Pacific Journal of Mathematics

ON THE GEOMETRY OF THE PAPPAS–RAPOPORT MODELS IN THE (AR) CASE

STÉPHANE BIJAKOWSKI AND VALENTIN HERNANDEZ

Volume 334 No. 1 January 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 334, No. 1, 2025
https://doi.org/10.2140/pjm.2025.334.107

ON THE GEOMETRY OF THE PAPPAS–RAPOPORT MODELS IN THE (AR) CASE

STÉPHANE BIJAKOWSKI AND VALENTIN HERNANDEZ

We study some integral model of P.E.L. Shimura varieties of type A for ramified primes. Precisely, we look at the Pappas–Rapoport model (or splitting model) of some unitary Shimura varieties for which there is ramification in the degree-2 CM extension. We show that the model isn't smooth, but that it is normal with Cohen–Macaulay special fiber. We study its special fiber by introducing a combinatorial stratification for which we can compute the closure relations. Even if there are "extra" components in the special fiber, we prove that those do not contribute to mod p modular forms in regular degree. We also study the interaction of the stratification with the natural stratification given by the vanishing of some partial Hasse invariants, in the case of signature $(1, n-1)$.

1. Introduction

In the last 50 years at least, Shimura varieties have played a central role in the Langlands program. Most of the time, these varieties can be thought of as moduli spaces of abelian varieties over $\mathrm{Spec}(\mathbb{Q})$. It turns out that for arithmetic applications it is sometimes desirable to have an integral structure on these spaces to be able to use their reduction modulo primes p. Such integral structures have been studied first by Deligne and Rapoport [6] and Katz and Mazur [9] for the modular curve and have been largely studied since then.

Obviously, there are a priori a lot of possible choices for these integral models, but there is a natural way to choose one by extending the moduli problem over $\mathrm{Spec}(\mathbb{Z})$ (or some localization of it). This *natural* strategy has been extensively studied and most of the results in the P.E.L. case can be found in works of Lan [11] as long as the moduli problem is *unramified*. An extra difficulty appears when we allow ramification in the moduli problem. In this case it has been realized a long time ago in works of Pappas and Rapoport (see, e.g., [12]) that the natural moduli problem has bad

MSC2020: 11G18, 14G35.
Keywords: Shimura varieties.

geometric properties. Pappas and Rapoport suggested to study a slightly different integral model than the *natural* one, by adding to the moduli problem (parametrizing abelian schemes) an extra linear data of a flag of the Hodge filtration (with some restricting properties). This model is referred to as the *splitting model*, or (as we call it) the Pappas–Rapoport model. In some sense, this model should be thought of as a blow-up of the natural model along some of its singularities. The goal of this article is to study the geometry of this model, and in particular of its special fiber.

Let us be more precise. As in [3], we focus on the P.E.L. cases of type A and C, allowing some ramification. More precisely, we consider quasisplit unitary or symplectic groups over a ramified number field. In case C, we studied all cases in [3], proving that the model is smooth, and the ordinary locus is dense in the special fiber. In case A, we have a CM field F over a totally real field F_0, and we studied in [3] the geometry of the model and its special fiber at p under the assumption that F/F_0 was unramified at p. In this case we showed also that the model is smooth and the μ-ordinary locus is dense in the special fiber. Both of these results were expected because of [13], and proved in some cases. It was clear, since the PhD thesis of Kramer [10] that the model could not be smooth if we allow F/F_0 to be ramified. We study the geometry of the special fiber of the Pappas–Rapoport model under this assumption (which is referred to as the (AR) case). We prove that the special fiber is stratified by an explicit poset with a combinatorial description, and in particular we have a description of the irreducible components of the special fiber. We prove the closure relations for this stratification. Note that the (generalized) Rapoport locus coincides with one (maximal) stratum of the special fiber (see [3, Definition 2.28]). Let us give a precise formulation when $F_0 = \mathbb{Q}$, i.e., when F is quadratic imaginary, and p is a ramified prime, π a uniformizer of $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Let a, b be integers with $a \leq b$, and let Y be the Pappas–Rapoport model over O_{F_p} (see Definition 2.1) for the unitary group $\mathrm{GU}(a, b)_{F/\mathbb{Q}}$, and let X be its special fiber. The entire point of the Pappas–Rapoport model in this case is that over Y, and thus over X, there is a locally direct factor $\omega_1 \subseteq \omega$ of rank a, where ω is the conormal sheaf of the universal abelian scheme. There is also a second locally direct factor $\omega_2 \subseteq \omega$ of rank b, which is obtained from ω_1 and the polarization (see Definition 2.4). The universal abelian scheme has an action of O_F, thus so does ω.

For every $0 \leq h \leq \ell \leq a$, set

$$X_{h,\ell} := \{x \in X \mid \dim \pi\omega = h,\ \dim \omega_1 \cap \omega_2 = \ell\}.$$

This is a locally closed subscheme of X. For example, $X_{a,a}$ is the (generalized) Rapoport locus. Our first result is the following:

Theorem 1.1. *Assume $p \neq 2$. For all $h \leq \ell$, the stratum $X_{h,\ell}$ is nonempty, smooth, and equidimensional of dimension $ab - \frac{(\ell-h)(\ell-h+1)}{2}$. Moreover we have the closure relations*

$$\overline{X_{h,\ell}} = \coprod_{0 \leq h' \leq h \leq \ell \leq \ell' \leq a} X_{h',\ell'}.$$

In particular X is not smooth, and the smooth locus is the union of the $X_{h,h}$ for $0 \leq h \leq a$. Moreover Y is flat over O_F, normal, and X is reduced and Cohen–Macaulay.

When $F_0 \neq \mathbb{Q}$, one has a similar description (the index of the stratification being more complicated), all computations reducing to the previous case. Let us stress that in the case of an unramified prime there is only one open stratum (the Rapoport locus) as in the cases considered in [3]. In particular, in the situation considered here (i.e., the (AR) case), there is no chance that the Rapoport locus or the μ-ordinary locus is Zariski dense.

We can then study how the stratification studied previously interacts with "classical" stratifications, for example, the one induced by the *partial* Hasse invariants. The general result is likely to be overly complicated for combinatorial reasons, so let us describe the situation when $F_0 = \mathbb{Q}$, $p \neq 2$ and $(a, b) = (1, n)$, $n \geq 1$. In this case the previous stratification gives only three strata, $R = X_{1,1}$, the Rapoport locus, $B = X_{0,0}$ the other open stratum and the intersection of their closures $P = X_{0,1}$. We have two partial Hasse invariants $\mathrm{hasse}_1, \mathrm{hasse}_2$ (see Definition 4.1) and we stratify further these three strata depending on the vanishing of these invariants. It turns out that there are restrictions on these vanishings, and we end up with six strata, refining the previous stratification:

$$R = R_0 \sqcup R_1 \sqcup R_2, \quad B = B_0 \sqcup B_1 \sqcup B_2, \quad P = P_0 \sqcup P_1 \sqcup P_2,$$

with $R_0 = X^{\mathrm{ord}}$ the μ-ordinary locus, R_0, P_0, B_0 the locus of nonvanishing of both the partial Hasse invariants, and R_2, P_2 the locus of vanishing of both partial Hasse invariants (see Section 4). We then have the following description, which may be surprising for $\overline{B_1}$ and when $n = 1$.

Theorem 1.2. *If $n = 1, 2$ the strata R_1, P_1 are empty. If $n = 1$ we have*

$$\overline{X^{\mathrm{ord}}} = X^{\mathrm{ord}} \cup P_0, \quad \overline{R_2} = R_2 \cup P_2, \quad \overline{B_0} = B_0 \cup B_1 \cup B_2 \cup P_0 \cup P_2,$$

while X^{ord}, R_2, B_0 are open, and P_0, P_2, B_1, B_2 are closed.

If $n \geq 2$, then B_2 and P_2 are closed, and we have the closure relations

$$\overline{X^{\text{ord}}} = X^{\text{ord}} \cup \bigcup_{i=1}^{2} R_i \cup \bigcup_{i=0}^{2} P_i, \quad \overline{R_2} = R_2 \cup P_2,$$

$$\overline{B_0} = \bigcup_{i=0}^{2} B_i \cup \bigcup_{i=0}^{2} P_i, \quad \overline{B_1} = B_1 \cup P_1 \cup P_2, \quad \overline{P_0} = \bigcup_{i=0}^{2} P_i.$$

If $n \geq 3$, one has

$$\overline{R_1} = \bigcup_{i=1}^{2} R_i \cup \bigcup_{i=1}^{2} P_i, \quad \overline{P_1} = P_1 \cup P_2.$$

We expect that this combinatorial description of the special fiber will relate to more classical geometric varieties, and hopefully that we will be able to prove some cohomological vanishing of modular forms using the geometry of the model. As a first step, we can already prove that the extra irreducible components in the special fiber, that is, those which are disjoint from the (generalized) Rapoport locus, do not contribute to modulo p modular forms in sufficiently regular weights. Namely assume $F_0 = \mathbb{Q}$ and let $\kappa = (k_1 \geq \cdots \geq k_a, \ell_1 \geq \cdots \geq \ell_b) \in \mathbb{Z}^{a+b}$ be a weight (see Section 3). Then we have the following result.

Theorem 1.3. *If $h < a$ and if we cannot find $\{i_1 < \cdots < i_{a-h}\} \subset \{1, \ldots, a\}$ such that*

$$k_{i_1} = \cdots = k_{i_{a-h}} \leq \ell_{b-h+1},$$

then $H^0(\overline{X_{h,h}}, \omega^\kappa) = 0$.

We hope to generalize this result to higher cohomology and less restrictive weights. In [14], similar results on the geometry of the mod p fibers of Shimura varieties are proven for the Pappas–Zhu model and its EKOR stratification. It would be interesting to know if our results are related to theirs. More recently, Zachos [15] has studied a similar problem under the assumption $F_0 = \mathbb{Q}$ restricting to the geometry around points of $X_{0,a}$, but has given an explicit blow-up of the splitting model which has semistable reduction, under the extra assumption that $(a, b) = (2, n-2)$.

2. Case of a quadratic imaginary F

2.1. ***Definition of the variety.*** Let F be a complex quadratic extension of $\mathbb{Q}$, and assume that p is ramified in F. Let F_p be the completion of F at p, and π a uniformizer of F_p. We write σ_1, σ_2 the embeddings of F_p into $\overline{\mathbb{Q}_p}$, and let us define $\pi_i = \sigma_i(\pi)$. Let a, b be integers with $a \leq b$, and define $m = a + b$.

Definition 2.1. Let Y be the moduli space over O_{F_p} whose R-points are tuples $(A, \lambda, \iota, \eta, \omega_1)$, where

- A is an abelian scheme over R of dimension m,
- λ is a polarization, principal at p,
- $\iota : O_F \to \mathrm{End}(A)$, making the Rosati involution and the complex conjugation compatible,
- η is a level structure away from p,
- $\omega_1 \subseteq \omega_A$ is a locally direct factor of rank a, stable by O_F,
- O_F acts by σ_1 on ω_1, and by σ_2 on ω_A/ω_1.

Remark 2.2. One might find the previous definition not symmetric on a and b. Actually Proposition 2.5 shows that the analogous moduli space obtained by exchanging σ_1, σ_2 and a, b is isomorphic to the previous one. This is the moduli-theoretic counterpart of the well-known isomorphism of unitary groups $U(a, b) \simeq U(b, a)$.

Let $\mathcal{E} = H^1_{dR}(A)$; it is a locally free sheaf on Y of rank $2m$. It has an action of O_F, and is locally free of rank m over $O_Y \otimes_{\mathbb{Z}} O_F$. The Hodge filtration is $\omega_A \subseteq \mathcal{E}$. The sheaf $\mathcal{E}$ has an action of O_{F_p}, and let $[a]$ be the action of a on $\mathcal{E}$ for every $a \in O_{F_p}$. The last condition implies that $([\pi] - \pi_1)\omega_1 = 0$ and $([\pi] - \pi_2)\omega \subseteq \omega_1$.

Thanks to the polarization, one has a perfect pairing on $\langle \cdot , \cdot \rangle$ on $\mathcal{E}$. The condition between the Rosati involution and the complex conjugation implies that for all $x, y \in \mathcal{E}$ one has

$$\langle [a] \cdot x, y \rangle = \langle x, [\bar{a}] \cdot y \rangle.$$

The Hodge filtration is totally isotropic for this pairing. The above relation implies that

$$\mathcal{E}[[\pi] - \pi_i]^{\perp} = \mathcal{E}[[\pi] - \pi_i],$$

where $\mathcal{E}[[\pi] - \pi_i]$ consists of the elements of $\mathcal{E}$ killed by $[\pi] - \pi_i$.

Remark 2.3. Since $\mathcal{E}[[\pi] - \pi_1]^{\perp} = \mathcal{E}[[\pi] - \pi_1]$, one has a perfect pairing between $\mathcal{E}[[\pi] - \pi_1]$ and $\mathcal{E}/\mathcal{E}[[\pi] - \pi_1]$. This last sheaf is isomorphic to $\mathcal{E}[[\pi] - \pi_2]$ via the multiplication by $[\pi] - \pi_1$. One has thus an induced pairing between $\mathcal{E}[[\pi] - \pi_1]$ and $\mathcal{E}[[\pi] - \pi_2]$, given by the formula

$$\{([\pi] - \pi_2)x, ([\pi] - \pi_1)y\} := \langle ([\pi] - \pi_2)x, y \rangle.$$

Definition 2.4. Let us define $\omega_2 \subseteq \mathcal{E}$ by the formula

$$\omega_2 = (([\pi] - \pi_2)^{-1}\omega_1)^{\perp}.$$

Proposition 2.5. *The sheaf ω_2 is locally free of rank b, and one has $\omega_2 \subseteq \omega$. One has*

$$([\pi]-\pi_2)\cdot\omega_2 = 0, \quad ([\pi]-\pi_1)\cdot\omega \subseteq \omega_2.$$

Proof. From the properties satisfied by ω_1, one has $\omega \subseteq ([\pi]-\pi_2)^{-1}\omega_1$. Taking the orthogonal of this relation (and using that $\omega^\perp = \omega$), one finds the relation $\omega_2 \subseteq \omega$.

One has $\mathcal{E}[[\pi]-\pi_2] \subseteq \omega_2^\perp$, and taking the orthogonal gives $\omega_2 \subseteq \mathcal{E}[[\pi]-\pi_2]$. In other words, $([\pi]-\pi_2)\cdot\omega_2 = 0$.

For the last point, we first claim that $([\pi]-\pi_1)\cdot\omega_1^\perp = \omega_2$. Indeed, let $x \in \omega_2$; since it belongs to $\mathcal{E}[[\pi]-\pi_2]$ there exists $x' \in \mathcal{E}$ such that $x = ([\pi]-\pi_1)x'$. Then

$$\begin{aligned} x \in \omega_2 &\iff \langle x, y\rangle = 0 && \text{for all } y \in ([\pi]-\pi_2)^{-1}\omega_1 \\ &\iff \langle x', ([\pi]-\pi_2)y\rangle = 0 && \text{for all } y \in ([\pi]-\pi_2)^{-1}\omega_1 \\ &\iff x' \in \omega_1^\perp. \end{aligned}$$

One thus has $([\pi]-\pi_1)\cdot\omega_1^\perp = \omega_2$, or equivalently $\omega_1^\perp = ([\pi]-\pi_1)^{-1}\omega_2$. The inclusion $\omega_1 \subseteq \omega$ then implies that $\omega \subseteq ([\pi]-\pi_1)^{-1}\omega_2$. In other words,

$$([\pi]-\pi_1)\cdot\omega \subseteq \omega_2. \qquad \square$$

2.2. ***Local model diagrams.*** We recall the theory of local models which is developed in [13, Section 15]. Here we assume $p \neq 2$. Let $\Lambda = \mathcal{O}_F^m$, with natural polarization $\langle x, y\rangle = \mathrm{tr}_{F/\mathbb{Q}}\big(\frac{1}{2\pi}\sum_{i=1}^m x_i c(y_{m-i+1})\big)$ for a choice of π such that $\mathrm{tr}(\pi) = 0$. We have a local model diagram

$$Y \leftarrow \widetilde{Y} \rightarrow \mathcal{N},$$

where $\widetilde{Y} = \mathrm{Isom}_{\mathcal{O}_F,\langle\cdot,\cdot\rangle}(\mathcal{E}, \Lambda\otimes O_Y)$, and $\mathcal{N}$ is the $\mathcal{O}_{F,p}$-scheme parametrizing, for any scheme S, couples

$$(\mathcal{F}_1, \mathcal{F}),$$

where $\mathcal{F} \subset \Lambda \otimes_{O_F} S$ is a locally direct factor, stable by $\mathcal{O}_F$, totally isotropic of rank $a+b$, and $\mathcal{F}_1 \subset \mathcal{F}$ is a locally direct factor of rank a, stable by $\mathcal{O}_F$, such that

- $([\pi]-\pi_1)(\mathcal{F}_1) \subset p\mathcal{F}$,
- $([\pi]-\pi_2)(\mathcal{F}) \subset \mathcal{F}_1$,

where the map $\widetilde{Y} \rightarrow Y$ is the natural one and is smooth (it is a torsor over a smooth algebraic group), and

$$\widetilde{Y} \rightarrow \mathcal{N}, \quad (A/S, \omega^1, i : \mathcal{E} \simeq \Lambda\otimes\mathcal{O}_S) \mapsto \omega^1 \subset \omega \subset \mathcal{E} \simeq \Lambda\otimes\mathcal{O}_S.$$

In particular $\widetilde{Y} \to \mathcal{N}$ is formally smooth by Grothendieck–Messing, and $\widetilde{Y} \to Y$ is a torsor under a smooth group scheme. We can thus reduce some of the geometry of Y to the one of $\mathcal{N}$. The analogs of Definition 2.4 and Proposition 2.5 remain true over $\mathcal{N}$ and there exists thus a second sheaf $\mathcal{F}_2$ over $\mathcal{N}$.

2.3. *Geometry of the special fiber.* Let X be the special fiber of Y and N the one of $\mathcal{N}$. Over X (resp. N), the sheaf $\mathcal{E}$ (resp. $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_N$) is locally free of rank m over $O_X[\pi]/\pi^2$. We now denote again by $\mathcal{E}$ the sheaf $\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{N}}$. As this is a model for the sheaf $\mathcal{E}$ on Y, we hope this abuse of notation will not cause confusion. The sheaves $\mathcal{F}_1, \mathcal{F}_2$ are in $\mathcal{E}[\pi]$, and contain $\pi \cdot \mathcal{F}$. The remainder of this section could be carried on Y, but a key reduction in the proof of Proposition 2.21 uses the extra symmetries of $\mathcal{N}$. We thus express everything on $\mathcal{N}$ but the previous local model diagram induces corresponding results on Y.

Remark 2.6. The sheaf $\mathcal{E}[\pi]$ is totally isotropic, but is endowed with a perfect modified pairing given by

$$\{\pi x, \pi y\} := \langle \pi x, y \rangle.$$

This pairing is *symmetric*; indeed since $\overline{\pi} = -\pi$ in the residue field of F, one has

$$\{\pi y, \pi x\} = \langle \pi y, x \rangle = \langle y, \overline{\pi} x \rangle = \langle \pi x, y \rangle = \{\pi x, \pi y\}.$$

If we want to denote the orthogonal of a subspace $\mathcal{G} \subset \mathcal{E}[\pi]$ for this new pairing, we denote it by $\mathcal{G}^{\perp'}$ to highlight the difference with the usual pairing $\langle \cdot , \cdot \rangle$, where we use the notation $\mathcal{G}^{\perp}$.

Definition 2.7. Let k be a field in characteristic p, and let $x \in N(k)$. Let us define the integers in the pair $(h(x), l(x))$ as the dimension of $\pi \cdot (\mathcal{F} \otimes k(x))$ and $(\mathcal{F}_1 \otimes k(x)) \cap (\mathcal{F}_2 \otimes k(x))$, respectively.

Remark 2.8. From the previous section, one gets that $\mathcal{F}_2$ is the orthogonal of $\mathcal{F}_1$ in $\mathcal{E}[\pi]$, for the modified pairing.

Proposition 2.9. *Let k be a field of characteristic p, and let $x \in X(k)$. Then*

$$0 \leq h(x) \leq l(x) \leq a.$$

The integers $h(x), l(x)$ will allow us to define a stratification on N. Indeed, one has, as a topological space,

$$N = \coprod_{0 \leq h \leq l \leq a} N_{h,l},$$

where $N_{h,l}$ consists in the points x with $(h(x), l(x)) = (h, l)$.

Proposition 2.10. *Let (h,l) be integers with $0 \le h \le l \le a$, and let $\overline{N_{h,l}}$ be the closure of $N_{h,l}$. Then*

$$\overline{N_{h,l}} \subseteq \coprod_{0 \le h' \le h \le l \le l' \le a} N_{h',l'}.$$

Proof. The integer h is equal to the dimension of $\pi \cdot \mathcal{F}$. It thus decreases by specialization. The integer l is equal to the dimension of

$$(\mathcal{F}_1 \otimes k(x)) \cap (\mathcal{F}_2 \otimes k(x)).$$

This quantity increases by specialization. □

In particular, the stratum $N_{0,a}$ is closed and the strata $N_{h,h}$ are open, for every $0 \le h \le a$.

2.4. ***Closure relations and geometry of strata.*** We assume that $p \neq 2$. Let us prove a first proposition about the closure relations for the strata.

Proposition 2.11. *Let (h,l) be integers with $0 \le h \le l \le a$. One has*

$$\overline{N_{h,l}} = \coprod_{0 \le h' \le h \le l \le l' \le a} N_{h',l'}.$$

Proof. The previous proposition gives the expected inclusion, we will now show the converse. Let us first prove that $N_{0,a}$ is in the closure of $N_{h,l}$ for every $0 \le h \le l \le a$. Let k be an algebraically closed field of characteristic p, and let $x \in N_{0,a}(k)$. We will prove that for any $h \le l$, one can find a generization of x which lies in $N_{h,l}$.

Since $(h(x), l(x)) = (0, a)$, one has the inclusions $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F} = \mathcal{E}[\pi]$ at x. One can find a basis $e_1, \dots, e_m$ for $\mathcal{E}$, where $m = a + b$, such that $\mathcal{F}_1$ is generated by $\pi e_1, \dots, \pi e_a$, and the matrix of the modified pairing on $\mathcal{E}[\pi]$ with respect to $\pi e_1, \dots, \pi e_m$ is

$$M = \begin{pmatrix} 0 & 0 & I_a \\ 0 & I_{b-a} & 0 \\ I_a & 0 & 0 \end{pmatrix}. \tag{1}$$

Moreover, one can arrange so that the matrix of the original pairing, with respect to the basis $e_1, \dots e_m, \pi e_1, \dots, \pi e_m$, is

$$\begin{pmatrix} 0 & -M \\ M & 0 \end{pmatrix}.$$

Let $\widetilde{\mathcal{E}}$ be $\mathcal{E} \otimes_{k(x)} k[[t]]$. We will investigate lifts of the modules $\mathcal{F}_1 \subseteq \mathcal{F}$ to $\widetilde{\mathcal{E}}$. Here and throughout, let $(\cdot)^{\mathrm{T}}$ denote the transpose of $(\cdot)$. First, let us define a lift $\widetilde{\mathcal{F}_1}$ of $\mathcal{F}_1$ inside $\widetilde{\mathcal{E}}[\pi]$ given by a matrix $(I_a \;\, 0 \;\, Y)^{\mathrm{T}}$ where Y is an $a \times a$ matrix with coefficients in $k[\![t]\!]$, whose reduction is 0 modulo t. One computes that the orthogonal of $\widetilde{\mathcal{F}_1}$ inside $\widetilde{\mathcal{E}}[\pi]$ is given by

$$\begin{pmatrix} I_a & 0 \\ 0 & I_{b-a} \\ -{}^tY & 0 \end{pmatrix}.$$

A lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ will thus contain the vectors $(0 \;\; I_{b-a} \;\; 0)^{\mathrm{T}}$, and be contained in $\pi^{-1}\widetilde{\mathcal{F}_1}$. We define $\widetilde{\mathcal{F}}$ as generated by the columns of the matrix

$$\begin{pmatrix} 0 & 0 & Z \\ 0 & 0 & 0 \\ 0 & 0 & YZ \\ I_a & 0 & 0 \\ 0 & I_{b-a} & 0 \\ Y & 0 & I_a \end{pmatrix},$$

where Z is an $a\times a$ matrix, and this matrix is considered with respect to the basis $e_1, \dots e_m, \pi e_1, \dots, \pi e_m$. The only condition to check is the isotropy of $\widetilde{\mathcal{F}}$. One gets the equations

$$Z^t = Z, \quad (Y + Y^t)Z = 0.$$

In generic fiber, the integer h will be equal to the rank of Z, and the integer l will be equal to the nullity of $Y + Y^t$. Choosing appropriate matrices Y, Z will indeed give a generization in $N_{h,l}$.

The general case can be deduced by a similar discussion: choose $h < \ell$ and $\pi e_1, \dots, \pi e_{a+b}$ a basis of $\mathcal{E}[\pi]$ such that $\pi e_1, \dots, \pi e_h$ is a basis of $\pi\mathcal{F}$, $\pi e_1, \dots, \pi e_\ell$ a basis of $\mathcal{F}_1 \cap \mathcal{F}_2$, $\pi e_1, \dots, \pi e_a$ is a basis of $\mathcal{F}_1$, $\pi e_1, \dots, \pi e_\ell, \pi e_{a+1}, \dots, \pi e_{b+a-\ell}$ a basis of $\mathcal{F}_2$, and $\pi e_1, \dots, \pi e_{a+b-h}$ is a basis of $\mathcal{F}[\pi]$ at $x \in N_{h,\ell}(k)$. We assume that the modified pairing is given by the $m\times m$ matrix (in this basis)

$$M = \begin{pmatrix} & & & & & I_h \\ & & & & I_{\ell-h} & \\ & & I_{a-\ell} & 0 & & \\ & & 0 & I_{b-\ell} & & \\ & I_{\ell-h} & & & & \\ I_h & & & & & \end{pmatrix}.$$

We will also assume that the matrix of the original pairing, with respect to the basis $e_1, \dots e_m, \pi e_1, \dots, \pi e_m$, is

$$\begin{pmatrix} 0 & -M \\ M & 0 \end{pmatrix}$$

and that the vectors $e_1, \dots, e_h$ are in $\mathcal{F}$.

We then set in $\widetilde{\mathcal{E}}[\pi]$ a lift $\widetilde{\mathcal{F}_1}$ of $\mathcal{F}_1$ by the columns of the matrix

$$\begin{pmatrix} I_h & & \\ & I_{\ell-h} & \\ & 0 & I_{a-\ell} \\ & 0 & \\ & Y & \\ & 0 & \end{pmatrix}$$

for some matrix $Y \in M_{\ell-h,\ell-h}(tk[\![t]\!])$. The set $\widetilde{\mathcal{F}}$ must then contain the vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ I_{b-l} \\ 0 \\ 0 \end{pmatrix}.$$

We then complete $\widetilde{\mathcal{F}}$ with the vectors $\binom{A}{B}$ with respect to the basis $e_1, \dots e_m$, $\pi e_1, \dots, \pi e_m$, where

$$A = \begin{pmatrix} 0 & I_h \\ Z & 0 \\ 0 & 0 \\ 0 & 0 \\ YZ & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_{l-h} & 0 \\ 0 & 0 \end{pmatrix},$$

where Z is an $(\ell-h)\times(\ell-h)$ matrix. Similarly as before, the isotropy of $\widetilde{F}$ implies the relations

$$Z = Z^t, \quad (Y + Y^t)Z = 0.$$

In generic fiber, h becomes $h + \operatorname{rk} Z$, and l becomes $l - \operatorname{rk}(Y + Y^t)$. If h', l' are integers with $h \leq h' \leq l' \leq l$, one can choose matrices Y, Z such that Z and $Y + Y^t$ have ranks $h' - h$ and $l - l'$, respectively. This will give a lift of x inside $N_{h',l'}$. □

Proposition 2.12. *For all $h \leq \ell$, the stratum $N_{h,l}$ is nonempty and smooth, and is equidimensional of dimension $ab - \frac{(l-h)(l-h+1)}{2}$.*

Proof. To prove that all the strata are nonempty, by the proposition above, it is enough to prove that $N_{0,a}$ is nonempty, which is obvious. Let us now compute the dimension of the stratum $N_{h,l}$. On this stratum, one has the sheaves

$$\pi\mathcal{F} \subseteq \mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{F}_1,$$

which are locally free of rank h, l, a, respectively. Deforming a point of $N_{h,l}$ inside $N_{h,l}$ thus consists in the following operations:

- Deform the sheaf $\pi\mathcal{F}$ in a sheaf $\widetilde{\mathcal{F}_0}$, which should be totally isotropic for the modified pairing.

- Deform the sheaf $\mathcal{F}_1 \cap \mathcal{F}_2$ inside the orthogonal of the previous one, which should also be totally isotropic (for the modified pairing).

- Deform the sheaf $\mathcal{F}_1$ inside the orthogonal of the previous one (for the modified pairing), asking that $\mathcal{F}_1/(\mathcal{F}_1 \cap \mathcal{F}_2) \cap (\mathcal{F}_1/(\mathcal{F}_1 \cap \mathcal{F}_2))^{\perp''} = \{0\}$, where $\perp''$ is the modified pairing descended to $(\mathcal{F}_1 \cap \mathcal{F}_2)^{\perp'}/(\mathcal{F}_1 \cap \mathcal{F}_2)$.

- Deform $\mathcal{F}$, which should contain the orthogonal of $\widetilde{\mathcal{F}_0}$ (for the modified pairing) and be contained in $\pi^{-1}\widetilde{\mathcal{F}_0}$, and be totally isotropic (for the original pairing). Note that[1] if we define $\mathcal{G} = (\widetilde{\mathcal{F}_0})^{\perp'}$ we have $\mathcal{G} \subset \mathcal{E}[\pi]$ thus $\pi\mathcal{G}^{\perp} = \mathcal{G}^{\perp'} = \widetilde{\mathcal{F}_0}$. But both $\mathcal{G}^{\perp}$ and $\pi^{-1}\widetilde{\mathcal{F}_0}$ contain $\mathcal{E}[\pi]$, and are equal after multiplying by π, thus $\mathcal{G}^{\perp} = \pi^{-1}\widetilde{\mathcal{F}_0}$. It is thus enough to deform the image of $\mathcal{F}$ in $\mathcal{G}^{\perp}/\mathcal{G}$.

In other words we look at the sequence of schemes

$$\begin{aligned}\mathrm{Gr}^{\mathrm{Sp}}\big(h, (\mathcal{F}^{\perp}/\mathcal{F}, \langle\cdot,\cdot\rangle)\big) \to U \to \mathrm{Gr}^{O}\big(l-h, ((\mathcal{F}_0)^{\mathrm{univ},\perp'}/\mathcal{F}_0^{\mathrm{univ}}, \{\cdot,\cdot\})\big) \\ \to \mathrm{Gr}^{O}\big(h, (\mathcal{E}[\pi], \{\cdot,\cdot\})\big),\end{aligned}$$

where $\mathrm{Gr}^{O}\big(k, (V, \{\cdot,\cdot\})\big)$ is the Grassmannian of totally isotropic subspace of rank k in a space V with symmetric pairing, and $\mathrm{Gr}^{\mathrm{Sp}}\big(k, (V, \langle\cdot,\cdot\rangle)\big)$ is the analogous one for an alternated pairing, $(\mathcal{F}_0)^{\mathrm{univ}}$ is the universal object of $\mathrm{Gr}^{O}\big(h, \big((\mathcal{E}[\pi], \{\cdot,\cdot\})\big)\big)$, and $U \subset \mathrm{Gr}^{O}(a-l, (\mathcal{F}_1 \cap \mathcal{F}_2)^{\mathrm{univ},\perp'}/(\mathcal{F}_1 \cap \mathcal{F}_2)^{\mathrm{univ}}, \{\cdot,\cdot\})$ is the open where the universal isotropic subspace F, which corresponds to $\mathcal{F}_1/(\mathcal{F}_1 \cap \mathcal{F}_2)$, satisfies $F^{\perp'} \cap F = \{0\}$, with $(\mathcal{F}_1 \cap \mathcal{F}_2)^{\mathrm{univ}}$ the (pullback of the) universal object of $\mathrm{Gr}^{O}\big(l-h, ((\mathcal{F}_0)^{\mathrm{univ}}, \{\cdot,\cdot\})\big)$.

All those Grassmannians are relatively smooth, and the first point gives a dimension $h(a+b-h) - \frac{h(h+1)}{2}$, the second one a relative dimension

$$(l-h)(a+b-l-h) - \frac{(l-h)(l-h+1)}{2},$$

the third one $(a-l)(b-l)$ and the last one $\frac{h(h+1)}{2}$. The total dimension is then

$$ab - \frac{(l-h)(l-h+1)}{2}.$$

Moreover if $x \in N_{h,l}(S)$ is an R-point for some ring R, and $S \to R$ is a square-zero thickening of $\mathbb{F}_p$-schemes, then to lift x it is enough to lift $\mathcal{F}_1 \subset \mathcal{F}$ satisfying all the

[1] $\pi\mathcal{G}^{\perp} = \mathcal{G}^{\perp'}$ if $\mathcal{G} \subset \mathcal{E}[\pi]$ and $\mathcal{H}^{\perp} = (\pi\mathcal{H})^{\perp'}$ if $\mathcal{E}[\pi] \subset \mathcal{H}$.

desired properties. But this is indeed possible as all the previous Grassmannians are formally smooth and the dimension is indeed the given one. □

We consider the group

$$G = \{g \in \mathrm{GL}_{a+b,k[X]/(X^2)}(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \mid \langle gx, y\rangle = \langle x, gy\rangle\}.$$

Then G acts naturally on N, and preserves each $N_{h,\ell}$.

Proposition 2.13. *G acts transitively on each $N_{h,\ell}$.*

Proof. Denote by $E_1, XE_1, \dots, E_d, XE_d$ the canonical (k-)basis of $\Lambda \otimes \mathbb{F}_p$, and by

$$J = \begin{pmatrix} & & & -1 \\ & & \iddots & \\ & -1 & & \\ 1 & & & \end{pmatrix}$$

the matrix of the pairing on $\Lambda \otimes \mathbb{F}_p$ in this basis (that is, $\langle X^{\varepsilon'} E_i, X^{\varepsilon} E_j\rangle = \pm\delta_{\varepsilon\neq\varepsilon'}\delta_{j=n-i+1}$, antisymmetrically with $\langle \pi x, y\rangle = -\langle x, \pi y\rangle = \langle x, \overline{\pi} y\rangle$). We start with a lemma.

Lemma 2.14. *Let $V = \Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ with the previous pairing.*

Assume we are given $e_1, \dots, e_h, Xe_{h+1}, \dots, Xe_{a+b-h}$ such that

$$\langle X^{\varepsilon'} e_i, X^{\varepsilon} e_j\rangle = 0, \quad \{Xe_i, Xe_j\} = \pm\delta_{j,d-i+1},$$

for all $\varepsilon, \varepsilon' \in \{0, 1\}$, i, j whenever it is defined. Then there exists a basis $e_1, \dots, e_m$ of V over $k[X]/X^2$ such that for all $i, j, \varepsilon, \varepsilon'$

$$\langle X^{\varepsilon} e_i, X^{\varepsilon'} e_j\rangle = \langle X^{\varepsilon} E_i, X^{\varepsilon'} E_j\rangle.$$

Proof of lemma. We construct $e_1, \dots, e_m$ inductively, $e_1, \dots, e_h$ being already constructed.

If $h = 0$, we take any e_1 lifting Xe_1 so that e_1 is orthogonal to $Xe_1, \dots, Xe_{a+b-1}$. This is possible (and automatic) as

$$\{Xe_1, Xe_j\} = \delta_{j,m} = \langle e_1, Xe_j\rangle,$$

for any lift.

Assume $e_1, \dots, e_k$, $1 \leq k \leq a + b - h - 1$, are already constructed. Then for any lift e_{k+1} of Xe_{k+1} we have

$$\langle e_{k+1}, Xe_i\rangle = \{Xe_{k+1}, Xe_i\} = \delta_{i,n-k} = \langle E_{k+1}, XE_i\rangle.$$

We need to assure that $\langle e_{k+1}, e_i\rangle = 0$ for $1 \le i \le k+1$. For $i = k+1$ this is automatic as $\langle \cdot, \cdot \rangle$ is symplectic. Choose a lift e^0_{k+1}. Now if

$$\langle e^0_{k+1}, e_i\rangle = a_i,$$

let v_i such that $\langle v_i, Xv\rangle = 0$ and $\langle v_i, e_j\rangle = \delta_{i,j} a_i$ for $v \in V$ and $j \le i$. This is possible as $\dim V[X] + k(e_1, \dots, e_i) = i + a + b \le 2(a + b)$ and the pairing is nondegenerate. Then set

$$e_{k+1} = e^0_{k+1} - \sum v_i.$$

This is indeed a lift of Xe_{k+1} as $v_i \in V[X] = V[X]^\perp$. This assures the existence of $e_1, \dots, e_{a+b-h}$. For the remaining vectors, let e_{d-i+1} such that $e_{d-i+1} \in (V[X] + k(e_1, \dots, \hat{e}_i, \dots, e_{d-i}))^\perp$ (by descending induction on i starting at $i = h$) and $\langle e_{d-i+1}, Xe_i\rangle = -1 = \langle E_{d-i+1}, XE_i\rangle$. This is possible by dimension reasons. Then automatically

$$\langle Xe_{d-i+1}, e_j\rangle = \delta_{j=i} = \langle XE_{d-i+1}, E_j\rangle. \qquad \square$$

For each (h, ℓ) define

$$\mathcal{F}_1^{0,h,\ell} := k(XE_1, \dots, XE_\ell, XE_{b+1}, \dots, XE_{b+a-\ell})$$

and

$$\mathcal{F}^{0,h,\ell} := k(E_1, \dots, E_h, XE_1, \dots, XE_{a+b-h}).$$

Then $(\mathcal{F}_1^{0,h,\ell}, \mathcal{F}^{0,h,\ell}) \in N_{h,\ell}(k)$. Assume $(\mathcal{F}_1, \mathcal{F}) \in N_{h,\ell}(k)$, k algebraically closed. Then there exists a $k[X]/(X^2)$-linearly independent family $e_1, \dots, e_h$ in $\mathcal{F}$ such that

$$\mathcal{F} = k(e_1, \dots, e_h) + k(Xe_1, \dots, Xe_{a+b-h}),$$

for some vectors $Xe_{a+1}, \dots, Xe_{a+b-h} \in \Lambda \otimes k[X]$. Up to changing the basis, we can assume that $\mathcal{F}_1$ is generated by $Xe_1, \dots, Xe_\ell, Xe_{b+1}, \dots, Xe_{b+a-\ell}$ and $\mathcal{F}_1^\perp = k(Xe_1, \dots, Xe_\ell, Xe_{\ell+1}, \dots, Xe_b)$. We now claim that all those vectors can be chosen to satisfy

$$(X^{\varepsilon'} e_i, X^\varepsilon e_j) = (X^{\varepsilon'} E_i, X^\varepsilon E_j).$$

Indeed, as $\pi\mathcal{F} = (Xe_1, \dots, Xe_a)$ is contained in $\mathcal{F}_1 \cap \mathcal{F}_1^{\perp'}$, which is of dimension ℓ, and $\mathcal{F}$ is totally isotropic, is suffices to choose $Xe_{a+1}, \dots, Xe_\ell \in \mathcal{F}_1 \cap \mathcal{F}_1^{\perp'}$, and then completing a basis of $\mathcal{F}_1^{\perp'}$ and $\mathcal{F}_1$ using $\dim \mathcal{F}_1 = a$, $\dim \mathcal{F}_1^{\perp'} = b$, we can assume that $\{Xe_{\ell+i}, Xe_{b+j}\} = \delta_{j,a-\ell-i+1}$ up to modifying the basis. Then any choice of the other vectors of $\mathcal{F}$ (up to reordering) works. In particular we are in the setting of the previous lemma and there thus exists a basis $e_1, \dots, e_m$ of V lifting the one constructed using $(\mathcal{F}_1, \mathcal{F})$. If g denotes the matrix sending E_i to e_i and XE_i to Xe_i then $g \in \mathrm{GL}_{a+b}(k[X]/(X^2)$. Moreover as

$$\langle X^a E_i, X^b E_j \rangle = \langle X^a e_i, X^b e_j \rangle,$$

we deduce that $g \in G$. Finally for $(\mathcal{F}_1^{0,h,\ell}, \mathcal{F}^{0,h,\ell}) \in N_{h,\ell}(k)$, $g(\mathcal{F}_1^0, \mathcal{F}^0) = (\mathcal{F}_1, \mathcal{F})$, proving the transitivity. □

Corollary 2.15. *The irreducible components of the scheme N are the closures of $N_{h,h}$ for $0 \leq h \leq a$. Each $N_{h,\ell}$ is connected.*

Proof. Proposition 2.11 assures that each irreducible component sits inside the closure of some $N_{h,h}$. Now the group

$$G = \{g \in \mathrm{GL}_{a+b,k[X]/(X^2)}(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \mid \langle gx, y \rangle = \langle x, gy \rangle\}$$

acts transitively on each $N_{h,\ell}$. If G were connected, thus so would be $N_{h,\ell}$ and we would be done. But G is not connected: as a scheme we have

$$G \simeq A_{a+b} \times O_{a+b},$$

with A_{a+b} the antisymmetric matrices and O_{a+b} the symmetric invertible ones, which is not connected and has two connected components. Indeed, a direct computation shows that in the basis $(E_1, \dots, E_m, XE_1, \dots, XE_m)$ we have $g \in G$ if and only if

$$\mathrm{Mat}(g) = \begin{pmatrix} M & 0 \\ M' & M \end{pmatrix}, \quad M \in O_{a+b}, \ M' \in A_{a+b}.$$

Now let g_1 correspond to $M' = 0$ and

$$M = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then $g_1 \in G$ but g_1 is not in the connected component of $I_{a+b} \in G$. In particular multiplication by g_1 identifies the two connected components of G. Now clearly g_1 acts trivially on $(\mathcal{F}_1^{0,h,\ell}, \mathcal{F}^{0,h,\ell}) \in N_{h,\ell}$ thus the action of G on N factors through a connected quotient, proving that $N_{h,\ell}$ is connected for all h, ℓ. □

Definition 2.16. We denote by $X_{h,\ell}$ the locally closed subscheme of X corresponding to $N_{h,\ell}$ through the local model diagram.

Corollary 2.17. *For all $h \leq \ell$, the stratum $X_{h,l}$ is nonempty and smooth, and is equidimensional of dimension $ab - \frac{(l-h)(l-h+1)}{2}$.*

Proof. To prove that all the strata are nonempty, by Proposition 2.11, it is enough to prove that $X_{0,a}$ is nonempty. Let $k = \overline{\mathbb{F}_p}$, and let E be an elliptic curve over k with complex multiplication by O_F such that the action of O_F on ω_E is given by σ_1, and let E^c be the same elliptic curve, but with the action of O_F twisted by the complex conjugation. Define $A = E^a \times (E^c)^b$, and let us choose a space $\omega_1 \subseteq \omega_A$ which is totally isotropic for the modified pairing. This gives a point in $X_{0,a}$. Grothendieck–Messing assures that the local rings of Y and $\mathcal{N}$ coincide, giving the dimension. □

Corollary 2.18. *The irreducible components of X are the closure of the irreducible components of the $X_{h,h}$, $0 \leq h \leq a$.*

2.5. ***Local rings.*** Assume $p \neq 2$.

Proposition 2.19. *The smooth locus of N is the union of the strata $N_{h,h}$ for $0 \leq h \leq a$.*

Proof. We have seen that the strata $N_{h,h}$ are open and smooth in the previous proposition, thus their union is included in the smooth locus. Assume now that $h < l$, and let $x \in N_{h,l}(k)$. We will prove that N is not smooth at x. The pairing on $\mathcal{E}[\pi]$ is given by a matrix

$$\begin{pmatrix} 0 & 0 & I_l \\ 0 & I_{a+b-2l} & 0 \\ I_l & 0 & 0 \end{pmatrix}$$

written with respect to a basis $\pi e_1, \dots, \pi e_{a+b}$, where $\mathcal{F}_1$ is spanned by $\pi e_1, \dots, \pi e_a$, $\mathcal{F}_1 \cap \mathcal{F}_2$ is spanned by $\pi e_1, \dots, \pi e_l$ and $\pi \mathcal{F}$ is spanned by $\pi e_{l-h+1}, \dots, \pi e_l$. One can also assume that $\mathcal{F}/\mathcal{F}_1$ is spanned by $\pi e_{a+1}, \dots, \pi e_{a+b-h}, e_{l-h+1}, \dots, e_l$. Let $\mathcal{E}'$ be $\mathcal{E} \otimes_k k[\varepsilon]/\varepsilon^2$. Let us define a lift $\mathcal{F}'$ of $\mathcal{F}$ to $\mathcal{E}'$. One can lift the basis on $\mathcal{E}$ to a basis $e'_1, \dots, e'_{a+b}$ in such a way that the matrix of the pairing is not changed. We define $\mathcal{F}'_1$ to be spanned by $\pi e'_1 + \varepsilon \pi e'_{a+b-l+1}, \pi e'_2, \dots, \pi e'_a$. We thus define $\mathcal{F}'/\mathcal{F}'_1$ to be spanned by $\pi e'_{a+1}, \dots, \pi e'_{a+b-l+1} + \varepsilon e'_1, \dots, \pi e'_{a+b-h}, e'_{l-h+1}, \dots, e'_l$. This gives a point $x' \in N(k[\varepsilon]/\varepsilon^2)$. We will now prove that this point cannot be lifted to $k[\varepsilon]/\varepsilon^3$. If it were the case, one would have a lift $\widetilde{\mathcal{E}}$ of $\mathcal{E}'$ to $k[\varepsilon]/\varepsilon^3$ together with a lift $\widetilde{\mathcal{F}}_1 \subset \widetilde{\mathcal{F}}$ of $\mathcal{F}'_1 \subset \mathcal{F}'$. In particular, there would exist an element in $\widetilde{\mathcal{F}_1}$ of the form $v_1 := \pi \widetilde{e_1} + \varepsilon \pi \widetilde{e_{a+b-l+1}} + \varepsilon^2 \pi u$. There would also be in $\widetilde{\mathcal{F}}$ an element of the form $v_2 := \pi \widetilde{e_{a+b-l+1}} + \varepsilon(\widetilde{e_1} + \varepsilon \widetilde{e_{a+b-l+1}}) + \varepsilon^2 v$ such that $\varepsilon^2 \pi v$ belongs to $\widetilde{\mathcal{F}_1}$. The (original) pairing between these two vectors is equal to

$$\langle v_1, v_2 \rangle = \{v_1, \pi v_2\} = 2\varepsilon^2 + \{\varepsilon^2 \pi \widetilde{e_1}, \pi v\}.$$

But $\{\varepsilon^2\pi\widetilde{e_1}, \pi v\} = 0$ modulo ε^3, since $\varepsilon^2\pi v$ belongs to $\widetilde{\mathcal{F}_1}$. This gives the desired contradiction, since $\widetilde{\mathcal{F}} \ni v_1, v_2$ should be totally isotropic. □

Remark 2.20. Even though the irreducible components of X are determined by the $X_{h,h}$ (namely they are locally the closure of $X_{h,h}$ on the local model), and $X_{h,h}$ is smooth, it is not true in general that the irreducible components are smooth. This is true when $a = 1$ [10], but already for $a = 2$, Zachos [15] computed that the irreducible component corresponding to $X_{1,1}$ is not smooth. Our computations prove that the irreducible component corresponding to $X_{h,h}$ is not smooth when $0 < h < a$. The variety X is thus not semistable when $a \geq 2$.

Proposition 2.21. *The scheme $\mathcal{N}$ (resp. Y) is flat over $\mathcal{O}_{F_p}$, normal and its special fiber is reduced and Cohen–Macaulay.*

Proof. The case of Y is deduced from $\mathcal{N}$ and the local model diagram; we thus focus on $\mathcal{N}$. What we have done before shows that N is reduced and the irreducible components of N are those of the various $N_{h,h}$. In particular,

$$\dim N = ab \quad \text{and} \quad \dim \mathcal{N} = ab + 1 = \dim N + \dim \mathcal{O}_F.$$

Thus to show that $\mathcal{N}$ is flat over $\mathcal{O}_F$ it is enough to show its generic fiber is dense. But N is endowed with an action of

$$G = \{g \in \mathrm{GL}_{a+b,k[X]/(X^2)}(\Lambda \otimes_{\mathbb{Z}_p} \mathbb{F}_p) \mid \langle gx, y\rangle = \langle x, gy\rangle\},$$

and G acts transitively on $N_{h,\ell}$ for each h, ℓ by Proposition 2.13. But G preserves the nonflat (resp. non-Cohen–Macaulay) locus, so as the flat (resp. Cohen–Macaulay) locus is open, it is enough to show that points of $N_{0,a}$ lift to characteristics zero (and are Cohen–Macaulay in N).

Claim 2.22. Let $x \in N_{0,a}(k)$. The local ring at x of $\mathcal{N}$ is given by (the localization at $Z = T = Y = \pi = 0$ of)

$$W_{\mathcal{O}}(k)[Z, X, Y]/(Z - {}^tZ, (Y + {}^tY + {}^tXX)Z - (\pi_1 - \pi_2)I_a),$$

where $W_{\mathcal{O}}(k) = W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_{F_p}$, Z is an (symmetric) $a\times a$-matrix, Y is an $a\times a$-matrix and X is a $(b-a)\times a$-matrix.

Assume the claim is true for the moment. The previous ring is flat over $\mathcal{O}_{F_p}$, with Cohen–Macaulay (and reduced) fibers by a result of [4, Theorem 2] and [5, Remarque 5.9]. Indeed, it is proven in [4] that, for k a field, the ring

$$k[Z, T]/(Z - {}^tZ, T - {}^tT, ZT),$$

where Z, T are two (symmetric) matrices of size a, is Cohen–Macaulay (and reduced) and the generic fiber is smooth. But if $\mathrm{Char}(k) \neq 2$, then the map

$$\frac{W_{\mathcal{O}}(k)[Z, X, T, A]}{((T + {}^tXX)Z - (\pi_1 - \pi_2)I_a)} \to \frac{W_{\mathcal{O}}(k)[Z, X, Y]}{((Y + {}^tY + {}^tXX)Z - (\pi_1 - \pi_2)I_a)},$$
$$T \mapsto Y + {}^tY,$$
$$A \mapsto Y - {}^tY,$$

where Z, T are symmetric size-a matrices and A is an antisymmetric size-a matrix, is an isomorphism. So

$$\begin{aligned}&\mathrm{Spec}\big(W_{\mathcal{O}}(k)[Z, X, Y](Z - {}^tZ, (Y + {}^tY + {}^tXX)Z - (\pi_1 - \pi_2)I_a)\big)\\ &\simeq \mathrm{Spec}\big(W_{\mathcal{O}}(k)[Z, X, T]/(Z - {}^tZ, T - {}^tT, (T + {}^tXX)Z - (\pi_1 - \pi_2)I_a)\big) \times \mathbb{A}^{\frac{a(a-1)}{2}}_{W_{\mathcal{O}}(k)}.\end{aligned}$$

Now, we have the isomorphism

$$\begin{aligned}&W_{\mathcal{O}}(k)[Z, X, T]/(Z - {}^tZ, T - {}^tT, (T + {}^tXX)Z - (\pi_1 - \pi_2)I_a)\\ &\qquad \to W_{\mathcal{O}}(k)[Z, X, T]/(Z - {}^tZ, T - {}^tT, TZ - (\pi_1 - \pi_2)I_a),\end{aligned}$$

given by

$$T \mapsto T - {}^tXX, X \mapsto X,$$

where the last ring is the one of [5] except that $(\pi_1 - \pi_2)$ appears instead of p and the special fiber is the same, so their argument that irreducible components in special fiber lifts carries over (changing p by $\pi_1 - \pi_2$). Thus the ring of the claim has reduced and Cohen–Macaulay fibers with every irreducible component being in characteristics zero, giving that the ring is flat. Thus $\mathcal{N}$ is flat over $\mathcal{O}_{F_p}$ with Cohen–Macaulay fibers. Moreover $\mathcal{N}$ is smooth in generic fiber, and N is generically regular as $X_{h,h}$ is smooth for all h, thus $\mathcal{N}$ is R1 and S2, thus normal by Serre's criterion. The same is true for Y using the local model diagram.

It only remains to prove the claim which is slightly more than what we did in the proof of Proposition 2.11 as we don't assume lifts are in special fiber anymore.

Proof of claim. As before, Λ is endowed with a $\mathcal{O}_{F_p} = \mathbb{Z}_p[X]/(X - \pi_1)(X - \pi_2)$-action, X acting by π, and a pairing, and $\Lambda[\pi - \pi_i] = (\pi - \pi_{3-i})\Lambda$ is totally isotropic as, e.g.,

$$\langle(\pi - \pi_2)x, (\pi - \pi_2)y\rangle = \langle\underbrace{(\pi - \pi_1)(\pi - \pi_2)}_{=0}x, y\rangle = 0.$$

One can assume that the matrix of the modified pairing on $(\Lambda \otimes k)[\pi]$ is

$$Q = \begin{pmatrix} 0 & 0 & I_a \\ 0 & I_{b-a} & 0 \\ I_a & 0 & 0 \end{pmatrix},$$

where $\mathcal{F}_1$ is spanned by $\pi e_1, \dots, \pi e_a \in \Lambda \otimes k$, and $\mathcal{F} = (\Lambda \otimes k)[\pi]$. Let R be a local $W_{\mathcal{O}}(k)$-algebra, with a surjective morphism $R \to k$, whose kernel is denoted by I. We investigate the possible lifts of $\mathcal{F}$ to $\Lambda \otimes_{\mathbb{Z}_p} R$. First one can assume that the matrix for the induced pairing between $\Lambda[\pi - \pi_1]$ and $\Lambda[\pi - \pi_2]$ is still given by the matrix Q, where $\Lambda[\pi - \pi_i]$ has the basis $(\pi - \pi_{3-i})e_1, \dots, (\pi - \pi_{3-i})e_{a+b}$. One can also assume that the matrix of the original pairing is

$$\begin{pmatrix} 0 & -Q \\ Q & 0 \end{pmatrix}$$

in the basis $(\pi - \pi_2)e_1, \dots, (\pi - \pi_2)e_{a+b}, e_1, \dots, e_{a+b}$.

One should lift $\mathcal{F}_1$ inside $\Lambda \otimes_{\mathbb{Z}_p} R[\pi - \pi_1]$. The latter is an R-module of rank $a+b$ with basis $(\pi - \pi_2)e_1, \dots, (\pi - \pi_2)e_{a+b}$. A lift of $\mathcal{F}_1$ is then spanned by the image of the matrix

$$\begin{pmatrix} I_a \\ X \\ Y \end{pmatrix},$$

with X, Y with coefficients in I. Let $\widetilde{f_1}, \dots, \widetilde{f_a}$ be the vectors defined by the column of this matrix, and $\widetilde{e_1}, \dots, \widetilde{e_a}$ be the vectors defined by this matrix in the basis $e_1, \dots, e_{a+b}$ (so that $\widetilde{f_a} = (\pi - \pi_2)\widetilde{e_a}$). One can then compute the orthogonal of $\mathcal{F}_1$ inside $\Lambda \otimes_{\mathbb{Z}_p} R[\pi - \pi_2]$ (for the modified pairing). It is defined by the matrix

$$\begin{pmatrix} I_a & 0 \\ 0 & I_{b-a} \\ -{}^tY & -{}^tX \end{pmatrix}$$

in the basis $(\pi - \pi_1)e_1, \dots, (\pi - \pi_1)e_{a+b}$. In particular, a possible lift of $\mathcal{F}$ will automatically contain the image of the matrix $(0 \;\; I_{b-a} \;\; -{}^tX)^{\mathrm{T}}$ in the basis $(\pi - \pi_1)e_1, \dots, (\pi - \pi_1)e_{a+b}$. Let $\widetilde{\mathcal{F}_1}$ be the lift of $\mathcal{F}_1$, and $\widetilde{\mathcal{F}}'$ be the free R-module obtained as the sum of $\widetilde{\mathcal{F}_1}$ and the previous module.

Now, we are left to lift $\mathcal{F}$ inside $(\pi - \pi_2)^{-1}\widetilde{\mathcal{F}_1}/\widetilde{\mathcal{F}}'$. A basis for this module consists in $(\pi - \pi_1)e_{b+1}, \dots, (\pi - \pi_1)e_{a+b}, \widetilde{e_1}, \dots, \widetilde{e_a}$, and a lift of $\mathcal{F}$ will be given by the image of a matrix of the form $\binom{I_a}{Z}$. All that is left to do is to check the isotropy condition for the lift of $\mathcal{F}$. The last vectors are automatically orthogonal to the vectors obtained by the matrix $(0 \;\; I_{b-a} \;\; -{}^tX)^{\mathrm{T}}$ in $\Lambda \otimes_{\mathbb{Z}_p} R[\pi - \pi_2]$. The orthogonality with $\widetilde{\mathcal{F}_1}$

gives the relation $(\pi_2 - \pi_1)I_a + (Y + {}^tY + {}^tXX)Z = 0$. Finally, the fact that these vectors should be pairwise orthogonal gives the relation ${}^tZ = Z$, hence the result. □

Remark 2.23. Even if there doesn't seem to be a reference in the literature for local model diagrams when $p = 2$, many of the previous constructions still apply when $p = 2$ using solely Grothendieck–Messing theory. Unfortunately in that case there are two distinct possibilities for the matrix of the modified pairing. When $p = 2$ the previous nonemptyness, smoothness and closure relations for the strata $X_{h,\ell}$ are not true anymore, and the results depends the shape of the modified pairing. In particular, there can be unexpected smoothness when $p = 2$ for those models!

3. Modular forms

3.1. *Sheaves.* We define $\mathcal{E}_i = \mathcal{E}[T - \pi_i]$, for $i = 1, 2$.

Proposition 3.1. *The sheaf* $\det \mathcal{E}$ *is trivial, and one has* $\det(\mathcal{E}_1) \simeq \det(\mathcal{E}_2)^{-1}$.

Proof. The follows from the fact that $\mathcal{E}$ has an alternate pairing, that $\mathcal{E}_1$ is totally isotropic, and that the multiplication by $T - \pi_1$ induces an isomorphism $\mathcal{E}/\mathcal{E}_1 \simeq \mathcal{E}_2$. Indeed, $\det \mathcal{E} = \det \mathcal{E}_1 \otimes \det(\mathcal{E}/\mathcal{E}_1)$, and the map

$$\mathcal{E}_1 \to \mathcal{E} \xrightarrow[\langle\cdot,\cdot\rangle]{\sim} \mathcal{E}^\vee \to \mathcal{E}_1^\vee$$

is zero as $\mathcal{E}_1$ is totally isotropic. We thus deduce an isomorphism $\mathcal{E}_1 \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_1)^\vee$, and finally

$$\det \mathcal{E} = \det \mathcal{E}_1 \otimes \det(\mathcal{E}_1)^{-1} = \mathcal{O}_S. \qquad \square$$

Proposition 3.2. *One has isomorphisms*

$$\det(\mathcal{E}_1) \simeq \det(\omega_1) \otimes \det(\omega_2)^{-1} \simeq \det(\omega/\omega_2) \otimes \det(\omega/\omega_1)^{-1}.$$

Proof. Inside $\mathcal{E}$ the orthogonal of ω_1 is $(T-\pi_1)^{-1}\omega_2$. Thus $\det((T-\pi_1)^{-1}\omega_2/\mathcal{E}_1) \simeq \det(\mathcal{E}_1/\omega_1)^{-1} \simeq \det(\mathcal{E}_1)^{-1} \otimes \det(\omega_1)$. Moreover, the multiplication by $T - \pi_1$ induces an isomorphism between $(T - \pi_1)^{-1}\omega_2/\mathcal{E}_1$ and ω_2.

For the second part, one uses

$$\det \omega = \det \omega_1 \otimes \det \omega/\omega_1 = \det \omega_2 \otimes \det \omega/\omega_2. \qquad \square$$

Proposition 3.3. *On the generalized Rapoport locus, one has an isomorphism* $\det(\omega/\omega_2) \simeq \det(\omega_1)$.

In the special fiber, one has an isomorphism $\mathcal{E}_1 \simeq \mathcal{E}_2$. *The sheaf* $\varepsilon := \det \mathcal{E}_1$ *satisfies* $\varepsilon^2 \simeq \mathcal{O}_S$.

3.2. ***Definition and vanishing modulo p.*** As we work in characteristic p, we will need to use an integral version of Schur functors. See also [7, Section 3.8]. For λ a character of $\mathbb{G}_m^r$, and M a rank-r free module over R, choose an isomorphism $M \simeq R^r$, and denote by $\mathcal{L}(\lambda)$ the sheaf on GL_r/B (B the upper triangular Borel of GL_r) whose sections are given by

$$\mathcal{L}(\lambda)(U) = \{f : \pi^{-1}(U) \to \mathbb{A}^1 \mid f(gb) = \lambda^{-1}(b)(g)\ \forall b \in B, g \in \pi^{-1}(U)\}.$$

Denote by $\mathcal{L}_M(\lambda)$ the sheaf on the flag variety $\mathcal{F}\ell(M)$ for M, given by $\phi_*\mathcal{L}(\lambda)$ after choosing an isomorphism $\phi : R^r \simeq M$ (inducing $\mathrm{GL}_r \simeq \mathrm{Isom}_R(R^r, M)$ and $\phi : \mathrm{GL}_r/B \simeq \mathcal{F}\ell(M)$). This is independent of the choice of ϕ. For a point $\underline{a} = (a_1 \geq \cdots \geq a_r) \in \mathbb{Z}^r$, with associated character of $T = \mathbb{G}_m^r \subset B$, denote by $M^{(a_1,\dots,a_r)}$ the global sections of $\mathcal{L}_M(\underline{a})$, i.e.,

$$M^{(a_1,\dots,a_r)} = H^0(\mathcal{F}\ell(M), \mathcal{L}_M(\underline{a})).$$

As $H^1(\mathcal{F}\ell(M), \mathcal{L}_M(\underline{a})) = 0$ (Kempf theorem; see [8, Proposition 4.5]), the formation of $M^{(a_1,\dots,a_r)}$ commutes with base change $R \to R'$, and thus the construction glues to a functor from the category of rank-r vector bundles on a scheme X to the category of vector bundles on X (of any rank) associating to $\mathcal{V}$ or rank r the vector bundle $\mathcal{V}^{(a_1,\dots,a_r)}$. We define $\underline{a}^\vee = (-a_r, \dots, -a_1)$.

Definition 3.4. Let k, l, r be three integers. A (scalar-valued) modular form of weight (k, l, r) is a section of the sheaf

$$(\det \omega_1)^k \otimes (\det \omega/\omega_1)^l \otimes (\det \mathcal{E}_1)^r.$$

More generally, given $\underline{k} = (k_1, \dots, k_a) \in \mathbb{Z}^a$, $\underline{\ell} = (\ell_1, \dots, \ell_b) \in \mathbb{Z}^b$ with $k_1 \geq \cdots \geq k_a$, $\ell_1 \geq \cdots \geq \ell_b$, we can consider the sheaf

$$\omega^{(\underline{k},\underline{\ell},r)} := \omega_1^{\underline{k}} \otimes (\omega/\omega_1)^{\underline{\ell}} \otimes (\det \mathcal{E}_1)^r.$$

A weight-$(\underline{k}, \underline{\ell}, r)$ modular form is a section of this sheaf.

Remark 3.5. In generic fiber we can remove the use of r, and we can replace ω/ω_1 by ω_2. In special fiber though, $\omega_1 = \omega_2$ up to a square zero sheaf. In special fiber, we can assume that $r = 0, 1$ by Proposition 3.3.

In special fiber we have the following vanishing result.

Proposition 3.6. *If* $-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1$ *is not decreasing* (*i.e., if* $k_1 > k_a$ *or* $k_a > \ell_b$), *then*

$$H^0(\overline{X_{0,0}}, \omega^{(\underline{k},\underline{\ell},r)}) = 0.$$

Proof. Let $x \in X_{0,0}$. Then above x we have $\omega_1 \subset \mathcal{E}[\pi] = \mathcal{E}_1 = \omega$. We look at $\mathrm{Gr}_{a,a+b}(\mathcal{E}[\pi])$ the Grassmannian of rank-a subbundles of $\mathcal{E}[\pi]$. Over it, we have a universal bundle $V_1 \subset \mathcal{E}[\pi]$, which induces an immersion $\mathrm{Gr}_{a,a+b}(\mathcal{E}_x[\pi]) \to \overline{X_{0,0}}$ mapping ω_1 to x. The pullback of $\omega = \mathcal{E}[\pi]$ to $\mathrm{Gr}_{a,a+b}$ is constant, and the pullback of the universal ω_1 on $\overline{X}$ is

$$V_1 =: \mathcal{O}(\underbrace{-1, 0, \dots, 0}_{a \text{ times}}, 0, \dots, 0)$$

(which corresponds to $\mathcal{O}(-1)$ on $\mathbb{P}^1$ when $a = b = 1$, up to twist by center). Thus, the pullback of ω/ω_1 is

$$\mathcal{E}[\pi]/V_1 =: \mathcal{O}(0, \dots, 0, \underbrace{0, \dots, 0, -1}_{b \text{ times}}),$$

(which corresponds to $\mathcal{O}(1)$ on $\mathbb{P}^1$ when $a = b = 1$, up to twist by the center). The restriction of a section of $\omega^{\underline{k},\underline{\ell},r}$ to $\mathrm{Gr}_{a,a+b}$ is then $\mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee)$. We remark that $\mathrm{Gr}_{a,a+b} \simeq P\backslash G$ for $G = \mathrm{GL}_{a+b}$ and P the standard parabolic of size a, b. We thus have a map $B\backslash G \xrightarrow{\pi} P\backslash G$ (for the upper triangular Borel B), and

$$\mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee) = \pi_* \mathcal{L}(-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1),$$

with $\mathcal{L}(-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1)$ the line bundle on $B\backslash G$. But

$$H^0(P\backslash G, \mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee)) = H^0(B\backslash G, \mathcal{L}(-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1)) = 0$$

under the assumption (see Proposition 3.7 and Lemma 3.8). This is true for all points of $X_{0,0}$, and thus we have the vanishing result. □

The following is well known:

Proposition 3.7. *Let G be a split reductive group in characteristic p. Let $B \subset P \subset G$ be a Borel and a parabolic subgroup, and T a torus of B. Define $\pi : G \to G/B$ and $f : G/B \to G/P$. Let $\lambda \in X(T)$ be a weight. Let $\mathcal{L}(\lambda)$ be the line bundle on G/B such that*

$$\mathcal{L}(\lambda)(U) = \{f : \pi^{-1}(U) \to \mathbb{A}^1 \mid f(gb) = \lambda^{-1}(b)(g)\ \forall b \in B, g \in \pi^{-1}(U)\},$$

and $\mathcal{L}_P(\lambda) = f_* \mathcal{L}(\lambda)$. Then λ is dominant if and only if

$$H^0(G/P, \mathcal{L}_P(\lambda)) \neq 0.$$

Proof. See [8, Section II.2] for the definitions. We have

$$H^0(G/P, \mathcal{L}_P(\lambda)) = H^0(G/B, \mathcal{L}(\lambda)).$$

But by [8, Proposition 2.6], λ is dominant if and only if $H^0(G/B, \mathcal{L}(\lambda)) \neq 0$. □

Lemma 3.8. *Let $G = \mathrm{GL}_{a+b}$, and P a standard parabolic with Levi $\mathrm{GL}_a \times \mathrm{GL}_b$. Let $\mathcal{W}$ be the universal direct factor and $\mathcal{V}$ the universal quotient on $X = G/P$. Then $\mathcal{W}^{\underline{k}} \otimes \mathcal{V}^{\underline{\ell}}$ coincides with $\mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee)$.*

Proof. In particular we need to prove that $\mathcal{V} = \mathcal{V}^{(1,0,\dots,0)} = \mathcal{L}_P(0, \dots, 0, -1)$ and $\mathcal{W} = \mathcal{L}_P(-1, 0, \dots, 0)$. But conversely, as $\mathcal{L}_P$ is compatible with tensor product (on sheaves) and sum (on characters; see [8, Chapter 4]), and as Schur functors commute with base change, it is enough to check this at the fiber over $1 \in G/P$ as a P-representation. But clearly $\mathcal{L}_P(0, \dots, 0, -1)^{\underline{\ell}} = \mathcal{L}_P(0, \dots, 0, -\ell_b, \dots, -\ell_1)$ as P-representation and similarly for $\mathcal{L}_P(-1, 0, \dots, 0)$. To prove that $\mathcal{V}$ coincides with $\mathcal{L}_P(0, \dots, 0, 1)$ and similarly for $\mathcal{W}$, recall that both are G-equivariant vector bundles, so we can check the isomorphism at the fiber above $1 \in G/P$. But it is clear that $\langle e_1, \dots, e_a\rangle = \mathcal{W}_1 = V_P(-1, 0, \dots, 0)$ as P-representation and similarly for $\mathcal{V}$. □

Now let $x \in X_{h,h}$ for some $h \le a$. We have

$$0 \subset \pi\omega_x \subset \omega_1 \subset \omega_x[\pi] \subset \mathcal{E}[\pi].$$

In particular ω_1 gives a point of $\mathrm{Gr}_{a-h,a+b-2h}(\omega_x[\pi]/\pi\omega_x)$, and there is a natural map

$$\mathrm{Gr}_{a-h,a+b-2h}(\omega_x[\pi]/\pi\omega_x) \to \overline{X_{h,h}}.$$

The pullback of $\pi\omega$, $\omega[\pi]$ to the Grassmannian is constant (by construction) and thus we have an extension

$$0 \to \pi\omega = \mathcal{O}^h \to \omega_1 \to \omega_1/\pi\omega = \mathcal{O}(0, \dots, 0, -1) \to 0,$$

and

$$0 \to \omega[\pi]/\omega_1 = \mathcal{O}(-1, 0, \dots, 0) \to \omega/\omega_1 \to \omega/\omega[\pi] \simeq \pi\omega \to 0,$$

where the sheaves $\mathcal{O}(k_1, \dots, k_{a+b-2h})$ are on $\mathrm{Gr}_{a-h,a+b-2h}(\omega_w[\pi]/\pi\omega_x)$, with notation as before. Thus we can use the previous strategy to prove the following.

Theorem 3.9. *Assume $h < a$. If we cannot find $a-h$ indexes $i_t \in \{1, \dots, a\}$ and $b-h$ indexes $j_s \in \{1, \dots, b\}$ such that*

$$k_{i_1} = \cdots = k_{i_{a-h}} \le \ell_{j_1} \le \cdots \le \ell_{j_{b-h}},$$

then

$$H^0(\overline{X_{h,h}}, \omega^{(\underline{k},\underline{\ell},r)}) = 0.$$

Remark 3.10. This is the case in particular if $\underline{k}$ is regular enough, or if $h+1$ weights of $\underline{k}$ are greater than $\underline{\ell}$. The most restrictive case to apply the theorem is when $h = a-1$, in which case we can apply it under the assumption $k_a > \ell_{b-a+1}$.

Proof. By what precedes, we can choose $x \in X_{h,h}$ and compute the global sections of $\omega^{(\underline{k},\underline{\ell},r)}$ on the associated Grassmannian $\mathrm{Gr}_x := \mathrm{Gr}_{a-h,a+b-2h}(\omega[\pi]/\omega)$ (seen as a closed subspace of $\overline{X_{h,h}}$). On this space, $\mathcal{E}$ is constant (the p-divisible group is fixed), thus we can forget about r. We define the following subgroups of GL_{a+b-h}:

$$M = \begin{pmatrix} \mathrm{GL}_h & & 0 \\ & \mathrm{GL}_{a+b-2h} & \\ 0 & & \mathrm{GL}_h \end{pmatrix} \supset P = \begin{pmatrix} \mathrm{GL}_h & & 0 & \\ & \mathrm{GL}_{a-h} & \star & \\ & 0 & \mathrm{GL}_{b-h} & \\ 0 & & & \mathrm{GL}_h \end{pmatrix}$$

and

$$P_{a-h,b-h} = \begin{pmatrix} \mathrm{GL}_{a-h} & \star \\ 0 & \mathrm{GL}_{b-h} \end{pmatrix} \subset \mathrm{GL}_{a+b-2h}\,.$$

We have an isomorphism $\mathrm{Gr}_{a-h,a+b-2h} := P_{a-h,b-h}\backslash \mathrm{GL}_{a+b-2h} \simeq P\backslash M =: \mathrm{Gr}$, and we will use the partial Borel–Weyl–Bott theorem on $P\backslash M = \mathrm{Gr}$. Denote by V the vector space of dimension $a+b$ on which M acts, it corresponds to a vector bundle $\mathcal{V}$ on Gr, which coincides with the pullback of ω to Gr. As representation of M, $V = V_0 \oplus V_1 \oplus V_2$, a sum of irreducible, and we need to compute the weights of the representation $V^{\underline{k},\underline{\ell}}$ (the Schur functor for GL_{a+b} associated to $(\underline{k},\underline{\ell})$) for the action of P. But as a representation of GL_{a+b}, $V^{\underline{k},\underline{\ell}}$ has weights $w\cdot(k_1,\dots,k_a,\ell_1,\dots,\ell_b)$, $w \in \mathfrak{S}_{a+b}$. Among those weights, the highest weights for the action of P are those of the form $w_1 w_2 \cdot (k_1,\dots,k_a,\ell_1,\dots,\ell_b)$ with $(w_1,w_2) \in \mathfrak{S}_a \times \mathfrak{S}_b$ and

$$\begin{aligned} w_1(1) \geq \cdots \geq w_1(h), \qquad & w_1(h+1) \geq \cdots \geq w_1(a), \\ w_2(1) \geq \cdots \geq w_2(b-h), \quad & w_2(b-h+1) \geq \cdots \geq w_2(b). \end{aligned}$$

Denote by ${}^PW$ this space. Thus, $\mathcal{V}^{\underline{k},\underline{\ell}}$ (and therefore $\omega^{(\underline{k},\underline{\ell})}$) is an extension of $\mathcal{L}_P(w_1w_2\cdot(-\underline{k},\underline{\ell}^\vee))$ for $w \in {}^PW$. But under the hypothesis none of these bundles have sections (Proposition 3.7), thus $H^0(\mathrm{Gr}, V^{\underline{k},\underline{\ell}}) = H^0(\mathrm{Gr}_x, \omega^{\underline{k},\underline{\ell},r}) = 0$. As this is true for any point $x \in X_{h,h}$, we deduce the result. □

4. Further strata for the case (1, *n*)

We consider the case where $(a,b) = (1,n)$, where $n \geq 1$ is an integer.

4.1. ***Definition of the invariants.*** We will define some invariants on the special fiber X. Let us recall that one has locally free sheaves ω_1 and ω_2 of rank 1 and n, respectively.

Definition 4.1. We define $b \in H^0(X, (\omega/\omega_2) \otimes \omega_1^{-1})$ given by the natural inclusion $\omega_1 \to \omega/\omega_2$.

Define $m \in H^0(X, \omega_1 \otimes (\omega/\omega_2)^{-1})$ given by the multiplication by $\pi : \omega/\omega_2 \to \omega_1$.

For $i \in \{1, 2\}$, we define $\mathrm{hasse}_i \in H^0((\omega/\omega_i)^{(p)} \otimes \omega_1^{-1})$ thanks to the map $\mathrm{hasse} : \mathcal{E}[\pi] \to (\omega/\omega_i)^{(p)}$, induced by the composition of the Verschiebung and the division by π.

We refer to [1, Definition 3.8] for more details about the definition of the maps hasse_i (the reference deals with the ordinary case, i.e., $a = b$).

Proposition 4.2. • *One has $bm = 0$ and $mb = 0$.*

- *If x is point of X with $b(x) = 0$, then* $\mathrm{hasse}_1(x) = 0$ *implies that* $\mathrm{hasse}_2(x) = 0$.
- *If x is point of X with $b(x) \neq 0$, then one cannot have* $\mathrm{hasse}_1(x) = 0$ *and* $\mathrm{hasse}_2(x) = 0$.

Remark 4.3. The stratification defined previously consists in three strata, according to whether the sections b and m are 0 or not.

Proof. Clearly $mb = 0$ as $\pi\omega_1 = 0$. Moreover, as $\pi\omega \subset \omega_2$ — as $\omega_1 \subset (\pi\omega)^{\perp'}$ because $(\pi\omega)^{\perp'} = \big(\pi(\omega + \mathcal{E}[\pi])\big)^{\perp'} = (\omega + \mathcal{E}[\pi])^{\perp}$ and this last space contains ω_1 as both ω and $\mathcal{E}[\pi]$ are totally isotropic — we have clearly that $bm = 0$.

For the second point, if $b = 0$ then $\omega_1 \subset \omega_2$ and thus if $\mathrm{hasse}_1 = 0$, that is, $\mathrm{hasse}(\omega_1) \subset \omega_1^{(p)}$ then $\mathrm{hasse}(\omega_1) \subset \omega_2^{(p)}$. For the last point, note that if x is a point, then $b \neq 0$ is equivalent to $\omega = \omega_1 \oplus \omega_2$ as ω_1 is of rank 1. Thus the vanishing of both hasse_1 and hasse_2 is equivalent to the vanishing of $\omega_1 \xrightarrow{\mathrm{hasse}} \omega^{(p)}$. But because $\omega_1 \oplus \omega_2 = \omega$, which is thus of π-torsion, hasse, which is surjective, induces an isomorphism $\mathcal{E}[\pi] \xrightarrow{\mathrm{hasse}} \omega^{(p)} = \mathcal{E}[\pi]^{(p)}$, and thus its restriction to ω_1 can't be zero. □

Let us now define the different strata that we will consider.

- The ordinary locus is $X^{\mathrm{ord}} = \{x \in X \mid m(x) \neq 0,\ \mathrm{hasse}_2(x) \neq 0\}$.
- $R_1 = \{x \in X \mid m(x) \neq 0,\ \mathrm{hasse}_1(x) \neq 0,\ \mathrm{hasse}_2(x) = 0\}$.
- $R_2 = \{x \in X \mid m(x) \neq 0,\ \mathrm{hasse}_1(x) = 0\}$.
- $B_0 = \{x \in X \mid b(x) \neq 0,\ \mathrm{hasse}_1(x) \neq 0,\ \mathrm{hasse}_2(x) \neq 0\}$.
- $B_1 = \{x \in X \mid b(x) \neq 0,\ \mathrm{hasse}_2(x) = 0\}$.
- $B_2 = \{x \in X \mid b(x) \neq 0,\ \mathrm{hasse}_1(x) = 0\}$.
- $P_0 = \{x \in X \mid m(x) = b(x) = 0,\ \mathrm{hasse}_2(x) \neq 0\}$.
- $P_1 = \{x \in X \mid m(x) = b(x) = 0,\ \mathrm{hasse}_1(x) \neq 0,\ \mathrm{hasse}_2(x) = 0\}$.
- $P_2 = \{x \in X \mid m(x) = b(x) = 0,\ \mathrm{hasse}_1(x) = 0\}$.

Proposition 4.4. *Let x be a point in X^{ord}. Then x is μ-ordinary in the sense of* [2]. *In particular, one has $A[\pi] \simeq \mu_p \times \mathbb{Z}/p\mathbb{Z} \times LT^{n-1}$.*

Remark 4.5. Here LT is defined in [2] before Définition 1.1.3, this is X_β with $\beta = (1)$ ($e = 2$ and $\mathcal{T}$ is a singleton).

4.2. ***The conjugate filtration.*** The Verschiebung induces a map $V : \mathcal{E} \to \omega^{(p)}$, which is compatible with the action of π.

Definition 4.6. We define the sheave $\mathcal{F}_i$, $i = 1, 2$, by the formula

$$\mathcal{F}_i := \pi \cdot V^{-1}\omega_i^{(p)}.$$

Proposition 4.7. *The sheaves $\mathcal{F}_i$, $i = 1, 2$, are locally free of rank* 1 *and n, and are included in $\mathcal{E}[\pi]$. Moreover, $\mathcal{F}_2$ is the orthogonal of $\mathcal{F}_1$ for the modified pairing.*

Proof. The sheaf $V^{-1}\omega_1^{(p)}$ is locally free of rank $n+2 = a+b+1$, and contains $\mathcal{E}[\pi]$. This implies that $\mathcal{F}_1$ is locally free of rank 1. One gets in a similar way the result for $\mathcal{F}_2$.

To prove the last part, one only needs to check that $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal. Let $x \in \mathcal{F}_1$ and $y \in \mathcal{F}_2$. By definition, there exist x', y' such that $x = \pi x'$ and $y = \pi y'$, and $Vx' \in \omega_1^{(p)}$, $Vy' \in \omega_2^{(p)}$. Since ω_1 and ω_2 are orthogonal for the modified pairing, one gets the relation $\{Vx', Vy'\} = 0$. The element Vx' is in $\mathcal{E}[\pi]^{(p)}$; there exists then $z \in \mathcal{E}^{(p)}$ such that $Vx' = \pi z$. Now one has

$$0 = \{Vx', Vy'\} = \{\pi z, Vy'\} = \langle z, Vy'\rangle = \langle Fz, y'\rangle.$$

But there exists a unit u such that $uFz = \pi x' = x$, this equality being in $\mathcal{F}/\pi\mathcal{F}$, where $\mathcal{F} = \operatorname{Ker} V$. There exists then $a \in \mathcal{F}$ such that $Fz = u^{-1}x + \pi a$. Thus $0 = \langle u^{-1}x + \pi a, y'\rangle = \{u^{-1}x, y\} - \langle a, y\rangle = \{u^{-1}x, y\}$. Indeed, since a and y belong to $\mathcal{F}$, which is totally isotropic, one must have $\langle a, y\rangle = 0$. One then observes that the quantity $\{u^{-1}x, y\}$ only depends on the class of u in O_F/π, and one concludes that $\{x, y\} = 0$. □

Proposition 4.8. *Let x be a point of X. Then the condition* $\mathrm{hasse}_2(x) = 0$ *is equivalent to $\omega_1 \subseteq \mathcal{F}_2$. The condition* $\mathrm{hasse}_1(x) = 0$ *is equivalent to $\omega_1 = \mathcal{F}_1$.*

4.3. ***Stratification when $n > 1$.*** First, we remark that R_1 and P_1 are empty if $n \leq 2$.

Proposition 4.9. *Assume that $n \leq 2$. Then R_1 and P_1 are empty.*

Proof. Assume that x is a point in R_1 or P_1. This implies that $\omega_1 \subseteq \mathcal{F}_2$. If $n = 1$, since $b(x) = 0$, one must have $\omega_1 = \omega_2$, hence $\mathcal{F}_1 = \mathcal{F}_2$ and then $\mathrm{hasse}_1(x) = 0$. This is a contradiction.

Assume now that $n=2$. Taking the orthogonal of the inclusion $\omega_1 \subseteq \mathcal{F}_2$ in $\mathcal{E}[\pi]$, one has $\mathcal{F}_1 \subset \omega_2$. As $b=0$ we have $\omega_1 \subset \omega_2$, thus $\omega_1^{(p)} \subset \omega_2^{(p)}$ and thus $\mathcal{F}_1 \subset \mathcal{F}_2$. In particular ω_1 and $\mathcal{F}_1$ are distinct isotropic lines, and ω_2 is the orthogonal of ω_1, thus one can see that the modified pairing induced on ω_2 is zero, which is not possible. □

Let us now state the principal result on the stratification of the variety. We will need the following remark.

Remark 4.10. Let $S_0 = \mathrm{Spec}(R)$ be a characteristic p scheme, and $T = \mathrm{Spec}(S)$, with $R = S/I$ for some ideal I a thickening of S_0, and assume T is of characteristic p again. Let G be a p-divisible group over S_0 and assume that $I^2 = 0$ in T and denote by $\mathcal{E}$ its crystal on the crystalline site $S_0/\mathrm{Spec}(\mathbb{Z}_p)$. Then by Grothendieck–Messing, lifting G to T is the same as lifting is Hodge filtration ω_G to $\mathcal{E}_T$. Assume $\widetilde{\omega}_G \subset \mathcal{E}_T$ is such a lift. Then as $I^2 = 0$ we claim that $\widetilde{\omega}_G^{(p)}$ doesn't depend on the lift. Indeed, let $w_1, w_2 \in \mathcal{E}_T$ which both lift $w \in \mathcal{E}_{S_0}$ and let $\underline{e}$ be a basis of $\mathcal{E}_T$ as an S-module. Then $w_2 = w_1 + M \cdot \underline{e}$ for some $M \in M_{2h}(I)$. Then $w_2 \otimes 1 = w_1 \otimes 1 + (M\underline{e}) \otimes 1 = w_1 \otimes 1 + \underline{e} \otimes M^{\sigma}$. But if $i \in I$, and $\sigma = \sigma_T$ is the Frobenius of T (which lifts the one of S_0) then $\sigma(i) = i^p \equiv 0$ in S, thus $w_2 = w_1$. In particular, in the previous situation as both F, V are maps on the crystal $\mathcal{E}$, we see that the lifts of $\mathcal{F}_1, \mathcal{F}_2$ don't depend on the lift of ω.

Theorem 4.11. *Assume that $n \geq 2$. The strata X^{ord} and B_0 are open and the strata P_2 and B_2 are closed. Moreover*

$$\overline{X^{\mathrm{ord}}} = X^{\mathrm{ord}} \cup \bigcup_{i=1}^{2} R_i \cup \bigcup_{i=0}^{2} P_i, \quad \overline{R_2} = R_2 \cup P_2,$$

$$\overline{B_0} = \bigcup_{i=0}^{2} B_i \cup \bigcup_{i=0}^{2} P_i, \quad \overline{B_1} = B_1 \cup P_1 \cup P_2, \quad \overline{P_0} = \bigcup_{i=0}^{2} P_i.$$

If $n \geq 3$, one has

$$\overline{R_1} = \bigcup_{i=1}^{2} R_i \cup \bigcup_{i=1}^{2} P_i, \quad \overline{P_1} = P_1 \cup P_2.$$

Proof. The fact that X^{ord} and B_0 are open is clear, as is the closeness of P_2. Let us prove the closure relations by looking at where we can specialize (for B_2) our deformed points of X.

- If x is a point of B_2, it can only specialize to a point in B_2 or P_2. We need to show that the latter cannot happen. Assume that a point x in $P_2(k)$ can be deformed to $k[\![X]\!]$,[2] such that the generization lies in B_2. Since $\text{hasse}_1=0$, $\omega_1=\mathcal{F}_1$ over $k[\![X]\!]$. If e_1 is a basis of ω_1, let $u=\{e_1,e_1\}$. The composition of V with the division by π defines a map $V_\pi:\mathcal{E}[\pi]\to\mathcal{E}[\pi]^{(p)}$; similarly, one has a map $F_\pi:\mathcal{E}[\pi]^{(p)}\to\mathcal{E}[\pi]$ given by the composition of the division by π and the Frobenius. These maps are well defined because the image of V is $\mathcal{E}[\pi]$. There exists a unit $u\in O_F^\times$ such that $F_\pi\circ V_\pi=u\,\mathrm{id}$. Since one has $\{F_\pi x,y\}=\{x,V_\pi y\}$, and $V_\pi e_1=\lambda e_1$ for some unit λ, one finds the equation $u=\lambda_0 u^p$, with $\lambda_0\in k^\times$. One gets a contradiction, since u must be nonzero and divisible by X.

• Let $x\in R_2(k)$. This implies that $\omega_1=\mathcal{F}_1$, and $\omega_2=\mathcal{F}_2$. One can find a basis $e_1,\dots,e_{n+1}$ of $\mathcal{E}[\pi]$ such that ω_1 is spanned by e_1, and ω_2 by $e_1,\dots,e_n$, and the modified pairing is given by the matrix

$$\begin{pmatrix}0&0&1\\0&J_{n-1}&0\\1&0&0\end{pmatrix},\quad\text{with }J_{n-1}=\begin{pmatrix}0&0&1\\0&\iddots&0\\1&0&0\end{pmatrix}. \tag{2}$$

One then looks for a lift to $k[\![T]\!]$, first of the Hodge filtration together with the extra data. The line ω_1 can be lifted to a line spanned by a vector $(1\ \ X\ \ y)^{\mathrm{T}}$. The vector needs to be isotropic, hence the condition

$$2y+{}^tXJ_{n-1}X=0.$$

Then we will look at the corresponding deformation step by step, i.e., successively from $k[T]/(T^n)\to k[T]/(T^{n-1})$ which is given by a square zero ideal. At each step, we have a p-divisible group G_n over $k[T]/(T^n)$ and by Remark 4.10, the deformation to G_{n+1} has a canonical lift of $\mathcal{F}_1$ and $\mathcal{F}_2$ which we can assume, if $\text{hasse}_1(G_n)=\text{hasse}_2(G_n)=0$ given by e_1 and $e_1,\dots,e_n$. The condition for the generization to be in R_2 is that at each step $X=0$, $y=0$. The condition for it to be in R_1 is $y=0$. Since $n\geq 2$, the point can always be lifted to a point in X^{ord}. If $n\geq 3$, it can be lifted to a point in R_1, but as in this case the two conditions (ω_1 totally isotropic and $\omega_1\subset\mathcal{F}_2$) make a nonsmooth condition, let us give a more precise argument: set $\widetilde{\mathcal{E}}=\mathcal{E}\otimes_k k[\![t]\!]$ and choose a lift of the basis such that the pairing is of the previous form, and reducing on $k[t]/(t^2)$ we have $\mathcal{F}_1$ given by e_1 and $\mathcal{F}_2$ by $e_1,\dots,e_n$, as before. Then set $\widetilde{\omega}_1$ spanned by $(1\ \ t\ \ 0_{n-1})^{\mathrm{T}}$. Then clearly $\widetilde{\omega}_1$ is totally isotropic, and reducing modulo t^2 we see that $\widetilde{\omega}_1\neq\mathcal{F}_1 \bmod t^2$, thus our deformed point is not in R_2 anymore, but we can't assure that the $k[\![t]\!]$-point is in R_1 at the moment. So assume we have lifted ω_1 to $k[t]/(t^n)$ to a point in R_1,

[2] In particular this means that we have a deformation of the Hodge filtration, and conversely a deformation of the Hodge filtration to $k[\![X]\!]$ induces step by step by Grothendieck–Messing a deformation of the p-divisible group.

and we assume that there is a basis of $\mathcal{E} \otimes_k k[t]/(t^n)$ such that ω_1 is spanned by $(1 \;\; t \;\; 0_{n-1})^{\mathrm{T}}$. We then choose a lift of this basis to $\mathcal{E} \otimes_k k[t]/(t^{n+1})$ such that the pairing as the same form (2). Then we simply set again $\widetilde{\omega}_1$ to be spanned by $(1 \;\; t \;\; 0_{n-1})^{\mathrm{T}}$. Inducting the argument gives the resulting point in R_1.

• The fact that any point $x \in P_0(k)$ can be deformed to a point in X^{ord} or B_0 follows from the previous sections.

• A point $x \in P_2(k)$ can be deformed to P_0 (and hence B_0, X^{ord}), and P_1 if $n \geq 3$, with exactly the same arguments as before, as we never used that $m \neq 0$. If we want to deform x to R_2, we can lift ω_1 "trivially" so that $\omega_1 \subset \omega_2 := \omega_1^{\perp'}$, and then deform ω/ω_2 so that $\pi\omega \subset \omega_1$ (by choosing elements as in Proposition 2.11). We can then deform to R_1 if $n \geq 3$. The point x can also be deformed to B_1, by lifting ω_1 inside $\mathcal{F}_2$, nonisotropically: concretely choose the isotrivial lift to $k[\![T]\!]$ of $\widetilde{\omega}$ of ω (here it means it is still of p-torsion, that is, $\widetilde{\omega} = \widetilde{\mathcal{E}}[\pi]$), and then inductively for each n, there is a canonical lift of $\mathcal{F}_1, \mathcal{F}_2$ from $k[T]/(T^{n-1})$ to $k[T]/(T^n)$ by Remark 4.10 if ω_1 and thus $\omega_2 = \omega_1^{\perp'}$ have been deformed to $k[T]/(T^{n-1})$. We thus have deformations of $\mathcal{F}_1, \mathcal{F}_2$ to $k[T]/(T^n)$ (orthogonal to each other for the modified pairing) and we choose a deformation of ω_1 still assuming $\widetilde{\omega}_1 \subset \mathcal{F}_2$. This is possible as the Grassmannian $\mathrm{Gr}_a(\mathcal{F}_2)$ is smooth. If n is big enough, as the condition of being totally isotropic is a closed condition which defines a proper closed subspace of $\mathrm{Gr}_a(\mathcal{F}_2)$, there exists a deformation of ω_1 which is not isotropic anymore. After this choice, any lift of ω_1 will do. The corresponding deformed p-divisible group is in B_1. Note that we have already proven that we can't deform from P_2 to B_2.

• A similar but easier argument shows that we can deform from B_2 to B_1, and it is easy to see that any element $x \in B_1(k)$ can be deformed to B_0.

• To finish the proof, let us remark that a point in R_1 can be deformed to X^{ord} if $n \geq 3$, by lifting ω_1 isotropically outside $\mathcal{F}_2$. Indeed, we have $\pi\omega = \omega_1 \subset \omega_2$ and by hypothesis $\mathcal{F}_1 \neq \omega_1 \subset \mathcal{F}_2$. In particular $\omega_2 \neq \mathcal{F}_2$. The divided pairing, on a basis $e_1, \dots, e_h$ such that e_1 generates ω_1 and $e_1, \dots, e_{h-1}$ generates ω_2, can be given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We thus look for a lift of ω_1 given by a vector $(1 \;\; X \;\; y)^{\mathrm{T}}$ with y, X with coefficients in $tk[\![t]\!]$. This lift is totally isotropic if $2y + {}^tXX = 0$. Let us prove that we can choose it away from $\mathcal{F}_2$. As mod t, $\omega_1 \subset \mathcal{F}_2 \neq \omega_2$, we have $e_1 \in \mathcal{F}_2$, there exist

$e_i \notin \mathcal{F}_2$ and as $n \geq 2$, there is a nonzero vector of the form

$$v = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} \in \mathcal{F}_2.$$

Thus if we set $\widetilde{\omega}_1$ generated by

$$v = \begin{pmatrix} 1 \\ tB+ta\delta_i \\ 0 \end{pmatrix} \notin \mathcal{F}_2$$

for a nonzero a, the condition of being totally isotropic is given by

$$t^2\Big(\sum_j b_j^2 + 2b_i a + a^2\Big) = 0.$$

If $\sum_j b_j^2$ is nonzero or if $b_i \neq 0$ we can find such a nonzero a. So assume that $\sum_j b_j^2 = 0$ but $b_i = 0$. As v is nonzero, there is j such that $b_j \neq 0$. If $e_j \notin \mathcal{F}_2$ then the previous argument applies. Otherwise $e_j \in \mathcal{F}_2$, and thus $w = v + ce_j \in \mathcal{F}_2$. But if we calculate its norm for the divided pairing, we have $\sum_i b_i^2 + 2cb_j + c^2 = 2cb_j + c^2$. But we can find c such that this is nonzero, and then reapply the previous argument with w instead of v.

- If $n \geq 3$, one checks that a point in P_1 can be deformed to P_0 (and then X^{ord}, B_0) by the exact same calculation. We can also deform from P_1 to B_1: mod t we have $\omega_1 \subset \omega_2 \neq \mathcal{F}_2$, thus up to choosing a basis as before we can set $\widetilde{w}_1$ mod t^2 generated by

$$\begin{pmatrix} 1 \\ tB \\ t \end{pmatrix} \in \widetilde{\mathcal{F}_2},$$

where

$$v = \begin{pmatrix} 0 \\ B \\ 1 \end{pmatrix} \in \mathcal{F}_2.$$

Then clearly $\widetilde{\omega}_1$ is not isotropic. Then assume that we have lifted ω_1 to $k[t]/(t^n)$, this gives a lift of $\mathcal{F}_2$ to $k[t]/(t^{n+1})$ and we choose any lift of ω_1 inside this. By induction, and Grothendieck–Messing, we get a point in B_1. We can also deform from P_1 to R_1: assume that we have lifted ω_1 to $k[t]/(t^n)$, inside $\mathcal{F}_2$, which has a canonical lift mod t^{n+1}. Then we want to deform ω_1 isotropically while staying in $\mathcal{F}_2$. But as $\omega_1^{\perp} \cap \mathcal{F}_2$ is nontrivial in special fiber, we can indeed find a lift of $\omega_1 \subset \mathcal{F}_2$ at each step which remains isotropic. ☐

4.4. ***Stratification when n = 1.*** We now suppose that $n = 1$. In this case P_1 and R_1 are empty. The situation is the following.

Theorem 4.12. *The strata* X^{ord}, R_2 *and* B_0 *are open. The strata* P_0, P_2 *and* B_i $(i = 1, 2)$ *are closed. Moreover*

$$\overline{X^{\text{ord}}} = X^{\text{ord}} \cup P_0, \quad \overline{R_2} = R_2 \cup P_2, \quad \overline{B_0} = \bigcup_{i=0}^{2} B_i \cup P_0 \cup P_2.$$

Proof. It is clear that X^{ord} and B_0 are open. As previously, any point of P_0 can be deformed to X^{ord} or B_0. It is easy to see that any point of B_i $(i = 1, 2)$ can be deformed to B_0.

Let us prove that R_2 is open. Let $x \in R_2(k)$, and let us investigate the possible lifts of x to a ring R. Over this ring, the space $\mathcal{F}_1$ lifts canonically. By assumption, the space ω_1 is equal to its $\mathcal{F}_1$ over k. Since any lift of ω_1 must be isotropic, and $\mathcal{E}[\pi]$ is a 2-dimensional space with a perfect pairing, we see that the space of totally isotropic lines in it is zero dimensional and reduced, thus one must have an equality $\widetilde{\omega}_1 = \mathcal{F}_1$. It is then not possible to lift x to a point in X^{ord}.

The same arguments show that a point in P_2 cannot be deformed into P_0 or X^{ord}. Similarly, if $x \in P_2$ is deformed over $k[[t]]$ in B_1 or B_2, for each n, modulo t^n this implies that we have canonical lifts $\mathcal{F}_1, \mathcal{F}_2$ modulo T^{n+1}. If $\omega_1 = \omega_2 \bmod t^n$, then $\mathcal{F}_1 = \mathcal{F}_2$, and if we deform in B_1 or B_2 (or any point such that $\text{hasse}_2 = 0$) we must have $\widetilde{\omega}_1 \subset \mathcal{F}_2$, but they have the same rank thus an equality, and thus $\widetilde{\omega}_1 = \mathcal{F}_1 = \mathcal{F}_2 = \widetilde{\omega}_2$. Thus actually the deformation remain in P_2. This proves that points of P_2 can only be possibly deformed to a point in R_2 or B_0. Conversely we can indeed deform to R_2 by only deforming ω/ω_2 to make it non-π-torsion as in proof of Proposition 2.11. To deform a point of P_2 to B_0, it is enough to deform $\omega_1 \subset \omega = \mathcal{E}[\pi]$ by a nontotally isotropic line. This is possible as this space is smooth (it is a projective space of dimension > 0). □

5. Case of a general CM field F

Let $(B, \star, V, \langle \cdot, \cdot \rangle, h)$ be P.E.L. datum (see [11]), so that $B/\mathbb{Q}$ is a finite-dimensional central semisimple $\mathbb{Q}$-algebra, with involution $\star$, center F.

Example 5.1. Let F_0 be a totally real field, and F/F_0 a CM field. Take $B = F$, $\star = c$ the complex conjugacy, $V = F^n$ and polarization by $(x, y) = xJc(y)$ for a hermitian matrix J. Let p be a prime. Then $B_{\mathbb{Q}_p} = \prod_{\pi_0 | p \in F} F \otimes_{F_0} F_{0,\pi}$. Everything splits over primes above p in F_0, thus for simplicity, we can assume that there is only one prime π_0 of F_0 above p. Let e, f be the ramification index, and the residual degree of π_0 over p. The case of unramified primes in F/F_0 is treated in [3], thus we can assume that π_0 ramifies in $F_p := F \otimes \mathbb{Q}_p$ and choose it so that $\pi_0 = \pi^2$ for some uniformizer π of F_p.

Now fix an integral P.E.L. datum $(\mathcal{O}_B, \star, \Lambda, \langle \cdot , \cdot \rangle)$, so that in particular $\mathcal{O}_B$ is a $\mathbb{Z}_{(p)}$-order in B, $\star$-stable and maximal over $\mathbb{Z}_p$, and $(\Lambda, \langle \cdot , \cdot \rangle) \otimes_{\mathbb{Z}} \mathbb{Q} = (V, \langle \cdot , \cdot \rangle)$.

Hypothesis 5.2. We assume the following:

(1) $B_{\mathbb{Q}_p}$ is a product of matrix algebras over finite extension of $\mathbb{Q}_p$.

(2) p is a *good* prime, i.e., $p \nmid [\Lambda^\sharp, \Lambda]$.

To simplify we assume that $\star$ is of the second kind on each simple factor of $(B, \star)$ (in particular we exclude factors of type D; see [3, Hypothesis 2.2]): factors of type (C) can be dealt with as in [3]. In most of what follows, we can treat simple factors separately, so that we will be able to assume $(B_{\mathbb{Q}_p}, \star)$ is a matrix algebra over its center or a product of two isomorphic matrix algebras over a field exchanged by $\star$. This second case is treated in [3]. So to fix notation we will often assume that $B_{\mathbb{Q}_p} = M_n(F_\pi)$, for some finite extension F_π of $\mathbb{Q}_p$, and we define $F_{\pi_0} = F_\pi^{\star=1}$: the extension F_π / F_{π_0} is of degree 2. We assume that the local field extension F_π / F_{π_0} is ramified (otherwise this is treated in [3] again). Fix a uniformizer π_0 of F_{π_0} and π of F_π so that $\pi^2 = \pi_0$. Let $F_{\pi_0}^{\mathrm{ur}}$ be the maximal unramified extension contained in F_{π_0}, and $\mathcal{T}$ the set of embeddings of $F_{\pi_0}^{\mathrm{ur}}$ into $\overline{\mathbb{Q}_p}$. For each $\tau \in \mathcal{T}$, let Σ_τ be the set of embeddings of F_{π_0} extending τ. We write $\Sigma_\tau = \{\sigma_{\tau,1}, \dots, \sigma_{\tau,e}\}$. For each $\tau \in \mathcal{T}$ and $1 \le i \le e$, let $\sigma_{\tau,i}^+$ and $\sigma_{\tau,i}^-$ be the embeddings of F_π extending $\sigma_{\tau,i}$: this is notation which will remain in force every time we (implicitly) choose a simple factor of $B_{\mathbb{Q}_p}$.

As in [3, Section 2.2], we can associate to the Shimura data a combinatorial data $(d_{j,\tau'})_{\tau'}$, (j corresponding to a choice of a simple factor) which, when we restrict to a simple factor of the previous type, is just a collection $(d_\sigma)_{\sigma : F_\pi \hookrightarrow \overline{\mathbb{Q}_p}}$ satisfying $d_{\sigma \circ c} = h - d_\sigma$ for a fixed value of $h \ge 1$ (which might depend on the simple factor). For simplicity we define for $\tau \in \mathcal{T}$, $i = 1, \dots, e$, $a_{\tau,i} := d_{\sigma_{\tau,i}^+}$, $b_{\tau,i} := d_{\sigma_{\tau,i}^-}$.

Definition 5.3. Let Y be the moduli space over O_{F_π} whose R-points are equivalence classes of tuples $(A, \lambda, \iota, \eta, \omega_1)$ up to $\mathbb{Z}_{(p)}^\times$-isogenies, where:

- A is an abelian scheme over R.
- λ is a $\mathbb{Z}_{(p)}^\times$-polarization.
- We have the map $\iota : O_B \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, making the Rosati involution and $\star$ compatible.
- η is a rational Λ-level structure outside p.

- For every simple factor $j = M_n(F_\pi)$ of $B_{\mathbb{Q}_p}$, there is an associated direct factor ω'_j of ω_A. By Morita equivalence, we have a $\mathcal{O}_{F_\pi}$-module $\omega_j = \bigoplus_\tau \omega_{\tau,j}$. We then require that

$$0 = \omega_\tau^{[0]} \subseteq \omega_\tau^{[1]} \subseteq \cdots \subseteq \omega_\tau^{[e]} = \omega_{\tau,j}$$

is a PR filtration, meaning that each $\omega_\tau^{[i]}$ is locally a direct factor, stable by O_{F_π}.
- The quotient $\omega_\tau^{[i]}/\omega_\tau^{[i-1]}$ is locally free of rank h.
- $O_{F_{\pi_0}}$ acts by $\sigma_{\tau,i}$ on $\omega_\tau^{[i]}/\omega_\tau^{[i-1]}$.
- The filtration is compatible with the polarization.
- For each i, $\omega_\tau^{[i-1]} \subseteq \omega_{\tau,1}^{[i]} \subseteq \omega_\tau^{[i]}$, where $\omega_{\tau,1}^{[i]}$ is locally a direct factor stable by O_{F_π}.
- $\omega_{\tau,1}^{[i]}/\omega_\tau^{[i]}$ is locally free of rank $a_{\tau,i}$, and O_{F_π} acts by $\sigma_{\tau,i}^+$ on it, and by $\sigma_{\tau,i}^-$ on the quotient $\omega_\tau^{[i+1]}/\omega_{\tau,1}^{[i]}$ (which is automatically locally free of rank $b_{\tau,i}$).

Let us be more precise about the compatibility with the polarization. One has a pairing on $\mathcal{E}$, and $\omega_{\tau,j}$ is totally isotropic for this pairing. The compatibility for the filtration is that

$$(\omega_\tau^{[i]})^\perp = Q_\tau^i(\pi_0)^{-1}\omega_\tau^{[i]}, \quad Q_\tau^i(T) = \prod_{t=i+1}^{e}(T - \sigma_{\tau,t}(\pi_0)),$$

and $Q_\tau = \prod_{i=1}^{e}(T - \sigma_{\tau,i}(\pi_0))$ is a minimal polynomial for π_0 in $\tau(F_{\pi_0}^{\mathrm{ur}})$. Let us define $\mathcal{E}_\tau^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1}\omega_\tau^{[i-1]}/\omega_\tau^{[i-1]}$. It is a locally free sheaf with an action of O_F, and an alternating perfect pairing. One has the subsheaves $\mathcal{F}_\tau^{[i]} := \omega_\tau^{[i]}/\omega_\tau^{[i-1]}$, which is totally isotropic for the previous pairing, and $\mathcal{F}_{\tau,1}^{[i]} := \omega_{\tau,1}^{[i]}/\omega_\tau^{[i-1]}$. We define $\mathcal{F}_{\tau,2}^{[i]} := (\pi - \sigma_{i,\tau}^-(\pi))(\mathcal{F}_{\tau,1}^{[i]})^\perp$.

Consider again a local model diagram as in Section 2, and assume therefore $p \neq 2$. The local model will split over direct factors of $B_{\mathbb{Q}_p}$, thus our main results will be proven for one factor at a time. Let F_π/F_{π_0}, e, f etc. as before, $M_n(\mathcal{O}_{F_\pi})$ the corresponding factor, and denote by Λ'_j the part of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$ corresponding to this simple factor and using Morita equivalence, so that $\Lambda = \sum_j \mathcal{O}_{F_\pi}^n \otimes_{\mathcal{O}_{F_\pi}} \Lambda'_j$. We have a diagram

$$Y \leftarrow \widetilde{Y} = \mathrm{Isom}(\mathcal{E}, \Lambda \otimes \mathcal{O}_S) \to \mathcal{N},$$

where the first map is a torsor over a smooth group scheme $\mathcal{G}$, and the second map is formally smooth and $\mathcal{G}$-equivariant by Grothendieck–Messing. Here $\mathcal{N}$ is a local model, see, e.g., [13, Section 14], analogous to the one in the proof of Proposition 2.21, parametrizing a PR-filtration $0 = F_\tau^{[0]} \subset F_\tau^{[1]} \subset \cdots \subset F_\tau^{[e]} = \Lambda_{\tau,j} \otimes \mathcal{O}_S$ in $\Lambda \otimes \mathcal{O}_S$, where

- each $F_\tau^{[i]}$ is a locally direct factor, stable by $\mathcal{O}_{F_\pi}$;

• each quotient $F_\tau^{[i]}/F_\tau^{[i-1]}$ is locally free of rank $h = a_{\tau,i} + b_{\tau,i}$ and $\mathcal{O}_{F_{\pi_0}}$ acts by $\sigma_{\tau,i}$ on it;

• the filtration is compatible with the polarization;

• for each i, there is a locally direct factor $F_{\tau,1}^{[i]}$ such that $F_\tau^{[i-1]} \subset F_{\tau,1}^{[i]} \subset F_\tau^{[i]}$, which is stable by $\mathcal{O}_{F_\pi}$;

• $F_{\tau,1}^{[i]}/F_\tau^{[i-1]}$ is a locally direct factor of rank $a_{\tau,i}$ and $\mathcal{O}_{F_\pi}$ acts through $\sigma_{\tau,i}^+$;

• $\mathcal{O}_{F_\pi}$ acts through $\sigma_{\tau,i}^-$ on $F_\tau^{[i]}/F_{\tau,1}^{[i]}$ (this is automatically locally free of rank $b_{\tau,i}$).

$F_\tau^{[i]}$ is obviously the analog of the $\omega_\tau^{[i]}$ in the definition of Y, and $F_{\tau,1}^{[i]}$ of $\omega_{\tau,1}^{[i]}$. We define

$$\mathcal{F}_\tau^{[i]} := F_\tau^{[i]}/F_\tau^{[i-1]} \quad \text{and} \quad \mathcal{F}_{\tau,1}^{[i]} = F_{\tau,1}^{[i]}/F_\tau^{[i-1]},$$

as on Y, and we define also $\mathcal{F}_{\tau,2}^{[i]} := (\pi - \sigma_{i,\tau}^-(\pi))(\mathcal{F}_{\tau,1}^{[i]})^\perp$ for the induced pairing on $\mathcal{E}_\tau^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1} F_\tau^{[i-1]}/F_\tau^{[i-1]}$.

Definition 5.4. Let k be an algebraically closed field of characteristic p. Let $x \in \mathcal{N}(k)$, and fix τ, i. We define the integers $h_\tau^{[i]}(x)$ and $l_\tau^{[i]}(x)$ as the dimensions, in $\mathcal{E}_\tau^{[i]} \otimes k(x)$, of $\pi \mathcal{F}_\tau^{[i]}$ and $\mathcal{F}_{\tau,1}^{[i]} \cap \mathcal{F}_{\tau,2}^{[i]}$, respectively.

Let $C := \{(h_\tau^{[i]}, l_\tau^{[i]})_{\tau \in \mathcal{T}, 1 \le i \le e} \mid 0 \le h_\tau^{[i]} \le l_\tau^{[i]} \le \min(a_{\tau,i}, b_{\tau,i})\}$. We define a stratification on $N = \mathcal{N} \times \operatorname{Spec}(k_F)$ by

$$N = \coprod_{c \in C} N_c,$$

where $N_c = \{x \in X(k) \mid (h_\tau^{[i]}(x), l_\tau^{[i]}(x)) = c\}$. Let $c = (h_\tau^{[i]}, l_\tau^{[i]})$ and $c' = (h_\tau^{[i]\prime}, l_\tau^{[i]\prime})$ be elements of C. We say that $c \le c'$ if, for all τ, i,

$$h_\tau^{[i]\prime} \le h_\tau^{[i]} \le l_\tau^{[i]} \le l_\tau^{[i]\prime}.$$

Theorem 5.5. *One has*

$$\overline{N_c} = \coprod_{c' \le c} N_{c'}.$$

Proof. We construct deformation of a point $x \in N_{c'}(k)$ one τ at a time as follows.

By the results of the previous section, we can deform both $F_{\tau,1}^{[1]} \subseteq F_\tau^{[1]}$ inside $\widetilde{\mathcal{E}}_\tau^{[1]} := \mathcal{E}_\tau^{[1]} \otimes_k k[[t]]$, with the deformation $\widetilde{F}_\tau^{[1]}$ of $F_\tau^{[1]}$ isotropic (for the divided pairing) and with $(h_\tau^{[1]}, l_\tau^{[1]}) = (h_\tau^{[1]\prime}, l_\tau^{[1]\prime})$. This is the result of Proposition 2.11. Then, look at $\pi^{2(e-1)} F_\tau^{[1]}/F_\tau^{[1]}$: this space has a natural lift $\pi^{2(e-1)} \widetilde{F}_\tau^{[1]}/\widetilde{F}_\tau^{[1]}$ inside $\widetilde{\mathcal{E}}/\widetilde{F}_\tau^{[1]}$. We then take an isotrivial lift of the filtration

$$\cdots \subset F_\tau^{[i-1]}/F_\tau^{[1]} \subset F_{\tau,1}^{[i]}/F_\tau^{[1]} \subset F_\tau^{[i]}/F_\tau^{[1]} \subset \cdots,$$

for $e \geq i \geq 1$ and then pull back to $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_k k((t))$ to get a full lift, and we get a point over $k((t))$ with new $\tilde{c}_\tau^{[1]} = c_\tau^{[1]'}$ but $\tilde{c}_\tau^{[i]} = c_\tau^{[i]}$ for $i \geq 2$. Then by induction, we can assume that for $1 \leq s \leq i$ we have

$$c_\tau^{[s]} := (h_\tau^{[s]}, l_\tau^{[s]}) = (h_\tau^{[s]'}, l_\tau^{[s]'}) =: c_\tau^{[s]'}$$

for our point over k. Then one deforms $F_{\tau,1}^{[i+1]}/F_\tau^{[i]} \subset F_\tau^{[i+1]}/F_\tau^{[i]}$ inside $\widetilde{\mathcal{E}}_\tau^{[i+1]}$ again using Proposition 2.11, and we do the same isotrivial lift for the rest of the filtration as when $i = 0$, to get the induction step, and thus the result. □

Theorem 5.6. *Y (resp. $\mathcal{N}$) is normal and flat over $\mathcal{O}_{F_\pi}$, and its special fiber is reduced and Cohen–Macaulay.*

Proof. Using the local model diagram, it is enough to see that $\mathcal{N}$ is flat, normal, and its special fiber is Cohen–Macaulay. Theorem 5.5 shows

$$N := \overline{\mathcal{N}} = \coprod_c N_c,$$

with expected (strong) closure relations. The proof of Proposition 2.12 for the maximal strata carries over and shows (doing one $F_{\tau,1}^{[i]}$ at a time) that maximal strata of N are smooth, thus reduced, and $\mathcal{N}$ is smooth in codimension 1. For each i, we have a space $\mathcal{N}_{\leq i}$ parametrizing locally direct factors $F_\tau^{[1]} \subset \cdots \subset F_\tau^{[i]} \subset \Lambda_{\tau,j} \otimes \mathcal{O}_S$ with the same properties as before, together with $F_{\tau,1}^{[k]}$ in $F_\tau^{[k]}/F_\tau^{[k-1]}$ of rank $a_{\tau,k}$ such that the actions of $\mathcal{O}_{F_\pi}$ are by $\sigma_{\tau,k}^+$ on $F_{\tau,1}^{[k]}/F_\tau^{[k-1]}$ and by $\sigma_{\tau,k}^-$ on the cokernel of the inclusion, for $k = 1, \ldots, i$. We have natural maps

$$\mathcal{N} = \mathcal{N}_{\leq e} \to \mathcal{N}_{\leq e-1} \to \cdots \to \mathcal{N}_{\leq 1} \to \operatorname{Spec}(\mathcal{O}_F) =: \mathcal{N}_{\leq 0}.$$

We will show inductively that $\mathcal{N}_{\leq i}$ is flat over $\mathcal{O}_{F_\pi}$, with Cohen–Macaulay fibers.

Because $\mathcal{N}$ and $\mathcal{N}_{\leq i}$ decompose naturally as products over the simple factors of $B_{\mathbb{Q}_p}$, and over the index τ, we can assume that there is only one factor and that $\mathcal{O}_B \otimes \mathbb{Z}_p = \mathcal{O}_F$ and that $\mathcal{T} = \{\tau\}$ so we suppress τ from the notation. Define $E^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1} F^{[i-1]}/F^{[i-1]}$, endowed with its own (perfect) pairing. By definition $\mathcal{N}_{\leq i}$ over $\mathcal{N}_{\leq i-1}$ parametrizes locally direct factors $F_1^{[i]}$ and $F^{[i]}$ of $E^{[i]}$ of respective ranks a_i and $h = a_i + b_i$, such that moreover $F_1^{[i]} \subset F^{[i]}$ and $F_1^{[i]} \subset E^{[i]}[\pi - \sigma_i^+(\pi)]$, and a compatibility for the polarization. We assume that $\mathcal{N}_{i-1}$ is flat over $\mathcal{O}_{F_\pi}$, with Cohen–Macaulay fibers. So let $U \subset \mathcal{N}_{\leq i-1}$ for $i \geq 1$ a small affine so that all $F^{[k]}$, $k < i$, and $E^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1} F^{[i-1]}/F^{[i-1]}$ are free. Now we claim that we can make the pairing of $E^{[i]}$ locally trivial. First, as it is perfect, we choose a basis so that it is of the form

$$\begin{pmatrix} 0 & \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_h \end{pmatrix} \\ \begin{pmatrix} -a_1 & & \\ & \ddots & \\ & & -a_h \end{pmatrix} & 0 \end{pmatrix},$$

with $a_i \in \mathcal{O}_S^\times$. But now, up to changing the basis vectors $(e_1, \dots, e_h, f_1, \dots, f_h)$ by $(e_1, \dots, e_h, a_1^{-1} f_1, \dots, e_h^{-1} f_h)$, it is of the desired form

$$\begin{pmatrix} 0 & \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} & 0 \end{pmatrix},$$

and thus $\mathcal{N}_{\leq i} \times_{\mathcal{N}_{\leq i-1}} U \simeq U \times_{\mathcal{O}_{F_\pi}} \mathcal{N}'$ where $\mathcal{N}'$ classifies $\mathcal{O}_{F_{\pi_0}}$ locally direct factors $(\mathcal{F}_1, \mathcal{F})$ inside $\mathcal{O}_{F_\pi}^{2h} \simeq \mathcal{O}_{F_{\pi_0}}[X]/(X^2 - \pi_0)^{2h}$ of respective ranks $a_i, a_i + b_i$ satisfying relations analogous to the one of $\mathcal{N}$ in the proof of Proposition 2.21. In particular, we can check that changing $\mathbb{Z}_p$ by $\mathcal{O}_{F_{\pi_0}}$ (and thus p by π_0) in the proof there we have that $\mathcal{N}'$ is flat over $\mathcal{O}_{F_\pi}$, with reduced Cohen–Macaulay fibers. Thus $\mathcal{N}' \times U$ is flat and Cohen–Macaulay over U, thus $\pi_i : \mathcal{N}_{\leq i} \to \mathcal{N}_{i-1}$ is flat and Cohen–Macaulay (with reduced fibers). Thus $\mathcal{N}_{\leq i}$ is flat and Cohen–Macaulay over $\mathcal{O}_{F_\pi}$ by the induction hypothesis and by induction $\mathcal{N} \to \mathcal{O}_{F_\pi}$ is flat with (reduced) Cohen–Macaulay fibers. Moreover, $\mathcal{N}$ is normal being smooth in codimension 1. □

Acknowledgements

We would like to thank Fabrizio Andreatta for suggesting to have a look at [14]. We would like to heartily thank Ioannis Zachos for pointing a mistake in a first version of this paper. The authors are part of the project ANR-19-CE40-0015 COLOSS.

References

[1] S. Bijakowski, "The compatibility with the duality for partial Hasse invariants", preprint, 2016. arXiv 1603.06874

[2] S. Bijakowski and V. Hernandez, "Groupes p-divisibles avec condition de Pappas–Rapoport et invariants de Hasse", *J. Éc. polytech. Math.* **4** (2017), 935–972. MR Zbl

[3] S. Bijakowski and V. Hernandez, "On the geometry of the Pappas–Rapoport models for PEL Shimura varieties", *J. Inst. Math. Jussieu* **22**:5 (2023), 2403–2445. MR Zbl

[4] C.-L. Chai and P. Norman, “Singularities of the $\Gamma_0(p)$-level structure”, *J. Algebraic Geom.* **1**:2 (1992), 251–278. MR Zbl

[5] P. Deligne and G. Pappas, “Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant”, *Compositio Math.* **90**:1 (1994), 59–79. MR Zbl

[6] P. Deligne and M. Rapoport, “Les schémas de modules de courbes elliptiques”, pp. 143–316 in *Modular functions of one variable*, *II* (Antwerp, Belgium, 1972), edited by P. Deligne and W. Kuyk, Lecture Notes in Math. **349**, Springer, 1973. MR Zbl

[7] W. Goldring, “Galois representations associated to holomorphic limits of discrete series”, *Compos. Math.* **150**:2 (2014), 191–228. MR Zbl

[8] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs **107**, American Mathematical Society, Providence, RI, 2003. MR Zbl

[9] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies **108**, Princeton University Press, 1985. MR Zbl

[10] N. Krämer, “Local models for ramified unitary groups”, *Abh. Math. Sem. Univ. Hamburg* **73** (2003), 67–80. MR Zbl

[11] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series **36**, Princeton University Press, 2013. MR Zbl

[12] G. Pappas and M. Rapoport, “Local models in the ramified case, I: The EL-case”, *J. Algebraic Geom.* **12**:1 (2003), 107–145. MR Zbl

[13] G. Pappas and M. Rapoport, “Local models in the ramified case, II: Splitting models”, *Duke Math. J.* **127**:2 (2005), 193–250. MR Zbl

[14] X. Shen, C.-F. Yu, and C. Zhang, “EKOR strata for Shimura varieties with parahoric level structure”, *Duke Math. J.* **170**:14 (2021), 3111–3236. MR Zbl

[15] I. Zachos, “Semi-stable models for some unitary Shimura varieties over ramified primes”, preprint, 2022. arXiv 2210.14043

Received May 14, 2024. Revised November 6, 2024.

STÉPHANE BIJAKOWSKI
CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ
ECOLE POLYTECHNIQUE
PALAISEAU
FRANCE

stephane.bijakowski@polytechnique.edu

VALENTIN HERNANDEZ
LABORATOIRE DE MATHÉMATIQUES D’ORSAY
UNIVERSITÉ PARIS-SACLAY
ORSAY
FRANCE

valentin.hernandez@math.cnrs.fr