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## AN ALBERTI'S RANK-ONE THEOREM FOR MARTINGALES

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**We prove that the polar decomposition of the singular part of a vector measure depends on its conditional expectations computed with respect to the  $q$ -regular filtration. This dependency is governed by a martingale analog of the so-called wave cone, which naturally corresponds to the result of De Philippis and Rindler about fine properties of PDE-constrained vector measures. As a corollary we obtain a martingale version of Alberti's rank-one theorem.**

The considerations of this paper are implicitly connected with a question in Banach space theory concerning finding suitable models of BV/Sobolev spaces that are easier to handle from the point of view of that field (for example, to investigate the bounded approximation property or the Dunford–Pettis property). In our findings we rely on a deep correspondence between Fourier analysis and martingale theory and show similar (in many ways) spaces with a nice additional combinatorial structure. However, in this article we focus on an analytic result that we treat as an indication that we are proceeding in a good direction.

The theorem in which we are interested is an analog of the result of De Philippis and Rindler concerning polar decomposition of PDE-constrained measures ([4]). The original theorem is this:

**Theorem 1** [4, Theorem 1.1]. *Let*

$$\mathcal{A}(D) = \sum_{|\alpha| \leq r} A_\alpha (\partial^\alpha), \quad A_\alpha \in M_{n \times m}(\mathbb{R}),$$

*be a constant-coefficient linear operator that maps  $\mathbb{R}^m$ -valued functions in  $N$  variables to  $\mathbb{R}^n$ -valued functions, with the principal symbol*

$$\mathbb{A}^r[\xi] = \sum_{|\alpha|=r} A_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^N.$$

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Suppose that a locally finite vector measure  $\mu \in \mathbf{M}(\Omega; \mathbb{R}^m)$ , where  $\Omega \subset \mathbb{R}^N$  is an arbitrary domain, satisfies

$$(1) \quad \mathcal{A}(D)\mu = 0 \quad \text{in } \mathcal{D}'(\Omega; \mathbb{R}^n).$$

Then,

$$(2) \quad \frac{d\mu}{d|\mu|}(x) \in \bigcup_{\xi \in \mathbb{R}^n \setminus \{0\}} \ker \mathbb{A}^r[\xi] \quad \text{for } |\mu_s|\text{-a.e. } x.$$

In (2) the symbol  $\frac{d\mu}{d|\mu|}(x)$  denotes the polar decomposition of  $\mu$  at  $x$ , i.e., the value of the Radon–Nikodym derivative of  $\mu$  with respect to its total variation at  $x$ . By  $\mu_s$  we denote the singular part of  $\mu$ . The set on the right-hand side of (2) is called the wave cone of  $\mathcal{A}$ . Defining a similar object associated with a filtration is perhaps the most rewarding outcome of our considerations. We will also derive a discrete variant of Alberti’s famous rank-one theorem [1].

We obtain our results on a specific metric space corresponding to the setting of  $q$ -regular martingales considered by Janson in [5] for the purpose of modeling real Hardy spaces. His martingale model can be realized on the probability space  $\Omega = (\mathbb{T}, \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n), \mu)$ , which we describe below.

Let  $q \geq 3$  be a fixed integer. The set  $\mathbb{T}$  will consist of all infinite paths of the infinite  $q$ -regular tree  $\mathcal{T}$  that begin from its root. To be more precise, the oriented graph  $\mathcal{T} = (V, E)$  is defined by the following properties:

- (1) It has a distinguished vertex, called the root.
- (2) It is an infinite directed and connected graph without cycles.
- (3) Each vertex has  $q$  outgoing edges.
- (4) Each vertex except the root has one incoming edge, the root has no incoming edges.

If  $v \rightarrow w$  in  $\mathcal{T}$ , then we refer to  $w$  as a child of  $v$  and write  $w^\uparrow = v$ . It will be convenient to enumerate the children of a vertex  $w$  with the numbers  $1, 2, \dots, q$  and fix such an enumeration.

By a path we mean an infinite directed sequence of vertices starting from the root, each succeeding being a child of the preceding. If  $x \in \mathbb{T}$  is a path, then we denote the  $n$ -th element of the corresponding sequence by  $x(n)$ . The set of atoms (of the  $n$ -th generation)  $\mathcal{A}\mathcal{F}_n$  consists of  $q^n$  sets of the form

$$\omega_v = \{x \in \mathbb{T} : x(n) = v\} \quad \text{for } v \text{ such that } d_{\mathcal{T}}(\text{root}, v) = n.$$

In other words, for a vertex  $v$  whose standard graph distance to the root is  $n$ , the set  $\omega_v$  consists of infinite paths that pass through  $v$  at position  $n$ . The collection of the sets  $\mathcal{A}\mathcal{F}_n$  forms a partition of  $\mathbb{T}$  and we put  $\mathcal{F}_n$  to be the  $\sigma$ -algebra generated by  $\mathcal{A}\mathcal{F}_n$ . Further, we tacitly transfer the tree structure from  $V$  to the set of all atoms

of all generations, writing  $\omega_v^\uparrow = \omega_w$  if  $v^\uparrow = w$ , etc. and we will not make any distinction between vertices of a tree and atoms. Finally, the measure  $\mu$  is simply the uniform measure on  $\mathbb{T}$ , i.e.,  $\mu(\omega) = q^{-n}$  for  $\omega \in \mathcal{AF}_n$ .

The space  $\mathbb{T}$  played an important role in modeling so-called Bourgain–Brezis inequalities, see [3] and [8]. See the first of these papers for more information about  $\mathbb{T}$ .

A sequence of  $\mathbb{R}^l$ -valued functions  $\{F_n\}_n$  is a martingale provided for any  $n \in \mathbb{N} \cup \{0\}$  the function  $F_n$  is  $\mathcal{F}_n$ -measurable and

$$(3) \quad \mathbb{E}(F_{n+1} \mid \mathcal{F}_n) = F_n.$$

Each finite  $\mathbb{R}^l$ -valued measure  $\nu \in \mathbf{M}(\mathbb{T}, \mathbb{R}^l)$  on  $\mathbb{T}$  generates a martingale  $\{\nu_n\}_n$  by the formula

$$\nu_n(x) = q^n \nu(\omega), \quad x \in \omega \in \mathcal{AF}_n.$$

**Definition 2.** Denote by  $\text{Diff}(\nu)$  the set of all matrices  $D_\omega \in M_{q \times l}(\mathbb{R})$  of the form

$$(4) \quad D_\omega = \begin{bmatrix} | & \dots & | \\ d_1 & \dots & d_q \\ | & \dots & | \end{bmatrix},$$

where  $\omega \in \mathcal{AF}_n$  and

$$d_i = \nu_{n+1}(\omega_i) - \nu_n(\omega),$$

and  $\omega_1, \dots, \omega_q \in \mathcal{AF}_{n+1}$  are all children of  $\omega$ .

**Definition 3.** Let  $W \subset M_{q \times l}(\mathbb{R})$  be a linear subspace. We denote the space  $\mathfrak{W}$  by the rule

$$(5) \quad \mathfrak{W} = \{\nu \in \mathbf{M}(\mathbb{T}, \mathbb{R}^l) : \text{Diff}(\nu) \subset W\}.$$

Intuitively, the subspace  $W$  will play a role similar to a constraint given by the operator  $\mathcal{A}$  in Theorem 1. Later this will become more evident when we will see concrete examples of  $W$  defined by expressions similar to the symbols of differential operators (see Corollary 13). Moreover, the spaces  $\mathfrak{W}$  and  $\{\mu \in \mathbf{M}(\mathbb{R}^N, \mathbb{R}^m) \mid \mathcal{A}(D)\mu = 0\}$  have many similarities. For example, they behave similarly under the action of Riesz potentials; see [3; 8]. Another confirmation of this principle comes from dimensional estimates for corresponding measures; see [2; 3; 9]. The informal explanation of the wonderful correspondence is that both the function and the martingale spaces admit certain invariance: the constrained spaces of measures are dilation and translation invariant, while the space  $\mathfrak{W}$  is invariant under shifts along the tree, or even, if  $W$  is invariant under certain group action, invariant under special rearrangements of the tree. Another important informal issue that emphasizes the

similarities is that a Littlewood–Paley decomposition  $f = \sum P_k f$ , where  $P_k$  are spectral projectors on the scales  $\sim 2^k$ , often behaves like a dyadic martingale.

By the martingale property (3), each row of the matrix (4) is a vector with zero mean. Therefore, we may restrict our attention to subspaces of the form  $W \subset \mathbb{R}_0^q \otimes \mathbb{R}^l$  only. Here  $\mathbb{R}_0^q$  denotes the subspace of  $\mathbb{R}^q$  of vectors with entries adding up to zero. In [3], the space  $\mathfrak{M}$  was called a martingale Sobolev space. The terminology “martingale BV-type space” seems more appropriate.

**Definition 4.** Let  $v \in \mathbb{R}^l$  and  $A \subset M_{q \times l}(\mathbb{R})$  be a subset of real  $q \times l$  matrices. We define the rank-one angle between  $v$  and  $A$  as

$$\gamma(v, A) := \inf\{|\angle(v \otimes w, m)|_{\text{HS}} : w \in \mathbb{R}_0^q, w \neq 0, m \in A\},$$

where

$$|\angle(A_1, A_2)|_{\text{HS}} = \arccos \frac{|\text{tr}(A_1^t A_2)|}{\|A_1\|_{\text{HS}} \|A_2\|_{\text{HS}}}, \quad A_1, A_2 \in M_{q \times l},$$

is the measure of the angle between matrices, computed with respect to the Hilbert–Schmidt norm  $\|\cdot\|_{\text{HS}}$ . Here,  $v \otimes w = v \cdot w^t$ .

**Definition 5.** We define the martingale wave cone of  $W$  as the set

$$\Lambda(W) = \{v \in \mathbb{R}^l : \exists w \in \mathbb{R}_0^q \setminus \{0\} \text{ such that } v \otimes w \in W\}.$$

In other words,

$$\Lambda(W) = \{v \in \mathbb{R}^l : \gamma(v, W) = 0\}.$$

Let us decompose  $v$  into an absolutely continuous and a singular part (with respect to the uniform measure on  $\mathbb{T}$ ):

$$v = v_{\text{abs}} + v_s.$$

Our main result is the following:

**Theorem 6.** *Let  $v \in \mathfrak{M}$  be a finite  $\mathbb{R}^l$ -valued measure. Then*

$$\frac{dv}{d|v|}(x) \in \Lambda(W) \quad \text{for } |v_s|\text{-a.e. } x.$$

### 1. Decomposition into flat and convex atoms

The proof of Theorem 6 relies on a combination of the ideas from [5] and [3]. The crucial tool is the decomposition of  $\mathbb{T}$  into parts corresponding to the so-called  $\varepsilon$ -convex and  $\varepsilon$ -flat atoms introduced in [3].

**Definition 7.** For a given  $\varepsilon \in [0, 1)$ , an atom  $\omega \in \mathcal{AF}_n$  is called  $\varepsilon$ -convex if

$$\mathbb{E}(\|v_{n+1}\| - \|v_n\|)1_\omega \geq \varepsilon \mathbb{E}\|v_n\|1_\omega.$$

If the reverse inequality holds, then  $\omega$  is  $\varepsilon$ -flat.

Here and in what follows, we use the standard Euclidean norm on  $\mathbb{R}^l$ . One may see that 0-flat atoms correspond to the case where the matrix  $D_w$  has rank one, because in this case the triangle inequality  $\mathbb{E}\|v_{n+1}\|1_\omega \geq \|v_n(\omega)\|$  turns into equality.

Let us denote by  $\mathcal{T}^\varepsilon$  the subgraph of  $\mathcal{T}$  generated by the vertices corresponding to  $\varepsilon$ -flat atoms. One can represent  $\mathcal{T}^\varepsilon = \cup \mathcal{T}_i^\varepsilon$ , where each  $\mathcal{T}_i^\varepsilon$  is a maximal by inclusion connected subgraph (a tree). It turns out that the singular part of  $\nu$  is carried by infinite paths of the trees  $\mathcal{T}_i^\varepsilon$ .

**Definition 8.** We call a point  $x \in \mathbb{T}$  an  $\varepsilon$ -leaf if there exists  $n_0 \in \mathbb{N}$  and a sequence of  $\varepsilon$ -flat atoms  $\{\omega_n\}_n$ ,  $\omega_n \in \mathcal{AF}_n$ , such that  $x \in \omega_n$  for all  $n \geq n_0$ . Let us denote by  $L(\varepsilon)$  the set of all  $\varepsilon$ -leaves and put

$$L := \bigcap_{\varepsilon > 0} L(\varepsilon).$$

For a subgraph  $G \subset \mathbb{T}$ , we denote  $L(\varepsilon, G)$  the set of all infinite paths of  $G$  that are restrictions of  $\varepsilon$ -leaves to  $G$ .

By Lemma 3.3 and Corollary 3.4 from [3], we get  $|v_s|(\mathbb{T} \setminus L(\varepsilon)) = 0$  for all  $\varepsilon > 0$  and so:

**Lemma 9.** *We have*

$$(6) \quad |v_s|(\mathbb{T} \setminus L) = 0.$$

We will need yet another classification of leaves.

**Definition 10.** We call  $x \in L$  a *big leaf* if there exists  $\beta > 0$  and a sequence of atoms

$$\omega_{n_1} \supset \omega_{n_2} \supset \cdots \supset \{x\}, \quad n_1 < n_2 < \dots$$

such that  $\omega_{n_k} \in \mathcal{AF}_{n_k}$  and

$$(7) \quad \frac{1}{q} \sum_{j=1}^q \|d_j^{(n_k)}\| \geq \beta \|v_{n_k}(\omega_{n_k})\|,$$

where

$$D_{\omega_k} = \begin{bmatrix} | & \cdots & | \\ d_1^{(n_k)} & \cdots & d_q^{(n_k)} \\ | & \cdots & | \end{bmatrix}.$$

Otherwise, we call  $x \in L$  a *small leaf*. The sets of small and big leaves will be denoted by  $L_s$  and  $L_b$ , respectively.

## 2. Proof of the main theorem

We need two algebraic lemmas. The first quantifies the “flattening effect” (cf. [3, Lemma 2.1]). The notation  $\pi_a x$  means projection of  $x \in \mathbb{R}^l$  onto the line spanned by  $a \in \mathbb{R}^l$ ,  $\pi_{a^\perp} x$  denotes the projection onto the orthogonal complement of  $a$ .

**Lemma 11.** *Let  $\omega \in \mathcal{AF}_n$  be an  $\varepsilon$ -flat atom with  $\varepsilon < 1$ , i.e.,*

$$\frac{1}{q} \sum_{j=1}^q \|a + d_j\| - \|a\| \leq \varepsilon \|a\|$$

for

$$D_\omega = \begin{bmatrix} | & \dots & | \\ d_1 & \dots & d_q \\ | & \dots & | \end{bmatrix}, \quad a = v_n(\omega).$$

Then, for all  $j = 1, 2, \dots, q$  we have  $\|\pi_{a^\perp} d_j\| \leq 2q\sqrt{\varepsilon}\|a\|$ .

*Proof.* From the triangle inequality and  $\sum_j d_j = 0$ , we have

$$(8) \quad \frac{1}{q} \sum_{j=1}^q \|a + \pi_a d_j\| \geq \|a\|.$$

This and the definition of an  $\varepsilon$ -flat atom imply that

$$(9) \quad \begin{aligned} \varepsilon \|a\| &\geq \frac{1}{q} \left( \sum_{j=1}^q \|a + d_j\| - \|a + \pi_a d_j\| \right) = \frac{1}{q} \sum_{j=1}^q \frac{\|\pi_{a^\perp}(d_j)\|^2}{\|a + d_j\| + \|a + \pi_a d_j\|} \\ &\geq \frac{1}{2q} \sum_{j=1}^q \frac{\|\pi_{a^\perp}(d_j)\|^2}{\|a + d_j\|} \geq \frac{1}{4q^2} \sum_{j=1}^q \frac{\|\pi_{a^\perp}(d_j)\|^2}{\|a\|}. \end{aligned}$$

The latter inequality follows from

$$(10) \quad \|a + d_j\| \leq \sum_{k=1}^q \|a + d_k\| \leq (1 + \varepsilon)q\|a\| \leq 2q\|a\|.$$

Thus, (9) yields  $\|\pi_{a^\perp} d_j\|^2 \leq 4q^2 \varepsilon \|a\|^2$  for all  $j = 1, 2, \dots, q$ .  $\square$

The second lemma uses the smoothness of the Euclidean norm [5, Lemma 10].

**Lemma 12.** *Suppose that  $\omega \in \mathcal{AF}_n$  and*

$$D_\omega = \begin{bmatrix} | & \dots & | \\ d_1 & \dots & d_q \\ | & \dots & | \end{bmatrix}, \quad a = v_n(\omega).$$

Let us assume that  $\gamma(a, \{D_\omega\}) \geq \eta > 0$  and  $\sum_{j=1}^q \|d_j\| \leq \delta \|a\|$  for some parameters  $\eta, \delta > 0$ . Then, for sufficiently small  $\delta$ , there exists  $p_0 = p_0(\delta, \eta)$  such that

$$(11) \quad \|a\|^p \leq \frac{1}{q} \sum_{j=1}^q \|a + d_j\|^p$$

for any  $p$  satisfying  $p_0 < p < 1$ .

*Proof.* Without loss of generality, we may assume  $\|a\| = 1$ . By duality,

$$(12) \quad \frac{\sqrt{\sum_{j=1}^q |\langle a, d_j \rangle|^2}}{\sqrt{\sum_{j=1}^q \|d_j\|^2}} = \sup_{\|\{\varepsilon_j\}\|=1} \frac{\sum_{j=1}^q \langle \varepsilon_j a, d_j \rangle}{\sqrt{\sum_{j=1}^q \|d_j\|^2}} \\ = \sup_{\|\{\varepsilon_j\}\|=1} \cos \left| \angle(a \otimes \{\varepsilon_j\}, D_\omega) \right|_{\text{HS}} \leq \cos \eta,$$

which leads to

$$(13) \quad \sum_{j=1}^q |\langle a, d_j \rangle|^2 \leq \cos^2 \eta \sum_{j=1}^q \|d_j\|^2.$$

Using the representation

$$(14) \quad \|a + d_j\|^p = (1 + 2\langle a, d_j \rangle + \|d_j\|^2)^{p/2},$$

and treating the  $d_j$  as small parameters, we apply Taylor's formula to the right-hand side of (11):

$$(15) \quad \sum_{j=1}^q \|a + d_j\|^p = \sum_{j=1}^q \left( 1 + \langle a, d_j \rangle + \frac{p}{2} \|d_j\|^2 + \frac{p(p-2)}{2} \langle a, d_j \rangle^2 \right) + O\left( \sum_{j=1}^q \|d_j\|^3 \right) \\ = q + \frac{p}{2} \sum_{j=1}^q \|d_j\|^2 + \frac{p(p-2)}{2} \sum_{j=1}^q \langle a, d_j \rangle^2 + O\left( \sum_{j=1}^q \|d_j\|^3 \right).$$

Using (13), we bound the right-hand side of (15) from below by

$$(16) \quad q + \frac{p}{2} \sum_{j=1}^q \|d_j\|^2 + \cos^2 \eta \frac{p(p-2)}{2} \sum_{j=1}^q \|d_j\|^2 + O\left( \sum_{j=1}^q \|d_j\|^3 \right).$$

Since  $|\cos \eta| < 1$ , the last expression is at least  $q$  provided that  $\delta$  is sufficiently small and  $p$  is sufficiently close to one. This justifies the desired inequality.  $\square$

*Proof of Theorem 6.* By the Besicovitch–Lebesgue differentiation theorem<sup>1</sup> we have that for  $|v|$ -a.e.  $x$

$$(17) \quad \lim_{n \rightarrow \infty} \angle \left( v_n(x), \frac{dv}{d|v|}(x) \right) = 0.$$

In particular, this is true for  $|v_s|$ -a.e.  $x \in L$ . By Lemma 9, it suffices to disprove that there exists  $\eta > 0$  such that

$$(18) \quad B_\eta = \left\{ x \in L : \gamma \left( \frac{dv}{d|v|}(x), \text{Diff}(v) \right) > \eta \right\}$$

has positive  $|v_s|$ -measure.

Let us decompose  $B_\eta = B^{(1)} \cup B^{(2)}$  into sets consisting of big and small leaves, respectively.

Step 1.  $|v_s|(B^{(1)}) = 0$ . Let  $x \in B^{(1)}$  be a big leaf and  $\{\omega_{n_k}\}, \{D_{\omega_{n_k}}\}$  and  $\beta$  be as in Definition 10. Put  $a = v_{n_k}(\omega_{n_k})$ . We will show that  $\gamma(a, \text{Diff}(v))$  is in fact arbitrarily small for sufficiently large  $k$ . Let us assume that  $\|a\| = 1$ . Then (7) and the Cauchy–Schwarz inequality yield

$$\sum_j \|d_j^{(n_k)}\|^2 \geq \frac{\beta^2}{q},$$

and Lemma 11 implies

$$\sum_j \|\pi_{a^\perp}(d_j^{(n_k)})\|^2 \leq (2q^2 \sqrt{\varepsilon})^2 = 4q^4 \varepsilon.$$

We have

$$\begin{aligned} & \sup_{v \in \mathbb{R}^q \setminus \{0\}} \frac{\text{tr}[(v \otimes a)^t D_{\omega_{n_k}}]}{\|v \otimes a\|_{\text{HS}} \|D_{\omega_{n_k}}\|_{\text{HS}}} \\ &= \sup_{v \in \mathbb{R}^q \setminus \{0\}} \frac{\sum_j v_j \langle a, d_j^{(n_k)} \rangle}{\|v\| \|a\| \sqrt{\sum_j \|d_j^{(n_k)}\|^2}} = \left( \frac{\sum_j |\langle a, d_j^{(n_k)} \rangle|^2}{\sum_j \|d_j^{(n_k)}\|^2} \right)^{\frac{1}{2}} \\ &= \left( \frac{\sum_j \|d_j^{(n_k)}\|^2 - \sum_j \|\pi_{a^\perp}(d_j^{(n_k)})\|^2}{\sum_j \|d_j^{(n_k)}\|^2} \right)^{\frac{1}{2}} \\ &\geq \left( 1 - \frac{4q^5 \varepsilon}{\beta^2} \right)^{\frac{1}{2}}. \end{aligned}$$

<sup>1</sup>We are applying a differentiation theorem on a special metric space, for example, Theorem 5.2 in [6]; the metric on  $\mathbb{T}$  is defined by  $d(x, y) = m^{-n(x, y)}$ , where  $n(x, y)$  is the smallest number  $n$  such that  $x(n) \neq y(n)$ .

Now it suffices to notice that for sufficiently large  $k$ ,  $\omega_{n_k}$  is  $\varepsilon$ -flat for arbitrarily small  $\varepsilon$ . Thus, from the preceding inequality, we have  $\gamma(a, \text{Diff}(v)) < \frac{\eta}{2}$ , provided  $k$  is sufficiently large. Consequently, by (17),  $\gamma\left(\frac{dv}{d|v|}(x), \text{Diff}(v)\right) \leq \frac{\eta}{2}$  for  $|v_s|$ -almost all  $x \in B^{(1)}$ . This leads to the desired assertion.

Step 2.  $|v_s|(B^{(2)}) = 0$ . Assume the contrary. Then, there exists  $\varepsilon > 0$  such that  $|v_s|(L(\varepsilon) \cap B^{(2)}) > 0$ . Consider the decomposition  $\mathcal{T}^{(\varepsilon)} = \bigcup_j \mathcal{T}_j^{(\varepsilon)}$ . One can find  $j$  such that  $L(\varepsilon, \mathcal{T}_j^{(\varepsilon)})$  has positive  $|v_s|$ -measure (by the disjointedness of those sets).

Assume first that the inequality reverse to (7) holds for all  $\omega \in \mathcal{T}_j^{(\varepsilon)}$  with constant corresponding to a suitable small  $\delta$  required in Lemma 12. Then, on the one hand we have the property that

$$(19) \quad \theta = v^{\mathcal{T}_j^{(\varepsilon)}} \llcorner L(\varepsilon, \mathcal{T}_j^{(\varepsilon)}) \quad \text{and} \quad v \llcorner L(\varepsilon, \mathcal{T}_j^{(\varepsilon)}) \text{ have the same singular part.}$$

Here  $v^{\mathcal{T}_j^{(\varepsilon)}}$  denotes the limit measure of the martingale whose evolution is restricted to the tree  $\mathcal{T}_j^{(\varepsilon)}$  (if we leave the tree, then we stop the martingale). On the other hand, for  $p < 1$  given by Lemma 12, the sequence  $\|\mathbb{E}(\theta | \mathcal{F}_n)\|^p$  is a positive submartingale, which, by Doob's theorem on the boundedness of the martingale maximal function in  $L_q$  with  $q > 1$ , implies that the maximal function of  $\theta$  is summable, and  $\theta$  lies in the martingale space  $H^1(\mathbb{R}^l)$  (for the details see p. 148 in [5]). Thus,  $\theta$  is absolutely continuous, which is a contradiction.

If the assumptions of Lemma 12 are not satisfied globally for all  $\omega \in \mathcal{T}_j^{(\varepsilon)}$ , then we use the fact that for each infinite path (which can be extended to some small leaf) they must be satisfied for atoms that are sufficiently far from the root. More precisely, for every such  $x$  there exists  $N$  such that for all  $n \geq N$  the inequality reverse to (7) holds, with sufficiently small constant, for  $\omega \in \mathcal{A}\mathcal{F}_n$  if  $x \in \omega$ .

From this we can cover  $L(\varepsilon, \mathcal{T}_j^{(\varepsilon)})$  by a countable union of disjoint sets of the form  $L(\varepsilon, \mathcal{T}'_k)$  where  $\mathcal{T}'_k$  are some trees and one of them falls under the scope of Lemma 12, gives rise to a measure satisfying (19), and its leaves form a set of positive  $|v_s|$ -measure, leading to a contradiction.  $\square$

### 3. A Martingale rank-one theorem

In this section we will present an analog of Alberti's famous rank-one theorem. For simplicity, we will formulate and prove only the two-dimensional special case. The extension to higher dimensions is straightforward.

We consider a specific space  $W$ . We assume  $q = m^2$  for some  $m \in \mathbb{N}$  and identify the set  $1, 2, \dots, q$  with the group  $(\mathbb{Z}/m\mathbb{Z})^2$ ; here  $\mathbb{Z}/m\mathbb{Z}$  is the group of residues modulo  $m$ . Then, the elements of  $M_{q \times l}$  are naturally identified with  $\mathbb{R}^l$  valued functions on the discrete torus  $(\mathbb{Z}/m\mathbb{Z})^2$ . We set  $l = 8$  and identify  $\mathbb{R}^l$  with the space of  $2 \times 2$  complex matrices. With this notation, define the space  $W$  by the

formula

$$(20) \quad W = \left\{ D \in \mathbb{R}_0^{m^2} \times \mathbb{R}^4 \left| \begin{array}{l} \exists f, g: (\mathbb{Z}/m\mathbb{Z})^2 \rightarrow \mathbb{C} \quad \forall i, j = 1, 2, \dots, m \\ D_{i,j} = \begin{pmatrix} f(i+1, j) - f(i, j) & f(i, j+1) - f(i, j) \\ g(i+1, j) - g(i, j) & g(i, j+1) - g(i, j) \end{pmatrix} \end{array} \right. \right\}.$$

The space  $\mathfrak{W}$  generated by this  $W$  resembles<sup>2</sup> the space of BV maps. In particular it is defined by matrices  $D_\omega$  whose discrete Fourier transforms resemble the symbol of gradient (see the proof below). Moreover, the next corollary may be thought of as a martingale version of Alberti's theorem from [1].

**Corollary 13.** *Let  $W$  be given by (20), let  $v \in \mathfrak{W}$ . Then,  $\frac{dv}{d|v|}$  is a matrix of rank one for  $|v_s|$  almost all  $x$ .*

*Proof.* To derive the corollary from Theorem 6, we need to show that any matrix in the martingale wave cone  $\Lambda(W)$  has rank one. We will describe the cone  $\Lambda(W)$  using the Fourier transform on  $(\mathbb{Z}/m\mathbb{Z})^2$  (see Section 7 in [7] for a more detailed exposition of similar material). We may describe  $W$  as

$$(21) \quad \{D \mid \forall \gamma \in (\mathbb{Z}/m\mathbb{Z})^2 \setminus \{0\} \quad \hat{D}(\gamma) \in \Omega(\gamma)\}, \quad \text{where } \Omega(\gamma) = \text{span} \left( \begin{pmatrix} e^{2\pi i \gamma_1/m} - 1 & e^{2\pi i \gamma_2/m} - 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e^{2\pi i \gamma_1/m} - 1 & e^{2\pi i \gamma_2/m} - 1 \end{pmatrix} \right).$$

Pick some  $v \in \Lambda(W)$  and let  $v \otimes w \in W$ , where  $w \in \mathbb{R}_0^{m^2} \setminus \{0\}$ . Then, with the notation  $\mathcal{F}$  for the Fourier transform,

$$\mathcal{F}[v \otimes w](\gamma) = v \otimes \hat{w}(\gamma).$$

Since  $w$  is not a constant function,  $\hat{w}(\gamma) \neq 0$  from some  $\gamma \in (\mathbb{Z}/m\mathbb{Z})^2 \setminus \{0\}$ . Then, by (21),

$$v = (e^{2\pi i \gamma_1/m} - 1, e^{2\pi i \gamma_2/m} - 1) \otimes (a, b),$$

where  $(a, b) \in \mathbb{C}^2$  is a nonzero vector. □

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<sup>2</sup>As noted by one of the referees, it would be interesting to explore whether standard facts about BV functions have analogs for martingales in this space. Does the coarea formula hold true, for example?

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