I and a y-compatible monomial order on R. Suppose that $\mathcal{G} = \{g_1, \dots, g_m\}$ is a Gröbner basis of I with respect to this monomial order, and for each $i = 1, \dots, m$, write $g_i = y^{d_i} q_i + r_i$, where $\mathrm{in}_y(g_i) = y^{d_i} q_i$ and y does not divide any term of q_i. It follows that $\mathrm{in}_y(I) = \langle y^{d_i} q_i \mid 1 \leq i \leq m \rangle$. We define the ideals

$$C_{y,I} = \langle q_1, \dots, q_m \rangle \quad \text{and} \quad N_{y,I} = \langle q_i \mid d_i = 0 \rangle.$$

It is important to observe that the ideals $C_{y,I}$ and $N_{y,I}$ do not depend on the choice of Gröbner basis, and in particular do not depend on the choice of y-compatible monomial order, since $C_{y,I} = \mathrm{in}_y(I) : \langle y \rangle^\infty$ and $N_{y,I} + \langle y \rangle = \mathrm{in}_y(I) + \langle y \rangle$ by [Knutson et al. 2009, Theorem 2.1]; see also [Klein and Rajchgot 2021, Section 2].

Definition 2.1. When $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, we call this a *geometric vertex decomposition* of I with respect to y. We say that the geometric vertex decomposition is *degenerate* if $\sqrt{C_{y,I}} = \sqrt{N_{y,I}}$ or if $C_{y,I} = \langle 1 \rangle$, and *nondegenerate* otherwise.

Recall that an ideal I is *unmixed* if I satisfies $\dim(R/I) = \dim(R/P)$ for all associated primes $P \in \mathrm{Ass}_R(R/I)$. We define the main object of study in this paper.

Definition 2.2. An ideal I of $R = \mathbb{K}[x_1, \dots, x_n]$ is *geometrically vertex decomposable* if I is unmixed and

(1) $I = \langle 1 \rangle$, or I is generated by a (possibly empty) subset of variables of R, or

(2) there exists a variable $y = x_j$ of R such that we have a geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, and the contractions of the ideals $C_{y,I}$ and $N_{y,I}$ to the ring $\mathbb{K}[x_1, \dots, \hat{x}_j, \dots, x_n]$ are geometrically vertex decomposable.

Thus, given a geometrically vertex decomposable ideal I, one can perform a geometric vertex decomposition with respect to some variable y to obtain a geometrically vertex decomposable ideal $C_{y,I}$ and a geometrically vertex decomposable ideal $N_{y,I}$. Each of these ideals can then be decomposed into their own geometrically vertex decomposable ideals, and so on, until all ideals have the form of item (1) of Definition 2.2. We refer to such a process of repeatedly performing geometric vertex decompositions, where all ideals at all stages are geometrically vertex decomposable, as a *geometric vertex decomposition process*. We note that an ideal I may have multiple different geometric vertex decomposition processes. Many well-known ideals are geometrically vertex decomposable. These include Stanley–Reisner ideals of vertex decomposable simplicial complexes, classes of generalized determinantal ideals (e.g., classical determinantal ideals, Schubert determinantal ideals, ideals of varieties of complexes), defining ideals of lower bound cluster algebras, and toric ideals of bipartite graphs [Cummings et al. 2023; Klein and Rajchgot 2021].

Our main results depend upon a relationship between the h-polynomials of R/I, $R/C_{y,I}$, and $R/N_{y,I}$. Recall that the *Hilbert series* of a graded R-module $M = \bigoplus_{i=1}^{\infty} M_i$ is the generating function

$$H_M(t) = \sum_{i=0}^{\infty} (\dim_{\mathbb{K}} M_i) t^i.$$

By the Hilbert–Serre theorem [Bruns and Herzog 1993, Corollary 4.1.8], the Hilbert series can be expressed as a rational function

$$H_M(t) = \frac{h_M(t)}{(1-t)^d},$$

where $h_M(t)$, the *h-polynomial* of M, is a polynomial with integer coefficients and $d = \dim M$. The following result, which is [Knutson et al. 2009, Theorem 2.1 (e)], relates the Hilbert series of an ideal to that of its geometric vertex decomposition; for completeness, we have included a proof.

Theorem 2.3. *Suppose that $I \subseteq R$ is a homogeneous ideal that has a geometric vertex decomposition with respect to y, that is,* $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. *Then*

$$H_{R/I}(t) = H_{R/(N_{y,I}+\langle y \rangle)}(t) + t H_{R/C_{y,I}}(t).$$

Proof. We have the short exact sequence

$$0 \longrightarrow \frac{R}{C_{y,I} \cap (N_{y,I} + \langle y \rangle)} \longrightarrow \frac{R}{C_{y,I}} \oplus \frac{R}{N_{y,I} + \langle y \rangle} \longrightarrow \frac{R}{C_{y,I} + N_{y,I} + \langle y \rangle} \longrightarrow 0.$$

Furthermore, note that $C_{y,I} + N_{y,I} + \langle y \rangle = C_{y,I} + \langle y \rangle$. Because $\mathrm{in}_y(I)$ and I have the same Hilbert series (since $\mathrm{in}_<(\mathrm{in}_y(I)) = \mathrm{in}_<(I)$), the Hilbert series of R/I

satisfies

$$H_{R/I}(t) = H_{R/(N_{y,I}+\langle y\rangle)}(t) + H_{R/C_{y,I}}(t) - H_{R/(C_{y,I}+\langle y\rangle)}(t).$$

Since y is a nonzero divisor on $R/C_{y,I}$, we have the short exact sequence

$$0 \longrightarrow \frac{R}{C_{y,I}}(-1) \xrightarrow{\times y} \frac{R}{C_{y,I}} \longrightarrow \frac{R}{C_{y,I}+\langle y\rangle} \longrightarrow 0,$$

which then implies that

$$H_{R/C_{y,I}}(t) - H_{R/(C_{y,I}+\langle y\rangle)}(t) = tH_{R/C_{y,I}}(t).$$

Consequently, $H_{R/I}(t) = H_{R/(N_{y,I}+\langle y\rangle)}(t) + tH_{R/C_{y,I}}(t)$, as desired. □

Note that Theorem 2.3 implies a relationship among the h-polynomials $h_{R/I}(t)$, $h_{R/(N_{y,I}+\langle y\rangle)}(t)$, and $h_{R/C_{y,I}}(t)$. To explicitly determine this relationship, one would need to know the dimension of the three rings. As shown below, we know the dimensions in the case that the homogeneous ideal $I \subseteq R$ is geometrically vertex decomposable. In fact, instead of the h-polynomial of $R/(N_{y,I}+\langle y\rangle)$, we use the h-polynomial of $R/N_{y,I}$.

Theorem 2.4. *Suppose that $I \subseteq R$ is a homogeneous ideal and R/I is equidimensional. If there is a variable y such that $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ is a nondegenerate geometric vertex decomposition, then the h-polynomial of R/I satisfies*

$$h_{R/I}(t) = h_{R/N_{y,I}}(t) + th_{R/C_{y,I}}(t).$$

Proof. Because R/I is equidimensional and the decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I}+\langle y\rangle)$ is nondegenerate, we may apply [Klein and Rajchgot 2021, Lemma 2.8] to conclude that $\mathrm{ht}(I) = \mathrm{ht}(C_{y,I}) = \mathrm{ht}(N_{y,I}) + 1$.

Since $\dim(R) = \dim(R/J) + \mathrm{ht}(J)$ for any ideal J in the polynomial ring R, we have

$$\dim(R/I) = \dim(R/C_{y,I}) = \dim(R/N_{y,I}) - 1 = \dim(R/(N_{y,I}+\langle y\rangle)) = d.$$

Therefore, by Theorem 2.3, we have

$$\frac{h_{R/I}(t)}{(1-t)^d} = \frac{h_{R/(N_{y,I}+\langle y\rangle)}(t)}{(1-t)^d} + \frac{th_{R/C_{y,I}}(t)}{(1-t)^d}.$$

To complete the proof, it suffices to show that $h_{R/(N_{y,I}+\langle y\rangle)}(t) = h_{R/N_{y,I}}(t)$. Because the generators of $N_{y,I}$ do not involve y, we then have a short exact sequence

$$0 \longrightarrow \frac{R}{N_{y,I}}(-1) \xrightarrow{\times y} \frac{R}{N_{y,I}} \longrightarrow \frac{R}{(N_{y,I}+\langle y\rangle)} \longrightarrow 0.$$

Consequently, the Hilbert series of $R/(N_{y,I} + \langle y \rangle)$ satisfies

$$\frac{h_{R/(N_{y,I}+\langle y\rangle)}(t)}{(1-t)^{d-1}} = H_{R'/N_{y,I}}(t) = (1-t)H_{R/N_{y,I}}(t) = \frac{(1-t)h_{R/N_{y,I}}(t)}{(1-t)^d}.$$

Comparing the numerator of both sides now gives the conclusion. □

3. Regularity of geometrically vertex decomposable ideals

We consider the (Castelnuovo–Mumford) regularity of geometrically vertex decomposable ideals. We derive a recursive formula for the regularity for this family that allows us to compute the regularity of various classes of ideals, e.g., Stanley–Reisner ideals of pure vertex decomposable simplicial complexes, and toric ideals of bipartite graphs in Section 6.

The (*Castelnuovo–Mumford*) *regularity* of a graded R-module M is given by

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\},$$

where $\beta_{i,j}(M)$ denotes the (i, j)-th graded Betti number that appears in the minimal graded free resolution of M. The following property, which relates the regularity to the degree of the h-polynomial in the Hilbert series, shall be of great use.

Lemma 3.1 [Vasconcelos 1998, Corollary B.28]. *Let $I \subseteq R$ be a homogeneous ideal such that R/I is Cohen–Macaulay. Then* $\operatorname{reg}(R/I) = \deg h_{R/I}(t)$.

Because all geometrically vertex decomposable ideals are Cohen–Macaulay, for this family of ideals, we can informally define the regularity of R/I to be the degree of the h-polynomial.

We come to the main result of this section, which describes a recursion between regularity values of a geometrically vertex decomposable I and the corresponding ideals $C_{y,I}$ and $N_{y,I}$. The proof relies on Theorem 2.4.

Theorem 3.2. *Suppose $I \subseteq R$ is a homogeneous and radical ideal such that R/I is Cohen–Macaulay. Suppose y is a variable such that* $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ *is a geometric vertex decomposition. If the decomposition is nondegenerate and $R/C_{y,I}$ and $R/N_{y,I}$ are Cohen–Macaulay, then*

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I)) = \max\{\operatorname{reg}(N_{y,I}), \operatorname{reg}(C_{y,I}) + 1\}.$$

Otherwise, if the decomposition is degenerate, then

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I)) = \operatorname{reg}(C_{y,I}) = \operatorname{reg}(N_{y,I}) \quad \text{if } C_{y,I} \neq \langle 1 \rangle$$

and

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I)) = \operatorname{reg}(N_{y,I}) \quad \text{if } C_{y,I} = \langle 1 \rangle.$$

Proof. Since R/I is Cohen–Macaulay, $\operatorname{reg}(R/I) = \deg h_{R/I}(t)$ by Lemma 3.1.

If the geometric vertex decomposition is nondegenerate and $R/C_{y,I}$ and $R/N_{y,I}$ are Cohen–Macaulay, then by Theorem 2.4 we have

$$\begin{aligned}\operatorname{reg}(R/I) = \deg h_{R/I}(t) &= \max\{\deg h_{R/N_{y,I}}(t),\ \deg h_{R/C_{y,I}}(t)+1\}\\ &= \max\{\operatorname{reg}(R/N_{y,I}),\ \operatorname{reg}(R/C_{y,I})+1\}.\end{aligned}$$

Note that the second equality follows from the fact the h-polynomial of a Cohen–Macaulay ring always has nonnegative coefficients (e.g., see [Stanley 1978, Corollary 3.11]), so there is no cancellation among the top-degree terms when the polynomials of Theorem 2.4 are added. To recover the statement of the theorem, use the fact that $\operatorname{reg}(R/J) = \operatorname{reg}(J) - 1$ for any proper homogeneous ideal J.

To show that $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}_y(I))$, note that, by [Klein and Rajchgot 2021, Corollary 4.11], we have $\operatorname{in}_y(I)$ is also Cohen–Macaulay. Since $\operatorname{in}_y(I)$ and I have the same Hilbert series (as the Hilbert series of $\operatorname{in}_<(\operatorname{in}_y(I)) = \operatorname{in}_<(I)$), we get

$$\operatorname{reg}(R/I) = \deg h_{R/I}(t) = \deg h_{R/\operatorname{in}_y(I)}(t) = \operatorname{reg}(R/\operatorname{in}_y(I)),$$

as desired.

Now suppose that the decomposition is degenerate. If $C_{y,I} = \langle 1\rangle$, we have $R/I \cong R/(N_{y,I} + \langle y\rangle)$, and thus the claim follows. Otherwise, if $C_{y,I} \neq \langle 1\rangle$, the result follows directly from the fact that $I = \operatorname{in}_y(I) = C_{y,I} = N_{y,I}$, as shown in [Klein and Rajchgot 2021, Proposition 2.4]. □

Corollary 3.3. *Theorem 1.1, part* (1) *is true.*

Proof. Let I be a geometrically vertex decomposable ideal. Then, by definition, there exists a variable y such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is a geometric vertex decomposition and $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable. Since geometrically vertex decomposable ideals are Cohen–Macaulay and radical by [Klein and Rajchgot 2021, Corollary 4.5 and Proposition 2.10], the result is now immediate from Theorem 3.2. □

The above result recovers the recursive formula for the regularity of the Stanley–Reisner ideal of a pure vertex decomposable complex. Since this is our only result concerning vertex decomposable simplicial complexes and Stanley–Reisner ideals, we point the reader to [Klein and Rajchgot 2021, Section 2.1] for notation and terminology that is not explained. We want to highlight that our result is for *pure* simplicial complexes; the following result can be seen as giving new proofs for special cases of [Hà and Woodroofe 2014, Theorem 4.2] and [Moradi and Khosh-Ahang 2016, Corollary 2.11]; in particular, [Hà and Woodroofe 2014; Moradi and Khosh-Ahang 2016] do not require the vertex decomposable simplicial complex to be pure.

Corollary 3.4. *Let Δ be a pure vertex decomposable simplicial complex, and let v be a shedding vertex of Δ. If I_Δ is the Stanley–Reisner ideal of Δ, then*

$$\operatorname{reg}(R/I_\Delta) = \max\{\operatorname{reg}(R/I_{\Delta_1}), \operatorname{reg}(R/I_{\Delta_2}) + 1\},$$

where $\Delta_1 = \operatorname{del}_\Delta(v)$ is the deletion of v, and $\Delta_2 = \operatorname{lk}_\Delta(v)$ is the link of v.

Proof. As pointed out in [Klein and Rajchgot 2021, Proposition 2.9], I_Δ is geometrically vertex decomposable, and from [Klein and Rajchgot 2021, Remark 2.5], we have a geometric vertex decomposition with $N_{y,I} + \langle y\rangle = I_{\Delta_1}$ and $C_{y,I} = I_{\operatorname{star}_\Delta(v)}$, where y is the variable corresponding to the vertex v. The decomposition is nondegenerate since v is a shedding vertex. Now, as $I_{\Delta_2} = I_{\operatorname{star}_\Delta(v)} + \langle y\rangle$ and since y is not in the support of $C_{y,I}$ and $N_{y,I}$, we have $\operatorname{reg}(C_{y,I}) = \operatorname{reg}(I_{\Delta_2})$ and $\operatorname{reg}(N_{y,I}) = \operatorname{reg}(I_{\Delta_1})$. Hence, the above formula follows from Theorem 3.2. □

Remark 3.5. Under the hypotheses that Δ is a pure simplicial complex, the decomposition $C_{y,I} \cap (N_{y,I} + \langle y\rangle)$ is degenerate if and only if Δ is a cone from v on Δ_2 (where y is the variable corresponding to v), and in this case, $\operatorname{reg}(I_\Delta) = \operatorname{reg}(I_{\Delta_1}) = \operatorname{reg}(I_{\Delta_2})$.

Remark 3.6. The proof of Corollary 3.4, as given in [Hà and Woodroofe 2014], uses tools from combinatorial topology, like the Mayer–Vietoris sequence. On the other hand, Corollary 3.4 is proved in [Moradi and Khosh-Ahang 2016] by first computing the projective dimension of $I_\Delta^\vee$, the corresponding Alexander dual of I_Δ, and then using the fact that this value equals $\operatorname{reg}(R/I_\Delta)$. Our proof of Corollary 3.4 provides an entirely new approach in the case of pure vertex decomposable simplicial complexes.

Example 3.7. The ideal $I = \langle y(zs - x^2), ywr, wr(z^2 + zx + wr + s^2)\rangle$ is geometrically vertex decomposable, with

$$C_{y,I} = \langle zs - x^2, wr\rangle \quad \text{and} \quad N_{y,I} = \langle wr(z^2 + zx + wr + s^2)\rangle,$$

and the geometric vertex decomposition is nondegenerate; see [Klein and Rajchgot 2021, Example 2.16]. The ideal $N_{y,I}$ is generated by one polynomial of degree 4, so $\operatorname{reg}(N_{y,I}) = 4$. For the ideal $C_{y,I}$, since its two generators have separate variables, we have

$$\operatorname{reg}(C_{y,I}) = \operatorname{reg}(\langle zs - x^2\rangle) + \operatorname{reg}(\langle wr\rangle) - 1 = 2 + 2 - 1 = 3.$$

Therefore, by Theorem 3.2, $\operatorname{reg}(I) = 4$.

Remark 3.8. As defined in [Klein and Rajchgot 2021, Definition 4.6], an ideal $I \subseteq R$ is called *weakly geometrically vertex decomposable* if I is unmixed and if

(1) $I = \langle 1\rangle$, or I is generated by a (possibly empty) subset of variables of R, or

(2) for some variable $y = x_j$ of R, there is a *degenerate* geometric vertex decomposition $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y\rangle)$, and the contraction of $N_{y,I}$ to the ring $\mathbb{K}[x_1, \ldots, \hat{x}_j, \ldots, x_n]$ is weakly geometrically vertex decomposable, or

(3) for some variable $y = x_j$ of R, there is a *nondegenerate* geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, the contraction of $C_{y,I}$ to the ring $\mathbb{K}[x_1, \ldots, \hat{x}_j, \ldots, x_n]$ is weakly geometrically vertex decomposable, and $N_{y,I}$ is radical and Cohen–Macaulay.

The proofs of Theorems 2.4 and 3.2 can be adapted easily to weakly geometrically vertex decomposable ideals. In these proofs, we only used the geometrically vertex decomposable property of I to obtain that the ideals I, $\mathrm{in}_y(I)$, $C_{y,I}$, and $N_{y,I}$ are Cohen–Macaulay, which is true by [Klein and Rajchgot 2021, Corollaries 4.8 and 4.11] without the geometrically vertex decomposable assumption. Furthermore, the height lemma [Klein and Rajchgot 2021, Lemma 2.8] and the fact that I is radical [Klein and Rajchgot 2021, Corollary 4.8] are also true in the weakly geometrically vertex decomposable setting. Therefore, we have the same recursive formula for weakly geometrically vertex decomposable ideals.

Example 3.9. The ideal $I = \langle y(zs - x^2), ywr, wr(x^2 + z^2 + wr + s^2) \rangle$ is weakly geometrically vertex decomposable, with

$$C_{y,I} = \langle zs - x^2, wr \rangle \quad \text{and} \quad N_{y,I} = \langle wr(x^2 + z^2 + wr + s^2) \rangle,$$

but I is not geometrically vertex decomposable; see [Klein and Rajchgot 2021, Example 4.10]. Nevertheless, by Remark 3.8, we still have

$$\mathrm{reg}(I) = \max\{\mathrm{reg}(N_{y,I}), \mathrm{reg}(C_{y,I}) + 1\} = \max\{4, 3 + 1\} = 4.$$

Remark 3.10. More generally, the proofs of Theorems 2.4 and 3.2 can be adapted easily to the case when we only require I to be a homogeneous ideal that possesses a geometric vertex decomposition and the ideals I and $N_{y,I}$ are Cohen–Macaulay. As in the proofs of Theorems 2.4 and 3.2, we only need that the ideals I, $\mathrm{in}_y(I)$, $C_{y,I}$, and $N_{y,I}$ are Cohen–Macaulay, which is true by [Klein and Rajchgot 2021, Corollaries 4.8 and 4.11], and the height lemma [Klein and Rajchgot 2021, Lemma 2.8] is also true when I is Cohen–Macaulay and the geometric vertex decomposition is nondegenerate. If, in addition, I is radical then the formula in the degenerate case also works by the same argument.

Example 3.11. Consider the ideal $I = \langle yz - xw, xy \rangle$. One can check that I is not geometrically vertex decomposable. Nevertheless, using the lexicographical order $x > y > z > w$, the Gröbner basis of I is $\{yz - xw, xy, y^2z\}$. One can check that I has a nondegenerate geometric vertex decomposition with $C_{x,I} = \langle y, w \rangle$ and $N_{x,I} = \langle y^2 z \rangle$. Since I and $N_{x,I}$ are Cohen–Macaulay, we get

$$\mathrm{reg}(R/I) = \max\{\mathrm{reg}(R/N_{x,I}), \mathrm{reg}(R/C_{x,I}) + 1\} = 2.$$

4. Multiplicity of geometrically vertex decomposable ideals

This short section considers the multiplicity of geometrically vertex decomposable ideals. As with Theorem 3.2, our results rely on using Theorem 2.4 to relate the h-polynomial of I to the h-polynomials of $C_{y,I}$ and $N_{y,I}$.

Definition 4.1. Let M be an R-module with Hilbert series $H_M(t) = h_M(t)/(1-t)^d$, where $d = \dim M$. Then the *multiplicity* of M is $e(M) = h_M(1)$.

The leading coefficient of the *Hilbert polynomial* of M is given by $e(M)/d!$, and when $M = R/I$, where I is the defining ideal of a projective variety, then $e(M)$ is the degree of the variety.

Theorem 4.2. *Let $I \subseteq R$ be a homogeneous and radical ideal such that R/I is equidimensional. Suppose there is a variable y such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition. If the decomposition is nondegenerate, then*

$$e(R/I) = e(R/N_{y,I}) + e(R/C_{y,I}).$$

Otherwise, if the decomposition is degenerate, we have

$$e(R/I) = e(R/N_{y,I}) = e(R/C_{y,I}) \quad \text{if } C_{y,I} \neq \langle 1 \rangle$$

and

$$e(R/I) = e(R/N_{y,I}) \quad \text{if } C_{y,I} = \langle 1 \rangle.$$

Proof. By Theorem 2.4, if the decomposition is nondegenerate, we have

$$h_{R/I}(t) = h_{R/N_{y,I}}(t) + t h_{R/C_{y,I}}(t).$$

Evaluating at $t = 1$ now gives the result. The result in the degenerate case again follows from $R/I \cong R/(N_{y,I} + \langle y \rangle)$ if $C_{y,I} = \langle 1 \rangle$, and $I = \operatorname{in}_y(I) = C_{y,I} = N_{y,I}$ (by [Klein and Rajchgot 2021, Proposition 2.4]) if $C_{y,I} \neq \langle 1 \rangle$. □

The following corollary is now immediate. Its proof is essentially the same as the proof of Corollary 3.3, so we omit it.

Corollary 4.3. *Theorem 1.1, part* (2) *is true.*

Example 4.4. (1) Referring to Example 3.7,

$$I = \langle y(zs - x^2),\ ywr,\ wr(z^2 + zx + wr + s^2) \rangle$$

is geometrically vertex decomposable with

$$C_{y,I} = \langle zs - x^2,\ wr \rangle \quad \text{and} \quad N_{y,I} = \langle wr(z^2 + zx + wr + s^2) \rangle.$$

Hence, $e(R/I) = e(R/N_{y,I}) + e(R/C_{y,I}) = 4 + 4 = 8$.

(2) Referring to Example 3.11, $I = \langle yz - xw, xy \rangle$ has a nondegenerate geometric vertex decomposition with $C_{x,I} = \langle y, w \rangle$ and $N_{x,I} = \langle y^2 z \rangle$. Since I and N are Cohen–Macaulay, the argument as in Remark 3.10 applies. Hence, $e(R/I) = e(R/N_{x,I}) + e(R/C_{x,I}) = 3 + 1 = 4$.

5. The a-invariant and the Hilbertian property

In this section, we study the a-invariant, as well as the related *Hilbertian property*, of a geometrically vertex decomposable ideal. As in Theorems 3.2 and 4.2, our results rely on relating the h-polynomial of I to the h-polynomials of $C_{y,I}$ and $N_{y,I}$.

We begin by recalling the definition of the a-invariant.

Definition 5.1. Let M be an R-module with Hilbert series $H_M(t) = h_M(t)/(1-t)^d$, where $d = \dim M$. Then the *a-invariant* of M is $a(M) = \deg h_M(t) - d$, that is, the degree of $H_M(t)$ as a rational function.

Among other things, the a-invariant equals the degree of the largest nonzero graded piece of the local cohomology module $H^{\dim M}_{\mathfrak{m}}(M)$, i.e.,

$$a(M) = \max\{t \mid \dim_k(H^{\dim M}_{\mathfrak{m}}(M))_t \neq 0\}.$$

If M is Cohen–Macaulay, it is well known that $a(M) = \operatorname{reg}(M) + \dim(M)$. In addition, for Cohen–Macaulay ideals, the a-invariant can also be used to characterize the following property.

Definition 5.2. A homogeneous ideal $I \subseteq R$ is *Hilbertian* if $HP_{R/I}(t) = HF_{R/I}(t)$ for all $t \geq 0$. The ideal I is *almost Hilbertian* if $HP_{R/I}(t) = HF_{R/I}(t)$ for all $t \geq 1$.

By using Serre's formula for $HF_{R/I}(t) - HP_{R/I}(t)$ (see, for instance, [Bruns and Herzog 1993, Theorem 4.4.3]), when R/I is Cohen–Macaulay,

$$HF_{R/I}(t) - HP_{R/I}(t) = (-1)^{\dim R/I} \dim_k(H^{\dim R/I}_{\mathfrak{m}}(R/I))_t \quad \text{for all } t \in \mathbb{Z}.$$

Consequently, the Hilbertian property can be defined by means of the a-invariant as follows.

Lemma 5.3. *A homogeneous, Cohen–Macaulay ideal $I \subseteq R$ is Hilbertian if and only if $a(R/I) < 0$. The ideal I is almost Hilbertian if $a(R/I) \leq 0$.*

We now have a result similar to Theorems 3.2 and 4.2.

Theorem 5.4. *Suppose that $I \subseteq R$ is a homogeneous and radical ideal such that R/I is Cohen–Macaulay. Suppose that y is a variable such that $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a geometric vertex decomposition. If the decomposition is nondegenerate and $R/C_{y,I}$ and $R/N_{y,I}$ are Cohen–Macaulay, then*

$$a(R/I) = \max\{a(R/N_{y,I}) + 1, a(R/C_{y,I}) + 1\}.$$

Otherwise, if the decomposition is degenerate, we have

$$a(R/I) = a(R/N_{y,I}) = a(R/C_{y,I}) \quad \text{if } C_{y,I} \neq \langle 1 \rangle$$

and

$$a(R/I) = a(R/N_{y,I}) \quad \text{if } C_{y,I} = \langle 1 \rangle.$$

Proof. Recall that, by Lemma 3.1, for any homogeneous ideal J such that R/J is Cohen–Macaulay, we have $\operatorname{reg}(R/J) = a(R/J) + \dim(R/J)$. So, if the geometric vertex decomposition is nondegenerate, then when applying Theorem 3.2 we have

$$\begin{aligned} &a(R/I) + \dim(R/I) \\ &\qquad = \max\{a(R/N_{y,I}) + \dim(R/N_{y,I}),\ a(R/C_{y,I}) + \dim(R/C_{y,I}) + 1\}. \end{aligned}$$

We have $\dim(R/I) = \dim(R/C_{y,I}) = \dim(R/N_{y,I}) - 1$, as shown in the proof of Theorem 3.2. The conclusion now follows.

The degenerate case uses the same reasoning as in the proof of Theorem 3.2. □

Remark 5.5. In Theorem 5.4, we are viewing $C_{y,I}$ and $N_{y,I}$ as ideals of R. But by Definition 2.2, we can also view these ideals as ideals of $R' = R/\langle y \rangle$. Since no generator of $N_{y,I}$ is divisible by y, we have $\dim(R/N_{y,I}) - 1 = \dim(R'/N_{y,I})$, and since no generator of $C_{y,I}$ is divisible by y, we have $\dim(R/C_{y,I}) - 1 = \dim(R'/C_{y,I})$. Theorem 5.4 thus implies that $a(R/I) = \max\{a(R'/N_{y,I}), a(R'/C_{y,I})\}$ if the geometric vertex decomposition $\operatorname{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is nondegenerate. If the decomposition is degenerate, then $a(R/I) = a(R'/N_{y,I}) - 1$.

Remark 5.6. There are situations where $a(R/I) = a(R/C_{y,I}) + 1$, and other situations where $a(R/I) = a(R/N_{y,I}) + 1$. Indeed, by Lemma 3.1, this is equivalent to saying that there are situations where the regularity of I is equal to $\operatorname{reg}(C_{y,I}) + 1$, and other situations where the regularity of I is equal to $\operatorname{reg}(N_{y,I})$ (see Theorem 3.2). We refer the readers to Example 6.15 for concrete examples using the results of Theorem 6.12 in Section 6.

Theorem 5.4 immediately implies Theorem 1.1 (3). The proof is omitted as it is essentially the same as the proof of Corollary 3.3.

Corollary 5.7. *Theorem 1.1* (3) *is true.*

Using Theorem 5.4 and Remark 5.5, we can easily prove that the a-invariant of a geometrically vertex decomposable ideal is always nonpositive. In particular, geometrically vertex decomposable ideals are almost Hilbertian.

Corollary 5.8. *Let $I \subset R$ be a proper homogeneous geometrically vertex decomposable ideal. Then $a(R/I) \leq 0$. In particular, I is almost Hilbertian.*

Proof. We induct on the number of variables in $R = \mathbb{K}[x_1, \ldots, x_n]$. If $n = 0$, the result is trivial. If $n = 1$, then the only proper homogeneous geometrically vertex decomposable ideals are $\langle x_1 \rangle$ and $\langle 0 \rangle$. Thus, the result holds.

More generally, consider a homogeneous geometrically vertex decomposable ideal $I \subset R = \mathbb{K}[x_1, \ldots, x_n]$, with $n \geq 2$. Then there is a variable $y = x_j$ and a geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. If this is a degenerate geometric vertex decomposition, then, by Remark 5.5 and the induction hypothesis, we have $a(R/I) = a(R'/N_{y,I}) - 1 < 0$. If $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a nondegenerate geometric vertex decomposition, then, by Remark 5.5, we have that $a(R/I) = \max\{a(R'/N_{y,I}), a(R'/C_{y,I})\}$. Hence the desired result follows from the induction hypothesis. □

Inspired by the definition of Hilbertian ideals, we next study geometrically vertex decomposable ideals for which the a-invariant is always negative. We single out the following class first, as the proof is straightforward.

Corollary 5.9. *Let $I \subset R$ be a proper homogeneous geometrically vertex decomposable ideal, and suppose that there is a minimal generating set of I that does not involve all the variables in R. Then $a(R/I) < 0$. In particular, I is Hilbertian.*

Proof. Suppose that there is some minimal generating set of I which does not involve the variable $y = x_i$. Then the geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is degenerate. Thus, by Remark 5.5 and Corollary 5.8, $a(R/I) = a(R'/N_{y,I}) - 1 < 0$. □

We can use this corollary to recover the result of [Stelzer and Yong 2023] that Schubert determinantal ideals are Hilbertian.

Example 5.10. Given a permutation $w \in S_n$, there is an associated generalized determinantal ideal $I_w \in \mathbb{K}[x_{ij}, 1 \leq i, j \leq n]$ called a *Schubert determinantal ideal*. By construction, x_{nn} doesn't appear in any term of any minimal generator of I_w. Hence by Corollary 5.9, I_w is Hilbertian.

We now consider a larger subclass of geometrically vertex decomposable ideals for which the a-invariant is always negative.

Definition 5.11. A proper homogeneous ideal I of $R = \mathbb{K}[x_1, \ldots, x_n]$ is *C-saturated geometrically vertex decomposable* if I is saturated and unmixed and

(1) I is generated by a (possibly empty) subset of variables of R, or

(2) there exists a variable $y = x_j$ of R and a y-compatible monomial order such that we have a degenerate geometric vertex decomposition

$$\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

where $N_{y,I}$ is C-saturated geometrically vertex decomposable, or

(3) there exists a variable $y = x_j$ of R and a y-compatible monomial order such that we have a nondegenerate geometric vertex decomposition

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

and the contractions of the ideals $C_{y,I}$ and $N_{y,I}$ to the ring $\mathbb{K}[x_1, \ldots, \hat{x}_j, \ldots, x_n]$ are C-saturated geometrically vertex decomposable.

Remark 5.12. Let I be a proper, saturated, homogeneous geometrically vertex decomposable ideal. We will now check that I is C-saturated geometrically vertex decomposable if and only if there is some geometric vertex decomposition process of I in which every $C_{y,I}$ ideal that appears in this decomposition process is *not* an irrelevant ideal (contracted to its appropriate polynomial ring). The forward direction is immediate by Definition 5.11.

Conversely, suppose that there is some decomposition process for I in which no ideal of the form $C_{y,I}$ is the irrelevant ideal (in its appropriate polynomial ring). To verify that I is C-saturated geometrically vertex decomposable, it suffices to check that every ideal of the form $N_{y,I}$ that appears in the given decomposition process is saturated. Furthermore, since $N_{y,I}$ is geometrically vertex decomposable, and hence radical, this is equivalent to checking that each $N_{y,I}$ ideal is not the irrelevant ideal. So, consider a geometric vertex decomposition $\text{in}_{x_i}(J) = C_{x_i,J} \cap (N_{x_i,J} + \langle x_i \rangle)$ in the given decomposition process, where J is one of the ideals that appears in the decomposition of I. Since $C_{x_i,J}$ is not the irrelevant ideal by our assumption, and $N_{x_i,J} \subseteq C_{x_i,J}$, we have that $N_{x_i,J}$ is also not the irrelevant ideal.

Example 5.13. Consider the homogeneous and unmixed ideal $I = \langle yz, x + z \rangle \subseteq \mathbb{K}[y, x, z]$. The given generators are a Gröbner basis for the y-compatible monomial order Lex with $y > x > z$. We have the geometric vertex decomposition

$$\text{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle),$$

where $C_{y,I} = \langle z, x \rangle$ and $N_{y,I} = \langle x + z \rangle$. Observe that $C_{y,I}$ is the irrelevant ideal in $\mathbb{K}[x, z]$. Alternatively, there are geometric vertex decompositions

$$\text{in}_x(I) = C_{x,I} \cap (N_{x,I} + \langle x \rangle) \quad \text{or} \quad \text{in}_z(I) = C_{z,I} \cap (N_{z,I} + \langle z \rangle).$$

In each case, $C_{x,I}$ or $C_{z,I}$ are irrelevant ideals (in their respective polynomial rings $\mathbb{K}[y, z]$ and $\mathbb{K}[x, y]$). Hence, for each potential decomposition process of I, one encounters irrelevant ideals. Thus, I is not C-saturated geometrically vertex decomposable.

The next result shows that C-saturated geometrically vertex decomposable ideals are always Hilbertian.

Proposition 5.14. *Let I be a proper homogeneous geometrically vertex decomposable ideal in a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$, with $n \geq 1$. If I is C-saturated geometrically vertex decomposable, then $a(R/I) < 0$. In particular, I is Hilbertian.*

Proof. Our argument is nearly identical to the proof of Corollary 5.8. We induct on the number of variables in $R = \mathbb{K}[x_1, \ldots, x_n]$. If $n = 1$, the only C-saturated geometrically vertex decomposable ideal is $\langle 0 \rangle$, and the result holds.

More generally, consider a homogeneous, C-saturated geometrically vertex decomposable ideal $I \subset R = \mathbb{K}[x_1, \ldots, x_n]$, with $n \geq 2$. Then there is a variable $y = x_j$ and a geometric vertex decomposition $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$. If this is a degenerate geometric vertex decomposition, then, by Remark 5.5 and the induction hypothesis, we have $a(R/I) = a(R'/N_{y,I}) - 1 < 0$ as $N_{y,I} \subseteq R'$ is C-saturated geometrically vertex decomposable.

If $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$ is a nondegenerate C-saturated geometric vertex decomposition then, by Remark 5.5, we have

$$a(R/I) = \max(a(R'/N_{y,I}), a(R'/C_{y,I})).$$

Hence the desired result follows from the induction hypothesis. □

Remark 5.15. A Stanley–Reisner ideal of a vertex decomposable simplicial complex is Hilbertian if and only if, in its vertex decomposition process, at every step (each step corresponds to removing one vertex from the vertex set) except the last one (where all simplicial complexes are at most one point), taking the link gives all nonempty simplicial complexes. Note that a connected vertex decomposable simplicial complex can have links that are $\{\varnothing\}$ in its vertex decomposition process. For example, consider the graph C_3, the three cycle. The link of any vertex is a simplicial complex consisting of two disconnected points, and the link of this simplicial complex is the complex $\{\varnothing\}$. Its Stanley–Reisner ideal $I = \langle xyz \rangle$ is vertex decomposable, but R/I is not Hilbertian.

6. Applications to toric ideals of graphs

Here we apply Theorems 3.2, 4.2 and 5.4 to study the invariants of toric ideals of (bipartite) graphs. By leveraging the result that the toric ideals of bipartite graphs are geometrically vertex decomposable (see [Cummings et al. 2023, Theorem 5.8]), we can give new proofs for a number of known results (e.g., [Almousa et al. 2022; Corso and Nagel 2009; Favacchio et al. 2021]) using our techniques.

6.1. ***Background on toric ideals of graphs.*** We begin with the relevant background on toric ideals of graphs. Let $G = (V, E)$ be a finite simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{e_1, \ldots, e_q\}$. If we need to highlight the graph, we sometimes write $V(G)$ and $E(G)$ for the vertices and edges of G. Abusing notation, we let x_i and e_j also denote variables, and let $\mathbb{K}[E] = \mathbb{K}[e_1, \ldots, e_q]$ and $\mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n]$. We define a $\mathbb{K}$-algebra homomorphism $\varphi : \mathbb{K}[E] \to \mathbb{K}[V]$ by $\varphi(e_i) = x_j x_k$, where $e_i = \{x_j, x_k\} \in E$. The kernel of φ, denoted I_G, is the *toric ideal of* G.

The ideal I_G is a toric ideal because it is a prime binomial ideal; for this fact and for more details about I_G, see [Herzog et al. 2018, Chapter 5] or [Villarreal 2015, Chapter 10]. A (nonminimal) set of generators of I_G can be described in terms of closed even walks of the graph. A sequence of distinct edges $\Gamma = (e_{i_1}, e_{i_2}, \ldots, e_{i_t})$ is a *walk* if $e_{i_j} \cap e_{i_{j+1}} \neq \varnothing$ for $1 \leq j \leq t-1$. The walk is *closed* if $e_{i_t} \cap e_{i_1} \neq \varnothing$. The walk is *even* if t is even. A closed walk Γ is a *cycle* if no edges in Γ are repeated. We can associate with every closed even walk $\Gamma = (e_{i_1}, \ldots, e_{i_{2m}})$ a binomial of the form

$$f_\Gamma = e_{i_1} e_{i_3} \cdots e_{i_{2m-1}} - e_{i_2} e_{i_4} \cdots e_{i_{2m}}.$$

Recall that a graph G is *bipartite* if the vertex set V can be partitioned into two disjoint sets, $V = V_1 \cup V_2$, such that every edge $e \in E$ satisfies $e \cap V_1 \neq \varnothing$ and $e \cap V_2 \neq \varnothing$. The next result now gives a set of generators for toric ideals of (bipartite) graphs. For integers $m, n \geq 1$, the *complete bipartite graph* $K_{m,n}$ is defined to be the graph with vertex set $V = \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$ and edge set $E = \{\{x_i, y_j\} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$.

Theorem 6.1 [Villarreal 1995, Proposition 3.1]. *If G is a finite simple graph with toric ideal I_G, then*

$$I_G = \langle f_\Gamma \mid \Gamma \text{ is a closed even walk of } G \rangle.$$

In addition, if G is bipartite, then $I_G = \langle f_\Gamma \mid \Gamma$ is a even cycle of $G\rangle$.

A binomial $f = u - v \in I_G$ is called *primitive* if there is no other binomial $g = u' - v'$ in I_G such that $u' | u$ and $v' | v$. A closed even walk (or cycle) Γ is a *primitive walk* (*or cycle*) if the corresponding binomial f_Γ is a primitive binomial. We have the following refinement of the previous result.

Theorem 6.2 [Ene and Herzog 2012, Proposition 5.19]. *If G is a finite simple graph with toric ideal I_G, then $\{f_\Gamma \mid \Gamma$ is a primitive walk$\}$ forms a universal Gröbner basis for I_G, and, in particular, forms a set of generators of I_G.*

For bipartite graphs, computing the a-invariant of $\mathbb{K}[E]/I_G$ is equivalent to computing the regularity of R/I_G.

Lemma 6.3. *Let $G = (V, E)$ be a finite simple bipartite graph with toric ideal I_G. Then*

$$a(\mathbb{K}[E]/I_G) = \operatorname{reg}(\mathbb{K}[E]/I_G) - (|V| - 1).$$

Proof. If G is bipartite, then $\mathbb{K}[E]/I_G$ is Cohen–Macaulay (see [Herzog et al. 2018, Corollary 5.26]). By Lemma 3.1, we have $\operatorname{reg}(\mathbb{K}[E]/I_G) = \deg h_{\mathbb{K}[E]/I_G}(t)$. By [Villarreal 2015, Corollary 10.1.21], $\dim(\mathbb{K}[E]/I_G) = |V| - 1$ when G is bipartite. Thus

$$a(\mathbb{K}[E]/I_G) = \deg h_{\mathbb{K}[E]/I_G}(t) - \dim(\mathbb{K}[E]/I_G) = \operatorname{reg}(\mathbb{K}[E]/I_G) - (|V| - 1),$$

as desired. □

6.2. ***Toric ideals of bipartite graphs.*** As our first application, we will show how Theorem 3.2 can be used to give a new proof for a result of [Almousa et al. 2022] about the regularity of bipartite graphs and their subgraphs.

We first recall some more relevant graph theory. Given a graph $G = (V, E)$, we say $H = (W, F)$ is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$. In the special case $H = (V, E \setminus \{e\})$ for some edge e, we write $G \setminus \{e\}$ to denote the graph G with the edge e removed. The *degree* of a vertex x is given by $\deg(x) = |\{y \in V \mid \{x, y\} \in E\}|$. An edge $e = \{x, y\} \in E$ is a *leaf* if $\deg(x) = 1$ or $\deg(y) = 1$, and a vertex x is *isolated* if $\deg(x) = 0$. If x is an isolated vertex of G, then it can be shown (e.g., see [Cummings et al. 2023, Lemma 3.2]) that $I_G = I_{G'}$, where $G' = (V \setminus \{x\}, E)$. Similarly, if e is a leaf of G, then $I_G = I_{G'}$, where $G' = G \setminus \{e\}$.

The next lemma applies to all toric ideals of graphs, not just bipartite graphs.

Lemma 6.4 [Cummings et al. 2023, Lemma 3.5]. *Let G be a finite simple graph with toric ideal I_G. If $<$ is any y-compatible monomial order with $y = e$ for some edge $e \in E$ of G, then $N_{y,I_G} = I_{G\setminus\{e\}}$.*

Now suppose that $\mathcal{G}$ is a family of graphs such that, for every $G \in \mathcal{G}$, the toric ideal I_G is geometrically vertex decomposable, and, for all $e \in E$, there is an e-compatible monomial order such that there is a geometric vertex decomposition with respect to e. Furthermore, suppose that $G \setminus \{e\} \in \mathcal{G}$ for any edge e of G. For such a family, we have the following result.

Theorem 6.5. *Let G be any graph in the family $\mathcal{G}$ given above. Then, for any subgraph H of G, we have*

(1) $\operatorname{reg}(I_H) \leq \operatorname{reg}(I_G)$,

(2) $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G)$, *and*

(3) $e(\mathbb{K}[E(G)]/I_H) \leq e(\mathbb{K}[E(G)]/I_G)$.

Proof. For any edge e, there is an e-compatible monomial order such that I_G has the decomposition $\operatorname{in}_e(I_G) = (N_{e,I_G} + \langle e\rangle) \cap C_{e,I_G}$, and, by Lemma 6.4, we can write the decomposition as

$$\operatorname{in}_e(I_G) = (I_{G\setminus\{e\}} + \langle e\rangle) \cap C_{e,I_G}.$$

As $G \setminus \{e\} \in \mathcal{G}$, $I_{G\setminus\{e\}}$ is geometrically vertex decomposable, and hence is Cohen–Macaulay and radical. Because I_G is geometrically vertex decomposable, we have $\operatorname{reg}(I_{G\setminus\{e\}}) \leq \operatorname{reg}(I_G)$ by Theorem 3.2 and Remark 3.10. Similarly, Theorem 5.4 gives

$$a(\mathbb{K}[E(G)]/I_{G\setminus\{e\}}) < a(\mathbb{K}[E(G)]/I_{G\setminus\{e\}}) + 1 \leq a(\mathbb{K}[E(G)]/I_G).$$

Moreover, by Theorem 4.2, we have $e(\mathbb{K}[E(G)]/I_{G\setminus\{e\}}) \leq e(\mathbb{K}[E(G)]/I_G)$, since multiplicity is a nonnegative integer.

Since $G \setminus \{e\}$ is a graph in $\mathcal{G}$, $I_{G\setminus\{e\}}$ again has a geometric vertex decomposition with respect to any edge f. Since any subgraph H of G can be obtained by removing edges and vertices, by repeating this argument (and possibly removing leaves and isolated vertices when needed), we get the desired conclusion. □

Remark 6.6. Note that we showed that $a(\mathbb{K}[E]/I_{G\setminus\{e\}}) \leq a(\mathbb{K}[E]/I_G) - 1$, when we remove the edge e from G. If $t = |E(G)| - |E(H)|$, that is, the number of edges we remove from H to form G, we actually have the stronger result $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G) - t$.

We can recover [Almousa et al. 2022, Theorem 6.11], which was proved using combinatorial techniques involving root polytopes. We also derive results about the a-invariant and multiplicity.

Theorem 6.7. *Let H be any subgraph of a bipartite graph G. Then*

(1) $\operatorname{reg}(I_H) \leq \operatorname{reg}(I_G)$,

(2) $a(\mathbb{K}[E(G)]/I_H) \leq a(\mathbb{K}[E(G)]/I_G)$, *and*

(3) $e(\mathbb{K}[E(G)]/I_H) \leq e(\mathbb{K}[E(G)]/I_G)$.

Proof. Let $\mathcal{G}$ be the family of bipartite graphs. By [Cummings et al. 2023, Theorem 5.8], for all $G \in \mathcal{G}$, the toric ideal I_G is geometrically vertex decomposable. It also follows that, for any $G \in \mathcal{G}$, we have $G \setminus \{e\} \in \mathcal{G}$, since removing edges does not destroy the bipartite property.

In addition, by [Cummings et al. 2023, Proposition 5.4], there is a geometric vertex decomposition

$$\operatorname{in}_e(I_G) = (I_{G\setminus\{e\}} + \langle e \rangle) \cap I_e^G,$$

where $\{e\}$ is a path-ordered matching of G and $I_e^G = I_{G\setminus\{e\}} + \langle M_e^G \rangle$, and

$$M_e^G = \{m \mid me - n \text{ is a binomial that corresponds to a cycle in } G\}.$$

Since any edge of a bipartite graph can be regarded as a path-ordered matching, the above geometric vertex decomposition holds for the toric ideal of any bipartite graph and any edge. Hence, Theorem 6.5 applies to any subgraph of H of G. □

Remark 6.8. Theorem 6.7 (1) gives another proof of [Almousa et al. 2022, Theorem 6.11] using the geometric vertex decomposability property of toric ideals of bipartite graphs. Villarreal pointed out to the third author that regularity could also be deduced from the edge polytope of G and Stanley's monotonicity property [1993, Theorem 3.3]. In particular, one can use the strategy given just after [Almousa et al. 2022, Question 6.12]. Another proof of Theorem 6.7 (1) can be found in the recent paper of Pinto and Villarreal [2023, Corollary 8.17] using normal monomial ideals. Additionally, there is a similar result to Theorem 6.7 (1) for the regularity of *induced* graphs that can be found in [Hà et al. 2019, Theorem 3.6].

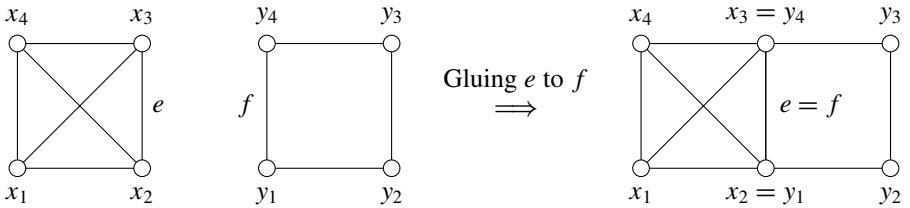

Figure 1. Two graphs G and C_4 glued along edges e and f.

6.3. ***Regularity and gluing cycles.*** We can use Theorem 3.2 to give a different proof for [Favacchio et al. 2021, Corollary 3.11] which describes how regularity behaves with respect to a "gluing" operation on graphs.

We first recall the notion of gluing a cycle to a graph along an edge, following [Favacchio et al. 2021, Construction 4.1]. A *cycle of length m* is the graph with vertex set $\{x_1, \ldots, x_m\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}, \{x_m, x_1\}\}$; we denote this graph by C_m. Let $G = (V, E)$ be any graph. Fix an edge $e \in E$ and an edge f of C_m. The graph H obtained from G by *gluing a cycle of length m* along an edge is the graph $G \bigcup_{e=f} C_m$, where we identify the edges and vertices of e and f. An example of gluing is given in Figure 1. When C_m has even length, the regularity of I_H, the toric ideal of the glued graph H, is related to that of I_G if I_G is geometrically vertex decomposable.

Theorem 6.9. *Suppose that G is a graph such that I_G is geometrically vertex decomposable in $\mathbb{K}[E(G)]$. Let H be the graph obtained from G by gluing a cycle of length $2d$ ($d \geq 2$) along an edge of G. Then*

$$\operatorname{reg}(\mathbb{K}[E(H)]/I_H) = \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-1).$$

Proof. Let $E(G) = \{e_1, \ldots, e_q\}$ denote the edges of G, and let $E(C) = \{f_1, \ldots, f_{2d}\}$ denote the edges of the cycle $C = C_{2d}$. We assume that the cycle is glued to G along f_{2d} and any edge of G. By [Cummings et al. 2023, Theorem 3.11] and its proof, I_H is geometrically vertex decomposable and, moreover, the geometric decomposition is given by

$$N_{y,I_H} = I_G \quad \text{and} \quad C_{y,I_H} = I_G + \langle f_3 f_5 \cdots f_{2d-1} \rangle,$$

where $y = f_1$, and some y-compatible monomial order. Since $I_G \subset \mathbb{K}[E(G)]$, $\sqrt{C_{y,I_H}} \neq \sqrt{N_{y,I_H}}$, and $C_{y,I_H} \neq \langle 1 \rangle$, the decomposition is nondegenerate.

Note that $\mathbb{K}[E(H)] = \mathbb{K}[E(G)] \otimes T$, where $T = \mathbb{K}[f_1, \ldots, f_{2d-1}]$. So

$$\begin{aligned}\operatorname{reg}(\mathbb{K}[E(H)]/(I_G + \langle f_3 f_5 \cdots f_{2d-1} \rangle)) &= \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + \operatorname{reg}(T/\langle f_3 f_5 \cdots f_{2d-1} \rangle) \\ &= \operatorname{reg}(\mathbb{K}[E(G)]/I_G) + (d-2).\end{aligned}$$

So, by Theorem 3.2, we have

$$\begin{aligned}\operatorname{reg}(\mathbb{K}[E(H)]/I_H) &= \max\{\operatorname{reg}(\mathbb{K}[E(G)]/I_G),\ \operatorname{reg}(\mathbb{K}[E(G)]/I_G)+(d-2)+1\}\\ &= \operatorname{reg}(\mathbb{K}[E(G)]/I_G)+(d-1),\end{aligned}$$

thus completing the proof. □

We can now derive the following corollary for gluing even cycles to graphs G such that I_G is geometrically vertex decomposable.

Corollary 6.10. *Suppose that G is a graph such that I_G is geometrically vertex decomposable in $\mathbb{K}[E(G)]$. Let H be the graph obtained from G by gluing a cycle of length $2d$ $(d \geq 2)$ along an edge of G. Then*

(1) $a(\mathbb{K}[E(H)]/I_H) = a(\mathbb{K}[E(G)]/I_G) - (d-1)$,

(2) $e(\mathbb{K}[E(H)]/I_H) = d \cdot e(\mathbb{K}[E(G)]/I_G)$.

Proof. Note that $|V(H)| = |V(G)| + (2d-1)$. By Lemma 6.3 and Theorem 6.9, we have

$$\begin{aligned}a(\mathbb{K}[E(H)]/I_H) &= \operatorname{reg}(\mathbb{K}[E(H)]/I_H) - (|V(H)|-1)\\ &= (\operatorname{reg}(\mathbb{K}[E(G)]/I_G)+(d-1)) - ((|V(G)|+(2d-2))-1)\\ &= a(\mathbb{K}[E(G)]/I_G) - (d-1),\end{aligned}$$

and hence part (1) follows.

For part (2), by Theorem 4.2,

$$e(\mathbb{K}[E(H)]/I_H) = e(\mathbb{K}[E(H)]/I_G) + e(\mathbb{K}[E(H)]/(I_G + \langle f_3 f_5 \cdots f_{2d-1}\rangle)).$$

Since $e(\mathbb{K}[E(H)]/I_G) = e(\mathbb{K}[E(G)]/I_G)$ and

$$\begin{aligned}&e(\mathbb{K}[E(H)]/(I_G + \langle f_3 f_5 \cdots f_{2d-1}\rangle))\\ &\qquad = e(\mathbb{K}[E(G)]/I_G) \cdot e(\mathbb{K}[f_1, f_2, \ldots, f_{2d-1}]/\langle f_3 f_5 \cdots f_{2d-1}\rangle)\\ &\qquad = (d-1)e(\mathbb{K}[E(G)]/I_G),\end{aligned}$$

it follows that $e(\mathbb{K}[E(H)]/I_H) = d \cdot e(\mathbb{K}[E(G)]/I_G)$, as desired. □

6.4. ***Regularity of toric ideals of Ferrers graphs.*** Recall that a *Ferrers graph* is a bipartite graph on the vertex set $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = \{u_1, u_2, \ldots, u_m\}$ such that $\{v_1, u_m\}$ and $\{v_n, u_1\}$ are edges, and if $\{v_i, u_j\}$ is an edge, then so are all the edges $\{v_k, u_l\}$ with $1 \leq k \leq i$ and $1 \leq l \leq j$. We also associate a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, to a Ferrers graph where $\lambda_i = \deg v_i$, and we denote the Ferrers graph by T_λ. See Figure 2 on page 384 for an example. We show how to use Theorem 3.2 to give a different proof to a result of Corso and Nagel [2009].

We first require a lemma; in the statement below, a bipartite graph is a *chordal bipartite* graph if every cycle of length ≥ 6 has a chord, that is, an edge that joins two nonconsecutive vertices of the cycle.

Lemma 6.11. *Suppose that T_λ is a Ferrers graph. Then T_λ is a chordal bipartite graph. Consequently, the toric ideal I_{T_λ} is generated by quadratics.*

Proof. Let $X = \{v_1, \ldots, v_n\}$ and $Y = \{u_1, \ldots, u_m\}$ be the partition of the vertices of T_λ. Suppose that $(v_{i_1}, u_{j_1}, v_{i_2}, u_{j_2}, \ldots, v_{i_s}, u_{j_s})$ are the vertices of a cycle of length $2s \geq 6$ in T_λ.

The three edges $\{v_{i_1}, u_{j_1}\}$, $\{v_{i_1}, u_{j_s}\}$, and $\{v_{i_s}, u_{j_s}\}$ appear in this cycle. Consider the indices of the two u vertices. If $j_1 < j_s$, then by the definition of T_λ, the edge $\{v_{i_s}, u_{j_1}\}$ is also an edge of T_λ. So the cycle has a chord. If $j_s < j_1$, note that $\{v_{i_2}, u_{j_1}\}$ is the next edge in the cycle. Since $j_s < j_1$, the edge $\{v_{i_2}, u_{j_s}\}$ is also an edge of T_λ. But then $(v_{i_1}, u_{j_s}, v_{i_2}, u_{j_1})$ is a four cycle of T_λ, that is, $\{v_{i_2}, u_{j_s}\}$ is a chord.

The final statement follows from the main result of [Ohsugi and Hibi 1999], which showed that the toric ideals of all chordal bipartite graphs are generated by quadratics. □

Theorem 6.12 [Corso and Nagel 2009, Proposition 5.7]. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a partition, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let T_λ be the associated Ferrers graph. Write $I_\lambda = I_{T_\lambda}$ and $R = \mathbb{K}[E(T_\lambda)]$.*

(1) *If $n = 1$ or $\lambda_2 = 1$, then $\operatorname{reg}(R/I_\lambda) = 0$.*

(2) *If $\lambda_2 \geq 2$, and suppose that $\lambda = (\lambda_1, \ldots, \lambda_s, 1, 1, \ldots, 1)$ where $\lambda_s \geq 2$, then*

$$\operatorname{reg}(R/I_\lambda) = \min\{s-1, \{\lambda_j + j - 3 \mid 2 \leq j \leq s\}\}.$$

Proof. If $n = 1$, we have $I_\lambda = \langle 0 \rangle$; hence $\operatorname{reg}(R/I_\lambda) = 0$. When $\lambda_2 = 1$, all the edges $\{v_i, u_1\}$ with $i \geq 2$ are leaves; hence, after removing them, we have $I_\lambda = \langle 0 \rangle$.

Now suppose that $\lambda_2 \geq 2$. Write $e_{ij} = \{v_i, u_j\}$, and let $e = e_{n\lambda_n} = \{v_n, u_{\lambda_n}\}$. Let $G = T_\lambda$ and $I_G = I_\lambda$. By [Cummings et al. 2023, Theorem 5.8], I_G is geometrically vertex decomposable since G is bipartite. Moreover, by [Cummings et al. 2023, Proposition 5.4], there is a geometric vertex decomposition

$$\operatorname{in}_e(I_G) = (I_{G\setminus\{e\}} + \langle e \rangle) \cap I_e^G,$$

where $I_e^G = I_{G\setminus\{e\}} + \langle M_e^G \rangle$ and $M_e^G = \{m_1 \mid m_1 e - m_2$ corresponds to a cycle in $G\}$ (the description of M_e^G appears directly after [Cummings et al. 2023, Lemma 5.1]).

We claim that $\langle M_e^G \rangle = \langle \{e_i \mid \{e, e_j, e_i, e_k\}$ is a cycle in $G\} \rangle$. Note that it suffices to verify that $\langle M_e^G \rangle \subseteq \langle \{e_i \mid \{e, e_j, e_i, e_k\}$ is a cycle in $G\} \rangle$, since the reverse containment is immediate. Suppose that the binomial $m_1 e - m_2$ corresponds to a cycle

of G. Since the cycle contains the edge $e = \{v_n, u_{\lambda_n}\}$, we can write this cycle as

$$(v_{i_1}, u_{j_1}, v_{i_2}, u_{j_2}, \ldots, v_{i_{s-1}}, u_{i_{s-1}}, v_n, u_{\lambda_n}),$$

where $i_s = n$ and $j_s = \lambda_n$. Furthermore, we denote the edges in the cycle as follows: $e_k = \{v_{i_k}, u_{j_k}\}$ for $k = 1, \ldots, s$ and $f_k = \{u_{j_k}, v_{i_{k+1}}\}$ for $k = 1, \ldots, s$, where $v_{i_{s+1}} = v_{i_1}$. Note that $e = e_{i_s}$, and we have $m_1 e - m_2 = e_1 e_2 \cdots e_{s-1} e - f_1 f_2 \cdots f_s$ with this notation. In this cycle, we now consider the three consecutive edges $e_{s-1} = \{v_{i_{s-1}}, u_{i_{s-1}}\}$, $f_{s-1} = \{u_{i_{s-1}}, v_n\}$, and $e = \{v_n, u_{\lambda_n}\}$. Since T_λ is a Ferrers graph, and since $i_{s-1} < n$, the edge $f = \{v_{i_{s-1}}, u_{\lambda_n}\}$ also belongs to T_λ. This gives a four cycle (e_{s-1}, f_{s-1}, e, f), and thus $e_{s-1}e - f_{s-1}f$ is a binomial that corresponds to a cycle of G. But this means that $e_{s-1} \in M_e^G$, and since e_{s-1} divides $m_1 = e_1 \cdots e_{s-1}$, we have that m_1 is in the ideal on the right-hand side.

Note that the graph $G \setminus \{e\}$ is the Ferrers graph on the same vertex set associated to the partition $\lambda'' = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n - 1)$. Hence, $N_{e,I_G} = I_{G\setminus\{e\}} = I_{\lambda''}$. On the other hand,

$$\langle M_e^G \rangle = \langle e_{ij} \mid 1 \le i \le n-1,\ 1 \le j \le \lambda_n - 1 \rangle.$$

Since $G\setminus\{e\}$ is a Ferrers graph, the generators of $I_{G\setminus\{e\}}$ are quadratics by Lemma 6.11. Moreover these generators are of the form $f = e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_1} e_{i_1 j_2}$, with $i_1 < i_2$ and $j_1 < j_2$. If $i_2 \le n-1$ and $j_2 \le \lambda_n - 1$, then we have $f \in \langle M_e^G \rangle$. If $i_2 = n$, then $j_1 < j_2 \le \lambda_n - 1$, and hence $f \in \langle M_e^G \rangle$. Therefore,

$$\begin{aligned} C_{e,I_G} &= I_{G\setminus\{e\}} + \langle M_e^G \rangle \\ &= \langle M_e^G \rangle + \langle e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_1} e_{i_1 j_2} \mid 1 \le i_1 < i_2 \le n-1,\ \lambda_n \le j_1 < j_2 \rangle. \end{aligned}$$

Note that the ideal $\langle e_{i_1 j_1} e_{i_2 j_2} - e_{i_2 j_1} e_{i_1 j_2} \mid 1 \le i_1 < i_2 \le n-1, \lambda_n \le j_1 < j_2 \rangle$ is the toric ideal of the Ferrers graph on the vertex set $X' = \{v_1, v_2, \ldots, v_{n-1}\}$ and $Y' = \{u_{\lambda_n}, u_2, \ldots, u_m\}$ associated to the partition

$$\lambda' = (\lambda_1 - \lambda_n + 1, \lambda_2 - \lambda_n + 1, \ldots, \lambda_{n-1} - \lambda_n + 1).$$

Thus, we can write $C_{e,I_G} = \langle M_e^G \rangle + I_{\lambda'}$, where the generators of the two ideals in the right-hand side are in separate sets of variables. Moreover, since $\langle M_e^G \rangle$ is generated by variables, $\operatorname{reg}(R/\langle M_e^G \rangle + I_{\lambda'}) = \operatorname{reg}(R/I'_\lambda)$. Thus, by Theorem 3.2,

$$\operatorname{reg}(R/I_\lambda) = \max\{\operatorname{reg}(R/I_{\lambda''}), \operatorname{reg}(R/I_{\lambda'}) + 1\},$$

where λ' and λ'' are defined as above.

We will now apply the above recursive formula to derive the formula of $\operatorname{reg}(R/I_\lambda)$ as claimed. First, if $\lambda = (\lambda_1, \ldots, \lambda_s, 1, 1, \ldots, 1)$ with $\lambda_s \ge 2$, then the edges $v_i u_1$ with $s+1 \le i \le n$ are all leaves; hence, we can remove them without changing the toric ideal. In other words, $\operatorname{reg}(I_\lambda) = \operatorname{reg}(I_{\tilde{\lambda}})$, where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_s)$. We will use

induction on n. Suppose $n = 2$ and $\lambda_1 \geq \lambda_2 \geq 2$. Then, by the recursive formula,

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1,\lambda_2)}) &= \max\{\operatorname{reg}(R/I_{(\lambda_1,\lambda_2-1)}),\ \operatorname{reg}(R/I_{(\lambda_1-\lambda_2+1)}) + 1\} \\ &= \max\{\operatorname{reg}(R/I_{(\lambda_1,\lambda_2-1)}),\ 1\}.\end{aligned}$$

By induction on λ_2, if $\lambda_2 = 2$, we have

$$\operatorname{reg}(R/I_{(\lambda_1,\lambda_2)}) = \max\{\operatorname{reg}(R/I_{(\lambda_1,1)}),\ 1\} = 1 = \min\{2-1,\ \lambda_2+2-3\},$$

and if $\lambda_2 > 2$, we have

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1,\lambda_2)}) &= \max\{\operatorname{reg}(R/I_{(\lambda_1,\lambda_2-1)}),\ 1\} \\ &= \max\{\min\{2-1,\ (\lambda_2-1)+2-3\}\},\ 1\} = 1 \\ &= \min\{2-1,\ \lambda_2+2-3\}.\end{aligned}$$

In both cases, the regularity agrees with the formula in our claim.

Now suppose that $n \geq 3$ and that the formula holds for any $k \leq n-1$ and for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. We will show that it holds for n and any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Now if $\lambda = (\lambda_1, \ldots, \lambda_s, 1, 1, \ldots, 1)$ with $\lambda_s \geq 2$, as shown above, we have $\operatorname{reg}(R/I_\lambda) = \operatorname{reg}(R/I_{\tilde{\lambda}})$, where $\tilde{\lambda} = (\lambda_1, \ldots, \lambda_s)$. Hence, if $s \leq n-1$, the result follows by the induction hypothesis. Also, if $\lambda_n = 1$, as $\operatorname{reg}(R/I_\lambda) = \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1})})$, the result again follows by the induction hypothesis. Thus, we can assume that $s = n$ and $\lambda_n \geq 2$. Then, by the recursive formula,

$$\operatorname{reg}(R/I_\lambda) = \max\{\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}),\ \operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) + 1\}.$$

Case 1: If $\lambda_n = 2$, then $\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}) = \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1})})$. So, by the induction hypothesis

$$\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}) = \min\{n-2,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq n-1\}\}.$$

Now if $\lambda_{n-1} \geq 3$, then, by the induction hypothesis,

$$\begin{aligned}&\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) + 1 \\ &\qquad = \min\{n-2,\ \{\lambda_j - \lambda_n + j - 2 \mid 2 \leq j \leq n-1\}\} + 1 \\ &\qquad = \min\{n-1,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq n-1\}\} \\ &\qquad \geq \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}).\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{reg}(R/I_\lambda) &= \operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)}) + 1 \\ &= \min\{n-1,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq n\}\},\end{aligned}$$

where the last equality holds since $\lambda_n + n - 3 = n - 1$.

Otherwise, if $\lambda_{n-1}=2$, assume that $\lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_{n-1}=2$ and $\lambda_k\geq 3$ for some $k\leq n-2$. Then $(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)=(\lambda_1-1,\ldots,\lambda_k-1,1,\ldots,1)$. Again, by the induction hypothesis,

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})+1 &= \min\{k-1,\ \{\lambda_j-1+j-3 \mid 2\leq j\leq k\}\}+1\\ &= \min\{k,\ \{\lambda_j+j-3 \mid 2\leq j\leq k\}\}\end{aligned}$$

if $k\geq 2$, or the regularity equals 0 if $k=1$. Moreover, for $k+1\leq j\leq n-1$, we have $k\leq \lambda_j+j-3\leq n-2$, and thus

$$\begin{aligned}\min\{k,\ \{\lambda_j+j-3 \mid 2\leq j\leq k\}\} &= \min\{k,\ \{\lambda_j+j-3 \mid 2\leq j\leq n-1\}\}\\ &\leq \min\{n-2,\ \{\lambda_j+j-3 \mid 2\leq j\leq n-1\}\}\\ &= \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}),\end{aligned}$$

where the last equality follows since $\lambda_j+j-3\leq n-2<n-1$ for $k+1\leq j\leq n-1$. Therefore,

$$\operatorname{reg}(R/I_\lambda)=\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)})=\min\{n-1,\ \{\lambda_j+j-3 \mid 2\leq j\leq n\}\},$$

where the last equality follows since $\lambda_n+n-3=n-1$.

Case 2: If $\lambda_n\geq 3$, by induction on λ_n, we have

$$\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)})=\min\{n-1,\ \{\lambda_j+j-3 \mid 2\leq j\leq n-1\},\ \lambda_n+n-4\}.$$

In addition, by the induction hypothesis (on n), we have

$$\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})=\min\{k-1,\ \{\lambda_j-\lambda_n+j-2 \mid 2\leq j\leq k\}\},$$

where $2\leq k\leq n-1$ is the maximum integer such that $\lambda_k\geq\lambda_n+1$ (hence, $\lambda_k-\lambda_n+1\geq 2$), or the regularity equals 0 if $k=1$. On the other hand, as $\lambda_j=\lambda_n$ for $j\geq k+1$, we have $\lambda_j-\lambda_n+j-2=j-2\geq k-1$ for all $j\geq k+1$. Thus,

$$\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})=\min\{k-1,\ \{\lambda_j-\lambda_n+j-2 \mid 2\leq j\leq n\}\}.$$

Therefore,

$$\begin{aligned}\operatorname{reg}(R/I_{(\lambda_1-\lambda_n+1,\ldots,\lambda_{n-1}-\lambda_n+1)})+1 &= \min\{k,\ \{\lambda_j-\lambda_n+j-1 \mid 2\leq j\leq n\}\}\\ &\leq \min\{n-1,\ \{\lambda_j+j-3 \mid 2\leq j\leq n\}\}\\ &= \operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)}),\end{aligned}$$

where the last equality holds since $\lambda_n+n-3>\lambda_n+n-4\geq n-1$. Therefore,

$$\operatorname{reg}(R/I_\lambda)=\operatorname{reg}(R/I_{(\lambda_1,\ldots,\lambda_{n-1},\lambda_n-1)})=\min\{n-1,\ \{\lambda_j+j-3 \mid 2\leq j\leq n\}\},$$

finishing our proof. □

Remark 6.13. If y is the variable that corresponds to the edge $e = \{v_n, \lambda_n\}$, by Theorem 2.3, we have

$$H_{R/I_\lambda}(t) = H_{R/(N_{y,I}+\langle y\rangle)}(t) + tH_{R/C_{y,I}}(t) = H_{R/I''_\lambda}(t) + tH_{R/I'_\lambda}(t).$$

This recovers [Corso and Nagel 2009, Lemma 5.3].

Remark 6.14. As shown in the proof of Theorem 6.12, we record the formulae for regularity of I_λ avoiding taking the maximum:

(1) If $\lambda_n = 2$ and $\lambda_{n-1} \geq 3$, then $\operatorname{reg}(I_\lambda) = \operatorname{reg}(C_{e,I_\lambda}) + 1 = \operatorname{reg}(I'_\lambda) + 1$.

(2) If $\lambda_{n-1} = \lambda_n = 2$, then $\operatorname{reg}(I_\lambda) = \operatorname{reg}(N_{e,I_\lambda}) = \operatorname{reg}(I''_\lambda)$.

(3) If $\lambda_n \geq 3$, then $\operatorname{reg}(I_\lambda) = \operatorname{reg}(N_{e,I_\lambda}) = \operatorname{reg}(I''_\lambda)$.

As mentioned in Remark 5.6, the following example shows that the a-invariant of a geometrically vertex decomposable ideal I can be either that of the C ideal or the N ideal of its decomposition, or equivalently, the regularity of I can be either $\operatorname{reg}(C) + 1$ or $\operatorname{reg}(N)$.

Example 6.15. • Consider the toric ideal $I = I_\lambda$ of the Ferrers graph associated to $\lambda = (3, 3, 3, 3)$. Then I has a nondegenerate geometric vertex decomposition with $N = I_{\lambda''}$ and $C = I_{\lambda'}$, where $\lambda'' = (3, 3, 3, 2)$ and $\lambda' = (1, 1, 1)$; see the proof of Theorem 6.12. Thus, by Theorem 6.12, $\operatorname{reg}(N) = 3$, $\operatorname{reg}(C) = 1$, and $\operatorname{reg}(I) = \operatorname{reg}(N) = 3 > \operatorname{reg}(C) + 1$. Note also that it is not hard to construct an example with a degenerate decomposition with $C = \langle 1\rangle$; in this case, we also have $\operatorname{reg}(I) = \operatorname{reg}(N) > \operatorname{reg}(C) + 1$.

• If $\lambda = (4, 4, 3, 2)$, we have $\lambda'' = (4, 4, 3, 1)$ and $\lambda' = (3, 3, 2)$. By Theorem 6.12, $\operatorname{reg}(N) = 3$, $\operatorname{reg}(C) = 3$, and $\operatorname{reg}(I) = \operatorname{reg}(C) + 1 = 4 > \operatorname{reg}(N)$.

• If $\lambda = (3, 3, 2, 2)$, we have $\lambda'' = (3, 3, 2, 1)$ and $\lambda' = (2, 2, 1)$. By Theorem 6.12, $\operatorname{reg}(N) = 3$, $\operatorname{reg}(C) = 2$, and $\operatorname{reg}(I) = \operatorname{reg}(C) + 1 = \operatorname{reg}(N) = 3$.

Remark 6.16. Two formulae for the regularity of R/I_λ are given in [Corso and Nagel 2009, Proposition 5.7]; however these two formulas to do not agree in general:

$$\min\{s-1,\ \{\lambda_j + j - 3 \mid 2 \leq j \leq s\}\} \neq \begin{cases} s-1 & \text{if } \lambda_s \geq 3, \\ \min\{j-1 \mid \lambda_j = 2\} & \text{if } \lambda_s = 2, \end{cases}$$

where s is the number of $\lambda_i \geq 2$. For example, let $s \geq 4$, $\lambda_1 = \cdots = \lambda_{s-1} = 3$ and $\lambda_s = 2$, then the left-hand side is 2, whereas the right-hand side is $s - 1 \geq 3$. Or if $\lambda_1 = \lambda_2 = \cdots = \lambda_s = 3$ and $s \geq 4$, then the left-hand side is 2 and the right-hand side is $s - 1 \geq 3$. This error occurs in [Corso and Nagel 2009] because the authors claim the equality between the two sides follows from the equality $\min\{\lambda_j + j - 3 \mid 2 \leq j \leq s\} = \lambda_s + s - 2$, which is not true in general. For a more specific example, consider the Ferrers graph T_λ for $\lambda = (3, 3, 3, 2)$ as given in Figure 2.

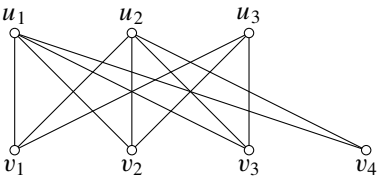

Figure 2. The graph T_λ for $\lambda = (3, 3, 3, 2)$.

Then the minimal graded free resolution of R/I_λ has Betti table

	0	1	2	3	4	5
total:	1	12	25	21	10	3
0:	1	-	-	-	-	-
1:	-	12	25	15	-	-
2:	-	-	-	6	10	3

In particular, $\operatorname{reg}(R/I_\lambda) = 2$.

Corollary 6.17. *With the same notation as in Theorem 6.12, we have*

(1) $a(R/I_\lambda) =$
$$\begin{cases} -(n+m-1) & \text{if } n = 1 \text{ or } \lambda_2 = 1, \\ -(n+m-1) + \min\{s-1, \{\lambda_j + j - 3 \mid 2 \le j \le s\}\} & \text{otherwise.} \end{cases}$$

(2) $$e(R/I_\lambda) = \sum_{j_{n-2}=\lambda_2-\lambda_n+1}^{\lambda_2} \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{j_{n-2}} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1.$$

Proof. Part (1) follows from Lemma 6.3 and Theorem 6.12, while part (2) follows by induction using Theorem 4.2, as

$$\begin{aligned} & e(R/I_\lambda) \\ & = e(R/I_{(\lambda_1,\dots,\lambda_{n-1},\lambda_n-1)}) + e(R/I_{(\lambda_1-\lambda_n+1,\dots,\lambda_{n-1}-\lambda_n+1)}) \\ & = \sum_{j_{n-2}=\lambda_2-\lambda_n}^{\lambda_2} \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{j_{n-2}} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1 + \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{\lambda_2-\lambda_n+1} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1 \\ & = \sum_{j_{n-2}=\lambda_2-\lambda_n+1}^{\lambda_2} \sum_{j_{n-3}=\lambda_2-\lambda_{n-1}+1}^{j_{n-2}} \cdots \sum_{j_1=\lambda_2-\lambda_3+1}^{j_2} j_1. \end{aligned}$$

□

Remark 6.18. Let $G = K_{n,m}$ be the complete bipartite graph. Then $I_G = I_\lambda$, where $\lambda = (m, m, m, \dots, m)$. Hence, $\operatorname{reg}(R/I_G) = \min\{n-1, m-1\} = \min\{n, m\} - 1$.

We recover the bounds for the regularity of bipartite graphs in [Almousa et al. 2022, Theorem 6.13] for connected bipartite graphs and [Biermann et al. 2017, Theorem 4.9] for chordal bipartite graphs, and the a-invariant of bipartite graphs in [Villarreal 2015, Proposition 11.5.1].

Corollary 6.19. *Let G be a connected bipartite graph with bipartition $V_1 \cup V_2$, with $|V_1| = n$ and $|V_2| = m$. Let $r = |\{v_i \in V_1 \mid \deg(v_i) = 1\}|$ and $s = |\{u_i \in V_2 \mid \deg(u_i) = 1\}|$. Then*

(1) $\operatorname{reg}(I_G) \leq \min\{n - r, m - s\}$, *and*

(2) $a(\mathbb{K}[E(G)]/I_G) \leq \min\{-m, -n\}$.

Proof. (1) Since r vertices in vertex set X and s vertices in vertex set Y belong to leaves, we can remove these vertices without changing the toric ideal I_G. Thus, we can assume that $r = s = 0$; that is G does not have any leaves. As G is a subgraph of $K_{n,m}$, by Theorem 6.7 and Remark 6.18, we have $\operatorname{reg}(I_G) \leq \operatorname{reg}(I_{K_{n,m}}) = \min\{n, m\}$ as desired.

(2) By Lemma 6.3, $a(\mathbb{K}[E(G)]/I_G) = \operatorname{reg}(\mathbb{K}[E(G)]/I_G) - (|V(G)| - 1)$. Because the graph is connected, $|V(G)| = m + n$. Also, by part (1), $\operatorname{reg}(\mathbb{K}[E(G)]/I_G) \leq \min\{n, m\} + 1$. The result now follows. □

We can now prove a very interesting property about toric ideals of bipartite graphs, which does not seem to have been observed before.

Theorem 6.20. *If G is a connected bipartite graph, then I_G is Hilbertian.*

Proof. As shown in Corollary 6.19 (2), the a-invariant of $\mathbb{K}[E(G)]/I_G$ is always negative; hence I_G is Hilbertian. □

6.5. ***Invariants for a class of bipartite graphs.*** In this section we apply Theorems 3.2, 5.4, and 4.2 to a family of toric ideals of bipartite graphs first studied in [Galetto et al. 2019] to further illustrate our techniques. The graphs studied in that paper were defined as follows.

Definition 6.21. Let d, r be integers such that $d \geq 1$ and $r \geq 3$. Let $G = G_{r,d}$ be the graph with $V(G) = \{x_1, x_2, y_1, \ldots, y_d, z_1, \ldots, z_{2r-3}\}$ and

$$E(G) = \{\{x_i, y_j\} \mid 1 \leq i \leq 2,\ 1 \leq j \leq d\} \cup \{\{x_1, z_1\}, \{z_1, z_2\}, \{z_2, z_3\}, \ldots, \{z_{2r-4}, z_{2r-3}\}, \{z_{2r-3}, x_2\}\}.$$

We label edges as follows: for $i = 1, \ldots, d$, let $a_i = \{x_1, y_i\}$ and $b_i = \{x_2, y_i\}$. Also, let $e_1 = \{x_1, z_1\}$, $e_{2r-2} = \{x_2, z_{2r-3}\}$, and let

$$e_2 = \{z_1, z_2\}, \quad e_3 = \{z_2, z_3\}, \quad \ldots, \quad e_{2r-3} = \{z_{2r-4}, z_{2r-3}\}.$$

Informally, the graph $G_{r,d}$ is constructed by starting with a complete bipartite graph $K_{2,d}$ (with $d \geq 1$). Then one connects the vertex x_1 to x_2 (the two vertices of degree d) with a path of length $2r - 2$. Figure 3 shows the graph $G_{6,5}$, where we start with the graph $K_{2,5}$ (the graph on $\{x_1, x_2, y_1, \ldots, y_5\}$), and then add a path of length $2 \cdot 6 - 2$ between x_1 and x_2, the two vertices of degree five in $K_{2,5}$. Figure 3 also illustrates our edge labeling.

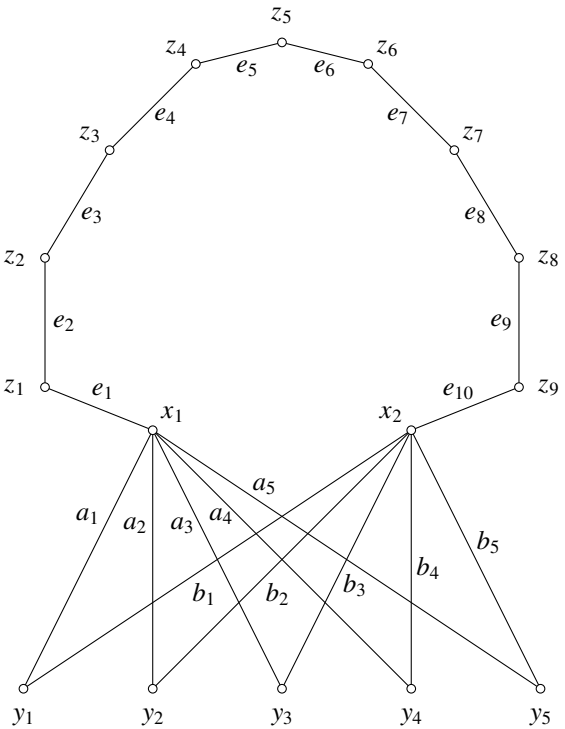

Figure 3. Illustration of $G_{r,d} = G_{6,5}$. This graph is $K_{2,5}$ with a path of length $2 \cdot 6 - 2 = 10$ connecting the two vertices of degree 5 in $K_{2,5}$.

Remark 6.22. Note Definition 6.21 still makes sense if we allow $r = 2$. However, if we allow $r = 2$, then we are adding a path of length $2r - 2 = 2$ between x_1 and x_2. Then the graph $G_{2,d}$ and the graph $K_{2,d+1}$ are isomorphic. It thus makes sense to restrict to the case $r \geq 3$.

The toric ideals of these graphs were studied in [Galetto et al. 2019] (see also [Nandi and Nanduri 2019] for a more general family); in particular, all the graded Betti numbers of I_G were determined, and consequently, one can determine the regularity (see [Galetto et al. 2019, Theorem 3.9]). The approach taken in that paper is to consider a careful analysis of the initial ideal of I_G and exploiting the fact that a universal Gröbner basis of I_G could be explicitly described.

Lemma 6.23 [Galetto et al. 2019, Corollary 3.3]. *Let d, r be integers such that $d \geq 1$ and $r \geq 3$. Then a universal Gröbner basis for $I_{G_{r,d}}$ is given by*

$$\{a_i b_j - a_j b_i \mid 1 \leq i < j \leq d\} \cup \{a_i e_2 e_4 \cdots e_{2r-2} - b_i e_1 e_3 \cdots e_{2r-3} \mid 1 \leq i \leq d\},$$

where, if $d = 1$, the first set is empty.

On the other hand, since $G = G_{r,d}$ is bipartite, we know that I_G is geometrically vertex decomposable. Using this fact, Lemma 6.23, and Theorem 3.2, we can give an alternative proof for the calculation of $\operatorname{reg}(I_{G_{r,d}})$.

Theorem 6.24. *Let $d \geq 1$ and $r \geq 3$ be integers, and let $G = G_{r,d}$. Then $\operatorname{reg}(I_G) = r$.*

Proof. We first consider the case that $d = 1$ and $r \geq 3$. In this case, $G_{r,1} = C_{2r}$, that is, the cycle on $2r$ vertices. By Lemma 6.23, $I_{G_{r,1}} = \langle a_1 e_2 \cdots e_{2r-2} - b_1 e_1 \cdots e_{2r-3} \rangle$ is generated by a single polynomial of degree r. The conclusion then follows.

We now proceed by induction on the tuple (d, r), where we assume the result holds for all graphs $G_{r',d'}$ with $d > d'$. Let $G = G_{r,d}$. Let $>$ denote the lexicographical monomial order on $\mathbb{K}[E(G)]$ with

$$a_d > a_{d-1} > \cdots > a_1 > e_1 > e_2 > \cdots > e_{2r-2} > b_d > \cdots > b_1.$$

If $y = a_d$, then $>$ is a y-compatible monomial order.

By Lemma 6.4, N_{y,I_G} is the toric ideal of $G \setminus \{a_d\}$. If we remove a_d from G, then b_d is a leaf of $G \setminus \{a_d\}$. Consequently, N_{y,I_G} is the toric ideal of the graph G with both edges a_d and b_d removed. But if we remove a_d and b_d from G, we obtain the graph $G_{r,d-1}$. Thus $N_{y,I_G} = I_{G_{r,d-1}}$.

By using the universal Gröbner basis of Lemma 6.23, we have

$$\begin{aligned} C_{y,I_G} &= \langle b_{d-1}, \ldots, b_1, e_2 e_4 \cdots e_{2r-2} \rangle + \langle a_i b_j - a_j b_i \mid 1 \leq i < j \leq d-1 \rangle \\ &\qquad + \langle a_i e_2 \cdots e_{2r-2} - b_i e_1 \cdots e_{2r-3} \mid 1 \leq i \leq d-1 \} \\ &= \langle b_{d-1}, \ldots, b_1, e_2 e_4 \cdots e_{2r-2} \rangle. \end{aligned}$$

The last equality follows from the fact that each term of the generators in the other two ideals is either divisible by some b_i with $i \in \{1, \ldots, d-1\}$ or $e_2 e_4 \cdots e_{2r-2}$. Consequently, C_{y,I_G} is a monomial ideal that is a complete intersection (since the monomials have disjoint support) with regularity

$$\underbrace{1 + \cdots + 1}_{d-1} + (r-1) - (d-1) = r - 1.$$

So, by Theorem 3.2 and induction, we have

$$\operatorname{reg}(I_G) = \max\{\operatorname{reg}(I_{G_{r,d-1}}), \operatorname{reg}(C_{y,I_G}) + 1\} = \max\{r, (r-1)+1\} = r,$$

as desired. □

We can now compute the a-invariant and the multiplicity of the rings $\mathbb{K}[E]/I_{G_{r,d}}$.

Corollary 6.25. *Let $d \geq 1$ and $r \geq 3$ be integers, and let $G = G_{r,d}$. Then*

(1) $a(\mathbb{K}[E(G)]/I_G) = 1 - d - r$, *and*

(2) $e(\mathbb{K}[E(G)]/I_G) = dr - (d-1)$.

Proof. We can use Theorem 5.4 to prove (1), but it is more direct to use Theorem 6.24, Lemma 6.3, and the fact that $G_{r,d}$ is a bipartite graph on $2+d+(2r-3)$ vertices.

To prove (2), we do induction on d. If $d=1$, then $G_{r,d}=C_{2r}$, and so $I_{G_{r,d}}$ is a principal ideal generated by a single generator of degree r. So $e(\mathbb{K}[E]/I_{G_{r,d}})=r$.

So suppose $d>1$. If $I=I_{G_{r,d}}$, then by Theorem 4.2, we have

$$e(\mathbb{K}[E]/I)=e(\mathbb{K}[E]/N_{y,I})+e(\mathbb{K}[E]/C_{y,I}).$$

As shown in the proof of Theorem 6.24, $C_{y,I}$ is a complete intersection generated by $d-1$ generators of degree one and one generator of degree $r-1$. Consequently,

$$e(\mathbb{K}[E]/C_{y,I})=1^{d-1}\cdot(r-1)=r-1.$$

On the other hand, as also shown in the proof of Theorem 6.24, $N_{y,I}=I_{G_{r,d-1}}$. Hence, by induction, $e(\mathbb{K}[E]/N_{y,I})=(d-1)r-(d-2)$. Thus

$$e(\mathbb{K}[E]/I)=(d-1)r-(d-2)+r-1=dr-(d-1). \qquad \square$$

Acknowledgements

The computer program `Macaulay2` was used for computations and examples; in particular, we made use of the package `GeometricDecomposability` [Cummings and Van Tuyl 2024]. The authors thank Patricia Klein for her comments and suggestions. The authors also thank the two anonymous referees for their detailed comments, which helped improve the presentation of the manuscript. Nguyễn's research is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2023.07. Part of this work was done while Nguyễn was visiting the Vietnam Institute for Advanced Study in Mathematics (VIASM). He thanks VIASM for its hospitality and support. Rajchgot's research is supported by NSERC Discovery Grant 2017-05732. Van Tuyl's research is supported by NSERC Discovery Grant 2019-05412.

References

[Abhyankar and Kulkarni 1989] S. Abhyankar and D. M. Kulkarni, "On Hilbertian ideals", *Linear Algebra Appl.* **116** (1989), 53–79. MR Zbl

[Almousa et al. 2022] A. Almousa, A. Dochtermann, and B. Smith, "Root polytopes, tropical types, and toric edge ideals", preprint, 2022. arXiv 2209.09851

[Biermann et al. 2017] J. Biermann, A. O'Keefe, and A. Van Tuyl, "Bounds on the regularity of toric ideals of graphs", *Adv. in Appl. Math.* **85** (2017), 84–102. MR Zbl

[Bruns and Herzog 1993] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993. MR Zbl

[Constantinescu and Gorla 2018] A. Constantinescu and E. Gorla, "Gorenstein liaison for toric ideals of graphs", *J. Algebra* **502** (2018), 249–261. MR Zbl

[Corso and Nagel 2009] A. Corso and U. Nagel, "Monomial and toric ideals associated to Ferrers graphs", *Trans. Amer. Math. Soc.* **361**:3 (2009), 1371–1395. MR Zbl

[Cummings and Van Tuyl 2024] M. Cummings and A. Van Tuyl, "The `GeometricDecomposability` package for Macaulay2", *J. Softw. Algebra Geom.* **14**:1 (2024), 41–50. MR Zbl

[Cummings et al. 2023] M. Cummings, S. Da Silva, J. Rajchgot, and A. Van Tuyl, "Geometric vertex decomposition and liaison for toric ideals of graphs", *Algebr. Comb.* **6**:4 (2023), 965–997. MR Zbl

[Cummings et al. 2024] M. Cummings, S. Da Silva, M. Harada, and J. Rajchgot, "Gröbner geometry for regular nilpotent Hessenberg Schubert cells", *J. Pure Appl. Algebra* **228**:7 (2024), art. id. 107648. MR Zbl

[Da-Silva and Harada 2023] S. Da-Silva and M. Harada, "Geometric vertex decomposition, Gröbner bases, and Frobenius splittings for regular nilpotent Hessenberg varieties", *Transform. Groups* (online publication July 2023), 1–36.

[D'Alì 2015] A. D'Alì, "Toric ideals associated with gap-free graphs", *J. Pure Appl. Algebra* **219**:9 (2015), 3862–3872. MR Zbl

[Dochtermann and Engström 2009] A. Dochtermann and A. Engström, "Algebraic properties of edge ideals via combinatorial topology", *Electron. J. Combin.* **16**:2 (2009), art. id. 2. MR Zbl

[Ene and Herzog 2012] V. Ene and J. Herzog, *Gröbner bases in commutative algebra*, Graduate Studies in Mathematics **130**, American Mathematical Society, Providence, RI, 2012. MR Zbl

[Favacchio et al. 2021] G. Favacchio, J. Hofscheier, G. Keiper, and A. Van Tuyl, "Splittings of toric ideals", *J. Algebra* **574** (2021), 409–433. MR Zbl

[Galetto et al. 2019] F. Galetto, J. Hofscheier, G. Keiper, C. Kohne, A. Van Tuyl, and M. E. U. Paczka, "Betti numbers of toric ideals of graphs: a case study", *J. Algebra Appl.* **18**:12 (2019), art. id. 1950226. MR Zbl

[Hà and Woodroofe 2014] H. T. Hà and R. Woodroofe, "Results on the regularity of square-free monomial ideals", *Adv. in Appl. Math.* **58** (2014), 21–36. MR Zbl

[Hà et al. 2019] H. T. Hà, S. K. Beyarslan, and A. O'Keefe, "Algebraic properties of toric rings of graphs", *Comm. Algebra* **47**:1 (2019), 1–16. MR Zbl

[Herzog et al. 2018] J. Herzog, T. Hibi, and H. Ohsugi, *Binomial ideals*, Graduate Texts in Mathematics **279**, Springer, Cham, Switzerland, 2018. MR Zbl

[Klein 2023] P. Klein, "Diagonal degenerations of matrix Schubert varieties", *Algebr. Comb.* **6**:4 (2023), 1073–1094. MR Zbl

[Klein and Rajchgot 2021] P. Klein and J. Rajchgot, "Geometric vertex decomposition and liaison", *Forum Math. Sigma* **9** (2021), art. id. e70. MR Zbl

[Klein and Weigandt 2022] P. Klein and A. Weigandt, "Bumpless pipe dreams encode Gröbner geometry of Schubert polynomials", *Sém. Lothar. Combin.* **86B** (2022), art. id. 84. MR Zbl

[Knutson et al. 2009] A. Knutson, E. Miller, and A. Yong, "Gröbner geometry of vertex decompositions and of flagged tableaux", *J. Reine Angew. Math.* **630** (2009), 1–31. MR Zbl

[Moradi and Khosh-Ahang 2016] S. Moradi and F. Khosh-Ahang, "On vertex decomposable simplicial complexes and their Alexander duals", *Math. Scand.* **118**:1 (2016), 43–56. MR Zbl

[Nandi and Nanduri 2019] R. Nandi and R. Nanduri, "Betti numbers of toric algebras of certain bipartite graphs", *J. Algebra Appl.* **18**:12 (2019), art. id. 1950231. MR Zbl

[Ohsugi and Hibi 1999] H. Ohsugi and T. Hibi, "Koszul bipartite graphs", *Adv. in Appl. Math.* **22**:1 (1999), 25–28. MR Zbl

[Pinto and Villarreal 2023] M. V. Pinto and R. H. Villarreal, "Graph rings and ideals: Wolmer Vasconcelos contributions", preprint, 2023. arXiv 2305.06270

[Provan and Billera 1980] J. S. Provan and L. J. Billera, "Decompositions of simplicial complexes related to diameters of convex polyhedra", *Math. Oper. Res.* **5**:4 (1980), 576–594. MR Zbl

[Stanley 1978] R. P. Stanley, "Hilbert functions of graded algebras", *Advances in Math.* **28**:1 (1978), 57–83. MR Zbl

[Stanley 1993] R. P. Stanley, "A monotonicity property of h-vectors and h^*-vectors", *European J. Combin.* **14**:3 (1993), 251–258. MR Zbl

[Stelzer and Yong 2023] A. Stelzer and A. Yong, "Schubert determinantal ideals are Hilbertian", preprint, 2023. arXiv 2305.12558

[Tatakis and Thoma 2011] C. Tatakis and A. Thoma, "On the universal Gröbner bases of toric ideals of graphs", *J. Combin. Theory Ser. A* **118**:5 (2011), 1540–1548. MR Zbl

[Vasconcelos 1998] W. V. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry*, Algorithms and Computation in Mathematics **2**, Springer, Berlin, 1998. MR Zbl

[Villarreal 1995] R. H. Villarreal, "Rees algebras of edge ideals", *Comm. Algebra* **23**:9 (1995), 3513–3524. MR Zbl

[Villarreal 2015] R. H. Villarreal, *Monomial algebras*, 2nd ed., CRC Press, Boca Raton, FL, 2015. MR Zbl

[Woodroofe 2009] R. Woodroofe, "Vertex decomposable graphs and obstructions to shellability", *Proc. Amer. Math. Soc.* **137**:10 (2009), 3235–3246. MR Zbl

Received February 21, 2024. Revised December 8, 2024.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 333 No. 2 December 2024