Pacific Journal of Mathematics

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Volume 334 No. 1 January 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 334, No. 1, 2025
https://doi.org/10.2140/pjm.2025.334.43

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We define a new conformal invariant on complete noncompact hyperbolic surfaces that can be conformally compactified to bounded domains in $\mathbb{C}$. We study and compute this invariant up to doubly connected surfaces. Our results give a new geometric criterion for choosing canonical representations of bounded domains in $\mathbb{C}$.

1. Introduction

Let (M, g_M) be a complete noncompact hyperbolic surface of finite topological type that can be conformally compactified to a bounded domain in $\mathbb{C}$. Consider all possible choices of $\Omega \subset \mathbb{C}$ such that M can be conformally compactified into them and are conformally equivalent to each other. Here we study a new canonical representation through proper geometric considerations.

By the Riemann mapping theorem, all simply connected domains that are not $\mathbb{C}$ are conformally equivalent to a round disk, which can be viewed as a canonical model. It is well known that any doubly connected region is conformally equivalent to a ring $\{z \in \mathbb{C}; \beta < |z| < 1\}$, where $0 \le \beta < 1$; for a reference, see [1]. When $\beta = 0$, the domain is simply a punctured disk. For multiply connected domains, see [1] for some classification results. These results lead to a complete characterization of the moduli space of such surfaces. For example, all n-connected ($n > 2$) bounded domains with nondegenerate boundary components form a $(3n-6)$-dimensional space. In this article, we propose a new geometric point of view.

The author acknowledges support from the Erwin and Peggy Kleinfeld Graduate Fellowship fund.
MSC2020: 53A05, 53C18.
Keywords: Liouville's equation, global coefficient, conformal geometry.

We set the for our construction. Let M be a complete noncompact hyperbolic surface and $\Omega \subset \mathbb{C}$ be a fixed conformal compactification of M. Then there exists a smooth function u on Ω such that $(M, e^{-2u} g_M)$ is isometric to (Ω, g_E). Let $v = e^{-u}$. Then u and v satisfy the equations

$$\begin{cases} \Delta u = e^{2u} & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

and

$$\begin{cases} v\Delta v = |\nabla v|^2 - 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-2}$$

The asymptotic behavior of v near $C^{3,\alpha}$ boundary components of $\partial\Omega$ is, according to [8; 13],

$$v(z) = d(z) - \tfrac{1}{2}\kappa(y) d(z)^2 + c_3(y) d(z)^3 + O(d^{3+\alpha}(z)), \tag{1-3}$$

where $d(z)$ is the distance from z to the boundary $\partial\Omega$, $y \in \partial\Omega$ is the point where $d(z) = |z - y|$, and $\kappa(y)$ is the curvature at y. In (1-3), $c_3(y)$ is the first global term, which depends on the global geometry of Ω. We remark that when $\partial\Omega$ is smooth, v is the special boundary defining function as in [7, Lemma 2.1].

We consider a scaling-free quantity following [13]. Assuming that the outermost boundary component $\mathcal{C}$ of Ω is $C^{3,\alpha}$, we consider for $\mathcal{C}$ the functional

$$\lambda(\Omega, v) = -\int_{\mathcal{C}} \mathrm{d}l \cdot \int_{\mathcal{C}} c_3(y)\, \mathrm{d}l(y)\,, \tag{1-4}$$

where the line integral is taken under the Euclidean metric and c_3 is defined as in (1-3).

The functional λ was first studied by Shen and Wang [13] in a slightly different setting. In our notation, Shen and Wang have proven the following.

Theorem 1.1 [13, Theorems 1.1, 1.2, 1.3]. *Let Ω be a bounded $C^{3,\alpha}$ domain, and v be a solution to* (1-2) *on Ω. Let λ be given as in* (1-4). *Then*

(a) $\lambda(\Omega, v) \geq 0$, *with equality if and only if Ω is a round disk*;

(b) *if Ω is multiply connected*, *then*

$$\lambda(\Omega, v) > \frac{2\pi^2}{3}. \tag{1-5}$$

Theorem 1.1 can be regarded as a rigidity and gap theorem and may also be interpreted as a type of positive mass theorem.

If M is multiply connected, each boundary component can be realized as the outermost boundary via a proper conformal transformation. Even with the fixed boundary component, (1-4) is not conformally invariant. We define the following renormalized volume-type invariant, which is also conformally invariant for the pair (M, g_M).

Definition 1.2. Let (M, g_M) be a complete, noncompact hyperbolic surface of finite topological type that can be conformally compactified to a bounded domain in $\mathbb{C}$. Let λ be given as in (1-4). Define

$$\Lambda(M, g_M) = \inf_{\Omega}\left\{ \lambda(\Omega, v) \;\middle|\; \begin{array}{l} (M, v^2 g_M) \text{ is isometric to } (\Omega, g_E) \text{ and the} \\ \text{outermost boundary component of } \Omega \text{ is } C^{3,\alpha} \end{array} \right\}. \tag{1-6}$$

By the Riemann mapping theorem, for any M we may find a compactification whose outermost boundary is smooth. Hence, Λ is well defined.

Our main result is the following.

Theorem 1.3. *Let (M, g_M) be a complete noncompact hyperbolic doubly connected surface with conformal compactification (Ω_0, g_E), where $\Omega_0 \subset \mathbb{R}^2$ is bounded. Let Λ be as in (1-6) and λ as in (1-4).Then*

$$\Lambda(M, g_M) = \frac{2\pi^2}{3}\left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right], \tag{1-7}$$

where $0 < \beta < 1$ is the exponent of the modulus of continuity of Ω_0, so that Ω_0 is biholomorphic to $B_1 - \overline{B}_\beta$.

Moreover, for any bounded Ω biholomorphic to Ω_0 with a $C^{3,\alpha}$ outermost boundary component, we have

$$\lambda(\Omega, v) \geq \frac{2\pi^2}{3}\left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right], \tag{1-8}$$

and equality holds if and only if Ω is the image of $B_1 - \overline{B}_\beta$ under a composition of translation and homotheties.

Our definition provides a new type of conformal invariant for two-dimensional hyperbolic conformally compact Einstein (CCE) manifolds. For higher dimensions, the renormalized volume is a preferred conformal invariant. However, it is topological in two dimensions for hyperbolic CCE manifolds. By [14, Corollary 3.5] and [10, Appendix A.1], the renormalized volume of a conformally compact hyperbolic surface (M, g_M) equals $-2\pi\chi(M)$. Definition 1.2 provides an alternative invariant to work on. Our construction focuses on the geometric meaning of the next term in corresponding expansion (1-3). We are hopeful that our invariant can serve as a substitute for the renormalized volume in two dimensions.

Theorem 1.3 extends Shen and Wang's work [13], in which they compute Λ when M is simply connected, and separate this case with multiply connected domains. Our result further characterizes doubly connected M. Theorem 1.3 shows Λ can be attained for doubly connected domains, specifically as $B_1 - \overline{B}_\beta$ up to a composition of translation and homotheties. In particular, when $\beta = 0$, Theorem 1.3 also gives the sharp case of Theorem 1.1(b).

An interesting byproduct of Theorem 1.3 is a new geometric interpretation of the exponent of modulus of continuity β.

For CCE manifolds in dimension 4, there is also some interest in studying particular compactifications; refer to [5], [2],[3] and [4] for an incomplete list.

Our approach is partly inspired by [13], though there are key differences. Shen and Wang use conformal transformations and compare the boundary integral (1-4) of the underlying domain with that of the punctured disk. Subsequently, techniques involving the Schwarzian derivative are applied. In our work, we use more refined models. Specifically, we use Fourier series to derive inequalities and establish rigidity results. In particular, our method can recover their results.

Note that by [9], solutions to (1-1) on $C^{3,\alpha}$ domains exist and are unique. Thus, the functional λ can be defined on bounded $C^{3,\alpha}$ domains $\Omega \subset \mathbb{C}$ without specifying the solution to (1-2) on Ω. For punctured domains, we prove the following existence result and include the proof in the appendix.

Proposition 1.4. *Let $\Omega \subset \mathbb{C}$ be a smooth domain. There exists a solution to* (1-1) *on $\Omega - \{p_1, \dots, p_n\}$ and it is unique among all solutions u satisfying the growth condition*

$$u(x) \geq -\log\left(-r_l \cdot |x - p_l| \cdot \log \frac{|x - p_l|}{r_l}\right), \tag{1-9}$$

near p_l for all $l = 1, \dots, k$, for $r_l > 0$ such that $B_{r_l}(p_l) \supset \Omega$.

We list some new directions. It is worthwhile to consider $\Lambda(M, g_M)$ for general multiply connected domains. The resulting compactification will be a new canonical classification of bounded domains in $\mathcal{C}$, see [1] for some known classifications. We are looking for correct models for multiply connected cases. For conformally compact hyperbolic surfaces of nonzero genus and conformally compact hyperbolic manifolds in general, defining $\Lambda(M, g_M)$ for such spaces would be interesting.

This paper is organized as follows. In Section 2 we prove a technical lemma. The proof of Theorem 1.3 is split into two parts. In Section 3, we prove the inequality in (1-8). In Section 4, we study when equality in (1-8) holds.

2. A technical tool

We prove a technical lemma that provides an analytic result on annuli. The corresponding result for discs is easier. See [12] for more details.

Theorem 2.1 (Kellogg–Warschawski Theorem [11, Theorem 3.6]). *Let f map B_1 conformally onto a domain bounded by a $C^{k,\alpha}$ Jordan curve. Then f has an extension in $C^{k,\alpha}(\overline{B}_1)$.*

Theorem 2.1 is a key ingredient in proving our lemma:

Lemma 2.2. *Let $f \in B_1 - \overline{B}_\beta \to \mathbb{C}$ be an orientation-preserving biholomorphic map such that $f(B_1 - \overline{B}_\beta) = \Omega - K$ for some compact subset K of some bounded open Ω. Then there exists a holomorphic $g : B_1 - \overline{B}_\beta \to \mathbb{C}$ such that $g^2 = 1/\partial_z f$.*

Proof. Fix $\beta < r < 1$. Let Ω_1 be the bounded component of the complement of $f(\partial B_r)$, and let $f_1 : \Omega_1 \to B_r$ be an orientation-preserving biholomorphic map, whose existence is guaranteed by the Riemann mapping theorem. By Theorem 2.1, f_1 extends to a smooth diffeomorphism between $\overline{\Omega}_1$ and $\overline{B}_r$. Consider $h = f_1 \circ f$, which is an orientation-preserving diffeomorphism from $\overline{B}_r - \overline{B}_\beta$ to $\overline{B}_r - f_1(K)$.

Since h restricts to a self-homeomorphism of ∂B_r,

$$0 = \frac{\partial}{\partial t}\big((h\bar{h})(e^{it})\big) = 2\Re\big(i\,\partial_z h(e^{it}) \cdot e^{it} \cdot \bar{h}(e^{it})\big) = 2\Im\left(\frac{\partial_z h(e^{it}) \cdot e^{it}}{h(e^{it})}\right). \tag{2-1}$$

By (2-1), there is $c : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{\partial_z h(e^{it}) \cdot e^{it}}{h(e^{it})} = c(t) > 0, \tag{2-2}$$

where the inequality follows from h being orientation-preserving. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed path defined by $\gamma(t) = e^{it}$. Then

$$\begin{aligned} \operatorname{Ind}_{\partial_z h(\gamma)}(0) &= \operatorname{Ind}_{h(\gamma)/\gamma}(0) \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\partial_z h(\gamma)\cdot \gamma_t/\gamma - h(\gamma)/\gamma^2 \cdot \gamma_t}{h(\gamma)/\gamma}\,\mathrm{d}t \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\partial_z h(\gamma)\cdot\gamma_t}{h(\gamma)} - \frac{\gamma_t}{\gamma}\,\mathrm{d}t \\ &= \operatorname{Ind}_{h(\gamma)}(0) - \operatorname{Ind}_\gamma(0) = 0, \end{aligned} \tag{2-3}$$

where Ind is the index of a point with respect to a closed path and the first equality follows from (2-2).

By [12, Theorem 13.11], (2-3) shows that $\log(\partial_z h)$ is well defined on $B_r - \overline{B}_\beta$. Since f_1 is defined on Ω_1, which is a simply connected domain, the same result from [12] implies that $\log \partial_z f_1$ is well defined on Ω_1. Hence

$$\partial_z f(z) = \frac{\partial_z h(z)}{\partial_z f_1(f(z))} = \exp\big(\log \partial_z h(z) - \log \partial_z f_1(f(z))\big), \tag{2-4}$$

for $z \in B_r - \overline{B}_\beta$. Invoking [12, Theorem 13.11] once again, (2-4) implies that $\operatorname{Ind}(\partial_z f(\gamma))(0) = 0$; therefore $\log \partial_z f$ is well defined on $B_1 - \overline{B}_\beta$. In particular, there exists a holomorphic $g : B_1 - \overline{B}_\beta \to \mathbb{C}$ such that $g^2 = 1/\partial_z f$. □

3. Doubly connected domains

In this section, we prove our main theorem. In the simply connected case, it has been shown in [13].

Let u be a solution to (1-1) on Ω. Let $f : B_1 - \overline{B}_\beta \to \Omega$ be a biholomorphic function, and u_β be the solution to (1-1) for $\Omega_\beta = B_1 - \overline{B}_\beta$, then $(\Omega, (f^{-1})^* e^{2u_\beta} g_E)$ is complete with Gaussian curvature -1, hence

$$u = (u_\beta - \ln|f_z|) \circ f^{-1} \tag{3-1}$$

is a solution to (1-1) in Ω. For $v = e^{-u}$ and $v_\beta = e^{-u_\beta}$, this relation becomes

$$v = (v_\beta \cdot |f_z|) \circ (f^{-1}). \tag{3-2}$$

Remark 3.1. We quote the following explicit solution from [13]. The solution to (1-2) on $B_1 - \overline{B}_\beta$ is explicitly written as

$$v_\beta(r) = -r \cdot \left(\frac{1}{\pi} \ln \beta\right) \cdot \sin \frac{\ln r}{1/\pi \cdot \ln \beta}. \tag{3-3}$$

At ∂B_1, it has expansion

$$v_\beta = d - \frac{1}{2} d^2 - \frac{1}{6}\left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right] d^3 + O(d^4), \tag{3-4}$$

where $d(x) = 1 - |x|$, and the first global term of v_β near ∂B_1 is

$$c_{\beta,3} = -\frac{1}{6}\left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right]. \tag{3-5}$$

The following lemma is proved in [13]; we include it here for completeness.

Lemma 3.2. *Let Ω be a bounded domain with $C^{3,\alpha}$ outermost boundary component $\mathcal{C}$. Let $f : B_1 - \overline{B}_\beta \to \Omega$ be a biholomorphic map. Let $c_3, c_{\beta,3}$ be the first global terms of the solutions v, v_β to (1-2) on Ω, $B_1 - \overline{B}_\beta$ near $\mathcal{C}$, ∂B_1 respectively. If f takes ∂B_1 to $\mathcal{C}$, then*

$$\int_{\mathcal{C}} -6c_3 \,\mathrm{d}l = \int_{\partial B_1} \frac{-6c_{\beta,3}}{|f_z|} + 2\pi \fint_{\partial B_1} \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} - 2\pi \fint_{\partial B_1} \frac{\partial}{\partial r} \frac{1}{|f_z|}. \tag{3-6}$$

Proof. By [13], the c_3 term in the expansion of v can be expressed, in a way independent of the distance function $d(x)$, as

$$-6c_3 = \partial_N \Delta v, \tag{3-7}$$

where N is the outward pointing normal and Δ is the Laplacian under metric (Ω, g_E).

Let w and z be the standard euclidean coordinates on Ω and $B_1 - \overline{B}_\beta$ respectively. Then

$$\begin{aligned} \int_{\mathcal{C}} -6c_3 \,\mathrm{d}l &= \int_{f(\partial B_1)} \partial_N \Delta_w v \,\mathrm{d}l = \int_{\partial B_1} f^*(\partial_N \Delta_w v \cdot \mathrm{d}l) \\ &= \int_{\partial B_1} ((f^{-1})_* \partial_N)(\Delta_w v \circ f) f^* \,\mathrm{d}l \\ &= \int_{\partial B_1} \frac{1}{|f_z|} \frac{\partial}{\partial r} \big[\Delta_{|f_z|^2 g_E}(v_\beta \cdot |f_z|)\big] |f_z| \,\mathrm{d}l \\ &= \int_{\partial B_1} \frac{\partial}{\partial r} \left[\frac{1}{|f_z|^2} \Delta_{g_E}(v_\beta \cdot |f_z|)\right] \mathrm{d}l \,. \end{aligned} \tag{3-8}$$

We can compute the terms involving v_β on ∂B_1 explicitly:

$$\partial_r v_\beta = -1, \quad \Delta_z v_\beta = -2, \quad \partial_r \Delta_z v_\beta = -6c_{\beta,3} = \left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right]. \tag{3-9}$$

Thus

$$\begin{aligned} \int_{\mathcal{C}} -6c_3 \,\mathrm{d}l &= \int_{\partial B_1} \left(\frac{\partial}{\partial r} \frac{1}{|f_z|}\right) \Delta v_\beta + \int_{\partial B_1} \frac{1}{|f_z|} \frac{\partial}{\partial r} \Delta v_\beta - 2 \int_{\partial B_1} \frac{\partial^2}{\partial r^2} v_\beta \frac{\partial}{\partial r} \frac{1}{|f_z|} \\ &\quad - 2 \int_{\partial B_1} \frac{\partial}{\partial r} v_\beta \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} + \int_{\partial B_1} \frac{\partial}{\partial r} v_\beta \Delta \frac{1}{|f_z|} + \int_{\partial B_1} v_\beta \frac{\partial}{\partial r} \Delta \frac{1}{|f_z|} \\ &= -6c_{\beta,3} \int_{\partial B_1} \frac{1}{|f_z|} + 2 \int_{\partial B_1} \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} - \int_{\partial B_1} \Delta \frac{1}{|f_z|} \\ &= \int_{\partial B_1} \frac{-6c_{\beta,3}}{|f_z|} + 2\pi \fint_{\partial B_1} \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} - 2\pi \fint_{\partial B_1} \frac{\partial}{\partial r} \frac{1}{|f_z|}. \end{aligned}$$

For the middle equality, we used the fact that f extends to a $C^{3,\alpha}$ diffeomorphism from $\overline{B}_1 - B_\beta$ onto its image. The regularity of the extension is an easy application of Theorem 2.1, which indicates the existence of a $C^{3,\alpha}$ diffeomorphism f_1 from the closure of the region bounded by the $C^{3,\alpha}$ curve $\mathcal{C}$ to $\overline{B}_1$. The composition $f_1 \circ f$ is then a map from $B_1 - \overline{B}_\beta$ to $B_1 - K$ for some compact $K \subset B_1$ that sends ∂B_1 to ∂B_1. An application of the Schwarz reflection principle [12, Theorem 11.14] extends $f_1 \circ f$ to a $C^{3,\alpha}$ function on $\overline{B}_1 - \overline{B}_\beta$. Composing $(f_1)^{-1}$ with this extension of $f_1 \circ f$ gives us the desired $C^{3,\alpha}$ extension of f. □

The next result is an improvement of Theorem 1.1(b).

Theorem 3.3. *Let Ω be a doubly connected domain whose outermost boundary component $\mathcal{C}$ is $C^{3,\alpha}$. If Ω is biholomorphic to $B_1 - \overline{B}_\beta$, then*

$$\lambda(\Omega, v) \geq \frac{2\pi^2}{3}\left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right]. \tag{3-10}$$

Proof. Let $f : B_1 - \overline{B}_\beta \to \Omega$ be an orientation-preserving biholomorphic map that takes ∂B_1 to $\mathcal{C}$. By Lemma 3.2, we have

$$\int_{\mathcal{C}} -6c_3 \,\mathrm{d}l = \int_{\partial B_1} \frac{-6c_{\beta,3}}{|f_z|} + 2\pi \fint_{\partial B_1} \frac{\partial^2}{\partial r^2}\frac{1}{|f_z|} - 2\pi \fint_{\partial B_1} \frac{\partial}{\partial r}\frac{1}{|f_z|}. \tag{3-11}$$

By Lemma 2.2, there is a holomorphic $g : B_1 - \overline{B}_\beta \to \mathbb{C}$ such that $g^2 = 1/f_z$.

Consider the Laurent expansion of g,

$$g = \sum_{k=-\infty}^{\infty} b_k z^k, \tag{3-12}$$

which leads to

$$\fint_{\partial B_r} |g^2| = \frac{1}{2\pi r}\int_{\partial B_r} |g^2| = \frac{1}{2\pi r}\int_{\partial B_r} g\cdot\bar{g} = \sum_{k=-\infty}^{\infty} |b_k|^2 r^{2k}. \tag{3-13}$$

Therefore,

$$\begin{aligned} r^2 \fint_{\partial B_r} \frac{\partial^2}{\partial r^2} g^2 - r \fint_{\partial B_r} \frac{\partial}{\partial r}|g^2| &= \left(r^2\frac{\partial^2}{\partial r^2} - r\frac{\partial}{\partial r}\right) \sum_{k=-\infty}^{\infty} |b_k|^2 r^{2k} \\ &= \sum_{k=-\infty}^{\infty} |b_k|^2 2k(2k-2) r^{2k} \geq 0. \end{aligned} \tag{3-14}$$

Letting $r \to 1$, we conclude from (3-11) and (3-14) that

$$\begin{aligned} \int_{\mathcal{C}} -6c_3 \,\mathrm{d}l &= \int_{\partial B_1} \frac{-6c_{\beta,3}}{|f_z|} + 2\pi \fint_{\partial B_1} \frac{\partial^2}{\partial r^2}\frac{1}{|f_z|} - 2\pi \fint_{\partial B_1} \frac{\partial}{\partial r}\frac{1}{|f_z|} \\ &\geq \int_{\partial B_1} \frac{-6c_{\beta,3}}{|f_z|}. \end{aligned} \tag{3-15}$$

Finally, by Hölder's inequality and (3-5),

$$\begin{aligned} \lambda(\Omega, v) &\geq \frac{1}{6}\left[\left(\frac{\pi}{\ln\beta}\right)^2 + 1\right] \cdot \int_{\partial B_1} \frac{1}{|f_z|}\,\mathrm{d}l \cdot \int_{\partial B_1} |f_z|\,\mathrm{d}l \\ &\geq \frac{1}{6}\left[\left(\frac{\pi}{\ln\beta}\right)^2 + 1\right]\left(\int_{\partial B_1} \frac{1}{|f_z|^{1/2}} \cdot |f_z|^{1/2}\right)^2 \\ &= \frac{2\pi^2}{3}\left[\left(\frac{\pi}{\ln\beta}\right)^2 + 1\right], \end{aligned} \tag{3-16}$$

completing the proof. □

The first part of Theorem 1.1(a) can be recovered by a modification of the proof of Theorem 3.3. First, for a simply connected domain Ω and biholomorphic

$f : B_1 \to \Omega$, Lemma 3.2 holds after we change $c_{\beta,3}$ to the first global term $c_{B_1,3}$ of the solution v_{B_1} to (1-2) on B_1 near ∂B_1. Next, v_{B_1} takes the form

$$v_{B_1} = \tfrac{1}{2}(1-r), \quad c_{B_1,3} = 0. \tag{3-17}$$

Then

$$\int_{\mathcal{C}} -6c_3 \,\mathrm{d}l = 2\pi \fint_{\partial B_1} \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} - 2\pi \fint_{\partial B_1} \frac{\partial}{\partial r} \frac{1}{|f_z|}. \tag{3-18}$$

which is nonnegative by (3-14). Thus, we conclude that for any simply connected domain Ω,

$$\lambda(\Omega, v) \geq 0. \tag{3-19}$$

The other part of Theorem 1.1(a) will be recovered in the following section.

4. Rigidity of doubly connected domains

In this section, we prove a rigidity theorem for doubly connected domains.

Let $f : B_1 - \overline{B}_\beta \to \Omega$ be an orientation-preserving biholomorphic map. Consider

$$A(t) = \frac{1}{2\pi e^t} \cdot \int_{\partial B_{e^t}(0)} \frac{1}{|f_z|}, \quad B(t) = A_{tt}(t) - 2A_t(t), \tag{4-1}$$

for $t \in (\ln \beta, 0)$.

Lemma 4.1. *Let $r = e^t$, then*

$$B(t) = r^2 \fint_{\partial B_r(0)} \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} - r \fint_{\partial B_r(0)} \frac{\partial}{\partial r} \frac{1}{|f_z|} \tag{4-2}$$

for $t \in (\ln \beta, 0)$.

Proof. For convenience, take

$$\phi(r) = \frac{1}{2\pi r} \int_{\partial B_r(0)} \frac{1}{|f_z|} \tag{4-3}$$

so that $\phi(r) = A(t)$. For fixed $r \in (\beta, 1)$, we have

$$\frac{\partial}{\partial t} A = e^t \cdot \left(\frac{\partial}{\partial r}\phi\right)(r) = \frac{r}{2\pi r} \int_{\partial B_r} \frac{\partial}{\partial r} \frac{1}{|f_z|} \,\mathrm{d}l\,, \tag{4-4}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial t^2} A &= e^t \cdot \left(\frac{\partial}{\partial r}\phi\right)(r) + e^{2t}\left(\frac{\partial^2}{\partial r^2}\phi\right)(r) \\ &= \frac{r}{2\pi r} \int_{\partial B_r} \frac{\partial}{\partial r} \frac{1}{|f_z|} \,\mathrm{d}l + \frac{r^2}{2\pi r} \int_{\partial B_r} \frac{\partial^2}{\partial r^2} \frac{1}{|f_z|} \,\mathrm{d}l\,. \end{aligned} \tag{4-5}$$

Thus the claim follows. □

By Lemma 3.2 and (4-2), we see that

$$\lim_{t\to 0} 2\pi \cdot B(t) = \int_{C} -6c_3 - \int_{\partial B_1} \frac{-6c_{\beta,3}}{|f_z|}, \tag{4-6}$$

which in a rough sense, measures the difference between $\lambda(\Omega, v)$ and $\lambda(B_1 - \overline{B}_\beta, v_\beta)$. Proposition 4.2 gives a description of the corresponding holomorphic functions f in the case when $B(0)$ vanishes.

Proposition 4.2. *Let $A(t)$ and $B(t)$ be defined by (4-1). Then*

$$B(t) \geq 0, \tag{4-7}$$

for all $t \in (\ln \beta, 0)$. If for some $t_0 \in (\ln \beta, 0)$ we have $B(t_0) = 0$, or if $\lim_{t\to 0} B(t) = 0$, then

$$B(t) = 0, \tag{4-8}$$

for all $t \in (\ln \beta, 0)$ and

$$f(z) = C_1 + C_2 z \quad \text{or} \quad f(z) = C_1 + \frac{C_2}{z + C_3}. \tag{4-9}$$

Proof. The nonnegativeness of B follows from (3-14) and (4-2).

Let $f : B_1 - \overline{B}_\beta \to \Omega$ be a biholomorphic orientation-preserving function. By Lemma 2.2, there is a holomorphic $g : B_1 - \overline{B}_\beta \to \mathbb{C}$ such that $g^2 = 1/f_z$. Consider the Laurent expansion of g:

$$g = \sum_{k=-\infty}^{\infty} b_k z^k. \tag{4-10}$$

By (3-14) and (4-2), we have

$$B(t) = \sum_{k=-\infty}^{\infty} |b_k|^2 2k(2k-2) r^{2k}. \tag{4-11}$$

In particular, for any $k \neq 0, 1$,

$$|b_k|^2 \leq r^{-2k} B(t), \tag{4-12}$$

for all $t \in (\ln \beta, 0)$, $r = e^t$. By the assumption of Proposition 4.2, either $B(t) = 0$ for some t or $\lim_{t\to 0} B(t) = 0$, it follows that,

$$\lim_{n\to\infty} |b_k|^2 = 0. \tag{4-13}$$

for $k \neq 0, 1$.

Then f_z is of the form

$$f_z = \frac{1}{(b_0 + b_1 z)^2}. \tag{4-14}$$

Suggesting that

$$f = \begin{cases} C + b_0^{-2} z & \text{if } b_1 = 0 \\ C - b_1^{-2} \cdot \big(b_0/b_1 + z\big)^{-1} & \text{if } b_1 \neq 0, \end{cases} \tag{4-15}$$

for some constant $C \in \mathbb{C}$.

It is easy to check that when (4-15) holds, $B(t) = 0$ for all $t \in (\ln r, 0)$. □

Theorem 4.3. *Let Ω be a doubly connected domain with $C^{3,\alpha}$ outermost boundary. If Ω is biholomorphic to $B_1 - \overline{B}_\beta$, and*

$$\lambda(\Omega, v) = \frac{2\pi^2}{3}\left[\left(\frac{\pi}{\ln \beta}\right)^2 + 1\right], \tag{4-16}$$

then Ω can be obtained from $B_1 - \overline{B}_\beta$ by a composition of translation and homotheties.

Proof. By the proof of Theorem 3.3 and (4-2), the equality (4-16) holds when $\lim_{t\to 0} B(t) = 0$ and Hölder's inequality in (3-16) is an equality. By Proposition 4.2, $\lim_{t\to 0} B(t) = 0$ forces the biholomorphic map $f : B_1 - \overline{B}_\beta \to \Omega$ to be a Möbius transformation; hence f_z is of the form

$$f_z = \frac{1}{(b_0 + b_1 z)^2}. \tag{4-17}$$

The Hölder's inequality in (3-16) becomes an equality when $|f_z|$ is a constant on ∂B_1, forcing either $b_0 = 0$ or $b_1 = 0$ in (4-17). We conclude that either $f = C_1 + C_2 z$ or $f = C_1 + C_2/z$ for $C_2 \neq 0$. Hence, Ω can be obtained from $B_1 - \overline{B}_\beta$ by a composition of translation and homotheties. □

The second part of Theorem 1.1(a) can be proved as an easy corollary of Proposition 4.2. Indeed, by Theorem 3.3 and the standard comparison theorem, $\lambda(\Omega, v) = 0$ implies that Ω is a simply connected domain. Let $f : B_1 \to \Omega$ be a biholomorphic map and define $A(t)$, $B(t)$ accordingly. Then Proposition 4.2 implies that f is a Möbius transformation, and therefore $f(B_1) = \Omega$ must be a disk.

Appendix

By [13], a solution to (1-1) on $B_r(p) - \{p\}$ is given by

$$u_{B_r(p)-\{p\}}(x) = -\log\Big(-\frac{|x-p|}{r} \cdot \log\Big(\frac{|x-p|}{r}\Big)\Big) - 2\ln r. \tag{A-1}$$

A solution to $B_1(p) - \overline{B}_r(p)$ is given by

$$u_{B_1(p)-\overline{B}_r(p)}(x) = -\log\left(\frac{|x-p|}{\pi}\cdot\log(1/r)\cdot\sin\Big(\frac{\pi\log(1/|x-p|)}{\log(1/r)}\Big)\right). \tag{A-2}$$

Then the growth condition (1-9) is equivalent to $u \geq u_{B_{r_l}(p_l)-\{p_l\}}$ for all $l = 1, \dots, n$.

Theorem A.1. *Let Ω be a bounded domain with smooth boundary. Let $p_1, \dots, p_n \in \Omega$ be a collection of points. Then there exists $w \in C^\infty(M)$ such that*

$$\begin{cases} \Delta w = e^{2w} & \text{in } \Omega - \{p_1, \dots, p_n\}, \\ w = \infty & \text{on } \partial\Omega \cup \{p_1, \dots, p_n\}, \end{cases} \tag{A-3}$$

and (1-9) is satisfied at $\{p_1, \dots, p_n\}$.

Proof. We follow the same idea as in [9]. Let u_Ω be the solution to (1-1) on Ω. Let A_m be an exhaustion of Ω by compact subsets with smooth boundary. Take $\Omega_m = A_m - \bigcup_{l=1}^n B_{1/m}(p_l)$. Let w_m be the solution to (1-1) on Ω_m. By the Maximum principle, w_m is a decreasing sequence. Additionally, w_m is bounded below by $u_{B_{r_l}(p_l)-\{p_l\}}$ for some r sufficiently large, with $l = 1, 2, \dots, n$.

Fix m. By the interior L^p estimate, [6, Theorem 9.11], for $k > m$, we have

$$\|w_k\|_{2,p;\Omega_m} \leq C(p, \Omega_m, \Omega_{m+1})\big(\|w_k\|_{p;\Omega_{m+1}} + \|e^{w_k}\|_{p;\Omega_{m+1}}\big) \tag{A-4}$$

where $\|w_k\|_{p;\Omega_{m+1}}$, $\|e^{w_k}\|_{p;\Omega_{m+1}}$ is bounded independent of k, since

$$\max(u_{B_{r_l}(p_l)-\{p_l\}}, u_\Omega) \leq w_k \leq w_{m+1} \tag{A-5}$$

for all l. This shows that w_k is a bounded sequence in $W^{2,p}(\Omega_m)$. Hence is bounded in $C^{1,\alpha}(\Omega_m)$.

By the interior Schauder estimate, [6, Corollary 6.3], we have

$$|w_k|_{2,\alpha;\Omega_m} \leq C(\alpha, \Omega_m, \Omega_{m+1})\big(|w_k|_{0;\Omega_{m+1}} + |e^{2w_k}|_{0,\alpha;\Omega_{m+1}}\big), \tag{A-6}$$

Hence, w_k is bounded in $C^{2,\alpha}(\Omega_m)$ independently of k. Taking a subsequence if necessary, we may assume that w_m converges in $C^2(\Omega_m)$ to some w such that

$$\Delta w = e^{2w} \tag{A-7}$$

in Ω_m and $\max(u_\Omega, u_{B_{r_l}(p_l)-\{p_l\}}) \leq w \leq w_{m+1}$.

Pushing m to ∞, w_m converges to a smooth function w satisfying $\Delta w = e^{2w}$ and

$$\max(u_\Omega, u_{B_{r_l}(p_l)-\{p_l\}}) < w. \tag{A-8}$$

Since $u_\Omega = \infty$ on $\partial\Omega$, and $u_{B_{r_l}(p_l)-\{p_l\}} = \infty$ on p_l for all l, we see that w is the desired function. □

Following [9], we shall establish the uniqueness of solution u to (1-1) on $\Omega - \{p_1, \dots, p_n\}$ satisfying (1-9) in two steps. First we study the asymptotic behavior of a solution near $\{p_1, \dots, p_n\}$. Then we apply maximum principle to the difference of two solutions.

We assume, without loss of generality, that

$$B_2(p_l) \subset \Omega - \{p_1, \dots, p_n\} \tag{A-9}$$

for all $l = 1, \dots, n$.

Proposition A.2. *Let u be a solution to (1-1) on $\Omega - \{p_1, \dots, p_n\}$. If u satisfies (1-9) at $\{p_1, \dots, p_n\}$, then there exists a constant C such that*

$$|u - \log\big(-|x - p_l| \cdot \log(|x - p_l|)\big)| < C \tag{A-10}$$

in a neighborhood of p_l for all $l = 1, \dots, n$.

Proof. Fix any l, and assume, without loss of generality, that $p_l = 0$. By the maximum principle,

$$u_{B_{r_l} - \{0\}} \le u \le u_{B_1 - \overline{B}_r} \tag{A-11}$$

on $B_1 - \overline{B}_r$, for any $0 < r < 1$. For $x \in B_1 - \overline{B}_r$, let $|x|$ be the distance to p_l. Take $r = |x|^k$; then

$$u_{B_1 - \overline{B}_r}(x) = -\log\bigg(\frac{-k|x|}{\pi} \cdot \log(|x|) \cdot \sin\Big(\frac{\pi}{k}\Big)\bigg). \tag{A-12}$$

Pushing k to ∞, we see that

$$u \le -\log\big(-|x| \cdot \log(|x|)\big) \tag{A-13}$$

near p_l. Now, the difference between the upper bound and the lower bound of u equals

$$-\log(-|x| \log|x|) + \log(-r|x| \log(|x|/r)) = \log \frac{r \log(|x|/r)}{\log|x|}, \tag{A-14}$$

which converges to 0 as $|x| \to 0$, proving the claim. □

Proof of Proposition 1.4. The existence of a solution follows from Theorem A.1. We follow the idea in [9]. Let u_1 and u_2 be two solutions satisfying (1-9) at $p_1, \dots, p_n$. By adding a constant if necessary, we may assume that $u_1, u_2 \ge 2\log 2$; then, for these two functions after modification, there exists some constant C such that

$$\Delta u_i = Ce^{2u_i} \tag{A-15}$$

on Ω for $i = 1, 2$.

For any $1/2 < s < 1$, by Proposition A.2,

$$\frac{su_1}{u_2} = \frac{su_1/(-\log(-|x-p_l|\cdot\log(|x-p_l|)))}{u_2/(-\log(-|x-p_l|\cdot\log(|x-p_l|)))} \to s < 1 \tag{A-16}$$

near p_l for all $l = 1, 2, \ldots, n$. By [8], near other nondegenerate boundary components, su_1/u_2 also converges to $s < 1$.

We claim that $su_1 \le u_2$. If not, let x_0 be the point where $su_1 - u_2$ attains its maximum. Then $(su_1 - u_2)(x_0) > 0$, and

$$0 \ge \Delta(su_1 - u_2)(x_0) = sCe^{2u_1} - Ce^{2u_2} \ge e^{2su_1} - e^{2u_2} > 0, \tag{A-17}$$

where the second inequality comes from [9, Lemma 4.3]. This leads to a contradiction.

Pushing s to 1, we get $u_1 \le u_2$. By switching u_1 and u_2, we get the inequality from other direction, thereby proving the claim. □

Acknowledgements

The author thanks Biao Ma for introducing him to the work of Shen and Wang [13]. The author also thanks Hao Fang, Lihe Wang and Biao Ma for helpful discussions, and Hao Fang for guidance in writing this paper.

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Received October 2, 2023.

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