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Let E/F be a quadratic extension of local nonarchimedean fields of characteristic zero and let D be a quaternion algebra over F containing E. We study a relation between the existence of twisted linear models on $\mathrm{GL}_n(D)$ and the local root numbers.

1. Introduction

Let E/F be a quadratic extension of local nonarchimedean fields of characteristic zero and let $\eta = \eta_{E/F} : F^\times/NE^\times \to \{\pm 1\}$ be the quadratic character associated to the extension E/F by local class field theory. Let $\omega : F^\times \to \mathbb{C}^\times$ and $\chi : E^\times \to \mathbb{C}^\times$ be two characters satisfying the condition $\chi^n|_{F^\times}\omega = 1$. Let A be a central simple algebra (CSA) over F of dimension $4n^2$ with a fixed embedding $E \to A$, and let B be the centralizer of E in A. Then B is a CSA of dimension n^2 over E. Put $G = A^\times$ and $H = B^\times$, both regarded as algebraic groups over F. Let Z be the center of G. Let π be an irreducible admissible representation of G whose central character is ω. We say that π is (H, χ^{-1})-distinguished if

$$\mathrm{Hom}_H(\pi, \chi^{-1}) \neq 0.$$

Here, χ^{-1} is regarded as a character of H by composing χ^{-1} with the reduced norm map $H \to E^\times$. Elements of $\mathrm{Hom}_H(\pi, \chi^{-1})$ are called (local) twisted linear periods, or (local) twisted linear models. Let π_0 be the representation of $\mathrm{GL}_{2n}(F)$

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which shares the same L-parameter as that of π and let $\pi_{0,E}$ be the base change of π_0 to $\mathrm{GL}_{2n}(E)$. The following conjecture of Prasad and Takloo-Bighash predicts when π is (H, χ^{-1})-distinguished in terms of the local epsilon factor.

Conjecture 1.1 [Prasad and Takloo-Bighash 2011]. *Let the notation be as above. If π is (H, χ^{-1})-distinguished, then the following two conditions hold*:

(1) *The Langlands parameter of π_0 takes values in* $\mathrm{GSp}_{2n}(\mathbb{C})$ *with similitude factor* $\chi^{-1}|_{F^\times}$.

(2) $\varepsilon(\pi_{0,E} \otimes \chi) = (-1)^r \eta(-1)^n \chi(-1)^n$, *where r is the split rank of G.*

Conversely, if π is a discrete series representation and satisfies the above conditions (1) *and* (2), *then π is (H, χ^{-1})-distinguished.*

Note that in [Prasad and Takloo-Bighash 2011] it is further assumed that the Jacquet–Langlands (JL) transfer $\pi_0 = \mathrm{JL}(\pi)$ is generic. This assumption is shown to be unnecessary in [Suzuki 2021]. In recent years, there has been much progress towards Conjecture 1.1. When $F = \mathbb{R}$, Conjecture 1.1 is established in [Suzuki and Tamori 2023] in all cases. When F is a finite extension of $\mathbb{Q}_p$ and χ is the trivial character, if either $p \neq 2$, or $G = \mathrm{GL}_n(D)$ where D is a quaternion algebra, then Conjecture 1.1 holds by a combination of [Sécherre 2024], [Suzuki 2021], [Xue 2021] and [Suzuki and Xue 2024]. For a general χ, Conjecture 1.1 is proved when π is a Steinberg representation, see [Chommaux 2019], and when $G = \mathrm{GL}_{2n}(F)$ and π is a depth-zero cuspidal representation, see [Chommaux and Matringe 2022].

In this paper, we study Conjecture 1.1 for $G = \mathrm{GL}_n(D)$ where D is a quaternion algebra and a general χ. Our first main result is a proof of the forward direction of the conjecture.

Theorem 1.2 (Theorem 5.1). *Let $G = \mathrm{GL}_n(D)$ where D is a quaternion algebra over F containing E, $H = \mathrm{GL}_n(E)$. The forward direction of* Conjecture 1.1 *holds.*

Theorem 1.2 is proved in Section 5, and is based on a relative trace formula proposed in [Xue and Zhang 2023]. We first prove Theorem 1.2 when π is supercuspidal, by combining a local-global argument and an involution method similar to that in [Xue 2021, Section 4]. If π is a discrete series representation, then π is a segment of the form

$$\{\rho\nu_\rho^{-(\ell-1)/2}, \ldots, \rho\nu_\rho^{(\ell-1)/2}\}$$

where ρ is an irreducible supercuspidal representation of $\mathrm{GL}_s(D)$, and $n = s\ell$. Then the discrete series case follows from this consideration and the supercuspidal case. Finally, using a classification of (H, χ^{-1})-distinguished representations, the general case of Theorem 1.2 follows from the case of discrete series.

Our second main result concerns the converse direction of Conjecture 1.1, and we prove it under some additional hypothesis.

Theorem 1.3 (Theorem 6.1). *Let D be a quaternion algebra over F containing E. Let π be a discrete series representation of $G = \mathrm{GL}_n(D)$ whose Jacquet–Langlands transfer $\pi_0 = \mathrm{JL}(\pi)$ to $\mathrm{GL}_{2n}(F)$ satisfies conditions* (1) *and* (2) *in* Conjecture 1.1. *Assume that $\chi|_{F^\times}$ is trivial, and that $\mathrm{BC}(\pi_0)$ is supercuspidal. Then π is (H, χ^{-1})-distinguished.*

When χ is trivial, the counterpart of Theorem 1.3 is proved in [Xue 2021] by studying certain minimal unipotent orbital integrals. The approach we take here, given in Section 6, is simpler and avoids the construction of orbital integrals attached to unipotent orbits. The main idea is that under the assumptions in Theorem 1.3, we can globalize π_0 to a cuspidal automorphic representation $\underline{\pi}_0$ of $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$ whose base change $\mathrm{BC}(\underline{\pi}_0)$ to $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{E}})$ is globally distinguished by $(\mathrm{GL}_n(\mathbb{A}_{\underline{E}})\times\mathrm{GL}_n(\mathbb{A}_{\underline{E}}), \underline{\chi}^{-1}_{H'})$ and $(\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}}), \underline{\eta})$. Here, we denote a global object by a letter with an underline, and $\underline{\chi}^{-1}_{H'}$ is the character on $\mathrm{GL}_n(\mathbb{A}_{\underline{E}}) \times \mathrm{GL}_n(\mathbb{A}_{\underline{E}})$ defined by $\underline{\chi}^{-1}_{H'}(h_1, h_2) = \underline{\chi}^{-1}(h_1\overline{h_2})$. We refer the reader to Section 6 for unexplained notations. The main difficulty is that we require $\mathrm{BC}(\underline{\pi}_0)$ to be distinguished by both $(\mathrm{GL}_n(\mathbb{A}_{\underline{E}}) \times \mathrm{GL}_n(\mathbb{A}_{\underline{E}}), \underline{\chi}^{-1}_{H'})$ and $(\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}}), \underline{\eta})$. To solve the problem, we move on to the Bessel periods for orthogonal groups and use a trace formula argument.

The condition that $\chi|_{F^\times}$ is trivial will be used only at this final globalization step. With this condition, we may view χ as a character of SO(2) and make use some known cases of the global Gross–Prasad conjecture for $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$. To treat the case of a general χ, we will need a Gross–Prasad-type conjecture for $\mathrm{GSpin}(2n+1)\times\mathrm{GSpin}(2)$. We hope that the argument in this paper could stimulate research in this direction.

We stick to the case $G = \mathrm{GL}_n(D)$ in this paper because we follow a relative trace formula approach and the relevant relative trace formula is only established when $G = \mathrm{GL}_n(D)$. Once we have the relative trace formula for all central simple algebras, it is possible to extend our argument to the more general situation.

We end the introduction by giving a brief overview of the structure of the rest of the paper. In Section 2, we review some basic facts about representations over local nonarchimedean field, and several local and global functorial lifts. In Sections 3 and 4, we recall from [Xue and Zhang 2023] the geometric side and the spectral side of the relative trace formula. Then the proof of Theorem 1.2 is given in Section 5, while the proof of Theorem 1.3 is given in Section 6.

2. Preliminaries

In this section, we let F be either a number field or a local nonarchimedean field of characteristic zero. Let E/F be a quadratic field extension, and let C be a central division algebra of dimension d^2 over F. Let $G_r = \mathrm{GL}_r(C)$ be the multiplicative group of $\mathrm{Mat}_r(C)$. If F is a number field, we denote by $\mathbb{A}_F$ its ring of adeles.

2.1. ***Representations over local nonarchimedean field.*** Let F be a local nonarchimedean field of characteristic zero. We first recall some basic facts about the local Langlands correspondence and the local Jacquet–Langlands transfer for $G_r = \mathrm{GL}_r(C)$. We refer the reader to [Aubert et al. 2016; Deligne et al. 1984] for more details.

An element $g \in \mathrm{GL}_{rd}(F)$ is called regular semisimple if the characteristic polynomial of g has distinct roots in an algebraic closure of F. There is a standard way of defining the characteristic polynomial for elements of $\mathrm{GL}_r(C)$ (see, for example, [Pierce 1982]). If $g' \in \mathrm{GL}_r(C)$, then the characteristic polynomial of g' has coefficients in F, and it is monic and has degree rd. The definition of a regular semisimple element of $\mathrm{GL}_r(C)$ is the same as for $\mathrm{GL}_{rd}(F)$. For $g \in \mathrm{GL}_r(C)$ and $g' \in \mathrm{GL}_{rd}(F)$, we say that g corresponds to g' if g and g' are regular semisimple and have the same characteristic polynomial. In this case, we write $g \leftrightarrow g'$. Let $\mathrm{Irr}(\mathrm{GL}_r(C))$ (resp. $\mathrm{Irr}(\mathrm{GL}_{rd}(F))$ denote the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_r(C)$ (resp. $\mathrm{GL}_{rd}(F)$), and let $\mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_r(C))$ (resp. $\mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_{rd}(F))$ denote the subset of discrete series representations of $\mathrm{GL}_r(C)$ (resp. $\mathrm{GL}_{rd}(F)$). For a representation π of $\mathrm{GL}_r(C)$ or $\mathrm{GL}_{rd}(F)$, we write θ_π for its character.

Theorem 2.1 [Deligne et al. 1984]. *There is a unique bijection*

$$\mathrm{JL} : \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_r(C)) \to \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_{rd}(F))$$

such that for $\pi \in \mathrm{Irr}_{\mathrm{disc}}(\mathrm{GL}_r(C))$, *we have*

$$(-1)^r \theta_\pi(g) = (-1)^{rd} \theta_{\mathrm{JL}(\pi)}(g')$$

for all $g \in \mathrm{GL}_r(C)$ *and* $g' \in \mathrm{GL}_{rd}(F)$ *such that* $g \leftrightarrow g'$.

Let $\mathrm{WD}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group of F. Let $\Phi(G_r)$ be the set of equivalence classes of L-parameters $\phi : \mathrm{WD}_F \to \mathrm{GL}_{rd}(\mathbb{C})$ which is relevant to G_r. Note that $\Phi(G_r) \subsetneq \Phi(\mathrm{GL}_{rd}(F))$ if G_r is not split.

The local Langlands correspondence (LLC) for $\mathrm{GL}_{rd}(F)$ established in [Harris and Taylor 2001; Henniart 2000; Scholze 2013] together with the Jacquet–Langlands transfer gives the LLC for G_r, which is a canonical bijective map

$$\mathrm{Irr}(G_r) \to \Phi(G_r).$$

It follows that we have a canonical injective map

$$\mathrm{rec}_{C,r} : \mathrm{Irr}(G_r) \to \Phi(\mathrm{GL}_{rd}(F)),$$

whose image is $\Phi(G_r)$. Note that for each $\phi \in \Phi(\mathrm{GL}_{rd}(F))$, its fiber $\mathrm{rec}_{C,r}^{-1}(\phi)$ is a singleton if $\phi \in \Phi(G_r)$, and empty otherwise. Given an irreducible representation π of G_r, we call the corresponding representation of $\mathrm{GL}_{rd}(F)$ which shares the same L-parameter as that of π the Jacquet–Langlands transfer of π, and denote it by $\mathrm{JL}(\pi)$.

Let E/F be a quadratic field extension. Given an irreducible admissible representation π of $\mathrm{GL}_n(F)$, we denote by $\mathrm{BC}(\pi)$ the local quadratic base change of π defined in [Arthur and Clozel 1989, Section 1.6], which is an irreducible admissible representation of $\mathrm{GL}_n(E)$.

Let τ be an irreducible admissible representation of the split $\mathrm{SO}_{2n+1}(F)$ (resp. $\mathrm{GSpin}_{2n+1}(F)$) with parameter $\phi_\tau : \mathrm{WD}_F \to \mathrm{Sp}_{2n}(\mathbb{C})$ (resp. $\mathrm{WD}_F \to \mathrm{GSp}_{2n}(\mathbb{C})$), and let $r : \mathrm{Sp}_{2n}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})$ (resp. $r : \mathrm{GSp}_{2n}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})$) be the embedding. An irreducible admissible representation π of $\mathrm{GL}_{2n}(F)$ is called a local functorial lift of τ if the parameter ϕ_π of π is given by $\phi_\pi = r \circ \phi_\tau$.

2.2. ***Automorphic representations.*** Let E/F be a quadratic extension of number fields with $\eta = \eta_{E/F}$ the quadratic character associated to the extension E/F via global class field theory. The global Jacquet–Langlands transfer is an injective map from the set of irreducible discrete series representations of $G_r(\mathbb{A}_F)$ to that of $\mathrm{GL}_{rd}(\mathbb{A}_F)$ [Badulescu 2008, Theorem 5.1]. We denote this map still by JL, since there is no chance of confusion. For the following fact about global base change lift, we refer the reader to [Arthur and Clozel 1989, Section 3, Theorem 4.2]. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\pi \ncong \pi \times \eta$. Then its base change to $\mathrm{GL}_n(\mathbb{A}_E)$, denoted by $\mathrm{BC}(\pi)$, exists and is unique, which is an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Moreover, by [Arthur and Clozel 1989, Section 3, Theorem 5.1], $\mathrm{BC}(\pi)_v = \mathrm{BC}(\pi_v)$ for all places v of F.

Let τ be an irreducible cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A}_F)$ and let π be an irreducible automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. We say that π is a (weak) functorial lift of τ if for almost all finite places v of F where τ_v is unramified, π_v is the local functorial lift of τ_v, and at every infinite place v, the infinitesimal character of π_v is determined, via the L-morphism, by that of τ_v. The existence of the global functorial lift from SO_{2n+1} to GL_{2n} is proven to hold by Arthur [2013] using the trace formula, and independently by Cai, Friedberg and Kaplan [2024] using the generalized doubling method and the converse theorem.

3. The geometric side: orbital integrals and smooth transfer

In this section, we assume that E/F is a quadratic field extension of either local fields or global fields of characteristic zero with the nontrivial Galois involution $g \mapsto \bar{g}$.

3.1. ***The split side.*** Let E/F be either local or global. Let $G' = \mathrm{Res}_{E/F}(\mathrm{GL}_{2n})$, $H' = \mathrm{Res}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_n)$, embedded in G' as diagonal blocks, and $H'' = \mathrm{GL}_{2n}(F)$.

The split center of G' is $Z' \cong \mathrm{GL}_{1,F}$, embedded in G' diagonally. Let

$$S' = \{g\bar{g}^{-1} \mid g \in G'\} \cong G'/H''.$$

This is a symmetric space on which H' acts by twisted conjugation. We say that an element in S' is semisimple if its H'-orbit is Zariski closed. We say that an element in S' is regular semisimple if its stabilizer in H' is a torus of dimension n, and it is in addition elliptic if its stabilizer is an anisotropic torus modulo the split center of G'. An element $g \in G'$ is semisimple (regular semisimple, elliptic, respectively) if $g\bar{g}^{-1}$ is so in S'. Let G'_{reg} and G'_{ell} denote the subsets of regular semisimple and elliptic elements of G'. We have the following classification of semisimple elements in $S'(F)$.

Lemma 3.1 [Xue and Zhang 2023]. *The following results hold.*

(1) *Every semisimple element in S' is in the H'-orbit of the form $s'(\alpha, n_1, n_2, n_3)$, where $n_1 + n_2 + n_3 = n$ is a partition of n, $\alpha \in \mathrm{GL}_{n_1}(E)$, $\alpha\bar{\alpha} \in \mathrm{GL}_{n_1}(F)$ is semisimple in the usual sense*, $\det(\alpha\bar{\alpha} - 1) \neq 0$, *and*

$$s'(\alpha, n_1, n_2, n_3) = \begin{pmatrix} \alpha & & & 1_{n_1} & & \\ & 0_{n_2} & & & 1_{n_2} & \\ & & 1_{n_3} & & & 0_{n_3} \\ 1_{n_1} - \alpha\bar{\alpha} & & & -\bar{\alpha} & & \\ & 1_{n_2} & & & 0_{n_2} & \\ & & 0_{n_3} & & & 1_{n_3} \end{pmatrix}.$$

(2) *Let $s' = s'(\alpha, n_1, n_2, n_3)$ be a semisimple element as in* (1). *It is regular semisimple if and only if $n_1 = n$ and $\alpha\bar{\alpha} \in \mathrm{GL}_n(F)$ is regular semisimple in the usual sense. It is elliptic if $\alpha\bar{\alpha} \in \mathrm{GL}_n(F)$ is elliptic in the usual sense.*

(3) *Two regular semisimple $s'(\alpha_1, n, 0, 0)$ and $s'(\alpha_2, n, 0, 0)$ are in the same H'-orbit if and only if α_1 and α_2 are twisted conjugate in* $\mathrm{GL}_n(E)$.

If $n_2 = n_3 = 0$, we write $s'(\alpha, n, 0, 0) = s'(\alpha)$.

Let $\boldsymbol{A}^n$ be the affine space of dimension n over F and let $q' : S' \to \boldsymbol{A}^n$ be the morphism

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \operatorname{tr} \wedge^i (2A\bar{A} - 1), \quad i = 1, \dots, n.$$

Then the map q' is a categorical quotient [Xue and Zhang 2023, Lemma 5.4].

Now we assume E/F is a quadratic extension of number fields, and suppose that $\eta : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ is the quadratic character attached to E/F by class field theory. We fix a character $\tilde{\eta} : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\tilde{\eta}|_{\mathbb{A}_F^\times} = \eta$. Let $f' \in \mathcal{C}_c^\infty(G'(\mathbb{A}_F))$

and let $x \in G'_{\mathrm{reg}}(\mathbb{A}_F)$ be regular semisimple. We define the orbital integral

$$O^{G'}(x, f') := \int_{(H'\times H'')_x(\mathbb{A}_F)\backslash(H'\times H'')(\mathbb{A}_F)} f'(h^{-1}xh'')(\chi_{H'}\chi^{-1}\tilde{\eta}^{-1})(h)(\chi\tilde{\eta})^{-1}(h^{-1}xh'')\,\mathrm{d}h\,\mathrm{d}h''.$$

Here, for $h = \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \in H'(F)$, $\chi_{H'}(h) = \chi(h_1\overline{h_2})$. This orbital integral is absolutely convergent for all regular semisimple x (see [Xue and Zhang 2023, Appendix A]).

The orbital integral $O^{G'}(g, f')$ can be simplified in the following way. Put

$$\tilde{f}'(g\bar{g}^{-1}) := \int_{H''(\mathbb{A}_F)} f'(gh)(\chi\tilde{\eta})^{-1}(gh)\,\mathrm{d}h.$$

Then $\tilde{f}' \in \mathcal{C}_c^\infty(S'(\mathbb{A}_F))$ and

$$O^{G'}(g, f') = O^{S'}(s', \tilde{f}') := \int_{H'_{s'}(\mathbb{A}_F)\backslash H'(\mathbb{A}_F)} \tilde{f}'(h^{-1}s'\bar{h})(\chi_{H'}\chi^{-1}\tilde{\eta}^{-1})(h)\,\mathrm{d}h, \quad s' = g\bar{g}^{-1}.$$

If v is a place of F, we define the local orbital integral by the same formula, but integrating over F_v-points instead, and the local orbital integral can be simplified in a similar way.

We now define a transfer factor at each place v for regular semisimple elements of $G'(F)$. We fix a purely imaginary element $\tau \in E^\times$ such that $\bar{\tau} = -\tau$. Let $x \in G'(F_v)$ be regular semisimple, and write

$$x\bar{x}^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in S'(F_v),$$

with $\alpha_i \in \mathrm{Mat}_n(E_v)$. Then we define

$$\kappa_v^{G'}(x) = \chi_v(\alpha_4)\,\tilde{\eta}_v(\tau\alpha_2).$$

3.2. ***The nonsplit side.*** Assume E/F is global. Let D be a quaternion algebra over F with fixed embedding $E \to D$. Let $G = \mathrm{GL}_n(D)$, $H = \mathrm{Res}_{E/F}\,\mathrm{GL}_n$, and $Z = \mathrm{GL}_{1,F}$. We fix an element $\epsilon \in NE^\times$ (resp. $F^\times\backslash NE^\times$) if D splits (resp. ramified) and the group G is realized as a subgroup of $\mathrm{GL}_{2n}(E)$ consisting of elements of the form

$$\begin{pmatrix} A & \epsilon B \\ \overline{B} & \overline{A} \end{pmatrix}, \quad A, B \in \mathrm{Mat}_n(E).$$

Then H consists of matrices of the form $\begin{pmatrix} A & \\ & \overline{A} \end{pmatrix}$, $A \in \mathrm{GL}_n(E)$.

For $g \in \mathrm{GL}_{2n}(E)$, we define an involution

$$\theta(g) = \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix} g \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}.$$

Then H is the group of fixed points of θ. We consider the following symmetric space

$$S = \{g\theta(g)^{-1} \mid g \in G\}$$

and then H acts on S via conjugation. We have an action of $H \times H$ on G by $(h_1, h_2) \cdot y = h_1^{-1} y h_2$. Similar to S', we say that an element in S is semisimple if its H-orbit is Zariski closed. It is regular semisimple if its stabilizer in H is a torus of dimension n. It is elliptic if furthermore its stabilizer in H is an elliptic torus modulo Z (over F). An element $g \in G$ is semisimple (regular semisimple, elliptic, respectively) if $g\theta(g)^{-1}$ is so in S. Let G_{reg} and G_{ell} denote the subsets of regular semisimple and elliptic elements of G.

Lemma 3.2 [Guo 1996]. *Every semisimple $g \in G(F)$ is in the $(H \times H)(F)$-orbit of*

$$g(\beta, n_1, n_2, n_3) = \begin{pmatrix} 1_{n_1} & & & \epsilon\beta & & \\ & 0_{n_2} & & & \epsilon 1_{n_2} & \\ & & 1_{n_3} & & & 0_{n_3} \\ \bar{\beta} & & & 1_{n_1} & & \\ & 1_{n_2} & & & 0_{n_2} & \\ & & 0_{n_3} & & & 1_{n_3} \end{pmatrix},$$

where $\beta \in \mathrm{GL}_{n_1}(E)$, $\beta\bar{\beta} \in \mathrm{GL}_{n_1}(F)$, and $\det(1-\epsilon\beta\bar{\beta}) \neq 0$. It is regular semisimple (resp. elliptic) if $n_2 = n_3 = 0$ and $\beta\bar{\beta}$ is regular semisimple (resp. elliptic) in $\mathrm{GL}_n(F)$ in the usual sense.

If $n_2 = n_3 = 0$, we write $s(\beta, n, 0, 0) = s(\beta)$ and $g(\beta, n, 0, 0) = g(\beta)$.

Let $g \in G$ and $s = g\theta(g)^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in S$. We define a morphism $q : G \to \boldsymbol{A}^n$ by

$$g \mapsto \operatorname{tr} \wedge^i A, \quad i = 1, \ldots, n.$$

Then q is a categorical quotient [Xue and Zhang 2023, Lemma 5.7].

Now we define orbital integrals. Let $y \in G_{\mathrm{reg}}(F)$ be regular semisimple and let $f \in \mathcal{C}_c^\infty(G(\mathbb{A}_F))$. We define the orbital integral

$$O^G(y, f) := \int_{(H\times H)_y(\mathbb{A}_F)\backslash(H\times H)(\mathbb{A}_F)} f(h_1^{-1} y h_2)\, \chi(h_1^{-1}h_2)^{-1} \,\mathrm{d}h_1 \,\mathrm{d}h_2.$$

This orbital integral is absolutely convergent for all regular semisimple y.

This orbital integral can be simplified as follows. Put

$$\tilde{f}(g\theta(g)^{-1}) = \int_{H(\mathbb{A}_F)} f(gh)\, \chi(gh)^{-1} \,\mathrm{d}h.$$

Then one can check that $\tilde{f} \in \mathcal{C}_c^\infty(S(\mathbb{A}_F))$ and

$$O^G(g, f) = O^S(s, \tilde{f}) := \int_{H_s(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \tilde{f}(h^{-1} s h) \,\mathrm{d}h, \quad s = g\theta(g)^{-1}.$$

If v is a place of F, the local orbital integral is defined similarly, and it can be simplified in a similar way.

The transfer factor is defined as follows. Let $y \in G_{\mathrm{reg}}(F_v)$ be a regular semisimple element, and write $y^{-1} = \begin{pmatrix} y_1 & \epsilon y_2 \\ \overline{y_2} & \overline{y_1} \end{pmatrix}$. Then at each place v we define

$$\kappa_v^G(y) = \chi_v(y_1).$$

3.3. ***Matching of test functions.*** Let F be global. Let $x \in G'(F)$ and let $y \in G(F)$ be regular semisimple elements, and write

$$x\bar{x}^{-1} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in S'(F), \quad y\theta(y)^{-1} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \in S(F),$$

where $\alpha_i, \beta_i \in \mathrm{Mat}_n(E)$. We say that x and y match if $2\alpha_1\overline{\alpha_1} - 1$ and β_1 have the same characteristic polynomial. We note that not all regular semisimple $x \in G'(F)$ match a regular semisimple $y \in G(F)$, and vice versa. The definition is similar when the field is local. We also note that there is a neighborhood of $1 \in G(F_v)$ such that every regular semisimple y in this neighborhood matches some $x \in G'(F_v)$.

For each place v of F, we define

$$\begin{aligned} &\mathcal{C}_c^\infty(G'(F_v))_0 \\ &\quad = \{ f' \in \mathcal{C}_c^\infty(G'(F_v)) \mid O^{G'}(x, f'_v) = 0 \text{ for all } x \text{ not matching any } y \in G(F_v) \}, \end{aligned}$$

and $\mathcal{C}_c^\infty(G(F_v))_0$ in a similar way. We say that two test functions $f' \in \mathcal{C}_c^\infty(G'(F_v))_0$ and $f \in \mathcal{C}_c^\infty(G(F_v))_0$ match if

$$\kappa_v^{G'}(x)\, O^{G'}(x, f'_v) = \kappa_v^G(y)\, O^G(y, f_v)$$

for all matching regular semisimple elements $x \in G'(F_v)$ and $y \in G(F_v)$. Two test functions $f' = \otimes f'_v \in \mathcal{C}_c^\infty(G'(\mathbb{A}_F))$ and $f = \otimes f_v \in \mathcal{C}_c^\infty(G(\mathbb{A}_F))$ match if $f'_v \in \mathcal{C}_c^\infty(G'(F_v))_0$ and $f_v \in \mathcal{C}_c^\infty(G(F_v))_0$ and they match for all places v of F.

We have the following results concerning the geometric side of the trace formula.

Theorem 3.3 [Xue and Zhang 2023, Theorem 2.1]. *Assume that v is nonarchimedean and nonsplit. For any $f'_v \in \mathcal{C}_c^\infty(G'(F_v))_0$, there is an $f_v \in \mathcal{C}_c^\infty(G(F_v))_0$ that matches it, and vice versa.*

Theorem 3.4 [Xue and Zhang 2023, Theorem 2.2]. *Let v be a nonsplit nonarchimedean odd place of F. Assume the quaternion algebra D splits at v and χ_v is unramified at v. Let $\mathfrak{o}_v$ be the ring of integers of F_v. We pick the measures on $G'(F_v)$ and $G(F_v)$ so that the volumes of $G'(\mathfrak{o}_v)$ and $G(\mathfrak{o}_v)$ are equal to* 1. *Then $\mathbf{1}_{G'(\mathfrak{o}_v)}$ and $\mathbf{1}_{G(\mathfrak{o}_v)}$ match.*

At the split places of F, the matching of test functions can be made explicit, as we explain below. Assume v is a split place of F. Then $G(F_v) = \mathrm{GL}_{2n}(F_v)$

and $G'(F_v) = \mathrm{GL}_{2n}(F_v) \times \mathrm{GL}_{2n}(F_v)$. We fix a measure on $\mathrm{GL}_{2n}(F_v)$ and then we have measures on $G'(F_v)$ and $G(F_v)$ under these identifications. The character η_v is trivial, so $\tilde{\eta}_v$ takes the form (η_0, η_0^{-1}) where η_0 is a character of $F_v^\times$. The character χ_v is of the form (χ_1, χ_2) where χ_1, χ_2 are characters of $F_v^\times$. Two regular semisimple elements $y \in G(F_v)$ and $(x_1, x_2) \in G'(F_v)$ match if $y = x_1 x_2^{-1}$. Let

$$f' = (f_1', f_2') \in \mathcal{C}_c^\infty(G'(F_v)) \cong \mathcal{C}_c^\infty(G(F_v)) \otimes \mathcal{C}_c^\infty(G(F_v))$$

and put

$$f(g) = \int_{\mathrm{GL}_{2n}(F_v)} f_1'(gh) f_2'(h) \chi_1(h)^{-1} \chi_2(h)^{-1}\, dh, \quad g \in \mathrm{GL}_{2n}(F_v).$$

Then the functions f' and f match [Xue and Zhang 2023, Lemma 2.3].

4. The spectral side: spherical characters

In this section, we assume that E/F is a quadratic extension of nonarchimedean local fields of characteristic zero and π is a (H, χ^{-1})-distinguished supercuspidal representation of G.

4.1. ***The split side.*** Let Π be an irreducible generic (unitary) representation of G'. We denote $\Pi^\vee$ the contragredient of Π, and Π^c the Galois conjugate of Π relative to E/F (i.e., $\Pi^c(g) = \Pi(\bar{g})$). It is known that $\Pi^c \cong \Pi$ if and only if $\Pi = \mathrm{BC}(\tau)$ for some irreducible admissible representation τ of $\mathrm{GL}_{2n}(F)$ [Arthur and Clozel 1989].

Following Chen and Sun [2020], let us recall the notion of good characters of H'. We use $|\cdot|_E$ to denote the normalized absolute value on E, and also use it to denote the character $t \mapsto |t|_E$ of $E^\times$. We say that a character $\xi : \mathrm{GL}_n(E) \to \mathbb{C}^\times$ is good if it is equal to $\alpha \circ \det$ for some character α of $E^\times$ such that the character $\alpha^{2r} \cdot |\cdot|_E^{-m}$ is not trivial for all $r \in \{\pm 1, \pm 2, \ldots, \pm n\}$ and for all $m \in \{1, 2, \ldots, 2n^2\}$. Note that ξ is good if and only if ξ^{-1} is good, and that all but finitely many characters of $\mathrm{GL}_n(E)$ are good. A character $\xi_0 \otimes \xi_1$ of $\mathrm{GL}_n(E) \times \mathrm{GL}_n(E)$ is said to be good if the character $\xi_0 \xi_1^{-1}$ of $\mathrm{GL}_n(E)$ is good.

We say that Π is $(H', \chi_{H'}^{-1})$-distinguished if $\mathrm{Hom}_{H'}(\Pi \otimes \chi_{H'}, \mathbb{C}) \neq 0$. Note that the character $\chi_{H'}$ is good because the restriction of $\chi\bar{\chi}^{-1}$ to $\mathrm{GL}_n(F)$ is trivial. Then the space $\mathrm{Hom}_{H'}(\Pi \otimes \chi_{H'}, \mathbb{C})$ is at most one-dimensional [Chen and Sun 2020]. We say that Π is $(H'', \chi^{-1}\eta)$-distinguished if $\mathrm{Hom}_{H''}(\Pi \otimes \chi\eta, \mathbb{C}) \neq 0$. It's known that the space $\mathrm{Hom}_{H''}(\Pi \otimes \chi\eta, \mathbb{C})$ is at most one-dimensional [Flicker 1991].

Let Q be the Shalika subgroup of $\mathrm{GL}_{2n}(E)$ consisting of matrices of the form

$$\begin{pmatrix} g & \\ & g \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \quad g \in \mathrm{GL}_n(E),\ x \in \mathrm{Mat}_n(E).$$

Let $\psi : E \to \mathbb{C}^\times$ be a nontrivial additive character and $\xi : E^\times \to \mathbb{C}^\times$ a multiplicative character. We define a character $\theta : Q(E) \to \mathbb{C}^\times$ by

$$\theta\left(\begin{pmatrix} g & \\ & g \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(\mathrm{tr}(x))\,\xi(\det(g)).$$

Let Π be an irreducible admissible representation of $\mathrm{GL}_{2n}(E)$ with central character ω, and assume that $\xi^n\omega = 1$. A functional $\lambda : V_\Pi \to \mathbb{C}$ is called a Shalika functional if λ satisfies

$$\lambda(\Pi(s)v) = \theta^{-1}(s)\,\lambda(v)$$

for every $s \in Q$ and v in the space of Π. It is known that the space of Shalika functionals on Π is at most one-dimensional [Jacquet and Rallis 1996].

Assume that Π has a nonzero Shalika functional, and let $\mathcal{V}$ be the corresponding Shalika model of Π corresponding to the characters ψ and $\xi = (\chi\chi^c)$. Let $\mathcal{W}$ be the Whittaker model of Π, defined with respect to the character ψ. We fix an isomorphism

$$\mathcal{W} \to \mathcal{V}, \quad W \mapsto \phi_W.$$

We also let $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{V}}$ be the Whittaker model and Shalika model of $\Pi^\vee$, defined by the character ψ^{-1} and ψ^{-1}, $(\chi\chi^c)^{-1}$ respectively. We also fix an isomorphism

$$\widetilde{\mathcal{W}} \to \widetilde{\mathcal{V}}, \quad W \mapsto \phi_W.$$

We consider the local Friedberg–Jacquet integral

$$Z^{\mathrm{FJ}}(s,\phi,\chi) = \int_{\mathrm{GL}_n(E)} \phi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right)\chi(a)\,|\det(a)|^{s-1/2}\,da, \quad \phi \in \mathcal{V}.$$

By [Friedberg and Jacquet 1993, Proposition 3.1], this integral converges absolutely for $\mathrm{Re}(s)$ large enough and has a meromorphic continuation to $\mathbb{C}$. Also, it is a holomorphic multiple of $L(s,\Pi\otimes\chi)$. Moreover, there is a $\phi \in \mathcal{V}$ such that the integral is equal to $L(s,\Pi\otimes\chi)$. Thus,

$$W \mapsto \ell'(W) = Z^{\mathrm{FJ}}\left(\tfrac{1}{2},\phi_W,\chi\right)$$

defines a nonzero element in $\mathrm{Hom}_{H'}(\Pi\otimes\chi_{H'},\mathbb{C})$.

Let P' be the mirabolic subgroup of G' and N' the standard upper triangular unipotent subgroup of G'. Define

$$\ell''(W) = \int_{N'\cap H''\backslash P'\cap H''} W(h)(\chi\eta)(h)\,dh.$$

Notice that the character $\chi\eta$ is trivial on $N'\cap H''$. The above integral is absolutely convergent and hence define a nonzero element in $\mathrm{Hom}_{H''}(\Pi\otimes\chi\eta,\mathbb{C})$.

We define a local spherical character on the split side as follows. For any $f' \in \mathcal{C}_c^\infty(G')$, put

$$I_\Pi(f') = \sum_W \ell'(\Pi(f')W)\,\overline{\ell''(W)},$$

where W ranges through an orthonormal basis of $\mathcal{W}$. This is called the spherical character of Π relative to $(H', \chi_{H'}^{-1})$ and $(H'', (\chi\eta)^{-1})$. A basic property of the spherical character I_Π is the representability by a locally integrable function, stated below.

Proposition 4.1. *I_Π is represented by a locally integrable function Θ_Π on G' which is locally constant on the regular semisimple locus, and satisfies*

$$\Theta_\Pi(hgh'') = \chi_{H'}(h)(\chi\eta)(h'')\,\Theta_\Pi(g), \quad h \in H'(F),\ h'' \in H''(F).$$

We will sketch a proof of Proposition 4.1 in Appendix.

We say that Π is (H', H'')-elliptic if there is an elliptic regular semisimple element $x \in G'$ such that x matches some $y \in G$ and $\Theta_\Pi(x) \neq 0$.

In Appendix, we will prove the following.

Theorem 4.2. *Assume Π is supercuspidal,*

$$\mathrm{Hom}_{H'}(\Pi \otimes \chi_{H'}, \mathbb{C}) \neq 0 \quad \textit{and} \quad \mathrm{Hom}_{H''}(\Pi \otimes \chi\eta, \mathbb{C}) \neq 0,$$

then Π is (H', H'')-elliptic.

We now define an involution on $\mathcal{C}_c^\infty(G')$ as follows. Let $f' \in \mathcal{C}_c^\infty(G')$. Define

$$f'^{\dagger}(g) = f'({}^t g^{-1})(\chi\chi^c)(g), \quad g \in G'. \tag{1}$$

A key observation of the relation between the spherical character I_Π and the involution defined in (1) is the following.

Lemma 4.3 [Xue and Zhang 2023, Lemma 3.5]. *For any $f' \in \mathcal{C}_c^\infty(G')$, we have*

$$I_\Pi(f'^{\dagger}) = \epsilon(\Pi \otimes \chi)\,\chi(-1)^n I_\Pi(f').$$

We also have the following relation between the matching of test functions and the involution defined in (1). Let $\varepsilon_D = \eta(\varepsilon) = \pm 1$.

Lemma 4.4 [Xue and Zhang 2023, Lemma 3.6]. *Let $f' \in \mathcal{C}_c^\infty(G'(F))_0$ and let $f \in \mathcal{C}_c^\infty(G(F))_0$ be matching test functions. Then $f'^{\dagger}$ and $\eta(-1)^n \varepsilon_D^n f$ match.*

4.2. *The nonsplit side.* Let π be an irreducible admissible unitary representation of G. We say that π is (H, χ^{-1})-distinguished if

$$\mathrm{Hom}_H(\pi, \chi^{-1}) \neq 0,$$

or equivalently,

$$\mathrm{Hom}_H(\pi \otimes \chi, \mathbb{C}) \neq 0.$$

In this case, this Hom space is one-dimensional [Lu 2023, Theorem 5.1]. We fix a nonzero element $\ell \in \mathrm{Hom}_H(\pi \otimes \chi, \mathbb{C})$. For $f \in \mathcal{C}_c^\infty(G)$, we define the spherical character attached to π by

$$J_\pi(f) = \sum_\phi \ell(\pi(f)\,\phi)\,\overline{\ell(\phi)},$$

where the sum ranges over an orthonormal basis of π. We note that π is (H, χ^{-1})-distinguished if and only if $\pi \otimes \eta$ is (H, χ^{-1})-distinguished, because η is trivial on H.

A test function of the form $f = f_1 * \overline{f_1^\vee}$, where $f_1^\vee(g) = f_1(g^{-1})$, is called of positive type.

Lemma 4.5 [Xue and Zhang 2023, Lemma 3.3]. *There is a positive-type test function $f \in \mathcal{C}_c^\infty(G(F))_0$ such that $J_\pi(f) > 0$.*

By [Guo 1998], the spherical character J_π is represented by a locally integrable function Θ_π on $G(F)$, which is locally constant on the regular semisimple locus, and satisfies the equivariant property $\Theta_\pi(h_1 g h_2) = \chi(h_1 h_2)\,\Theta_\Pi(g)$. We say that π is H-elliptic if $\Theta_\pi(y) \neq 0$ for some elliptic regular semisimple element $y \in G(F)$ and y matches some $x \in G'(F)$. Similar to Theorem 4.2, we have the ellipticity of π.

Proposition 4.6 [Xue and Zhang 2023, Proposition 3.4]. *If π is supercuspidal and $\mathrm{Hom}_H(\pi \otimes \chi, \mathbb{C}) \neq 0$, then π is H-elliptic.*

4.3. ***Global arguments.*** We recall the relative trace formula from [Xue and Zhang 2023] and use it to prove certain results. We recall that $A = \mathrm{Mat}_n(D)$, $G = \mathrm{GL}_n(D)$, $H = \mathrm{GL}_n(E)$. We will use an underline to denote a global object. We have the following globalization results.

Lemma 4.7. *There exist the following data*:

(1) *A quadratic extension of number fields $\underline{E}/\underline{F}$ that splits at all archimedean places, a set of inert finite places S with $|S| = n$, and a nonarchimedean inert place v_0 of $\underline{F}$ such that $\underline{E}_{v_0}/\underline{F}_{v_0}$ is isomorphic to E/F. We denote by $\underline{\eta}$ the quadratic character attached to $\underline{E}/\underline{F}$ by global class field theory.*

(2) *A CSA $\underline{A}$ over $\underline{F}$ with an embedding $\underline{E} \to \underline{A}$ whose centralizer is $\underline{B}$, with the property that $(\underline{A}_{v_0}, \underline{B}_{v_0})$ is isomorphic to (A, B), the invariant of $\underline{A}_v$ is $\frac{1}{2n}$ if $v \in S$, and $\underline{A}_v \cong \mathrm{Mat}_{2n}(\underline{F}_v)$ for all $v \notin S \cup \{v_0\}$.*

(3) *A character $\underline{\chi} : \underline{E}^\times \backslash \mathbb{A}_{\underline{E}}^\times \to \mathbb{C}^\times$ such that $\underline{\chi}_{v_0} = \chi$, and $\underline{\chi}_v$ is unramified at finite places outside $S \cup \{v_0\}$.*

Proof. The existence of items (1) and (2) can be found in [Xue 2021, Lemma 3.4]. The existence of item (3) is clear. □

With the $\underline{A}$ and $\underline{B}$ as in Lemma 4.7, we let $\underline{G}=\underline{A}^\times$ and $\underline{H}=\underline{B}^\times$, both viewed as algebraic groups over $\underline{F}$. Let $\underline{Z}$ be the center of $\underline{G}$. Now we globalize the representation π.

Lemma 4.8. *With the* $\underline{A}$, $\underline{B}$ *and* $\underline{\chi}$ *as in* Lemma 4.7, *we can find an irreducible cuspidal automorphic representation* $\underline{\pi}$ *of* $\underline{G}(\mathbb{A}_{\underline{F}})$ *with central character* $\underline{\omega}$, *such that the period integral*

$$\int_{\underline{H}(\underline{F})\underline{Z}(\mathbb{A}_{\underline{F}})\backslash\underline{H}(\mathbb{A}_{\underline{F}})}\varphi(h)\,\underline{\chi}(h)\,dh$$

is not identically zero, where $\underline{\pi}_{v_0}\cong\pi$, $\underline{\pi}_v$ *is a* $(\underline{H}(\underline{F}_v),\underline{\chi}_v^{-1})$*-distinguished representation for* $v\in S$, $\underline{\pi}_w$ *is a supercuspidal* $(\underline{H}(\underline{F}_w),\underline{\chi}_w^{-1})$*-distinguished representation for some split place* w *of* $\underline{F}$, *and* $\underline{\pi}_u$ *is unramified at all other nonarchimedean places* u.

Proof. This follows from [Prasad and Schulze-Pillot 2008, Theorem 4.1]. We note that $\underline{H}/\underline{Z}$ has no $\underline{F}$-rational characters. □

Let us now recall the relative trace formula developed in [Xue and Zhang 2023]. We start with the nonsplit side. Let σ be an irreducible cuspidal automorphic representation of $\underline{G}(\mathbb{A}_{\underline{F}})$. For $\varphi\in V_\sigma$, we define a global period

$$P_{\underline{\chi}}(\varphi)=\int_{\underline{Z}(\mathbb{A}_{\underline{F}})\underline{H}(\underline{F})\backslash\underline{H}(\mathbb{A}_{\underline{F}})}\varphi(h)\,\underline{\chi}(h)\,dh.$$

We say that σ is globally $(\underline{H}(\mathbb{A}_{\underline{F}}),\underline{\chi}^{-1})$-distinguished if $P_{\underline{\chi}}(\varphi)$ is not identically zero. We define a global distribution

$$J_\sigma(f)=\sum_{\varphi}P_{\underline{\chi}}(\pi(f)\,\varphi)\overline{P_{\underline{\chi}}(\varphi)},\qquad f\in\mathcal{C}_c^\infty(\underline{G}(\mathbb{A}_{\underline{F}})),\tag{2}$$

where φ runs through an orthonormal basis of σ. Then σ is globally $(\underline{H}(\mathbb{A}_{\underline{F}}),\underline{\chi}^{-1})$-distinguished if and only if $J_\sigma\neq 0$.

Now we consider the split side. We denote

$$\underline{G}'=\mathrm{Res}_{\underline{E}/\underline{F}}\,\mathrm{GL}_{2n},\quad \underline{H}'=\mathrm{Res}_{\underline{E}/\underline{F}}(\mathrm{GL}_n\times\mathrm{GL}_n),\quad \underline{H}''=\mathrm{GL}_{2n}.$$

Let $\underline{Z}'$ be the center of $\underline{G}'$ and $\underline{Z}_{2n}$ be the center of $\underline{H}''$. Let σ' be an irreducible automorphic representation of $\underline{G}'(\mathbb{A}_{\underline{F}})$. For $\varphi\in V_{\sigma'}$, we define the global periods

$$P'_{\underline{\chi}}(\varphi)=\int_{\underline{Z}'(\mathbb{A}_{\underline{F}})\underline{H}'(\underline{F})\backslash\underline{H}'(\mathbb{A}_{\underline{F}})}\varphi\left(\begin{pmatrix}h_1&\\&h_2\end{pmatrix}\right)\underline{\chi}(h_1\overline{h_2})\,dh_1\,dh_2,$$

and

$$P''_{\underline{\chi}\underline{\eta}}(\varphi)=\int_{\underline{Z}_{2n}(\mathbb{A}_{\underline{F}})\underline{H}''(\underline{F})\backslash\underline{H}''(\mathbb{A}_{\underline{F}})}\varphi(h)(\underline{\chi}\,\underline{\eta})(h)\,dh.$$

We define $\underline{\chi}_{\underline{H}'}$ on $\underline{H}'(\mathbb{A}_{\underline{F}})$ by $\underline{\chi}_{\underline{H}'}(h') = \underline{\chi}(h_1\overline{h_2})$ for $h' = (h_1, h_2) \in \underline{H}'(\mathbb{A}_{\underline{F}})$. We say that σ' is globally $(\underline{H}'(\mathbb{A}_{\underline{F}}), \underline{\chi}_{\underline{H}'}^{-1})$-distinguished if $P'_{\underline{\chi}}$ is not identically zero. The linear period $P'_{\underline{\chi}}$ is not identically zero if and only if $L\big(\frac{1}{2}, \sigma' \otimes \underline{\chi}\big) \neq 0$ and $L(s, \sigma', \wedge^2 \otimes \underline{\chi}\,\underline{\chi}^c)$ has a pole at $s = 1$. The latter condition implies that $\sigma'^{\vee} \cong \sigma' \otimes \underline{\chi}\,\underline{\chi}^c$. Similarly, we say σ' is globally $(\underline{H}''(\mathbb{A}_{\underline{F}}), (\underline{\chi}\,\underline{\eta})^{-1})$-distinguished if $P''_{\underline{\chi}\underline{\eta}}$ is not identically zero. The linear period $P''_{\underline{\chi}\underline{\eta}}$ is not identically zero if and only if the Asai L-function $L(s, \sigma' \otimes \underline{\chi}, \mathrm{As}^-)$ has a pole at $s = 1$. This condition implies that $\sigma'^{\vee} \otimes \underline{\chi}^{-1} \cong \sigma'^{c} \otimes \underline{\chi}^c$. We refer the reader to [Xue and Zhang 2023, Section 3.1] for more details.

We recall the following result.

Lemma 4.9 [Xue and Zhang 2023, Lemma 3.2]. *Let σ' be an irreducible automorphic cuspidal representation of $\underline{G}'(\mathbb{A}_{\underline{F}})$. If neither of $P'_{\underline{\chi}}$ and $P''_{\underline{\chi}\underline{\eta}}$ is identically zero, then there is an irreducible cuspidal automorphic representation τ of $\underline{H}''(\mathbb{A}_{\underline{F}})$ such that $\sigma' = \mathrm{BC}(\tau)$. Moreover, $L\big(\frac{1}{2}, \sigma' \otimes \underline{\chi}\big) \neq 0$, and $L(s, \tau, \wedge^2 \otimes \underline{\chi}|_{\mathbb{A}_{\underline{F}}^{\times}})$ has a simple pole at $s = 1$.*

The following theorem is due to the works in [Jacquet and Shalika 1990; Asgari and Shahidi 2006; 2014; Hundley and Sayag 2016].

Theorem 4.10. *Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$ with central character ω_τ. Let $\psi : \underline{F}\backslash\mathbb{A}_{\underline{F}} \to \mathbb{C}^\times$ be a nontrivial additive character and $\xi : \underline{F}^\times\backslash\mathbb{A}_{\underline{F}}^\times \to \mathbb{C}^\times$ a multiplicative character such that $\xi^n = \omega_\tau$. The following are equivalent*:

(i) *There is a $\varphi \in V_\tau$ and $g \in \mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$ such that the global Shalika period*

$$\int_{[\mathrm{Mat}_n]}\int_{\mathbb{A}_{\underline{F}}^\times \mathrm{GL}_n(\underline{F})\backslash \mathrm{GL}_n(\mathbb{A}_{\underline{F}})} (\tau(g)\,\varphi)\left(\begin{pmatrix} h & \\ & h\end{pmatrix}\begin{pmatrix} 1_n & X \\ & 1_n\end{pmatrix}\right) \xi^{-1}(\det(h))\,\psi^{-1}(\mathrm{tr}(X))\,dh\,dX \neq 0.$$

(ii) *Let S be a finite set of places outside including the archimedean ones of which τ, ψ, ξ are unramified. The twisted partial exterior square L-function*

$$L^S(s, \tau, \wedge^2 \otimes \xi^{-1}) := \prod_{v \notin S} L(s, \tau_v, \wedge^2 \otimes \xi_v^{-1})$$

has a pole at $s = 1$.

(iii) *τ is the transfer of a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_{\underline{F}})$ whose central character is equal to ξ.*

For $f' \in \mathcal{C}_c^\infty(\underline{G}'(\mathbb{A}_{\underline{F}}))$, put

$$I_{\sigma'}(f') = \sum_{\varphi} P'_{\underline{\chi}}(\sigma'(f')\,\varphi)\,\overline{P''_{\underline{\chi}\underline{\eta}}(\varphi)},$$

where φ sums over an orthonormal basis of σ'. This is a global spherical character attached to σ'.

Given global test functions $\boldsymbol{f} = \otimes f_v \in \mathcal{C}_c^\infty(\underline{G}(\mathbb{A}_{\underline{F}}))$ and $\boldsymbol{f}' = \otimes f'_v \in \mathcal{C}_c^\infty(\underline{G}'(\mathbb{A}_{\underline{F}}))$, we say that $\boldsymbol{f}$ and $\boldsymbol{f}'$ match if $f_v \in \mathcal{C}_c^\infty(\underline{G}(\underline{F}_v))_0$ and $f'_v \in \mathcal{C}_c^\infty(\underline{G}'(\underline{F}_v))_0$ and they match for all places v of $\underline{F}$.

Proposition 4.11. *Let σ be an irreducible cuspidal automorphic representation of $\underline{G}(\mathbb{A}_{\underline{F}})$ with central character $\underline{\omega}$ such that $\sigma_{v_0} \cong \pi$ where σ_v is a $(\underline{H}(\underline{F}_v), \underline{\chi}_v^{-1})$-distinguished representation of $\underline{G}(\underline{F}_v)$ if $v \in S$; σ_w is a supercuspidal $(\underline{H}(\underline{F}_w), \underline{\chi}_w^{-1})$-distinguished representation for some split place w of K. Let $\sigma' = \mathrm{JL}(\sigma)$ be the Jacquet–Langlands transfer of σ to $\underline{H}''(\mathbb{A}_{\underline{F}})$, and $\mathrm{BC}(\sigma')$ the base change of σ' to $\underline{G}'(\mathbb{A}_{\underline{F}})$. Suppose that $\boldsymbol{f} = \otimes f_v \in \mathcal{C}_c^\infty(\underline{G}(\mathbb{A}_{\underline{F}}))$ and $\boldsymbol{f}' = \otimes f'_v \in \mathcal{C}_c^\infty(\underline{G}'(\mathbb{A}_{\underline{F}}))$ match. Assume that if $v \in S$, f_v is a positive type test function in $\mathcal{C}_c^\infty(\underline{G}(\underline{F}_v))_0$ supported sufficiently close to 1 such that $J_{\sigma_v}(f_v) \neq 0$, that f_{v_0} is supported in the elliptic locus, and that $f_w = f'_w$ is an essential matrix coefficient of σ_w. Then we have*

$$I_{\mathrm{BC}(\sigma')}(\boldsymbol{f}') = J_\sigma(\boldsymbol{f}) + J_{\sigma\otimes\eta}(\boldsymbol{f}). \tag{3}$$

The proof essentially follows from [Xue and Zhang 2023, Section 4.1]. For completeness, we reproduce the proof here, after we introduce some necessary notations and tools from [Beuzart-Plessis et al. 2021].

Proof. Let $\mathcal{A}$ be a complex algebra. Recall that a multiplier is a complex linear map $\mu\star : \mathcal{A} \to \mathcal{A}$ that commutes with the left and right multiplications in $\mathcal{A}$. The space of multipliers of $\mathcal{A}$ is denoted by $\mathrm{Mul}(\mathcal{A})$.

Let v be an archimedean place of $\underline{F}$, $\mathcal{S}(\underline{G}(\underline{F}_v))$ the Schwartz space of $\underline{G}(\underline{F}_v)$, $\mathfrak{t}_v$ the complexified Cartan subalgebra of $\underline{G}(\underline{F}_v)$, $\mathfrak{t}_v^*$ the dual space of $\mathfrak{t}_v$, and $\mathcal{Z}_{\underline{G}(\underline{F}_v)} \cong \mathbb{C}[\mathfrak{t}_v]^{W_v}$ the center of the universal enveloping algebra of $\underline{G}(\underline{F}_v)$. We write $\underline{\chi}_v = (\chi_1, \chi_2)$. Then the character $\chi_1\chi_2$ of $\underline{F}_v^\times$ defines an element $\underline{a}_v \in \mathfrak{t}_v^*$.

Let $\mathcal{M}_v$ be the space of holomorphic functions on $\mathfrak{t}_v^*$ defined in [Beuzart-Plessis et al. 2021, Definition 2.8 (3)] (note that the notation is $\mathcal{M}_\theta^\sharp(\mathfrak{h}_\mathbb{C}^*)$ in [Beuzart-Plessis et al. 2021]). We refer the reader to [Beuzart-Plessis et al. 2021, Theorem 2.13] for the following property. There is an algebra homomorphism

$$\mathcal{M}_v \to \mathrm{Mul}\big(\mathcal{S}(\underline{G}(\underline{F}_v))\big), \quad \mu \mapsto \mu\star$$

such that

$$\sigma(\mu \star f) = \mu(\lambda_\sigma)\,\sigma(f)$$

for every $f \in \mathcal{S}(\underline{G}(\underline{F}_v))$ and every irreducible admissible representation σ of $\underline{G}(\underline{F}_v)$, where λ_σ is the infinitesimal character of σ. Let ι_v be the involution on $\mathcal{M}_v$ such that $\iota_v(\mu)(z) = \mu(-\underline{a}_v - z)$ for all $z \in \mathfrak{t}_v^*$. Let $\mathcal{M}_v^+$ be the subspace of $\mathcal{M}_v$ consisting of elements invariant under ι_v. Note that the map $f \mapsto f^\vee\chi_1\chi_2$, where

$f^\vee(g) = f(g^{-1})$, is an involution on $\mathcal{S}(\underline{G}(\underline{F}_v))$. Define

$$\mathcal{S}(\underline{G}(\underline{F}_v))^+ = \{f \in \mathcal{S}(\underline{G}(\underline{F}_v)) : f^\vee \chi_1\chi_2 = f\}.$$

Then for $f \in \mathcal{S}(\underline{G}(\underline{F}_v))^+$ and $\mu \in \mathcal{M}_v^+$, we have $\mu \star f \in \mathcal{S}(\underline{G}(\underline{F}_v))^+$. We denote $\mathcal{Z}_{\underline{G}} = \prod_{v|\infty} \mathcal{Z}_{\underline{G}(\underline{F}_v)}$, $\lambda_\infty = \otimes_{v|\infty}\lambda_v$, $\mathcal{M} = \prod_{v|\infty} \mathcal{M}_v$, and $\mathcal{M}^+ = \prod_{v|\infty} \mathcal{M}_v^+$.

We let $\mathtt{S}$ be a finite set of finite places of $\underline{F}$ such that if $v \notin \mathtt{S}$, then $\underline{E}_v/\underline{F}_v$, σ_v and $\underline{\chi}_v$ are all unramified. Recall that S is a finite set of inert finite places as in Lemma 4.7. Denote $\mathtt{T} = \mathtt{S} \cup S$ and let

$$\mathcal{H}_{\underline{G}}^{\mathtt{T}} = \otimes_{v\nmid\infty, v\notin\mathtt{T}} \mathcal{H}_v = \otimes_{v\nmid\infty, v\notin\mathtt{T}} \mathcal{C}_c^\infty\big(\underline{G}(\mathfrak{o}_{\underline{F}_v})\backslash\underline{G}(\underline{F}_v)/\underline{G}(\mathfrak{o}_{\underline{F}_v})\big)$$

be the spherical Hecke algebra away from $\mathtt{T}$, where $\mathfrak{o}_{\underline{F}_v}$ is the ring of integers of $\underline{F}_v$ at a finite place $v \notin \mathtt{T}$. Let $K = \prod_{v\nmid\infty} K_v$ be a fixed compact open subgroup such that $K_v = \underline{G}(\mathfrak{o}_{\underline{F}_v})$ if $v \notin \mathtt{S}$. We denote by $\mathcal{C}_c^\infty(\underline{G}(\mathbb{A}_{\underline{F}}))_K$ the subalgebra of $\mathcal{C}_c^\infty(\underline{G}(\mathbb{A}_{\underline{F}}))$ of bi-K-invariant functions.

The above objects also have their counterparts for $\underline{G}'$. We keep v an archimedean place of $\underline{F}$, and we have an algebra of homomorphic functions $\mathcal{M}'_v$, which is identified with $\mathcal{M}_v \otimes \mathcal{M}_v$. Put

$$\mathcal{M}_v'^+ = \mathcal{M}_v^+ \otimes \mathcal{M}_v^+, \quad \mathcal{M}' = \prod_{v|\infty}\mathcal{M}'_v, \quad \mathcal{M}'^+ = \prod_{v|\infty}\mathcal{M}_v'^+.$$

We have $\mathcal{S}(\underline{G}'(\underline{F}_v))^+ = \mathcal{S}(\underline{G}(\underline{F}_v))^+ \otimes \mathcal{S}(\underline{G}(\underline{F}_v))^+$. The universal enveloping algebra $\mathcal{Z}_{\underline{G}'}$ is identified with $\mathcal{Z}_{\underline{G}} \otimes \mathcal{Z}_{\underline{G}}$, and the spherical Hecke algebra away from $\mathtt{T}$ is

$$\mathcal{H}_{\underline{G}'}^{\mathtt{T}} = \otimes_{v\nmid\infty, v\notin\mathtt{T}} \mathcal{H}_{\underline{G}',v} = \mathcal{H}_{\underline{G}}^{\mathtt{T}} \otimes \mathcal{H}_{\underline{G}}^{\mathtt{T}}.$$

We have a base change homomorphism $\mathrm{bc} : \mathcal{Z}_{\underline{G}'} \otimes \mathcal{H}_{\underline{G}'}^{\mathtt{T}} \to \mathcal{Z}_{\underline{G}} \otimes \mathcal{H}_{\underline{G}}^{\mathtt{T}}$ given by the usual multiplication in $\mathcal{Z}_{\underline{G}}$ and $\mathcal{H}_{\underline{G}}^{\mathtt{T}}$. Let $K' = \prod_{v\nmid\infty} K'_v$ be a fixed compact open subgroup such that $K'_v = \underline{G}'(\mathfrak{o}_{\underline{F}_v})$ if $v \notin \mathtt{S}$. Let $\mathcal{C}_c^\infty(\underline{G}'(\mathbb{A}_{\underline{F}}))_{K'}$ be the subalgebra of $\mathcal{C}_c^\infty(\underline{G}'(\mathbb{A}_{\underline{F}}))$ of bi-K'-invariant functions.

Suppose that $\lambda = (\lambda_\infty, \lambda^{\infty,\mathtt{T}})$ is the character of $\mathcal{Z}_{\underline{G}} \otimes \mathcal{H}_{\underline{G}}^{\mathtt{T}}$ associated to σ, and let $\lambda' = \lambda \circ \mathrm{bc} = (\lambda, \lambda)$. Then λ' is the character of $\mathcal{Z}_{\underline{G}'} \otimes \mathcal{H}_{\underline{G}'}^{\mathtt{T}}$ associated to $\mathrm{BC}(\sigma')$ (see [Arthur and Clozel 1989, Chapter 1, Section 5]). Let

$$L_0^2(\underline{G}(\underline{F})\backslash\underline{G}(\mathbb{A}_{\underline{F}})/K, \underline{\omega})[\lambda] \quad \text{and} \quad L_0^2(\underline{G}'(\underline{F})\backslash\underline{G}'(\mathbb{A}_{\underline{F}})/K', \underline{\omega}')[\lambda']$$

be the maximal quotients of the spaces

$$L_0^2(\underline{G}(\underline{F})\backslash\underline{G}(\mathbb{A}_{\underline{F}})/K, \underline{\omega}) \quad \text{and} \quad L_0^2(\underline{G}'(\underline{F})\backslash\underline{G}'(\mathbb{A}_{\underline{F}})/K', \underline{\omega})$$

on which $\mathcal{Z}_{\underline{G}} \otimes \mathcal{H}_{\underline{G}}^{\mathtt{T}}$ and $\mathcal{Z}_{\underline{G}'} \otimes \mathcal{H}_{\underline{G}'}^{\mathtt{T}}$ acts by λ and λ' respectively. Then we have

$$L_0^2(\underline{G}(\underline{F})\backslash\underline{G}(\mathbb{A}_{\underline{F}})/K, \underline{\omega})[\lambda] = \sigma \oplus (\sigma \otimes \underline{\eta}),$$
$$L_0^2(\underline{G}'(\underline{F})\backslash\underline{G}'(\mathbb{A}_{\underline{F}})/K', \underline{\omega}')[\lambda'] = \mathrm{BC}(\sigma').$$

Let

$$\mathrm{bc}:\mathcal{M}'\otimes\mathcal{H}^{\mathrm{T}}_{\underline{G}'}=(\mathcal{M}\otimes\mathcal{H}^{\mathrm{T}}_{\underline{G}})\otimes(\mathcal{M}\otimes\mathcal{H}^{\mathrm{T}}_{\underline{G}})\to\mathcal{M}\otimes\mathcal{H}^{\mathrm{T}}_{\underline{G}}$$

be the map given by multiplication.

By [Xue and Zhang 2023, Proposition 3.7], there exist elements $\mu'\in\mathcal{M}'^{+}\otimes\mathcal{H}^{\mathrm{T}}_{\underline{G}'}$ and $\mu=\mathrm{bc}(\mu')\in\mathcal{M}^{+}\otimes\mathcal{H}^{\mathrm{T}}_{\underline{G}}$ such that, for all $f\in\mathcal{C}^\infty_c(\underline{G}(\mathbb{A}_{\underline{F}}))$, we have that:

- $R(\mu'\star f')$ maps $L^2(\underline{G}'(\underline{F})\backslash\underline{G}'(\mathbb{A}_{\underline{F}})/K',\underline{\omega}')$ into $\mathrm{BC}(\sigma')$ for all $f'\in\mathcal{C}^\infty_c(\underline{G}'(\mathbb{A}_{\underline{F}}))$.
- $\mathrm{BC}(\sigma')(\mu'\star f')=\mathrm{BC}(\sigma')(f')$ for all $f'\in\mathcal{C}^\infty_c(\underline{G}'(\mathbb{A}_{\underline{F}}))_{K'}$.
- $R(\mu\star f)$ maps $L^2(\underline{G}(\underline{F})\backslash\underline{G}(\mathbb{A}_{\underline{F}})/K,\underline{\omega})$ into $\sigma\oplus(\sigma\otimes\underline{\eta})$ for all $f\in\mathcal{C}^\infty_c(\underline{G}(\mathbb{A}_{\underline{F}}))$.
- $\sigma(\mu\star f)=\sigma(f)$ for all $f\in\mathcal{C}^\infty_c(\underline{G}(\mathbb{A}_{\underline{F}}))_K$.

We emphasize that the multipliers μ' and μ are in the "plus" subspaces. Let $\boldsymbol{f}=\otimes f_v\in\mathcal{C}^\infty_c(\underline{G}(\mathbb{A}_{\underline{F}}))$ and $\boldsymbol{f}'=\otimes f'_v\in\mathcal{C}^\infty_c(\underline{G}'(\mathbb{A}_{\underline{F}}))$ be the matching test functions given in Proposition 4.11. We now use the test functions $\mu'\star\boldsymbol{f}'$ and $\mu\star\boldsymbol{f}$, which still match. We conclude that

$$I_{\mathrm{BC}(\sigma')}(\boldsymbol{f}')=J_\sigma(\boldsymbol{f})+J_{\sigma\otimes\eta}(\boldsymbol{f}).\qquad\square$$

As an application of Proposition 4.11, we have the following result relating a (H,χ^{-1})-distinguished supercuspidal representation of G to a simultaneously $(H',\chi^{-1}_{H'})$-distinguished and $(H'',\chi^{-1}\eta)$-distinguished representation of G' through the Jacquet–Langlands transfer.

Theorem 4.12. *Let π be an irreducible (H,χ^{-1})-distinguished supercuspidal representation of G with central character ω. Let $\pi_{0,E}$ be the base change of π_0. Then $\pi_{0,E}$ is both $(H',\chi^{-1}_{H'})$-distinguished and $(H'',\chi^{-1}\eta)$-distinguished.*

Proof. Since π is (H,χ^{-1})-distinguished, by Lemma 4.5, there exists some positive-type test function $f\in\mathcal{C}^\infty_c(G)$ such that $J_\pi(f)>0$. By Lemma 4.8, we can find a globally $(\underline{H}(\mathbb{A}_{\underline{F}}),\underline{\chi}^{-1})$-distinguished cuspidal representation $\underline{\pi}$ of $\underline{G}(\mathbb{A}_{\underline{F}})$ and a test function $\boldsymbol{f}=\otimes f_v\in\mathcal{C}^\infty_c(\underline{G}(\mathbb{A}_{\underline{F}}))$ with $\boldsymbol{f}_{v_0}=f$ so that they satisfy the conditions of Proposition 4.11. Let $\boldsymbol{f}'=\otimes f'_v\in\mathcal{C}^\infty_c(\underline{G}'(\mathbb{A}_{\underline{F}}))$ be a function that matches $\boldsymbol{f}$. Since the test function f is of positive type, by Proposition 4.11, we have $I_{\mathrm{BC}(\sigma')}(\boldsymbol{f}')>0$, where $\sigma'=\mathrm{JL}(\underline{\pi})$. Since $\pi_{0,E}$ is the local component of $\mathrm{BC}(\sigma')$, we conclude that $\pi_{0,E}$ is both $(H',\chi^{-1}_{H'})$-distinguished and $(H'',\chi^{-1}\eta)$-distinguished. $\square$

5. The forward direction

In this section, following a similar argument as in [Xue 2021, Section 4], we prove Theorem 1.2, which is restated as follows. Recall that we fix $\varepsilon\in NE^\times$ (resp. $F^\times\backslash NE^\times$) if D splits (resp. ramifies) and the group G is realized as a subgroup of $\mathrm{GL}_{2n}(E)$ which consists of elements of the form $\left[\begin{smallmatrix}\alpha & \varepsilon\beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right]$, $\alpha,\beta\in\mathrm{GL}_n(E)$. Let $\varepsilon_D=1$ (resp. $\varepsilon_D=-1$) if D splits (resp. ramifies).

Theorem 5.1. *Let π be an irreducible (H, χ^{-1})-distinguished representation of G such that its Jacquet–Langlands transfer $\pi_0 = \mathrm{JL}(\pi)$ is generic with central character ω. Then the following two conditions hold*:

(1) *The Langlands parameter of π_0 takes values in $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude factor $\chi^{-1}|_{F^\times}$.*

(2) $\varepsilon(\pi_{0,E} \otimes \chi) = \varepsilon_D^n \eta(-1)^n \chi(-1)^n$.

5.1. ***The supercuspidal case.*** We first prove Theorem 5.1 under the assumption that π is supercuspidal.

Proof of Theorem 5.1 assuming π is supercuspidal. Assume that π be an irreducible (H, χ^{-1})-distinguished supercuspidal representation of G. We keep the notations from the proof of Theorem 4.12. In particular, we have matching test functions $\boldsymbol{f} = \otimes f_v \in \mathcal{C}_c^\infty(\underline{G}(\mathbb{A}_{\underline{F}}))$ with $f_{v_0} = f$ so that they satisfy the conditions of Proposition 4.11 and $\boldsymbol{f}' = \otimes f'_v \in \mathcal{C}_c^\infty(\underline{G}'(\mathbb{A}_{\underline{F}}))$, so that

$$I_{\mathrm{BC}(\sigma')}(\boldsymbol{f}') = J_{\underline{\pi}}(\boldsymbol{f}) + J_{\underline{\pi}\otimes\underline{\eta}}(\boldsymbol{f}) > 0.$$

Recall that $\sigma' = \mathrm{JL}(\underline{\pi})$ is the Jacquet–Langlands transfer of $\underline{\pi}$. By Lemma 4.9, $L(s, \sigma', \wedge^2 \otimes \underline{\chi}|_{\mathbb{A}_{\underline{F}}^\times})$ has a simple pole at $s = 1$. Since π_0 is the local component of $\sigma' = \mathrm{JL}(\underline{\pi})$, by Theorem 4.10 we conclude that the Langlands parameter of π_0 takes values in $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude factor $\chi^{-1}|_{F^\times}$.

Now we move on to compute the local root number $\varepsilon(\pi_{0,E} \otimes \chi)$. The main idea is to use the involution defined in (1). Note that $\varepsilon_{D_v}^n \eta_v(-1)^n f_v$ and $f_v'^{\dagger}$ also match, by Lemma 4.4. Define a global test function $\boldsymbol{f}'^{\dagger}$ by

$$\boldsymbol{f}'^{\dagger} = \otimes_{w \neq v_0} f'_w \otimes f'^{\dagger}_{v_0}.$$

Now we use Lemma 4.3 to conclude that

$$\varepsilon(\pi_{0,E} \otimes \chi)\chi(-1)^n I_{\mathrm{BC}(\sigma')}(\boldsymbol{f}') = \varepsilon_D^n \eta(-1)^n \big(J_\sigma(\boldsymbol{f}) + J_{\sigma\otimes\eta}(\boldsymbol{f})\big) \neq 0.$$

Thus

$$\varepsilon(\pi_{0,E} \otimes \chi) = \varepsilon_D^n \eta(-1)^n \chi(-1)^n.$$

This completes the proof of Theorem 5.1 assuming π is supercuspidal. □

5.2. ***The general case.*** Before we proceed to the general case, we recall some generalities on representations of $\mathrm{GL}_r(C)$, where C be a central division algebra of dimension d^2 over F. Let $\rho_1, \ldots, \rho_s$ be irreducible representations of $\mathrm{GL}_{r_1}(C), \ldots, \mathrm{GL}_{r_s}(C)$ respectively, and we denote by

$$\rho_1 \times \cdots \times \rho_s$$

the normalized induced representation of $\mathrm{GL}_r(C)$, associated to the usual standard upper triangular parabolic subgroup corresponding to the partition $r = r_1 + \cdots + r_s$.

We denote by ν the absolute value of the reduced norm of any CSA. Suppose that $r = s\ell$ and ρ is a supercuspidal representation of $\mathrm{GL}_s(C)$. We consider the case $C = F$ first. Then ρ is a supercuspidal representation of $\mathrm{GL}_s(F)$, and the representation

$$\rho \times \rho\nu \times \cdots \times \rho\nu^{\ell-1}$$

has a unique irreducible quotient which is a discrete series representation of $\mathrm{GL}_r(F)$. Moreover, any discrete series representation of $\mathrm{GL}_r(F)$ is obtained in this way. In general, assume that $\rho' = \mathrm{JL}(\rho)$ is the Jacquet–Langlands transfer of ρ to $\mathrm{GL}_{sd}(F)$. Then it is an irreducible quotient of $\tau \times \cdots \times \tau\nu^{q-1}$. Set $\nu_\rho = \nu^q$. Again, the normalized parabolic induction

$$\rho \times \rho\nu_\rho \times \cdots \times \rho\nu_\rho^{\ell-1}$$

has a unique quotient which is a discrete series representation of $\mathrm{GL}_r(C)$. Moreover, all discrete series representation of $\mathrm{GL}_r(C)$ arise in this way. We call such a representation a segment, and denote it by $\Delta = \{\rho, \rho\nu_\rho, \dots, \rho\nu_\rho^{\ell-1}\}$.

Let $\mathrm{WD}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group of F. To each irreducible representation π' of $\mathrm{GL}_{2n}(F)$, one can associate a Weil–Deligne representation

$$\phi_{\pi'} : \mathrm{WD}_F \to \mathrm{GL}_{2n}(\mathbb{C})$$

by the local Langlands correspondence. If π' is supercuspidal, then $\phi_{\pi'}$ is an irreducible representation of W_F and is trivial on $\mathrm{SL}_2(\mathbb{C})$. If π' is a segment of

$$\{\tau, \dots, \tau\nu^{\ell-1}\},$$

then $\phi_{\pi'} = \phi_\tau \boxtimes \mathrm{Sym}^{\ell-1}$, where ϕ_τ is an irreducible representation of W_F associated to τ, and $\mathrm{Sym}^{\ell-1}$ is the unique irreducible algebraic ℓ-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. Moreover, the local root number of π' is given by

$$\varepsilon(\pi') = \varepsilon(\phi_{\pi'}) = \varepsilon(\phi_\tau)^\ell \det(-\mathrm{Frob}|\phi_\tau^{I_F})^{\ell-1},$$

where I_F is the inertia subgroup of W_F, and $\phi_\tau^{I_F}$ stands for the subspace of ϕ_τ on which I_F acts trivially. We note that if ϕ_τ is not one-dimensional, then $\phi_\tau^{I_F} = 0$.

Using the classification of (H, χ^{-1})-distinguished representations, the proof of Theorem 5.1 reduces to the case of discrete series. Assume that π is (H, χ^{-1})-distinguished. By [Suzuki 2023, Proposition 3.4], π is a quotient of $\Delta_1 \times \cdots \times \Delta_s$ where each Δ_i is an irreducible discrete series representation of $\mathrm{GL}_{n_i}(D)$ with $n_1 + \cdots + n_s = n$, and there is an involutive permutation $\zeta \in \mathfrak{S}_s$ such that:

- $n_{\zeta(i)} = n_i$ for each i.
- If $\zeta(i) = i$, then n_i is even and Δ_i is (H_i, χ^{-1})-distinguished. Here H_i denotes the centralizer of $E^\times$ in $\mathrm{GL}_{n_i}(D)$.
- If $\zeta(s) \neq i$, then $\Delta_{\zeta(i)} \cong \Delta_i^\vee \cdot \chi^{-1}$.

Using this description, we have the following result.

Lemma 5.2 [Suzuki 2023, Corollary 3.5]. Theorem 5.1 *for discrete series implies* Theorem 5.1.

Proof of Theorem 5.1 assuming π is a discrete series representation. Assume that π is a discrete series representation. We write π as a segment

$$\{\rho\nu_\rho^{-(\ell-1)/2}, \dots, \rho\nu_\rho^{(\ell-1)/2}\},$$

where ρ is an irreducible supercuspidal representation of $\mathrm{GL}_s(D)$, $s\ell = n$, $\nu_\rho = \nu^q$, and ν is the absolute value of the reduced norm. The case $s = 1$ and ρ being one-dimensional is proved in [Chommaux 2019], and hence from now on we assume ρ is not one-dimensional.

First we assume ℓ is even. This implies n is even and hence $\varepsilon_D^n \eta(-1)^n \chi(-1)^n = 1$. Then by the proof of [Broussous and Matringe 2021, Proposition 5.6], we have $\rho \cong \chi^{-1}|_{F^\times} \otimes \rho^\vee$. Write $\pi_0 = \mathrm{JL}(\pi)$ as a segment

$$\{\tau\nu^{-(\ell'-1)/2}, \dots, \tau\nu^{(\ell'-1)/2}\},$$

where τ is a supercuspidal representation of $\mathrm{GL}_{2n/\ell'}(F)$, $\tau \cong \chi^{-1}|_{F^\times} \otimes \tau^\vee$, and ℓ' is even. Let ϕ_τ be the representation of the Weil group of F associated to τ. Since ρ is not one-dimensional, so ϕ_τ is not one-dimensional, and hence $\phi_\tau^{I_F} = 0$. Then

$$\varepsilon(\pi_{0,E} \otimes \chi) = \varepsilon(\pi_0 \otimes \chi|_{F^\times})\eta(-1)^n = \varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^{\ell'}\eta(-1)^n = 1,$$

which proves the theorem in this case.

Now assume ℓ is odd. Again, by the proof of [Broussous and Matringe 2021, Proposition 5.6], we have ρ is (H_s, χ^{-1})-distinguished, where H_s is the centralizer of $E^\times$ in $\mathrm{GL}_s(D)$. Let $\rho' = \mathrm{JL}(\rho)$ be the Jacquet–Langlands transfer of ρ to $\mathrm{GL}_{2s}(F)$. Write ρ' as a segment

$$\{\tau\nu^{-(a-1)/2}, \dots, \tau\nu^{(a-1)/2}\},$$

and write $\pi_0 = \mathrm{JL}(\pi)$ as a segment

$$\{\tau\nu^{-(\ell a-1)/2}, \dots, \tau\nu^{(\ell a-1)/2}\}.$$

If a is even, then the same computation as in the case ℓ being even gives that

$$\varepsilon(\pi_{0,E} \otimes \chi) = \varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^{\ell a}\eta(-1)^n = \eta(-1)^n.$$

If a is odd, then both a and ℓa are odd, and

$$\varepsilon(\pi_{0,E} \otimes \chi) = \varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^{\ell a}\eta(-1)^n.$$

In both cases, we have

$$\varepsilon(\pi_{0,E} \otimes \chi) = \varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^{\ell a}\eta(-1)^n.$$

On the other hand, by Theorem 5.1 for the supercuspidal case, we get

$$\varepsilon(\mathrm{BC}(\rho')\otimes\chi)=\varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^a\,\eta(-1)^s=\varepsilon_D^s\,\eta(-1)^s\chi(-1)^s.$$

Since $n=s\ell$ and ℓ is odd, n and s have the same parity, and thus we conclude that

$$\varepsilon(\pi_{0,E}\otimes\chi)=\varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^{\ell a}\,\eta(-1)^n=\varepsilon(\phi_{\tau\otimes\chi|_{F^\times}})^a\,\eta(-1)^n=\varepsilon_D^n\,\eta(-1)^n\chi(-1)^n.$$

This completes the proof of Theorem 5.1. □

6. The converse direction

The goal of this section is to prove Theorem 1.3, which we restate below. We recall the following setup:

- E/F is a quadratic extension of local nonarchimedean fields of characteristic zero.
- D is a quaternion algebra over F containing E.
- $G=\mathrm{GL}_n(D)$, and $H=\mathrm{Res}_{E/F}\,\mathrm{GL}_{n,F}$ regarded as a subgroup of G.
- Put $\epsilon_D=1$ (resp. $\epsilon_D=-1$) if D splits (resp. ramifies).

Theorem 6.1. *Assume $\chi|_{F^\times}$ is trivial. Suppose π_0 is a discrete series representation of $\mathrm{GL}_{2n}(F)$ satisfying*

(1) *the Langlands parameter of π_0 takes values in $\mathrm{Sp}_{2n}(\mathbb{C})$,*

(2) $\varepsilon(\pi_{0,E}\otimes\chi)=\epsilon_D^n\,\eta(-1)^n$.

Assume in addition that $\mathrm{BC}(\pi_0)$ is supercuspidal. Then the Jacquet–Langlands transfer $\pi=\mathrm{JL}(\pi_0)$ of π_0 to $G=\mathrm{GL}_n(D)$ is (H,χ^{-1})-distinguished.

In the rest of this section, we assume the conditions in Theorem 6.1 are satisfied.

6.1. *Global arguments.* In this subsection, we prove a globalization result. We will usually denote a global object by a letter with an underline. Let $\underline{E}/\underline{F}$ be a quadratic extension of number fields such that it is split at all archimedean places, and there is a finite inert place v_0 of $\underline{F}$ such that $\underline{E}_{v_0}/\underline{F}_{v_0}=E/F$. Let $\underline{\eta}:\underline{F}^\times\backslash\mathbb{A}_{\underline{F}}^\times\to\{\pm1\}$ be the quadratic character attached to $\underline{E}/\underline{F}$ via class field theory. Let Σ_1 be a finite set of inert places not containing v_0. We fix another finite split place v_1. We globalize χ to a character $\underline{\chi}$ of $\mathbb{A}_{\underline{E}}^\times$ such that $\underline{\chi}|_{\mathbb{A}_{\underline{F}}^\times}$ is trivial. Let $\underline{\psi}:\underline{F}\backslash\mathbb{A}_{\underline{F}}\to\mathbb{C}$ be a fixed nontrivial additive character. Put $\underline{G}'=\mathrm{Res}_{\underline{E}/\underline{F}}(\mathrm{GL}_{2n,\underline{F}})$, $\underline{H}'=\mathrm{Res}_{\underline{E}/\underline{F}}(\mathrm{GL}_{n,\underline{F}}\times\mathrm{GL}_{n,\underline{F}})$, and $\underline{H}''=\mathrm{GL}_{2n,\underline{F}}$.

Proposition 6.2. *There is an irreducible cuspidal automorphic representation $\underline{\pi}_0$ of $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$ such that*:

(1) $\underline{\pi}_{0_{v_0}} = \pi_0$, $\underline{\pi}_{0_{v_1}}$ *is supercuspidal*, $\underline{\pi}_{0_{v_1}} \not\cong \underline{\pi}_{0_{v_1}} \otimes \underline{\eta}_{v_1}$.

(2) *If v is a nonsplit place and $\underline{\pi}_{0_v}$ is not supercuspidal, then all data at v are unramified.*

(3) $\mathrm{BC}(\underline{\pi}_0)$ *is globally distinguished by* $(\underline{H}', \underline{\chi}_{\underline{H}'}^{-1})$ *and* $(\underline{H}'', \underline{\eta})$.

The existence of such a $\underline{\pi}_0$ requires an argument similar to the one in [Ichino et al. 2017, Proposition A.4] and [Sakellaridis and Venkatesh 2017, Theorem 16.3.2], and we present it in Section 6.2.

We also globalize CSAs.

Lemma 6.3. *There is a CSA $\underline{A}$ over $\underline{F}$ containing $\underline{E}$ with the following properties*:

(1) *$\underline{A}$ splits at all split places.*

(2) $\underline{A}_{v_0} = \mathrm{Mat}_n(D)$.

(3) *If the split rank of $\underline{G}_v$ is r_v, then* $\varepsilon(\underline{\pi}_{0,E_v} \otimes \chi_v) = (-1)^{r_v} \underline{\eta}_v(-1)^n$.

Proof. First we assume n is odd. We let $w_1, w_2, \ldots, w_{2k}$ be the places where

$$\varepsilon(\underline{\pi}_{0,E_{w_i}} \otimes \chi_{w_i}) \underline{\eta}_{w_i}(-1)^n = -1$$

and assume $w_1 = v_0$. Then we take a CSA A_i at w_i such that $A_1 = \mathrm{Mat}_n(D)$, $\mathrm{inv}(A_i) = -\mathrm{inv}(A_{n+i})$ for $i = 1, \ldots, n$, and the split rank of $A_i^\times$'s are all odd (note that the split rank of $A_i^\times$ and $A_{n+i}^\times$ are the same). Then we take $\underline{A}$ such that $\underline{A}_{w_i} = A_i$ and $\underline{A}_v = \mathrm{Mat}_{2n}(\underline{F}_v)$ for all other places v. Now we assume that n is even. Let the invariant of $\mathrm{Mat}_n(D)$ be a/b, then $b|n$. We take two places w_1, w_2, such that

$$\varepsilon(\underline{\pi}_{0,E_{w_i}} \otimes \chi_{w_i}) \underline{\eta}_{w_i}(-1)^n = 1$$

and take A_1 and A_2 whose invariants are $-a/2b$. Let $w_3, \ldots, w_{2k}$ be places where

$$\varepsilon(\underline{\pi}_{0,E_{w_i}} \otimes \chi_{w_i}) \underline{\eta}_{w_i}(-1)^n = -1$$

for $3 \le i \le 2k$, and we choose $A_3, \ldots, A_{2k}$ as in the odd case. Then we take $\underline{A}$ such that $\underline{A}_{w_i} = A_i$, and $\underline{A}_v = \mathrm{Mat}_{2n}(\underline{F}_v)$ for all other v. □

Put $\underline{G} = \underline{A}^\times$.

Let $\underline{f}' = \otimes f_v'$ be a test function on $\underline{G}'(\mathbb{A}_{\underline{F}})$ that is unramified if the data at that place is unramified. At the split place, we take any test function. If v is a nonsplit place with some ramified data, then either $v = v_0$ or $\mathrm{BC}(\underline{\pi}_{0_v})$ is supercuspidal by Proposition 6.2. If $v = v_0$, then by the assumption of Theorem 6.1, $\mathrm{BC}(\underline{\pi}_0)_{v_0} = \mathrm{BC}(\pi_0)$ is supercuspidal, hence in both cases $\mathrm{BC}(\underline{\pi}_{0_v})$ is supercuspidal. By Theorem 4.2, $\mathrm{BC}(\underline{\pi}_{0_v})$ is elliptic. Let f_v'' be a test function supported in the elliptic locus such that $I_{\underline{\pi}_{0_v}}(f_v'') \neq 0$. Let

$$f_v' = f_v'' + \varepsilon(\underline{\pi}_{0,E_v} \otimes \chi_v) f_v''^{\dagger},$$

where $f_v''^{\dagger}$ is the involution defined in (1). By Lemma 4.3, we have

$$I_{\mathrm{BC}(\underline{\pi}_{0_v})}(f_v') = I_{\mathrm{BC}(\underline{\pi}_{0_v})}(f_v'') + \varepsilon(\underline{\pi}_{0,E_v} \otimes \chi_v)^2 I_{\mathrm{BC}(\underline{\pi}_{0_v})}(f_v'') = 2I_{\mathrm{BC}(\underline{\pi}_{0_v})}(f_v'') \neq 0.$$

Moreover, by assumption (2) of Theorem 6.1 and by the proof of [Xue and Zhang 2023, Lemma 4.1], we have $O^{\underline{G}'}(g, f_v') = 0$ if $g \in \underline{G}'(\underline{F}_v)$ does not match any element in $\underline{G}(\underline{F}_v)$. If v is any other (nonsplit) place and $\mathrm{BC}(\underline{\pi}_{0_v})$ is supercuspidal, we take f_v' to be an essential matrix coefficient of $\underline{\pi}_{0_v}$. By the proof of [Xue and Zhang 2023, Lemma 4.1], again we have $O^{\underline{G}'}(g, f_v') = 0$ if $g \in \underline{G}'(\underline{F}_v)$ does not match any element in $\underline{G}(\underline{F}_v)$. Here, by the orbital integral $O^{\underline{G}'}(g, f_v')$ we mean $O^{\underline{G}'}(g, f_{v,1}')$ where $f_{v,1}' \in \mathcal{C}_c^\infty(\underline{G}'(\underline{F}_v))$ is a function such that

$$\int_{Z'(\underline{F}_v)} f_{v,1}'(zg)\, \omega_{\mathrm{BC}(\underline{\pi}_0)}(z)\, dz = f_v'(g).$$

Let $\underline{f}$ be a test function on $\underline{G}(\mathbb{A}_{\underline{F}})$ that matches $\underline{f}'$. By the relative trace formula identity of [Xue and Zhang 2023], we have

$$I_{\mathrm{BC}(\underline{\pi}_0)}(\underline{f}') = J_{\mathrm{JL}(\underline{\pi}_0)}(\underline{f}) + J_{\mathrm{JL}(\underline{\pi}_0)\otimes\underline{\eta}}(\underline{f}). \tag{4}$$

By the choice of test function $\underline{f}'$, we have $I_{\mathrm{BC}(\underline{\pi}_0)}(\underline{f}') \neq 0$. Thus

$$J_{\mathrm{JL}(\underline{\pi}_0)}(\underline{f}) + J_{\mathrm{JL}(\underline{\pi}_0)\otimes\underline{\eta}}(\underline{f}) \neq 0.$$

So either $J_{\mathrm{JL}(\underline{\pi}_0)}(\underline{f}) \neq 0$ or $J_{\mathrm{JL}(\underline{\pi}_0)\otimes\underline{\eta}}(\underline{f}) \neq 0$. This means that either $\mathrm{JL}(\underline{\pi}_0)$ is globally $(\underline{H}, \underline{\chi}^{-1})$-distinguished, or $\mathrm{JL}(\underline{\pi}_0) \otimes \underline{\eta}$ is globally $(\underline{H}, \underline{\chi}^{-1})$-distinguished. We conclude that either $\pi = \mathrm{JL}(\underline{\pi}_0)_{v_0}$ is (H, χ^{-1})-distinguished, or $\pi \otimes \underline{\eta}_{v_0}$ is (H, χ^{-1})-distinguished. Since $\underline{\eta}_{v_0}$ is trivial on H, it follows that π being (H, χ^{-1})-distinguished is equivalent to $\pi \otimes \underline{\eta}_{v_0}$ being (H, χ^{-1})-distinguished. Hence π is (H, χ^{-1})-distinguished.

6.2. *Proof of Proposition 6.2.* We are given a self-dual discrete series representation π_0 of $\mathrm{GL}_{2n}(F)$ with Langlands parameter

$$\rho_{\pi_0} : \mathrm{WD}_F \to \mathrm{Sp}_{2n}(\mathbb{C}),$$

where $\mathrm{WD}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ is the Weil–Deligne group. The output we need is an irreducible cuspidal automorphic representation $\underline{\pi}_0$ of $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$ satisfying the three conditions in Proposition 6.2. We do this in several steps. Put $\Sigma = \Sigma_1 \cup \{v_0\}$.

Let X_n be a $(2n+1)$-dimensional vector space over $\underline{F}$ equipped with a nondegenerate symmetric bilinear form $(\cdot\,,\cdot)$ of Witt index n. Take a maximal isotropic subspace X_n^+ of X_n of dimension n. Then we can write

$$X_n = X_n^+ + \underline{F}e + X_n^-,$$

where X_n^- is an isotropic subspace dual to X_n^+ and e is an anisotropic vector orthogonal to $X_n^+ + X_n^-$. We assume that $(e, e) = 1$. Let $\{e_1, \dots, e_n\}$ and $\{e_{-1}, \dots, e_{-n}\}$ be bases for X_n^+ and X_n^- respectively so that

$$(e_i, e_{-j}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

We denote the special orthogonal group $\mathrm{SO}(X_n) = \mathrm{SO}(2n+1)$ by $\underline{\widetilde{G}}$. When we write elements in $\underline{\widetilde{G}}$ as matrices, we will employ the ordered basis

$$\{e_{-1}, \dots, e_{-n}, e, e_n, \dots, e_1\}.$$

We take $\lambda \in \underline{F}^\times$ so that $\underline{E} = \underline{F}(\sqrt{\lambda})$. Let L be a subspace of X_n spanned by e_{-n}, e, e_n. We fix an $e_\lambda \in L(\underline{F})$ such that $(e_\lambda, e_\lambda) = \lambda$. Let $\underline{P}'$ be the maximal parabolic subgroup of $\underline{\widetilde{G}}$ preserving the isotropic subspace spanned by

$$\{e_{-1}, e_{-2}, \dots, e_{-n+1}\},$$

with a Levi decomposition $\underline{P}' = \underline{M}'\underline{S}'$, where $\underline{M}'$ is the Levi subgroup. Then

$$\underline{M}' = \left\{ \begin{pmatrix} a & & \\ & h & \\ & & a^* \end{pmatrix} : a \in \mathrm{GL}_{n-1},\ h \in \mathrm{SO}(L) \right\},$$

where $a^* = J_{n-1}{}^t a^{-1} J_{n-1}$ with J_{n-1} the $(n-1) \times (n-1)$ matrix with ones on the antidiagonal and zeros everywhere else, and $\underline{S}'$ consists of elements of the form

$$s' = \begin{pmatrix} 1_{n-1} & A & B \\ & 1_3 & A' \\ & & 1_{n-1} \end{pmatrix},$$

where

$$A' = -J_3\, {}^t A J_{n-1} \quad \text{and} \quad {}^t A' J_3 A' + J_{n-1} B + {}^t B J_{n-1} = 0.$$

We define a character $\underline{\psi}_\lambda$ of $\underline{S}'(\underline{F}) \backslash \underline{S}'(\mathbb{A}_{\underline{F}})$ by

$$\underline{\psi}_\lambda \begin{pmatrix} 1_{n-1} & A & B \\ & 1_3 & A' \\ & & 1_{n-1} \end{pmatrix} = \underline{\psi}((Ae_\lambda, e_{n-1})).$$

Let U_{n-1} be the group of upper unipotent matrices in GL_{n-1}. For $u \in U_{n-1}$, we denote

$$\check{u} = \begin{pmatrix} u & & \\ & I_3 & \\ & & u^* \end{pmatrix} \in \underline{P}'.$$

Let $\underline{S} = \underline{S}'\underline{S}''$ be the unipotent subgroup of $\underline{P}'$, where $\underline{S}'' = \{\check{u} : u \in \mathrm{GL}_{n-1}\}$. We extend the character ψ_λ to $\underline{S}(\mathbb{A}_{\underline{F}})$ by

$$\psi_\lambda(\check{u}) = \psi(u_{1,2} + \cdots + u_{n-2,n-1}), \quad u \in U_{n-1}(\mathbb{A}_{\underline{F}}).$$

Let $\underline{D}_\lambda$ be the subgroup of $\widetilde{\underline{G}}$ defined by

$$\underline{D}_\lambda = \left\{ \begin{pmatrix} 1_{n-1} & & \\ & h & \\ & & 1_{n-1} \end{pmatrix} : h \in \mathrm{SO}(L),\ he_\lambda = e_\lambda \right\}.$$

and let $\underline{R}_\lambda = \underline{D}_\lambda \underline{S}$ be the Bessel subgroup. Note that the elements of $\underline{D}_\lambda(\mathbb{A}_{\underline{F}})$ stabilizes ψ_λ by conjugation, and that

$$\underline{D}_\lambda(\underline{F}) \cong \mathrm{SO}(\underline{E}) \cong \underline{E}^\times/\underline{F}^\times.$$

We define a character $\underline{\nu}_{\lambda,\underline{\chi}}$ on $\underline{R}_\lambda(\mathbb{A}_{\underline{F}})$ by

$$\underline{\nu}_{\lambda,\underline{\chi}}(ts) = \underline{\chi}(t)\,\underline{\psi}_\lambda(s), \quad t \in \underline{D}_\lambda(\mathbb{A}_{\underline{F}}),\ s \in \underline{S}(\mathbb{A}_{\underline{F}}).$$

We say that a representation Π of $\mathrm{SO}(2n+1, \mathbb{A}_{\underline{F}})$ has a global Bessel model of type $(\lambda, e_\lambda, \underline{\psi}^{-1}, \underline{\chi}^{-1})$ if the global Bessel period

$$P_{\lambda,e_\lambda,\underline{\psi},\underline{\chi}}(\phi) := \int_{\underline{R}_\lambda(\underline{F})\backslash \underline{R}_\lambda(\mathbb{A}_{\underline{F}})} \phi(r)\,\underline{\nu}_{\lambda,\underline{\chi}}(r)\,dr$$

is not identically zero on Π.

Since the local Gross–Prasad conjecture [Gross and Prasad 1994] is known for $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$ (see [Waldspurger 2012; Mœglin and Waldspurger 2012]), π_0 descents to a unique representation σ of $\mathrm{SO}(2n+1, F)$ which has a Bessel-model, i.e., $\mathrm{Hom}_{\underline{R}_\lambda(F)}(\sigma \otimes (\chi\psi), \mathbb{C}) \neq 0$. At a place $v \in \Sigma_1$, we also fix a supercuspidal representation τ_v of $\mathrm{SO}(2n+1, \underline{F}_v)$ whose functorial lift to $\mathrm{GL}_{2n}(\underline{F}_v)$ is supercuspidal. We may moreover assume that at the place v_1, the image of the functorial lift is not isomorphic to its twist by $\underline{\eta}_{v_1}$.

Let $\mathcal{A}^B$ be the set of cuspidal automorphic representations Π of $\mathrm{SO}(2n+1, \mathbb{A}_{\underline{F}})$ satisfying the following conditions:

(1) The functorial lifting of Π to $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$ is cuspidal.

(2) Π has a global Bessel model of type $(\lambda, e_\lambda, \underline{\psi}^{-1}, \underline{\chi}^{-1})$.

(3) Π is unramified at all inert places not in Σ.

Recall the notion of weak containment in [Bekka et al. 2008, Appendix F].

Lemma 6.4. *The space* $L^2(\underline{R}_{\lambda\Sigma}\backslash\widetilde{\underline{G}}_\Sigma, \underline{\nu}_{\lambda,\underline{\chi}}^{-1})$ *is weakly contained in* $\bigoplus_{\Pi\in\mathcal{A}^B}\Pi_\Sigma$.

Assuming Lemma 6.4 for now, we conclude that there is a sequence of $\Pi^{(\ell)} \in \mathcal{A}^B$, $\ell = 1, 2, \dots$, such that

$$\Pi_\Sigma^{(\ell)} \to \rho \otimes \bigotimes_{v \in \Sigma_1} \tau_v$$

in the Fell topology. Moreover, the same argument as in the proof of [Beuzart-Plessis 2021a, Proposition 3.6.1] gives that $\Pi_\Sigma^{(\ell)}$ is tempered when ℓ is large. The restriction of the Fell topology to the tempered spectrum coincides with the usual topology on the tempered spectrum (see the Remark after [Beuzart-Plessis 2021b, Proposition 4.11]) and square integrable representations are isolated points. It follows that

$$\Pi_\Sigma^{(\ell)} = \rho \otimes \bigotimes_{v \in \Sigma_1} \tau_v,$$

when ℓ is sufficiently large. For this $\Pi^{(\ell)}$, let $\underline{\pi}_0$ be its functorial lifting to $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{F}})$. Now we explain that $\underline{\pi}_0$ is the desired representation. We just need to explain condition (3) in Proposition 6.2. Note that $\underline{\pi}_0$ is cuspidal and $\underline{\pi}_0 \not\cong \underline{\pi}_0 \otimes \underline{\eta}$. Hence its base change $\mathrm{BC}(\underline{\pi}_0)$ to $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{E}})$ exists and is unique, which is a cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A}_{\underline{E}})$ [Arthur and Clozel 1989]. Since $\Pi^{(\ell)}$ has a global Bessel model, by [Jiang and Zhang 2020, Theorem 5.7] on the global Gross–Prasad conjecture, we have $L\big(\tfrac{1}{2}, \mathrm{BC}(\underline{\pi}_0) \otimes \underline{\chi}\big) \neq 0$. This implies that $\mathrm{BC}(\underline{\pi}_0)$ is globally distinguished by $(\underline{H}', \underline{\chi}_{\underline{H}'}^{-1})$. On the other hand, since $\underline{\pi}_0$ comes from the functorial lifting of a cuspidal representation of $\mathrm{SO}(2n+1, \mathbb{A}_F)$, $L^S(s, \underline{\pi}_0, \wedge^2)$ has a simple pole at $s = 1$ (see [Ginzburg et al. 2001, Theorem 1]). By [Shahidi 1997, Theorem 1.1], $L^S(s, \underline{\pi}_0, \mathrm{Sym}^2)$ is nonzero at $s = 1$. Observe that

$$\begin{aligned} L^S(s, \mathrm{BC}(\underline{\pi}_0) \otimes \underline{\chi}\underline{\eta}, \mathrm{As}^-) &= L^S(s, \underline{\pi}_0, \mathrm{Sym}^2 \otimes \underline{\chi}|_{\mathbb{A}_{\underline{F}}^\times}) L^S(s, \underline{\pi}_0, \wedge^2 \otimes \underline{\chi}|_{\mathbb{A}_{\underline{F}}^\times}\underline{\eta}) \\ &= L^S(s, \underline{\pi}_0, \mathrm{Sym}^2) L^S(s, \underline{\pi}_0, \wedge^2 \otimes \eta). \end{aligned}$$

It follows that $L^S(s, \mathrm{BC}(\underline{\pi}_0) \otimes \underline{\chi}\underline{\eta}, \mathrm{As}^-)$ has a pole at $s = 1$. Thus $\mathrm{BC}(\underline{\pi}_0)$ is globally distinguished by $(\underline{H}'', \underline{\eta})$.

It remains to prove Lemma 6.4.

Proof of Lemma 6.4. By [Bekka et al. 2008, Lemma F.1.3], it suffices to prove that for any $\varepsilon > 0$, any compact $\Omega \subset \underline{\widetilde{G}}_\Sigma$, and all $\alpha \in \mathcal{C}_c^\infty(\underline{\widetilde{G}}_\Sigma)$ such that

$$\int_{\underline{\widetilde{G}}(\underline{F}_\Sigma)} \int_{\underline{R}_\lambda(\underline{F}_\Sigma)} \alpha(hg) \overline{\alpha(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(h)\, dh\, dg \neq 0,$$

there are finitely many $\varphi \in \oplus_{\Pi \in \mathcal{A}^B} \Pi_\Sigma$ such that

$$\left| \int_{\underline{\widetilde{G}}(\underline{F}_\Sigma)} \int_{\underline{R}_\lambda(\underline{F}_\Sigma)} \alpha(hgy) \overline{\alpha(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(h)\, dh\, dg - \sum_\varphi \langle \Pi_\Sigma(y)\, \varphi, \varphi \rangle \right| < \varepsilon \tag{5}$$

for all $y \in \Omega$.

We now fix a split place w and a supercuspidal representation τ_w of

$$\mathrm{SO}(2n+1, \underline{F}_w),$$

whose functorial transfer to $\mathrm{GL}_{2n}(\underline{F}_w)$ is still supercuspidal. Let

$$\phi = \otimes \phi_v \in \mathcal{C}_c^\infty(\mathrm{SO}(2n+1, \mathbb{A}_{\underline{F}}))$$

be a test function such that $\phi_\Sigma = \alpha,\ \phi_v$ is the unit in the spherical Hecke algebra if v is inert and not in Σ, and a suitable test function if v splits. In particular, we require ϕ_w is the matrix coefficient of τ_w. Let

$$K_\phi(h, g) = \sum_{\gamma \in \mathrm{SO}(2n+1, \underline{F})} \phi(h^{-1}\gamma g)$$

be the usual kernel function and put

$$K_\phi(g) = \int_{\underline{R}_\lambda(\underline{F}) \backslash \underline{R}_\lambda(\mathbb{A}_{\underline{F}})} K_\phi(h, g) \underline{\nu}_{\lambda, \underline{\chi}}(h)\, dh.$$

We then compute

$$\langle R(y) K_\phi, K_\phi \rangle. \tag{6}$$

Spectrally, we have

$$K_\phi(h, g) = \sum_{\Pi} \sum_{\varphi \in \Pi} \Pi(\phi)\, \varphi(h)\, \overline{\varphi(g)},$$
$$K_\phi(g) = \sum_{\Pi \in \mathcal{A}^B} \sum_{\varphi \in \Pi} P_{\lambda, e_\lambda, \underline{\psi}, \underline{\chi}}(\Pi(\phi)\, \varphi)\, \overline{\varphi(g)}.$$

Thus,

$$\begin{aligned} (7) \quad & \langle R(y) K_\phi, K_\phi \rangle \\ &= \Big\langle \sum_{\Pi \in \mathcal{A}^B} \sum_{\varphi \in \Pi} P_{\lambda, e_\lambda, \underline{\psi}, \underline{\chi}}(\Pi(\phi)\, \varphi) \overline{\Pi(y)\, \varphi},\ \sum_{\Pi \in \mathcal{A}^B} \sum_{\varphi \in \Pi} P_{\lambda, e_\lambda, \underline{\psi}, \underline{\chi}}(\Pi(\phi)\, \varphi)\, \bar{\varphi} \Big\rangle \\ &= \sum_{\Pi \in \mathcal{A}^B} \sum_{\varphi \in \Pi} P_{\lambda, e_\lambda, \underline{\psi}, \underline{\chi}}(\Pi(\phi)\, \varphi)\, \overline{P_{\lambda, e_\lambda, \underline{\psi}, \underline{\chi}}(\Pi(\phi)\, \varphi)} \langle \overline{\Pi(y)\, \varphi}, \bar{\varphi} \rangle. \end{aligned}$$

Here, the sum is over $\mathcal{A}^B$ because of our choices of the test functions, in particular at w and the inert places.

Geometrically, we have

$$\int_{\underline{\widetilde{G}}(\underline{F}) \backslash \underline{\widetilde{G}}(\mathbb{A}_{\underline{F}})} \int_{[\underline{R}_\lambda]^2} \sum_{\gamma_1, \gamma_2 \in \underline{\widetilde{G}}} (\underline{F})\, \phi(h_1^{-1}\gamma_1 g y)\, \overline{\phi(h_2^{-1}\gamma_2 g)}\, \underline{\nu}_{\lambda, \underline{\chi}}(h_1)\, \overline{\underline{\nu}_{\lambda, \underline{\chi}}(h_2)}\, dh_1\, dh_2\, dg.$$

By collapsing the sum over γ_2 with the dg integration, and making a change of variable $g \mapsto h_2\, g$, we obtain

$$\int_{\underline{\widetilde{G}}(\mathbb{A}_{\underline{F}})} \int_{[\underline{R}_\lambda]^2} \sum_{\gamma \in \underline{\widetilde{G}}(\underline{F})} \phi(h_1^{-1}\gamma h_2\, gy)\, \overline{\phi(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(h_1)\, \overline{\underline{\nu}_{\lambda,\underline{\chi}}(h_2)}\, dh_1\, dh_2\, dg.$$

We consider the $\underline{R}_\lambda \times \underline{R}_\lambda$ action on $\underline{\widetilde{G}}$ given by

$$(r_1, r_2) \cdot g = r_1^{-1} g r_2.$$

For $\gamma \in \underline{\widetilde{G}}(\underline{F})$, we denote $(\underline{R}_\lambda \times \underline{R}_\lambda)(\underline{F}) \cdot \gamma$ for the orbit of γ. Then the integral is equal to

$$\sum_{\gamma} \int_{[(\underline{R}_\lambda \times \underline{R}_\lambda)_\gamma]} \underline{\nu}_{\lambda,\underline{\chi}}(r_1 r_2^{-1})\, dr_1\, dr_2 \cdot J(\gamma, \phi). \tag{8}$$

Here, γ runs over the representatives for $\underline{R}_\lambda(\underline{F}) \backslash \underline{\widetilde{G}}(\underline{F}) / \underline{R}_\lambda(\underline{F})$, $(\underline{R}_\lambda \times \underline{R}_\lambda)_\gamma$ is the stabilizer of γ, and

$$J(\gamma, \phi) := \int_{\underline{\widetilde{G}}(\mathbb{A}_{\underline{F}})} \int_{(\underline{R}_\lambda \times \underline{R}_\lambda)_\gamma(\mathbb{A}_{\underline{F}}) \backslash (\underline{R}_\lambda \times \underline{R}_\lambda)(\mathbb{A}_{\underline{F}})} \phi(h_1^{-1}\gamma h_2\, gy)\, \overline{\phi(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(h_1 h_2^{-1})\, dh_1\, dh_2\, dg.$$

Note that the integral

$$\int_{[(\underline{R}_\lambda \times \underline{R}_\lambda)_\gamma]} \underline{\nu}_{\lambda,\underline{\chi}}(r_1 r_2^{-1})\, dr_1\, dr_2$$

vanishes unless $\underline{\nu}_{\lambda,\underline{\chi}}(r_1 r_2^{-1})$ is trivial on $[(\underline{R}_\lambda \times \underline{R}_\lambda)_\gamma]$, in which case the integral is equal to $\mathrm{vol}([(\underline{R}_\lambda \times \underline{R}_\lambda)_\gamma])$. Also, we have

$$J(1_{2r+1}, \phi) = \int_{\underline{\widetilde{G}}(\mathbb{A}_{\underline{F}})} \int_{\underline{R}_\lambda(\mathbb{A}_{\underline{F}})} \phi(rgy)\, \overline{\phi(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(r)\, dr\, dg,$$

and the right-hand side further decomposes into

$$\left(\int_{\underline{\widetilde{G}}(\underline{F}_\Sigma)} \int_{\underline{R}_\lambda(\underline{F}_\Sigma)} \alpha(hgy)\, \overline{\alpha(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(h)\, dh\, dg \right) \cdot \prod_{v \notin \Sigma} \int_{\underline{\widetilde{G}}(\underline{F}_v)} \int_{\underline{R}_\lambda(\underline{F}_v)} \phi_v(hgy)\, \overline{\phi_v(g)}\, \underline{\nu}_{\lambda,\underline{\chi}}(h)\, dh\, dg.$$

We keep all other places intact but only one ϕ_v, and shrink its support to a sufficiently small neighborhood of identity. We can choose the support of ϕ_v so small that it has empty intersection with the orbit $(\underline{R}_\lambda \times \underline{R}_\lambda)(\underline{F}) \cdot \gamma$ if the orbit $(\underline{R}_\lambda \times \underline{R}_\lambda)(\underline{F}) \cdot \gamma$ does not contain the identity. Then for all $y \in \Omega$, all the terms in (8) vanish except the term corresponding to $\gamma = 1_{2r+1}$. Thus, the geometric expansion gives

$$\left(\int_{\widetilde{\underline{G}}(\underline{F}_\Sigma)} \int_{\underline{R}_\lambda(\underline{F}_\Sigma)} \alpha(hgy)\,\overline{\alpha(g)}\,\underline{\nu}_{\lambda,\underline{\chi}}(h)\,dh\,dg\right) \tag{9}$$

$$\mathrm{vol}([(\underline{R}_\lambda \times \underline{R}_\lambda)_{1_{2r+1}}]) \cdot \prod_{v \notin \Sigma} \int_{\widetilde{\underline{G}}(\underline{F}_v)} \int_{\underline{R}_\lambda(\underline{F}_v)} \phi_v(hgy)\,\overline{\phi_v(g)}\,\underline{\nu}_{\lambda,\underline{\chi}}(h)\,dh\,dg.$$

By comparing (9) with (7), and truncating the spectral side (7) to a finite sum, we obtain (5), as desired. □

Appendix: Ellipticity of supercuspidal representations of G'

The goal of this section is to prove Proposition 4.1 and Theorem 4.2. The argument is standard and it is very close to the classical theory of Harish-Chandra. The main part of the proof also appeared in [Xue 2022] in a different setting. Here, we will only sketch the main argument of the proof since a detailed proof is quite long and deviates significantly from the main goal of this paper. We refer the reader to [Xue 2022; Guo 1998; Kottwitz 2005] for more details of similar setups.

A.1. ***The Lie algebra of G'/H''.*** Recall that we have a symmetric space

$$S' = \{g\bar{g}^{-1} \mid g \in G'\} \cong G'/H''$$

on which H' acts by twisted conjugation. Let

$$\mathfrak{s}' = \left\{ \begin{pmatrix} & X \\ Y & \end{pmatrix} \mid X, Y \in \mathrm{Mat}_n(E)^- \right\},$$

where $\mathrm{Mat}_n(E)^-$ denotes the matrices in $\mathrm{Mat}_n(E)$ with purely imaginary entries. This is viewed as an algebraic variety over F, and it is isomorphic to the tangent space of S' at the point represented by the identity element in G'. The stabilizer of 1 in H' is isomorphic to $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$, which acts on $\mathfrak{s}'$ by conjugation.

Let $g' \in G'$ and let $s' = g\bar{g}^{-1} \in S'$. The tangent space of S' at the point s' is identified with

$$T_{s'} = \{gY\bar{g}^{-1} \mid Y \in \mathfrak{s}'\}.$$

The tangent space of the H'-orbit of s' is identified with

$$TO_{s'} = \{Xs' - s'\overline{X} \mid X \in \mathfrak{h}'\}.$$

We fix an inner product on $T_{s'}$ via

$$\langle gY_1\,\bar{g}^{-1}, gY_2\,\bar{g}^{-1}\rangle = \mathrm{tr}\, Y_1Y_2.$$

Note that this inner product is $H'_{s'}$-invariant. Let

$$N_{s'} = \{Ys' \mid \theta'(Y) = -Y,\ Ys' = -s'\overline{Y}\}.$$

Then we have an orthogonal decomposition

$$T_{s'} = TO_{s'} \oplus N_{s'}$$

and hence $N_{s'}$ is the sliced representation at s'.

Now we describe more concretely the sliced representations at s'. Let

$$s' = s'(\alpha, n_1, n_2, n_3)$$

be a semisimple element in S'. The stabilizer of $s'(\alpha, n_1, n_2, n_3)$ in H' is equal to

$$H_1' \times H_2' \times H_3' = (\mathrm{GL}_{n_1,E})_{\alpha,\mathrm{twisted}} \times \mathrm{GL}_{n_2,E} \times (\mathrm{GL}_{n_3,F} \times \mathrm{GL}_{n_3,F}).$$

The sliced representation at $s'(\alpha, n_1, n_2, n_3)$ can be identified as $V_1' \oplus V_2' \oplus V_3'$ where

$$V_1' = \{A \in \mathrm{Mat}_{n_1}(E) : \alpha \bar{A} = A\bar{\alpha}\}, \quad V_2' = \mathrm{Mat}_{n_2}(E), \quad V_3' = \mathrm{Mat}_{n_3}(E)^- \oplus \mathrm{Mat}_{n_3}(E)^-.$$

We have the following descriptions in the three extreme cases where $n_1 = n$ (Case (i)), $n_2 = n$ (Case (ii)), and $n_3 = n$ (Case (iii)).

Case (i). Assume $n_1 = n,\ n_2 = n_3 = 0$. Then $s' = \begin{pmatrix} \alpha & 1 \\ 1-\alpha\bar{\alpha} & -\bar{\alpha} \end{pmatrix}$. The embedding of $(\mathrm{GL}_{n_1,E})_{\alpha,\mathrm{twisted}}$ in H' is given by

$$h \mapsto \begin{pmatrix} h & \\ & \bar{h} \end{pmatrix}.$$

The embedding of V_1' in $N_{s'}$ is given by

$$A \mapsto \begin{pmatrix} & A \\ -\bar{A}(1-\alpha\bar{\alpha}) & \end{pmatrix} s'.$$

Case (ii). Assume $n_2 = n,\ n_1 = n_3 = 0$. Then $s' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. The embedding of $\mathrm{GL}_{n_2,E} = \mathrm{GL}_{n,E}$ in H' is given by

$$h \mapsto \begin{pmatrix} h & \\ & \bar{h} \end{pmatrix}.$$

The embedding of $V_2' = \mathrm{Mat}_n(E)$ into $N_{s'}$ is given by

$$A \mapsto \begin{pmatrix} & A \\ -\bar{A} & \end{pmatrix} s' = \begin{pmatrix} A & \\ & -\bar{A} \end{pmatrix}.$$

Case (iii). Assume $n_3 = n,\ n_1 = n_2 = 0$. Then $s' = 1$. The embedding of $\mathrm{GL}_{n_3,F} \times \mathrm{GL}_{n_3,F} = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n,F}$ in H' is given by

$$(h_1, h_2) \mapsto \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix}.$$

The embedding of $V_3' = \mathrm{Mat}_n(E)^- \oplus \mathrm{Mat}_n(E)^-$ into $N_{s'}$ is given by

$$(A, B) \mapsto \begin{pmatrix} & A \\ B & \end{pmatrix} s' = \begin{pmatrix} & A \\ B & \end{pmatrix}.$$

A.2. ***Semisimple descent of orbital integrals.*** We recall the notion of the analytic Luna slice. This is a very general notion, so temporarily we let G be a reductive group which acts on a smooth affine variety X. Let $x \in X(F)$ be G-semisimple (i.e., the G-orbit of x is Zariski closed, or equivalently $G(F)x \subset X(F)$ is closed in the analytic topology, see [Aizenbud and Gourevitch 2009, Theorem 2.3.8]). Let $N^X_{Gx,x}$ be the normal space of Gx at x. Write $N_x = N^X_{Gx,x}$. Then by [Aizenbud and Gourevitch 2009, Theorem 2.3.17], there exists the analytic Luna slice at x, denoted by (U, p, ψ, M, N_x) (or simply by (U, p, ψ)), where:

- U is an open $G(F)$-invariant analytic neighborhood of x in $X(F)$.
- p is an $G(F)$-equivariant analytic retraction $p : U \to G(F)\, x$ and $M = p^{-1}(x)$.
- ψ is an $G(F)_x$-equivariant analytic embedding $M \to N_x$ with an open saturated image such that $\psi(x) = 0$.

Here saturated means that $M = \psi^{-1}(\psi(M))$. Let $y \in p^{-1}(x)$ and $z := \psi(y)$. Then we have (see [Aizenbud and Gourevitch 2009, Corollary 2.3.19]):

- $(G_x)_z = G_y$.
- $N^X_{Gy,y} = N^{N_x}_{G_x(F)z,z}$ as $G(F)_y$ spaces.
- y is G-semisimple if and only if z is G_x-semisimple.

Now we describe the semisimple descent of orbital integrals.

Proposition A.1 [Zhang 2014]. *Let χ be a character of $G(F)$. Let $x \in X(F)$ be G-semisimple, and let (U, p, ψ, M, N_x) be an analytic Luna slice at x. Then there exists a neighborhood $\mathcal{U} \subset \psi(M)$ of 0 in N_x with the following properties:*

- *To every $f \in \mathcal{C}_c^\infty(X(F))$, there is an $f_x \in \mathcal{C}_c^\infty(N_x(F))$ such that for all semisimple $z \in \mathcal{U}$ with $z = \psi(y)$ such that χ is trivial on $G_y(F)$, we have*

$$\int_{G_y(F)\backslash G(F)} f(gy)\,\chi(g)\,dg = \int_{G_y(F)\backslash G_x(F)} f_x(gz)\,\chi(g)\,dg. \tag{10}$$

- *Conversely, given any $f_x \in \mathcal{C}_c^\infty(N_x(F))$, there exists an $f \in \mathcal{C}_c^\infty(X(F))$ such that* (10) *holds if $z \in \mathcal{U}$ is semisimple, $z = \psi(y)$, and χ is trivial on $G_y(F)$.*

A.3. ***Orbital integrals on the sliced representations.*** In this subsection, we define orbital integrals on the sliced representations. Let $s' = s'(\alpha, n_1, n_2, n_3)$ be a semisimple element in S'. Recall that the sliced representation at s' is isomorphic to

$$(H_1', V_1') \times (H_2', V_2') \times (H_3', V_3'),$$

where

$$H_1' = (\mathrm{GL}_{n_1,E})_{\alpha,\mathrm{twisted}}, \quad H_2' = \mathrm{GL}_{n_2,E}, \quad H_3' = \mathrm{GL}_{n_3,F} \times \mathrm{GL}_{n_3,F}.$$

and

$$V_1' = \{A \in \mathrm{Mat}_{n_1}(E) : \alpha\bar{A} = A\bar{\alpha}\}, \quad V_2' = \mathrm{Mat}_{n_2}(E), \quad V_3' = \mathrm{Mat}_{n_3}(E)^- \oplus \mathrm{Mat}_{n_3}(E)^-.$$

We can speak of orbital integrals on each component.

A.3.1. *Orbital integrals on* (H_1', V_1'). The group H_1' acts on V_1' by twisted conjugation. Let L be the centralizer of $\alpha\bar{\alpha}$ in $\mathrm{GL}_{n_1}(F)$. Then H_1' is an inner form of L. The map $V_1' \to \mathfrak{h}_1'$ given by $X \mapsto X\bar{\alpha}$ is an isomorphism of representations of H_1', where H_1' acts on $\mathfrak{h}_1'$ by conjugation. We may speak of semisimple and regular semisimple elements in V_1'. For regular semisimple $X \in V_1'$ and any test function $f' \in \mathcal{C}_c^\infty(V_1)$, we define an orbital integral

$$O^{V_1'}(X, f') = \int_{H_{1,X}'\backslash H_1'} f'(h^{-1}X\bar{h})\,\mathrm{d}h,$$

where $H_{1,X}' = \{h \in H_1' \mid h^{-1}X\bar{h} = X\}$.

A.3.2. *Orbital integrals on* (H_2', V_2'). Note that V_2' is isomorphic to $\mathrm{Mat}_{n_2}(E)$, and H_2' acts on V_2' by twisted conjugation. An element $X \in V_2'$ is regular semisimple if $X\bar{X}$ is in $\mathrm{GL}_{n_2}(E)$ and is regular semisimple in the usual sense. For regular semisimple $X \in V_2'$ and any test function $f' \in \mathcal{C}_c^\infty(V_2')$, we define an orbital integral

$$O^{V_2'}(X, f') = \int_{H_{2,X}'\backslash H_2'} f'(h^{-1}X\bar{h})\,\chi(h^{-1}\bar{h})\,\mathrm{d}h,$$

where $H_{2,X}' = \{h \in H_2' \mid h^{-1}X\bar{h} = X\}$.

A.3.3. *Orbital integrals on* (H_3', V_3'). The group H_3' acts on V_3' by conjugation. An element $(X_1, X_2) \in V_3'$ is semisimple or regular semisimple if $\begin{pmatrix} & X_1 \\ X_2 & \end{pmatrix}$ is so in $\mathrm{Mat}_{2n_3}(E)$. For $(X_1, X_2) \in V_3'$ regular semisimple and any $f' \in \mathcal{C}_c^\infty(V_3')$, we define an orbital integral

$$O^{V_3'}((X_1, X_2), f') = \int_{H_{3,(X_1,X_2)}'\backslash H_3'} f'(h_1^{-1}X_1h_2, h_2^{-1}X_2\,h_1)\,\eta(h_1h_2)\,\mathrm{d}h_1\,dh_2.$$

Here, $H_{3,(X_1,X_2)}' = \{(h_1, h_2) \in H_3' \mid (h_1^{-1}X_1h_2, h_2^{-1}X_2\,h_1) = (X_1, X_2)\}$.

A.4. ***Outline of the proof of Proposition 4.1.*** Recall that for an elliptic element $g \in G'$ we have defined an orbital integral

$$O^{G'}(g, f') := \int_{(H'\times H'')_g(F)\backslash(H'\times H'')(F)} f'(h^{-1}gh'')(\chi_{H'}\chi^{-1}\tilde{\eta}^{-1})(h)(\chi\tilde{\eta})^{-1}(h^{-1}gh'')\,\mathrm{d}h\,\mathrm{d}h''.$$

Let $\mathcal{D}(G')^{(H',\chi_{H'}),(H'',\chi\tilde{\eta})}$ be the space of left $(H', \chi_{H'})$ and right $(H'', \chi\tilde{\eta})$-invariant distributions on G'. It's clear that $O^{G'}(g, \cdot) \in \mathcal{D}(G')^{(H',\chi_{H'}),(H'',\chi\tilde{\eta})}$.

Proposition A.2. *The set $\{O^{G'}(g, \cdot) : g \in G'$ is regular semisimple$\}$ is weakly dense in $\mathcal{D}(G')^{(H',\chi_{H'}),(H'',\chi\tilde{\eta})}$; i.e., if $f' \in \mathcal{C}_c^\infty(G')$ and $O^{G'}(g, f') = 0$ for all regular semisimple $g \in G'$, then $\lambda(f') = 0$ for all $\lambda \in \mathcal{D}(G')^{(H',\chi_{H'}),(H'',\chi\tilde{\eta})}$.*

Recall that

$$\mathfrak{s}' = \left\{ \begin{pmatrix} & X \\ Y & \end{pmatrix} \mid X, Y \in \mathrm{Mat}_n(E)^- \right\},$$

where $\mathrm{Mat}_n(E)^-$ denotes the matrices in $\mathrm{Mat}_n(E)$ with purely imaginary entries. This is viewed as an algebraic variety over F, and it is isomorphic to the tangent space of S' at the point represented by the identity element in G'. The stabilizer of 1 in H' is isomorphic to $\mathsf{H}' = \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$, which acts on $\mathfrak{s}'$ by conjugation. If we identify $\mathfrak{s}'$ with $\mathrm{Mat}_n(E)^- \oplus \mathrm{Mat}_n(E)^-$, then we have an action of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ on $\mathfrak{s}'$ by

$$(h_1, h_2) \cdot (X, Y) = (h_1 X h_2^{-1}, h_2 Y h_1^{-1}).$$

An element in $\mathfrak{s}'$ is called semisimple or regular semisimple if it is so in $\mathrm{Mat}_{2n}(E)$. The locus of semisimple and regular semisimple elements in $\mathfrak{s}'$ are denoted by $\mathfrak{s}'_{\mathrm{ss}}$ and $\mathfrak{s}'_{\mathrm{reg}}$ respectively.

Given $\gamma = (X, Y) \in \mathfrak{s}'_{\mathrm{ss}}$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$, we define an orbital integral

$$O(\gamma, \eta, f') = \int_{\mathsf{H}'_\gamma \backslash \mathsf{H}'} f'(h_1 X h_2^{-1}, h_2 Y h_1^{-1}) \eta(h_1 h_2)\, dh_1\, dh_2,$$

where $\mathsf{H}'_\gamma = \{(h_1, h_2) \in \mathsf{H}' \mid (h_1, h_2) \cdot \gamma = \gamma\}$. This integral is absolutely convergent.

Suppose that $\mathcal{D}(\mathfrak{s}')^{\mathsf{H}',\eta}$ is the space of (H', η)-invariant distributions on $\mathfrak{s}'$. Then $O(\gamma, \eta, \cdot) \in \mathcal{D}(\mathfrak{s}')^{\mathsf{H}',\eta}$ for all regular semisimple γ in $\mathfrak{s}'$.

We fix an H'-invariant inner product on $\mathfrak{s}'$ via $\langle Y_1, Y_2 \rangle = \operatorname{tr} Y_1 Y_2$, where the product and trace on the right-hand side are taken in $\mathrm{Mat}_{2n}(E)$. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$, we define its Fourier transform by

$$\hat{f}'(X) = \int_{\mathfrak{s}'} f'(Y) \langle X, Y \rangle \, dY.$$

Hence we can speak of the Fourier transform of distributions on $\mathfrak{s}'$.

Proposition A.3. *Let $\gamma \in \mathfrak{s}'$ be regular semisimple. The Fourier transform of the distribution $O(\gamma, \eta, \cdot)$ is represented by a locally integrable (H', η)-invariant function on $\mathfrak{s}'$. This function is locally constant on $\mathfrak{s}'_{\mathrm{reg}}$.*

We will define nilpotent orbital integrals of $\mathfrak{s}'$ and prove the following result.

Proposition A.4. *The Fourier transforms of nilpotent orbital integrals are represented by locally integrable functions on $\mathfrak{s}'$. These functions are locally constant on $\mathfrak{s}'_{\mathrm{reg}}$.*

Proposition 4.1 will follow from Proposition A.4. We will give more details in Sections A.5–A.10. In particular, since the tangent space $\mathfrak{s}'$ is isomorphic to the tangent space treated in [Xue 2022], the results in Sections A.5–A.8 summarize the results given in [Xue 2022].

A.5. *The nilpotent cone.* Let $\mathcal{N} \subset \mathfrak{s}'$ be the nilpotent cone in $\mathfrak{s}'$, which is the closed subvariety of $\mathfrak{s}'$ consisting of all elements whose orbit closure contains $0 \in \mathfrak{s}'$. We call an element or an H'-orbit in $\mathcal{N}$ a nilpotent element or a nilpotent orbit respectively. In this subsection, we classify the nilpotent orbits which support an (H', η)-invariant distribution.

Lemma A.5. *The nilpotent cone $\mathcal{N}$ consists of elements in $\mathfrak{s}'$ that are nilpotent in $\mathrm{Mat}_{2n}(E)$ in the usual sense.*

Let $V = V^+ \oplus V^-$ be a $\mathbb{Z}/2\mathbb{Z}$-graded F-vector space with homogeneous components $V^\pm$ and $\dim_F V^\pm = n$. Then

$$\mathfrak{s}' \cong \mathrm{Hom}(V^+, V^-) \oplus \mathrm{Hom}(V^-, V^+), \quad \mathsf{H}' \cong \mathrm{GL}(V^+) \times \mathrm{GL}(V^-).$$

The nilpotent cone in $\mathfrak{s}'$ consists of pairs of endomorphism $\xi = (X, Y) \in \mathrm{End}(V)$ with $X \in \mathrm{Hom}(V^+, V^-)$ and $Y \in \mathrm{Hom}(V^-, V^+)$ such that XY is nilpotent (and hence YX is also nilpotent).

Let $\theta \in \mathsf{H}'$ be the element which acts on $V^\pm$ by ± 1. Then θ acts on $\mathfrak{gl}(V)$ by sending $Z \in \mathfrak{gl}(V)$ to $\mathrm{Ad}(\theta)Z = \theta Z \theta^{-1}$. Then $\mathfrak{h}$ and $\mathfrak{s}'$ are eigenspaces of $\mathrm{Ad}(\theta)$ with eigenvalues 1 and -1 respectively.

Let $\xi = (X, Y) \in \mathcal{N}$ with $\xi^s = 0$. Then we have the filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{s-1} \subset W_s = V, \tag{11}$$

where $W_i = \ker \xi^i$. Note that V can be viewed as an $F[\xi]$-module and it is a direct sum of indecomposable $F[\xi]$-modules. By [Kraft and Procesi 1979], we can choose the generators of these submodules to be homogeneous. Let U be such an indecomposable submodule of dimension a over F. We can choose a homogeneous element $u \in U$ such that $u, \xi u, \xi^2 u, \ldots, \xi^{a-1} u$ form a F-basis of U. Then for each i, we have

$$W_i = W_i^+ \oplus W_i^-, \quad W_i^\pm = W_i \cap V^\pm.$$

It follows that we have two filtrations

$$0 = W_0^\pm \subset W_1^\pm \subset \cdots \subset W_{s-1}^\pm \subset W_s^\pm = V^\pm. \tag{12}$$

We remark that the filtration in (11) is strictly increasing while the two filtrations in (12) may not be strictly increasing.

Let $r_i^? = \dim_F W_i^?/W_{i-1}^?$ where ? stands for $+$, $-$, or empty. Note that $r_i \geq r_{i+1}$ for all i because ξ induces an injective map $W_{i+1}/W_i \to W_i/W_{i-1}$. We also have $r_i^\pm \geq r_{i+1}^\mp$ for all i. Let $P = MN$ be the parabolic subgroup of $\mathrm{GL}(V)$ stabilizing the filtration in (11), and let $P^+ = M^+N^+$ be the parabolic subgroup of H' stabilizing both filtrations in (12). We have

$$M^+ \cong \prod_{i=0}^{s-1} \mathrm{GL}(W_{i+1}^+/W_i^+) \times \prod_{i=0}^{s-1} \mathrm{GL}(W_{i+1}^-/W_i^-),$$

and

$$P \cap \mathsf{H}' = P^+, \quad M \cap \mathsf{H}' = M^+, \quad N \cap \mathsf{H}' = N^+.$$

We have the following classification of nilpotent orbits.

Lemma A.6. *The set of nilpotent orbits in $\mathcal{N}$ is in one-to-one correspondence with the set of two sequences of integers $r_i^\pm$ with $i = 1, \dots, s$ such that*

$$\begin{gathered} n = r_1^\pm + \cdots + r_s^\pm, \quad r_1^\pm \geq r_2^\mp \geq r_3^\pm \geq \dots, \\ r_1^+ + r_1^- > r_2^+ + r_2^- > \cdots > r_s^+ + r_s^- > 0. \end{gathered} \tag{13}$$

Note that we have two chains of injective maps induced by ξ:

$$W_s^\epsilon/W_{s-1}^\epsilon \hookrightarrow \cdots \hookrightarrow W_3^\mp/W_2^\mp \hookrightarrow W_2^\pm/W_1^\pm \hookrightarrow W_1^\mp, \tag{14}$$

where $\epsilon = +$ or $-$ depending on the parity of s. We call an integer i with $1 \leq i \leq s-1$ a jump if $\dim_F W_{i+1}^\pm/W_i^\pm < \dim_F W_i^\mp/W_{i-1}^\mp$ (the inequality hold for at least the $+$ one or the $-$ one, it does not have to hold for both filtrations). We call the integer s a jump if $\dim_F W_s^\epsilon/W_{s-1}^\epsilon \neq 0$. The following lemma establishes a necessary condition for an orbit to support an (H', η)-invariant distribution.

Lemma A.7. *Suppose that the orbit represented by ξ supports an (H', η)-invariant distribution. Then all jumps are even integers. This means that we have the strict inequality $r_i^\epsilon > r_{i+1}^\epsilon$ ($\epsilon = +$ or $-$) in* (13) *only when i is even.*

Our next goal is to show that the condition in the above lemme is also sufficient for an orbit to support an (H', η)-invariant distribution. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{s}'$ represented by an element ξ. Then attached to ξ, we have a parabolic subgroup $P = MN$ of $\mathrm{GL}(V)$, a parabolic subgroup $P^+ = M^+N^+$ of H', and two sequences of integers $r_i^\pm$ satisfying (13). Let $2i_1 < \cdots < 2i_a$ be the set of all jumps in the sequence $r_1^+ \geq r_2^- \geq \dots$, and let $2j_1 < \cdots < 2j_b$ be the set of all jumps in the sequence $r_1^- \geq r_2^+ \geq \dots$. The space $\mathfrak{n} \cap \mathfrak{s}'/[\mathfrak{n}, \mathfrak{n}]$ is isomorphic to

$$\bigoplus_{i=1}^{2i_a} \mathrm{Hom}(W_{i+1}^{(-1)^i}/W_i^{(-1)^i}, W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-1}}) \oplus \bigoplus_{i=1}^{2j_b} \mathrm{Hom}(W_{i+1}^{(-1)^{i+1}}/W_i^{(-1)^{i+1}}, W_i^{(-1)^i}/W_{i-1}^{(-1)^i}).$$

We write an element in $\mathfrak{n} \cap \mathfrak{s}'/[\mathfrak{n}, \mathfrak{n}]$ as

$$m(x_1, \dots, x_{2i_a}; y_1, \dots, y_{2j_b})$$

with

$$x_i \in \mathrm{Hom}(W_{i+1}^{(-1)^i}/W_i^{(-1)^i}, W_i^{(-1)^{i-1}}/W_{i-1}^{(-1)^{i-1}}),$$
$$y_i \in \mathrm{Hom}(W_{i+1}^{(-1)^{i+1}}/W_i^{(-1)^{i+1}}, W_i^{(-1)^i}/W_{i-1}^{(-1)^i}).$$

Note that if i is odd, then both $r_{i+1}^\pm = r_i^\mp$ since all jumps are even integers. Moreover, the map induced by ξ:

$$\xi|_{W_{i+1}^\pm/W_i^\pm} : W_{i+1}^\pm/W_i^\pm \to W_i^\mp/W_{i+1}^\mp$$

is an isomorphism. To ease notation, we denote $\xi_i^\mp = \xi|_{W_{i+1}^\pm/W_i^\pm}$. Put

$$\det\nolimits_{2i-1}^+(x_{2i-1}) = \det x_{2i-1}(\xi_{2i-1}^+)^{-1}, \quad \det\nolimits_{2i-1}^-(y_{2i-1}) = \det y_{2i-1}(\xi_{2i-1}^-)^{-1},$$

and

$$\begin{aligned}&\det\nolimits_{\mathfrak{n}}(m)\\ &\quad = \det\nolimits_1^+(x_1)\det\nolimits_3^+(x_3)\dots\det\nolimits_{2i_a-1}^+(x_{2i_a-1})\det\nolimits_1^-(y_1)\det\nolimits_3^-(y_3)\dots\det\nolimits_{2j_b-1}^-(y_{2j_a-1}).\end{aligned}$$

Let $\mathfrak{n}'$ be the subspace of $\mathfrak{n} \cap \mathfrak{s}'$ generated by $[\mathfrak{n}, \mathfrak{n}] \cap \mathfrak{s}'$ and

$$\bigoplus_{i \text{ even}} \mathrm{Hom}(W_{i+1}^+/W_i^+, W_i^-/W_{i-1}^-) \oplus \bigoplus_{i \text{ even}} \mathrm{Hom}(W_{i+1}^-/W_i^-, W_i^+/W_{i-1}^+).$$

For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$, we define a function $\tilde{f}' \in \mathcal{C}_c^\infty(\mathfrak{n} \cap \mathfrak{s}'/\mathfrak{n}')$ as

$$\tilde{f}'(m) = \int_{\mathfrak{n}'} f'(m+u)\, du. \tag{15}$$

This is a function in the variables $m = (x_1, x_3, \dots, x_{2i_a-1}; y_1, y_3, \dots, y_{2j_b-1})$. Let $\underline{s} = (s_1, s_3, \dots, s_{2i_a-1})$ and $\underline{t} = (t_1, t_3, \dots, t_{2j_b-1})$ be complex numbers, and define

$$\begin{aligned}\det\nolimits_{\mathfrak{n},\underline{s},\underline{t}}(m) = {} & |\det\nolimits_1^+(x_1)|^{s_1}|\det\nolimits_3^+(x_3)|^{s_3}\dots|\det\nolimits_{i_a-1}^+(x_{2i_a-1})|^{s_{2i_a-1}}\\ & |\det\nolimits_1^-(y_1)|^{t_1}|\det\nolimits_3^-(y_3)|^{t_3}\dots|\det\nolimits_{j_b-1}^-(y_{2i_a-1})|^{t_{2j_b-1}}.\end{aligned}$$

Consider the integral

$$Z(\underline{s}, \underline{t}, \eta, \tilde{f}') = \int \tilde{f}'(m)\, \eta(\det\nolimits_{\mathfrak{n}}(m)) \det\nolimits_{\mathfrak{n},\underline{s},\underline{t}}(m)\, dm,$$

where the domain of integration is $\mathfrak{n} \cap \mathfrak{s}'/\mathfrak{n}'$, which is identified with

$$\bigoplus_{i \text{ odd}} \mathrm{Hom}(W_{i+1}^-/W_i^-, W_i^+/W_{i-1}^+) \oplus \bigoplus_{i \text{ odd}} \mathrm{Hom}(W_{i+1}^+/W_i^+, W_i^-/W_{i-1}^-).$$

The integral $Z(\underline{s}, \underline{t}, \eta, \tilde{f}')$ is convergent when the real parts of s_i and of t_i are large enough, and it has a meromorphic continuation to $\mathbb{C}^{i_a+j_b}$, which is holomorphic at the points where all s_i's and t_i's are integers. Put

$$\tilde{\mu}_{\mathcal{O}}(f') = Z(\underline{s}, \underline{t}, \eta, \tilde{f}')|_{s_i=r_i^-,\, t_i=r_i^+ \text{ for all } i}.$$

Then for any $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$ and any $p \in P^+$, we have

$$\tilde{\mu}_{\mathcal{O}}(\mathrm{Ad}(p) f') = \delta_{P^+}(p)\, \eta(\det(p))\, \tilde{\mu}_{\mathcal{O}}(f').$$

Now we choose an open compact subgroup K of H' so that $\mathsf{H}' = P^+K$. Define

$$f'_K(\gamma) = \int_K f'(\gamma^k)\, \eta(\det(k))\, dk, \quad \mu_{\mathcal{O}}(f') = \tilde{\mu}_{\mathcal{O}}(f'_K).$$

Lemma A.8. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{s}'$. The distribution on $\mathfrak{s}'$ given by $f' \mapsto \mu_{\mathcal{O}}(f')$ is (H', η)-invariant. Moreover, the linear form $\mu_{\mathcal{O}}$ extends the (H', η)-invariant distribution on $\mathcal{O}$ to an (H', η)-invariant distribution on $\mathfrak{s}'$.*

As a consequence, we have:

Corollary A.9. *A nilpotent orbit $\mathcal{O}$ supports an (H', η)-invariant distribution if and only if the necessary condition in* Lemma A.7 *is satisfied. If $\mathcal{O}$ supports an (H', η)-invariant distribution, then the distribution extends to an (H', η)-invariant distribution on $\mathfrak{s}'$.*

We say that a nilpotent orbit that supports an (H', η)-invariant distribution (or any element of the orbit) is visible. Let $\mathcal{N}_0 \subset \mathcal{N}$ be the subset of $\mathcal{N}$ consisting of visible nilpotent orbits. Then the set

$$\{\mu_{\mathcal{O}} \mid \mathcal{O} \subset \mathcal{N}_0\}$$

is an orthonormal basis of the space of (H', η)-invariant distribution on $\mathfrak{s}'$ supported on $\mathcal{N}$.

Let $d_{\mathcal{O}} = \dim_F N^+$. We have the following homogeneity property of nilpotent orbital integrals.

Lemma A.10. *Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$. For any $t \in F^\times$, we put $f'_t(X) = f'(t^{-1}X)$. Let $\mathcal{O} \subset \mathcal{N}_0$. Then*

$$\mu_{\mathcal{O}}(f'_t) = |t|^{d_{\mathcal{O}}} \eta(t)^n \mu_{\mathcal{O}}(f'), \quad \mu_{\mathcal{O}}(\hat{f}'_t) = |t|^{2n^2-d_{\mathcal{O}}} \eta(t)^n \mu_{\mathcal{O}}(\hat{f}').$$

A.6. ***Orbital integrals.*** Now we define orbital integrals on the entire space $\mathfrak{s}'$, not necessarily on semisimple or nilpotent orbits.

Let $\gamma \in \mathfrak{s}'$ and let $\gamma = \gamma_s + \gamma_n$ be its Jordan decomposition where γ_s is semisimple and γ_n is nilpotent, both in the usual sense. Then $\gamma_s, \gamma_n \in \mathfrak{s}'$. Since $\gamma_s\gamma_n = \gamma_n\gamma_s$, we have $\gamma_n \in \mathfrak{s}'_{\gamma_s}$, and is nilpotent in $\mathfrak{s}'_{\gamma_s}$. Assume that γ_n is visible in $\mathfrak{s}'_{\gamma_s}$ and denote its orbit by $\mathcal{O}_{\gamma_n}$. Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$ and $h \in \mathsf{H}'$. Put

$$f_1'(h) = \mu_{\mathcal{O}_{\gamma_n}}\big(f'(h^{-1}(\gamma_s + \cdot\,)\,h)\big).$$

As a function of $h \in \mathsf{H}'$, f_1' is compactly supported on $\mathsf{H}'_{\gamma_s}\backslash\mathsf{H}'$. Also, for any $y \in \mathsf{H}'_{\gamma_s}$, we have $f_1'(yh) = \eta(\det y))f_1'(h)$. We define an orbital integral

$$O(\gamma, \eta, f') = \int_{\mathsf{H}'_{\gamma_s}\backslash\mathsf{H}'} f_1'(h)\,\eta(\det(h))\,dh.$$

This integral is absolutely convergent. Moreover, if the restriction of f' to the orbit of γ is compactly supported, then $O(\gamma, \eta, f')$ agrees with the integral on the orbit of γ.

Recall that by Proposition A.1, we have an analytic Luna slice (U, p, ψ) at γ. Given $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$, we define

$$f'_{\gamma_s}(\xi) = \int_{\mathsf{H}'} f'(h^{-1}\psi^{-1}(\xi)\,h)\,\eta(\det(h))\,\alpha(h)\,dh, \quad \xi \in \omega_\gamma,$$

where $\omega_\gamma \subset \psi(p^{-1}(\gamma))$ and $\alpha \in \mathcal{C}_c^\infty(\mathsf{H}')$ are given in [Xue 2022, Proposition 2.1]. Then $f'_{\gamma_s} \in \mathcal{C}_c^\infty(\mathfrak{s}_{\gamma_s})$.

Lemma A.11. *Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$. Then $\mu_{\mathcal{O}_{\gamma_n}}(f'_{\gamma_s}) = \mathcal{O}(\gamma, \eta, f')$.*

Lemma A.12. *If γ_n is not visible in $\mathfrak{s}'_{\gamma_s}$, then the orbit of γ in $\mathfrak{s}'$ does not support any (H', η)-invariant distributions.*

A.7. ***The germ expansion.*** We have the canonical germ expansion of orbital integrals.

Proposition A.13. *There is a unique (H', η)-invariant real-valued function $\Gamma_{\mathcal{O}}$ for each nilpotent orbit $\mathcal{O} \subset \mathcal{N}_0$ satisfying the following properties.*

(i) *For any $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$, there is an H'-invariant neighborhood $U_{f'}$ of $0 \in \mathfrak{s}'$ such that*

$$O(\gamma, \eta, f') = \sum_{\mathcal{O}\subset\mathcal{N}_0} \Gamma_{\mathcal{O}}(\gamma)\,\mu_{\mathcal{O}}(f'). \tag{16}$$

(ii) *Let $t \in F^\times$ and $\xi \in \mathfrak{s}'_{\mathrm{reg}}$. Then*

$$\Gamma_{\mathcal{O}}(t\gamma) = |t|^{-d_{\mathcal{O}}}\eta(t)^n\,\Gamma_{\mathcal{O}}(\gamma).$$

The function $\Gamma_{\mathcal{O}}$ is called the Shalika germ for the nilpotent orbit $\mathcal{O}$.

Now we consider the Shalika germ expansion around an arbitrary semisimple $\gamma \in \mathfrak{s}'$. Let $\gamma \in \mathfrak{s}'$ be a fixed semisimple element, and let

$$H''_\gamma = \{g \in H'(F) \mid g^{-1}\gamma g = \gamma\} \quad \text{and} \quad \mathsf{H}'_\gamma = \mathsf{H}' \cap H''_\gamma.$$

Let $\mathfrak{h}''_\gamma$ and $\mathfrak{h}'_\gamma$ be the Lie algebras of H''_γ and H'_γ respectively. Let $\mathfrak{s}'_\gamma$ be such that $\mathfrak{h}''_\gamma = \mathfrak{h}'_\gamma \oplus \mathfrak{s}'_\gamma$. Then the space $\mathfrak{s}'_\gamma$ with an action of H'_γ is isomorphic to $\mathfrak{s}'_1 \times \mathfrak{s}'_2$ with an action of $\mathsf{H}'_1 \times \mathsf{H}'_2$. We refer the reader to [Xue 2022, Section 2] for the explicit description of the action of $\mathsf{H}'_1 \times \mathsf{H}'_2$ on $\mathfrak{s}'_1 \times \mathfrak{s}'_2$. Let $\{\xi_1, \dots, \xi_r\}$ be a complete set of representatives of nilpotent elements in $\mathfrak{s}'_\gamma$ and $\xi_i \in \mathcal{O}_i$. Let Γ_i^γ be the Shalika germ on $\mathfrak{s}'_\gamma$ for the nilpotent orbit $\mathcal{O}_i$. As an application of Proposition A.13, we have:

Corollary A.14. *Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}')$. There exists a neighborhood $U_{f'}$ of γ in $\mathfrak{s}'_\gamma$ so that for any $\xi \in U_{f'} \cap \mathfrak{s}'_{\mathrm{reg}}$, we have*

$$O(\xi, \eta, f') = \sum_{i=1}^{r} \Gamma_i^\gamma(\xi)\, O(\gamma + \xi_i, \eta, f').$$

A.8. ***Linear independence of Shalika germs.*** Similar to [Kottwitz 2005, Section 24], we have the linear independence of Shalika germs, which is closely related to the density of regular semisimple orbital integrals on $\mathfrak{s}'$.

Proposition A.15. (1) *The Shalika germs $\Gamma_{\mathcal{O}}$, $\mathcal{O} \subset \mathcal{N}_0$, are linearly independent. They are not identically zero in any neighborhood of* 0*. If $\mathcal{O} = \{0\}$ is the minimal nilpotent orbit, then $\Gamma_0(\gamma) = 0$ if γ is not elliptic in $\mathfrak{s}'$.*

(2) *The set of regular semisimple orbital integrals is weakly dense in $\mathcal{D}(\mathfrak{s}')^{\mathsf{H}',\eta}$.*

As a consequence of the above proposition, by making use of Howe's finiteness theorem [Rader and Rallis 1996, Theorem 6.7] for $\mathfrak{s}'$, we have:

Corollary A.16. *The Fourier transform $\hat{\mu_{\mathcal{O}}}$ of $\mu_{\mathcal{O}}$ is represented by a locally integrable function in $\mathfrak{s}'$ for all $\mathcal{O} \subset \mathcal{N}_0$, which we will also denote by $\hat{\mu_{\mathcal{O}}}$.*

A.9. ***Regular semisimple orbital integrals on G'.*** We establish results on the level of G'. Let ω be an (H', η)-invariant neighborhood of $0 \in \mathfrak{s}'$, and let Ω be a neighborhood of $1 \in S'$ such that the exponential map $\exp : \omega \to \Omega$ is defined and is a homeomorphism. Let $f' \in \mathcal{C}_c^\infty(G')$. We define a function $f'_\sharp \in \mathcal{C}_c^\infty(\omega)$ by requiring that

$$\int_{H''} f'(gh)(\chi\tilde{\eta}^{-1})(gh)\, dh = f'_\sharp(\gamma)$$

if $g\bar{g}^{-1} = \exp(\gamma)$, and we extend $f'_\sharp$ to a function in $\mathcal{C}_c^\infty(\mathfrak{s}')$ by extension by zero. Let $u_1, \dots, u_r, u_{r+1}, \dots, u_s$ be a complete set of representatives of unipotent orbits in G'. Let $\mathcal{O}_i$ be the nilpotent orbits in $\mathfrak{s}'$ represented by $\exp^{-1}(u_i\overline{u_i}^{-1})$, and we may label them so that $\mathcal{O}_i$ is visible precisely when $1 \le i \le r$. Then u_i represents a unipotent orbit in G' which supports a left $(H', \chi_{H'}^{-1})$ and right $(H'', \chi^{-1}\tilde{\eta})$-invariant distribution precisely when $1 \le i \le r$. If $f' \in \mathcal{C}_c^\infty(G')$, then

$$O^{G'}(u_i, f') = \mu_{\mathcal{O}_i}(f'_\sharp).$$

We call the unipotent elements $u_1, \dots, u_r$ or their orbits visible. We have the Shalika germ expansion of orbital integrals on G'.

Proposition A.17. *Let $f' \in \mathcal{C}_c^\infty(G')$. There is a neighborhood $U_{f'} \subset \Omega$ of $1 \in S'$ such that if $g \in G'$ is regular semisimple in G' with $g\bar{g}^{-1} = \exp(\gamma)$ where $\gamma \in \omega$, then*

$$O^{G'}(g, f') = \sum_{i=1}^{r} \Gamma_{\mathcal{O}_i}(\gamma)\, \mu_{\mathcal{O}_i}(f'_\sharp).$$

As a consequence of Proposition A.17, we have Proposition A.2.

A.10. ***The Spherical character on G'.*** Here we state the germ expansion for the spherical character I_Π.

Let $s' = g\bar{g}^{-1} \in S'$ be a semisimple element, and consider the map

$$\mathsf{H}' \times H'_{s'} \times \mathsf{H}' \to H', \quad (h_1, g, h_2) \mapsto h_1 s' g h_2.$$

Let $U_{s'}$ be the open subset of $H'_{s'}$ consisting of elements $g \in H'_{s'}$ such that the above map is submersive at $(1, g, 1)$, and let $\Omega_{s'}$ be the subset $\mathsf{H}' s' U_{s'} \mathsf{H}'$. Then $U_{s'}$ is a bi-$\mathsf{H}_{s'}$-invariant neighborhood of 1 in $H'_{s'}$, and $\Omega_{s'}$ is an open and bi-H' invariant neighborhood of 1 in H'. By the standard theory of Harish-Chandra, there exists a surjective map

$$\mathcal{C}_c^\infty(\mathsf{H}' \times U_{s'} \times \mathsf{H}') \to \mathcal{C}_c^\infty(\Omega_{s'}), \quad \alpha \mapsto f'_\alpha,$$

satisfying the property that

$$\int_{\mathsf{H}' \times U_{s'} \times \mathsf{H}'} \alpha(h_1, g, h_2)\, \beta(h_1 s' g h_2)\, dh_1\, dg\, dh_2 = \int_{\Omega_{s'}} f'_\alpha(g)\, \beta(g)\, dg$$

for all $\beta \in \mathcal{C}_c^\infty(\Omega_{s'})$. Then there is a unique left $\mathsf{H}_{s'}$-invariant and right $(\mathsf{H}_{s'}, \eta)$-invariant distribution $J_{s'}$ on $H'_{s'}$, such that

$$I_\Pi(f'_\alpha) = J_{s'}(\beta_\alpha)$$

for all $\alpha \in \mathcal{C}_c^\infty(\mathsf{H}' \times U_{s'} \times \mathsf{H}')$, where

$$\beta_\alpha(g) = \int_{\mathsf{H}'} \int_{\mathsf{H}'} \alpha(h_1, g, h_2)\, \eta(\det h_2)\, dh_1\, dh_2, \quad g \in H'_{s'}.$$

We have the germ expansion of I_Π.

Proposition A.18. *There are constants $c_{\mathcal{O}}$ for each visible nilpotent orbit $\mathcal{O}$ in the nilpotent cone $\mathcal{N}_{s'}$, such that*

$$I_\Pi(f'_\alpha) = \sum_{\mathcal{O}\subset\mathcal{N}_{s'}} c_{\mathcal{O}}\,\hat{\mu_{\mathcal{O}}}(\beta_{\alpha,\sharp})$$

for all $\alpha \in \mathcal{C}_c^\infty(\mathsf{H}' \times U_{s'} \times \mathsf{H}')$.

We have Proposition 4.1 as a consequence of Proposition A.18 and Corollary A.16.

A.11. ***Proof of Theorem 4.2.*** We follow the argument in [Xue and Zhang 2023, Appendix B] closely.

Recall that we have a transfer factor

$$\kappa^{G'}(g) = \chi^{-1}(\alpha_4)\,\tilde{\eta}(\tau\alpha_2),$$

where

$$g\bar{g}^{-1} = \begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix} \in S'(F).$$

Let $\tilde{\Theta}_\Pi(g) = \kappa^{G'}(g)\,\Theta_\Pi(g)$.

Recall the explicit description of the intertwinings from [Beuzart-Plessis 2018, Section 4.2], when Π is supercuspidal. For a linear form $\ell' \in \mathrm{Hom}_{H'}(\Pi\otimes\chi_{H'}, \mathbb{C})$,

$$\ell'(v)\,\overline{\ell'(w)} = \int_{Z'(F)\backslash H'(F)} \langle v, \Pi(h^{-1})\,w\rangle \chi_{H'}(h)\,dh. \tag{17}$$

Similarly, for $\ell'' \in \mathrm{Hom}_{H''}(\Pi\otimes\chi\tilde{\eta}, \mathbb{C})$,

$$\ell''(v)\,\overline{\ell''(w)} = \int_{Z'(F)\backslash H''(F)} \langle v, \Pi(h'')\,w\rangle (\chi\tilde{\eta})^{-1}(h'')\,\mathrm{d}h''. \tag{18}$$

Lemma A.19. *Let Π be supercuspidal, and suppose* $\mathrm{Hom}_{H'}(\Pi\otimes\chi_{H'}, \mathbb{C}) \neq 0$ *and* $\mathrm{Hom}_{H''}(\Pi\otimes\chi\tilde{\eta}, \mathbb{C}) \neq 0$. *Let $v, w \in \Pi$ and $f'(g) = \langle v, \Pi(g)\,w\rangle$ be the matrix coefficient of* Π. *Then*

$$\kappa^{G'}(g)\,O^{G'}(g, f') = (\chi\tilde{\eta})^{-1}(g)\,\tilde{\Theta}_\Pi(g)\,\ell'(v)\,\overline{\ell''(w)}. \tag{19}$$

Proof. It suffices to prove that for any $\varphi \in \mathcal{C}_c^\infty(G')$ supported in the elliptic locus, we have

$$\int_{G'} \varphi(g)\kappa^{G'}(g)\,O^{G'}(g, f')\,dg = \int_{G'} \varphi(g)(\chi\tilde{\eta})^{-1}(g)\,\tilde{\Theta}_\Pi(g)\,dg\cdot\ell'(v)\,\overline{\ell''(w)}. \tag{20}$$

Note that the right-hand side of (20) is equal to $I_\Pi(\varphi\kappa^{G'}(\chi\tilde{\eta})^{-1})\cdot\ell'(v)\,\overline{\ell''(w)}$.

Since $(H'\times H'')_g$ is an anisotropic torus modulo the split center Z' of G', up to a nonzero constant depending only on the choice of the measures, the orbital integral $O^{G'}(g, f')$ is equal to

$$\int_{Z'(F)\backslash(H'\times H'')(F)} f'(h^{-1}gh'')(\chi_{H'}\chi^{-1}\tilde{\eta}^{-1})(h)(\chi\tilde{\eta})^{-1}(h^{-1}gh'')\,\mathrm{d}h\,\mathrm{d}h''.$$

Since φ is supported in the elliptic locus, the left-hand side of (20) is equal to

$$\int_{G'}\int_{Z'(F)\backslash(H'\times H'')(F)} \varphi(g)\kappa^{G'}(g)\langle v, \Pi(h^{-1}gh'')\,w\rangle(\chi_{H'}\chi^{-1}\tilde{\eta}^{-1})(h)(\chi\tilde{\eta})^{-1}(h^{-1}gh'')\,\mathrm{d}h\,\mathrm{d}h''\,dg.$$

This integral is absolutely convergent. Changing the order of integration, we conclude that the left-hand side of (20) equals

$$\int_{Z'(F)\backslash(H'\times H'')(F)} \langle v, \Pi(h^{-1})\,\Pi(\overline{\varphi\kappa^{G'}(\chi\tilde{\eta})^{-1}})\,\Pi(h'')\,w\rangle(\chi_{H'}\chi^{-1}\tilde{\eta}^{-1})(h)(\chi\tilde{\eta})^{-1}(h^{-1}h'')\,\mathrm{d}h\,\mathrm{d}h''.$$

This simplifies to

$$\int_{Z'(F)\backslash(H'\times H'')(F)} \langle v, \Pi(h^{-1})\,\Pi(\overline{\varphi\kappa^{G'}(\chi\tilde{\eta})^{-1}})\,\Pi(h'')\,w\rangle\chi_{H'}(h)(\chi\tilde{\eta})^{-1}(h'')\,\mathrm{d}h\,\mathrm{d}h''. \tag{21}$$

Since Π is admissible, there are vectors $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ in Π such that

$$\Pi(\overline{\varphi\kappa^{G'}(\chi\tilde{\eta})^{-1}})(v) = \sum_{i=1}^{r}\langle v, v_i\rangle\,w_i.$$

Then (21) equals

$$\sum_{i=1}^{r}\int_{Z'(F)\backslash(H'\times H'')(F)}\langle v, \Pi(h^{-1})\,w_i\rangle\langle v_i, \Pi(h'')\,w\rangle\chi_{H'}^{-1}(h)(\chi\tilde{\eta})^{-1}(h'')\,\mathrm{d}h\,\mathrm{d}h''.$$

Now we apply (17) and (18) to conclude that the left-hand side of (20) is equal to

$$\sum_{i=1}^{r}\ell'(v)\,\overline{\ell'(w_i)}\,\ell''(v_i)\,\overline{\ell''(w)} \tag{22}$$

On the other hand, we have

$$\begin{aligned} I_\Pi(\varphi\kappa^{G'}(\chi\tilde{\eta}^{-1}) &= \sum_u \overline{\ell'\big(\Pi(\overline{\varphi\kappa^{G'}(\chi\tilde{\eta})^{-1}})\,u\big)\,\overline{\ell''(u)}} \\ &= \sum_u \overline{\ell'\bigg(\sum_{i=1}^{r}\langle u, v_i\rangle\,w_i\bigg)\,\overline{\ell''(u)}} \\ &= \sum_u\sum_{i=1}^{r}\overline{\langle u, v_i\rangle}\,\overline{\ell'(w_i)}\,\ell''(u) = \sum_{i=1}^{r}\overline{\ell'(w_i)}\,\ell''(v_i). \end{aligned}$$

Thus the right-hand side of (20) is equal to (22). □

Proof of Theorem 4.2. Let γ be in a small neighborhood of $0 \in \mathfrak{s}'$, and take $g \in G'$ such that $g\bar{g}^{-1} = \exp(\gamma)$. Similar to [Rader and Rallis 1996, Theorem 7.11], we have a character expansion

$$\tilde{\Theta}_\Pi(g) = \sum_{\mathcal{O}} c_{\mathcal{O}}\, \hat{\mu_{\mathcal{O}}}(\gamma), \tag{23}$$

where $\hat{\mu_{\mathcal{O}}}$ is a locally integrable function on $\mathfrak{s}'$ which represents the Fourier transform of the orbital integral $\mu_{\mathcal{O}}$ as in Corollary A.16. Note that g is elliptic if and only if γ is elliptic. To prove that Π is (H', H'')-elliptic, it suffices to show that $\tilde{\Theta}_\Pi(g) \neq 0$ for some elliptic $g \in G'$ which is sufficiently close to 1. Since $\mathcal{O} = \{0\}$ is the only nilpotent orbit with $\hat{\mu_{\mathcal{O}}}(t\gamma) = \hat{\mu_{\mathcal{O}}}(\gamma)$ for all $\gamma \in \mathfrak{s}'$ and $t \in F^\times$, it suffices to show that $c_0 \neq 0$.

Let $v, w \in \Pi$ and $f'(g) = \langle v, \Pi(g)w\rangle$ be the matrix coefficient of Π such that

$$\int_{Z'(F)\backslash H'(F)} f'(h^{-1})\, \chi_{H'}(h)\, dh \cdot \int_{Z'(F)\backslash H''(F)} f'(h'')(\chi\tilde{\eta})^{-1}(h'')\, \mathrm{d}h'' \neq 0.$$

By (17) and (18), the above inequality is equivalent to

$$\ell'(v)\, \overline{\ell'(w)}\, \ell''(v)\, \overline{\ell''(w)} \neq 0.$$

Now we consider both sides of (19) when g is sufficiently close to 1. By the Shalika germ expansion of orbital integral (see Proposition A.17) and the character expansion of spherical character (see (23)), we have

$$\sum_{\mathcal{O}} \kappa^{G'}(g)\, \Gamma_{\mathcal{O}}(\gamma)\, \mu_{\mathcal{O}}(f'_\sharp) = \sum_{\mathcal{O}} (\chi\tilde{\eta})^{-1}(g)\, c_{\mathcal{O}}\, \hat{\mu_{\mathcal{O}}}(\gamma)\, \ell'(v)\, \overline{\ell''(w)}.$$

The only terms in both sides of the expansion that are invariant under the scaling $\gamma \mapsto t\gamma$ are the terms corresponding to $\mathcal{O} = \{0\}$. It follows from the homogeneity property of $\Gamma_{\mathcal{O}}$ and $\hat{\mu_{\mathcal{O}}}$ that

$$\kappa^{G'}(g)\, \Gamma_0(\gamma)\, \mu_0(f'_\sharp) = (\chi\tilde{\eta})^{-1}(g)\, c_0\, \hat{\mu}_0(\gamma)\, \ell'(v)\, \overline{\ell''(w)}. \tag{24}$$

By our choice of f', we have

$$\mu_0(f'_\sharp) = \int_{H''(F)} f'(h'')(\chi\tilde{\eta})(h'')\, dh'' \neq 0.$$

If $c_0 = 0$, then from (24) we deduce that $\Gamma_0(\gamma) = 0$ if γ is elliptic in a neighborhood of 0. By Proposition A.15, we also know that $\Gamma_0(\gamma') = 0$ if γ' is not elliptic. Hence Γ_0 is identically zero in a neighborhood of 0. This contradicts Proposition A.15. Therefore, we conclude that $c_0 \neq 0$. This finishes the proof of Theorem 4.2. □

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