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We prove that various morphisms related to the six Grothendieck operations on sheaves become isomorphisms when restricted to (weakly) constructible sheaves. To this end, we first study some properties of weakly cohomologically constructible sheaves. We then deduce several compatibilities of the six operations in the context of (weakly) $\mathbb{R}$-constructible sheaves.

1. Introduction

Let $\boldsymbol{k}$ be a commutative unital ring of finite global dimension and $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ be the bounded derived category of sheaves on a good topological space X. There are some classical morphisms which are, in general, not isomorphisms, such as

$$\mathrm{R}\mathscr{H}om(F_1, F_2) \overset{\mathrm{L}}{\otimes} F_3 \to \mathrm{R}\mathscr{H}om(F_1, F_2 \overset{\mathrm{L}}{\otimes} F_3),$$
$$f^!G_1 \overset{\mathrm{L}}{\otimes} f^{-1}G_2 \to f^!(G_1 \overset{\mathrm{L}}{\otimes} G_2)$$

for $F_1, F_2, F_3 \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$, $G_1, G_2 \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ and $f : X \to Y$ a continuous map. We will prove here that, under suitable hypotheses of (weak) constructibility, these morphisms become isomorphisms.

We prove the following results in the categories of real analytic manifolds and weakly $\mathbb{R}$-constructible sheaves (see Theorems 4.3, 4.5 and 4.8).

Suppose that $f : X \to Y$ is a morphism of real analytic manifolds. Also let $F_1, F_2, K \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$, $G, M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$ with F_1 being $\mathbb{R}$-constructible and K, M locally constant. Then we have the following isomorphisms:

$$f^!G \otimes f^{-1}M \xrightarrow{\sim} f^!(G \otimes M),$$
$$f^{-1}\mathrm{R}\mathscr{H}om(M, G) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(f^{-1}M, f^{-1}G),$$
$$\mathrm{R}\mathscr{H}om(F_1, F_2) \otimes K \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(F_1, F_2 \otimes K),$$
$$\mathrm{R}\mathscr{H}om(K, F_2) \otimes F_1 \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(K, F_2 \otimes F_1).$$

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For direct images, we need slightly stronger assumptions: Consider a morphism $f : X_\infty \to Y_\infty$ of b-analytic manifolds (see [4] for this notion) and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$ be weakly $\mathbb{R}$-constructible up to infinity and M be locally constant. We prove the following isomorphisms:

$$\mathrm{R}f_* F \otimes M \xrightarrow{\sim} \mathrm{R}f_*(F \otimes f^{-1}M),$$
$$\mathrm{R}f_! \,\mathrm{R}\mathscr{H}om(f^{-1}M, F) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(M, \mathrm{R}f_! F).$$

We start by introducing the notion (implicitly already defined in [2, Section 3.4]) of weakly cohomologically constructible sheaves and the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$ of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ consisting of such objects. On a real analytic manifold, the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$ contains the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ of weakly $\mathbb{R}$-constructible objects.

We prove first that $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$ is triangulated. Then our main tool is that for an object F of this category, for $x \in X$ and $L \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$, one has functorial isomorphisms

$$\mathrm{R}\mathscr{H}om(L_X, F)_x \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(L, F_x), \quad \mathrm{R}\Gamma_x F \otimes L \xrightarrow{\sim} \mathrm{R}\Gamma_x(F \otimes L_X),$$

where L_X denotes the constant sheaf associated with L.

The motivation for this note came through the work [1], where field extensions for sheaves are considered: In this context, given a field extension $\boldsymbol{k} \subset \boldsymbol{l}$, there are natural functors of extension and coextension of scalars from $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ to $\mathrm{D}^{\mathrm{b}}(\boldsymbol{l}_X)$, which are given by $F \mapsto F \otimes \boldsymbol{l}_X$ and $F \mapsto \mathrm{R}\mathscr{H}om(\boldsymbol{l}_X, F)$, respectively. It is important to study the question of compatibility of these functors with the six Grothendieck operations on constructible sheaves. It turns out that many of the desired functorialities of [1] indeed follow from the setup developed here, but our approach is more general in multiple aspects: We allow $\boldsymbol{k}$ to be a (suitable) commutative ring, we only require some of the sheaves to be *weakly* constructible, and we replace $\boldsymbol{l}_X$ by any locally constant sheaf.

2. Preliminaries

Throughout the paper, we work in a given universe $\mathcal{U}$. All limits and colimits (in particular, products and direct sums) are assumed to be small. Recall that $\boldsymbol{k}$ is a commutative unital ring of finite global dimension. We assume that all topological spaces are "good", that is, Hausdorff, locally compact, countable at infinity and of finite flabby dimension.

For a topological space X as above, one denotes by $\mathrm{Mod}(\boldsymbol{k}_X)$ the Grothendieck abelian category of sheaves of $\boldsymbol{k}$-modules and by $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ its bounded derived category.

We mainly follow the notations of [2]. In particular:

- ω_X denotes the dualizing complex and D_X the duality functor $\mathrm{R}\mathscr{H}om(\,\cdot\,, \omega_X)$.
- D denotes the duality functor on $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$.

• $a_X : X \to \{\mathrm{pt}\}$ denotes the unique map from X to a one-point space. Hence, for $F \in \mathrm{D^b}(\boldsymbol{k}_X)$, one has $\mathrm{R}\Gamma(X; F) \simeq \mathrm{R}a_{X*}F$.

• For $L \in \mathrm{D^b}(\boldsymbol{k})$, and for Z locally closed in X, one denotes by L_{XZ} the constant sheaf L_Z on Z extended by 0 on $X \setminus Z$. When Z is closed, we shall often simply denote by L_Z the sheaf L_{XZ}.

• For $x \in X$, denoting by $i_x : \{x\} \hookrightarrow X$ the embedding, and for $F \in \mathrm{D^b}(\boldsymbol{k}_X)$, one denotes as usual by $F_x = i_x^{-1}F$ its stalk at x. One also sets $\mathrm{R}\Gamma_x F = i_x^! F$. (We sometimes identify $i_x^{-1}F$ and $i_{x*}i_x^{-1}F$ as well as $i_x^! F$ and $i_{x*}\mathrm{R}\Gamma_x F$.)

• We often denote by K a locally constant sheaf on X and by M a locally constant sheaf on Y. As already mentioned, if $L \in \mathrm{D^b}(\boldsymbol{k})$, we denote by L_X or L_Y the associated constant sheaf on X or Y.

• Recall (see [2, Exercise I.30]) that $L \in \mathrm{D^b}(\boldsymbol{k})$ is *perfect* if it is isomorphic to a bounded complex of finitely generated projective $\boldsymbol{k}$-modules. If L is perfect, then so is $\mathrm{D}(L)$ and the morphism $L \to \mathrm{DD}(L)$ is an isomorphism. We shall denote by $\mathrm{D}^{\mathrm{b}}_f(\boldsymbol{k})$ the full triangulated category of $\mathrm{D^b}(\boldsymbol{k})$ consisting of perfect objects.

Ind-objects. We shall make use of ind-objects; see [2, Section 1.11] for a short exposition. For a more detailed study, including new results that we shall use here, see [3, Chapter 6, Section 8.6 and Chapter 15]. Let us recall a few facts that we need, skipping some delicate questions of universes.

If $\mathscr{C}$ is a category, one denotes by $\mathrm{Ind}(\mathscr{C})$ the category of ind-objects of $\mathscr{C}$, a full subcategory of the category $\mathscr{C}^\wedge$ of functors from $\mathscr{C}^{\mathrm{op}}$ to *Set*. Recall from [3, Section 6.1] that:

- The natural functor $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is fully faithful.
- The category $\mathrm{Ind}(\mathscr{C})$ admits small filtrant colimits, denoted by "colim".
- Let $\mathscr{I}$ be a small and filtrant category and $\alpha : \mathscr{I} \to \mathscr{C}$ a functor. Let $T : \mathscr{C} \to \mathscr{C}'$ be a functor. Then $T(\text{“colim”}\,\alpha) \simeq \text{“colim”}(T \circ \alpha)$.

Now we assume that $\mathscr{C}$ is abelian. Recall from [3, Theorem 8.6.5] that:

- The category $\mathrm{Ind}(\mathscr{C})$ is abelian and the fully faithful functor $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ is exact.
- Small filtrant colimits are exact in $\mathrm{Ind}(\mathscr{C})$.

One should be aware that even if $\mathscr{C}$ is a Grothendieck category, $\mathrm{Ind}(\mathscr{C})$ does not admit enough injectives in general.

Let $\mathscr{I}$ be a small category and $\alpha : \mathscr{I} \to \mathscr{C}$ be a functor. As already mentioned, one denotes by "colim" α its colimit in $\mathrm{Ind}(\mathscr{C})$. Note that if $\mathscr{C}$ admits colimits, denoted by colim, there is a natural morphism in $\mathrm{Ind}(\mathscr{C})$:

$$\text{“colim”}\,\alpha \to \operatorname{colim}\alpha.$$

But this morphism is not an isomorphism in general. However:

- If "colim" α belongs to $\mathscr{C}$, then "colim" $\alpha \xrightarrow{\sim} \operatorname{colim} \alpha$ (see [2, Corolarry 1.11.7]). In this case, if $T : \mathscr{C} \to \mathscr{C}'$ is a functor, then "colim"$(T \circ \alpha) \xrightarrow{\sim} T(\operatorname{colim} \alpha)$. Therefore, "colim"$(T \circ \alpha)$ belongs to $\mathscr{C}'$ and hence is isomorphic to $\operatorname{colim}(T \circ \alpha)$.

3. Weakly cohomologically constructible sheaves

We consider here a slight generalization of the notion of cohomologically constructible sheaves (see [2, Definition 3.4.1]).

Definition 3.1. Let $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$. We say that F is *weakly cohomologically constructible* if for all $x \in X$, one has the isomorphisms

$$\text{“}\operatorname*{colim}_{x \in U}\text{”}\, \mathrm{R}\Gamma(U; F) \xrightarrow{\sim} F_x, \quad \mathrm{R}\Gamma_x F \xrightarrow{\sim} \text{“}\lim_{x \in U}\text{”}\, \mathrm{R}\Gamma_c(U; F).$$

We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ consisting of weakly cohomologically constructible objects.

Recall that F is cohomologically constructible if F is weakly cohomologically constructible and moreover F_x and $\mathrm{R}\Gamma_x F$ are perfect objects of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$ for all $x \in X$. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\boldsymbol{k}_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$ consisting of cohomologically constructible objects.

Remark 3.2. As explained in [2, Remark 4.3.2], the isomorphisms in Definition 3.1 hold as soon as the objects $\text{“}\operatorname*{colim}_{x \in U}\text{”}\, \mathrm{R}\Gamma(U; F)$ and $\text{“}\lim_{x \in U}\text{”}\, \mathrm{R}\Gamma_c(U; F)$ are representable.

Proposition 3.3. *The category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$ *is triangulated.*

Proof. (i) Remark first that for $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$, $x \in X$ and $j \in \mathbb{Z}$, one has

$$\text{“}\operatorname*{colim}_{x \in U}\text{”}\, H^j(U; F) \xrightarrow{\sim} H^j(F)_x.$$

(ii) Clearly, if $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$, then so is the shifted sheaf $F[j]$ for $j \in \mathbb{Z}$.

(iii) Consider a distinguished triangle $F' \to F \to F'' \xrightarrow{+1}$ in $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ and assume that $F', F'' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$. Let $x \in X$ and let U be an open neighborhood of x. We get the morphism of long exact sequences in the abelian category $\mathrm{Mod}(\boldsymbol{k})$:

$$\begin{array}{ccccccccc} \cdots \to & H^j(U; F') & \to & H^j(U; F) & \to & H^j(U; F'') & \to & H^{j+1}(U; F') & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H^j(F')_x & \longrightarrow & H^j(F)_x & \longrightarrow & H^j(F'')_x & \longrightarrow & H^{j+1}(F')_x & \longrightarrow \cdots \end{array} \tag{3-1}$$

Applying the functor $\text{“colim”}_{x\in U}$ and using [3, Theorem 8.6.5], the first line gives rise to the long exact sequence in the abelian category Ind(Mod($\boldsymbol{k}$)):

$$\cdots \to H^j(F')_x \to \underset{x\in U}{\text{“colim”}}\, H^j(U; F) \to H^j(F'')_x \to H^{j+1}(F')_x \to \cdots$$

We apply [3, Lemma 15.4.6], following its notations, to the category $\mathscr{C} = \mathrm{Mod}(\boldsymbol{k})$. Consider the morphism

$$\varphi : \underset{x\in U}{\text{“colim”}}\, \mathrm{R}\Gamma(U; F) \to F_x.$$

It follows from (3-1) that $IH^j(\varphi)$ is an isomorphism for all $j \in \mathbb{Z}$ and therefore φ is an isomorphism by [3, Lemma 15.4.6].

(iv) The proof for $\mathrm{R}\Gamma_x F$ is the same and we do not repeat it. □

Proposition 3.4 (see [2, Proposition 3.4.3]). *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$. Then we have $\mathrm{D}_X F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$. Moreover, one has the isomorphisms $\mathrm{R}\Gamma_x \mathrm{D}_X F \simeq \mathrm{D}(F_x)$ and $(\mathrm{D}_X F)_x \simeq \mathrm{D}(\mathrm{R}\Gamma_x F)$.*

The case of $\mathrm{D}^{\mathrm{b}}_{\mathrm{cc}}(\boldsymbol{k}_X)$ is treated in [2, Proposition 3.4.3]. When reading carefully the proof, one checks that the hypotheses that F_x and $\mathrm{R}\Gamma_x F$ are perfect are not used in the proof of the statement of Proposition 3.4. Hence, the proof given in the same work indeed already proves this more general statement and we shall not repeat the argument here.

Proposition 3.5. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$, $L \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$ and let $x \in X$. Then $F \overset{\mathrm{L}}{\otimes} L_X$ and $\mathrm{R}\mathscr{H}om(L_X, F)$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$. Moreover, one has the isomorphisms*

$$\mathrm{R}\mathscr{H}om(L_X, F)_x \xrightarrow{\sim} \mathrm{RHom}(L, F_x), \quad (\mathrm{R}\Gamma_x F) \overset{\mathrm{L}}{\otimes} L \xrightarrow{\sim} \mathrm{R}\Gamma_x(F \otimes L_X),$$

Proof. (i) One has

$$\begin{aligned}\mathrm{RHom}(L, F_x) &\simeq \mathrm{RHom}\Big(L, \underset{x\in U}{\text{“colim”}}\, \mathrm{R}\Gamma(U; F)\Big)\\ &\simeq \underset{x\in U}{\text{“colim”}}\, \mathrm{RHom}(L, \mathrm{R}\Gamma(U; F))\\ &\simeq \underset{x\in U}{\text{“colim”}}\, \mathrm{R}\Gamma(U; \mathrm{R}\mathscr{H}om(L_X, F)).\end{aligned}$$

The last isomorphism follows from

$$\mathrm{RHom}(L, \mathrm{R}\Gamma(U; F)) \simeq \mathrm{R}\Gamma(U; \mathrm{R}\mathscr{H}om(L_X, F)),$$

which is true for any $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ (as recalled later in (4-3)).

This proves that $\text{“colim”}_{x\in U}\, \mathrm{R}\Gamma(U; \mathrm{R}\mathscr{H}om(L_X, F))$ is representable as well as the first isomorphism.

(ii) One has

$$(\mathrm{R}\Gamma_x F) \overset{\mathrm{L}}{\otimes} L \simeq \Big(\text{“}\varinjlim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;\, F)\Big) \overset{\mathrm{L}}{\otimes} L$$
$$\simeq \text{“}\varinjlim_{x\in U}\text{”}(\mathrm{R}\Gamma_c(U;\, F) \overset{\mathrm{L}}{\otimes} L) \simeq \text{“}\varinjlim_{x\in U}\text{”}\, \mathrm{R}\Gamma_c(U;\, F \overset{\mathrm{L}}{\otimes} L_X).$$

The last isomorphism follows from

$$\mathrm{R}\Gamma_c(U;\, F) \overset{\mathrm{L}}{\otimes} L \simeq \mathrm{R}\Gamma_c(U;\, F \overset{\mathrm{L}}{\otimes} L_X),$$

which is true for any $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ (as recalled later in (4-3)).

This proves that “$\varinjlim_{x\in U}$” $\mathrm{R}\Gamma_c(U;\, F \otimes L_X)$ is representable as well as the second isomorphism. □

4. Weakly ℝ-constructible sheaves

The property of being weakly cohomologically constructible is not stable by the six operations. That is why we shall consider weakly ℝ-constructible sheaves instead. Hence, from now on, all manifolds and morphisms of manifolds will be real analytic.

Let X be a real analytic manifold. As already mentioned, we denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$) the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ consisting of weakly ℝ-constructible (resp. ℝ-constructible) complexes on X.

Recall that $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ if and only if its microsupport $\mathrm{SS}(F)$ is contained in a conic subanalytic isotropic subset of T^*X and this is equivalent to the fact that $\mathrm{SS}(F)$ is a conic subanalytic Lagrangian subset.

If $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$, then $F_1 \overset{\mathrm{L}}{\otimes} F_2$ and $\mathrm{R}\mathscr{H}om(F_1, F_2)$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$. Moreover, if $f : X \to Y$ is a morphism of real analytic manifolds and $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$, then $f^{-1}G$ and $f^!G$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$. If $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ and f is proper on $\mathrm{supp}(F)$, then $\mathrm{R}f_! F \xrightarrow{\sim} \mathrm{R}f_* F$ belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$. This follows from [2, Proposition 8.4.6].

Finally, recall (see [2, Section 8.4]) that $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ is ℝ-constructible if for any $x \in X$, F_x is perfect.

The following lemma about the relation between weak ℝ-constructibility and weak cohomological constructibility is well known. Namely, it follows from [2, Lemma 8.4.7] as in the proof of Proposition 8.4.9 in [2].

Lemma 4.1. *Weakly ℝ-constructible sheaves are weakly cohomologically constructible. In other words, the category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ *is a full triangulated subcategory of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{wcc}}(\boldsymbol{k}_X)$.

The following proposition will be an important ingredient in many of the isomorphisms we prove below. We give a direct proof for weakly ℝ-constructible sheaves, but we also refer to [5, Corolarry 2.2.6] for more general results in this direction.

Proposition 4.2. *The functor* $\prod_{x\in X}\mathrm{R}\Gamma_x(\,\cdot\,)\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)\to \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$ *is conservative.*

Proof. Let $F\in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$. We need to show that $\mathrm{R}\Gamma_x F\simeq 0$ for all $x\in X$ implies $F\simeq 0$.

(i) First, assume that F is locally constant. Since the problem is local, we may assume that F is constant, that is, $F=L_X$ for some $L\in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$. Let $d\geq 1$ be the dimension of X. (The statement is clear for $\dim X=0$.) Consider an open ball (in a local chart) $B_{x_0}(\varepsilon)$ centered at x_0 with radius $\varepsilon>0$. Then we consider the distinguished triangle

$$\mathrm{R}\Gamma_{x_0}F\to \mathrm{R}\Gamma_{B_{x_0}(\varepsilon)}F\to \mathrm{R}\Gamma_{B_{x_0}(\varepsilon)\setminus\{x_0\}}F\xrightarrow{+1}.$$

After taking global sections, the second and third objects in this triangle become $\mathrm{R}\Gamma(B_{x_0}(\varepsilon);L_X)\simeq L$ and $\mathrm{R}\Gamma(B_{x_0}(\varepsilon)\setminus\{x_0\};L_X)\simeq L\oplus L[-(d-1)]$, respectively. (Note that $B_{x_0}(\varepsilon)$ is contractible and $B_{x_0}(\varepsilon)\setminus\{x_0\}$ is homotopy equivalent to a sphere S^{d-1}, the boundary of $B_{x_0}(\varepsilon)$.) Therefore, the hypothesis $\mathrm{R}\Gamma_{x_0}L_X\simeq 0$ implies $L\simeq L\oplus L[-(d-1)]$, and hence $L\simeq 0$.

(ii) In the general case, consider a subanalytic stratification $X=\bigsqcup_\alpha X_\alpha$ such that F is locally constant on the strata. Let d be the dimension of X and let us argue by induction on d, assuming the result is proved for manifolds of dimension $d-1$. By (i), $F\simeq 0$ on the open strata. Let Z be a stratum of maximal dimension on which $F|_Z$ is (possibly) not zero. Denote by $j_Z\colon Z\hookrightarrow X$ the embedding and let $x\in Z$. Since F is supported by Z (in a neighborhood of Z), $F\simeq j_{Z*}j_Z^{-1}F$. Then

$$\begin{aligned}\mathrm{R}\Gamma_x F\simeq \mathrm{R}\mathscr{H}om(\boldsymbol{k}_{Xx},F)&\simeq \mathrm{R}\mathscr{H}om(\boldsymbol{k}_{Xx},j_{Z*}j_Z^{-1}F)\\&\simeq j_{Z*}\mathrm{R}\mathscr{H}om(j_Z^{-1}\boldsymbol{k}_{Xx},j_Z^{-1}F)\simeq j_{Z*}\mathrm{R}\Gamma_x(F|_Z).\end{aligned}$$

If $\mathrm{R}\Gamma_x F\simeq 0$, we get $\mathrm{R}\Gamma_x(F|_Z)\simeq 0$. Hence $F\simeq 0$ by the induction hypothesis. □

Inverse images.

Theorem 4.3. *Let* $f\colon X\to Y$ *be a morphism of real analytic manifolds. Let* $M,G\in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$ *and assume that* M *is locally constant. Then*:

(a) $f^{-1}\mathrm{R}\mathscr{H}om(M,G)\xrightarrow{\sim}\mathrm{R}\mathscr{H}om(f^{-1}M,f^{-1}G)$.

(b) $f^!G\overset{\mathrm{L}}{\otimes}f^{-1}M\xrightarrow{\sim}f^!(G\overset{\mathrm{L}}{\otimes}M)$.

Proof. Since the problem is local on Y, we may assume that $M=L_Y$ is the constant sheaf associated with some $L\in\mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$. Hence $f^{-1}L_Y\simeq L_X$.

(a) Let $x\in X$ and set $y=f(x)$. Applying Proposition 3.5, one gets

$$\begin{aligned}\mathrm{R}\mathscr{H}om(L_X,f^{-1}G)_x&\simeq \mathrm{R}\mathscr{H}om(L,(f^{-1}G)_x)\\&\simeq \mathrm{R}\mathscr{H}om(L,G_y)\\&\simeq \mathrm{R}\mathscr{H}om(L_Y,G)_y\simeq (f^{-1}\mathrm{R}\mathscr{H}om(L_Y,G))_x.\end{aligned}$$

(b) Remark first that for any sheaf H on Y, one has $\mathrm{R}\Gamma_x(f^!H) \simeq \mathrm{R}\Gamma_y H$. Then using Proposition 3.5, one has

$$\mathrm{R}\Gamma_x(f^!G \overset{\mathrm{L}}{\otimes} L_X) \simeq (\mathrm{R}\Gamma_x f^!G) \overset{\mathrm{L}}{\otimes} L \simeq (\mathrm{R}\Gamma_y G) \overset{\mathrm{L}}{\otimes} L,$$
$$\mathrm{R}\Gamma_x f^!(G \overset{\mathrm{L}}{\otimes} L_Y) \simeq \mathrm{R}\Gamma_y(G \overset{\mathrm{L}}{\otimes} L_Y) \simeq (\mathrm{R}\Gamma_y G) \overset{\mathrm{L}}{\otimes} L.$$

Set $A = f^!G \overset{\mathrm{L}}{\otimes} L_X$ and $B = f^!(G \overset{\mathrm{L}}{\otimes} L_Y)$. We have proved that the morphism $A \to B$ induces for all $x \in X$ an isomorphism $\mathrm{R}\Gamma_x A \simeq \mathrm{R}\Gamma_x B$. Then $A \simeq B$ by Proposition 4.2. □

Tensor product and hom. We shall make use of the following well-known result. We nonetheless give a proof for the reader's convenience.

Lemma 4.4. *Let $L, M \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$ and let $N \in \mathrm{D}^{\mathrm{b}}_f(\boldsymbol{k})$. Then*

$$\mathrm{RHom}(L, M) \overset{\mathrm{L}}{\otimes} N \xrightarrow{\sim} \mathrm{RHom}(L, M \overset{\mathrm{L}}{\otimes} N).$$

Proof. By hypothesis, we may represent N by a bounded complex of projective modules of finite rank.

(i) Assume first that $N = P$ is concentrated in a single degree. If P is of finite rank, there exists an integer n and an epimorphism $\boldsymbol{k}^n \twoheadrightarrow P$. If moreover P is projective, then this epimorphism has a retract and we get $\boldsymbol{k}^n \simeq P \oplus Q$. This proves the result in this case.

(ii) Now assume that N is represented by the complex $0 \to P^0 \to \cdots \to P^m \to 0$ with all P^j's projective of finite rank. Here we assume for simplicity in the notations that P^0 is in degree 0. Assume that the result is proved for complexes of amplitude $\leq m$. Let us use the so-called "stupid truncation". Denote by N_0 the complex $0 \to P^0 \to \cdots \to P^{m-1} \to 0$ and by $u : N \to N_0$ the natural morphism. We have an exact sequence of complexes $0 \to P^m[-m] \to N \xrightarrow{u} N_0 \to 0$ and it follows that the triangle $P^m[-m] \to N \xrightarrow{u} N_0 \xrightarrow{+1}$ is distinguished. Arguing by induction on m, the proof is complete. □

Let X and Y be real analytic manifolds. As usual, one denotes by q_1 and q_2 the projections from $X \times Y$ to X and Y, respectively. One denotes by $\delta : X \to X \times X$ the diagonal morphism. One denotes by $\overset{\mathrm{L}}{\boxtimes}$ the external tensor product

$$F \overset{\mathrm{L}}{\boxtimes} G := q_1^{-1} F \overset{\mathrm{L}}{\otimes} q_2^{-1} G.$$

Recall [2, Proposition 3.4.4] that for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ and $G \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$, one has

$$\mathrm{D}_X F \overset{\mathrm{L}}{\boxtimes} G \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(q_1^{-1} F, q_2^! G). \tag{4-1}$$

We also have the following isomorphism for $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ and $G_1, G_2 \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$:

$$(F_1 \overset{\mathrm{L}}{\boxtimes} F_2) \overset{\mathrm{L}}{\otimes} (G_1 \overset{\mathrm{L}}{\boxtimes} G_2) \simeq (F_1 \overset{\mathrm{L}}{\otimes} G_1) \overset{\mathrm{L}}{\boxtimes} (F_2 \overset{\mathrm{L}}{\otimes} G_2). \tag{4-2}$$

Theorem 4.5. *Let $K, F_1 \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$, with K locally constant and let $F_2 \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$. Then*:

(a) $\mathrm{R}\mathscr{H}om(K, F_1) \overset{\mathrm{L}}{\otimes} F_2 \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(K, F_1 \overset{\mathrm{L}}{\otimes} F_2)$.

(b) $\mathrm{R}\mathscr{H}om(F_2, F_1) \overset{\mathrm{L}}{\otimes} K \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(F_2, F_1 \overset{\mathrm{L}}{\otimes} K)$.

Proof. We may assume that $K = L_X$ is the constant sheaf associated with $L \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$. The fact that $F_1 \overset{\mathrm{L}}{\otimes} F_2$ and $\mathrm{R}\mathscr{H}om(L_X, F_1 \overset{\mathrm{L}}{\otimes} F_2)$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ follows from [2, Proposition 8.4.6].

(a) Let $x \in X$. One has

$$\begin{aligned}(\mathrm{R}\mathscr{H}om(L_X, F_1) \overset{\mathrm{L}}{\otimes} F_2)_x &\simeq \mathrm{R}\mathscr{H}om(L_X, F_1)_x \overset{\mathrm{L}}{\otimes} (F_2)_x \\ &\simeq \mathrm{RHom}(L, (F_1)_x) \overset{\mathrm{L}}{\otimes} (F_2)_x \\ &\simeq \mathrm{RHom}(L, (F_1)_x \overset{\mathrm{L}}{\otimes} (F_2)_x) \\ &\simeq (\mathrm{R}\mathscr{H}om(L_X, F_1 \overset{\mathrm{L}}{\otimes} F_2))_x.\end{aligned}$$

The second and fourth isomorphisms follow from Proposition 3.5 and the third one from Lemma 4.4.

(b) One has

$$\begin{aligned}\mathrm{R}\mathscr{H}om(F_2, F_1) \overset{\mathrm{L}}{\otimes} L_X &\simeq \delta^!(\mathrm{D}_X F_2 \overset{\mathrm{L}}{\boxtimes} F_1) \overset{\mathrm{L}}{\otimes} \delta^{-1}(\boldsymbol{k}_X \overset{\mathrm{L}}{\boxtimes} L_X) \\ &\simeq \delta^!((\mathrm{D}_X F_2 \overset{\mathrm{L}}{\boxtimes} F_1) \overset{\mathrm{L}}{\otimes} (\boldsymbol{k}_X \overset{\mathrm{L}}{\boxtimes} L_X)) \\ &\simeq \delta^!((\mathrm{D}_X F_2 \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_X) \overset{\mathrm{L}}{\boxtimes} (F_1 \overset{\mathrm{L}}{\otimes} L_X)) \simeq \mathrm{R}\mathscr{H}om(F_2, F_1 \overset{\mathrm{L}}{\otimes} L_X).\end{aligned}$$

Here, the second isomorphism follows from Theorem 4.3(b). The other ones follow from (4-1) and (4-2). □

Direct images. Let $f : X \to Y$ be a morphism of real analytic manifolds. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ and let $M \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ be locally constant. One can ask if the morphism

$$\mathrm{R}f_* F \overset{\mathrm{L}}{\otimes} M \to \mathrm{R}f_*(F \overset{\mathrm{L}}{\otimes} f^{-1}M)$$

is an isomorphism. The answer is negative in general, even if we require F to be constructible, thanks to an example in [1, Remark 4.4].

However, there is a positive answer when considering sheaves *constructible up to infinity*. Before proving the result for general direct images, let us establish it in the particular case of open embeddings.

Lemma 4.6. *Let $j : U \hookrightarrow X$ be the open embedding of a subanalytic relatively compact open subset U of X. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_U)$ and assume that there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ with $j^{-1}G \simeq F$. Let $K \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ be locally constant. Then*:

(a) $\mathrm{R}j_!\mathrm{R}\mathscr{H}om(j^{-1}K, F) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(K, \mathrm{R}j_!F)$.

(b) $\mathrm{R}j_*F \overset{\mathrm{L}}{\otimes} K \xrightarrow{\sim} \mathrm{R}j_*(F \overset{\mathrm{L}}{\otimes} j^{-1}K)$.

Proof. As above, we may assume that $K = L_X$ is the constant sheaf associated with $L \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$. Let $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ be such that $j^{-1}G \simeq F$. Then

$$\mathrm{R}j_*F \simeq \mathrm{R}\Gamma_U G \simeq \mathrm{R}\mathscr{H}om(\boldsymbol{k}_{XU}, G) \quad \text{and} \quad \mathrm{R}j_!F \simeq G_U \simeq \boldsymbol{k}_{XU} \otimes G.$$

(a) Note that $j^{-1}\mathrm{R}\mathscr{H}om(L_X, G) \simeq \mathrm{R}\mathscr{H}om(j^{-1}L_X, F)$. Using Theorem 4.5(a), we get

$$\begin{aligned}\mathrm{R}j_!\mathrm{R}\mathscr{H}om(j^{-1}L_X, F) &\simeq \boldsymbol{k}_{XU} \otimes \mathrm{R}\mathscr{H}om(L_X, G)\\ &\simeq \mathrm{R}\mathscr{H}om(L_X, G \otimes \boldsymbol{k}_{XU}) \simeq \mathrm{R}\mathscr{H}om(L_X, \mathrm{R}j_!F).\end{aligned}$$

(b) Using Theorem 4.5(b), we have

$$\begin{aligned}\mathrm{R}j_*F \overset{\mathrm{L}}{\otimes} L_X &\simeq \mathrm{R}\mathscr{H}om(\boldsymbol{k}_{XU}, G) \overset{\mathrm{L}}{\otimes} L_X\\ &\simeq \mathrm{R}\mathscr{H}om(\boldsymbol{k}_{XU}, G \overset{\mathrm{L}}{\otimes} L_X)\\ &\simeq \mathrm{R}j_*j^{-1}(G \otimes L_X) \simeq \mathrm{R}j_*(F \overset{\mathrm{L}}{\otimes} j^{-1}L_X).\end{aligned}$$ □

Recall the following notions extracted from [4].

Definition 4.7. A b-analytic manifold X_∞ is a pair $(X, \hat{X})$ with $X \subset \hat{X}$ an open embedding of real analytic manifolds such that X is relatively compact and subanalytic in $\hat{X}$. One writes $j_X : X \hookrightarrow \hat{X}$ for the inclusion.

A morphism $f : X_\infty = (X, \hat{X}) \to Y_\infty = (Y, \hat{Y})$ of b-analytic manifolds is a morphism of real analytic manifolds $f : X \to Y$ such that the graph Γ_f of f in $X \times Y$ is subanalytic in $\hat{X} \times \hat{Y}$.

Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$. One says that F is *weakly constructible up to infinity* or simply *weakly b-constructible* if $j_{X!}F$ (or, equivalently, $\mathrm{R}j_{X*}F$) belongs to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{\hat{X}})$. One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_X)$ consisting of weakly b-constructible objects.

Theorem 4.8. *Let $f : X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$ and let $M \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ be locally constant. Then $\mathrm{R}f_!F$ and $\mathrm{R}f_*F$ belong to $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$ and*

(a) $\mathrm{R}f_!\,\mathrm{R}\mathscr{H}om(f^{-1}M, F) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om(M, \mathrm{R}f_!F)$,

(b) $(\mathrm{R}f_*F) \overset{\mathrm{L}}{\otimes} M \xrightarrow{\sim} \mathrm{R}f_*(F \overset{\mathrm{L}}{\otimes} f^{-1}M)$.

Proof. Since the problem is local on Y, we may assume that $M = L_Y$ is the constant sheaf associated with some $L \in \mathrm{D^b}(\boldsymbol{k})$. Set for short $Z := \hat{X} \times \hat{Y}$ and denote by q_1 and q_2 the first and second projection from Z to $\hat{X}$ and $\hat{Y}$, respectively. Denote by $\Gamma_f \subset Z$ the graph of f. Note that Γ_f is subanalytic in Z by definition, and relatively compact in Z since it is contained in the relatively compact subset $X \times Y$.

One has

$$\mathrm{R}f_! F \simeq j_Y^{-1} \mathrm{R}q_{2!}(q_1^{-1} j_{X!} F \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f}), \quad \mathrm{R}f_* F \simeq j_Y^! \mathrm{R}q_{2*} \mathrm{R}\mathscr{H}om(\boldsymbol{k}_{\Gamma_f}, q_1^! \mathrm{R}j_{X*} F).$$

Note that the supports of $q_1^{-1} j_{X!} F \otimes \boldsymbol{k}_{\Gamma_f}$ and $\mathrm{R}\mathscr{H}om(\boldsymbol{k}_{\Gamma_f}, q_1^! \mathrm{R}j_{X*} F)$ are contained in $\overline{\Gamma}_f$ and hence compact in Z.

(a) Let us apply the functor $\mathrm{R}\mathscr{H}om(L_Y, \cdot\,)$ to the first isomorphism. We get

$$\begin{aligned}
\mathrm{R}\mathscr{H}om(L_Y, \mathrm{R}f_! F) &\simeq \mathrm{R}\mathscr{H}om\big(j_Y^{-1} L_{\hat{Y}}, j_Y^{-1} \mathrm{R}q_{2*}(q_1^{-1} j_{X!} F \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f})\big) \\
&\simeq j_Y^{-1} \mathrm{R}\mathscr{H}om\big(L_{\hat{Y}}, \mathrm{R}q_{2*}(q_1^{-1} j_{X!} F \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f})\big) \\
&\simeq j_Y^{-1} \mathrm{R}q_{2*} \mathrm{R}\mathscr{H}om(L_{\hat{X} \times \hat{Y}}, q_1^{-1} j_{X!} F \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f}) \\
&\simeq j_Y^{-1} \mathrm{R}q_{2*}\big(\mathrm{R}\mathscr{H}om(q_1^{-1} L_{\hat{X}}, q_1^{-1} j_{X!} F) \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f}\big) \\
&\simeq j_Y^{-1} \mathrm{R}q_{2*}\big(q_1^{-1} \mathrm{R}\mathscr{H}om(L_{\hat{X}}, j_{X!} F) \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f}\big) \\
&\simeq j_Y^{-1} \mathrm{R}q_{2*}\big(q_1^{-1} j_{X!} \mathrm{R}\mathscr{H}om(L_X, F) \overset{\mathrm{L}}{\otimes} \boldsymbol{k}_{\Gamma_f}\big) \\
&\simeq \mathrm{R}f_! \, \mathrm{R}\mathscr{H}om(f^{-1} L_Y, F).
\end{aligned}$$

The second and fifth isomorphisms use Theorem 4.3(a), the fourth isomorphism uses Theorem 4.5(a), and the sixth isomorphism uses Lemma 4.6(a). On the other hand, the third isomorphism is classical (see [2, (2.6.15)]).

(b) The proof is completely analogous, using parts (b) of the statements mentioned above instead. □

Recall the following isomorphisms which hold for any $F \in \mathrm{D^b}(\boldsymbol{k}_X)$, $L \in \mathrm{D^b}(\boldsymbol{k})$ and any open $U \subset X$. One has

$$\begin{gathered}
\mathrm{R}\Gamma_c(U; F \overset{\mathrm{L}}{\otimes} L_X) \xrightarrow{\sim} \mathrm{R}\Gamma_c(U; F) \overset{\mathrm{L}}{\otimes} L, \\
\mathrm{R}\Gamma(U; \mathrm{R}\mathscr{H}om(L_X, F)) \simeq \mathrm{RHom}(L, \mathrm{R}\Gamma(U; F)).
\end{gathered} \tag{4-3}$$

Corollary 4.9. *Let $F \in \mathrm{D^b_{w\mathbb{R}c}}(\boldsymbol{k}_X)$ and let $L \in \mathrm{D^b}(\boldsymbol{k})$. Let U be an open relatively compact subanalytic subset of X. Then*:

(a) $\mathrm{R}\Gamma_c(U; \mathrm{R}\mathscr{H}om(L_X, F)) \xrightarrow{\sim} \mathrm{RHom}(L, \mathrm{R}\Gamma_c(U; F))$.

(b) $\mathrm{R}\Gamma(U, F \overset{\mathrm{L}}{\otimes} L_X) \simeq \mathrm{R}\Gamma(U; F) \overset{\mathrm{L}}{\otimes} L$.

Proof. Apply Theorem 4.8 to the sheaf $F|_U$ with $X_\infty = (U, X)$, $Y_\infty = (\mathrm{pt}, \mathrm{pt})$ and $f = a_U$. □

In particular, when $X_\infty = (X, \hat{X})$ is b-analytic and $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$, one obtains

$$\mathrm{R}\Gamma_c(X; \mathrm{R}\mathscr{H}om(L_X, F)) \xrightarrow{\sim} \mathrm{RHom}(L, \mathrm{R}\Gamma_c(X; F)),$$
$$\mathrm{R}\Gamma(X; F \overset{\mathrm{L}}{\otimes} L_X) \simeq \mathrm{R}\Gamma(X; F) \overset{\mathrm{L}}{\otimes} L.$$

Duality. From our above result, we obtain slight generalizations of well-known statements about the behavior of the duality functor with respect to direct and inverse images.

For any $G \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ and a continuous $f : X \to Y$, one has $\mathrm{D}_X f^{-1} G \simeq f^! \mathrm{D}_Y G$. In general, the isomorphism $\mathrm{D}_X f^! G \simeq f^{-1} \mathrm{D}_Y G$ does not hold, but it holds if f is a morphism of real analytic manifolds and $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$ (see [2, Exercise VIII.3(ii)]). The following corollary generalizes this statement.

Corollary 4.10. *Let $f : X \to Y$ be a morphism of real analytic manifolds. Let $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$ and assume that $M \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ is locally constant. Then*

$$\mathrm{D}_X f^!(G \overset{\mathrm{L}}{\otimes} M) \simeq f^{-1} \mathrm{D}_Y(G \overset{\mathrm{L}}{\otimes} M).$$

Proof. One has the chain of isomorphisms

$$\begin{aligned} \mathrm{D}_X f^!(G \overset{\mathrm{L}}{\otimes} M) &\simeq \mathrm{R}\mathscr{H}om(f^! G \overset{\mathrm{L}}{\otimes} f^{-1} M, \omega_X) \\ &\simeq \mathrm{R}\mathscr{H}om(f^{-1} M, \mathrm{D}_X(f^! G)) \\ &\simeq \mathrm{R}\mathscr{H}om(f^{-1} M, f^{-1} \mathrm{D}_Y G) \\ &\simeq f^{-1} \mathrm{R}\mathscr{H}om(M, \mathrm{D}_Y G) \simeq f^{-1} \mathrm{D}_Y(G \overset{\mathrm{L}}{\otimes} M). \end{aligned}$$

Here, we have used Theorem 4.3(b) in the first isomorphism. Note also that the third isomorphism uses the classical fact (as mentioned above) that $\mathrm{D}_X f^! G \simeq f^{-1} \mathrm{D}_Y G$ for $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_Y)$. □

Recall that for any $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$, one has $\mathrm{D}_Y \mathrm{R}f_! F \simeq \mathrm{R}f_* \mathrm{D}_X F$. On the contrary, the isomorphism $\mathrm{R}f_! \mathrm{D}_X F \simeq \mathrm{D}_Y \mathrm{R}f_* F$ does not hold in general. However, it holds under the assumption that f is a morphism of real analytic manifolds and that F and $\mathrm{R}f_! \mathrm{D}_X F$ are $\mathbb{R}$-constructible complexes (see [2, Exercise VIII.3(iii)]). In particular, this means that it holds if $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$, since the six functors preserve $\mathbb{R}$-constructibility in the b-analytic setting (see [4, Corolarry 2.13]). The following corollary is a generalization of the latter statement.

Corollary 4.11. *Let $f : X_\infty \to Y_\infty$ be a morphism of b-analytic manifolds. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$ and let $M \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_Y)$ be locally constant. Then*

$$\mathrm{R}f_! \mathrm{D}_X(F \overset{\mathrm{L}}{\otimes} f^{-1} M) \simeq \mathrm{D}_Y(\mathrm{R}f_* F \overset{\mathrm{L}}{\otimes} M).$$

Proof. One has the chain of isomorphisms

$$\begin{aligned} \mathrm{R}f_!\mathrm{D}_X(F \overset{\mathrm{L}}{\otimes} f^{-1}M) &\simeq \mathrm{R}f_!\,\mathrm{R}\mathscr{H}om(f^{-1}M, \mathrm{D}_X F) \\ &\simeq \mathrm{R}\mathscr{H}om(M, \mathrm{R}f_!\mathrm{D}_X F) \\ &\simeq \mathrm{R}\mathscr{H}om(M, \mathrm{D}_Y \mathrm{R}f_* F) \simeq \mathrm{D}_Y(\mathrm{R}f_* F \overset{\mathrm{L}}{\otimes} M). \end{aligned}$$

Here, we have used Theorem 4.8(a) in the second isomorphism. Note that in the third isomorphism we have used the fact that $\mathrm{R}f_!\mathrm{D}_X F \simeq \mathrm{D}_Y \mathrm{R}f_* F$ for $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\boldsymbol{k}_{X_\infty})$, as mentioned above. ☐

Field extensions and microsupport. Consider the case where $\boldsymbol{k}$ is a field and $\boldsymbol{l}$ is a field extension of $\boldsymbol{k}$. Then, of course, all preceding results apply in particular with $K = \boldsymbol{l}_X$ and $M = \boldsymbol{l}_Y$, i.e., taking the constant sheaf with stalk $\boldsymbol{l}$ as the locally constant sheaves in the above statements. This is the case of interest in [1], where extension of scalars for sheaves of vector spaces is studied. Our above results apply, however, to more general situations, since we put weaker restrictions on the objects involved, as described in the introduction.

Remark 5.1.5 in [2] asserts that if *for* denotes the forgetful functor

$$for : \mathrm{D}^{\mathrm{b}}(\boldsymbol{l}_X) \to \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$$

and if $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{l}_X)$, then the microsupport of F and that of $for(F)$ are the same. However, this result does not mean that for $F \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$, $\mathrm{SS}(F) = \mathrm{SS}(F \otimes \boldsymbol{l}_X)$. We shall discuss this last point now.

Remark first that for $F, K \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ with K locally constant, one has

$$\mathrm{SS}(K \overset{\mathrm{L}}{\otimes} F) \subset \mathrm{SS}(F), \quad \mathrm{SS}(\mathrm{R}\mathscr{H}om(K, F)) \subset \mathrm{SS}(F).$$

Indeed, by [2, Proposition 5.4.14] one has

$$\mathrm{SS}(K \overset{\mathrm{L}}{\otimes} F) \subset T^*_X X + \mathrm{SS}(F) = \mathrm{SS}(F)$$

and similarly with $\mathrm{R}\mathscr{H}om(K, F)$.

Finally, let us state the following useful result, well known to specialists, which says that the converse inclusion is true over a field.

Proposition 4.12. *Assume that $\boldsymbol{k}$ is a field. Let $F, K \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_X)$ with K locally constant, X connected and $K \neq 0$. Then*

$$\mathrm{SS}(K \otimes F) = \mathrm{SS}(F), \quad \mathrm{SS}(\mathrm{R}\mathscr{H}om(K, F)) = \mathrm{SS}(F).$$

Proof. The problem is local and we may assume that $K = L_X$ is the constant sheaf associated with $L \in \mathrm{D}^{\mathrm{b}}(\boldsymbol{k})$. Then $L \simeq \oplus_j H^j(L)\,[-j]$. Since the microsupport of a finite direct sum is the union of microsupports of the summands (which is a direct consequence of the definition of the microsupport [2, Definition 5.1.2]), we may assume that $L \in \mathrm{Mod}(\boldsymbol{k})$. In this case, there exists $L' \in \mathrm{Mod}(\boldsymbol{k})$ such that $L \simeq \boldsymbol{k} \oplus L'$ (since every vector space has a basis), and hence $L_X \simeq \boldsymbol{k}_X \oplus L'_X$. Therefore, $L_X \otimes F \simeq F \oplus G$ with $G = L'_X \otimes F$. Since again $\mathrm{SS}(F \oplus G) = \mathrm{SS}(F) \cup \mathrm{SS}(G)$, the result for $L_X \otimes F$ follows. The case of $\mathrm{R}\mathscr{H}om(L_X, F)$ is similar. $\square$

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