Pacific Journal of Mathematics

SEVERAL REMARKS ON TENSOR RANK COMPUTATION

Yaroslav Shitov

Volume 334 No. 1 January 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 334, No. 1, 2025
https://doi.org/10.2140/pjm.2025.334.143

SEVERAL REMARKS ON TENSOR RANK COMPUTATION

YAROSLAV SHITOV

The *Waring rank* of a homogeneous polynomial f is the smallest number $\operatorname{wr} f$ for which f is the sum of $\operatorname{wr} f$ powers of linear forms. We show

$$\operatorname{wr}(f \cdot g) \leqslant (\operatorname{wr} f) \cdot (\operatorname{wr} g) \cdot \max\{\deg f, \deg g\} \quad \text{if} \;\; \operatorname{wr} f > 1, \;\operatorname{wr} g > 1$$

and answer a question of Teitler. We also discuss further questions of Koiran and Schaefer regarding the algorithmic computation of tensor ranks.

The problem of tensor decomposition has a long history and growing popularity in the literature [4; 14; 29]. Both the classical version of this problem and its symmetric counterpart (which is also known as the *Waring decomposition*) are hard to solve [13; 30], so, in some special cases or even relatively small particular instances, the question can be attractive and important [2; 32]. This note deals with three questions on the topic regarding, more specifically,

- the Waring rank of the product of polynomials (by Teitler [37]),
- the computation of commuting extensions of matrices (by Koiran [21]),
- the recognition of tensors of bounded rank (by Schaefer and Štefankovič [27] and Schaefer, Cardinal and Miltzow [28]).

Section 1 contains some basics on the topic, and Section 2 gives a general upper bound on Waring ranks of products of polynomials. In Section 3, we switch to algorithmic questions in the nonsymmetric case. Namely, (1) we discuss the proof of the hardness of extending a given family of matrices to a family of diagonalizable commuting matrices, as suggested by Koiran [21], and (2) we revisit standard methods of the recognition of tensors of bounded rank as in the questions of Schaefer and Štefankovič [27] and Schaefer, Cardinal and Miltzow [28], and we further discuss these problems over different fields.

1. Preliminaries

A three-way *tensor* T can be thought of as an $m \times n \times p$ array of coordinates taken in some field $\mathbb{F}$. This tensor T is said to be of the *rank one* if there are vectors

$$(u, v, w) \in \mathbb{F}^m \times \mathbb{F}^n \times \mathbb{F}^p$$

MSC2020: 14N07, 15A27, 15A69.
Keywords: tensor rank, Waring decomposition, algorithms.

such that $T = u \otimes v \otimes w$, that is, one has $T_{ijk} = u_i \cdot v_j \cdot w_k$ for all indexing triples (i, j, k) pointing to entries of T [13]. An $n \times n \times n$ tensor is called *symmetric* if the value T_{ijk} remains the same for any permutation of the indexes (i, j, k).

Definition 1.1 [9; 35]. Let $\mathbb{K} \supseteq \mathbb{F}$ be fields, and let T be a tensor over $\mathbb{F}$. The *rank* $\operatorname{rk}_{\mathbb{K}} T$ is the smallest integer r such that T is the sum of r rank-one tensors over $\mathbb{K}$. If T is symmetric, then the *symmetric rank* $\operatorname{srk}_{\mathbb{K}} T$ is the smallest integer s such that T is a linear combination of s symmetric rank-one tensors over $\mathbb{K}$.

One of the most relevant cases is when $\mathbb{K} = \mathbb{C}$, and, in this case, the notion of symmetric rank is essentially equivalent to that of the *Waring rank*.

Definition 1.2 [4; 23]. The *Waring rank* of a homogeneous polynomial f is the smallest number $\operatorname{wr}(f)$ so that f is the sum of $\operatorname{wr}(f)$ powers of linear forms over $\mathbb{C}$.

Indeed, the Waring rank of f equals the symmetric rank of a tensor representing f, which shows that the corresponding notions are equivalent over $\mathbb{C}$ [4; 23]. The following standard statement is a tensorial analogue of the Gaussian elimination.

Definition 1.3 [13; 21]. Let T be an $n_1 \times n_2 \times n_3$ tensor, let $\chi \in \{1, 2, 3\}$ and $e \in \{1, \dots, n_\chi\}$. The e-th χ*-slice* is the matrix obtained from (T_{ijk}) by removing every coordinate (i, j, k) which does not have the index e at the χ-th position.

Proposition 1.4 (the substitution method [13; 15; 35]). *Let T be an $n_1 \times n_2 \times n_3$ tensor over a field $\mathbb{F}$, let $\chi \in \{1, 2, 3\}$ and $i \in \{1, \dots, n_\chi\}$. If the i-th χ-slice of T is a rank-one matrix m, then* $\operatorname{rk}_{\mathbb{F}} T$ *equals one plus the smallest rank of any tensor obtained by removing the i-th χ-slice of T and subsequent addition of scalar multiples of m to the remaining χ-slices of the resulting tensor.*

We are ready proceed with the main content of the article.

2. The Waring rank of the product

In relation to algorithmic questions on matrix multiplication [10; 29; 34], a notable part of contemporary work deals with the behavior of ranks under basic operations such as the tensor product [6], Kronecker product [8], direct sum [29; 31] and many others [7; 30]. The symmetric counterpart of the problem involves polynomials, and the natural algebraic operations over them are the sum and product. Indeed, the Waring ranks of the sums are widely studied [12; 24; 36], and a particular well-known conjecture posited the additivity of the Waring rank under the sum of polynomials with disjoint variable families [4], but it turned out to be false [33].

How does the Waring rank behave under the product? This question does not seem to have attracted any amount of attention comparable to its analogue with the sum, but the following well-known result can be a good starting point.

Theorem 2.1 [4, Proposition 3.1]. *If $a_1 \leqslant \cdots \leqslant a_k$ are positive integers, then*

$$\mathrm{wr}((x_1)^{a_1} \cdot (x_2)^{a_2} \cdots (x_k)^{a_k}) = (a_2+1)(a_3+1)\cdots(a_k+1).$$

Later on, a MathOverflow user, SMD, proposed a question as to whether the Waring rank is a submultiplicative function with respect to the product of polynomials [37]. Using the above theorem, Teitler demonstrated the upper bound

$$\mathrm{wr}(f \cdot g) \leqslant (\mathrm{wr}\, f) \cdot (\mathrm{wr}\, g) \cdot \max\{\deg f + 1, \deg g + 1\} \tag{2-1}$$

and gave several further examples on the same web page [37]. In particular, one has $\mathrm{wr}(x^d) = \mathrm{wr}(y^e) = 1$ and $\mathrm{wr}(x^d y^e) = \max\{d+1, e+1\}$, which shows that wr is not a submultiplicative function with respect to the product, and, in addition, the bound (2-1) can be tight in some cases [37]. However, $\mathrm{wr}(xy) = \mathrm{wr}(uv) = 2$ and

$$\mathrm{wr}(xyuv) = 8 < \mathrm{wr}(xy) \cdot \mathrm{wr}(uv) \cdot \max\{\deg(xy)+1, \deg(uv)+1\} = 2 \cdot 2 \cdot 3 = 12,$$

so (2-1) can be strict [37]. These considerations suggest the following.

Question 2.2 (by Teitler [37]). Let f and g be homogeneous polynomials such that there is equality in (2-1). Does it follow that either $\mathrm{wr}(f) = 1$ or $\mathrm{wr}(g) = 1$?

In this section, we improve the bound (2-1) and give a positive answer to Question 2.2. The following important statement was suggested in an anonymous review report. In fact, this is a version of the well-known *apolarity lemma* of Iarrobino and Kanev [16] as adapted for applications with binary polynomials.

Lemma 2.3 (see [16]). *The Waring rank of a binary polynomial $m(x, y)$ equals the smallest number k of pairwise noncollinear differentiation operators $(h_1, \ldots, h_k)$ defined as $h_i = a_i\, \partial/\partial x + b_i\, \partial/\partial y$ such that $h_1 \circ h_2 \circ \cdots \circ h_k(m) = 0$.*

This statement has an additional benefit of being sufficiently elementary, so we believe that the omission of its well-known proof is not an obstacle even for those interested readers who had no previous knowledge of apolarity theory. We also get a further lemma that appeared in the first draft of the article with a long proof. As pointed out in review, several important special cases of the below lemma are consequences of known results, including Proposition 7.2 in [23]. Another relevant case $\alpha \neq 0$ in (2-2) below appears in [24, Theorem 3.1]. However, it seems natural to consider the corresponding cases (2-2) and (2-3) as follows.

Lemma 2.4. *If x, y are variables, and $d > e$ are positive integers, then*

$$\mathrm{wr}(x^d y^e + \alpha y^{d+e}) \leqslant d, \tag{2-2}$$

$$\mathrm{wr}(\beta x^{2d} + x^d y^d + \gamma y^{2d}) \leqslant d \tag{2-3}$$

hold for some $\alpha, \beta, \gamma \in \mathbb{C}$.

The following corollary is now immediate.

Corollary 2.5. *If x, y are variables, and $d \geqslant e$ are positive integers, then there exists $\pi = \pi(d, e) \in \mathbb{C}$ such that, for all $\alpha, \beta \in \mathbb{C}$ with $\beta \neq 0$ and $\alpha\beta = \pi$, one has*

$$\operatorname{wr}(\alpha x^{d+e} + x^d y^e + \beta y^{d+e}) \leqslant d.$$

Proof. This follows from Lemma 2.4 by the substitutions $(x, y) \to (\lambda x, y)$. In particular, the case $\pi = 0$ corresponds to (2-2) and an option $\beta\gamma = 0$ in (2-3). ☐

We give a Waring decomposition of a polynomial related to Question 2.2.

Lemma 2.6. *Let d, e, m, n be positive integers such that $d \geqslant e$ and $\min\{m, n\} \geqslant 2$. If $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ are disjoint variable families, then*

$$\operatorname{wr}\big(((x_1)^d + \cdots + (x_m)^d) \cdot ((y_1)^e + \cdots + (y_n)^e)\big) \leqslant dmn. \tag{2-4}$$

Proof. As per Corollary 2.5, the possible cases are $\pi(d, e) = 0$ and $\pi(d, e) \neq 0$.

If $\pi(d, e) = 0$, we take a generic family of complex numbers (α_{ij}) satisfying

$$\alpha_{1j} + \alpha_{2j} + \cdots + \alpha_{mj} = 0, \tag{2-5}$$

for all $j \in \{1, \dots, n\}$. Since $m \geqslant 2$, the numbers (α_{ij}) are all nonzero, and hence

$$\operatorname{wr}((x_i)^d (y_j)^e + \alpha_{ij} (y_j)^{d+e}) \leqslant d \tag{2-6}$$

is valid by Corollary 2.5. According to (2-5), the polynomial in (2-4) equals the sum of all polynomials in (2-6) over $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$.

In the case $\pi(d, e) = \pi \neq 0$, we take $A, B \in \mathbb{C}$ so that $A \cdot B = \pi$, and we define

$$\alpha_{\hat{\imath}\hat{\jmath}} = A, \quad \alpha_{\hat{\imath}n} = A(1-n), \quad \alpha_{m\hat{\jmath}} = \frac{A}{1-m}, \qquad \alpha_{mn} = \frac{A(1-n)}{1-m},$$
$$\beta_{\hat{\imath}\hat{\jmath}} = B, \quad \beta_{\hat{\imath}n} = \frac{B}{1-n}, \qquad \beta_{m\hat{\jmath}} = B(1-m), \quad \beta_{mn} = \frac{B(1-m)}{1-n}$$

with $\hat{\imath} \in \{1, \dots, m-1\}$ and $\hat{\jmath} \in \{1, \dots, n-1\}$. Using Corollary 2.5, we get

$$\operatorname{wr}(\alpha_{ij}(x_i)^{d+e} + (x_i)^d (y_j)^e + \beta_{ij}(y_j)^{d+e}) \leqslant d, \tag{2-7}$$

with $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$. As in the case $\pi(d, e) = 0$, the polynomial in (2-4) is the sum of all polynomials in (2-7), so the desired bound follows. ☐

Remark 2.7. The case $\min\{m, n\} = 1$ of Lemma 2.6 is not needed in our main results, but we refer to [5, Lemma 4.5(3)] for the corresponding computation.

We need one easy statement to proceed with the main result of the section.

Observation 2.8. If $(z_1, \dots, z_t)$ and $(\zeta_1, \dots, \zeta_\tau)$ are disjoint variable families, and ψ is a linear mapping $z_1\mathbb{C} + \cdots + z_t\mathbb{C} \to \zeta_1\mathbb{C} + \cdots + \zeta_\tau\mathbb{C}$, then, for any homogenous polynomial $f(z_1, \dots, z_t)$, one has $\operatorname{wr} f(\psi(z_1), \dots, \psi(z_t)) \leqslant \operatorname{wr} f(z_1, \dots, z_t)$.

Proof. This is well known: If $f(z_1, \dots, z_t) = g_1(z_1, \dots, z_t) + \dots + g_k(z_1, \dots, z_t)$, then

$$f(\psi(z_1), \dots, \psi(z_t)) = g_1(\psi(z_1), \dots, \psi(z_t)) + \dots + g_k(\psi(z_1), \dots, \psi(z_t)),$$

and, if $g_i(z_1, \dots, z_t)$ is a power of a linear form, then so is $g_i(\psi(z_1), \dots, \psi(z_t))$. □

Theorem 2.9. *If* f, g *are homogeneous polynomials with* $\mathrm{wr}(f) \geqslant 2$ *and* $\mathrm{wr}(g) \geqslant 2$, *then* $\mathrm{wr}(f \cdot g) \leqslant (\mathrm{wr}\, f) \cdot (\mathrm{wr}\, g) \cdot \max\{\deg f, \deg g\}$.

Proof. We can assume $\deg f = d \geqslant e = \deg g$. If $\mathrm{wr}(f) = m$, $\mathrm{wr}(g) = n$, then

$$f = (\ell_1(\zeta_1, \dots, \zeta_k))^d + \dots + (\ell_m(\zeta_1, \dots, \zeta_k))^d$$

and

$$g = (\ell_{m+1}(\zeta_1, \dots, \zeta_k))^e + \dots + (\ell_{m+n}(\zeta_1, \dots, \zeta_k))^e$$

with some linear forms $(\ell_1, \dots, \ell_{m+n})$. We define the linear mapping ψ

$$(x_1\mathbb{C} + \dots + x_m\mathbb{C}) + (y_1\mathbb{C} + \dots + y_n\mathbb{C}) \to \zeta_1\mathbb{C} + \dots + \zeta_k\mathbb{C}$$

with the formulas $x_i \to \ell_i$ and $y_j \to \ell_{m+j}$ with $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, and the result follows from Lemma 2.6 and Observation 2.8. □

Indeed, Theorem 2.9 gives a substantial improvement on (2-1) in the relevant case $\mathrm{wr}(f) \neq 1$, $\mathrm{wr}(g) \neq 1$, and it indicates a positive resolution of Question 2.2.

3. Two algorithmic questions

A family $Z = (Z_1, \dots, Z_p)$ of $n \times n$ matrices is said to be a *commuting extension* of a family $A = (A_1, \dots, A_p)$ of $m \times m$ matrices if $n \geqslant m$ and, in addition,

- A_t is the upper left submatrix of Z_t with any $t \in \{1, \dots, p\}$,
- the matrices in Z pairwise commute [22].

In algebraic complexity, this notion is related to works of Strassen [35] and further recent studies of Koiran [20; 21; 22]. Commuting extensions also appear in linear algebra [19], numerical analysis [11], quantum physics [3] and other disciplines [22].

Problem 3.1 [20, Problem 1]. Does a given family of $n \times n$ matrices admit a commuting extension of a given size $r \times r$?

In fact, the works of Koiran [20; 21; 22] are mostly focused on the variation of Problem 3.1 that takes an additional requirement that the matrices in the resulting commuting extension should be diagonalizable [20, page 3]. He poses the question of the algorithmic complexity of both these versions and presumes that the problem is NP-hard; see [22, page 4] and [21, pages 8 and 33]. Indeed, Theorem 11 in [21] states that, for an arbitrary infinite ground field $\mathbb{K}$ and any $n \times n \times p$ tensor Q having the invertible 3-slice as in Example 3.3 below, the rank $\mathrm{rk}_{\mathbb{K}}\, Q$ equals the smallest order of the matrices in any commuting extension Z of the 3-slices of Q over $\mathbb{K}$ such that Z has diagonalizable matrices only. Koiran suggests the following question.

Question 3.2 [21, pages 8 and 33]. Is it NP-hard to compute the rank of an $n\times n\times p$ tensor that is promised to have at least one invertible slice?

The property of having an invertible generic 3-slice is a feature of the tensors in the known reductions in [27] and in [30], so the answer to Question 3.2 is positive. The reduction of Håstad [13] may not have this feature as it uses $m\times n\times k$ tensors with different m, n, k, but the following reduction is valid in the general case.

Example 3.3. Let T be an $m\times n\times k$ tensor with entries in a field $\mathbb{F}$. We define the $(m+n)\times(m+n)\times(k+1)$ tensor $Q(T)$ which has one of the 3-slices equal to the unity matrix, and the k remaining 3-slices of $Q(T)$ have the form

$$q_i = \begin{pmatrix} O_{m\times m} & t_i \\ O_{n\times m} & O_{n\times n} \end{pmatrix}$$

with $(t_1, \dots, t_k)$ being the family of all 3-slices of T. Then

$$\operatorname{rk}_{\mathbb{F}} Q(T) = \operatorname{rk}_{\mathbb{F}} T + m + n.$$

Proof. Subsequently use Proposition 1.4 with the 1-slice substitutions involving the first m slices and the 2-slice substitutions with the last n ones. These transformations leave the initial tensor untouched, and hence we get a lower bound $\operatorname{rk}_{\mathbb{F}} Q(T) \geqslant \operatorname{rk}_{\mathbb{F}} T + m + n$. The corresponding upper bound $\operatorname{rk}_{\mathbb{F}} Q(T) \leqslant \operatorname{rk}_{\mathbb{F}} T + m + n$ is immediate because the removal of the slice equal to the $(m+n)\times(m+n)$ unity matrix gives a copy of T yet again. □

Clearly, one of the 3-slices of $Q(T)$ is the unity matrix, and hence $Q(T)$ has an invertible 3-slice. Therefore, Example 3.3 is a reduction from the general tensor rank computation to the version of it in which one of the slices has to be invertible. Therefore, we get an easier resolution of Question 3.2, and, in addition, Example 3.3 underlines the difference between Problem 3.1 and its version with diagonalizable matrices. Indeed, the 3-slices of $Q(T)$ commute pairwise, which means that the order of their smallest commuting extension is $m+n$. However, with an additional assumption that the matrices in the extension are diagonalizable, the smallest order can grow much faster as the maximal rank of an $n\times n\times n$ tensor is $\Theta(n^2)$; see [35].

We conclude with comments regarding the approach of [30].

Remark 3.4. In several works with different coauthors, Schaefer repeats the question on the existence of a polynomial time algorithm detecting $m\times n\times k$ tensors of the rank at most r, for any fixed value of r [27; 28]. Of course, this question is equivalent to a possibility of polynomial time detection of $(r+1)\times(r+1)\times(r+1)$ tensors of the rank at most r over the same ground field; see also [1]. (Indeed, it is immediate that, for every $\chi \in \{1, 2, 3\}$, the χ-slices of a tensor of the rank at most r should span the subspace of the dimension at most r [1], and hence the standard Gaussian elimination either shows that the rank of the initial tensor is at least $r+1$ or reduces the consideration to the $(r+1)\times(r+1)\times(r+1)$ blocks of that initial tensor. Moreover, if the arithmetic operations can be performed in the initial field $\mathbb{F}$ as in Definition 1.1 in polynomial time, then the same Gaussian elimination would allow one to find the appropriate $(r+1)\times(r+1)\times(r+1)$ block in polynomial time even if r was not fixed, but, in any case, if r is fixed, then the number of all $(r+1)\times(r+1)\times(r+1)$ blocks is polynomial.) Since the standard algorithms of quantifier elimination halt in time polynomial in the size of the data times a function of r [25], we get that the problem of detecting rational tensors with real or complex rank at most r is fixed-parameter tractable. In fields different from $\mathbb{R}$ and $\mathbb{C}$, the possibility of polynomial time recognition of tensors of the rank at most r may depend on the initial choice of r. Indeed, for every ground field $\mathbb{F}$ allowing the algorithmic computation of the arithmetic operations and for every system S of polynomial equations over $\mathbb{F}$, there is an algorithm that constructs a tensor $T(S)$ and an integer $r(S)$ so that S admits a solution over $\mathbb{F}$ if and only if $\operatorname{rk}_{\mathbb{F}} T(S) \leqslant r(S)$ [30]. For instance, as the field $\overline{\mathbb{Q}}(t_1, t_2)$ has the undecidable Diophantine theory [18], we have that, for some sufficiently large fixed r, there is no algorithm that detects tensors of the rank at most r with respect to $\overline{\mathbb{Q}}(t_1, t_2)$.

A review report contained a further question about particular values of r for which the problem of detecting tensors of the rank at most r is undecidable. The answer depends on the choice of the ground field, and, as explained above, in the most standard setting over $\mathbb{R}$ or $\mathbb{C}$, the problem is decidable and even fixed-parameter tractable. Concerning Diophantine equations, the most well-studied case is the ring of integers $\mathbb{Z}$, and, in the answer section of [26] with a further reference to an older article [17], Vladimir Reshetnikov gives an explicit family of polynomials with integer coefficients which has bounded number of variables and total degree so that the lack of their solutions cannot be tested by any algorithm. Using the reduction in Section 4 in [30], one can proceed with an explicit $r \in \mathbb{Z}$ and tensor T so that T can have rank greater than r over $\mathbb{Z}$ (which means that T cannot be written as the sum of r decomposable tensors with integer entries), but this property cannot be tested by any algorithm. In addition, the reduction in [30] is explicit and polynomial, and one can expect a resulting value of the order such as $r \sim 10^{10}$ or smaller. On the other hand, for small values such as $r = 1$ or $r = 2$, the algorithms of checking whether a given tensor has the rank at most r over $\mathbb{Z}$ are immediate.

Acknowledgment

I thank the author of the review report. It was really helpful.

References

[1] M. Aliabadi and S. Friedland, "On the complexity of finding tensor ranks", *Commun. Appl. Math. Comput.* **3**:2 (2021), 281–289. MR Zbl

[2] M. Bläser, "On the complexity of the multiplication of matrices of small formats", *J. Complexity* **19**:1 (2003), 43–60. MR Zbl

[3] D. K. Burgarth, P. Facchi, V. Giovannetti, H. Nakazato, S. Pascazio, and K. Yuasa, "Exponential rise of dynamical complexity in quantum computing through projections", *Nat. Commun.* **5** (2014), art. id. 5173.

[4] E. Carlini, M. V. Catalisano, and A. V. Geramita, "The solution to the Waring problem for monomials and the sum of coprime monomials", *J. Algebra* **370** (2012), 5–14. MR Zbl

[5] E. Carlini, M. V. Catalisano, and A. Oneto, "Waring loci and the Strassen conjecture", *Adv. Math.* **314** (2017), 630–662. MR Zbl

[6] L. Chen and S. Friedland, "The tensor rank of tensor product of two three-qubit W states is eight", *Linear Algebra Appl.* **543** (2018), 1–16. MR Zbl

[7] M. Christandl and J. Zuiddam, "Tensor surgery and tensor rank", *Comput. Complexity* **28**:1 (2019), 27–56. MR Zbl

[8] M. Christandl, A. K. Jensen, and J. Zuiddam, "Tensor rank is not multiplicative under the tensor product", *Linear Algebra Appl.* **543** (2018), 125–139. MR Zbl

[9] P. Comon, G. Golub, L.-H. Lim, and B. Mourrain, "Symmetric tensors and symmetric tensor rank", *SIAM J. Matrix Anal. Appl.* **30**:3 (2008), 1254–1279. MR Zbl

[10] D. Coppersmith and S. Winograd, "Matrix multiplication via arithmetic progressions", *J. Symbolic Comput.* **9**:3 (1990), 251–280. MR Zbl

[11] I. Degani, J. Schiff, and D. J. Tannor, "Commuting extensions and cubature formulae", *Numer. Math.* **101**:3 (2005), 479–500. MR Zbl

[12] M. Fedorchuk, "Direct sum decomposability of polynomials and factorization of associated forms", *Proc. Lond. Math. Soc.* (3) **120**:3 (2020), 305–327. MR Zbl

[13] J. Håstad, "Tensor rank is NP-complete", *J. Algorithms* **11**:4 (1990), 644–654. MR Zbl

[14] C. J. Hillar and L.-H. Lim, "Most tensor problems are NP-hard", *J. ACM* **60**:6 (2013), art. id. 45. MR Zbl

[15] J. E. Hopcroft and L. R. Kerr, "On minimizing the number of multiplications necessary for matrix multiplication", *SIAM J. Appl. Math.* **20** (1971), 30–36. MR Zbl

[16] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics **1721**, Springer, 1999. MR Zbl

[17] J. P. Jones, "Universal Diophantine equation", *J. Symbolic Logic* **47**:3 (1982), 549–571. MR Zbl

[18] K. H. Kim and F. W. Roush, "Diophantine undecidability of $\mathbb{C}(t_1, t_2)$", *J. Algebra* **150**:1 (1992), 35–44. MR Zbl

[19] D. P. Kimsey and H. J. Woerdeman, "Minimal normal and commuting completions", *Int. J. Inf. Syst. Sci.* **4**:1 (2008), 50–59. MR Zbl

[20] P. Koiran, "On tensor rank and commuting matrices", preprint, 2020. arXiv 2006.02374

[21] P. Koiran, "An efficient uniqueness theorem for overcomplete tensor decomposition", preprint, 2024. arXiv 2404.07801

[22] P. Koiran, "On the uniqueness and computation of commuting extensions", preprint, 2024. arXiv 2401.01302

[23] J. M. Landsberg and Z. Teitler, "On the ranks and border ranks of symmetric tensors", *Found. Comput. Math.* **10**:3 (2010), 339–366. MR Zbl

[24] L. Brustenga i Moncusí and S. K. Masuti, "The Waring rank of binary binomial forms", *Pacific J. Math.* **313**:2 (2021), 327–342. MR Zbl

[25] J. Renegar, "On the computational complexity and geometry of the first-order theory of the reals, III: Quantifier elimination", *J. Symbolic Comput.* **13**:3 (1992), 329–352. MR Zbl

[26] V. Reshetnikov, Is the set of undecidable problems decidable?, November 2011, available at https://mathoverflow.net/q/82140.

[27] M. Schaefer and D. Štefankovič, "The complexity of tensor rank", *Theory Comput. Syst.* **62**:5 (2018), 1161–1174. MR Zbl

[28] M. Schaefer, J. Cardinal, and T. Miltzow, "The existential theory of the reals as a complexity class: a compendium", preprint, 2024. arXiv 2407.18006

[29] A. Schönhage, "Partial and total matrix multiplication", *SIAM J. Comput.* **10**:3 (1981), 434–455. MR Zbl

[30] Y. Shitov, "How hard is the tensor rank?", preprint, 2016. arXiv 1611.01559

[31] Y. Shitov, "Counterexamples to Strassen's direct sum conjecture", *Acta Math.* **222**:2 (2019), 363–379. MR Zbl

[32] Y. Shitov, "The Waring rank of the 3×3 permanent", *SIAM J. Appl. Algebra Geom.* **5**:4 (2021), 701–714. MR Zbl

[33] Y. Shitov, "Higher rank substitutions for tensor decompositions, I: Direct sum conjectures", preprint, 2023, available at vixra.org/abs/2307.0100.

[34] V. Strassen, "Gaussian elimination is not optimal", *Numer. Math.* **13** (1969), 354–356. MR Zbl

[35] V. Strassen, "Rank and optimal computation of generic tensors", *Linear Algebra Appl.* **52/53** (1983), 645–685. MR Zbl

[36] Z. Teitler, "Sufficient conditions for Strassen's additivity conjecture", *Illinois J. Math.* **59**:4 (2015), 1071–1085. MR Zbl

[37] Z. Teitler, Reply to "Is the Waring rank homogeneous polynomials sub-multiplicative?", January 2018, available at https://mathoverflow.net/q/291044.

Received October 3, 2024. Revised January 5, 2025.

YAROSLAV SHITOV
MOSCOW
RUSSIA
yaroslav-shitov@yandex.ru